1. *(40 points - 8 each)* Evaluate these definite and indefinite integrals.

(a) $\int xe^{3x}\,dx$

$u = x \qquad dv = e^{3x}dx$

$du = dx \qquad v = \frac{1}{3}e^{3x}$

$= \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx = \boxed{\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C}$

(b) $\frac{1}{2}\int_1^{\sqrt{2}} 2x(x^2-1)^4\,dx$

$u = x^2 - 1 \qquad du = 2x\,dx$

If $x = 1$, then $u = 0$

If $x = \sqrt{2}$, then $u = 1$.

$= \frac{1}{2}\int_0^1 u^4\,du = \frac{1}{10}u^5\Big|_0^1 = \frac{1}{10}(1)^5 - \frac{1}{10}(0)^5$

$= \boxed{\frac{1}{10}}$

(c) $\int_0^{\pi/6} \sec^4(2t)\tan^3(2t)\,dt = \int_0^{\pi/6} \tan^3(2t)\cdot\sec^2(2t)\left[\sec^2(2t)\,dt\right]$

$= \frac{1}{2}\int_0^{\pi/6} \tan^3(2t)\cdot\left[\tan^2(2t)+1\right]\left[2\sec^2(2t)\,dt\right] \qquad u = \tan(2t)$

$du = 2\sec^2(2t)\,dt$

$t = 0 \Rightarrow u = 0$

$t = \frac{\pi}{6} \Rightarrow u = \sqrt{3}$

$= \frac{1}{2}\int_0^{\sqrt{3}} u^3[u^2+1]\,du = \frac{1}{2}\int_0^{\sqrt{3}}(u^5+u^3)\,du$

$= \frac{1}{12}u^6 + \frac{1}{8}u^4\Big|_0^{\sqrt{3}} = \frac{1}{12}\left((\sqrt{3})^2\right)^3 + \frac{1}{8}\left((\sqrt{3})^2\right)^2 = \frac{27}{12} + \frac{9}{8}$

$= \frac{9}{4} + \frac{9}{8} = \frac{18}{8} + \frac{9}{8} = \boxed{27/8}$

(Method 2)

$$\int_0^{\pi/6} \sec^4(2t)\cdot\tan^3(2t)\,dt = \frac{1}{2}\int_0^{\pi/6} \sec^3(2t)\tan^2(2t)\left[2\sec(2t)\tan(2t)\right]dt$$

$u = \sec(2t) \qquad du = 2\sec(2t)\tan(2t)\,dt$

If $x=0$, then $u=1$. If $x=\frac{\pi}{6}$, then $u=2$.

$$= \frac{1}{2}\int_0^{\pi/6} \sec^3(2t)\cdot(\sec^2(2t)-1)\left[2\sec(2t)\tan(2t)\right]dt$$

$$= \frac{1}{2}\int_1^2 u^3(u^2-1)\,du = \frac{1}{2}\int_1^2 (u^5-u^3)\,du = \frac{1}{12}u^6 - \frac{1}{8}u^4\Big|_1^2$$

$$= \left(\frac{2^6}{12} - \frac{16}{8}\right) - \left(\frac{1}{12} - \frac{1}{8}\right) = \frac{64}{12} - 2 - \frac{1}{12} + \frac{1}{8}$$

$$= \frac{21}{4} - 2 + \frac{1}{8} = \frac{42}{8} - \frac{16}{8} + \frac{1}{8} = \boxed{\frac{27}{8}}$$

(d) $\displaystyle\int \frac{5x-2}{(x+2)(x-2)}\,dx$

$$\frac{5x-2}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$5x-2 = A(x-2) + B(x+2)$$

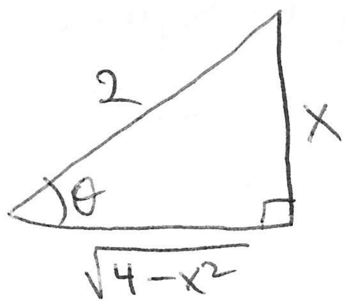
$x=2)\quad 8 = 4B \quad \boxed{B=2}$

$x=-2)\quad -12 = -4A \quad \boxed{A=3}$

$$\int \frac{5x-2}{(x+2)(x-2)}dx = 3\int \frac{dx}{x+2} + 2\int \frac{dx}{x-2}$$

$$= \boxed{3\ln|x+2| + 2\ln|x-2| + C}$$

(e) $\displaystyle\int \sqrt{4-x^2}\,dx$



$2\sin\theta = x$

$2\cos\theta\, d\theta = dx$

$2\cos\theta = \sqrt{4-x^2}$

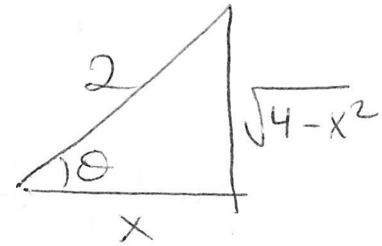
$$= \int (2\cos\theta)(2\cos\theta\, d\theta) = 4\int \cos^2\theta\, d\theta = 4\int \frac{1+\cos(2\theta)}{2}d\theta$$

$$= 2\theta + 2\left(\frac{\sin(2\theta)}{2}\right) = 2\theta + 2\sin\theta\cos\theta$$

$$= 2\arcsin\left(\frac{x}{2}\right) + 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)$$

$$= \boxed{2\arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C}$$

(Method 2) $\int \sqrt{4-x^2}\,dx$



$2\cos\theta = x \qquad 2\sin\theta = \sqrt{4-x^2}$

$-2\sin\theta\,d\theta = dx$

$$\int \sqrt{4-x^2}\,dx = -4\int \sin^2\theta\,d\theta = -4\int \frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= -2\left[\theta - \frac{\sin(2\theta)}{2}\right] = -2\theta + 2\sin\theta\cos\theta$$

$$= -2\arccos\left(\frac{x}{2}\right) + 2\left(\frac{\sqrt{4-x^2}}{2}\right)\left(\frac{x}{2}\right)$$

$$= \boxed{-2\arccos\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C}$$

2. *(10 points - 5 each)* The velocity function $v(t) = 2\cos\left(\dfrac{\pi t}{6}\right)$ is given for a particle moving (back and forth) on a straight line. Assume $v(t)$ is measured in meters per second. Find the following:

(a) The displacement of the particle between time $t = 0$ and time $t = 4$.

$$2\int_0^4 \cos(\pi t/6)\,dt = \frac{12}{\pi}\sin\left(\frac{\pi t}{6}\right)\Big|_0^4$$

$$= \frac{12}{\pi}\sin\left(\frac{2\pi}{3}\right) - 0 = \frac{12}{\pi}\frac{\sqrt{3}}{2} = \boxed{\frac{6\sqrt{3}}{\pi}\ m}$$

(b) The total distance traveled by the particle between time $t = 0$ and time $t = 4$.

$$\int_0^4 |2\cos(\pi t/6)|\,dt = 2\int_0^3 \cos(\pi t/6)\,dt - 2\int_3^4 \cos\left(\frac{\pi t}{6}\right)dt$$

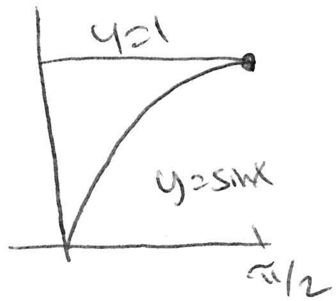
$$= \frac{12}{\pi}\sin\left(\frac{\pi t}{6}\right)\Big|_0^3 - \frac{12}{\pi}\sin\left(\frac{\pi t}{6}\right)\Big|_3^4$$

$$= \left[\frac{12}{\pi} - 0\right] - \left[\frac{6\sqrt{3}}{\pi} - \frac{12}{\pi}\right]$$

$$= \frac{12}{\pi} - \frac{6\sqrt{3}}{\pi} + \frac{12}{\pi} = \boxed{\frac{24 - 6\sqrt{3}}{\pi}\ m}$$

3. *(20 points)* Let $\mathcal{R}$ be the region bound by $y = \sin x, y = 1$ and $x = 0$.

(a) *(5 points)* Sketch the region $\mathcal{R}$ and compute its area.



$$\int_0^{\pi/2} (1-\sin x)\,dx = x + \cos(x)\Big|_0^{\pi/2} =$$

$$\left[\frac{\pi}{2}+0\right]-\left[0+1\right] = \boxed{\frac{\pi}{2}-1}$$

(b) *(15 points)* Using any method that you want, set up integrals (but do **NOT** evaluate) to find:

i. the volume of the solid obtained by revolving $\mathcal{R}$ about the x-axis.

Washers (or) Shells

$$\pi\int_0^{\pi/2}[1^2-\sin^2 x]\,dx \qquad 2\pi\int_0^1 y\cdot\arcsin(y)\,dy$$

ii. the volume of the solid obtained by revolving $\mathcal{R}$ about the y-axis.

$$\pi\int_0^1 (\arcsin(y))^2\,dy \quad \text{(or)} \quad 2\pi\int_0^{\pi/2} x(1-\sin(x))\,dx$$

iii. the volume of the solid obtained by revolving $\mathcal{R}$ about $y = 1$.

$$\pi\int_0^{\pi/2}(1-\sin x)^2\,dx \quad \text{(or)} \quad 2\pi\int_0^1 (1-y)\arcsin(y)\,dy$$

(either answer is acceptable)

4. *(10 points - 5 each)* **Differentiate** the following functions.

(a) $F(x) = \int_{-1}^{x} \ln(5t-1)\ dt$

$$\frac{d}{dx}F(x) = \boxed{\ln(5x-1)}$$

(b) $G(x) = \int_{0}^{x^2} \frac{3t}{2+\sin(t^2)} dt$ Let $g(x) = \int_0^x \frac{3t}{2+\sin(t^2)} dt$

Then $G(x) = g(x^2)$, so

$$\frac{d}{dx}G(x) = \frac{d}{dx}g(x^2) = g'(x^2)\cdot 2x = \frac{3x^2}{2+\sin(x^4)}\cdot 2x$$

$$= \boxed{\frac{6x^3}{2+\sin(x^4)}}$$

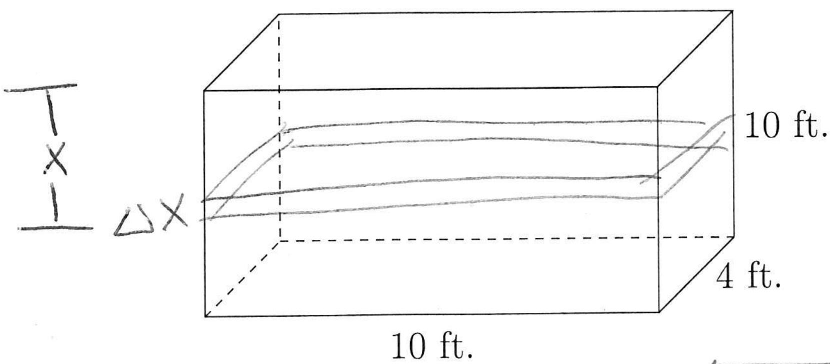
5. *(5 points)* Does $\int_1^\infty \frac{1}{x^2}\, dx$ converge? If so, compute its value.

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t$$

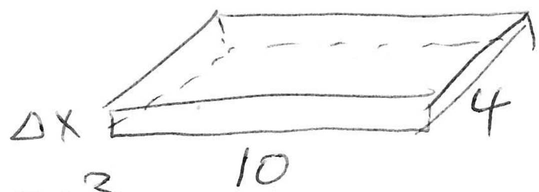
$$= \lim_{t\to\infty} 1 - \frac{1}{t} = \boxed{1}$$

Yes, it converges to 1

6. *(10 points)* A large aquarium is in the shape of a rectangular box as shown below. The aquarium is full of water but is filthy and needs to be cleaned. Find the amount of work needed to empty the aquarium (use the fact that water weights 62.5 lbs/cubic foot and that 62.5 = 125/2).



A Slice looks like



$$V_{slice} = 40\,\Delta x \text{ ft}^3$$

$$F_{slice} = \frac{125}{2} \cdot 40\,\Delta x = 125 \cdot 20\,\Delta x \text{ lbs}$$

$$D_{slice} = x \qquad W_{slice} = 125 \cdot 20 \cdot x\,\Delta x$$

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} W_{slice} = \int_0^{10} 125 \cdot 20 x\, dx$$

$$= 125 \cdot \frac{20}{2} x^2 \Big|_0^{10} = 1250 x^2 \Big|_0^{10}$$

$$= 1250 \cdot 100 = \boxed{125{,}000 \text{ ft-lbs}}$$

7. *(5 points)* Suppose $\int_0^5 f(x)\,dx = 5$, $\int_0^8 g(x)\,dx = 10$ and $\int_5^8 g(x)\,dx = 3$.

Compute $\int_0^5 \Big(2f(x) + 3g(x)\Big)\,dx$.

$$= 2\int_0^5 f(x)\,dx + 3\int_0^5 g(x)\,dx = 2[5] + 3\left[\int_0^8 g(x)\,dx - \int_5^8 g(x)\,dx\right] = 10 + 3[10-3]$$

$$= 10 + 21 = \boxed{31}$$

8. **Extra Credit** *(5 points)* Compute the volume of **one** of the solids from problem 3b. You can pick any one of the three, but indicate which one you have chosen. (The integral must be set up correctly for credit)

i $\pi\int_0^{\pi/2}(1-\sin^2 x)\,dx = \pi\int_0^{\pi/2}\cos^2 x\,dx = \pi\int_0^{\pi/2}\frac{1+\cos(2x)}{2}\,dx$

$$= \frac{\pi}{2}\left[x + \frac{\sin(2x)}{2}\right]\Big|_0^{\pi/2} = \frac{\pi}{2}\left[\left(\frac{\pi}{2}+0\right)-(0)\right] = \boxed{\frac{\pi^2}{4}}$$

ii $2\pi\int_0^{\pi/2} x(1-\sin(x))\,dx = 2\pi\left[x(x+\cos(x))\Big|_0^{\pi/2} - \int_0^{\pi/2}(x+\cos(x))\,dx\right]$

$u = x \quad dv = (1-\sin x)\,dx$

$du = dx \quad v = x + \cos(x)$

$$= 2\pi\left[\frac{\pi^2}{4} - \left[\frac{1}{2}x^2 + \sin(x)\right]\Big|_0^{\pi/2}\right]$$

$$= 2\pi\left[\frac{\pi^2}{4} - \frac{\pi^2}{8} - 1\right] = 2\pi\left[\frac{\pi^2}{8} - 1\right] = \boxed{\frac{\pi^3}{4} - 2\pi}$$

iii $\pi\int_0^{\pi/2}(1 - 2\sin(x) + \sin^2 x)\,dx = \pi\int_0^{\pi/2}\left(1 - 2\sin(x) + \frac{1-\cos(2x)}{2}\right)dx$

$$= \pi\left[\frac{3x}{2} + 2\cos(x) - \frac{\sin(2x)}{4}\right]\Big|_0^{\pi/2} = \pi\left[\frac{3\pi}{4} - 2\right] = \boxed{\frac{3\pi^2}{4} - 2\pi}$$