Math 19B Quiz: August 24, 2012 Solutions

Show all work and make sure to give exact answers. Good luck, and enjoy your weekend!

Determine if the series converge or diverge and **justify** your answer (5 points each).
Extra Credit: Determine the value of any series which converges (2 points each).

The first bullet point lists the solution I intended (so this quiz would just cover 11.2 and 11.3) but I have included all possible solutions (and why other tests don't apply).

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n} = \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots$

- This is a **Geometric Series** with $a = 1/3$ and $r = -1/3$. Since $|r| < 1$ this series converges to
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n} = \frac{a}{1-r} = \frac{1/3}{1-(-1/3)} = \frac{1/3}{4/3} = \frac{1}{4}$$
Note that this method is the only way to find the value that this series converges to, but not the only way to show convergence.
- This is an **Alternating Series**, so we can apply the alternating series test.
  (a) $a_{n+1} \leq a_n$ (where $a_n = 1/3^n$). Since $3^{n+1} = 3 \cdot 3^n$,
  $$\frac{1}{3^{n+1}} = \frac{1}{3} \cdot \frac{1}{3^n} < \frac{1}{3^n}$$
  which shows that $\{a_n\}$ is a decreasing sequence.
  (b) It is clear that $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1}}{3^n} = 0$.
  Hence by the alternating series test this series converges.
- Using the **Roots Test**,
$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^{n+1}}{3^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{3}\right)^n} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3}$$
Since $L < 1$ we know the series converges by the Roots Test.
- Using the **Ratio Test**,
$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}}{3^{n+1}} \cdot \frac{3^n}{(-1)^{n+1}}\right| = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3}$$
Since $L < 1$ we know the series converges by the Ratio Test.
- Since $\displaystyle\lim_{n\to\infty} a_n = 0$, the divergence test does not apply. Since $(-1)^{n+1}/3^n$ is not positive, the integral test and comparison tests do not apply.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

- This looks like a function that we know how to integrate, so lets try the **Integral Test** (with $f(x) = \frac{1}{1+x^2}$). To use the integral test we need to know that f is continuous, positive, and decreasing. The first two conditions are clearly true. Observe that

$$f'(x) = \frac{-2x}{(1+x^2)^2} < 0 \text{ when } x \geq 1$$

  which shows that f is decreasing. Hence we can use the integral test.

$$\int_1^{\infty} \frac{dx}{1+x^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \arctan(x)\Big|_1^t = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

  Since the integral converges, we know that the series converges. We know that the value of the series is between 0 and $\pi/4$ but can't find its value.

- This looks similar to a p-series which is one of our building blocks. Hence we should consider a **comparison** to $\sum 1/n^2$. We can use a comparison since both are positive series. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

  the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges. Note the direction of the above inequality. We could not make this argument if the direction were reversed.

- Since this series and the series $\sum 1/n^2$ both have positive terms we can also use a **limit comparison**. Observe that

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^2+1} \cdot \frac{n^2}{1} = 1$$

  Since c is finite and $c > 0$ the limit comparison tells us these series do the same thing. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, p=2) meaning $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

- The test for divergence does not apply since $\lim\limits_{n\to\infty} a_n = 0$. The ratio test can be applied but gives $L = 1$ (which is inconclusive). This is not an alternating series (so that does not apply) and since all the terms are positive there is no reason to consider $\sum |a_n|$ (since it is the same thing).

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

- This looks like a function that we know how to integrate, so lets try the **Integral Test** (with $f(x) = \frac{x}{1+x^2}$). To use the integral test we need to know that f is continuous, positive, and decreasing. The first two conditions are clearly true. Observe that

$$f'(x) = \frac{1 \cdot (1+x^2) - 2x \cdot x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} < 0 \text{ when } x > 1$$

  which shows that f is decreasing. Hence we can use the integral test.

$$\int_1^{\infty} \frac{x}{1+x^2}\, dx = \lim_{t\to\infty} \int_1^t \frac{x}{1+x^2}\, dx = \lim_{t\to\infty} \frac{1}{2} \ln(1+x^2)\Big|_1^t = \infty$$

  Since the integral diverges, we know that the series diverges.

- This looks similar to a p-series which is one of our building blocks. Hence we should consider a comparison to $\sum 1/n$. While both are positive series, since the harmonic series diverges, so the inequality

$$\frac{1}{1+n} < \frac{1}{n}$$

  is in the wrong direction to prove divergence. Hence a normal comparison will not work. However, we can still use a **limit comparison**. Observe that

$$c = \lim_{nto\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1+n} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

  Since c is finite and $c > 0$ the limit comparison says both series do the same thing. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (it's the harmonic series, or equivalently a p-series, p=1) so by limit comparison we know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n}$ diverges.

- The test for divergence does not apply since $\lim\limits_{n\to\infty} a_n = 0$. The ratio test can be applied but gives $L = 1$ (which is inconclusive). This is not an alternating series (so that does not apply) and since all the terms are positive there is no reason to consider $\sum |a_n|$ (since it is the same thing).

4. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$

- This is a **Geometric Series** with $a = 3/2$ and $r = 3/2$. Since $|r| \geq 1$ it diverges.
- Since $\lim\limits_{n\to\infty}(3/2)^n \neq 0$ this series diverges by the **Divergence Test**.
- Since this is of the form $\sum(a_n)^n$ we can use the **Roots Test**.

$$L = \lim_{n\to\infty}\sqrt[n]{\left(\frac{3}{2}\right)^n} = \lim_{n\to\infty}\frac{3}{2} = \frac{3}{2} > 1$$

  meaning that the series diverges.
- We can also use the **Ratio Test**.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{3}{2}\right)^{n+1}\cdot\left(\frac{2}{3}\right)^n = \lim_{n\to\infty}\frac{3}{2} = \frac{3}{2} > 1$$

  meaning that the series diverges.
- The Integral Test does not apply since $(3/2)^n$ is not decreasing. It would be silly to do a comparison since this is already one of our building blocks. This is not an alternating series (so that does not apply) and since all the terms are positive there is no reason to consider $\sum|a_n|$ (since it is the same thing).