1. *(40 points - 10 each)* Evaluate the following integrals:

(a) $\frac{1}{2}\int_0^{\frac{\sqrt{\pi}}{2}} 2x\cos(x^2)\,dx$

$u = x^2$ $\quad$ $x = \frac{\sqrt{\pi}}{2} \Rightarrow u = \frac{\pi}{4}$

$du = 2x\,dx$ $\quad$ $x = 0 \Rightarrow u = 0$

$$= \frac{1}{2}\int_0^{\pi/4} \cos(u)\,du = \frac{1}{2}\sin(u)\Big|_0^{\pi/4}$$

$$= \frac{1}{2}\frac{\sqrt{2}}{2} - 0 = \boxed{\frac{\sqrt{2}}{4}}$$

(b) $\int x \sin x\,dx$

$u = x$ $\quad$ $dv = \sin x\,dx$

$du = dx$ $\quad$ $v = -\cos x$

$$= -x\cdot\cos(x) + \int \cos x\,dx$$

$$= \boxed{-x\cdot\cos(x) + \sin(x) + C}$$

(c) $\int \left[t^3 - e^t + \cos(2t) + \frac{1}{t} - \frac{1}{1+t^2}\right] dt$

$$= \boxed{\frac{1}{4}t^4 - e^t + \frac{\sin(2t)}{2} + \ln|t| - \arctan(t) + C}$$

(d) $\int_0^{\frac{\pi}{3}} \sin^2\theta \, d\theta$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$= \int_0^{\pi/3} \left(\frac{1}{2} - \frac{\cos(2\theta)}{2}\right) d\theta = \frac{1}{2}\theta - \frac{\sin(2\theta)}{4}\Big|_0^{\pi/3}$$

$$= \left[\frac{\pi}{6} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right] - [0-0]$$

$$= \boxed{\frac{\pi}{6} - \frac{\sqrt{3}}{8}}$$

2. *(30 points - 15 each)* Evaluate the following integrals:

(a) $\displaystyle\int \frac{5x^2+x+3}{x^3+x}\,dx$

$$\frac{5x^2+x+3}{(x^2+1)x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

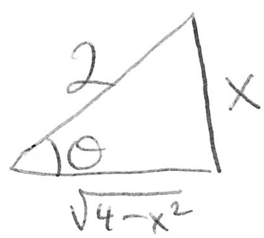
$$\begin{aligned} 5x^2+x+3 &= A(x^2+1) + (Bx+C)x \\ &= Ax^2 + A + Bx^2 + Cx \\ &= (A+B)x^2 + Cx + A \end{aligned}$$

$A=3,\ C=1,\ B=2$

$$\int\left(\frac{3}{x} + \frac{2x+1}{x^2+1}\right)dx = \int \frac{3}{x}\,dx + \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \boxed{3\ln|x| + \ln|x^2+1| + \arctan(x) + C}$$

(b) $\displaystyle\int \frac{x^2}{\left(\sqrt{4-x^2}\right)^3}\,dx$



$2\sin\theta = x$

$2\cos\theta\,d\theta = dx$

$2\cos\theta = \sqrt{4-x^2}$

$$\int \frac{4\sin^2\theta}{8\cos^3\theta}\cdot 2\cos\theta\,d\theta = \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$$

$$= \int(\sec^2\theta - 1)\,d\theta = \tan\theta - \theta$$

$$= \boxed{\frac{x}{\sqrt{4-x^2}} - \arcsin\left(\frac{x}{2}\right) + C}$$

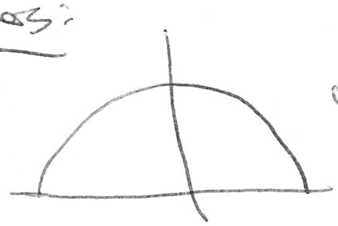
3. *(20 points)* Using integration (either the method of washers or the method of shells), verify that the volume of a sphere with radius r is

$$V = \frac{4}{3}\pi r^3$$

Hint: The equation of a circle with radius r, centered at the origin is

$$x^2 + y^2 = r^2$$

Washers:



$y = \sqrt{r^2 - x^2}$

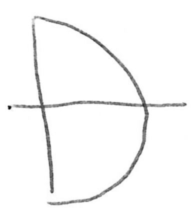
$$V = \pi \int_{-r}^{r} \left(\sqrt{r^2-x^2}\right)^2 dx = \pi \int_{-r}^{r} (r^2 - x^2)\,dx$$

$$= 2\pi \int_0^r (r^2-x^2)\,dx = 2\pi \left[r^2x - \tfrac{1}{3}x^3\right]\Big|_0^r$$

$$= 2\pi\left[r^3 - \tfrac{1}{3}r^3\right] = 2\pi\left[\tfrac{2}{3}r^3\right] = \tfrac{4}{3}\pi r^3$$

Shells:

$R = x \quad H = 2\sqrt{r^2-x^2}$



$$V = 2\pi \int_0^r x \cdot 2\sqrt{r^2-x^2}\,dx$$

$u = r^2 - x^2$

$du = -2x\,dx$

$$= -2\pi \int_{r^2}^{0} \sqrt{u}\,du = +2\pi \int_0^{r^2} \sqrt{u}\,du$$

$$= 2\pi \cdot \tfrac{2}{3} u^{\frac{3}{2}}\Big|_0^{r^2} = \tfrac{4}{3}\pi (r^2)^{3/2} = \tfrac{4}{3}\pi r^3$$

4. (a) *(10 points)* Show that $\int_1^\infty \frac{x}{e^x}\,dx$ converges by finding its value.

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$\int_1^\infty xe^{-x}dx = \lim_{t\to\infty}\int_1^t xe^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}x\Big|_1^t + \int_1^t e^{-x}dx\right]$$

$$= \lim_{t\to\infty}\left[-e^{-t}\cdot t + \frac{1}{e} + -e^{-x}\Big|_1^t\right]$$

$$= \frac{1}{e} + \lim_{t\to\infty}\left[-e^{-t} + \frac{1}{e}\right] = \frac{1}{e} + \frac{1}{e} = \boxed{\frac{2}{e}}$$

(b) *(10 points)* Show that $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges.

Since we ~~first~~ just found the integral $\int_1^\infty f(x)dx$ ($f(x) = \frac{x}{e^x}$) the integral test seems like a good test. Can we use it? $f(x)$ is clearly positive and continuous. Observe that

$$f'(x) = \frac{e^x - xe^x}{(e^x)^2} = \frac{e^x(1-x)}{(e^x)^2} = \frac{1-x}{e^x} < 0 \text{ when } x > 1.$$

Meaning f is decreasing. Since $\int_1^\infty f(x)dx$ converges, So does $\sum_{n=1}^{\infty} \frac{n}{e^n}$.

(c) *(Extra Credit, 5 points)* Find the sum of the infinite series in (b).

$$\sum_{n=1}^{\infty} \frac{n}{e^n} = \frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \frac{4}{e^4} + \cdots$$

$$= \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} + \cdots = \frac{1/e}{(1-1/e)} = \frac{1}{e-1}$$

$$+ \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} + \cdots = \frac{1/e^2}{(1-1/e)} = \frac{1}{e}\left(\frac{1}{e-1}\right)$$

$$+ \frac{1}{e^3} + \frac{1}{e^4} + \cdots = \frac{1/e^3}{(1-1/e)} = \frac{1}{e^2}\left(\frac{1}{e-1}\right)$$

$$= \frac{1}{e-1} + \frac{1}{e}\left(\frac{1}{e-1}\right) + \frac{1}{e^2}\left(\frac{1}{e-1}\right) + \cdots$$

$$= \frac{1}{e-1} \cdot \frac{1}{1-1/e} \cdot \frac{e}{e} = \frac{e}{(e-1)(e-1)} = \boxed{\frac{e}{(e-1)^2}}$$

5. *(10 points)* If the **work** required to stretch a spring 1 foot beyond its natural length is 12 foot-pounds, how much work is needed to stretch it 9 inches beyond its natural length?

$$W = \int_0^1 K \cdot x\,dx = 12 = \frac{Kx^2}{2}\Big|_0^1 = \frac{K}{2}$$

$$\Rightarrow K = 24 \ (\text{lbs}/\text{ft})$$

$$W = \int_0^{3/4} K \cdot x\,dx = 24 \int_0^{3/4} x\,dx = 12x^2 \Big|_0^{3/4}$$

$$= 12\left(\frac{9}{16}\right) = 3 \cdot \frac{9}{4} = \boxed{\frac{27}{4} \text{ ft-lbs}}$$

6. *(20 points - 10 each)* Test for absolute convergence, conditional convergence or divergence for the following series. Be sure to indicate which test you are using, explain why you can use it, and state your conclusion.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n}}$

Test for absolute convergence:

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n+\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$. Looks like $\sum_{n=1}^{\infty} \frac{1}{n}$.

Since both have positive terms, use limit comparison.

$c = \lim_{n\to\infty} \left| \frac{1}{n+\sqrt{n}} \cdot \frac{n}{1} \right| = \lim_{n\to\infty} \left(\frac{n}{n+\sqrt{n}} \right) = \lim_{n\to\infty} \left(\frac{1}{1+\sqrt{\frac{1}{n}}} \right) = 1$

Since $c>1$ and c is finite, these series do the same thing. Since $\sum 1/n$ diverges (p-series, $p=1$), so does $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$. Thus $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n}}$ does not converge absolutely. →

(b) $\sum_{n=1}^{\infty} \frac{\cos(n^2)}{1+n^2}$

Does this converge absoluetly?

$\sum_{n=1}^{\infty} \left| \frac{\cos(n^2)}{1+n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos(n^2)|}{1+n^2}$

$\frac{|\cos(n^2)|}{1+n^2} < \frac{1}{n^2}$

Both $\sum_{n=1}^{\infty} \frac{|\cos(n^2)|}{1+n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are positive series. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \frac{|\cos(n^2)|}{1+n^2}$. Hence $\sum_{n=1}^{\infty} \frac{\cos(n^2)}{1+n^2}$ converges absoluetly.

Does it converge conditionally? This is an alternating series, so apply alternating series test. Let $f(x) = \frac{1}{x+\sqrt{x}} = (x+\sqrt{x})^{-1}$

- $f'(x) = -(x+\sqrt{x})^{-2} \cdot (1+\frac{1}{2}x^{-1/2}) < 0$
- $\lim_{n\to\infty} \frac{1}{n+\sqrt{n}} = 0$

Hence by the alternating series test this series converges. Thus $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n}}$ converges conditionally.

7. *(20 points - 10 each)* Find a power series representation for the following functions using any method you like. Write your answer using sigma notation. You do **not** need to find the radius and interval of convergence.

(a) $f(x) = \ln(1 + x)$

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)}$$

$$= 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\int \frac{1}{1+x}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \ln(x+1)$$

$$x=0) \quad C = \ln(1) = 0$$

$$\Rightarrow \boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}$$

(b) $g(x) = x \cdot \sin(2x)$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$$

$$x \cdot \sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1} \cdot x}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n+1} x^{2n+2}}{(2n+1)!}$$

8. *(20 points - 10 each)* Find the radius and interval of convergence of the following power series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+4)^n}{n \cdot 7^n}$

$$L = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{(x+4)^{n+1}}{(n+1)\cdot 7^{n+1}} \cdot \frac{n\cdot 7^n}{(x+4)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x+4)}{7}\cdot\frac{n}{n+1}\right| = \frac{|x+4|}{7}\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \frac{|x+4|}{7}$$

$L<1 \Rightarrow |x+4|<7 \Rightarrow \boxed{\text{Radius} = 7}$

$x=3)\ \displaystyle\sum_{n=1}^{\infty}\frac{(x+4)^n}{n\cdot 7^n} = \sum_{n=1}^{\infty}\frac{7^n}{n\cdot 7^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ Diverges (Harmonic)

$x=-11)\ \displaystyle\sum_{n=1}^{\infty}\frac{(x+4)^n}{n\cdot 7^n} = \sum_{n=1}^{\infty}\frac{(-7)^n}{n\cdot 7^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ Converges (alt harmonic)

Hence the interval of convergence is $\boxed{[-11, 3)}$

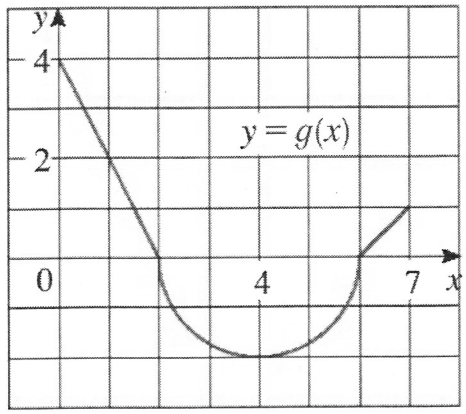
(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)!(3x)^n}{(2n)!}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)!(3x)^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{(n+1)!(3x)^n}\right|$$

$$= |3x|\lim_{n\to 0}\left|\frac{n+1}{(2n+2)(2n+1)}\right| = 0$$

Since $L<0$ for any x, $\boxed{\text{Radius} = \infty}$ and interval of convergence is $(-\infty, \infty)$.

9. *(10 points - 2 each)* The graph of g consists of two straight lines and a semi-circle, shown below. Let $f(x) = \int_2^x g(t)\,dt$.



Compute:

(a) $f(6)$ $= \int_2^6 g(t)\,dt = -\pi(2)^2 = \boxed{-4\pi}$

(b) $f(2)$ $= \int_2^2 g(t)\,dt = \boxed{0}$

(c) $f(0)$ $= \int_2^0 g(t)\,dt = -\int_0^2 g(t)\,dt$

$= -\frac{1}{2}(2)(4) = \boxed{-4}$

(d) $f'(1)$ $= g(1) = \boxed{2}$

(e) $f'(4)$ $= g(4) = \boxed{-2}$

10. *(10 points - 5 each)* Let C be the curve $y = 1 + e^x$ from $x = 0$ to $x = 1$.

(a) Set up, but do **not** evaluate, an integral to find the arc length of C.

$y' = e^x$

$$\int_0^1 \sqrt{1+e^{2x}}\,dx$$

(b) Setup, but do **not** evaluate, an integral to find the surface area of the object obtained by revolving C around the x-axis.

$$2\pi \int_0^1 (1+e^x)\sqrt{1+e^{2x}}\,dx$$

(c) *(Extra Credit, 5 points)* Compute the arc length of C. The integral from part (a) must be set up correctly.

$$\int_0^1 \sqrt{1+e^{2x}}\,dx =$$

$$\int_{\sqrt{2}}^{\sqrt{1+e^2}} u \cdot \frac{u}{u^2-1}\,du$$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{u^2}{u^2-1}\,du$$

$u = \sqrt{1+e^{2x}}$

$du = \frac{1}{\cancel{2}}(1+e^{2x})^{-1/2} \cdot \cancel{2}e^{2x}\,dx$

$= \dfrac{e^{2x}}{\sqrt{1+e^{2x}}}\,dx$

$= \dfrac{u^2-1}{u}\,dx$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1}{u^2-1}\right) du$$

$$= u \Big|_{\sqrt{2}}^{\sqrt{1+e^2}} + \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{1}{(u+1)(u-1)}\, du$$

$$= u \Big|_{\sqrt{2}}^{\sqrt{1+e^2}} + \frac{1}{2}\int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{du}{u-1} - \frac{1}{2}\int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{du}{u+1}$$

$$= u \Big|_{\sqrt{2}}^{\sqrt{1+e^2}} + \left(\frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1|\right)\Big|_{\sqrt{2}}^{\sqrt{1+e^2}}$$

$$= u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| \Bigg|_{\sqrt{2}}^{\sqrt{1+e^2}}$$

$$= \boxed{\sqrt{1+e^2} + \frac{1}{2}\ln\left|\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right| + \\ -\sqrt{2} - \frac{1}{2}\ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|}$$