1 a) $u = \arcsin(x)$ $\quad dv = dx$

$$du = \frac{dx}{\sqrt{1-x^2}} \qquad v = x$$

$$\int_0^1 \arcsin(x)\,dx = x\arcsin(x)\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\arcsin(x)\Big|_0^1 + \lim_{t\to 1^-} \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$$

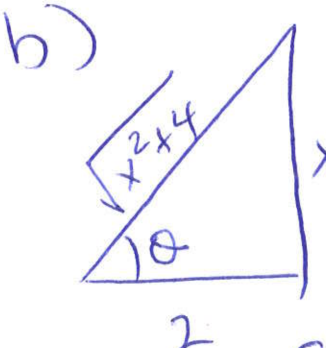
$w = 1-x^2 \quad dw = -2x\,dx$

$$= \frac{\pi}{2} + \frac{1}{2}\lim_{t\to 1^-}\int_1^{1-t^2} \frac{du}{\sqrt{u}}$$

$$= \frac{\pi}{2} + \lim_{t\to 1^-}\sqrt{u}\Big|_1^{1-t^2} = \frac{\pi}{2} + \lim_{t\to 1^-}\sqrt{1-t^2} - 1$$

$$= \boxed{\frac{\pi}{2} - 1}$$

b)



$2\tan\theta = x \qquad 2\sec\theta = \sqrt{x^2+4}$

$2\sec^2\theta\, d\theta = dx$

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta|$$

1 $$= \boxed{\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C}$$

c) $\frac{1}{2}\int_3^4 \frac{2x}{x^2-4}\,dx$

$u = x^2-4$
$du = 2x\,dx$
$x=3 \Rightarrow u=5$
$x=4 \Rightarrow u=12$

$= \frac{1}{2}\int_5^{12} \frac{du}{u}$

$= \frac{1}{2}\ln(12) - \frac{1}{2}\ln(5) = \boxed{\frac{1}{2}\ln(12/5)}$

(It is also possible to do this problem using partial fractions, but that's more work.)

d) $\int_2^3 \frac{dx}{(x-1)^{4/3}}$ $\quad u = x-1 \quad du = dx$

$= \int_1^2 \frac{du}{u^{4/3}} = \int_1^2 u^{-4/3}\,du = -3u^{-1/3}\Big|_1^2$

$= -3\left[2^{-1/3} - 1\right] = \boxed{3\left[1-2^{-1/3}\right]}$

e) $\int \tan^5\theta \cdot \sec^3\theta \, d\theta = \int \tan^4\theta \cdot \sec^2\theta \, (\tan\theta \cdot \sec\theta) \, d\theta$

$= \int (\sec^2\theta - 1)^2 \cdot \sec^2\theta \, (\tan\theta \cdot \sec\theta) \, d\theta$ $\quad u = \sec\theta$, $du = \sec\theta \tan\theta \, d\theta$

$= \int (u^2-1)^2 \cdot u^2 \, du = \int (u^4 - 2u^2 + 1) \cdot u^2 \, du$

$= \int (u^6 - 2u^4 + u^2) \, du = \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3$

$= \frac{1}{7}\sec^7\theta - \frac{2}{5}\sec^5\theta + \frac{1}{3}\sec^3\theta + C$

f) $u = \sqrt{t} = t^{1/2}$

$du = \frac{1}{2}t^{-\frac{1}{2}} dt = \frac{1}{2\sqrt{t}} dt = \frac{1}{2u} dt \Rightarrow dt = 2u\,du$

$\int e^{\sqrt{t}} dt = 2\int u e^u du$ $\quad w = u$, $dw = du$ $\quad dv = e^u du$, $v = e^u$

$2\int u e^u du = 2ue^u - 2\int e^u du = 2ue^u - 2e^u$

$= \boxed{2\sqrt{t}\, e^{\sqrt{t}} - 2e^{\sqrt{t}} + C}$

② a) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$ $\quad \lim_{n\to\infty} (-1)^n \frac{n}{n+2} \neq 0$, so

the series diverges by the divergence test.

b) First lets see if the series converges absoluetly.

$\sum_{n=1}^{\infty} \frac{n}{n^2+2}$ has positive terms, as does $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$c = \lim_{n\to\infty} \frac{n}{n^2+2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+2} = 1.$$

Since c is finite and $c>0$, by limit comparsion these series do the same thing. Since the harmonic series diverges, so does $\sum_{n=1}^{\infty} \frac{n}{n^2+2}$.

Hence $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$ does not converge ~~[illegible]~~ absoluetly. Does it converge conditionally?

It is an alternating series.

- $\lim_{n\to\infty} \frac{n}{n^2+2} = 0$ ✓

- $f(x) = \frac{x}{x^2+2} \Rightarrow f'(x) = \frac{x^2+2-2x^2}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2} < 0$ when $x > \sqrt{2}$ ✓

hence this series converges by the alternating series test, meaning $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+2}$ converges conditionally.

c) Does this converge absolutely?

Look at $\sum_{n=1}^{\infty} \frac{n}{n^3+2}$. This looks like $\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$. Since both have positive terms we can use limit comparison.

$$c = \lim_{n\to\infty} \left| \frac{n}{n^3+2} \cdot \frac{n^2}{1} \right| = \lim_{n\to\infty}\left(\frac{n^3}{n^3+2}\right) = 1$$

Since c is finite and $c>0$ these series do the same thing. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p=2$) so does $\sum_{n=1}^{\infty} \frac{n}{n^3+2}$. Hence $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^3+2}$ converges absolutely.

d) This looks like a function I know how to integrate, so lets consider the integral test. Let $f(x) = x^2 e^{-x^3}$. We need to check that f is positive, continuous, and decreasing. It's clearly positive and continuous. Observe that

$$f'(x) = 2xe^{-x^3} + x^2 \cdot e^{-x^3} \cdot (-3x^2)$$
$$= 2xe^{-x^3} - 3x^4 e^{-x^3} = xe^{-x^3}(2-3x^3)$$

which is eventually less than zero (so f is decreasing). Hence the integral test is applicable.

$$\int_1^\infty x^2 e^{-x^3} dx = \frac{1}{3}\int_1^\infty 3x^2 e^{-x^3} dx \qquad \begin{array}{l} u = x^3 \\ du = 3x^2 dx \end{array}$$
$$= \frac{1}{3} \lim_{t\to\infty} \int_1^t 3x^2 e^{-x^3} dx = \frac{1}{3} \lim_{t\to\infty} \int_1^{t^3} e^{-u} du$$
$$= \frac{1}{3} \lim_{t\to\infty} -e^{-u}\Big|_1^{t^3} = \frac{1}{3} \lim_{t\to\infty} \left[e^{-1} - e^{-t^3}\right]$$
$$= \frac{1}{3e}$$

Since $\int_1^\infty x^2 e^{-x^3} dx$ converges so does $\sum_{n=1}^\infty n^2 e^{-n^3}$ by the integral test.

e) The factorial suggests using the ratio test.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}\cdot(n+1)^2}{(n+1)!}\cdot\frac{n!}{3^n\cdot n^2}\right|$$

$$= 3\lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+1)\cdot n^2}\right| = 3\lim_{n\to\infty}\left|\frac{n+1}{n^2}\right| = 0$$

Since $L<1$, this series converges by the ratio test.

f) $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 2^n\cdot 2^{-1}}{5^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot 2^{n-1}}{5\cdot 5^{n-1}} = \sum_{n=1}^{\infty}\frac{1}{5}\cdot\left(-\frac{2}{5}\right)^{n-1}$$

This is a geometric series ($a=\frac{1}{5}$, $r=-2/5$). Since $|r|<1$ this series converges.

g) The n^{th} power suggests the roots test

$$L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n^2+1}{3n^2+7n+2}\right)^n} = \lim_{n\to\infty}\left(\frac{2n^2+1}{3n^2+7n+2}\right) = \frac{2}{3}$$

Since $L<1$, this series converges by the roots test.

h) This is a geometric series $\left(r=\frac{5}{4}\right)$. Since $|r|\geq 1$, this series diverges.

i) This series ~~contains both pos~~ has only positive terms since $\frac{1}{n} \leq 1 < \frac{\pi}{2}$ (meaning $\sin(\frac{1}{n}) > 0$ for all $n \geq 1$).

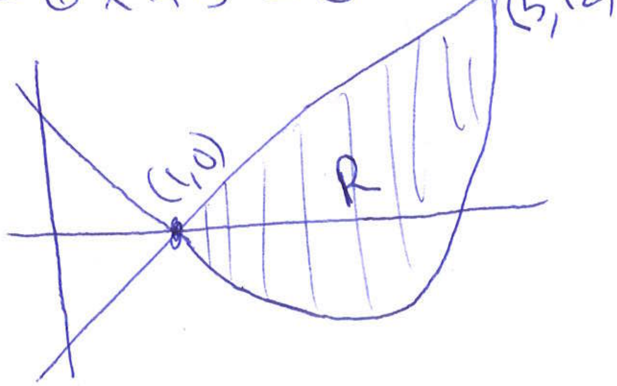
Since $\sum \frac{1}{n^2}$ is a positive series we can use a comparison.

$$\frac{\sin(1/n)}{n^2} < \frac{1}{n^2}$$

and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p=2$), so does $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^2}$

3) $x^2-3x+2=3x-3$ $\quad (x-1)(x-5)=0$

$x^2-6x+5=0$ $\quad x=1$ or $x=5$



a) $$\int_1^5 \left[(3x-3)-(x^2-3x+2)\right]dx$$

b) $V = \pi \int (R_{out}^2 - R_{in}^2)\, dx$ (washers)

$R_{out} = (3x-3) - (-2) = 3x - 1$

$R_{in} = (x^2 - 3x + 2) - (-2) = x^2 - 3x + 4$

$$V = \pi \int_1^5 \left[(3x-1)^2 - (x^2-3x+4)^2 \right] dx$$

c) $r = x$ $\quad h = (3x-3) - (x^2 - 3x + 2)$ (shells)

$= -x^2 + 6x - 5$

$$V = 2\pi \int_1^5 x(-x^2 + 6x - 5)\, dx$$

4 a) $\int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$ $\quad (f'(x) = \cos(x))$

b) $2\pi \int_0^{\pi} \sin(x) \cdot \sqrt{1 + \cos^2 x}\, dx$

5 $F = kx$ $\qquad 20 = k\left(\frac{1}{10}\right) \Rightarrow k = 200\ N/m$

$$W = \int_0^{1/5} 200x\, dx = 100x^2 \Big|_0^{1/5} = \frac{100}{25} = \boxed{4\ J}$$

6) a) $g(0) = \int_0^0 f(t)\,dt = 0$ (by properties of the definite integral)

$g(1) = 1 \cdot 2 = 2$

$g(2) = 2 + 2 + \frac{1}{2}(1)(2) = 5$

$g(3) = 5 + \frac{1}{2}(1)(4) = 7$

$g(6) = 7 - [\frac{1}{2}(2)(2) + 1(2) + \cancel{\ }]$

$= 7 - [2 + 2 + \cancel{\ }] = 3$

b) g is increasing when $g'(x) > 0$.

$g'(x) = f(x) > 0$ when $\boxed{0 \leq x < 3}$

c) $g'(x) = 0$ when $x = 3$ and when $x = 7$

The max is at $x = 3$ since g is concave down at $x = 3$ ($g''(x) = f'(x) < 0$ @ $x = 3$)

7) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\int_0^{\infty} \frac{1}{1+x^2}\,dx = 2\lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$

$= 2\lim_{t\to\infty} \arctan(x)\Big|_0^t = 2\lim \arctan(t)$

$= 2\left(\frac{\pi}{2}\right) = \boxed{\pi}$

10

8 a) This is geometric, $a = x$, $r = -x^2$

$$\frac{x}{1-(-x^2)} = x - x^3 + x^5 - x^7 + \cdots = \sum_{n=0}^{\infty} x(-x^2)^n$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n x^{2n+1}}$$

b) We know $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\arctan(x) = \int \frac{dx}{1+x^2} = C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$x = 0 \Rightarrow C = 0$, so

$$\boxed{\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}}$$

c) We know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Hence $e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}}$

d) Recall $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(Note $\sin(x)$ is an odd function, so power series only contains odd terms)

Hence $\sin(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{2n+1}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4}}{(2n+1)!}}$

9) a) Ratio Test

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1} \cdot (-1)^{n+1}}{(n+1)^2 \cdot 5^{n+1}} \cdot \frac{n^2 \cdot 5^n}{x^n \cdot (-1)^n} \right|$$

$$= \frac{|x|}{5} \lim_{n \to \infty} \left| \frac{n^2}{(n+1)^2} \right| = \frac{|x|}{5}$$

$L < 1 \Rightarrow |x| < 5$ Hence radius of convergence $= \boxed{5}$

When $x = -5$, $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (-5)^n}{n^2 \cdot 5^n} = \sum_{n=1}^{\infty} \frac{5^n}{n^2 \cdot 5^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Converges (p-series, $p=2$).

When $x = 5$, $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 5^n}{n^2 \cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely

(see series above) meaning this series is convergent.

Hence the interval of convergence is $-5 \leq x \leq 5$ (aka $[-5,5]$)

b) Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)\cdot 4^{n+1}}\cdot\frac{n\cdot 4^n}{(x+2)^n}\right|$$

$$= \frac{|x+2|}{4}\cdot\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \frac{|x+2|}{4}$$

$L<1 \Rightarrow |x+2|<4$, Radius of convergence $= \boxed{4}$

When $x=2$, $\sum_{n=1}^{\infty}\frac{(2+2)^n}{n\cdot 4^n} = \sum_{n=1}^{\infty}\frac{4^n}{n\cdot 4^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ Diverges

since this is the harmonic series

When $x=-6$, $\sum_{n=1}^{\infty}\frac{(-6+2)^n}{n\cdot 4^n} = \sum_{n=1}^{\infty}\frac{(-4)^n}{n\cdot 4^n} = \sum_{n=1}^{\infty}\frac{(-1)\cdot 4^n}{n\cdot 4^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

which converges since this is the alternating harmonic.

Hence the interval of convergence is $-6 \le x < 2$ $\left([-6,2)\right)$

c) Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}\cdot(x-2)^{n+1}}{(n+3)!}\cdot\frac{(n+2)!}{2^n\cdot(x-2)^n}\right|$$

$$= 2|x-2|\lim_{n\to\infty}\left|\frac{1}{n+3}\right| = 0$$

Hence this converges for all x, meaning the radius is ∞ and the interval of convergence is $(-\infty, \infty)$

d) (Ratio Test)

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}}\cdot\frac{\sqrt{n+3}}{2^n(x-3)^n}\right|$$

$$= 2|x-3|\cdot\lim_{n\to\infty}\left|\frac{\sqrt{n+3}}{\sqrt{n+4}}\right| = 2|x-3|$$

$L<1 \Rightarrow |x-3| < \frac{1}{2}$, so radius of converge is $\frac{1}{2}$.

When $x = 3.5$, $\displaystyle\sum_{n=0}^{\infty}\frac{2^n\cdot(3.5-3)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty}\frac{2^n\cdot(\frac{1}{2})^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+3}}$

This looks like $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$. Since both are positive series we can use limit comparison.

$$c = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\sqrt{n}}{\sqrt{n+3}}\right) = 1$$

Since c is finite and $c>0$, both series do the same thing. Since $\displaystyle\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges (p-series, $p=\frac{1}{2}$), so does $\displaystyle\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+3}}$.

When $x = 2.5$, $\displaystyle\sum_{n=0}^{\infty}\frac{2^n\cdot(2.5-3)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty}\frac{2^n\cdot(-\frac{1}{2})^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+3}}$

This is an alternating series, so we can use the alternating series test. Since $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n+3}} = 0$ and $\left\{\frac{1}{\sqrt{n+3}}\right\}$ is decreasing $\left(f(x) = (x+3)^{-1/2} \quad f'(x) = -\frac{1}{2}(x+3)^{-3/2} < 0\right)$ this series converges by the alternating series test.

Hence the interval of convergence is $\frac{5}{2} \le x < \frac{7}{2}$ (aka $\left[\frac{5}{2}, \frac{7}{2}\right)$)