

Math 19B

Final

Study Aid

June 6, 2011

General advise.

- **Get plenty of practice.** There's a lot of material in this section - try to do as many examples and as much of the homework as possible to get some practice. Just reading this review or just reading the chapter in the text/notes won't prepare you - you have to actually do some examples to make it sink in. Chapter 11.7 has a bunch of good examples for practice on applying the convergence/divergence tests.
- **Older material.** The final is comprehensive. Make sure you leave some time to review the earlier chapters - the past exams and your own midterms are useful study aids. Keep in mind the examples you spent time on your homework doing which haven't appeared on a midterm yet. These are likely topics for the final. Try to review some of the past homeworks too.
- **Differentiation recap.** There's a fair amount of differentiation required in this chapter - you'll need it to apply L'Hopital's theorem and also to compute Taylor series (from scratch). Brush up on your skills - in particular make sure you know the quotient rule, chain rule and product rule - they're all in the text if you want to look them up.
- **Be careful applying the convergence tests.** In the exam a lot of people will apply the convergence tests from this chapter incorrectly. Make sure you know under what conditions a given test can be used and what conclusions it generates. A lot of the tests are inconclusive in certain situations. If you're not sure if you're applying the test correctly read the statement of the theorem again or go and see a TA / Ed to check. In the exam be sure to state which theorem / test you are using and why it applies (i.e. show that the conditions of the test are met).

Chapter 11.1 - Sequences.

Definitions

- **Sequence.** A *sequence* is an assignment of a real number a_n to each natural number $n \in \mathbb{N}$ (\mathbb{N} stands for the 'natural numbers' or the 'counting numbers' 1,2,3,4, ...). The notations we use to denote a sequence are $\{a_1, a_2, a_3, \dots\}$, $\{a_n\}_{n=1}^{\infty}$, $\{a_n\}_{n \in \mathbb{N}}$ or (most commonly) just $\{a_n\}$.
- **Limits, convergence and divergence.** A sequence $\{a_n\}$ has a *limit* L , which we denote

$$“a_n \rightarrow L \text{ as } n \rightarrow \infty” \quad \text{or} \quad \lim_{n \rightarrow \infty} a_n = L,$$

if we can make the terms a_n as close to L as we like by making n sufficiently large. A sequence may or may not have a limit. If it does then we say that the sequence *converges* or 'the limit *exists*'. If it doesn't we say the sequence *diverges* or 'the limit does not exist'.

- **Factorial, $n!$, $0!$** If n is a natural number, the symbol $n!$ (pronounced ' n factorial') is short hand for

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots 2 \cdot 1.$$

For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. By convention (and slightly counter-intuitively), we define $0! = 1$.

Note that $n!$ *only* makes sense if n is a natural number or zero: $\left(\frac{1}{2}\right)!$ is nonsense.

Theory

- **The ‘associated function’ theorem.** If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$ for every $n \in \mathbb{N}$ then $\lim_{n \rightarrow \infty} a_n = L$.
- **The squeeze theorem.** If $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ then $\lim_{n \rightarrow \infty} b_n = L$.
- **Continuous function theorem.** If $\lim_{n \rightarrow \infty} a_n = L$ and the function f is continuous at L then

$$\lim_{n \rightarrow \infty} f(a_n) = f(L).$$

Remarks.

- **Finding limits of sequences.** Throughout this chapter you’ll need to know how to compute limits. You can state, without proof, the following results in the exam:

$$\begin{array}{ll} \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) = 0 & \lim_{n \rightarrow \infty} (\ln n) \rightarrow \infty \\ \lim_{n \rightarrow \infty} (a^{-n}) = 0 & \lim_{n \rightarrow \infty} (a^n) \rightarrow \infty \\ \lim_{n \rightarrow \infty} (e^{-n}) = 0 & \lim_{n \rightarrow \infty} (e^n) \rightarrow \infty \\ \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1 & \lim_{n \rightarrow \infty} x^{1/n} = 1 \ (x > 0) \\ \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 & \lim_{n \rightarrow \infty} x^n = 0 \ (|x| < 1) \end{array}$$

for any constant $a > 1$. In addition to these main results there are two approaches that are often used to compute limits.

- (i) **L’Hopital’s theorem.** L’Hopital’s theorem states that if $f(x)$ is differentiable at c (usually c will be ∞ in our examples) and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$, or $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = \pm\infty$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

provided this limit exists or is infinite.

Informally, we use L’Hopital’s rule for limits of the form ‘ $0/0$ ’ or ‘ $\pm\infty/\pm\infty$ ’. We may also apply L’Hopital’s theorem to tackle limits of the form ‘ $0 \cdot \infty$ ’, ‘ 1^∞ ’ and ‘ ∞^0 ’ by using a bit of algebraic manipulation to reduce it to a limit of the form ‘ $0/0$ ’ or ‘ $\pm\infty/\pm\infty$ ’. Here’s an example:

$$\begin{array}{ll} \lim_{x \rightarrow \infty} x \sin(1/x) & \text{limit is of the form } \infty \cdot 0 \\ = \lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} & \text{limit is of the form } 0/0 \\ = \lim_{x \rightarrow \infty} \frac{-x^{-2} \cos(1/x)}{-x^{-2}} & \text{applying L’Hopital’s} \\ = \lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right) & \\ = \cos\left(\lim_{x \rightarrow \infty} \left(\frac{1}{x}\right)\right) = \cos(0) = 1. & \end{array}$$

Note that by the ‘associated function theorem’, the above calculation shows that the *sequence* $\{\ln n/n\}_{n \in \mathbb{N}}$ (which technically isn’t a differentiable function) converges to the same limit 0.

(ii) **Limits of quotients.** The other common case you’ll need to know is how to take limits such as

$$\lim_{n \rightarrow \infty} \frac{\sqrt[3]{3n^3 + n^2 - 5}}{2n + 8}.$$

To approach these, first determine the degree of the numerator and the denominator. If the numerator has larger degree than the denominator then the limit tends to infinity and does not exist. Conversely, if the denominator has larger degree than the numerator then the limit will be zero. In the case where both numerator and denominator have the same degree, pick the highest power of n from the numerator and the denominator (taking into account radicals) and divide the numerator and denominator by this. In the example above, both the numerator and denominator have the same degree (1) so:

$$\lim_{n \rightarrow \infty} \frac{\sqrt[3]{3n^3 + n^2 - 5}}{2n + 8} = \lim_{n \rightarrow \infty} \frac{(\sqrt[3]{3n^3 + n^2 - 5})/n}{(2n + 8)/n} = \lim_{n \rightarrow \infty} \frac{\sqrt[3]{3 + n^{-1} - 5n^{-3}}}{2 + 8n^{-1}} = \frac{\sqrt[3]{3}}{2}.$$

- **Continuous function theorem.** Almost all the functions we see in this course are continuous. This theorem says it’s ok to ‘push limits through continuous functions’. For example, since \ln is continuous we have

$$\lim_{n \rightarrow \infty} \ln \left(\frac{n+1}{n} \right) = \ln \left(\lim_{n \rightarrow \infty} \frac{n+1}{n} \right) = \ln(1) = 0.$$

Chapter 11.2 - Series.

Definitions.

- **Infinite series and convergence.** Consider a sequence $\{a_n\}$. Adding together all the a_n :

$$\sum_{n=1}^{\infty} a_n$$

gives what we call a *series*. You may think of it as an ‘infinite sum’.

If we just add together the first n terms in the sequence we obtain the n^{th} partial sum, denoted s_n :

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n.$$

If the limit $\lim_{n \rightarrow \infty} s_n$ *exists* then

$$s = \lim_{n \rightarrow \infty} s_n$$

is called the *sum* and the series $\sum a_n$ is said to *converge*. We use the short-hand

$$s = \sum_{n=1}^{\infty} a_n$$

to express this fact. If the limit $\lim_{n \rightarrow \infty} s_n$ does not exist then the series is said to *diverge*.

Theory.

- **Geometric series.** The specific series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is called the *geometric series*. It converges (i.e. you get a numerical answer for this ‘infinite sum’) if $|r| < 1$ and diverges if $|r| \geq 1$. If it converges, we can calculate its sum using the formula:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

- **The divergence test.** For *any* series $\sum_{n=1}^{\infty} a_n$, if $\lim_{n \rightarrow \infty} a_n \neq 0$ then the series diverges.
- **Sums of series.** If $\sum a_n$ and $\sum b_n$ are **convergent** series then so too are $\sum c a_n$ (for any constant c), $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ with:

$$\begin{aligned}\sum_{n=1}^{\infty} c a_n &= c \sum_{n=1}^{\infty} a_n \\ \sum_{n=1}^{\infty} (a_n + b_n) &= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \\ \sum_{n=1}^{\infty} (a_n - b_n) &= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n.\end{aligned}$$

- **The harmonic series.** This will be mentioned again later (it is a p -series with $p = 1$). The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

diverges.

- **Telescoping series....** Yeah, know how to do these! I’ll fill out some comments here when I have time....

Remarks.

- **Sequences vs. series.** Be sure to differentiate between the two - a series is an ‘infinite sum’, a sequence is just an infinite list of (ordered) terms.
- **When to use the divergence test. (Always!)** A good proportion of this chapter (and hence your exam!) is dedicated to the methods we use to decide when a series will or will not converge. This test only addresses the later case but is by far the easiest to implement of all the tests you’ll come across. Therefore, the divergence test should **always** be the *first thing you check* when you’re attempting to test for (convergence)/divergence.

Note that if $\lim_{n \rightarrow \infty} a_n = 0$ then the divergence test is **inconclusive** - in particular, it does *not* tell you that the series will converge (there are lots of examples, such as the harmonic series, where $\lim a_n = 0$ but the series still diverges).

- **The geometric sum is v.useful!** Finding the sum of a series is usually extremely difficult, even in the cases where we know that it converges (and therefore the sum must exist). (In general it is an impossible task, the best we can usually achieve is an *approximation* using numerical methods - see later). The geometric series however, is one of the few instances when the sum can be calculated exactly. If a question asks you to *compute* the sum of an infinite series (exactly, rather than by approximations) you will either have to use this formula or the series will be a 'telescoping example'.

Notice that the geometric series is characterized by the fact that each term ' a_n ' in the series is obtained by multiplying the previous term a_{n-1} by some value r . Whenever a series has this property it is a geometric series and you can manipulate it so that the formula above may be used to find its sum (provided $|r| < 1$).

As a final note, you should be aware that the geometric series contains its own convergence / divergence test - calculating the ratio of successive terms in a geometric series will give you the value ' r '. A geometric series always converges if $|r| < 1$ and always diverges if $|r| \geq 1$ (- compare this to the 'ratio test' below).

- **Recognize sums of series.** Throughout this chapter it will sometimes be useful to 'split' series into a sum of two series or to factor out a constant to facilitate determining convergence and/or to use the geometric series formula above. There is an example contained in the comparison test remarks that uses this property.

Note also that just as for differentiation and integration, multiplication by a constant has little effect on a series - you can just 'factor it out' to the front and 'forget' about it.

Chapter 11.3 - The integral test.

Theory.

- **Integral test.** This gives a practical method for determining if a series converges or diverges. If f is a continuous, **positive**, decreasing function on $[1, \infty)$ such that $f(n) = a_n$, then:

(i) if $\int_1^\infty f(x) dx$ is convergent then $\sum_{n=1}^\infty a_n$ is convergent,

(ii) if $\int_1^\infty f(x) dx$ is divergent then $\sum_{n=1}^\infty a_n$ is divergent.

- **The p-series.** The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is very important. So important it has its own name: the p-series. You should memorize the fact that it converges for $p > 1$ and diverges if $p \leq 1$. Note that for $p = 1$ it is just the *harmonic series*, which as we already noted, diverges. You can prove the conditions under which this series converges and diverges using the integral test above - it boils down to the fact that the improper integral

$$\int_1^\infty \frac{1}{x^p} dx$$

converges if $p > 1$ and diverges if $p \leq 1$.

Remarks.

- **When to use the integral test.** Use it when a) you are asked to determine the convergence or divergence of a function, b) when the divergence test (Chap 11.2) has been inconclusive and c) when the a_n terms look like a function you can integrate.

- **Finding f and ‘start values’.** The function $f(x)$ you choose for the integral test will be obvious - just replace n ’s with x ’s in the ‘ a_n ’ terms. Also, it does not matter if the series fails to start at $n = 1$, simply alter the integral accordingly. Both of these remarks are illustrated in determining the convergence / divergence of

$$\sum_{n=3}^{\infty} ne^{-n^2}.$$

Method: we just check whether the integral

$$\int_3^{\infty} xe^{-x^2} dx$$

converges or diverges.

- **It’s only a test!** The integral test *just* checks for divergence or convergence of a series. If the series converges then the sum is **not** the value of the improper integral. If you are specifically asked to calculate the sum of a series you should probably be using the geometric series formula or applying the ‘telescoping’ method.

Chapter 11.4 - Comparison tests.

Theory.

- **The comparison test.** If $\sum a_n$ and $\sum b_n$ are series with **positive** terms then:

- if $\sum b_n$ converges and $a_n \leq b_n$ for all n , then $\sum a_n$ converges,
- if $\sum b_n$ diverges and $a_n \geq b_n$ for all n , then $\sum a_n$ diverges.

Note that this is exactly the same setup for the comparison test for improper integrals.

- **The limit comparison test.** If $\sum a_n$ and $\sum b_n$ are series with **positive** terms and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where $0 < c < \infty$, then either both series diverge or both series converge.

Remarks.

- **When to use them.** First off, check that the divergence test (chap 11.2) doesn’t apply and that you can’t integrate the terms in the series (otherwise just use the integral test). The comparison tests are probably the hardest tests to implement in practice - only apply them when you have to.

The idea behind these comparison tests is to compare the series we are given to a simpler series which we know either converges or diverges. The only series you are expected to know the convergence / divergence of are the **p-series** and the **geometric series**, so these are the only two series you will use in the comparison. Of the two, the p-series comparison is ‘more commonly’ used - or it is in your homework and examples at least... We therefore use the comparison test when the series ‘approximately looks like’ (!) either a geometric series or (more commonly) a p-series. To be more specific, and to give a common example, you would attempt to make use of a *p-series comparison* when determining the convergence / divergence of series of the form:

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)}$$

where the quotient is positive for all values of n and $f(n)$ and $g(n)$ are either a) polynomials in n , b) polynomials raised to some (constant) power (possibly $1/2$ - i.e. a square root) or c) sequences that

are *bounded* for $n \geq 0$ (for example $\sin(n)$, $\cos^2(n)$, e^{-n} etc...). You would consider using a *geometric series comparison* if the terms in the series you're interested in contain powers of n (although keep in mind that the 'root test' or 'ratio test' may be more appropriate in this instance - see later). For example:

$$\sum_{n=1}^{\infty} \frac{n}{2n^3 + 1}, \quad \frac{1 + n + n^2}{\sqrt[3]{1 + n^2 + n^6}}, \quad \sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right) e^{-n}, \quad \sum_{n=1}^{\infty} \frac{(n-1) \sin^2(n)}{n^2 \sqrt{n}}$$

should all be compared to p-series (note that in the last two examples e^{-n} and $\sin^2(n)$ are bounded above by 1). These series:

$$\sum_{n=1}^{\infty} \frac{4 + 2^n}{3^n}, \quad \sum_{n=1}^{\infty} \frac{1 + 3^n}{1 + 4^n}, \quad \sum_{n=1}^{\infty} \frac{1 + \sin n}{10^n}$$

should all be compared to geometric series.

- **Which series to compare to?** If you are making a p-series comparison, then choose the highest power in both the numerator and the denominator of the function. Then simplify and try to use this as your p-series comparison (it won't always work but it gives you a starting point). For example, for

$$\sum_{n=1}^{\infty} \frac{1 + n + n^2}{\sqrt[3]{1 + n^2 + 8n^{12}}} \quad \text{use} \quad \sum_{n=1}^{\infty} \frac{n^2}{\sqrt[3]{8n^{12}}} = \sum_{n=1}^{\infty} \frac{n^2}{2n^4} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

for the comparison (which converges).

In other cases it is not so clear cut.

- **Comparison vs limit comparison... Which test to use?** As a general rule of thumb, the limit test is usually easier to implement and saves you having to 'fudge' constants to get the inequalities to work out. It may however leave you with a difficult limit to calculate (review the remarks in Chap 11.1 if necessary). If it is obvious that the terms of one series are larger or smaller than the other, the comparison test will probably be quicker and easier. When in doubt pick one, see if it works out and if not try the other. Practice a few examples so that you're used to using both approaches. If neither approach works see the comments below!

Chapter 11.5 - Alternating series test.

Definitions

- **Alternating series.** An *alternating* series is a series in which consecutive terms alternate between positive and negative:

$$\sum_{i=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \dots \quad \text{with } b_n > 0.$$

Theory.

- **Alternating series test.** If an alternating series $\sum (-1)^{n+1} b_n$ satisfies

$$(i) \quad b_{n+1} \leq b_n \text{ for all } n,$$

$$(ii) \quad \lim_{n \rightarrow \infty} b_n = 0,$$

then the series is convergent.

Remarks.

- **When to use the alternating series test.** This test should be *considered* for determining the convergence of *any series that is alternating*. However, see the comments below concerning absolute convergence...

In practice you should check the second condition of the alternating series test first. If condition two does **not** hold (i.e. $\lim b_n \neq 0$) then you don't have to go any further - the series will **diverge** by the divergence test (Chap 11.2). On the other hand, if the first condition does not hold this **does not** show that the series diverges - it just means the test is inconclusive. The reason for this is that the first condition can be weakened slightly - see below.

- **Weakened alternating series test.** This is not too important but you should be aware that the first condition of the alternating series test can be weakened slightly - all that is important is that $b_{n+1} \leq b_n$ *eventually* - i.e. for all n sufficiently large or, in other words, for all n bigger than some large number N .
- **Showing $b_{n+1} < b_n$.** Differentiation can be used to show that $b_{n+1} < b_n$. For example, consider the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 1}.$$

To show that the sequence $b_n = n^2/(n^3 + 1)$ is decreasing, it suffices to show that the related function $f(x) = x^2/(x^3 + 1)$ is decreasing (for sufficiently large x values). The function $f(x)$ is decreasing wherever $f'(x) < 0$. Since

$$f'(x) = \frac{x(2 - x^3)}{(x^3 + 1)^2}$$

we see that $f(x)$ is decreasing for $2 - x^3 < 0$ or, in other words, for $x > \sqrt[3]{2}$. We conclude that b_n is decreasing (i.e. $b_{n+1} \leq b_n$) for all $n \geq 2$. Therefore (by the comments above) the alternating series test applies.

Chapter 11.6 - Absolute convergence, the ratio and root tests.

Definitions.

- **Absolutely convergent.** A series $\sum a_n$ is called *absolutely convergent* if $\sum |a_n|$ is convergent.
- **Conditionally convergent.** A series $\sum a_n$ is called *conditionally convergent* if it is convergent but not absolutely convergent. In other words, $\sum a_n$ is conditionally convergent if $\sum a_n$ is convergent but $\sum |a_n|$ is divergent.

Theory.

- **Absolute convergence implies convergence.** If a series is absolutely convergent then it is convergent.
- **The ratio test.** Let $\sum a_n$ be any series.

- (i) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- (ii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- (iii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, then the ratio test is **inconclusive**.

- **The root test.** Let $\sum a_n$ be any series.

- (i) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- (ii) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- (iii) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, the root test is inconclusive.

Remarks.

- **Absolute convergence ‘stronger’ than convergence.** Be aware that if a function is not absolutely convergent then it may still be conditionally convergent.
- **Alternatives to the alternating series test.** Because absolute convergence implies (usual) convergence then just because a series is alternating doesn’t necessarily mean we have to use the alternating series test. We might instead find it easier to apply the ratio test or the comparison test or the integral test to $\sum |a_n|$ in order to show *absolute convergence* - which in turn implies convergence.
To prove that an alternating series is *conditionally convergent* you need to show that a) $\sum |a_n|$ is divergent (so that the series does not converge *absolutely*), and b) that $\sum |a_n|$ is convergent. The latter will require the alternating series test.
- **When to use the ratio test.** Series that involve factorials or constants being raised to a power of n - for example:

$$\sum_{n=1}^{\infty} \frac{2^k}{k!}, \quad \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^4}, \quad \sum_{n=3}^{\infty} \frac{(-10)^n}{n!},$$

are all conveniently tested using the ratio test.

- **When not to use the ratio test.** The ratio test will not work on rational functions or algebraic rational functions (rational functions with square roots etc). For these problems use the comparison test for a suitable p-series - see ‘when to use the comparison tests’.
- **When to use the root test.** Pretty much only when a_n is of the form $(b_n)^n$ (and isn’t a geometric series!), for example:

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n, \quad \sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}, \quad \sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}.$$

Chapter 11.7 - Strategy for testing series.

This section is very useful - it’s like chapter 7.5 for integration; there are lots of problems and you have to work out which divergence / convergence test is appropriate for the series. Read through the ‘strategy for testing series’ on page 721. This is excellent practice for the exam since when you come to take your test it is unlikely that you’ll be told which test to use. Try and have a look at as many of these problems as possible. They test you on all the convergence / divergence tests you have learnt so far and get you used to quickly spotting which test is appropriate. Practice is the key!

Chapter 11.8 - Power series.

Definitions.

- **Power series.** A *power series about a* is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \dots$$

where x is a variable, c_n are constants - (called the *coefficients of the power series*) and a is a constant. Note that a could be zero in which case we have a ‘power series about the origin’:

$$\sum_{n=0}^{\infty} c_n(x-0)^n = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

- **Radius of convergence, interval of convergence.** Given a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

the *interval of convergence* is the set of all x values for which the series converges. By the theorem below, this will always be an interval; hence the name ‘interval of convergence’. The *radius of convergence* is the largest number R such that the series converges for every x on the interval $(a-R, a+R)$ (i.e. for every x with $a-R < x < a+R$).

Theory.

- **Convergence of power series.** For a power series about a , i.e.

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

there are only three possibilities for the values of x that will make the power series converge:

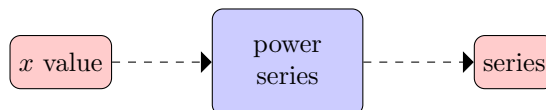
- The series *only* converges when $x = a$. In this case the interval of convergence is $[a, a]$ (i.e. just the point a) and the radius of convergence is $R = 0$.
- The series converges for *every* value of x . In this case the interval of convergence is $(-\infty, \infty)$ (i.e. all real numbers) and we say the radius of convergence is ‘infinite’ : $R = \infty$.
- The series converges for all x less than a distance r from a - that is, for all x such that $|x-a| < r$ - and diverges for all x a distance greater than r from a - that is for all x such that $|x-a| > r$. In this case the radius of convergence is $R = r$ and the interval of convergence is

$$(a-r, a+r) \quad \text{or} \quad (a-r, a+r] \quad \text{or} \quad [a-r, a+r) \quad \text{or} \quad [a-r, a+r]$$

depending on whether or not the power series converges at $x = a-r$ and/or $x = a+r$ (which we call the *endpoints of the radius of convergence* or just ‘the *endpoints*’).

Remarks.

- **Comments on the definitions.** What is a power series and how is it different to a ‘usual’ series? The key difference is that a power series is a *function* of x . The way this function ‘works’ is that for each value of x you input, it outputs a *series* (in the usual sense):



For example, considering the *power series*

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n},$$

then plugging in say, $x = 4$, gives the *series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

It is important to make this distinction between power series and series. The interval of convergence is exactly the domain of this function - the values of x that output a convergent series.

- **Determining the radius of convergence.** To find the interval / radius of convergence of a power series $\sum c_n(x-a)^n$ just apply the ratio test (or occasionally you'll need the geometric series test) to the terms $a_n = c_n(x-a)^n$ in the power series. For example, for the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

then $a_n = x^n/n!$ and so

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = 0$$

for every x . The ratio test therefore implies the power series converges for every x (recall that the ratio test says that the series will converge if $\lim |a_{n+1}/a_n| < 1$ - we've shown this happens for all values of x). This means the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.

- **Take care with 'endpoints'!** When using the geometric series test the end points of the interval of convergence are not included. When using the ratio test you must test the end points of the interval of convergence 'manually' - the power series may converge there or it may not. To determine this, plug the endpoints into your power series and, using an appropriate convergence / divergence test from chapters 11.2 to 11.6, determine whether the resulting *series* (it's just a normal series once you have chosen an x value) converges or diverges.

Here's an example. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}.$$

We set $a_n = n(x+2)^n/3^{n+1}$ and find that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) \frac{|x+2|}{3} = \frac{|x+2|}{3}.$$

Therefore, by the ratio test, the power series will converge when

$$\frac{|x+2|}{3} < 1$$

- that is, when $|x+2| < 3$ (note that this says that the radius of convergence is 3). Another way of expressing this interval is $-5 < x < 1$. The problem is, the ratio test is inconclusive at the 'endpoints' $x = -5$ and $x = 1$ and so we must check these 'manually'. This means checking the two series

$$\sum_{n=1}^{\infty} \frac{n(-5+2)^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{n(-3)^n}{3^{n+1}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n(1+2)^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{n(3)^n}{3^{n+1}}$$

for convergence or divergence independently. The best test to use on both of these is the divergence test - for both series we have $\lim a_n \neq 0$ and so neither series converges. Hence the interval of convergence is $(-5, 1)$ (in interval notation). *Before* we tested the end points, any of the following intervals *could* have been the interval of convergence: $(-5, 1)$, $[-5, 1)$, $(-5, 1]$, $[-5, 1]$ (recall that 'square brackets' indicate that you include the endpoint - i.e. $(-5, 1]$ is the interval $-5 < x \leq 1$).

Chapter 11.9 - Representations of functions as power series.

Theory.

- **Differentiating and integrating power series.** If a power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$ then the function defined by

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

is differentiable and integrable on the interval of convergence (or ‘domain’ of $f(x)$) $(a-R, a+R)$. Furthermore,

$$(i) \quad f'(x) = \sum_{n=0}^{\infty} n c_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$

$$(ii) \quad \int f(x) dx = \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \dots$$

and the *radius* of convergence of (i) and (ii) is still R - although *convergence at the endpoints may change* (the *interval* of convergence could be different)!

This theorem says that on the interval of convergence you find the derivative (respectively integral) of the power series by just differentiating (resp. integrating) *term by term*.

Remarks.

I’ve combined the remarks of this section with those from the next section - see below.

Chapter 11.10 - Taylor (and Maclaurin) series.

Theory.

- **Taylor series theorem.** Suppose $f(x)$ is **any** function that can be differentiated infinitely many times at a point $x = a$. Then for some radius of convergence R , $f(x)$ has a power series expansion about a :

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad |x-a| < R$$

where the coefficients c_n are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

The power series above is called the *Taylor series expansion* of $f(x)$ at a . If we take the Taylor series expansion at $a = 0$ we get what is called the *Maclaurin* series of $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots \quad |x| < R.$$

Remarks

- **Notation.** The notation $f^{(n)}(a)$ in the Taylor series expansion means the n^{th} derivative of f evaluated at $x = a$:

$$f^{(n)}(a) = \left. \frac{d^n f}{dx^n}(x) \right|_{x=a} = \frac{d^n f}{dx^n}(a).$$

- **Why Taylor series?** Taylor series are important because the theory of integration is incomplete: unlike differentiation, there are continuous (perfectly nice looking functions!) that either we can't integrate or simply don't have an antiderivate. One example is e^{x^2} . Taylor series, which *can be integrated* on their interval of convergence, (see the 'differentiating and integrating power series theorem' from the previous chapter) provide one approach to partially get around this problem.
- **Some examples.** Computing Taylor series is not hard, it's all just computation (although you may want to revise your differentiation skills!). Here are some common Taylor series (centered at 0) along with their radius of convergence - you should be able to deduce all these straight from the definition:

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots & R = 1 \\ e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots & R = \infty \\ \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots & R = \infty \\ \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots & R = \infty.\end{aligned}$$

You probably won't be expected to memorize these but you should be able to derive them.

As an example of calculating Taylor series expansions, consider the problem of finding the Maclauren series expansion (i.e the Taylor series expansion about $x = 0$) of $f(x) = \ln|x+1|$. We have:

$$\begin{aligned}f(x) &= \ln|x+1| \\ f'(x) &= \frac{1}{x+1} \\ f''(x) &= -\frac{1}{(x+1)^2} \\ f'''(x) &= \frac{2}{(x+1)^3} \\ f^{(4)}(x) &= \frac{-3 \cdot 2}{(x+1)^4} \\ &\dots \\ f^{(n)}(x) &= \frac{(-1)^{n+1}n!}{(x+1)^n}.\end{aligned}$$

Hence the Taylor series of this function is

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= f(0) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n!}{n! (0+1)^n} x^n \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} x^n \\ &= x - x^2 + x^3 - x^4 + \dots\end{aligned}$$

This series is geometric with common ratio $r = (-x)$. The Taylor series therefore converges if, and only if, $|r| = |x| < 1$.

- **Short-cuts for computing Taylor expansions.** The method above for computing Taylor series is very time-consuming. In your exam you *may* be given a Taylor series and then asked to use it to

compute the expansion of a related function. For example, using the $\cos x$ expansion above, we have

$$x \cos\left(\frac{1}{2}x^2\right) = x \left(\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}x^2\right)^{2n}}{(2n)!} \right) = x \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)! 2^{2n}} \right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)! 2^{2n}}.$$

Since the radius of convergence of the Taylor series of $\cos x$ is $R = \infty$ then the radius of convergence of this power series is also $R = \infty$. However, this is a special case - in general the ‘new’ series **will not** have the same radius of convergence as the original series. Suppose, for example, the radius of convergence of $\cos x$ had been $R = 8$ in the example above (it’s not, but just suppose), then the series converges for $|x| < 8$. We replaced this x with an $\frac{1}{2}x^2$ and so the new series converges when $|\frac{1}{2}x^2| < 8$, that is, for $|x| < 4$. Hence the radius of convergence of the new series we derived would be $R = 4$. Sorry that’s a bad example, it’s getting late! Haha...

- **More short-cuts!** Another common way of generating Taylor series (without doing all the long calculations) is to use the ‘differentiation and integration of power series theorem’ from the last chapter. Note that the theorem tells us that radius of convergence remains unchanged after integrating or differentiating (although the *interval* of convergence may be different - i.e. the end points may no longer be convergent / divergent). Here’s an example:

The Taylor series of $1/(1-x)$ about 0 (given above) has radius of convergence $R = 1$. Thus for $|x| < 1$ we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

(note that this is just the usual formula for the sum of the geometric series). Differentiating this gives the Taylor expansion of $1/(1-x)^2$ on $|x| < 1$:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$$

Similarly, integrating gives

$$\ln|1-x| = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

which is valid for $|x| < 1$. The radius of convergence of both these last two series is equal to the first, $R = 1$.

- **Calculating infinite sums.** If we have a Taylor series expansion $\sum c_n x^n$ for a function $f(x)$ we can use it to find the value of specific infinite series. For any value $x = k$ such that k belongs to the interval of convergence then $\sum c_n k^n = f(k)$. For example, the power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges for all x . Therefore, choosing $x = 1$ we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = e.$$