Math 19B Quiz: August 10, 2012 Solutions

Show all work and make sure to give exact answers. Good luck, and enjoy your weekend!

For all problems let $\mathcal{R}$ be the region bound by $y = x^2$ and $y = x$.

1. *(5 points)* Find the area of $\mathcal{R}$.

   Method 1: Integrate with respect to x.

$$\int_0^1 (x - x^2)\, dx = \frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \boxed{\frac{1}{6}}$$

   Method 2: Integrate with respect to y.

$$\int_0^1 (\sqrt{y} - y)\, dy = (\frac{2}{3}y^{3/2} - \frac{1}{2}y^2)\Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \boxed{\frac{1}{6}}$$

2. *(5 points)* Find the volume of the solid obtained by revolving $\mathcal{R}$ around $y = -1$.

   Method 1: Washers (integrating with respect to x).

   $R_{out} = x - (-1) = x + 1$ and $R_{in} = x^2 - (-1) = x^2 + 1$, so

$$\begin{aligned}
V &= \pi \int_0^1 (R_{out}^2 - R_{in}^2)\, dx = \pi \int_0^1 \left[(x^2 + 2x + 1) - (x^4 + 2x^2 + 1)\right]\, dx \\
&= \pi \int_0^1 (-x^4 - x^2 + 2x)\, dx = \pi \left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 + x^2\right]\Big|_0^1 \\
&= \pi \left[-\frac{1}{5} - \frac{1}{3} + 1\right] = \pi \left[-\frac{3}{15} - \frac{5}{15} + \frac{15}{15}\right] = \boxed{\frac{7\pi}{15}}
\end{aligned}$$

   Method 2: Cylindrical Shells (integrating with respect to y).

   $R = y - (-1) = y + 1$ and $l = \sqrt{y} - y$, so

$$\begin{aligned}
V &= 2\pi \int_0^1 R \cdot l\, dx = 2\pi \int_0^1 (y + 1)(\sqrt{y} - y)\, dy = 2\pi \int_0^1 (y^{3/2} - y^2 + y^{1/2} - y)\, dy \\
&= 2\pi \left[\frac{2}{5}y^{5/2} - \frac{1}{3}y^3 + \frac{2}{3}y^{3/2} - \frac{1}{2}y^2\right]\Big|_0^1 = 2\pi \left[\frac{2}{5} - \frac{1}{3} + \frac{2}{3} - \frac{1}{2}\right] \\
&= 2\pi \left[\frac{12}{30} + \frac{10}{30} - \frac{15}{30}\right] = 2\pi \left[\frac{7}{30}\right] = \boxed{\frac{7\pi}{15}}
\end{aligned}$$

3. *(5 points)* Find the volume of the solid obtained by revolving $\mathcal{R}$ around the y-axis.

Method 1: Cylindrical Shells (integrating with respect to x).

$R = x$ and $l = x - x^2$, so

$$\begin{aligned} V &= 2\pi \int_0^1 R \cdot l\, dx = 2\pi \int_0^1 x(x - x^2)\, dx = 2\pi \int_0^1 (x^2 - x^3)\, dx \\ &= 2\pi \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right] \Big|_0^1 = 2\pi \left[\frac{1}{3} - \frac{1}{4}\right] \\ &= 2\pi \left[\frac{4}{12} - \frac{3}{12}\right] = \boxed{\frac{\pi}{6}} \end{aligned}$$

Method 2: Washers (integrating with respect to y).

$R_{out} = \sqrt{y}$ and $R_{in} = y$, so

$$\begin{aligned} V &= \pi \int_0^1 [R_{out}^2 - R_{in}^2]\, dy = \pi \int_0^1 [(\sqrt{y})^2 - y^2]\, dy = \pi \int_0^1 [y - y^2]\, dy \\ &= \pi \left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right] \Big|_0^1 = \pi \left[\frac{1}{2} - \frac{1}{3}\right] = \pi \left[\frac{3}{6} - \frac{2}{6}\right] = \boxed{\frac{\pi}{6}} \end{aligned}$$

4. *(5 points)* Let $\mathcal{R}$ be the base of a solid. Cross-sections perpendicular to the x-axis are squares. Find the the volume of this solid.

Since the cross-sections are squares, we know that $A = l^2$ (where l is the length of the side of the square). We need to find l as a function of x.

Observe that $l = x - x^2$, so

$$\begin{aligned} V &= \int_0^1 A(x)\, dx = \int_0^1 (x - x^2)^2\, dx = \int_0^1 (x^2 - 2x^3 + x^4)\, dx \\ &= \left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right] \Big|_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{10}{30} - \frac{15}{30} + \frac{6}{30} = \boxed{\frac{1}{30}} \end{aligned}$$