Nonlinear Noise Reduction

Computational Physics Seminar on Analyzing Biomedical Signals

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Outline

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Definitions and scope

So far:

- Systems evolve deterministically

$$\mathbf{y}_{n+1} = f(\mathbf{y}_n).$$

- Timeseries are projections of the true state

$$y_n = s(\mathbf{y}_n).$$

Definitions and scope

- Measurement noise:

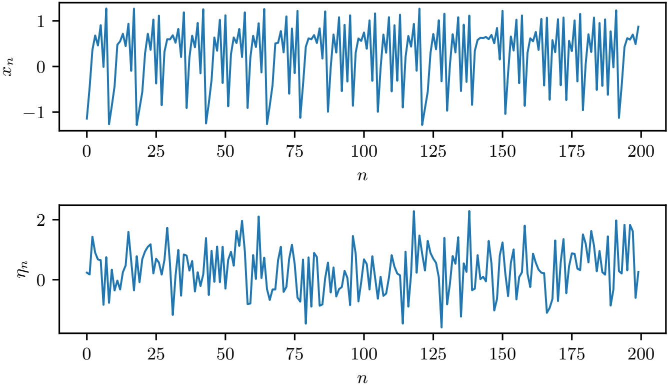
$$x_n = s(\mathbf{y}_n) + \eta_n$$

- Dynamic noise:

$$\mathbf{y}_{n+1} = f(\mathbf{y}_n + \eta_n)$$

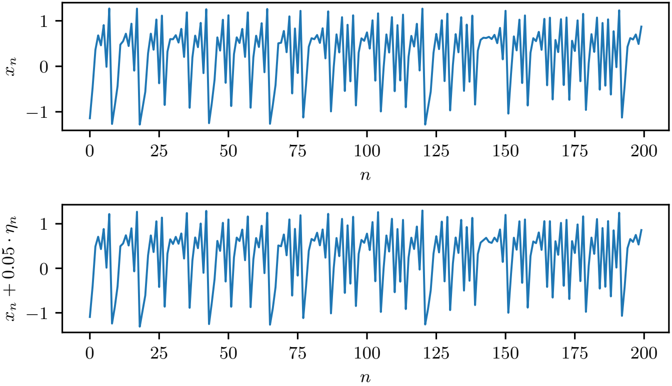
In this talk, we only focus on measurement noise.

Why does classical filtering not work?



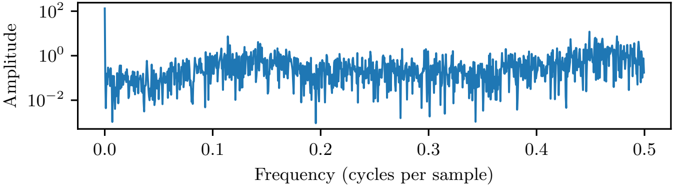
Timeseries of the x-coordinate of the Hénon map for 200 steps (above) and of white noise with the same mean and standard derivation (below).

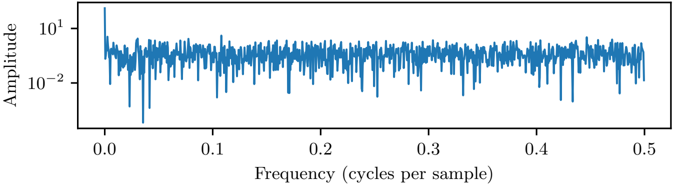
Why does classical filtering not work?



Timeseries of the Hénon map without noise (above) and with 5% relative noise (below).

Why does classical filtering not work?





Power spectrum of the Hénon map (above) and of white noise with the same mean and standard derivation (below) for 2000 steps.

The problem

Problem:

- Spectral analysis and filtering are of limited use for reducing noise in chaotic systems. ($\rightarrow$ broadband)

Idea:

- Make use of higher dimensional structures in the state space of dynamical systems.

Outline

Takens' theorem

- **Goal:** Reconstruct the phase space vectors from a scalar timeseries.
- **Takens' theorem:** Use delay vectors

$$x_n \to \mathbf{x}_n := \left(x_n, x_{n+\tau}, \ldots, x_{n+(m-1)\tau}\right)$$

Manifold hypothesis

- **Assumption:** Noise-free dynamics $\mathbf{y}_n$ lie on submanifold $\mathcal{A}$ with

$$\dim \mathcal{A} = D < m.$$

- Noise scatters points $\mathbf{x}_n$ away from $\mathcal{A}$.

Recap: Hénon map

Defined by

$$x_{n+1} = 1 - ax_n^2 + y_n$$

$$y_{n+1} = bx_n,$$

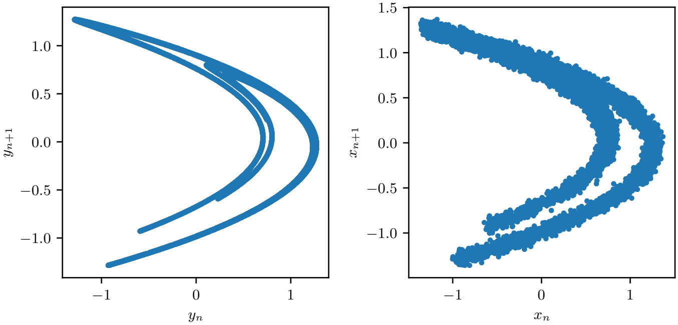
with parameters

$$a = 1.4 \quad \text{and} \quad b = 0.3,$$

and starting value

$$(x_0, y_0) = (0, 0).$$

Hénon attractor with and without noise



Noise-free reconstructed Hénon attractor for 10000 steps (left) and reconstructed phase space from data with 5% relative noise (right).

General strategy

Geometric idea:

- Identify the manifold $\mathcal{A}$ of the attractor.
- Move noisy points $\mathbf{x}_n$ towards $\mathcal{A}$.

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A simple method: local averaging

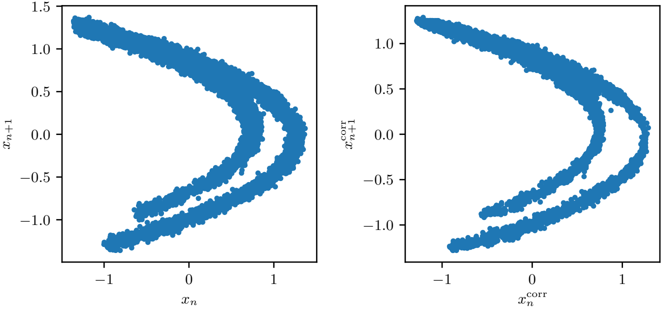
Proposed by Schreiber 1993:

1. Find neighbourhood $\mathcal{U}_{n,\varepsilon} = \{k \mid \|\mathbf{x}_k - \mathbf{x}_n\| < \varepsilon\}$ around $\mathbf{x}_n$.
2. Compute corrected vectors

$$x^{\mathrm{corr}}_{n+m/2} = \frac{1}{|\mathcal{U}_{n,\varepsilon}|} \sum_{k \in \mathcal{U}_{n,\varepsilon}} x_{k+m/2}.$$

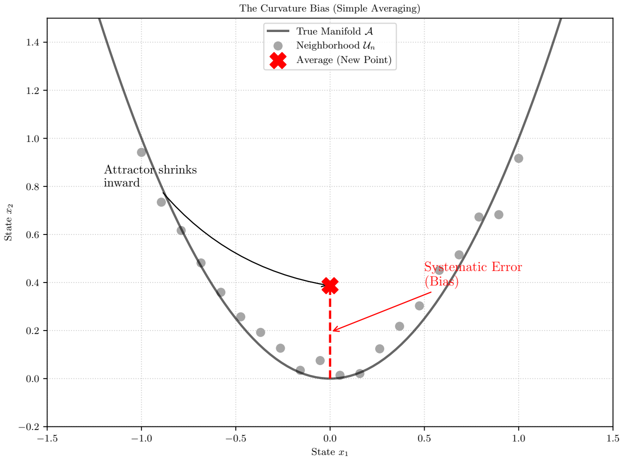
3. Replace all x_n.
4. Decrease $\varepsilon \to \mathrm{RMS}(x^{\mathrm{corr}}_n - x_n)$.
5. Iterate.

Simple noise reduction on the Hénon attractor



Noisy reconstructed Hénon attractor for 10000 steps (left) and attractor after applying the simple noise reduction algorithm for five iterations with $\varepsilon_0 = 15\%$.

Limitations



Curvature bias: In curved regions, the average pulls points inside the curve, shrinking the attractor.

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The projective method

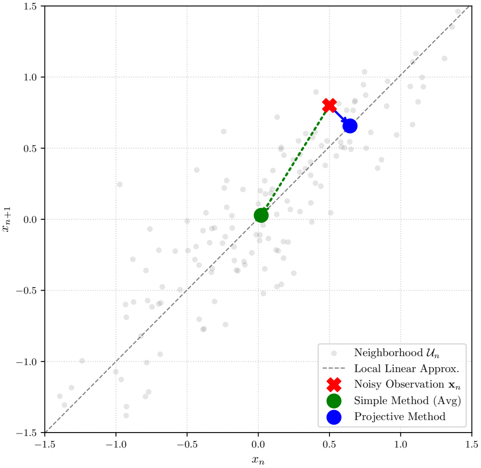
Assumption:

- The true D-dim attractor $\mathcal{A}$ is smooth enough to be locally approximated linearly.

Idea: (Grassberger et al. 1993)

1. Find (hyper-)plane that locally approximates the attractor in an ε-neighbourhood of $\mathbf{x}_n$.
2. Project $\mathbf{x}_n$ onto that subspace.
3. Reduce ε.
4. Iterate.

The projective method



Schematic visualization of how both methods apply noise correction to a point outside the attractor.

Constraints

Ideally:

$$F(\mathbf{y}_n) = 0$$

Approximation:

$$\mathbf{a}^{(n)} \cdot (\mathbf{x} - \overline{\mathbf{x}}^{(n)}) \approx 0.$$

Finding the subspace

For a fixed $\mathbf{x}_n$ with neighbourhood $\mathcal{U}_n$:

1. Covariance matrix:

$$C_{ij} = \frac{1}{|\mathcal{U}_n|} \sum_{k \in \mathcal{U}_n} x_{k+i} x_{k+j} - \overline{x_{k+i}} \cdot \overline{x_{k+j}}$$

2. Pricipal Component Analysis (PCA):

$$\sigma_1^2 \geq \ldots \geq \sigma_m^2 \text{ eigenvalues with eigenvectors } \mathbf{v}_1, \ldots, \mathbf{v}_m$$

3. Identify subspace:
   - Large eigenvalues $\rightarrow$ tangent
   - Small eigenvalues $\rightarrow$ orthogonal

Projecting onto the subspace

- Noise directions (small eigenvalues):

$$\mathbf{v}_{D+1}, \dots, \mathbf{v}_m \leftrightarrow \text{ orthogonal to the attractor}$$

- Correction:

$$\delta \mathbf{x}_n = -\sum_{k=D+1}^{m} (\mathbf{x}_n \cdot \mathbf{v}_k) v_k$$

The consistency problem

- The scalar x_n appears in m delay vectors

$$\mathbf{x}_n = (x_n, \ldots, x_{n+m-1})$$

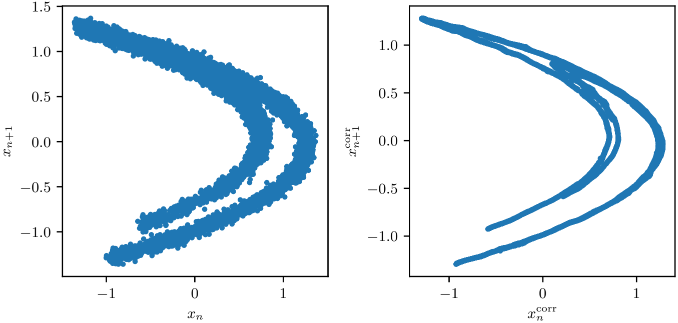
$$\vdots$$

$$\mathbf{x}_{n-m+1} = (x_{n-m+1}, \ldots, x_n).$$

- **Problem:** Projecting $\mathbf{x}_n$ independently gives m different corrections for x_n.
- **Solution:** Average

$$x_n^{\mathrm{corr}} = x_n + \frac{1}{m} \sum_{j=0}^{m-1} \delta x_{n-j}^{(j)}$$

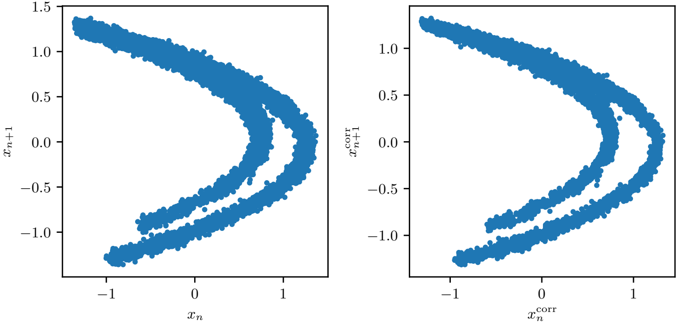
Projective noise reduction on the Hénon attractor



Noisy reconstructed Hénon attractor for 10000 steps (left) and attractor after applying the first order noise reduction algorithm for three iterations with $m = 5$ and $D = 2$.

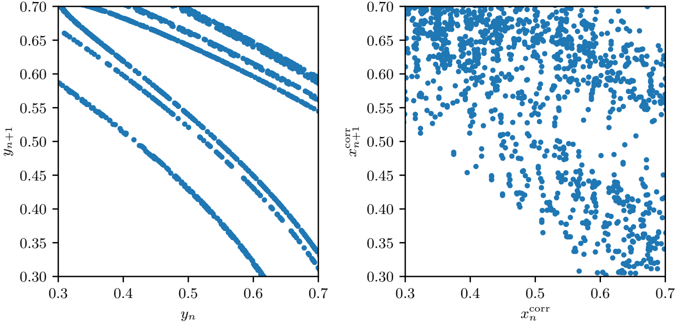
Outline

Performance of the zero-order algorithm



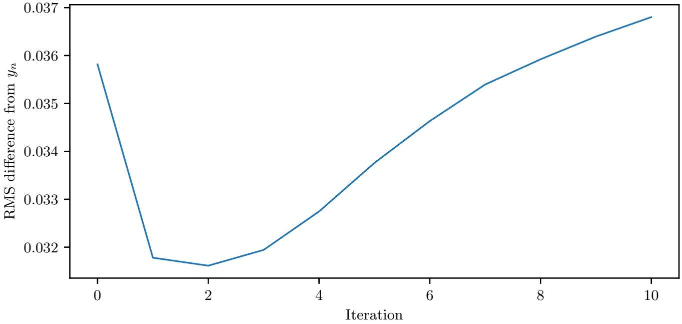
Noisy reconstructed Hénon attractor for 10000 steps (left) and attractor after applying the zero order noise reduction algorithm for two iterations with $m = 5$ and $D = 2$.

Performance of the zero-order algorithm



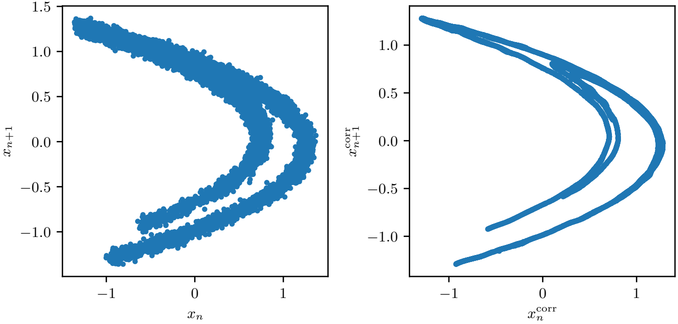
Zoomed view of the noise-free reconstructed Hénon attractor for 10000 steps (left) and attractor after applying the zero order noise reduction algorithm for two iterations with $m = 2$ and $\varepsilon_0 = 10\%$.

Change per iteration of the zero-order algorithm



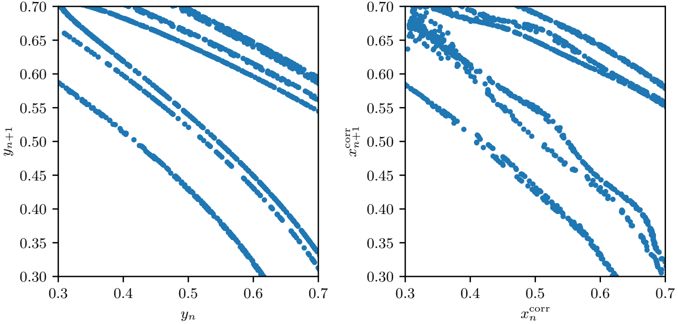
RMS difference of the true noise free timeseries y_n and the corrected timeseries x_n^{corr} for 10 iterations.

Performance of the first order algorithm



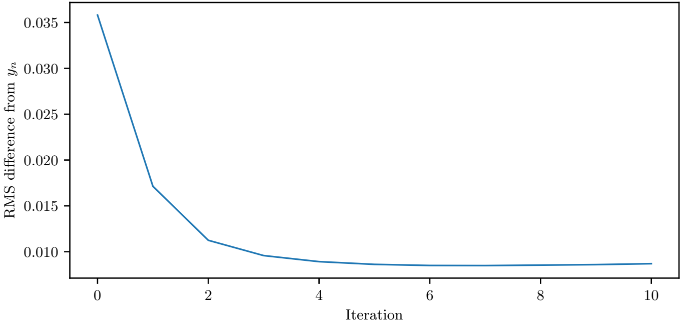
Noisy reconstructed Hénon attractor for 10000 steps (left) and attractor after applying the first order noise reduction algorithm for five iterations with $m = 5$ and $D = 2$.

Performance of the first order algorithm



Zoomed view of the noise-free reconstructed Hénon attractor for 10000 steps (left) and attractor after applying the first order noise reduction algorithm for five iterations with $m = 5$ and $D = 2$.

Change per iteration of the first order algorithm



RMS difference of the true noise free timeseries y_n and the corrected timeseries x_n^{corr} for 10 iterations.

Sources: Software

The software used to generate the datasets as well as apply the algorithms is the TISEAN package first published by Hegger, Kantz, and Schreiber 1999. The plots shown in this presentation were produced using the Python packages Matplotlib and Numpy. The Code can be found in the sources below:

- **TISEAN:** `https://www.mpipks-dresden.mpg.de/tisean/Tisean_3.0.1/index.html`
- **Presentation:** `https://github.com/lebega/ws2025-physics650-seminar-analyzing-biomedical-signals`

Sources: Literature

- Grassberger, Peter et al. (1993). "On noise reduction methods for chaotic data". In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 3.2, pp. 127–141.
- Hegger, Rainer, Holger Kantz, and Thomas Schreiber (1999). "Practical implementation of nonlinear time series methods: The TISEAN package". In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 9.2, pp. 413–435.
- Schreiber, Thomas (1993). "Extremely simple nonlinear noise-reduction method". In: *Physical Review E* 47.4, p. 2401.