Dimension Increase in Filtered Chaotic Signals

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An increase in fractal dimension is shown to occur in filtered chaotic signals. The dependence of the Lyapunov dimension on the filter parameters is used to predict the behavior of the information dimension which is directly evaluated for two experimental systems: the NMR laser and Rayleigh-Bénard convection. Good quantitative agreement with the theoretical predictions is found. The understanding of the role of filtering not only clarifies aspects relevant in the calculation of fractal dimensions, but also yields an indirect but precise way to evaluate Lyapunov exponents.

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Experimental chaotic signals are more and more frequently characterized by the evaluation of the fractal dimension of the underlying strange attractor, reconstructed in a suitable embedding space. In order to increase the signal-to-noise ratio, the experimental data are often filtered, with use of either analog or digital techniques. Moreover, the finite instrumental bandwidth of the measuring apparatus may produce similar effects. Here we show that filtering processes introduce additional Lyapunov exponents and may cause an increase of the fractal dimension $D(q)$. The results of direct evaluations of the information dimension for two sets of experimental data are also presented. The increase of $D(q)$ with decreasing filter-bandwidth η confirms quantitatively the theoretical predictions. A precise estimate of the Lyapunov exponents is also obtained.

Consider a physical system modeled by evolution equations of the form $\dot{\mathbf{u}}(t)=\mathbf{F}(\mathbf{u})$, where $\mathbf{u}(t)$ is the state vector in phase space. The nonlinear differentiable function $\mathbf{F}(\mathbf{u})$ determines the time behavior of $\mathbf{u}(t)$. In the case of an experimental system, a single component $x(t)$ of $\mathbf{u}(t)$ is usually measured, and its values are recorded as a scalar time series $\{x(t)\}$. We concentrate, for simplicity, on an ideal linear low-pass filter, whose action can be described by our adding to the original differential model the further equation

$$\dot{z}(t)=-\eta z(t)+x(t), \tag{1}$$

where $z(t)$ is the output of the filter and η the cutoff frequency. Since $z(t)$ is coupled to the system through $x(t)$, a faithful description of the dynamics can be obtained by the reconstruction of the attractor in an E-dimensional "embedding" space through points

$$\mathbf{z}_i \equiv \{z(t_i), z(t_i+\Delta t), \ldots, z(t_i+(E-1)\Delta t)\},$$

Δt being a suitably chosen sampling time. If we indicate with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_E$ the Lyapunov exponents of the original system, the Lyapunov dimension $D_{\rm L}$ of the unfiltered attractor is given by $D_{\rm L}=j+\sum_1^j \lambda_k/|\lambda_{j+1}|$, where j is the largest index for which the sum over k is nonnegative.[1] If the filter is present, a new Lyapunov exponent $\lambda_f=-\eta$ should be taken into account, while the others remain unaffected, as a result of the form of Eq. (1). As a consequence, the dimension $D_{\rm L}$ remains unchanged as long as $\eta \ge |\lambda_{j+1}|$. To discuss what happens when the latter inequality is no longer satisfied, we assume, for simplicity, that only three Lyapunov exponents (one of which is equal to 0) determine the information dimension of the unfiltered signal, i.e., that the relevant exponents are $\{\lambda_1, 0, \lambda_3, -\eta\}$, where the order depends on the magnitude of η. It is then easily seen that the Lyapunov dimension $D_{\rm L}(\eta)$ exhibits, as a function of η, the following behavior:

$$D_{\rm L}(\eta)=\begin{cases} 2+\lambda_1/|\lambda_3| & \text{for } -\eta<\lambda_3, \\ 2+\lambda_1/\eta & \text{for } \lambda_3<-\eta<-\lambda_1, \\ 3+(\lambda_1-\eta)/|\lambda_3| & \text{for } -\lambda_1<-\eta<0. \end{cases} \tag{2}$$

In the first region, corresponding to large η, the variable z can be adiabatically eliminated and coincides with x, apart from a scale factor. When the value $-\eta$ increases beyond λ_3, $\lambda_f=-\eta$ becomes the third Lyapunov exponent in the list and takes the place of $|\lambda_3|$ in the expression for $D_{\rm L}$. Finally, in the third region, η is so small that $\lambda_1-\eta$ is still nonnegative and the dimension becomes larger than 3, increasing linearly for $0<\eta<\lambda_1$. The largest increment (equal to 1) is obtained in the degenerate case $\eta=0$, when $z(t)$ is no longer a stationary process. In fact, $z(t)$ reduces to the integral of the chaotic signal $x(t)$ whose power spectrum $S(\omega)$ usually does not vanish faster than ω, for $\omega \rightarrow 0$. The first corner point in the curve $D_L(\eta)$ (T_1 in Fig. 1 corresponding to $\eta = -\lambda_3$) can be interpreted as a first-order phase transition, within the framework of the thermodynamical formalism for dissipative dynamical systems.[2] More generally, when we deal with multipole filters or with higher-dimensional attractors, the relative positions of the real parts of the filter's poles η_j and the system's Lyapunov exponents determine a piecewise differentiable function $D_L(\eta_j)$, analogous to that described above.

A particularly instructive example is provided by the simplest version of a high-pass filter, obtained by the substition of $\dot{x}(t)$ for $x(t)$ in Eq. (1). Since the real attractor is properly reconstructed from both $x(t)$ and $\dot{x}(t)$, the fractal dimension of the output $z(t)$ is the same in the two cases. This is confirmed by numerical simulations performed on the Hénon map, at the standard parameter values,[3] with $x_{n+1} - x_n$ in place of $\dot{x}(t)$ and the discrete filter equation $z_{n+1} = \gamma z_n + x_{n+1} - x_n$ (in which γ plays the role of $e^{-\eta\Delta t}$). The estimates were obtained with $n = 10^6$ points and $\gamma = 0$ and 0.8. However, an important difference exists between the spectral properties of the outputs of the two filters since they contain only the low- or high-frequency components of the original signal. Therefore, we have an example of the complexity of the relations (if any) between the shape of power spectra and the structure of probability distributions of strange attractors. Notice also that the high-pass filter leads to an apparent paradox in the limit $\eta \rightarrow 0$. On the one hand, from Eq. (2), one would expect a dimension increase of one unit while, on the other hand, the dimension should remain constant since the output of the filter coincides, for $\eta = 0$, with $x(t)$. The contradiction is removed by the observation that, for $\eta \ll \lambda_1$, two distinct time scales exist: a short one, in which $z(t)$ essentially follows $x(t)$, and a long (asymptotic) one, in which the effect of the filter is relevant. The second regime is reached in longer times as η tends to zero and, eventually, never occurs.

With the assumption that the Kaplan-Yorke conjecture $D_L = D(1)$ holds,[4] relation (2) can be compared with the results of accurate measurements of the information dimension $D(1)$ of both signals $x(t)$ and $z(t)$ for different values of η. In the case of systems of information dimension between 2 and 3, this comparison constitutes an excellent test for the consistency and accuracy of the numerical procedure used for the determination of the fractal dimension. The results of our measurements, performed on two different experimental systems with use of the fixed-mass method,[5] are discussed in the following paragraphs. Moreover, by fitting the experimental data by relation (2), one can estimate indirectly the values of the Lyapunov exponents. This indirect estimation can be more precise than procedures based on the evaluation of the metric entropy $K(1)$. When one follows direct approaches to the estimation of the metric entropy $K(1)$ (see Broggi and co-workers[6] and Grassberger[7]), it is in fact necessary to reach very large embedding dimensions in order to obtain a value sufficiently asymptotic in time and to get rid of spurious fluctuations. In comparison to direct estimates of Lyapunov exponents,[8] we only recall that the present method, based on the evaluation of fractal dimensions, does not require the proper setting of many parameters. This requirement was mentioned by the same authors[8] as being needed for a sufficient approximation of the linearized flow to be achieved.

In our algorithm,[6] m reference points $\mathbf{y}_j$ are chosen on the attractor, at random with respect to the natural measure. The distances $\delta = \delta_{j,k,E}(n)$ between the point $\mathbf{y}_j$ and its kth nearest neighbor, chosen among the n generic data points x, are then computed in the E-dimensional embedding space. The information dimension $D(1)$ is evaluated by use of the asymptotic relation

$$D(1) \sim -\frac{\log(n/k)}{\langle \log\delta_{k,E}(n)\rangle}, \tag{3}$$

for $n \rightarrow \infty$, $k \rightarrow \infty$, $(n/k) \rightarrow \text{const.}$, $\forall E > 2D(1)+1$. The actual computation of $D(1)$ is performed by our first plotting, as a function of $\log(n)$, the quantity

$$\langle \log\delta_{k,E}(n)\rangle = \frac{1}{m}\sum_{j=1}^{m} \log\delta_{j,k,E}(n),$$

for chosen values of embedding dimensions E and nearest-neighbor order k, and then computing the slopes of the regression lines of these curves, in selected intervals of $\log(n)$. In our numerical experiments, particular efforts were devoted to the evaluation of the magnitude of the uncertainty in the measurement of $D(1)$, arising either from intrinsic fluctuations (lacunarity of the attractor) or from statistical or numerical problems. The error bounds given in the figures are based not only on the mean square error in the computation of the slopes of the above-mentioned regression lines, but also on the spread of the curves $D_E(n,k)$ versus E or n, around the extrapolated asymptotic value $D(1)$. To eliminate the effects due to the nonstationarity of the signal (slow experimental drifts), the indices of the points (i.e., vectors of order E) in the embedding space were disordered according to a random permutation of the original (natural) sequence. These and other techniques used to increase the reliability of the measurements are extensively described in Ref. 6.

The first experimental system we considered is the NMR laser[9] with externally modulated cavity Q factor. The output signal, proportional to the transverse magnetization M_v of the spin system, was recorded in a moderately chaotic regime. The experimental data set consisted of a string of 249984 integers, sampled with twelve-bit resolution, about 30 times per period of the external forcing field. Since the original sampling frequency is high (as necessary for a proper numerical filtering of the data), only one point out of four of the original string was utilized for the measurement of the fractal dimension (see Ref. 6). Consequently, the calculation was performed with $n=55489$ data points, $m=5000$ reference points, embedding dimensions E up to 50, and nearest-neighbor order k ranging from 5 to 50. The value $D(1)=2.35\pm0.05$ was obtained. The experimental data were then numerically filtered according to Eq. (1), with the choice of several different values of $\tilde{\eta}=\eta\Delta t$, with $\Delta t=3.34\times10^{-4}$ s. A suitable number of points at the beginning of the filter's output string were also eliminated, to get rid of transient effects. For the reasons mentioned above, a post-sampling process was carried out by keeping one point out of four, five, six, or seven in the original sequence, according to the value of $\tilde{\eta}$. The results of these computations are displayed in Fig. 1. For $\tilde{\eta}\geq\tilde{\eta}_3\cong0.0243$, the filtered signal has the same dimension as the original one; in the region $0.0085\cong\tilde{\eta}_1\leq\tilde{\eta}\leq\tilde{\eta}_3\cong0.0243$ a sudden, nonlinear increase in $D(1)$ is observed; finally, for $\tilde{\eta}\leq\tilde{\eta}_1\leq0.0085$, the dimension increases almost linearly with $\tilde{\eta}$. Notice that the accuracy of the evaluation of $D(1)$ is systematically poorer around the transition points T_1 and T_2, where a critical slowing-down effect appears.[2] The dashed curve in the figure shows the behavior of the Lyapunov dimension $D_L(\tilde{\eta})$, as given by Eqs. (2) for values of the Lyapunov exponents $\tilde{\lambda}_1=\lambda_1\Delta t=0.0085$, $\tilde{\lambda}_2=\lambda_2\Delta t=0$, and $\tilde{\lambda}_3=\Delta t=-0.0243$. The value of the positive exponent should be compared with the numeric result $K(1)=0.013\pm0.005$ obtained by the application of the constant-mass method.[6]

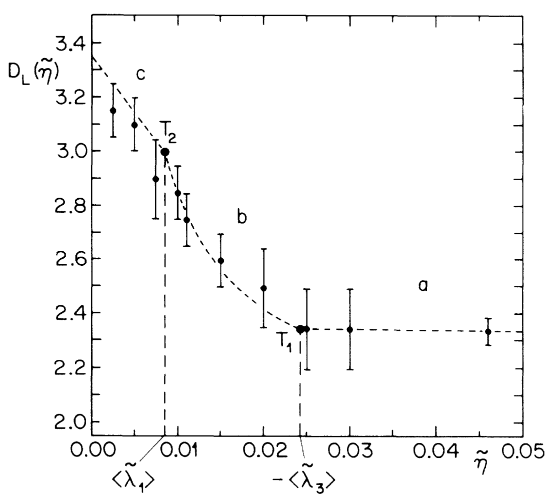

FIG. 1. Estimates of the information dimension $D(1)$ of the NMR-laser signal, processed with a low-pass filter, for various values of the cutoff frequency $\tilde{\eta}$ (small dots with error bars). The dashed line shows the analytical Lyapunov dimension $D_L(\tilde{\eta})$, representing the best fit to the numerical estimates, obtained for values of the average Lyapunov exponents $\langle\tilde{\lambda}_1\rangle=0.0085$, $\langle\tilde{\lambda}_2\rangle=0$, and $\langle\tilde{\lambda}_3\rangle=-0.0243$. The two transition points are indicated by T_1 and T_2 (big dots). All quantities are in units of the sampling time $\Delta t=3.34\times10^{-4}$ s.

The second experimental signal we analyzed refers to a Rayleigh-Bénard instability in silicone oil with a Prandtl number around 30.[10] The aspect ratio of the cell was 2 (the other two sides being equal) and the Rayleigh number was set to about 400 times the critical value. The horizontal temperature gradient at the center of the cell was recorded, with a twelve-bit resolution and sampling frequency of 4 Hz (corresponding to roughly 80 times the main Fourier frequency of the system). The data string, consisting of 512000 points, was analyzed as before, yielding a dimension $D(1)=2.50\pm0.05$. After filtering, even though the evaluations became more difficult (lack of asymptoticity), it was possible to observe a dimension increase to $D(1)\approx3$.

The measurements described in the previous paragraphs give evidence that filtering processes may increase the dimension of a chaotic signal. In particular, we have considered linear low- and high-pass filters and systems having three Lyapunov exponents, for which, by means of the Kaplan-Yorke relation, one can predict such an increase to occur if the cutoff frequency η of the filter is smaller than the modulus of the third Lyapunov exponent. The behavior of the measured dimension as a function of η is, in the studied cases, in good agreement with the theoretical analysis. Although our results have been obtained for numerically filtered data, the same effects should appear if the signal is processed during the experimental acquisition.[11] Although we concentrated on the simple, single-pole filter described by Eq. (1), we must recall that as long as the effect of filtering can be described by a differential equation and the Kaplan-Yorke formula holds, an analogous dimension increase is to be expected. A quantitative description of the phenomenon can be obtained by comparison of the magnitude of the (*a priori* unknown) Lyapunov exponents with the real parts of the filter's poles. In case no quantitative description is needed, it is sufficient to vary the filter's parameters around suitably assigned values. The independence of the fractal dimension of such changes indicates that we are in the region where the filter plays no role in the determination of the geometrical structure of the attractor [i.e., the region indicated by (a) in the figure]. Finally, notice that our indirect estimate of the Lyapunov exponents, obtained by fitting the computed dimensions as shown in Fig. 1, is in many cases more precise and reliable than any direct computation of these quantities making use of the same amount of data.

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