5.5 Law of the unconscious statistician (LOTUS)

As we saw from the St. Petersburg paradox, $E(g(X)) \neq g(E(X))$ if function g is not linear. So to calculate $E(g(X))$, we follow these steps:

- Get the distribution of $g(X)$ since its a random variable.
- Calculate $E(g(X))$ using the definition of expectation

The Law of the unconscious statistician allows us to calculate $E(g(X))$ directly without going through those steps.

Theorem 5.5.1 (LOTUS)

If X is a discrete r.v and g is a function from $\mathbb{R}$ to $\mathbb{R}$, then:

$$E(g(X)) = \sum_x g(x) \cdot \underbrace{P(X=x)}_{\text{PMF of } X}$$

where the sum is taken over all possible values of X.

Relation to $E(X)$: Recall that $E(X) = \sum_x x \cdot P(X=x)$, going from $E(X)$ to $E(g(X))$ only requires changing x to $g(x)$ in definition, hence the "unconscious"

Proof: Let X has support $0, 1, 2, \ldots$ with probabilities $p_0, p_1, p_2, \ldots$

so PMF is $P(X=n) = p_n$

Let X^3 has support $0^3, 1^3, 2^3, \ldots$ with probabilities $p_0, p_1, p_2, \ldots$

So: $$\begin{cases} E(X) = \sum_{n=0}^{\infty} n\, p_n \\ E(X^3) = \sum_{n=0}^{\infty} n^3\, p_n \end{cases}$$