## 5.6 Variance

One important application of LOTUS is for finding the variance of random variable. The variance tells us how spread out the distribution is

**Definition 5.6.1** (Variance and Standard Deviation)

The variance of an r.v $X$ is:

$$Var(X) = E(X - EX)^2$$

This means expectation of r.v $(X - EX)^2$, not $(E(X-EX))^2$ (which is 0 by linearity)

The square root of the variance is called the standard deviation (SD):

$$SD(X) = \sqrt{Var(X)}$$

Why use average squared difference instead of just the average difference?

- The average difference, denoted $E(X - EX)$, always equals 0 since positive and negative deviations cancel each other out.
- Squaring the deviations ensure both positive and negative contribute to overall variability

Why not use absolute value, $E|X - EX|$, since achieves the same goal as squaring?

- The absolute function is not differentiable at 0
- Squared differences also connected to geometry via distance formula and Pythagorean

Another formula of variance which is easier to work with when doing actual calculations is:

**Theorem 5.6.2** (Another formula of variance)

For any r.v $X$,

$$Var(X) = E(X^2) - (EX)^2$$

Proof: Let $\mu = EX$. Expand $(E-\mu)^2$ using linearity:

$$Var(X) = E(X-\mu)^2 = E(X^2 - 2\mu X + \mu^2) = E(X^2) - 2\mu \underbrace{EX}_{\mu} + \mu^2 = E(X^2) - \mu^2$$

Properties of variance:

- $Var(X + c) = Var(X)$ for any constant $c$ (shifting distribution)
- $Var(cX) = c^2 Var(X)$ for any constant $c$
- $Var(X+Y) = Var(X) + Var(Y)$ if $X$ and $Y$ are independent (proof in chapter 7)
- $Var(X) \geq 0$, with equality i.o.i $P(X=a) = 1$ for some constant $a$. In other words, the only r.v.s that have zero variance are constants (degenerate r.v.s)

Example 5.6.3 (Geometric and Negative Binomial variance)

- Let $X \sim Geom(p)$, we know that $E(X) = \frac{q}{p}$, by LOTUS:

$$E(X^2) = \sum_{k=0}^{\infty} k^2 P(X=k) = \sum_{k=0}^{\infty} k^2 p q^k = \sum_{k=1}^{\infty} k^2 p q^k$$

- We'll find simpler $E(X^2)$ using similar tactic to how we find the expectation, starting with the geometric series:

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

$$\Leftrightarrow \sum_{k=1}^{\infty} k q^{k-1} = \frac{1}{(1-q)^2} \quad \langle \text{take derivative both sides respect to } q \rangle$$

$$\Leftrightarrow \sum_{k=1}^{\infty} k q^k = \frac{q}{(1-q)^2} \quad \langle \text{multiply both sides by } q \rangle$$

$$\Leftrightarrow \sum_{k=1}^{\infty} k^2 q^{k-1} = \frac{1+q}{(1-q)^3} \quad \langle \text{take derivative again} \rangle$$

$$\Leftrightarrow \sum_{k=1}^{\infty} k^2 p q^k = \frac{q(1+q)}{p^2} \quad \langle \text{multiply both sides by } pq \rangle$$

So, $E(X^2) = \sum_{k=1}^{\infty} k^2 p q^k = \frac{q(1+q)}{p^2}$

Finally, $Var(X) = E(X^2) - (EX)^2 = \frac{q(1+q)}{p^2} - \left(\frac{q}{p}\right)^2 = \frac{q}{p^2}$

$$\boxed{Var(X) = \frac{q}{p^2} \text{ is variance of Geometric and Negative Binomial}}$$

Example 5.6.4 (Binomial variance)

- Let $X \sim Bin(n,p)$, it can also be written as $X = I_1 + I_2 + \dots + I_n$ where $I_j$ is the indicator of the $j^{th}$ trial being a success. So:

$$Var(I_j) = E(I_j^2) - (E(I_j))^2 = p - p^2 = p(1-p)$$

- Since $I_j$ are independent, we can add variances to get variance of their sum:

$$Var(X) = Var(I_1) + Var(I_2) + \dots + Var(I_n)$$

$$= np(1-p)$$

$\boxed{Var(X) = np(1-p) \text{ is variance of Binomial}}$