6.3 Practice Problems

Problem 1a:

a) A stick is broken into 3 pieces by picking 2 points independently and uniformly along the stick.
What is the probability that the 3 pieces can be assembled into a triangle ?

- Let X, Y be 2 breakpoints on the stick. $X, Y \overset{i.i.d}{\sim} \text{Uniform}(0,1)$
- A triangle of 3 lines a, b, c can only be formed if $a, b, c \in \left(0, \frac{1}{2}\right)$
- There are 2 scenarios:
  - When $x > y$:

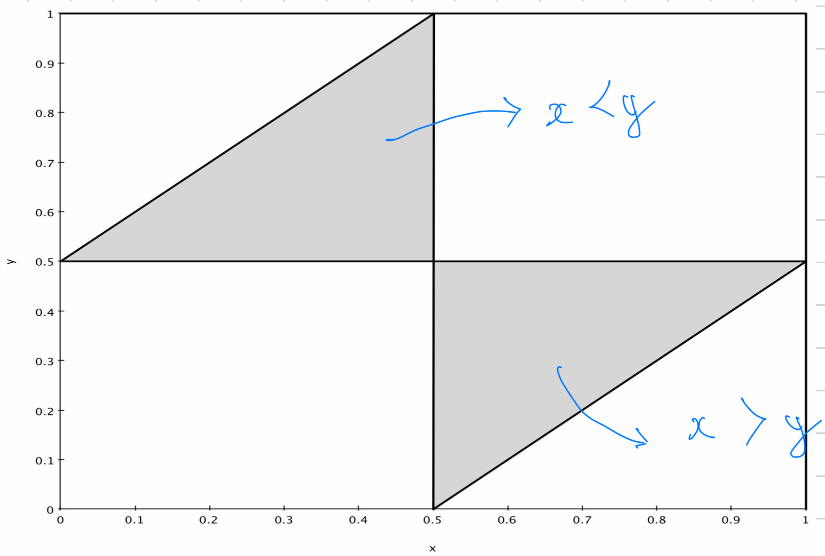
    Then the 3 sides are: $y,\ x-y,\ 1-x$.

    $$P(\text{triangle}) = P\left(y < \tfrac{1}{2},\ x-y < \tfrac{1}{2},\ 1-x < \tfrac{1}{2}\right)$$
    $$= P\left(y < \tfrac{1}{2},\ x < y + \tfrac{1}{2},\ x > \tfrac{1}{2}\right)$$
    $$= 1/8$$

  - When $x < y$:

    Then the 3 sides are: $x,\ y-x,\ 1-y$.

    $$P(\text{triangle}) = P\left(x < \tfrac{1}{2},\ y-x < \tfrac{1}{2},\ 1-y < \tfrac{1}{2}\right)$$
    $$= P\left(x < \tfrac{1}{2},\ y < x + \tfrac{1}{2},\ y > \tfrac{1}{2}\right)$$
    $$= 1/8$$



Problem 1b:

Three legs of a round table are positioned uniformly and independently. What is the probability that the table will stand

Let A, B, C be arc lengths from one leg to another, assume the table circumference is 1:

$$P(\text{table falls}) = P(\text{3 legs are all in some semicircle})$$
$$= P(\text{at least one of } A, B, C \text{ is } > \tfrac{1}{2})$$
$$= \frac{3}{4}$$

$$\Rightarrow P(\text{table stand}) = 1 - P(\text{table falls}) = \frac{1}{4}$$

Problem 2:

Two fair dice are rolled, with outcomes X and Y respectively.

a) Compute covariance of $X+Y$ and $X-Y$

$$\mathrm{Cov}(X+Y, X-Y)$$
$$= \mathrm{Cov}(X,X) \underbrace{- \mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X)}_{0} - \mathrm{Cov}(Y,Y)$$
$$= \mathrm{Cov}(X,X) - \mathrm{Cov}(Y,Y)$$
$$= \mathrm{Var}(X) - \mathrm{Var}(Y)$$
$$= 0 \quad (\text{Since } X \text{ and } Y \text{ i.i.d})$$

b) Are $X+Y$ and $X-Y$ independent?

No, consider the case where $X+Y = 12$ then $X = Y = 6$:

$$P(X-Y=0 \mid X+Y=10) = 1 \neq P(X-Y=0)$$

Problem 3: A coin with probability of heads $p = \frac{1}{3}$ is flipped repeatedly

a) What is the expected number of flips before pattern HT appears?

- Let
  - W_{HT} be number of flips before HT appears
  - W_H be number of flips before the first Head. $W_H \sim FS(\frac{1}{3})$
  - W_T be number of flips before the first Tail. $W_T \sim FS(\frac{2}{3})$

- By linearity: $E(W_{HT}) = E(W_H) + E(W_T)$

$$= \frac{1}{p_H} + \frac{1}{p_T}$$

$$= \frac{1}{1/3} + \frac{1}{2/3}$$

$$= 4,5$$

b) What is expected number of flips until pattern HH appears?

- Let W_{HH} be number of flips until HH.
- Condition on the first flip:

$$E(W_{HH}) = E(W_{HH} \mid \text{first flip } H) \cdot P(\text{first flip } H) + E(W_{HH} \mid \text{first flip } T) \cdot P(\text{first flip } T)$$

$$= E(W_{HH} \mid \text{first flip } H) \cdot \frac{1}{3} + E(W_{HH} \mid \text{first flip } T) \cdot \frac{2}{3}$$

- For first term:

$$E(W_{HH} \mid \text{first flip } H) = E(W_{HH} \mid \text{first flip } H, \text{ second flip } H) \cdot \frac{1}{3} + E(W_{HH} \mid \text{first flip } H, \text{ second flip } T) \cdot \frac{2}{3}$$

$$= 2 \cdot \frac{1}{3} + (1 + E(W_{HH})) \cdot \frac{2}{3}$$

- For second term:

$$E(W_{HH} \mid \text{first flip } T) = 1 + E(W_{HH})$$

- Plugging these 2 terms into original equation:

$$E(W_{HH}) = \left(2 \cdot \frac{1}{3} + (1 + E(W_{HH})) \cdot \frac{2}{3}\right) \cdot \frac{1}{3} + 1 + E(W_{HH})$$

$$\Rightarrow E(W_{HH}) =$$

$$x = \left(\frac{2}{3} + \frac{2}{3} + \frac{2}{3}x\right)\frac{1}{3} + 1$$

$$= \frac{4}{9} + \frac{4}{9}x + 1$$