5.4 Indicator random variables and the fundamental bridge

Recall that an indicator r.v I_A for event A is $\begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$

So $I_A \sim \text{Bern}(p)$. Some useful properties of indicator r.v.s are:

Theorem 5.4.1 (Indicator r.v properties)

Let A and B be events. The following properties hold:

1. $(I_A)^k = I_A$ for any positive integer k
2. $I_{A^c} = 1 - I_A$
3. $I_{A \cap B} = I_A \cdot I_B$
4. $I_{A \cup B} = I_A + I_B - I_A \cdot I_B$

Indicator r.v.s provide a link between probability and expectation; this link is called <u>fundamental bridge</u>.

Theorem 5.4.2 (Fundamental bridge between probability and expectation)

There is a 1-to-1 correspondence between events and indicator r.v.s, and the probability of event A is the expected value of its indicator r.v I_A:

$$P(A) = E(I_A)$$

Proof: By definition of expectation for discrete random variables:

$$E(I_A) = 1 \cdot P(A) + 0 \cdot P(A^c) = P(A)$$

<u>What is the applications of fundamental bridge?</u>

- Allow us to express any probability as expectation
- Find the expectation of the indicators that made up a discrete r.v, then using linearity, obtain the expectation of the discrete r.v

Example 5.4.3 (Putnam problem)

Consider a permutation $a_1, a_2, \dots, a_n$ of $1, 2, \dots, n$ has a local maximum at j if $a_j > a_{j-1}$ and $a_j < a_{j+1}$. For example, 4 2 5 3 6 1 has 3 local maxima, at position 1, 3 and 5.

The Putnam exam posted the following question, for $n \geq 2$, what is the average number of local maxima of a random permutation of $1, 2, \ldots, n$ with all $n!$ permutations equally likely?

- Let $\begin{cases} N \text{ be r.v represent number of local maxima in a permutation.} \\ I_1, \ldots, I_n \text{ be indicator r.v.s, where } I_j = \begin{cases} 1 & \text{if there is local maximum at } j \\ 0 & \text{otherwise} \end{cases} \end{cases}$

- We are interested in $E(N)$, which equals $E\left(\sum_{j=1}^{n} I_j\right)$

  since $N = \sum_{j=1}^{n} I_j$

- Now, consider a permutation $a_1 \; \underbrace{a_2 \; a_3 \; a_4} \; a_5$ for example

These numbers have probability $\frac{1}{3}$ of being local maxima since there are 2 neighbors

These 2 has probability $\frac{1}{2}$ of being local maxima since there is only 1 neighbor

By linearity: $E\left(\sum_{j=1}^{n} I_j\right) = 2 \cdot \frac{1}{2} + (n-2)\frac{1}{3} = \frac{n+1}{3}$