5.8 Expectation of a continuous random variable

- The definition of expectation for continuous r.v.s is analogous to the definition for discrete r.v.s ; we just replace the sum with an integral and PMF with the PDF.

Definition 5.8.1 (Expectation of continuous random variables)

The expected value of a continuous r.v X with PDF f is:
$$E(X) = \int_{-\infty}^{\infty} x \, f(x) \, dx.$$

Note: The integral is taken over the entire real line, but if the supports of X is not the entire real line, then we only need to take integral on the support.

- Linearity of expectation also holds for continuous r.v.s
- LOTUS also holds for continuous r.v.s, replacing the sum with an integral and PMF with PDF.

Theorem 5.8.2 (LOTUS, for continuous r.v.s)

If X is a continuous r.v with PDF f and g is a function from $\mathbb{R}$ to $\mathbb{R}$ then:
$$E(g(X)) = \int_{-\infty}^{\infty} g(x) \, f(x) \, dx$$

Example 5.8.3 (Mean and variance of Uniform r.v)

- The mean is very intuitive: the PDF is constant, so the mean should be the midpoint of (a, b). We can verify this using the definition of expected value for continuous r.v. :
$$\boxed{E(U) = \int_a^b x \cdot \frac{1}{b-a} \, dx = \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{a+b}{2}} \to \text{mean of Uniform}$$

- Finding the variance: $Var(U) = \underbrace{E(U^2)} - (E(U))^2$
$$E(U^2) = \int_a^b x^2 \frac{1}{b-a} \, dx = \frac{1}{3} \cdot \frac{b^3 - a^3}{b-a}$$

So: $$\boxed{Var(U) = \frac{1}{3} \cdot \frac{b^3 - a^3}{b-a} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}} \to \text{variance of Uniform}$$

Example 5.8.4 (Mean and variance of Normal r.v.)

- Lets verify $N(\mu, \sigma^2)$ indeed have mean μ and variance σ^2.
- Consider r.v Z that has Standard Normal distribution. By symmetry, its mean must be 0, we can also check this by looking at $E(Z)$.

$$E(Z) = \int_{-\infty}^{\infty} z \cdot f(z)\, dz = \int z \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2}\, dz$$

Since $g(z) = z e^{-z^2/2}$ is an odd function, meaning $g(-z) = -g(z)$, the area in $(-\infty, 0)$ cancel the area in $(0, \infty)$, therefore $E(Z) = 0$.

- For the variance, we can use LOTUS, we still dealing with Standard Normal distributed r.v Z:

$$\begin{aligned} Var(Z) = E(Z^2) - (EZ)^2 &= E(Z^2) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2}\, dz \\ &= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-z^2/2}\, dz \end{aligned}$$

Last step uses $g(x) = z^2 e^{-z^2/2}$ is an even function, meaning $g(-x) = g(x)$, using integration by parts, we can arrive at: $Var(Z) = 1$

- Now that we proved that r.v $Z \sim N(0,1)$ indeed has mean $\mu = E(Z) = 0$ and variance $\sigma^2 = Var(Z) = 1$, we can extend this to $X \sim N(\mu, \sigma^2)$

For $X \sim Unif(\mu, \sigma^2)$, we can write:

$$X = \mu + \sigma Z, \quad \text{with } Z \sim N(0,1)$$

Then: $\begin{cases} E(X) = \mu + \sigma \cdot 0 = \mu \\ Var(X) = \sigma^2 Var(Z) = \sigma^2 \end{cases}$

Example 5.8.5 (Mean and variance of an Exponential r.v)

- Start by finding mean and variance of an r.v $X \sim Expo(1)$

Mean: $E(X) = \int_{-\infty}^{\infty} x e^{x}\, dx = 1$

Variance:
$$\begin{aligned} Var(X) &= E(X^2) - (EX)^2 \\ &= \int_0^{\infty} x^2 e^{-x}\, dx - 1 \\ &= 2 - 1 = 1 \end{aligned}$$

- Now let $Y = \frac{X}{\lambda} \sim Expo(\lambda)$. Then:

$$\boxed{\begin{aligned} E(Y) &= \frac{1}{\lambda} E(X) = \frac{1}{\lambda} \\ Var(Y) &= \frac{1}{\lambda^2} Var(X) = \frac{1}{\lambda^2} \end{aligned}}$$

→ Mean and variance of Exponential

Example 5.8.6 (Blissville and Blotchville)

Fred lives in Blissville, where buses arrive on time, with time between buses fixed at 10 minutes. Having lost his watch, Fred arrives at the bus stop at a uniformly random time on a certain day

a) What is the distribution of how long Fred has to wait for the next bus? What is the average time that Fred has to wait?

The distribution is Uniform on $(0, 10)$, so the mean is 5 minutes

↳ Because:
- An event happens at regular intervals, and r.v can occur at any point between these intervals.
- Equally likely outcomes of r.v occurs on the interval
- Independence of r.v occurs on the interval

b) Given that the bus has not yet arrived after 6 minutes, what is the probability that Fred has to wait at least 3 more minutes?

Let W be the waiting time. Then:

$$P(W \geqslant 6+3 \mid W > 6) = \frac{P(W \geqslant 9, W > 6)}{P(W > 6)}$$

$$= \frac{P(W \geqslant 9)}{P(W > 6)} = \frac{1/10}{4/10} = \frac{1}{4}$$

So, Fred's waiting time is not memoryless. Because condition on waiting 6 minutes already, there is $\frac{1}{4}$ chance that he'll have to wait for another 3 minutes, while if he has just arrived, there would be $P(W \geqslant 3) = \frac{7}{10}$ chance that he'll have to wait 3 minutes.

c) Now the bus in another town, Blotchville, arrived at exponential rate with mean 10 minutes. Fred arrives at the bus stop at random time, not knowing how long ago the previous bus came.
What is the distribution of Fred's waiting time for the next bus?
What is the average time that Fred has to wait?

- By memoryless property, the distribution is Exponential with parameter $\lambda = \frac{1}{10}$ (and mean 10 minutes). Regardless of how long Fred has waited how much longer the next bus arrives is the same as when Fred has just showed up.
- The average waiting time is 10 minutes.

d) Fred's friend make a statement "You arrive at a uniform instant between the previous bus arrival and the next bus arrival. The average length of that interval is 10 minutes, but since you equally likely to arrive at any time in that interval, your average waiting time is only 5 minutes". Is this statement correct?

- The statement is incorrect, as Fred's friend is making a mistake explained in Warning 5.1.3, of replacing a r.v with its expected value, thereby ignoring the variability in inter arrival times.
- Yes, the average interval between 2 buses is 10 minutes, but Fred is more likely to arrive at the longer intervals than shorter intervals. This is called length-biasing