## Homework Problems:

**Problem 1:** A circle with a random radius $R \sim \text{Unif}(0,1)$ is generated. Let $A$ be its area. Find the PDF of $A$.

. CDF of $A$ is: $F(a) = P(A \leq a)$

$$= P(\pi R^2 \leq a) \quad \langle A \text{ is area of circle with } R \text{ radius} \rangle$$

$$= P\left(R \leq \sqrt{a/\pi}\right) \quad \text{for } 0 < \sqrt{\tfrac{a}{\pi}} < 1$$

$$= \sqrt{\frac{a}{\pi}} \quad \text{for } 0 < a < \pi \quad (\text{and } 0 \text{ otherwise})$$

. PDF of $A$ is: $f(a) = F'(a) = \dfrac{d\,F(a)}{dx} = \dfrac{d\sqrt{a/\pi}}{dx}$

$$= \frac{1}{2\sqrt{\pi a}}$$

**Problem 2:** Let $U \sim \text{Unif}(0,1)$. Suppose that we want to construct $X$ as a function of $U$, such that $X \sim \text{Expo}(\lambda)$. Define $X$.

. According to Universality of Uniform part 1:

$X = F^{-1}(U)$ is a r.v with CDF $F$

. Since $X \sim \text{Expo}(\lambda)$, $F(U) = 1 - e^{-\lambda U}$

. Now we find the inverse of $F(U)$, $F^{-1}(U)$:

$$F(U) = 1 - e^{-\lambda U}$$

$$\Leftrightarrow e^{-\lambda U} = 1 - u \quad \langle \text{set } u = F(U) \rangle$$

$$\Leftrightarrow \ln(e^{-\lambda U}) = \ln(1-u)$$

$$\Leftrightarrow -\lambda U = \ln(1-y)$$

$$\Leftrightarrow U = -\frac{1}{\lambda}\ln(1-u)$$

Therefore, $F^{-1}(u) = -\frac{1}{\lambda}\ln(1-u)$

Hence, $F^{-1}(U) = -\frac{1}{\lambda}\ln(1-U)$

Problem 3: The distance between 2 points need to be measured in meters. The true distance between the points is 10 meters, but due to measurement error we can't get the exact number. Instead, we will observe a value of $10 + \varepsilon$, where the error $\varepsilon$ is distributed $N(0, 0.04)$. Find the probability that the observed distance is with 0.4 meters of the true distance (10 meters). Give an approx numerical answer.

We need to find $P(-0.4 < \varepsilon < 0.4)$, 1 way to get approximate answer is to use 68 - 95 - 99.7 rule since $\varepsilon \sim N(0, 0.04)$

. $P(-0.4 < \varepsilon < 0.4) = P(5 \times (-0.4) < 5\varepsilon < 5 \times 0.4)$

$= P(-2 < 5\varepsilon < 2)$

$= 0.95$

Problem 4: Emails arrive in an inbox according to a Poisson process with rate 20 emails/hour. Let $T$ be the time at which the third email arrives, measured in hours after a certain fixed starting time. Find $P(T > 0.1)$ without using calculus.

. Let $\begin{cases} N_t & \text{be number of email arrived in period } (0, t]. \\ T_t & \text{be the time at which } t \text{ email arrives.} \end{cases}$

. According to count - time duality and $N_t \sim Pois(\underbrace{\lambda t}_{2})$ $\langle \lambda . t = 20 \times (0.1) = 2 \rangle$

$P(T_3 > 0.1) = P(N_t \leq 2)$

$= P(N_t = 0) + P(N_t = 1) + P(N_t = 2)$

$= e^{-2} + e^{-2} . 2 + e^{-2} . 2^2/2!$

$\approx 0.6767$