Unit 2: THE SVD

Orthogonal vectors

Dot product

With $x, y \in \mathbb{C}^m$

$$x^H y = \bar{x}^T y = \overline{x^T} y = \sum_{i=0}^{m-1} \bar{x}_i y_i$$

Hw 2.2.1.1

ALWAYS. $\overline{x^H y} = y^H x$

Hw 2.2.1.2

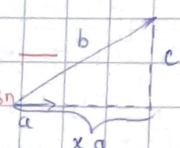
ALWAYS: $x^H x$ is real-valued

Orthogonal vectors

Let $x, y \in \mathbb{C}^m$. x and y are orthogonal if:

$$x^H y = 0$$

Component in the direction of a vector



(T to H if $a \in \mathbb{C}$)

- $\chi a = \left(\frac{a a^T}{a^T a}\right) b$
- $c = Ib - \left(\frac{a a^T}{a^T a}\right) b = \left(I - \frac{a a^T}{a^T a}\right) b$

Hw 2.2.2.1

$$\left(\frac{a a^H}{a^H a}\right)\left(\frac{a a^H}{a^H a}\right) = \frac{a a^H}{a^H a} \quad \text{ALWAYS}$$

Orthogonally projecting the orthogonal projection yields the projection itself

Hw 2.2.2.2

$$\left(\frac{a a^H}{a^H a}\right)\left(I - \frac{a a^H}{a^H a}\right) = 0 \quad \text{ALWAYS}$$

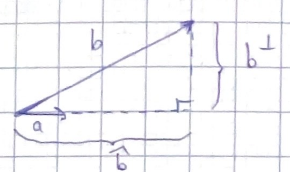
If you: • first, orthogonally projecting onto the space orthogonal to vector a.
• then you orthogonally projecting the resulting vector that a, you would gets zero

Hw 2.2.2.3

Let $a, b \in \mathbb{C}^n$, $\hat{b} = \frac{a a^H}{a^H a} b$, $b^\perp = b - \hat{b}$

ALWAYS: $\hat{b}^H b^\perp = 0$

A ~~bb~~ better note for component in the direction of a vector



$$\hat{b} = \left(\frac{a a^T}{a^T a}\right) b = b - b^\perp$$

$$b^\perp = \left(I - \frac{a a^T}{a^T a}\right) b = b - \hat{b}$$

Othonormal vectors

Let $u_0, u_1, \dots, u_{n-1} \in \mathbb{C}^m$. These vectors are mutually orthonormal if

$$u_i^H u_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Hw 2.2.3.1

ALWAYS: With $u \neq 0 \in \mathbb{C}^m$

$\frac{u}{\|u\|_2}$ ~~has~~ has unit length

The standard basis vectors and any of it subset are mutually orthonormal vectors

$$e_i^H e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

For n vectors of size m to be mutually orthonormal, $n \leq m$, because these vectors are linearly independent so there cannot be more than m.

Orthonormal matrix

Let $Q \in \mathbb{C}^{m \times n}$ (with $n \leq m$). Then Q is an orthonormal matrix if:

$$Q^H Q = I$$

Hw 2.2.3.2

Q is an orthonormal matrix if and only if $q_0, q_1, \dots q_{n-1}$ are mutually orthonormal

Hw 2.2.3.3

SOMETIMES : $Q^H Q = I$ then $QQ^H = I$

Only true if Q is square, because then $Q^H Q = I \Rightarrow Q^{-1} = Q^H$

And since $QQ^{-1} = I \Rightarrow QQ^H = I$

Unitary matrix

Let $U \in \mathbb{C}^{m\times m}$. Then U is unitary matrix if and only if: $U^H U = I$

Note: • Unitary matrices are always square. Sometimes the term orthogonal matrix is used, especially if the matrix is real valued.

• Unitary matrix preserves its length

Hw 2.2.4.2

$Q \in \mathbb{C}^{m\times m}$ be a unitary matrix

ALWAYS: $Q^{-1} = Q^H$ and $QQ^H = I$

Hw 2.2.4.3

TRUE: If U is unitary. So is U^H

Hw 2.2.4.4

ALWAYS: Let $U_0, U_1 \in \mathbb{C}^{m\times m}$ both be unitary

$U_0 U_1$ is unitary

Hw 2.2.4.5 important

ALWAYS: Let $U_0, U_1, \ldots, U_{k-1} \in \mathbb{C}^{m\times m}$ all be unitary

$U_0 U_1 \ldots U_{k-1}$ is unitary

Hw 2.2.4.6

Let $U \in \mathbb{C}^{m\times m}$ be a unitary matrix, $x \in \mathbb{C}^m$

$\|Ux\|_2 = \|x\|_2$

Theorem preserves length

If $A \in \mathbb{C}^{m\times m}$ preserves length ($\|Ax\|_2 = \|x\|_2$) then A is unitary

Hw 2.2.4.7

If U is unitary then $\|U\|_2 = 1$

Hw 2.2.4.8

If U is unitary then $\kappa_2(U) = 1$

Hw 2.2.4.9

Let $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$ be unitary and $A \in \mathbb{C}^{m \times n}$, then

- $\| U^H A \|_2 = \| A \|_2$
- $\| A V \|_2 = \| A \|_2$
- $\| U^H A V \|_2 = \| A \|_2$

Hw 2.2.4.10

Let $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$ be unitary and $A \in \mathbb{C}^{m \times n}$, then

- $\| U^H A \|_F = \| A \|_F$
- $\| A V \|_F = \| A \|_F$
- $\| U^H A V \|_F = \| A \|_F$

Examples of unitary matrices

ROTATION Hw 2.2.5.1

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$ (rotation matrix)

is a unitary matrix (or orthorgonal matrix)

Hw 2.2.5.2

$\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$

REFLECTION Reflection matrix

Given u orthogonal to the mirror, $x \in \mathbb{R}^m$ the linear transformation that represents a reflection is $M(x) = x - 2(u^T x)u$

$\Rightarrow$ Matrix that represents $M(x)$ is $I - 2uu^T$

Conclusion:

Rotation and Reflection are unitary matrices

Change of orthonormal basis

Write a vector in the lense of the orthonormal basis

- We know that a vector can be written as
  $x = x_0 e_0 + \dots x_{m-1} e_{m-1}$
- If we have U represents all the orthonormal ~~matr~~ vectors
  $x = Ix = UU^T x = U \begin{pmatrix} u_0^H x \\ \vdots \\ u_{m-1}^H x \end{pmatrix}$
- Also; $x = \alpha_0 u_0 + \dots \alpha_{m-1} u_{m-1}$
  $\Rightarrow \alpha_i = u_i^H x$

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Matrix-vector multiplication sensitivity

Hw 2.2.7.1

Let $A \in \mathbb{C}^{n\times n}$ be nonsingular and $x \in \mathbb{C}^n$ a nonzero vector. With $y = Ax$ and $y + \delta y = A(x + \delta x)$

$$\frac{\|\delta y\|}{\|y\|} \leq \|A\| \, \|A^{-1}\| \, \frac{\|\delta x\|}{\|x\|}$$

The SVD theorem

Lemma 2.3.1.3

Given $A \in \mathbb{C}^{m\times n}$, with $1 \leq n \leq m$ and $A \neq 0$, there exist unitary matrices $\tilde{U} \in \mathbb{C}^{m\times m}$ and $\tilde{V} \in \mathbb{C}^{n\times n}$ such that:

$$A = \tilde{U} \left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & B \end{array}\right) \tilde{V}^H, \text{ where } \sigma_1 = \|A\|_2$$

Hw 2.3.1.1

Let $A \in \mathbb{C}^{m\times n}$ with $A = \left(\begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & B \end{array}\right)$ and assume that $\|A\|_2 = \sigma_1$

ALWAYS $\|B\|_2 \leq \sigma_1$

Hw 2.3.1.2

Let $\Sigma = \text{diag}(\sigma_0, \dots, \sigma_{n-1})$:

ALWAYSS: $\|\Sigma\|_2 = \max_{i=0}^{n-1} |\sigma_i|$

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Hw 2.3.1.3

Let $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$ be unitary matrices and $A, B \in \mathbb{C}^{m\times n}$ with $B = UAV^H$

Rmb: the product of 2 unitary matrices of the same size is also unitary

Hw 2.3.1.4

Let $A \in \mathbb{C}^{m\times n}$ with $n < m$, $A = U\Sigma V^H$

ALWAYS: $A^H = V\Sigma^T U^H$

(Σ could be retangle therefore $\Sigma^T \neq \Sigma$)

Conclusion

With $A = U\Sigma V^H$.
When solving $y = Ax$
We can instead solve $\hat{y} = \Sigma \hat{x}$
Where $\hat{y} = U^H y$ and $\hat{x} = V^H x$ to decrease error rate

$$\|A\|_F = \|U\Sigma V^H\| = \sqrt{\sigma_0^2 + \dots + \sigma_{n-1}^2}$$

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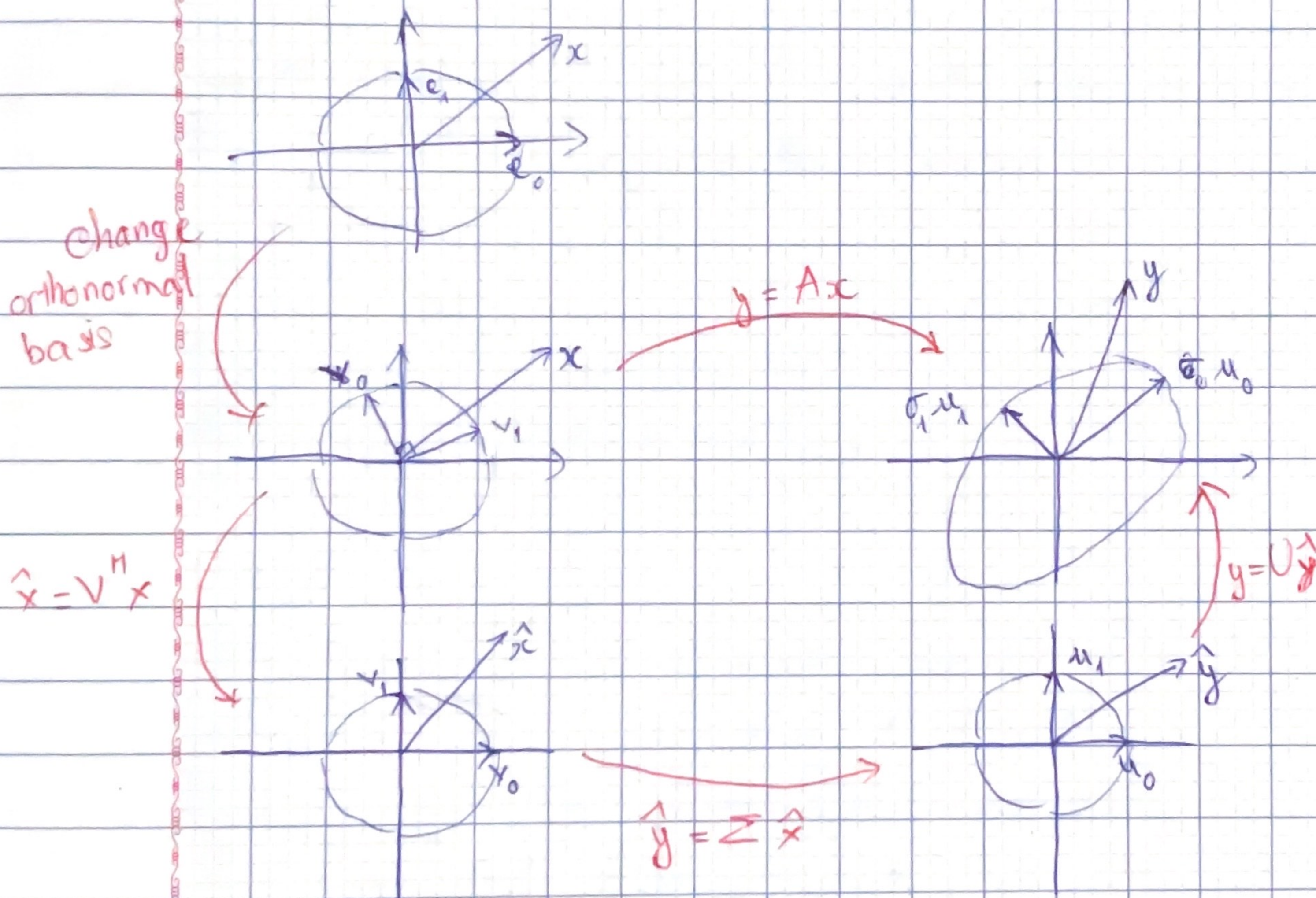
Geometric interpretation



Reduced Singular Value Decomposition

~~Cor~~ Corollary 2.3.4.1

Let $A \in \mathbb{C}^{m\times n}$ and $r = \text{rank}(A)$. There exist orthonormal matrix $U_L \in \mathbb{C}^{m\times r}$, $V_L \in \mathbb{C}^{n\times r}$ and $\Sigma_{TL} \in \mathbb{R}^{r\times r}$ that $\Sigma_{TL} = \text{diag}(\sigma_0, \dots, \sigma_{r-1})$ and $\sigma_0 \geq \dots \geq \sigma_{r-1}$ so that:

$$A = U_L \Sigma_{TL} V_L^H = \sum_{i=0}^{r-1} u_i \sigma_i v_i^H$$

Remark 2.3.4.3

Matrix A with rank r can be written as a linear combinations of r outer products

$$A = \sum_{i=0}^{r-1} u_i \sigma_i v_i^H = \sigma_0 + \sigma_1 + \dots + \sigma_{r-1}$$

SVD of nonsingular matrices

Hw 2.3.5.1

Let $A \in \mathbb{C}^{m\times m}$ and $A = U\Sigma V^H$ be its SVD

A is nonsingular if and only if Σ is nonsingular

Hw 2.3.5.2

Let $A \in \mathbb{C}^{m\times n}$ and $A = U\Sigma V^H$ be its SVD

A is nonsingular if and only if $\sigma_{m-1} \neq 0$

Hw 2.3.5.3

Let $A \in \mathbb{C}^{m\times m}$ be nonsingular and $A = U\Sigma V^H$ be its SVD

SOMETIMES: SVD of A^{-1} is $V\Sigma^{-1}U^H$

(Only when $\sigma_0 = \sigma_1 = \dots = \sigma_{m-1}$)

Hw 2.3.5.4

$$A^{-1} = (v_{m-1} | \dots | v_0) \begin{pmatrix} 1/\sigma_{m-1} & & \\ & \ddots & \\ & & 1/\sigma_0 \end{pmatrix} (u_{m-1} | \dots | u_0)^H$$

Hw 2.3.5.5

Let $A \in \mathbb{C}^{m\times m}$ be nonsingular

$$\|A^{-1}\|_2 = \frac{1}{\min_{\|x\|_2 = 1} \|Ax\|_2}$$

Best rank-k approximation

Theorem 2.3.6.1

Given $A \in \mathbb{C}^{m \times n}$, $A = U \Sigma V^H$. Assume the entries on the main diagonal of Σ are $\sigma_0, \ldots, \sigma_{\min(m,n)-1}$ with $\sigma_0 \geq \ldots \geq \sigma_{\min(m,n)-1}$. Given k such that $0 \leq k \leq \min(m,n)-1$, partition:

$$U = (U_L \mid U_R) \quad \Sigma = \left(\begin{array}{c|c} \Sigma_{TL} & 0 \\ \hline 0 & \Sigma_{BR} \end{array}\right), \quad V = (V_L \mid V_R)$$

Then:

$$B = U_L \Sigma_{TL} V_L^H$$

is the matrix $\in \mathbb{C}^{m \times n}$ that is closest to A in the following sense:

$$\| A - B \|_2 = \min_{\substack{C \in \mathbb{C}^{m \times n} \\ \text{rank}(C) \leq k}} \| A - C \|_2$$

In other words, B is the matrix that has rank at most k that is closest to A. Also:

$$\| A - B \|_2 = \begin{cases} \sigma_k & \text{if } k < \min(m,n) \\ 0 & \text{otherwise} \end{cases}$$

Hw 2.3.6.1

Given $A \in \mathbb{C}^{m \times n}$, $A = U \Sigma V^H$ be its SVD

$$A v_j = \sigma_j u_j \quad \text{with } 0 \leq j \leq \min(m,n)$$

Assignment 2:

These are true for $k_2(A)$ Hw 2.5.1.7

- $k_2(A) = \| A \|_2 \, \| A^{-1} \|_2$
- $k_2(A) = \dfrac{\sigma_0}{\sigma_{m-1}}$
- $k_2(A) = \dfrac{u_0^H A v_0}{u_{m-1}^H A v_{m-1}}$
- $k_2(A) = \dfrac{\max_{\|x\|_2 = 1} \| Ax \|_2}{\min_{\|x\|_2 = 1} \| Ax \|_2}$