Unit 1: Probability, Counting and Story Proofs

A Set Theory Dictionary

Events and occurences

English	Sets
Sample space	S
the empty set (impossible event)	$\emptyset$
s is a possible outcome	$s \in S$
A is an event	$A \subseteq S$
A occured	$s_{actual} \in A$
Something must happen	$s_{actual} \in S$

New events from old events

English	Sets
A or B (inclusive)	$A \cup B$
A and B	$A \cap B$
not A	A^c
A or B, but not both	$(A \cap B^c) \cup (A^c \cap B)$
at least one of $A_1, \dots A_n$	$A_1 \cup \dots \cup A_n$ or $\bigcup_{j=1}^{n} A_j$
all of $A_1, \dots A_n$	$A_1 \cap \dots \cap A_n$ or $\bigcap_{i=1}^{n} A_i$

Relationships between ~~sets~~ events

English	Sets
A implies B	$A \subseteq B$
A and B are mutually exclusive	$A \cap B = \emptyset$
$A_1, \dots A_n$ are a partition of S	$A_1 \cup \dots \cup A_n = S$, $A_i \cap A_j = \emptyset$, $i \neq j$

Naive definition of probability

Let A be an event for an experiment with a finite sample space S. The naive probability of A is:

$$P_{naive}(A) = \frac{|A|}{|S|} = \frac{\text{number of outcomes favor } A}{\text{total number of outcomes } S}$$

where $|A|$ is the size of A

Multiplication rule

With 2 sub-experiments A, B. A has a outcomes, B has b outcomes, then the compound experiment has $a.b$ outcomes

Sampling with replacement

Consider n objects and making k choices from them, one at a time with replacement, then there are n^k possible outcomes

Sampling without replacement

Consider n objects and making k choices from them, one at a time without replacement, then there are $n(n-1)\dots(n-k+1)$ possible outcomes $(k \leq n)$

Binomial coefficient (or Combination)

For any nonnegative integers k and n, the binomial coefficient $\binom{n}{k}$, read as "n choose k", is the number of subsets of size k for a set of size n

For $k \leq n$:

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!}$$

For $k > n$:

$$\binom{n}{k} = 0$$

Story proof for Vandermonde's identity

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$$

Consider a group of m peacocks and n toucans, a set of size k birds will be chosen. There are $\binom{m+n}{k}$ possibilities for this set of birds. If there are j peacocks in the set, then there must be k-j toucans in the set. The right side of Vandermonde's identity sum up the cases for j

General definition of probability

A probability space cosists of sample space S, probability function P, takes event $A \subseteq S$ and return $P(A)$ $(0 \leq P(A) \leq 1)$. P is:

1. $P(\emptyset) = 0$, $P(S) = 1$
2. If $A_1, A_2, \dots$ are disjoint events, then

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)$$

Properties of Probability

For any events A and B:

1. $P(A^c) = 1 - P(A)$
2. If $A \subseteq B$, then $P(A) \leq P(B)$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Inclusion - Exclusion for n events

For any events $A_1 \dots A_n$

$$P\left(\bigcup_{i=0}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \dots + (-1)^{n+1} P(A_1 \cap \dots \cap A_n).$$

1.3 Practice Problems

Problem 1a

a) How many ways to split 12 people into 3 teams, where one team has 2 people, and the other two teams have 5 people each

- Pick 2 people out of 12 : $\binom{12}{2}$
- Pick 5 people out of (12-2) : $\binom{10}{5}$
- Divide by 2 since the two teams of 5 are ~~indistingus~~ the same, adjust for overcounting.

$\Rightarrow \binom{12}{2}\binom{10}{5} \times \frac{1}{2!} = 8316$

Problem ~~2a~~ 1b

How many ways to split 12 ppl into 3 teams, each has 4

- Pick 4 ppl out of 12 : $\binom{12}{4}$
- Pick 4 ppl out of (12 - 4) : $\binom{8}{4}$
- Divide by 3! since 3 teams are similar

$\Rightarrow \binom{12}{4}\binom{8}{4} \times \frac{1}{3!} = 5775$

Problem 2

What is the story proof for $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

$\Rightarrow$ Consider n+1, with 1 of them being "the president". The RHS pick k ppl out of n+1 ppl. The LHS counts the same thing in different way, by considering two disjoint cases : the president is not in the group and the president is in the group.

Probem 3a

P(the total after rolling 4 dice is 21) ?

- U (all possibilities) = 6^4 (roll dice 4 times)
- There are 3 events with total = 21 $(x_1 + x_2 + x_3 + x_4 = 21)$
  - ~~$x_1\ x_2\ x_3\ x_4$~~ {5, 5, 5, 6}
    $\Rightarrow$ 4 ways (to place "6" in 4 positions)
  - {4, 5, 6, 6}
    $\Rightarrow$ 12 ways (to place "4" and "5" in 4 positions)
  - {3, 6, 6, 6}
    $\Rightarrow$ 4 ways (to place "3" in 4 positions)

Therefore, $P = \frac{4 + 12 + 4}{6^4} \approx 0,0154$

P(the total after rolling 4 dice is 22) ?

- $U = 6^4$
- There are 2 events with total = 22
  - {6, 5, 5, 6} $\Rightarrow$ 6 ways (to place two "6" in 4 positions)
  - {6, 4, 6, 6} $\Rightarrow$ 4 ways (to place "4" in 4 positions)

Therefore, $P = \frac{6 + 4}{6^4} \approx 0,0077$

Problem 3b

P(a random 2-letter word is palindrome)

x_1, x_2 represents the word | $U = 24^2$

- 24 ways to choose x_1
- 1 way to choose x_2

$\Rightarrow$ 24 ways $\Rightarrow P = \frac{24}{24^2}$

P(a random 3-letter word is palindrome)

$U = 24^2$ (^{2} bc the letter in the center is not important)

x_1, x_2, x_3 represents the word

- 24 ways to choose x_1
- 1 way to choose x_2 (anything since its not important)
- 1 way to choose x_3 (similar to x_1)

$\Rightarrow P = \frac{24}{24^2}$

Problem 4

There are N elks, of which n are captured and tagged. Then return n elks to the population and capture m elks. What is the propability that exatcly k of the m elks were previously tagged?

Hypergeometric distributon Formula

$$\boxed{P = \frac{\binom{n}{k}\binom{N-n}{m-k}}{\binom{N}{m}}}$$

$\binom{n}{k}$: ways to choose k elks out of n tagged initially

$\binom{N-n}{m-k}$: ways to choose remaining m-k elks out of M-n that not tagged

$\binom{N}{m}$: ways to choose m elk out of the population

1.4 Homework Problems

Problem 1a

a) How many 7 digit phone numbers are possible, assuming the first digit can't be 0 or 1?

$x_0 x_1 x_2 x_3 \ldots x_6$ denotes the phone number

- x_0 has 8 ways (cant choose 0 or 1)
- $x_1 \ldots x_6$ has 10^6 ways

$\Rightarrow 8 \cdot 10^6 = 8{,}000{,}000$

Problem 1b

How many 7-digit phone numbers possible, now also that the number is not allowed to start with 911

$x_0 x_1 \ldots x_6$ denotes the phone number

- $x_0 x_1 x_2$ has 1 way to choose (911 specifically)
- $x_3 \ldots x_6$ has 10^4 ways

$\Rightarrow$ There are 1×10^4 ways to choose numbers start with 911

So there are $\underbrace{8{,}000{,}000 \text{ (previous hw)} - 10^4}_{= 7{,}990{,}000}$ numbers possible.

Problem 2a

How many possible outcomes for the individual games are there, such that overall player A ends up with 3 wins, 2 draws, 2 losses?

- Multinominal Approach

$$P = \frac{7!}{3!\,2!\,2!} \cdot \left(\frac{1}{3}\right)^7 \text{ where}$$

7!: assume 7 outcomes could be distinct

3!: within those 7! arrangements, there are many repeats caused by shuffling the 3 wins

Same goes for 2!

- Sequential Approach

$P = C(7,3) \cdot C(4,2)$

→ Select 3 wins out of 7 slots

→ Select 2 wins out of 4 slots

Problem 2b

How many possible outcomes for the individual games are there, such that A ends up with 4 points, B ends up with 3 points

There are 4 scenrios that A get 4 points :

- $4W, 3L, 0D : \frac{7!}{4!\,0!\,3!} = 35$
- $3W, 2D, 2L : \frac{7!}{3!\,2!\,2!} = 210$
- $2W, 4D, 1L : \frac{7!}{2!\,4!\,1!} = 105$
- $1W, 6D, 0L : \frac{7!}{1!\,6!\,0!} = 7$

In total, 357 outcomes

Problem 2c

Assume they play best-of-7 match, where the match will end when either player has 4 points or when 7 games have been played, whichever is first. How many possible outcomes such that the match lasts for 7 games and A wins by a score of 4 to 3 ?

Basically we couting outcomes that A ~~win~~ doesn't lose the 7^{th} game

$g_1\ g_2\ g_3\ g_4\ g_5\ g_6\ g_7 \rightarrow$ W or D

- When g_7 is W: $\frac{6!}{3!\,0!\,3!} + \frac{6!}{2!\,2!\,2!} + \frac{6!}{1!\,4!\,1!} + \frac{6!}{0!\,6!\,0!}$
- When g_7 is D: $\frac{6!}{3!\,1!\,2!} + \frac{6!}{2!\,3!\,1!} + \frac{6!}{1!\,5!\,0!}$

The total is 267

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Problem 3

3 ppl get in elevator at 1^{st} floor of a 10 floor building. What is the probability the buttons for 3 consecutive floors are pressed ?

- $U = \cancel{9^3}\ \cancel{\binom{9}{3}}\ 9^3$ (choose 2-10 so 9 choices with ~~no~~ repeats)
- 7 ways to choose 3 consecutive floors (2-3-4 - 8-9-10)

~~So the $P = \frac{7}{\binom{9}{3}} = \frac{1}{12}$~~

- But also, ABC chosing 2-3-4 is different than CBA chosing 2-3-4 so 3!

$\Rightarrow P = \frac{7 \cdot 3!}{9^3} = \frac{7}{729}$

Problem 4 Decide if order matters

a) Ways to choose 5 of 10 ppl is > 6 of 10 ppl

b) Ways to break ~~ppl~~ 10 ppl into 2 teams of 5 < ways to break 10 ppl into team of 6 and 4

$$\binom{10}{5}\frac{1}{2!} < \binom{10}{6}$$

c) ~~Prop~~ Probability 3 ppl born on 1/2 < Probability 3 ppl born in 1/2, 2/2, 3/2 respectively

Problem 5

Martin and Gale plays "toss the coin", Martin wins if sequence of HH occurs, Gale wins if sequence of TH occurs. The game keep going until find the winner.

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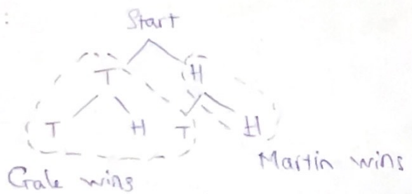
Analyze visually :



$\Rightarrow$ P(Martin wins) $= \frac{1}{4} < \frac{1}{2}$

Martin is less likely to win because as soon as Tail appears TH will definitely occur before HH.