LINEAR PROGRAMMING

Motivation example:

	Book	Calculator
Sales	\$20	\$18
Cost	\$5	\$4
Time	5 minutes	15 minutes

- Monthly cost must not exceed \$ 27 000
- How many books and calculators should the company makes to maximize profit? Determine max profit in 30 day period?

Let
- B = books in a month
- C = calculators in a month
- P = profits in a month

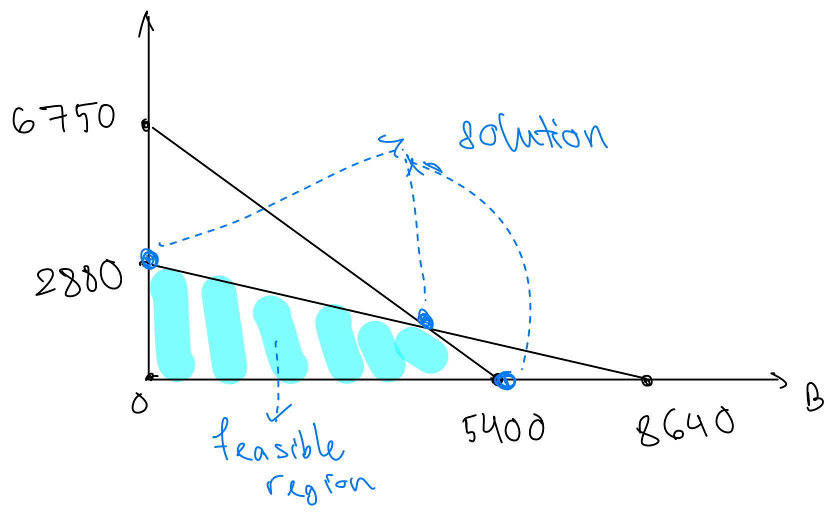
Constraints:

$$\begin{cases} 5B + 4C \leq 27000 & (1) \\ 5B + 15C \leq 43\,200 & (2) \end{cases}$$

Objective: $\max P = 15B + 14C$ (3)

Solution:

1. Find intersects (1) and (2) on the axis



- $B = 0$ in (1) $\Rightarrow C = 6750$
- $C = 0$ in (1) $\Rightarrow B = 5400$
- $B = 0$ in (2) $\Rightarrow C = 2880$
- $C = 0$ in (2) $\Rightarrow B = 8640$

2. Find intersection of (1) and (2):

Solve
$$\begin{cases} 5B + 4C = 27000 \\ 5B + 15C = 43200 \end{cases}$$

$$\Rightarrow \begin{cases} B = 4221 \\ C = 1473 \end{cases}$$

3. Find solution to objective

$$P = \underset{B,C}{\operatorname{argmax}} (15B + 14C) : (B,C) = \begin{cases} (0, 2880) \\ (4221, 1473) \\ (5400, 0) \end{cases}$$

$\Rightarrow P = 83\,937$ with $(B,C) = (4221, 1473)$

Linear Program Components

1. Decision Variables

$x_1, x_2, \ldots, x_n \in \mathbb{R}$

2. Linear Constraints

$$\sum_{j=1}^{d} a_{ij} x_j \quad \begin{matrix} = \\ \leq \\ \geq \end{matrix} \quad b_i$$ where a_{ij}, b_i are constants (part of the input)

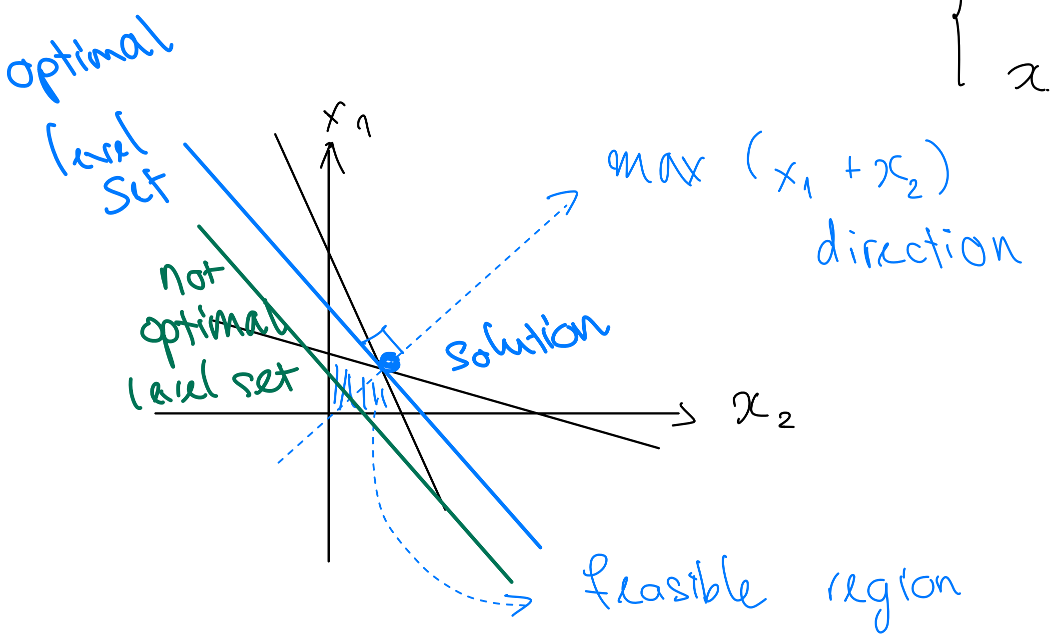
3. Objective function

$$\max / \min \sum_{j=1}^{n} c_j x_j$$

$\Rightarrow$ If you have these 3 components defined, then you have a linear program

Example 1: Consider this LP : $\max (x_1 + x_2)$

$$\begin{cases} 2x_1 + x_2 \leq 1 \\ x_1 + 2x_2 \leq 1 \end{cases}$$



$\Rightarrow$ Optimal at the "level set" that is "furthest out" in the feasible region

In general:

- feasible region = intersection of half spaces

  Recall: in $\mathbb{R}^n$ problem space, solution is a line / plane of dimension $(n-1)$. If we talk ab inequalities, then we have a half space divided by that solution line / plane

- Level sets = parallel hyperplanes
- Optimal solution = furthest point in feasible region in direction c

Edge cases

1. Infeasible linear programming

Constraints: $x_1 \leq -5$

$x_1 \geq 0$

Objective: Maximize x_1

2. Unbounded linear programming (opt. value $= \pm\infty$)

Constraint: $x_1 > 0$

Objective: Maximize x_1

"Feasible" means satisfying all the constraints

Slack variable:

These statements are similar:

Constraint: $x_1 + x_2 \leq 7$ $\forall x_1, x_2 \geq 0$ $\equiv$ Constraint: $x_1 + x_2 + x_3 = 7$ $\forall x_1, x_2, x_3 \geq 0$

Linear Programming Problem

Maximize / Minimize a linear function subject to linear constraints

For example: Maximize $c^T x$ subject to: $\begin{cases} Ax \leq b \\ x \geq 0 \end{cases}$

where:
$$\begin{cases} x \in \mathbb{R}^n : & n \times 1 \text{ vector} \\ c \in \mathbb{R}^n : & n \times 1 \text{ vector} \\ A \in \mathbb{R}^{m \times n} : & m \times n \text{ matrix} \\ b \in \mathbb{R}^m : & m \times 1 \text{ matrix} \end{cases}$$

Purpose:

Linear Programming is a tool we used to answer the question "How do we know we are done" when designing an algorithm

Other examples

- Inequalities
- Max-flow min-cut with constraints (capacities)

Applications

1. Max-flow

- Decision variables : $f_e : e \in E$ (flows)
- Constraints :
  - flow in = flow out for any vertices not s or t

$$\sum_{e \in \delta^-(v)} f_e = \sum_{e \in \delta^+(v)} f_e$$

  - capacity constraint

$$0 \leqslant f_e$$
$$f_e \leqslant c_e$$

- Objective function :

  Maximize flow going out of the source

$$\max \sum_{e \in \delta^+(s)} f_e$$

2. Min-cost flow

- Decision variables : $f_e : e \in E$
- Constraints :
  - At each vertex, inflow - outflow equals the supply / demand of that vertex :

$$\sum_{in(v)} f_e - \sum_{out(v)} f_e = b_v$$

where $\begin{cases} b_v > 0 & : \text{vertex } v \text{ is "supply" vertex} \\ b_v < 0 & : \text{vertex } v \text{ is "demand" vertex} \\ b_v = 0 & : \text{vertex } v \text{ is "intermediate" vertex} \end{cases}$

  - Capacity constraints
- Objective func :

$$\min \sum c_e f_e$$

Duality of Linear Program

Any linear program has a dual.

For example: Given this linear program

"Maximize $c^T x$ subject to $\begin{cases} Ax \leq b \\ x \geq 0 \end{cases}$"

Its "dual" is:

"Minimize $y^T b$ subject to $\begin{cases} A^T y \geq c \\ y \geq 0 \end{cases}$"

where $\begin{cases} y \in \mathbb{R}^m \\ c \in \mathbb{R}^n \\ A \in \mathbb{R}^{m \times n} \\ b \in \mathbb{R}^m \end{cases}$

Dual of dual is Primal

"Max $c^T x$, constraints $\begin{cases} Ax \leq b \\ x \geq 0 \end{cases}$" $\equiv$ "Min $-c^T x$, constraints $\begin{cases} -Ax \geq -b \\ x \geq 0 \end{cases}$"

primal-dual

primal-dual

"Min $y^T b$, constraints $\begin{cases} A^T y \geq c \\ y \geq 0 \end{cases}$" $\equiv$ "Max $-y^T b$, constraints $\begin{cases} -A^T y \leq -c \\ y \geq 0 \end{cases}$"

Weak Duality Theorem

Given Primal: "Maximize $c^T x$ subject to $Ax \leq b$ and $x \geq 0$"

Dual: "Minimize $y^T b$ subject to $A^T y \geq c$ and $y \geq 0$"

If x is feasible in Primal, y is feasible in Dual

$\Rightarrow \quad c^T x \leq y^T b$

Proof:

$$A^T y \geq c$$

$$\Leftrightarrow \quad y^T A \geq c^T$$

$$\Leftrightarrow \quad y^T A x \geq c^T x \qquad \langle x \geq 0 \rangle$$

Since $b \geq Ax \Leftrightarrow y^T b \geq y^T A x$

$$\Rightarrow \quad y^T b \geq c^T x$$

Collary 1: Weak Duality

If x is feasible in Primal, y is feasible in Dual,

and $c^T x = y^T b$

$\Rightarrow$ x is optimal for Primal, y is optimal for Dual

Proof: (by contradiction)

Assume $\exists\ y'$ that is "more optimal" in Dual

$$\Rightarrow \quad y'^T b < y^T b$$
$$= c^T x$$

$\Rightarrow$ Contradicts the condition $y'^T b = c^T x$

Collary 2: Weak Duality

If the Primal is feasible and unbounded, then Dual is infeasible

Strong Duality Theorem

Given Primal: "Maximize $c^T x$ subject to $Ax \leqslant b$ and $x \geqslant 0$"

Dual: "Minimize $y^T b$ subject to $A^T y \geqslant c$ and $y \geqslant 0$"

If either
- (1) Both Primal and Dual are feasible
- (2) Only Primal is feasible and bounded

Then:
- Both Primal and Dual are feasible and bounded
- Primal and Dual have the same objective function

Combinations of Primal - Dual Pairs

There are 3 possible categories for any LP

- Feasible and bounded
- Feasible and unbounded
- Infeasible

The possible combinations of Primal - Dual are:

1. Primal is feasible and bounded $\Rightarrow$ Dual is feasible and bounded
2. Primal is feasible and unbounded $\Rightarrow$ Dual is infeasible
3. Primal is infeasible $\Rightarrow$ - Dual is feasible and unbounded
   - Dual is infeasible

Upper bound

Case 1: Concrete numbers

Given this LP: $\max\ (x_1 + x_2)$

$$\text{s.t} \begin{cases} 4x_1 + x_2 \leq 2 \\ x_1 + 2x_2 \leq 1 \end{cases}$$

We can have thesse upper bounds:

- $x_1 + x_2 \leq 4x_1 + x_2 \leqslant 2$ because $(1,1) \leq (4,1)$
- $x_1 + x_2 \leq x_1 + 2x_2 \leqslant 1$ $(1,1) \leqslant (1,2)$
- $x_1 + x_2 \leqslant \frac{1}{7}(4x_1 + x_2) + \frac{3}{7}(x_1 + 2x_2)$

$$\leqslant \frac{1}{7} \cdot 2 + \frac{3}{7} \cdot 1$$

$$= 5/7 \Rightarrow \text{tightest bound}$$

Case 2: No-concrete numbers

Consider this LP: $\max \sum_{j=1}^{n} c_j x_j \rightarrow c^T x$

$$\text{s.t} \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i \quad \forall\ 1 \leqslant i \leqslant m$$

$\rightarrow Ax \leq b$

Similar to case 1, we can deduce the upper bound as follows.

Suppose $\exists$ $y_1, y_2, \dots y_m \geqslant 0$ that satisfy:

$$\sum_{i=1}^{m} y_i a_{ij} \geqslant c_j \quad \forall\ 1 \leqslant j \leqslant n \rightarrow A^T y \geqslant c$$

Then: $\sum_{j=1}^{n} c_j x_j \leqslant \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j$

$$= \sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j \right)$$

$$\leqslant \sum_{i=1}^{m} y_i b_i \rightarrow y^T b$$

Conclusion: $\forall\ y \geqslant 0$ s.t. $(*)$: the mentioned LP problem holds the optimal solution is upper-bounded by $y^T b$.

$\rightarrow$ We try to get the "tighest" bound, so objective func $= \min(y^T b)$

So: $\max(c^T x)$ s.t $Ax \leq b$ $\rightleftarrows$ $\min(y^T b)$ s.t $A^T y \geqslant c$

$\Rightarrow$ Proof for duality of Linear Program

Denote $\begin{cases} P: & LP \quad \max(c^T x) \quad s.t \quad Ax \leq b & \text{Primal} \\ D: & LP \quad \min(y^T b) \quad s.t \quad A^T x \geq c & \text{Dual} \end{cases}$

Then by construction (the green highlight particularly)

$$\text{Optimal }(P) \leq \text{Optimal }(D)$$
$$\text{Primal} \leq \text{Dual}$$

Sanity check Case 1:

Primal

$P: \max(x_1 + x_2)$

$$s.t \begin{cases} 4x_1 + x_2 \leq 2 \\ x_1 + 2x_2 \leq 1 \end{cases}$$

$\Rightarrow$ Tightest upper-bound:

$$x_1 + x_2 \leq 5/7$$

Dual

$D: \min(2y_1 + y_2)$

$$s.t \begin{cases} 4y_1 + y_2 \leq 1 \\ y_1 + y_2 \leq 1 \end{cases}$$

$\Rightarrow$ Tightest lower-bound:

$$2y_1 + y_2 \geq 5/7$$

Since: $\begin{cases} \text{Optimal }(P) \leq \text{Optimal }(D) \\ P \leq 5/7 \leq D \end{cases}$

$\Rightarrow$ Optimal solution for both P and D is $5/7$

Duality in Max-Flow

Alternative LP: Maximize Paths instead of edges

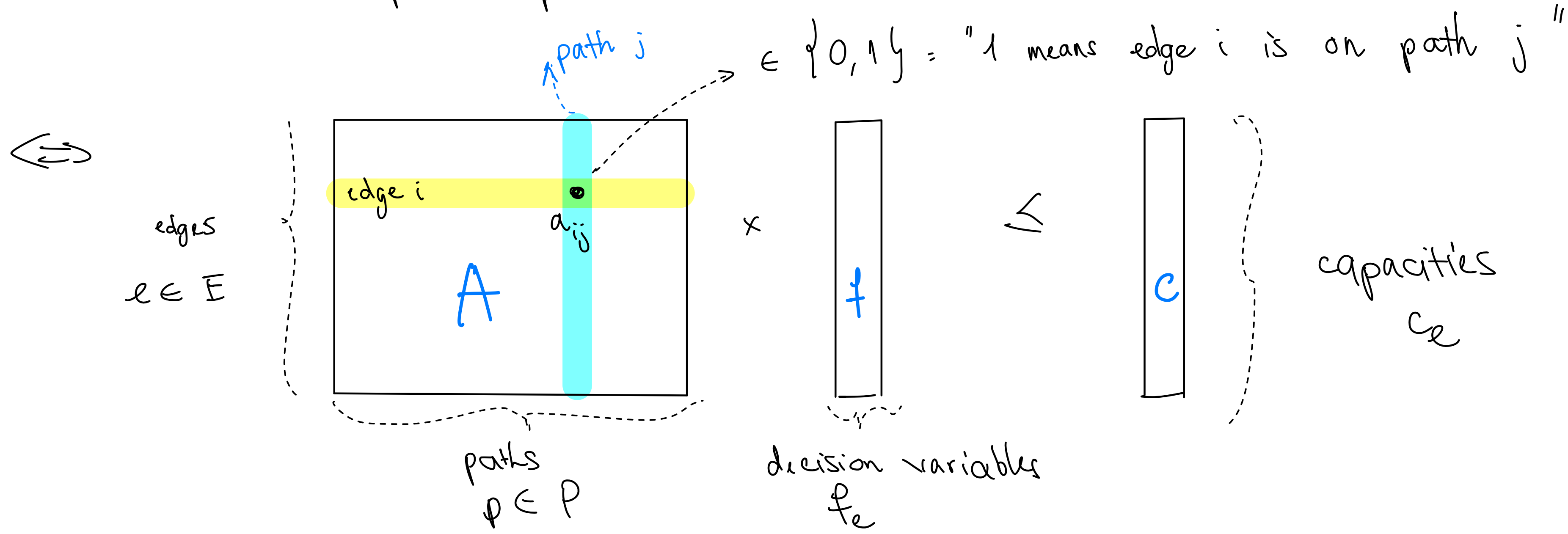
- Decision variables: $f_e \quad \forall e \in E$
- Constraints:
  - Capacity constraint: "for all path p s.t. edge e belongs to, the total flow should not exceed the capcity"
  $$\sum_{p \in P : e \in p} f_e \leq c_e \quad \forall e \in E$$
  - Flow conservations constraint is automatically satisfied by formulation of the Alternative LP
- Objective: Maximize flows of all s-t paths
$$\max \sum_{p \in P : e \in p} f_e$$

Think of this Alternative LP in terms of matrix:

Lets think of the constraint:

$$\sum_{p \in P:\, e \in p} f_e \leq c_e \quad \forall e \in E$$

$\Leftrightarrow$



We try to maximize: $\mathbb{1}^T f$ where $\mathbb{1}$: vector size $(1 \times P)$; f: vector size $(1 \times P)$

Dual of Alternative LP

Let variable l_e correspond to every f_e s.t $l_e \geq 0$

Then in theory, the dual is:

" Min $l^T c$ s.t. $A^T l \geq \mathbb{1}$ "

$$\sum_{e \in E} c_e \cdot l_e \quad \forall e \in E$$

$$\sum_{e \in p} l_e \geq 1 \quad \forall p \in P$$

$\Leftrightarrow$ This inequality holds for every path

$\Rightarrow$ Also holds for the "shortest path"

Intepretion of the Dual: Compute nonnegative lengths on the edges so that if you compute the shortest path, it has length at least 1.

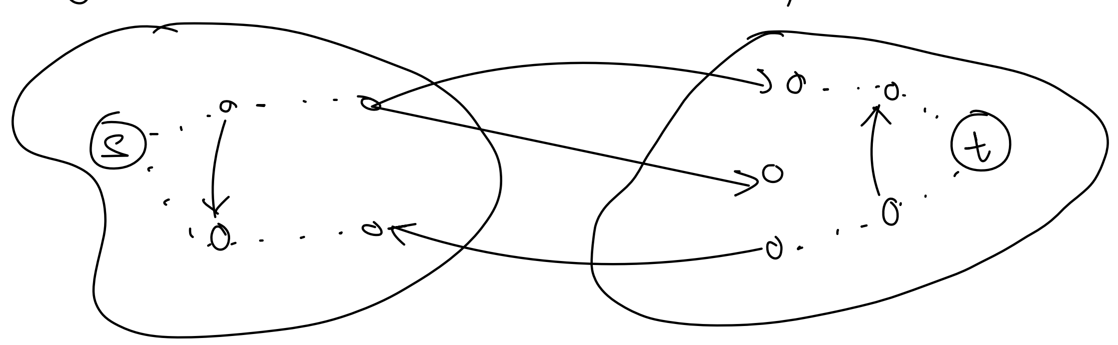
$\Rightarrow$ Objective function meaning: min $l^T c$ - minimizing the "volume" of "a pipe" of length ≥ 1 and width = capacity

Observations

- Dual's constraints $\sim$ Primal's decision variabes
  $l_e \ \forall p \in P$ $\qquad$ $f_e \ \forall p \in P$
- Dual's decision variables $\sim$ Primal's constraints
  $l_e \ \forall e \in E$ $\qquad$ $f_e \ \forall e \in E$

Connection Min-Capacity Cut and the Dual

Fix your favorite s-t cut (A, B), there are 3 kind of edges: internal, forward, backward



<u>Set</u>: $l_e = \begin{cases} 1 & : \text{edge "sticking out" of } A \\ 0 & : \text{otherwise} \end{cases}$

<u>Claim</u>: $\forall$ possible cuts, the result is "feasible" solution of Dual

In other words: Any s-t path in any cut will have length of at least 1

$$\sum_{e \in P} l_e \geq 1 \quad \forall\, p \in P$$

<u>Proof Claim</u>: Since $l_e = 1$ for edge "stick out" of A

$\Rightarrow$ That edge must belong to some s-t path $(p \in P)$

$\Rightarrow$ Any s-t path created by the cut (A, B) have length of at least 1 (you cannot have a path of all 0-length edges)

<u>ctive function</u>: Instead of minimizing the "volume" in the Dual, we minimizing only the capacity of the cut

$$\text{Min} \sum_{e \in P} c_e l_e \qquad (\text{Dual Obj})$$

$$= \text{Min} \sum_{e \in \delta^+(A)} c_e \qquad \left(\text{Replace} \begin{cases} l_e = 1 & : e \in \delta^+(A) \\ l_e = 0 & : e \notin \delta^+(A) \end{cases} \right)$$

$$= \text{Min Capacity of cut } (A, B)$$

<u>Conclusion</u>: • For every s-t cut, we can extract a feasible solution to the Dual (minimizing "volume") with the same objective function (minimizing capacity)

$\Rightarrow$ Min Cut Capacity is a "special case" of the Dual

$\Rightarrow$ Optimal (Dual) $\leq$ Min cut capacity

Main Takeaways: Max flow Min Cut as LP

$$\text{Max flow} = \text{Optimal (Primal)} \leq \text{Optimal (Dual)} = \text{Min Volume} \quad (1)$$
$$\leq \text{Min Cut Capacity} \quad (2)$$

(1): By construction

(2): Explained in previous page

→ Weak Duality

Stronger takeaway: Using Max flow Min Cut theorem

We know that in Maxflow Min Cut theorem

$$\text{Max flow} = \text{Min Cut Capacity}$$

So, we can strengthen our conclusion above to say that:

$$\text{Max flow} = \text{Optimal (Primal)} = \text{Optimal (Dual)}$$
$$= \text{Min Cut Capacity}$$

→ Strong Duality