Practice Problems:

Problem 1: The Rayleigh distribution has PDF:

$$f(x) = x e^{-x^2/2}, \quad x > 0$$

Let X have the Rayleigh distribution

a) Find $P(1 < X < 3)$

. Find CDF $F(x) = \int_{-\infty}^{x} f(x)\, dx$

$$= \int_{-\infty}^{x} x e^{-x^2/2}\, dx$$

$$= 1 - e^{-x^2/2}$$

. $P(1 < X < 3) = P(X < 3) - P(X < 1)$

$$= F(3) - F(1)$$

$$= 1 - e^{-3^2/2} - (1 - e^{-2^2/2})$$

$$= e^{-3^2/2} + e^{-2^2/2} \approx 0.5954$$

b) Find the first quartile, median, and third quartile of X; these are defined to be values q_1, q_2, q_3 such that $P(X \leq q_j) = \frac{j}{4}$ for $j = 1, 2, 3$.

. $P(X < q_1) = F(q_1) = \frac{1}{4} = 0.25$

$$\Leftrightarrow 1 - e^{-q_1^2/2} = 0.25$$

$$\Rightarrow q_1 \approx 0.7585$$

. $P(X < q_2) = F(q_2) = \frac{1}{2} = 0.5$

$$\Leftrightarrow 1 - e^{-q_2^2/2} = 0.5$$

$$\Rightarrow q_2 \approx \pm 1.1774$$

. $P(X < q_3) = F(q_3) = \frac{3}{4} = 0.75$

$$\Leftrightarrow 1 - e^{-q_3^2/2} = 0.75$$

$$\Rightarrow q_3 \approx \pm 1.6651$$

Problem 2: Let $Z \sim N(0,1)$ and $X = Z^2$. Then the distribution of X is called Chi-Square with 1 degree of freedom. This distribution appears in many statistical methods. Find a good number of approximation to $P(1 \leq X \leq 4)$ using facts of the Normal distribution, without query a calculator / computer / table ?

$$P(1 \leq X \leq 4) = P(1 \leq Z^2 \leq 4)$$
$$= P(1 \leq Z \leq 2 \quad \text{or} \quad -2 \leq Z \leq -1)$$
$$= P(-2 \leq Z \leq 2) - P(-1 \leq Z \leq 1)$$
$$= 2P(1 \leq Z \leq 2)$$
$$= 2(\phi(2) - \phi(1)) \simeq 0.27$$

Problem 3: The Pareto distribution with parameter $\alpha > 0$ has PDF $f(x) = \frac{\alpha}{x^{\alpha+1}}$ for $x \geq 1$ (and 0 otherwise). This distribution is often used in Statical modeling.

a) Find the CDF of a Pareto r.v with parameter α.

. $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} \frac{\alpha}{x^{\alpha+1}}\,dx = \int_{-\infty}^{x} \alpha x^{-(\alpha+1)}\,dx = \alpha \int_{-\infty}^{x} x^{-(\alpha+1)}\,dx$

$$= \alpha\left(\frac{x^{-(\alpha+1)+1}}{-(\alpha+1)+1}\right) + C = \alpha\left(\frac{x^{-\alpha}}{-\alpha}\right) + C = -x^{-\alpha} + C = -\frac{1}{x^{\alpha}} + C$$

. Given that $f(x) = \begin{cases} \frac{\alpha}{x^{\alpha+1}} & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow \begin{cases} F(x) = \frac{-1}{x^{\alpha}} + C & \text{if } x \geq 1 \\ F(x) = 0 & \text{otherwise} \end{cases}$

We might assume that the integral $= 0$ at $x = 1$ to ensure continuity and smooth transition :

$$-\frac{1}{1^{\alpha}} + C = 0$$
$$\Rightarrow C = 1$$

So the integral from $x \geq 1$ is: $\int_{-\infty}^{x} \frac{\alpha}{x^{\alpha+1}}\,dx = -\frac{1}{x^{\alpha}} + 1$ for $x > 1$

b) Suppose that for a simulation you want to run, you need to generate i.i.d Pareto (α) random variables. Suppose you have a computer that know how to generate Unif (0,1) r.v but not Pareto r.v.

Let $U \sim \text{Unif}(0,1)$. Give a function of U that has the Pareto (α) distribution.

. Part 1 of the universality of Uniform says that $F^{-1}(U) \sim \text{Pareto}(\alpha)$ with $U \sim \text{Unif}(0,1)$

so we first inverse CDF to get F^{-1}:

$$F^{-1}(u) = \frac{1}{(1-u)^{1/\alpha}}$$

so $X = \frac{1}{(1-U)^{1/\alpha}} \sim \text{Pareto}(\alpha)$

Problem 4: A post office has 2 clerks. Alice enters the post office while 2 other customers, Bob and Claire, are being served by the 2 clerks. She is next in line. Assume that the time a clerk ~~spende~~ spends serving a customer has the Expo (λ) distribution. What is the probability that Alice is the last of the 3 customers to be done serving?

Alice begins to be served when either Bob or Claire leaves.

By memoryless property, the additional time needed to serve whichever of Bob or Claire is still there is Expo (λ) and the time it takes to serve Alice is also Expo (λ),

So by symmetry, the probability is $\frac{1}{2}$ that Alice is the last to be done serving