REPRESENTING POLYNOMIALS

2 ways to represent polynomials

There are 2 ways: coefficient form and point-value form

- Coefficient form:

  For a polynomial $A(x) = \sum_{j=0}^{n-1} a_j x^j$. The coefficient representation is vector $a = (a_0, a_1, \ldots a_{n-1})$

- Point-value form:

  For a polynomial $A(x) = \sum_{j=0}^{n-1} a_j x^j$. The point-value representation is a set of n point-value pairs:

  $\{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$

  such that $y_j = A(x_0)$

Convert from Coefficient Form to Point-value Form, and vice versa:

- From Coefficient Form to Point-value Form

  Problem: Given vector $a = (a_0, a_1, \ldots, a_{n-1})$ represents a polynomial $A(x) = \sum_{j=0}^{n-1} a_j x^j$. Find a set of n distinct points such that $y_j = A(x_0)$.

  ⇒ This is actually the process of evaluating $A(x)$ at n distinct points.

  ⇒ Using Horner's Method, this takes $O(n^2)$

  ⇒ Using Fast Fourier Transform, this takes $O(n \log n)$

  → discussed in "Discrete Fourier Transform"

- From Point-value Form to Coefficient Form

  Problem: Given a set of n distinct point-value pairs:

  $\{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$

  Find the vector $a = (a_0, a_1, \ldots, a_{n-1})$ represent the coefficients of polynomial $A(x) = \sum_{j=0}^{n-1} a_j x^j$

  ⇒ This is actually the process of interpolation $A(x)$ given n distincts pairs.

$\Rightarrow$ There are 3 ways to interpolation:

- <u>Inverse Vander-monde matrix:</u>

  Solve $a = V^{-1} y$, where $V = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & & & & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & & x_{n-1}^{n-1} \end{bmatrix}$

  This cost $O(n^3)$

- <u>Lagrange's formula:</u>

  $$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - y_j)}{\prod_{j \neq k} (x_k - j)}$$

  This cost $O(n^2)$

- <u>Inverse Fast Fourier Eval</u> (Inverse - FFT)

  Since Polynomial Evaluation and Interpolation are inverse operations, we can perform interpolation using Inverse-FFT

  This cost $O(n \log n)$