5.7 Poisson

Recall that we were briefly introduced the Poisson in Unit 3, and saw it in the Poisson process context in Unit 4. Now we will go more in depth

Definition 5.7.1 (Poisson distribution)

An r.v X has the Poisson distribution with parameter λ, where $\lambda > 0$, if the PMF of X is:

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$

We write this as $X \sim \text{Pois}(\lambda)$

Example 5.7.2 (Poisson expectation and variance)

- Let $X \sim \text{Pois}(\lambda)$, the mean is:

$$\begin{aligned} E(X) &= e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} \\ &= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda e^{-\lambda} e^{\lambda} = \lambda \end{aligned}$$

So $\boxed{E(X) = \lambda \text{ is mean of Poisson distribution}}$

- Now consider the variance: $\text{Var}(X) = E(X^2) - \underbrace{(EX)}_{\text{mean}}^2$
  - First we find $E(X^2)$:

$$E(X^2) = \sum_{k=0}^{\infty} k^2 P(X=k) = e^{-\lambda} \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!}$$

  - Using the same method we used to get variance of Geometric r.v, we have:

$$E(X^2) = \lambda(1+\lambda)$$

So $\boxed{\text{Var}(X) = E(X^2) - (EX)^2 = \lambda(1+\lambda) - \lambda^2 = \lambda \text{ is variance of Poisson distribution}}$

What are the applications of Poisson distribution?

Poisson distribution are often used in situations where we are counting number of successes in a fixed region or time period. With large number of trials, and small probability of success.

For example, these r.v.s could follow Poisson distribution:

- The number of emails received in an hour. Imagine slice 1 hour into milliseconds, the probability that you receive email in that millisecond is small, while there is large number of milliseconds in an hour.
- The number chips in a chocolate chip cookie. Imagine divide a chocolate chip cookie into smaller cubes, the probability of the chip inside a cube is small, while there is large number of cubes.
- The number of earthquakes in a year. Follow the same reasoning as above.

What does parameter λ represents?

λ is interpreted as the rate of occurrence of these rare events. In the examples above, λ could be:

- 20 emails per hour
- 10 chips per cookie
- 2 earthquakes per year

Poisson approximation

- Let $A_1, A_2, A_3, \dots, A_n$ be events with $p_j = P(A_j)$.
- When n is large, p_j are small, and A_j are independent or weakly dependent:

> Let $X = \sum_{j=1}^{n} I(A_j)$ count how many of A_j occur.
> Then X is approximately $\text{Pois}(\lambda)$, with $\lambda = \sum_{j=1}^{n} p_j$

Poisson paradigm (or Law of rare events)

Note: Don't confuse X being Binomial distributed, as Binomial r.v.s requires the constituent indicator r.v.s have the same success probability p, and don't require the event being "rare" (p being low) nor n being large.

What are the conditions for Poisson paradigm?

The conditions are fairly flexible:

- The n trials can have different success probabilities, as long as they are low.
- The trials don't have to be independent, though they shouldn't be too dependent.

This makes Poisson distribution a popular model for data values that are nonnegative integers

Example 5.7.4 (Birthday problem continued)

- We have m people and make the usual assumptions about birthdays. Then each pair of people of $p = \frac{1}{365}$ of matching birthday, and there are $\binom{m}{2}$ pairs.
- Let X be number of birthday matches. $X \sim \text{Pois}(\lambda)$ since there is small chance $\left(\frac{1}{365}\right)$ any 2 people would have matching birthday, while the number of pairs are large $\binom{m}{2}$. Then the probability of at least 1 match is:

$$P(X \geq 1) = 1 - P(X=0) \approx 1 - e^{-\lambda} \quad , \text{with } \lambda = \binom{m}{2}\frac{1}{365}$$

- Note that in this problem, we should only care about 2 things:
  - $p = \frac{1}{365}$, which is the probability of success, and can be different
  - $\binom{m}{2}$, total number of "trials" for successful birthday match.

Example 5.7.5 (Near-birthday problem)

What if we want to find the number of people required in order to have 50-50 chance that 2 people would have birthdays within one day of each other (i.e., on the same day or one day apart)?

- The Poisson paradigm still applies, the probability that any 2 people having birthdays within one day of each other is $\frac{3}{365}$, and there are $\binom{m}{2}$ pairs.
- Let X be the number of birthdays are within one day of each other

then $X \sim \text{Pois}(\lambda)$ with $\lambda = \binom{n}{2} \frac{3}{365}$

- Then the probability of at least 1 match within one day is:

$$P(X \geq 1) = 1 - P(X=0) \approx 1 - e^{-\lambda}$$

From this, we can work out that with $n = 14$ or more would give us $P(X \geq 1)$ approximately $\frac{1}{2}$

Theorem 5.7.6 (Sum of independent Poissons)

If $X \sim \text{Pois}(\lambda_1)$ and $Y \sim \text{Pois}(\lambda_2)$, and X is independent of Y, then:

$$X + Y \sim \text{Pois}(\lambda_1 + \lambda_2)$$

Proof: To get PMF of $X+Y$, condition on X and use LOTP:

$$\begin{aligned}
P(X+Y=k) &= \sum_{j=0}^{k} P(X+Y=k \mid X=j)\, P(X=j) \\
&= \sum_{j=0}^{k} P(Y=k-j)\, P(X=j) \\
&= \sum_{j=0}^{k} \frac{e^{-\lambda_2} \lambda_2^{k-j}}{(k-j)!} \frac{e^{-\lambda_1} \lambda_1^{j}}{j!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda_1^{j} \lambda_2^{k-j} \\
&= \frac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^k}{k!}
\end{aligned}$$

So $X + Y \sim \text{Pois}(\lambda_1 + \lambda_2)$. If there are 2 different types of events occurring at rates λ_1 and λ_2, independently, then the overall event rate is $\lambda_1 + \lambda_2$