The following statements are equivalent about $A \in \mathbb{R}^{n \times n}$

- A is nonsingular
- A is invertible
- A^{-1} exists
- $AA^{-1} = A^{-1}A = I$
- A represents a linear transformation that is a bijection
- $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$
- $Ax = 0$ implies that $x = 0$
- $Ax = e_j$ has a solution for all $j \in \{0, \dots, n-1\}$
- The determinant of A is nonzero: $\det(A) \neq 0$
- LU with partial pivoting doesn't break down
- $C(A) = \mathbb{R}^n$
- A has linearly independent columns
- $\mathcal{N}(A) = \{0\}$
- $\operatorname{rank}(A) = n$

- $A = U_L \Sigma_{TL} V^H \quad (V_L = V)$

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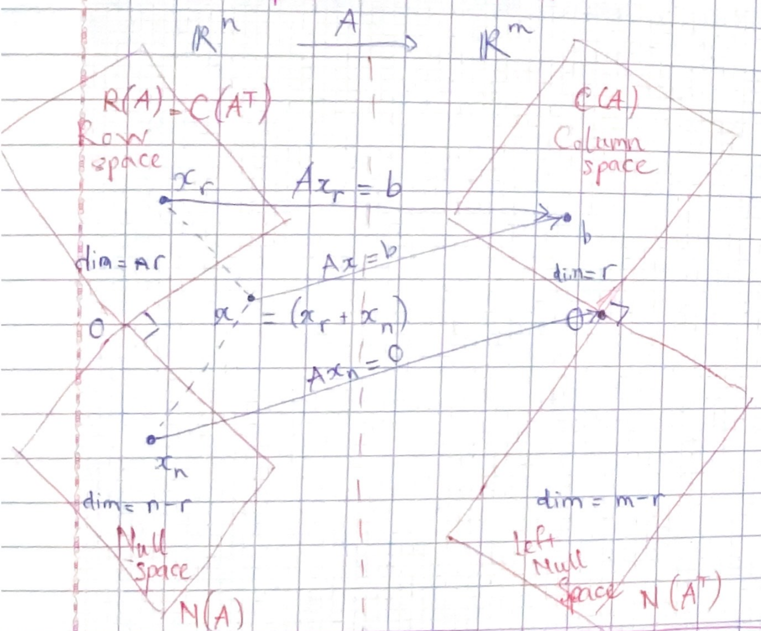
Subspaces cheat sheet



Eigenvectors and Eigenvalue

$$Ax = \lambda x$$
$$\iff \det(A - \lambda I) = 0$$

- Eigenvectors ~~are~~ stay on its span when A tranceform, eiganvalue says how much it scale
- Eigenvectors can be use as new basis:

Say v_1, v_2 are 2 eigen: $(v_1, v_2)^{-1} A (v_1, v_2)$ = diagonal matrix

I Orthogonality

HW 1.1.1.5

- When is a upper triangular matrix singular?

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0,n-1} \\ 0 & a_{11} & & a_{1,n-1} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} \end{bmatrix}$$

An upper triangular matrix singular when one of the elements in its diagonal is 0.

Because det(A) is ~~its~~ product of its diagonal

And $\underbrace{\det(A) = 0}_{\text{means A does not have an inverse}} \iff A$ is singular

MOTIVATION

We have $Ux = b$

and $f(b) = x$. In the computer, it return an approximate solution $\hat{x}$

$\Rightarrow \hat{f}(b) = \hat{x}$ (the computer function)

$\Rightarrow \hat{b} = U\hat{x}$ (find b based of approx x)

$\Rightarrow f(\hat{b}) = \hat{x}$

Can see: $\frac{\|\hat{b} - b\|}{\|b\|}$ is small while $\frac{\|\hat{x} - x\|}{\|x\|}$ can be large

Unit 1: NORMS

Absolute value norms

For $|\cdot|: \mathbb{C} \to \mathbb{R}$, $\alpha = \alpha_r + \alpha_c i$

$$|\alpha| = \sqrt{\alpha_r^2 + \alpha_c^2}$$

Also: $|\alpha| = \sqrt{\bar{\alpha}\alpha}$ where $\bar{\alpha} = \alpha_r - \alpha_c i$

The absolute value function has these properties

- $\alpha \neq 0 \Rightarrow |\alpha| > 0$ ($|\cdot|$ is positive definite)
- $|\alpha\beta| = |\alpha|\,|\beta|$ ($|\cdot|$ is homogeneous)
- $|\alpha + \beta| \leq |\alpha| + |\beta|$ ($|\cdot|$ obeys triangle inequality)

Things to rmb

- $\alpha\beta = \beta\alpha$
- $\overline{\bar{\alpha}\beta} = \bar{\beta}\alpha$
- $\bar{\alpha}\alpha \in \mathbb{R}$
- $|\bar{\alpha}| = |\alpha|$

$\bar{a} + a = Real(a)$

$\cancel{\bar{a}\,\bar{a}}$

$\bar{a}\,a = |a|^2$

Vectors norms

Vector norm measure the magnitude (length) of a vector, it has these properties

For $\nu: \mathbb{C}^m \to \mathbb{R}$, $x, y \in \mathbb{C}^m$, $\alpha \in \mathbb{R}$

- $x \neq 0 \Rightarrow \nu(x) > 0$ (positive definite)
- $\nu(\alpha x) = |\alpha|\,\nu(x)$ (homogeneous)
- $\nu(x + y) \leq \nu(x) + \nu(y)$ (triangle inequality)

Vector 2-norm

For $\|.\|_2 : \mathbb{C}^m \to \mathbb{R}$ is defined for $x \in \mathbb{C}^m$:

$$\|x\|_2 = \sqrt{|x_0|^2 + \dots |x_{m-1}|^2} = \sqrt{\sum_{i=0}^{m-1} |x_i|^2}$$

or

$$\|x\|_2 = \sqrt{\bar{x}_0 x_0 + \dots \bar{x}_{m-1} x_{m-1}} = \sqrt{\sum_{i=0}^{m-1} \bar{x}_i x_i}$$

Cauchy-Schwarz inequality

Let $x, y \in \mathbb{C}^m$. Then:

$$|x^H y| \leq \|x\|_2 \|y\|_2$$

Equivalent statements:

$$\|x\|_2^2 = x^H x$$

Vector 1-norm

The vector 1-norm, $\|.\|_1 : \mathbb{C}^m \to \mathbb{R}$, is defined $x \in \mathbb{C}^m$

$$\|x\|_1 = |x_0| + \dots |x_{m-1}| = \sum_{i=0}^{m-1} |x_i|$$

Vector ∞-norm

The vector ∞-norm, $\|.\|: \mathbb{C}^m \to \mathbb{R}$, is defined $x \in \mathbb{C}^m$

$$\|x\|_\infty = \max(|x_0|, \dots, |x_{m-1}|) = \max_{i=0}^{m-1} |x_i|$$

Unit ball

Given norm $\|.\| : \mathbb{C}^m \to \mathbb{R}$, the unit ball with respect to $\|.\|$ is the set $\{x \mid \|x\| = 1\}$

$\forall \quad \|x\| = 1$ (short - form)

This is the set of vectors with norm equal to 1

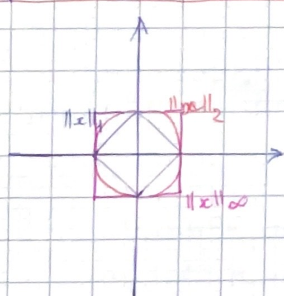
Equivalence of vector norms

	$\|x\|_1 \leq \sqrt{m}\ \|x\|_2$	$\|x\|_1 \leq m \|x\|_\infty$
$\|x\|_2 \leq \|x\|_1$		$\|x\|_2 \leq \sqrt{m}\ \|x\|_\infty$
$\|x\|_\infty \leq \|x\|_1$	$\|x\|_\infty \leq \|x\|_2$	

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$$

$m \geq$; $\sqrt{m} \geq$; $\sqrt{m} \geq$

$\cancel{\|x\|_\infty \leq}\ \|x\|_2$



Matrix Norms

Matrix norms

Let $\nu : \mathbb{C}^{m \times n} \to \mathbb{R}$. Then ν is a matrix norm if for all $A, B \in \mathbb{C}^{m \times n}$ and all $\alpha \in \mathbb{C}$:

- $A \neq 0 \Rightarrow \nu(A) > 0$
- $\nu(\alpha A) = |\alpha|\ \nu(A)$
- $\nu(A + B) \leq \nu(A) + \nu(B)$

The Frobenius norm

$\|.\|_F : \mathbb{C}^{m \times n} \to \mathbb{R}$ is defined for $A \in \mathbb{C}^{m \times n}$

$$\|A\|_F = \sqrt{\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |\alpha_{i,j}|^2} = \sqrt{\begin{array}{l} |\alpha_{0,0}|^2 + \dots + |\alpha_{0,n-1}|^2 \\ + \\ |\alpha_{m-1,0}|^2 + \dots + |\alpha_{m-1,n-1}|^2 \end{array}}$$

Hermitian transpose

- $\overline{A}^T = \overline{A^T}$
- $|\overline{\alpha}| = |\alpha|$
- If $A \in \mathbb{R}^{m \times n}$, $A^H = A^T$
- If $x \in \mathbb{C}^m$, then x^H is defined consistent with how we have used it before
- If $\alpha \in \mathbb{C}$, then $\alpha^H = \overline{\alpha}$

Also: $(AB)^H = B^H A^H$; $\overline{a}^H = a^T$

$x^H x = \|x\|_2^2$

Induced matrix norms

Let $\|.\|_\mu : \mathbb{C}^m \to \mathbb{R}$ and $\|.\|_\nu : \mathbb{C}^n \to \mathbb{R}$ be vector norms. Define $\|.\|_{\mu,\nu} : \mathbb{C}^{m \times n} \to \mathbb{R}$ by

$$\|A\|_{\mu,\nu} = \max_{x \neq 0} \frac{\|Ax\|_\mu}{\|x\|_\nu}$$

or

$$\|A\|_{\mu,\nu} = \max_{\|x\|_\nu = 1} \|Ax\|_\mu$$

Hw 1.3.4.1

p-norm $\|\cdot\|_p : \mathbb{C}^n \to \mathbb{R}$, induced matrix $|||\cdot|||_p : \mathbb{C}^{m\times n} \to \mathbb{R}$

$$|||A||| = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

We can ~~say~~ prove: $|||y||| = \|y\|_p$, $y \in \mathbb{C}^m$

Matrix p-norm

Matrix p-norm $\|\cdot\|_p : \mathbb{C}^{m\times n} \to \mathbb{R}$

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

or $\|A\|_p = \max_{\|x\|_p = 1} \|Ax\|_p$

Note: . matrix 2-norm is difficult to evaluate

. matrix 1-norm, ∞-norm, Forobenius norm are straightforward and cheap to compute, $O(mn)$

Matrix 2-norm

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2 = 1} \|Ax\|_2$$

computing 2-norm matrix involve finding the max eigenvalue which is non-trivial if m or $n > 3$

2 x 2 matrix

For some special matrices, 2-norm is easy to compute, like matrix 2x2

$$\boxed{\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 = \max(|\delta_0|, |\delta_1|)}$$

To prove this, we use an important technique:

To prove: $\max_{x \in D} f(x) = \alpha$, given $f : D \to \mathbb{R}$

We first prove: $\max_{x \in D} f(x) \leq \alpha$, find $y \in D : f(y) = \alpha$

then: $\max_{x \in D} f(x) \geq \alpha$

For example: prove $\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 = \max(|\delta_0|, |\delta_1|)$

- Prove $\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 \leq \max(|\delta_0|, |\delta_1|)$

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2^2 = \max_{\|x_2\|_2 = 1} \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \right\|_2^2$$

$$= \max_{\|x\|_2 = 1} \left\| \begin{pmatrix} \delta_0 x_0 \\ \delta_1 x_1 \end{pmatrix} \right\|_2^2$$

$$= \max_{\|x\|_2 = 1} \left[(\delta_0 x_0)^2 + (\delta_1 x_1)^2 \right]$$

$$= \max_{\|x\|_2 = 1} \left(|\delta_0|^2 |x_0|^2 + |\delta_1|^2 |x_1|^2 \right)$$

$$\leq \max_{\|x\|_2 = 1} \left[\max(|\delta_0|, |\delta_1|)^2 \left[|x_0|^2 + |x_1|^2 \right] \right]$$

$$= \max(|\delta_0|, |\delta_1|)$$

- Choose $j \in \{0, 1\}$ such that $|\delta_j| = \max(|\delta_0|, |\delta_1|)$

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 = \max_{\|x\|_2 = 1} \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \right\|_2$$

$$\geq \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} e_j \right\|_2 = \|\delta_j e_j\|_2$$

$$= |\delta_j| \, \|e_j\|_2 = |\delta_j| = \max(|\delta_0|, |\delta_1|)$$

Diagonal matrix

$$\|D\|_2 = \max_{j=0}^{m-1} |\delta_j|$$

1.3.5.2

Let $y \in \mathbb{C}^m$ and $x \in \mathbb{C}^n$

$$\|yx^H\|_2 = \|y\|_2 \|x\|_2$$

1.3.5.3

$\|a_j\|_2 \leqslant \|A\|_2$ with $A \in \mathbb{C}^{m\times n}$

a_j is the j column

1.3.5.4

- $\|A\|_2 = \max_{\|x\|_2 = \|y\|_2 = 1} |y^H A x|$
- $\|A^H\|_2 = \|A\|_2$
- $\|A^H A\|_2 = \|A\|_2^2$

1.3.5.5

$$\|A_{ij}\|_2 \leqslant \|A\|_2$$

Matrix 1-norm

$$\|A\|_1 = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{\|x\|_1 = 1} \|Ax\|_1$$

Matrix infinity norm

$$\|A\|_\infty = \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{\|x\|_\infty = 1} \|Ax\|_\infty$$

1.3.6.1

Let $A \in \mathbb{C}^{m\times n}$ and partition $A = (a_0 | \dots | a_{n-1})$

$$\|A\|_1 = \max_{0 \leqslant j < n} \|a_j\|_1$$

1.3.6.2

Let $A \in \mathbb{C}^{m\times n}$ and partition $A = \begin{pmatrix} \tilde{a}_0^T \\ \vdots \\ \tilde{a}_{m-1}^T \end{pmatrix}$

$$\|A\|_\infty = \max_{0 \leqslant i < m} \|\tilde{a}_i\|_1 \quad (= \max(|\alpha_{i0}| + |\alpha_{i1}| + \dots + |\alpha_{i,n-1}|))$$

Equivalent of matrix norms

1.3.7.2

Given $A \in \mathbb{C}^{m \times n}$,

$$\|A\|_2 \leq \|A\|_F$$

	$\|A\|_1 \leq \sqrt{m}\,\|A\|_2$	$\|A\|_1 \leq m\,\|A\|_\infty$	$\|A\|_1 \leq \sqrt{m}\,\|A\|_F$
$\|A\|_2 \leq \sqrt{n}\,\|A\|_1$		$\|A\|_2 \leq \sqrt{m}\,\|A\|_\infty$	$\|A\|_2 \leq \|A\|_F$
$\|A\|_\infty \leq n\,\|A\|_1$	$\|A\|_\infty \leq \sqrt{n}\,\|A\|_2$		$\|A\|_\infty \leq \sqrt{n}\,\|A\|_F$
$\|A\|_F \leq \sqrt{n}\,\|A\|_1$	$\|A\|_F \leq ?\,\|A\|_2$	$\|A\|_F \leq \sqrt{m}\,\|A\|_\infty$	

Subordinate matrix norms

A matrix norm $\|\cdot\|: \mathbb{C}^{m \times n} \to \mathbb{R}$ is said to be subordinate to vector norms $\|\cdot\|_\mu : \mathbb{C}^m \to \mathbb{R}$ and $\|\cdot\|_\nu : \mathbb{C}^n \to \mathbb{R}$ if, all $x \in \mathbb{R}^n$

$$\|Ax\|_\mu \leq \|A\|\,\|x\|_\nu$$

If $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$ are the same norms, $\cancel{\|\cdot\|_\mu : \mathbb{C}^m \to \mathbb{R}}$ then $\|\cdot\|$ is said to be subordinate to the given vector norm.

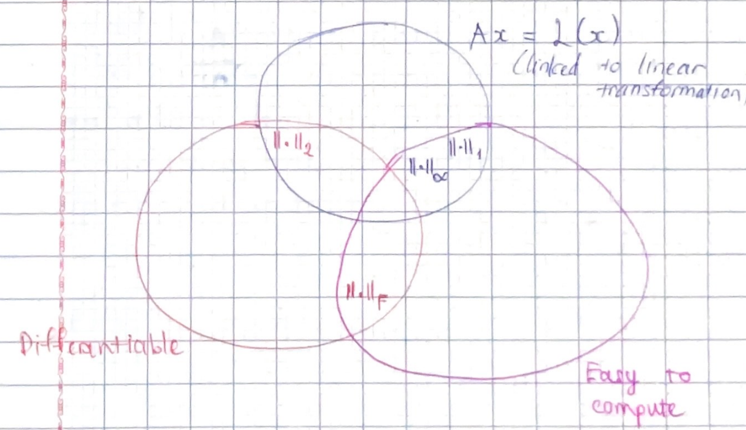
Consistent matrix norm

A matrix norm is consistent if is defined for all m and n, using the same formula for all m and n, ($\|\cdot\|: \mathbb{C}^{m \times n} \to \mathbb{R}$)

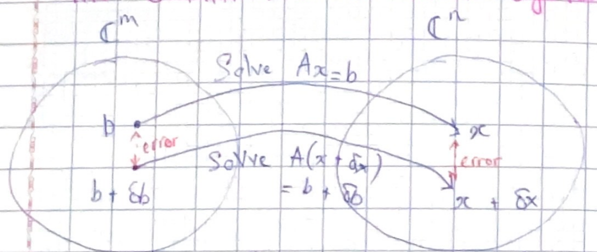
Submultiplicative matrix norm

A consistent matrix norm $\|\cdot\|: \mathbb{C}^{m \times n} \to \mathbb{R}$ is said to be submultiplicative if it satisfies

$$\|AB\| \leq \|A\|\,\|B\|$$



THE Condition of a linear system



Question is: $\dfrac{\|\delta x\|}{\|x\|} \overset{?}{=} \dfrac{\|\delta b\|}{\|b\|}$

(~~relati~~ error x relative to x) (error b relative to b)

Can be proven that

$$\boxed{\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\|A\|\,\|A^{-1}\|}_{\kappa(A)} \frac{\|\delta b\|}{\|b\|}}$$

The error in b cannot be amplified more than $\kappa(A)$ when there is error in x.
$\kappa(A)$ is the condition number of matrix A

1.4.1.1

$$\|I\| = \max_{\|x\|=1} \|Ix\| = \max_{\|x\|=1} \|x\| = 1$$

1.4.1.2

$$\boxed{\kappa(A) = \|A\|\,\|A^{-1}\| \geq 1}$$

This show that there are choices for δb and b so that best case scenario is $\dfrac{\|\delta x\|}{\|x\|} \leq \dfrac{\|\delta b\|}{\|b\|}$

and worst is $\dfrac{\|\delta x\|}{\|x\|} \leq \|A\|\,\|A^{-1}\|$

1.4.2.1

$\alpha = -14.24123$, $\hat{\alpha} = -14.24723$

$\log_{10}\left(\dfrac{|\alpha - \hat{\alpha}|}{|\alpha|}\right) \approx -3.4$

Computing $\log_{10}$ tells you approximately how many decimal digits are accurate: 3.4 digits

~~Figure 1.2.1.2 tells us about the dimension of these subspaces~~

~~$\dim(\mathcal{R}(A)) = \dim(\mathcal{C}(A)) = r$~~

~~$\dim(\mathcal{N}(A)) = n - r$~~

~~$\dim(\mathcal{N}(A^H)) = m - r$~~

~~with $A \in \mathbb{C}^{m\times n}$~~

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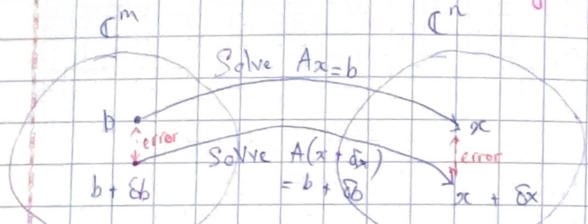
THE Condition of a linear system



Question is: $\dfrac{\|\delta x\|}{\|x\|} = ? \dfrac{\|\delta b\|}{\|b\|}$

(~~relat~~ error x relative to x) (error b relative to b)

Can be proven that

$$\boxed{\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\|A\| \, \|A^{-1}\|}_{k(A)} \frac{\|\delta b\|}{\|b\|}}$$

The error in b cannot be amplified more than $k(A)$ when there is error in x.
$k(A)$ is the condition number of matrix A

1.4.1.1

$$\|I\| = \max_{\|x\|=1} \|Ix\| = \max_{\|x\|=1} \|x\| = 1$$

1.4.1.2

$$\boxed{k(A) = \|A\| \, \|A^{-1}\| \geq 1}$$

This show that there are choices for δb and b so that best case scenario is $\dfrac{\|\delta x\|}{\|x\|} \leq \dfrac{\|\delta b\|}{\|b\|}$

and worst is $\dfrac{\|\delta x\|}{\|x\|} \leq \|A\| \, \|A^{-1}\|$

1.4.2.1

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~~Figure 1.2.12 tells us about the dimension of these subspaces~~

~~$\dim(R(A)) = \dim(C(A)) = r$~~
~~$\dim(N(A)) = n - r$~~
~~$\dim(N(A^H)) = m - r$~~
~~with $A \in \mathbb{C}^{m \times n}$~~