The Stationary Behaviour of Fluid Limits of Reversible Processes is Concentrated on Stationary Points

Jean-Yves Le Boudec

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Abstract

Assume that a stochastic process can be approximated, when some scale parameter gets large, by a fluid limit (also called "mean field limit", or "hydrodynamic limit"). A common practice, often called the "fixed point approximation" consists in approximating the stationary behaviour of the stochastic process by the stationary points of the fluid limit. It is known that this may be incorrect in general, as the stationary behaviour of the fluid limit may not be described by its stationary points. We show however that, if the stochastic process is reversible, the fixed-point approximation is indeed valid. More precisely, we assume that the stochastic process converges to the fluid limit in distribution (hence in probability) at every fixed point in time. This assumption is very weak and holds for a large family of processes, among which many mean field and other interaction models. We show that the reversibility of the stochastic process implies that any limit point of its stationary distribution is concentrated on stationary points of the fluid limit. If the fluid limit has a unique stationary point, it is an approximation of the stationary distribution of the stochastic process.

1. Fluid Limit (= mean field limit) for Interacting Objects

Model: N interacting objects, discrete time, finite state space S, object n is in state $X_n^N(k) \in S$ at time step k; $\left(X_1^N(k), \ldots, X_N^N(k)\right)$ is Markov, $M_i^N(k) = \frac{1}{N}\sum_{n=1}^{N} 1_{X_{n(k)}^N=1}$ is the occupancy measure, with $M_i^N(k) \in E = \mathcal{P}(S)$

Scaling: $W^N(k) :=$ #objects that do a transition at step k, $\mathbb{E}\left(W^N(k)\right) = O(1)$ and $\mathbb{E}(W^N(k)^2) = o(N)$

Drift: $f^N(m) := \mathbb{E}(M^N(k+1) - M^N(k) | M^{N(k)} = m)$; $f(m) := \lim_{N\to\infty} N f^N(m)$

Fluid limit: $m: [0, +\infty) \to E$ is solution of ODE $\frac{dm(t)}{dt} = f\left(m(t)\right)$

Theorem: $\sup_{0\le t \le T} \|M^N(\lfloor Nt \rfloor) - m(t)\| \to 0$ in probability [Benaim 2008]

Example: Dormant Malware [Benaim 2008]

Model: Every time step, one node (`S' Susceptible, `D' Dormant or `A' Active) is picked and triggers a transition with this proba:

case	prob
1	$D\delta_D$
2	$D\lambda\frac{ND-1}{N-1}$
3	$A\beta\frac{D}{h+D}$
4	$A\delta_A$
5	$S(\alpha_0+rD)$
6	$S\alpha$

1. Recovery
   - D -> S
2. Mutual upgrade
   - D + D -> A + A
3. Infection by active
   - D + A -> A + A
4. Recovery
   - A -> S
5. Recruitment by Dormant
   - S + D -> D + D

   Direct infection
   - S -> D
6. Direct infection
   - S -> A

Occupancy measure is

$M^N(k) = (D^N(k), A^N(k), S^N(k))$

with

$S^N(k) + D^N(k) + A^N(k) = 1$

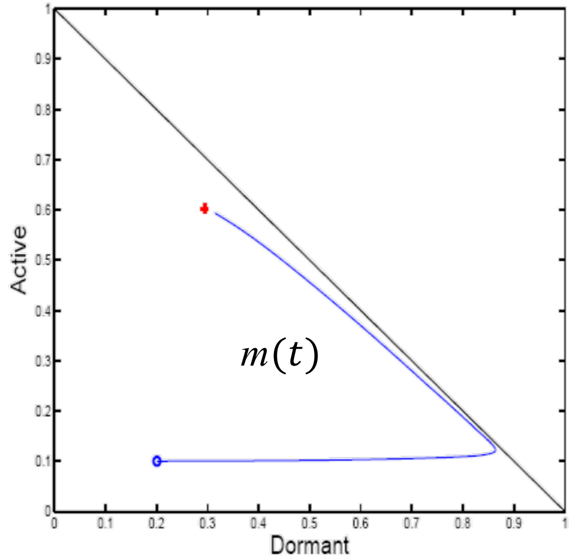
and e.g. $D^N(k)$ = proportion of nodes in state `D'

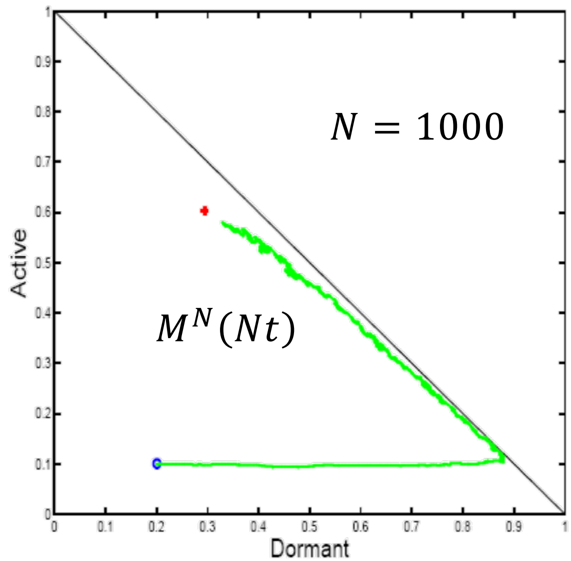
Fluid limit is ODE:

$$\frac{\partial D}{\partial t} = -\delta_D D - 2\lambda D^2 - \beta A\frac{D}{h+D} + (\alpha_0 + rD)S$$

$$\frac{\partial A}{\partial t} = 2\lambda D^2 + \beta A\frac{D}{h+D} - \delta_A A + \alpha S$$

$$\frac{\partial S}{\partial t} = \delta_D D + \delta_A A - (\alpha_0 + rD)S - \alpha S$$





Example: El Botellon [Rowe 2003]

Model: N people in total, N_i are in square i.
At every step:
pick one person at random; say she is in square i;

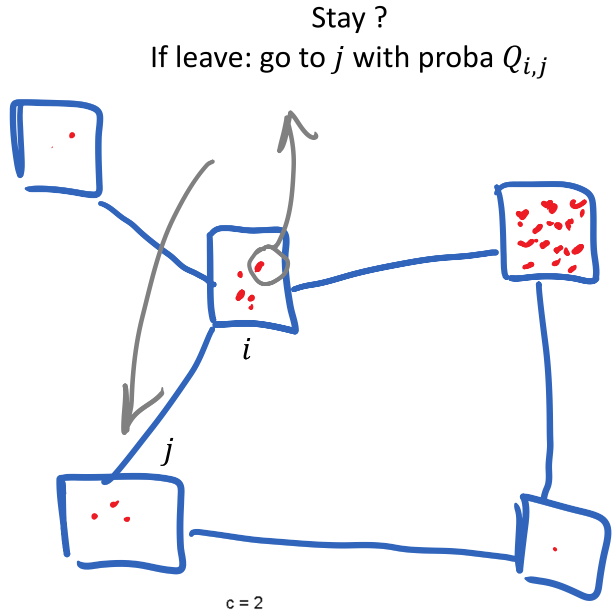
proba she leaves square is $\left(1-\frac{s}{N}\right)^{N_i-1}$ Socialization factor

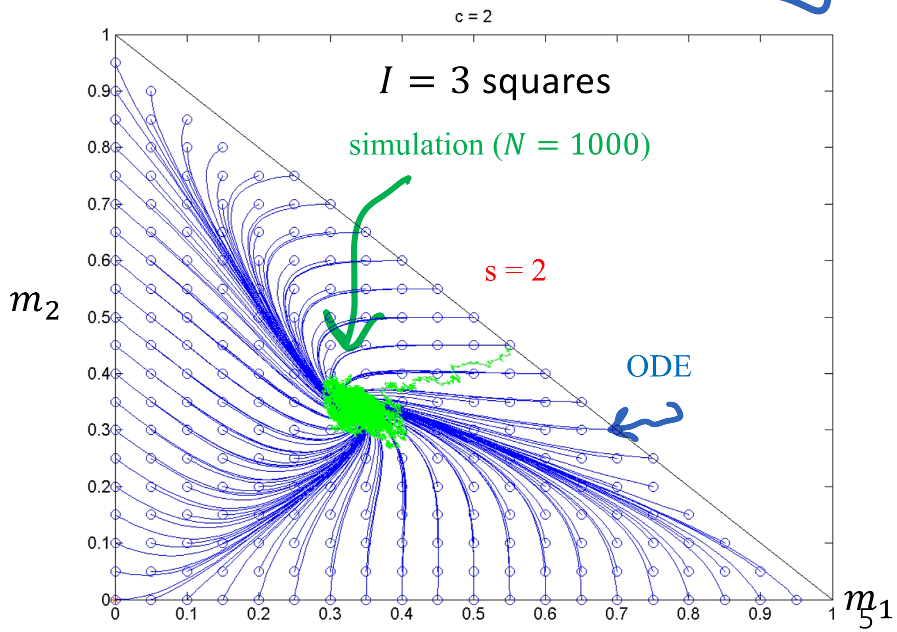
Occupancy measure is $M^N(k) = \left(\frac{N_i(k)}{N}\right)_{i=1:N}$

Fluid limit is ODE:

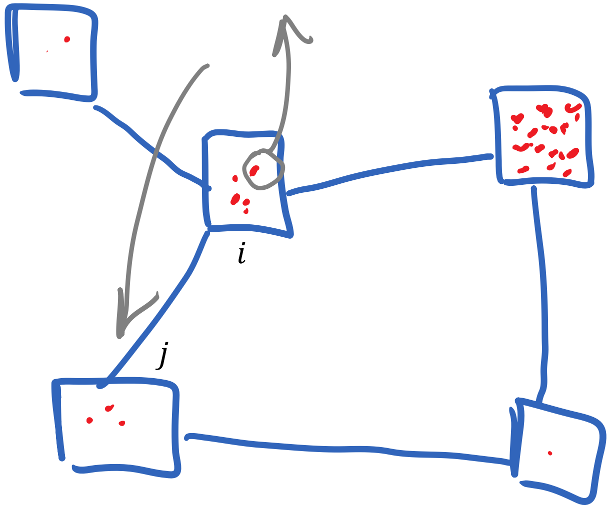
$$\frac{dm_i}{dt} = -m_i e^{-sm_i} + \sum_j m_j\, e^{-sm_j} Q_{i,j}$$

and for large N: $M^N(k) \approx m\left(\frac{k}{N}\right)$





El Botellon, Continuous Time Version



$Z^N(t) = \left(N_i(t)\right)_{i=1:N}$ is a continuous time Markov chain with $\sum_i N_i\,(t) = N$.

Transition matrix: $A_{\vec{n},\vec{n}-\vec{e}_i+\vec{e}_j} = \frac{1}{N} n_i \left(1-\frac{s}{N}\right)^{n_i-1} Q_{i,j}$
where λ is a constant (sampling rate).

$M^N(k)$ is the uniformization of $Z^N(t)$ with sampling rate $\lambda = N$
i.e. $M^N(k)$ is state of $Z^N(t)$ sampled at k^{th} tick of a sampling Poisson process with rate N [Rubino 2014].

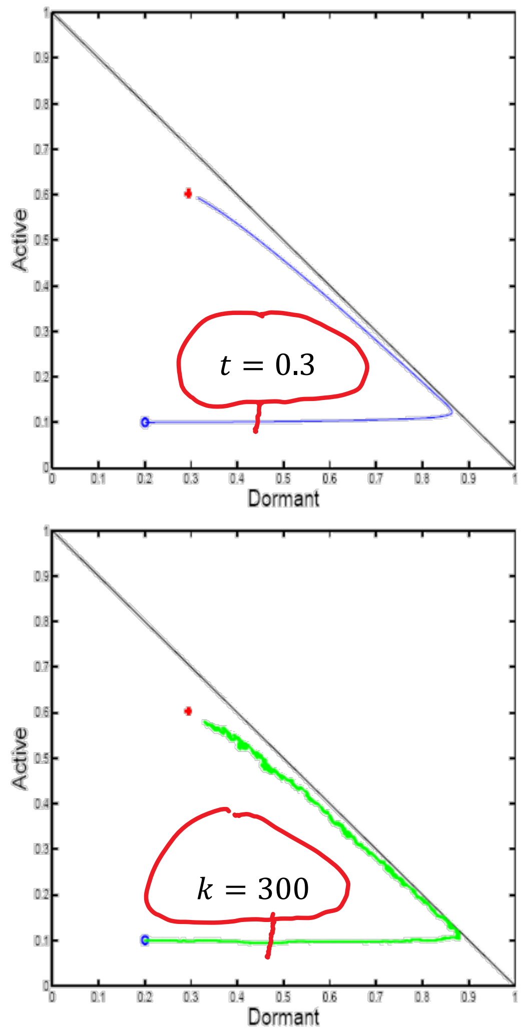
$Y^N(t) \coloneqq \frac{1}{N} Z^N(t) \to m(t)$ same fluid limit and same convergence result as discrete time model ("Kurtz's theorem", [Le Boudec 2013]).

Fluid Limit and Decoupling Assumption

For the generic model of interacting objects [Sznitman 1991]:

- $M^N \to$ deterministic limit m (fluid limit)
  $\Leftrightarrow$
  any ℓ objects are asymptotically mutually independent (ℓ is fixed and $N \to \infty$)

- and then $m\left(\frac{k}{N}\right) \approx$ law of state of one object $X_1^N(k)$



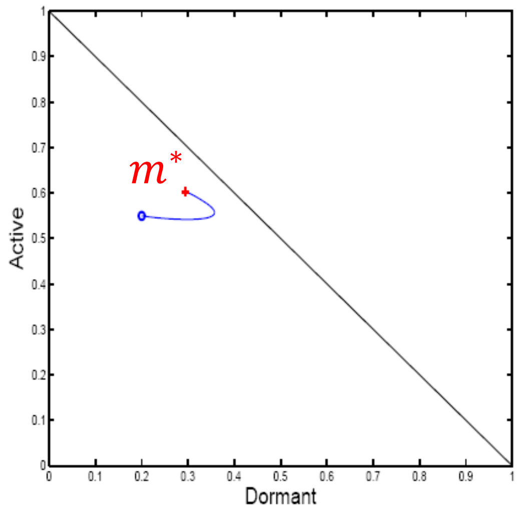
2. Stationary Regime

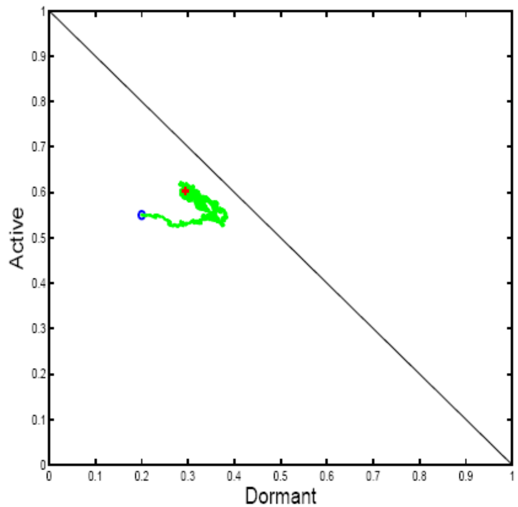
Assume system with finite N is ergodic (hence has a unique stationary regime).

Decoupling assumption says $m(k/N)$ approximates the law $\pi_1^N(k)$ of one object at step k. We are looking for $m^* = \lim_{k\to\infty} \pi_1^N(k)$

Now $\frac{dm(t)}{dt} = f\big(m(t)\big)$, thus m^*should satisfy $f(m^*) = 0$. This is called the "fixed-point assumption."

Commonly used: e.g. "Bianchi's formula" for 802.11 [Bianchi 1998] for El Botellon [Rowe 2003], for alternate routing in [Kelly 1991] etc.



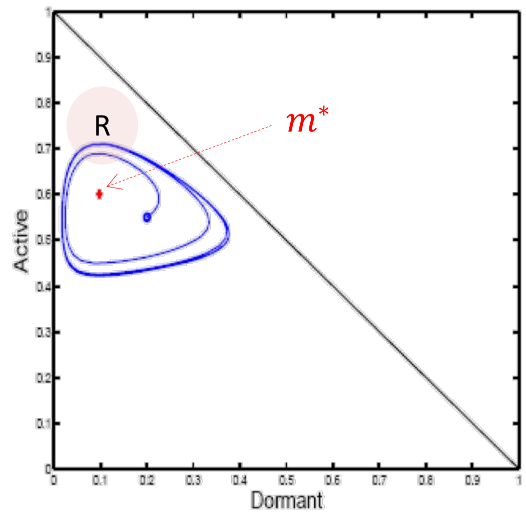


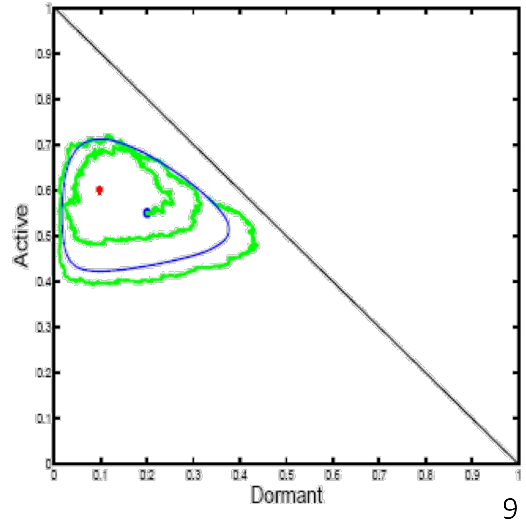
When fixed point assumption fails

- Here, fluid limit has a unique fixed point $f(m^*) = 0$ but m^* is not the large time approximation of stationary distribution – instead, there is a limit cycle.
- Assume you are in stationary regime (simulation has run for a long time) and you observe that one node, say $n = 1$, is in state 'A' – It is more likely that $m(t)$ is in region R. Thus, it is more likely that some other node, say $n = 2$, is also in state 'A'. I.e. decoupling assumption fails here in stationary regime – objects states are correlated.
- We can't interchange limits:

$$
\begin{array}{ccccc}
\mathbb{P}\left(X_1^N\left(\frac{t}{N}\right) = i \text{ and } X_2^N\left(\frac{t}{N}\right) = j\right) & & \xrightarrow{\quad t \to \infty \quad} & & \pi_{i,j}^N \\
\downarrow N \to \infty & & & & \downarrow N \to \infty \\
m_i(t)m_j(t) & \xrightarrow{t \to \infty} & m_i^* m_j^* & \neq & \int \mu_i \mu_j \Pi(d\mu)
\end{array}
$$

$h = 0.1$ (instead of 0.3)



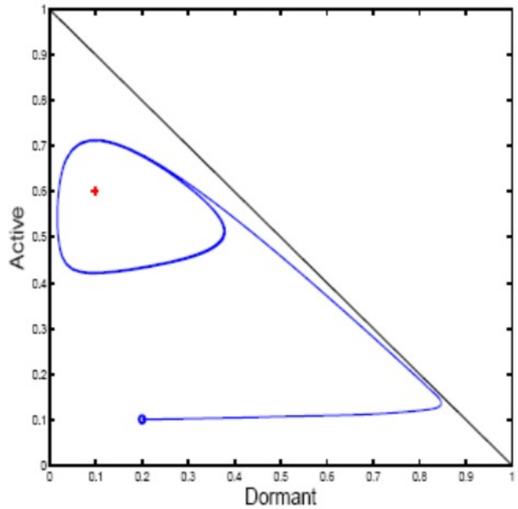


Long run behaviour : randomness comes back

[Benaim 2008] Any limit point of the stationary distribution is in the Birkhoff center of the ODE.

Birkhoff center = closure of set of points m s.t. $m \in \omega(m)$
Omega limit $\omega(m)$ = set of limit points of orbit starting at m

Here: Birkhoff center = limit cycle $\bigcup$ fixed point.



With more work, we could show that there is a probability distribution Π on the limit cycle that is invariant under the semi-flow of the ODE, and such that $\lim_{N\to\infty} \lim_{t\to\infty} \mathbb{E}\left(h\left(M^N\left(\frac{t}{N}\right)\right)\right) = \int h(\mu)\Pi(d\mu)$

Large N limit of stationary regime is not deterministic (unlike for transient regime).

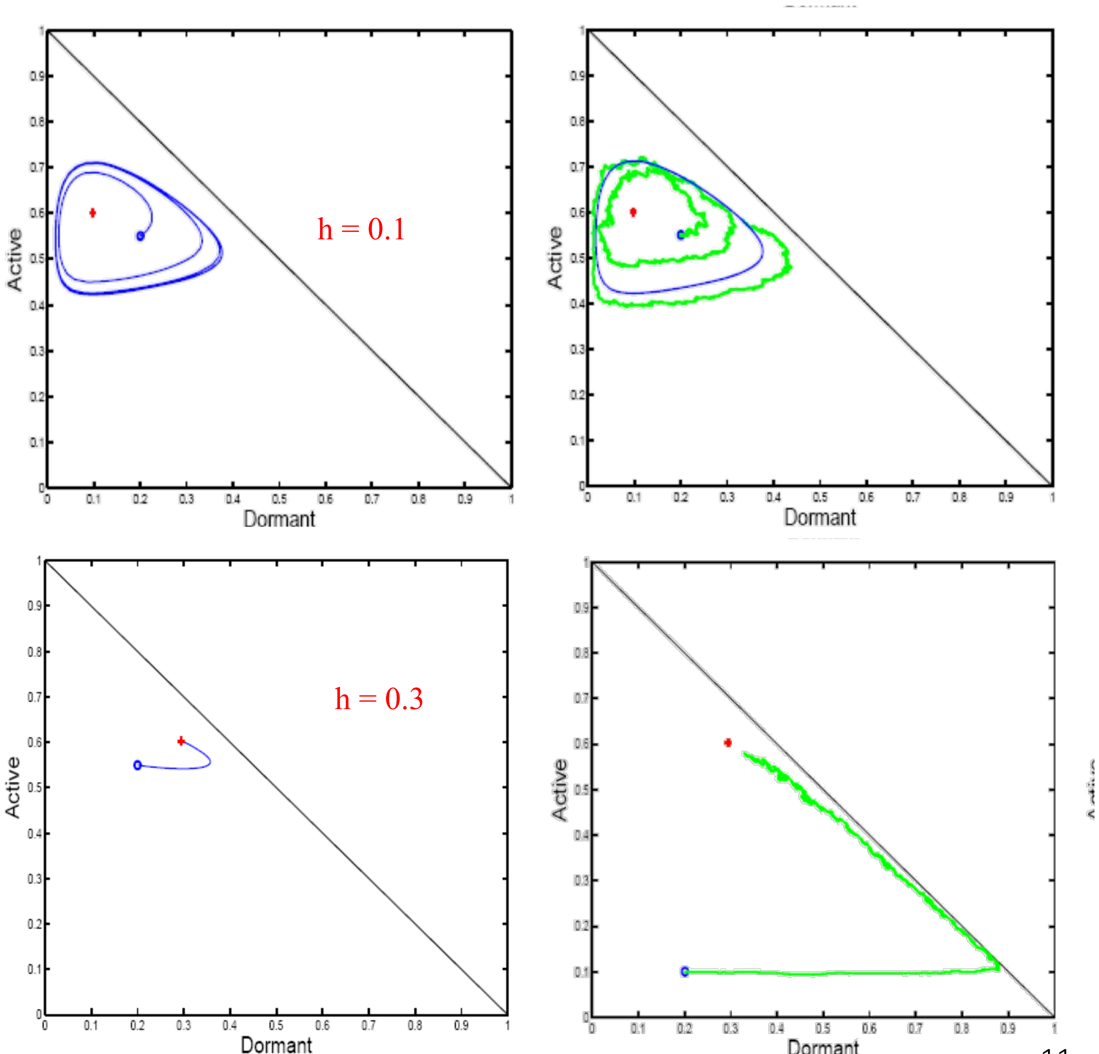
A Simple Case

[Benaim 2008]

If ODE has a unique fixed point to which all trajectories converge, then the stationary distribution of M^N converges to a Dirac mass at this fixed point and the fixed point assumption is valid.

Fixed point assumption is not valid

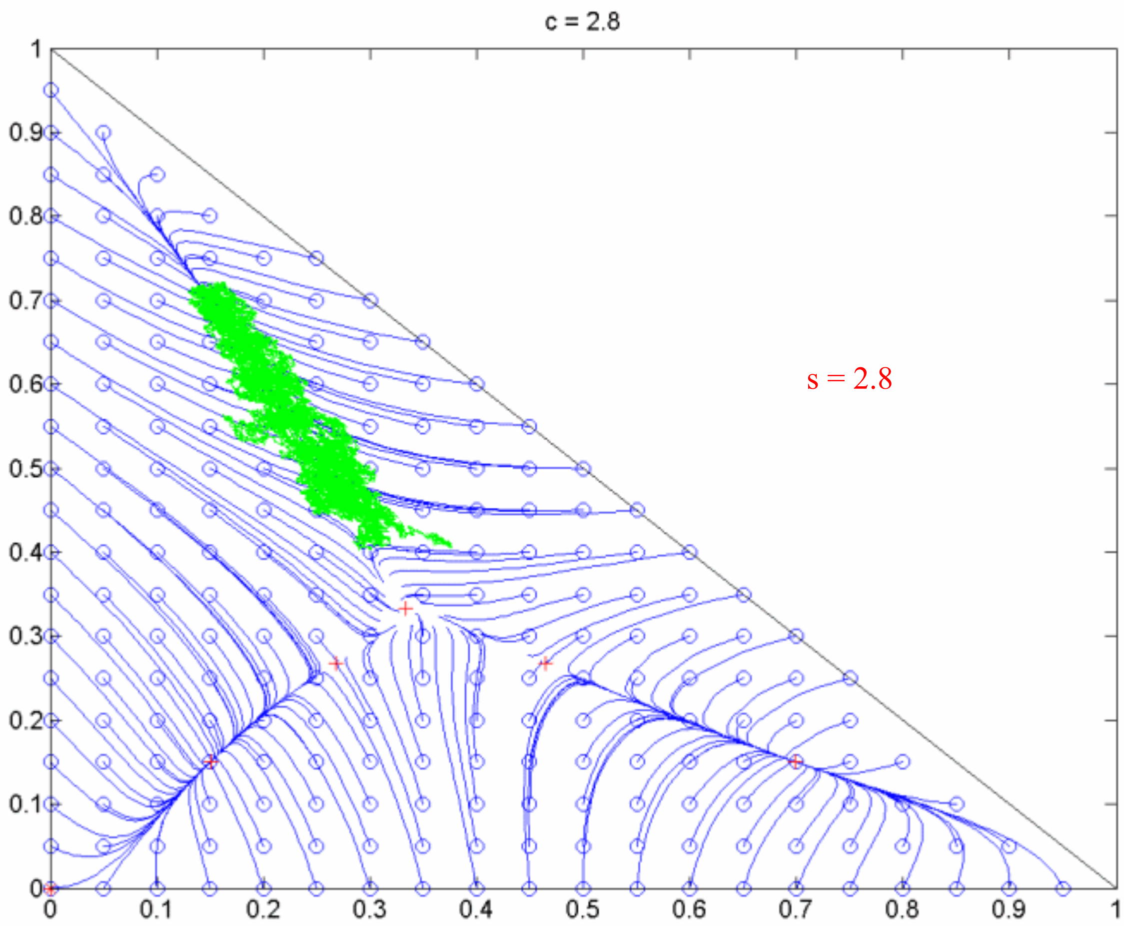
Fixed point assumption is valid

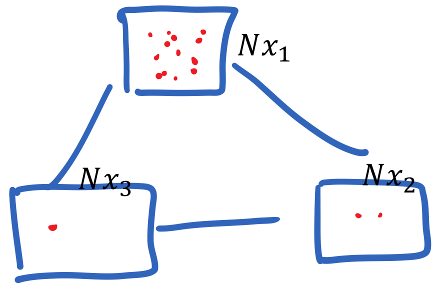


Example with El Botellon

Multiple fixed points, so the fixed point method breaks.

Object states are correlated.





3. Time Reversibility

Strong form: A random process Y defined for $t \geq 0$ is time-reversible iff for any $0 \leq t_1 < t_2 < \cdots < t_k$,

$$\big(Y\,(t_1), \ldots, Y(t_k)\big) \sim \big(Y\,(t_k), \ldots, Y(t_1)\big)$$

where $\sim$ means same distribution. This implies that Y is stationary.

Weak form: Y is reversible under Π iff for any test function h:

$$\int \mathbb{E}(h\big(y, Y(t)\big)|Y(0) = y)\Pi(dy) = \int \mathbb{E}(h(Y(t), y)|Y(0) = y)\Pi(dy)$$

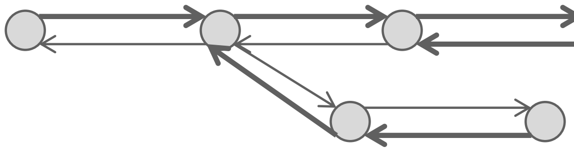
Y time-reversible (strong form) and Π is law of $Y(0) \Rightarrow Y$ is reversible under Π.

Ergodic discrete time Markov chain is time-reversible iff $\Pi_i Q_{i,j} = \Pi_j Q_{j,i}$, $\forall i, j$, where Q is transition probability matrix and Π stationary probability.

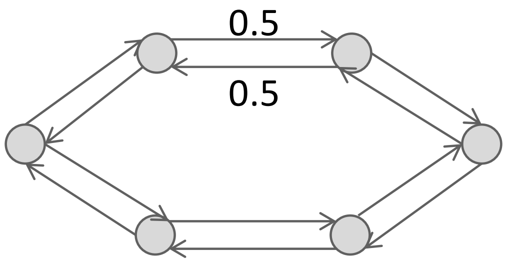
Ergodic continuous time Markov chain is time-reversible iff $\Pi_i A_{i,j} = \Pi_j A_{j,i}$, $\forall i, j$, where A is infinitesimal generator and Π stationary probability.

Reversible

- Markov chain on a tree



- Discrete Time Markov chain with symmetric transition proba $P_{i,j} = P_{j,i}$

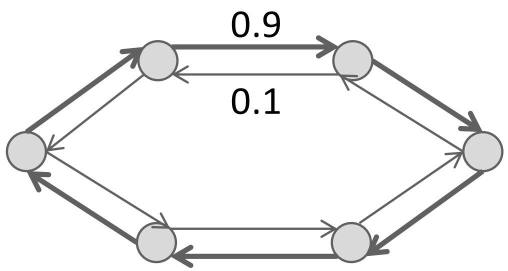


Reversible

- Birth and Death process, M/M/1 queue
- Product-form queuing network when the chain of visited stations is reversible [Le Boudec 87].
- El Botellon when neighbours are visited with equal proba.

Non-Reversible

- This Markov chain on a ring with unequal transition probabilities

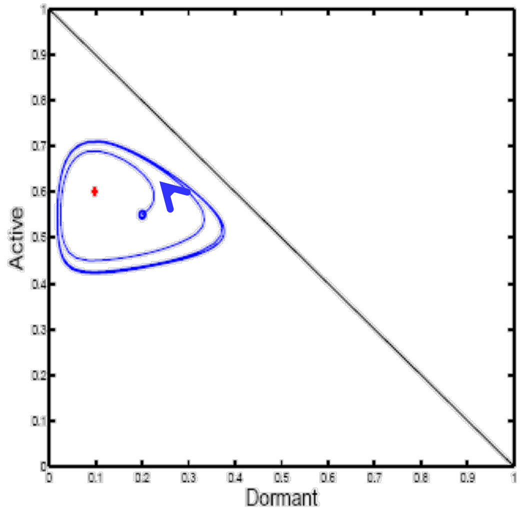


Reversible Semi-Flows

Semi-flow: mapping $[0, +\infty) \times E \to E, (t, y) \to \varphi_t(y)$
s.t. $\varphi_0(y) = y$ and $\varphi_{s+t}(y) = \varphi_s(\varphi_t(y))$.

Stationary point y: $\varphi_t(y) = y, \forall t \geq 0$.

Example: ODE: $\varphi_t(m_0) = m(t)$ where $\frac{dm(t)}{dt} = f(m(t))$
with $m(0) = m_0$; stationary m_0: $f(m_0) = 0$ (= fixed point)



This semi-flow is not reversible under the invariant probability of the limit-cycle: if we reverse time, it turns in the other direction.

Semi-flow φ is reversible under probability Π iff

$$\int_E h(y, \varphi_t(y))\Pi(dy) = \int_E h(\varphi_t(y), y)\Pi(dy)$$

i.e. trajectories in stationary regime are statistically same under time reversal.

Concentration on stationary points: If semi-flow $(t, y) \to \varphi_t(y)$ is continuous w.r.t. y and is reversible under Π, then $\Pi(\text{set of stationary points}) = 1$. [Le Boudec 2013]

Fluid Limit of Reversible Processes

Theorem [Le Boudec 2013] If

1) Y^N is reversible under Π^N
2) $Y^N(t) \to \varphi_t$ in distribution in the following sense:

   $\forall t \geq 0$, conditional law of $Y^N(t)|Y^N(0) = y^N(0) \xrightarrow{N \to \infty}$ dirac mass at $\varphi_t(y_0)$

   whenever $y^N(0) \xrightarrow{N \to \infty} y_0$
3) Π is a limit point of Π^N

then fluid limit φ is reversible under Π, hence $\Pi(\text{stationary points of } \varphi) = 1$

Examples: 2) holds whenY^N $=$ occupancy measure for N interacting objects

Corollary:

If in addition, E is compact and semi-flow φ has a unique stationary point point m^*, then $\lim_{N \to \infty} \Pi^N = \delta_{m^*}$ (for weak topology)

Examples: E is compact when state space of one object is finite

In the reversible case, no need to show that all trajectories converge to m^* !

$s < 2.7456$

All trajectories of ODE converge to the unique fixed point

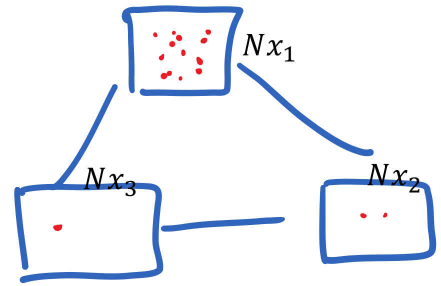
Fixed-point method is valid

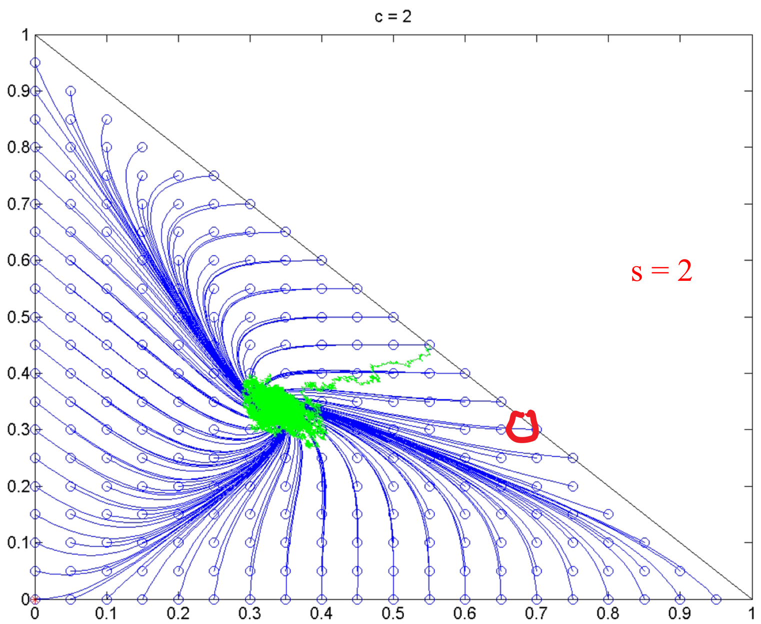
The proba that a person is in square i is $\approx \frac{1}{3}$

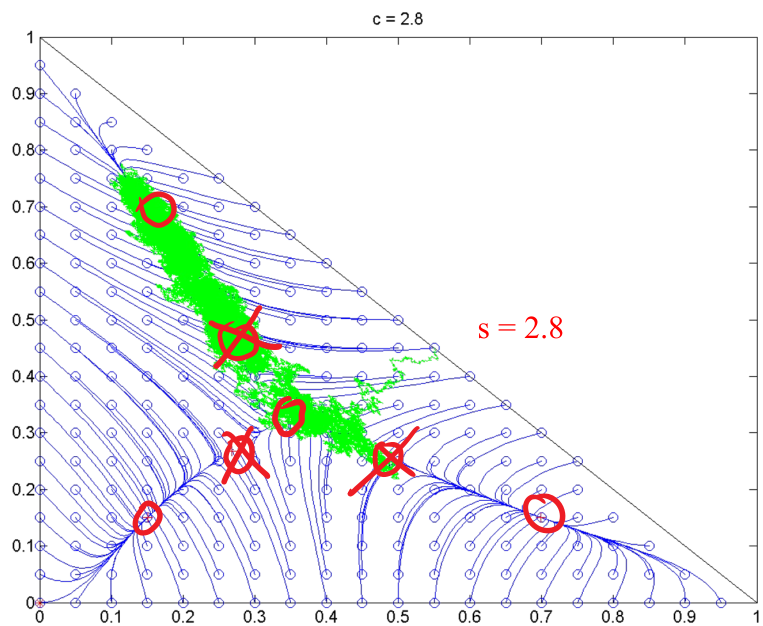
$s > 2.7456$

ODE has four stable fixed points

The occupancy measure is concentrated on the four stable fixed points (metastability)







Conclusion

Mean-field / fluid approximation gives a large N deterministic approximation of interacting objects over finite horizons.

In stationary regime, the fluid limit may appear to be random.

The (common) assumption that stationary points of the fluid limit characterize its stationary behaviour is wrong, in general, but is true if pre-limit processes are time-reversible.

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