Understanding the Simulation of Mobility Models with Palm Calculus

Jean-Yves Le Boudec
EPFL, I&C
Lausanne, Switzerland

Tutorial
Performance'05

Joint work with Milan Vojnovic and Darryl Veitch

- **This tutorial is available at**

  (with animation)
  http://ica1www.epfl.ch/perfeval/slides/leb-perf05.ppt

  (for printing)
  http://ica1www.epfl.ch/perfeval/slides/leb-perf05.pdf

- **The full text is available at**

  http://lcawww.epfl.ch/Publications/LeBoudec/LeBoudecV04.pdf

Motivation

- Simulation of mobility models often cause subtle problems
  - decay of average speed
  - difference between long term and initial distribution of nodes
  - sometimes instability
- Difficulty is in the nature of the models, not in simulation/programming technique
- Palm calculus is a key tool to master such complexities

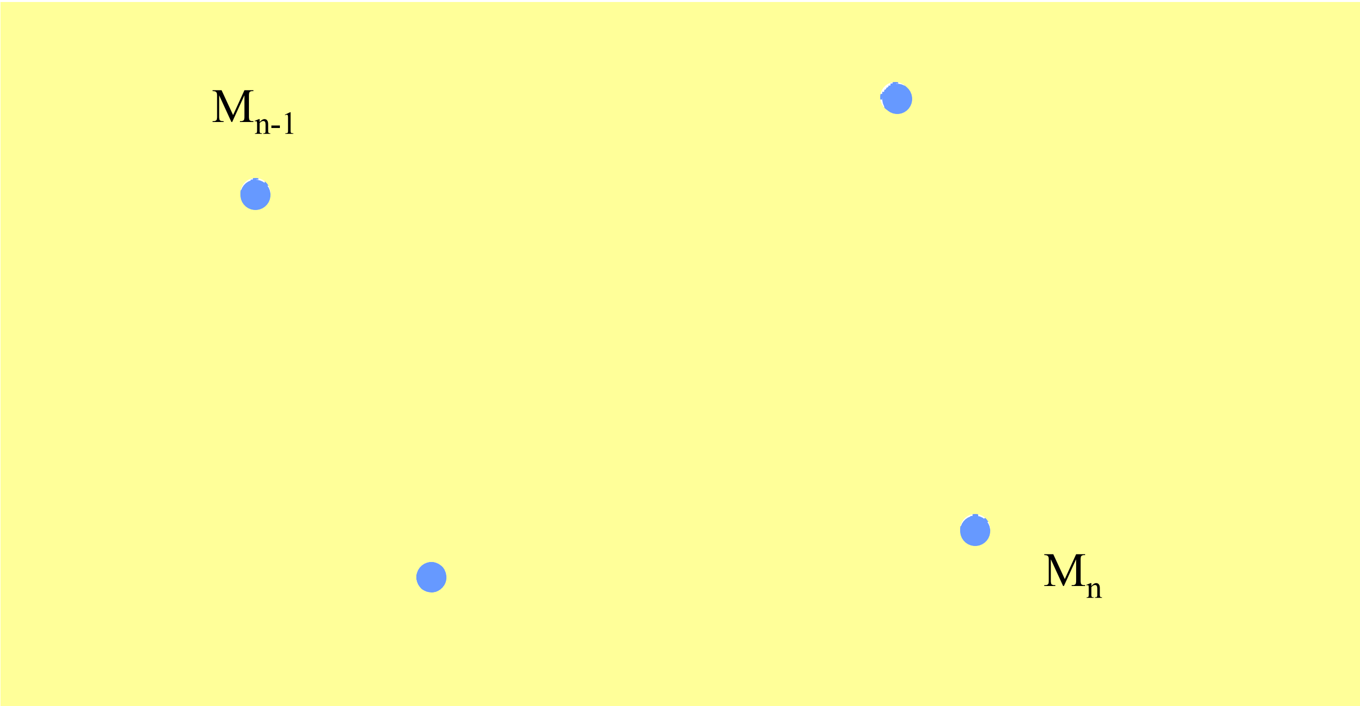
Contents

1. The Random Waypoint and Random Trip Models

The Random Waypoint Model

In its simplest form:

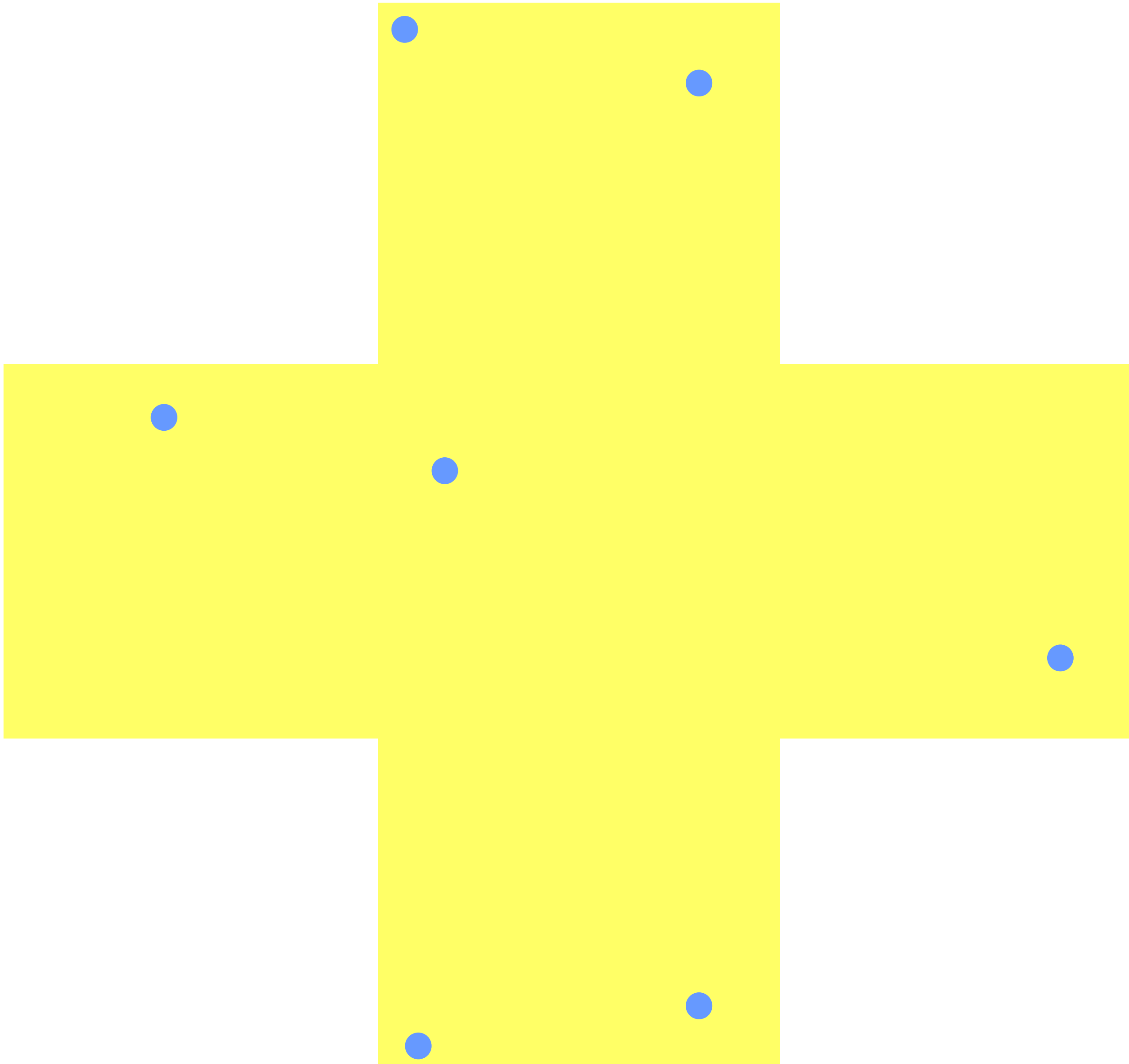
- Mobile picks next waypoint M_n uniformly in area, independent of past and present
- Mobile picks next speed V_n uniformly in $[v_{min}; v_{max}]$
- independent of past and present
- Mobile moves towards M_n at constant speed V_n



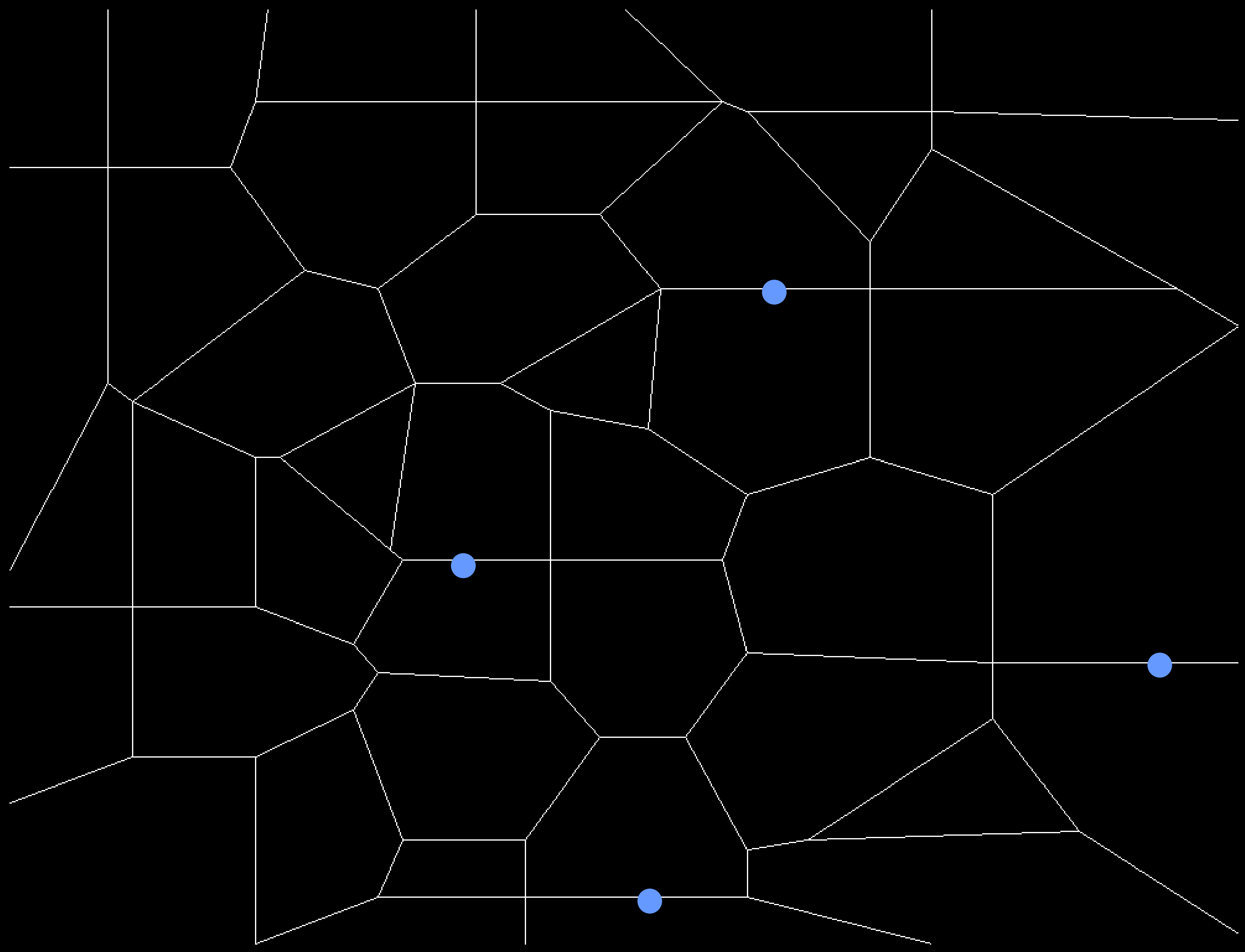
The Random Trip model

- Random Waypoint is a special case of Random Trip [L-Vojnovic-Infocom05]:
  - mobile picks a path in a set of paths and a speed
  - at end of path, mobile picks a new path and speed
  - evolution is a Markov process

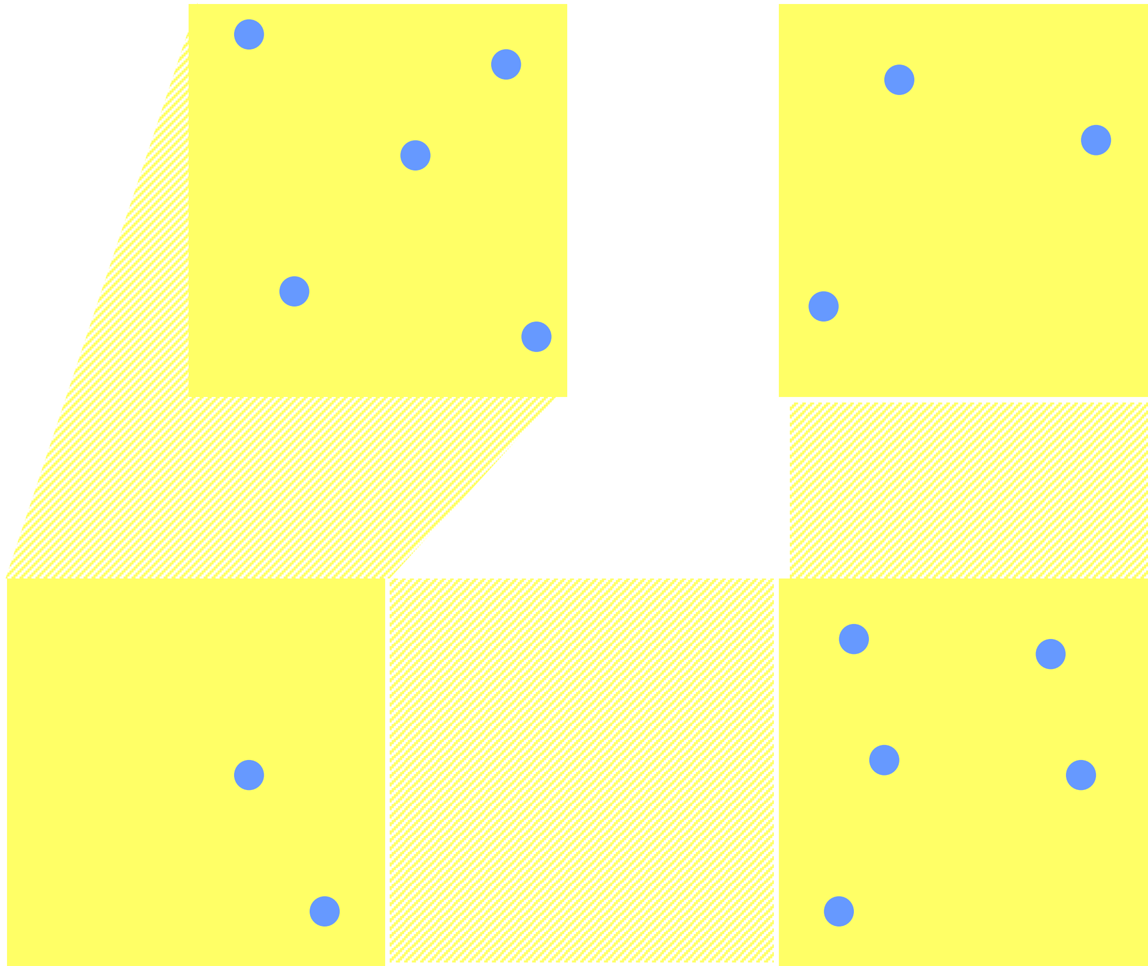
- Examples of random trip models in the next slides



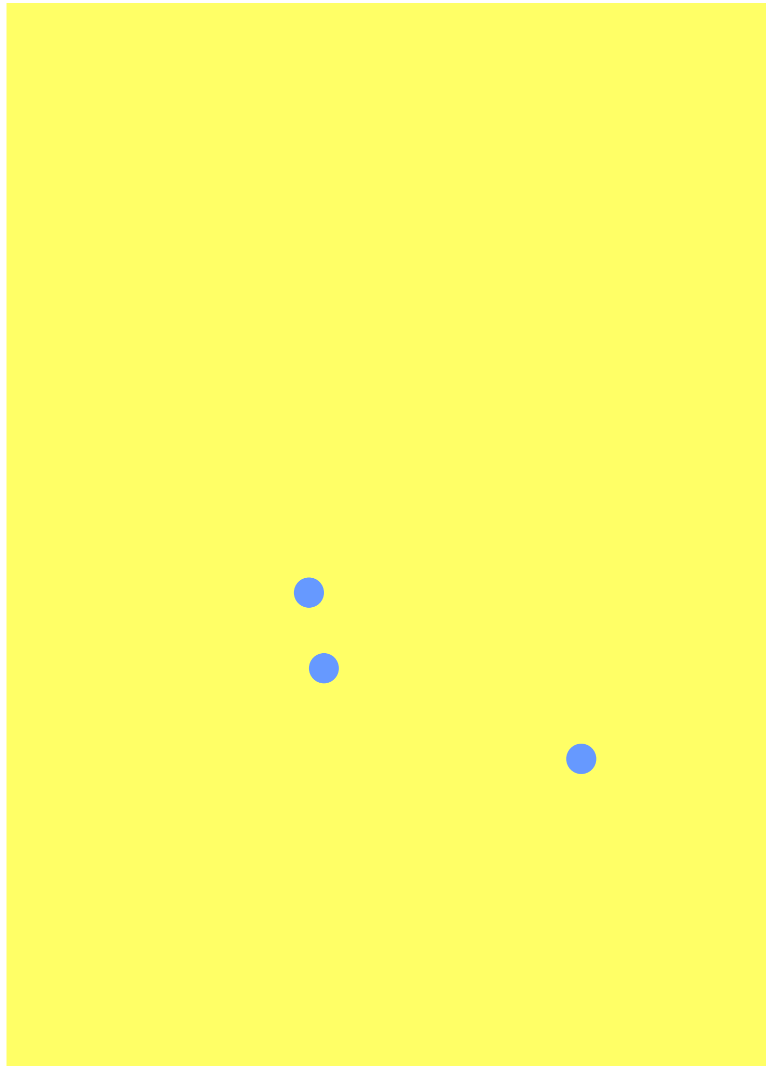
RWP with pauses on general connected domain



City-Section



Restricted RWP (Blažević et al, 2004)



Random Walk with Reflection

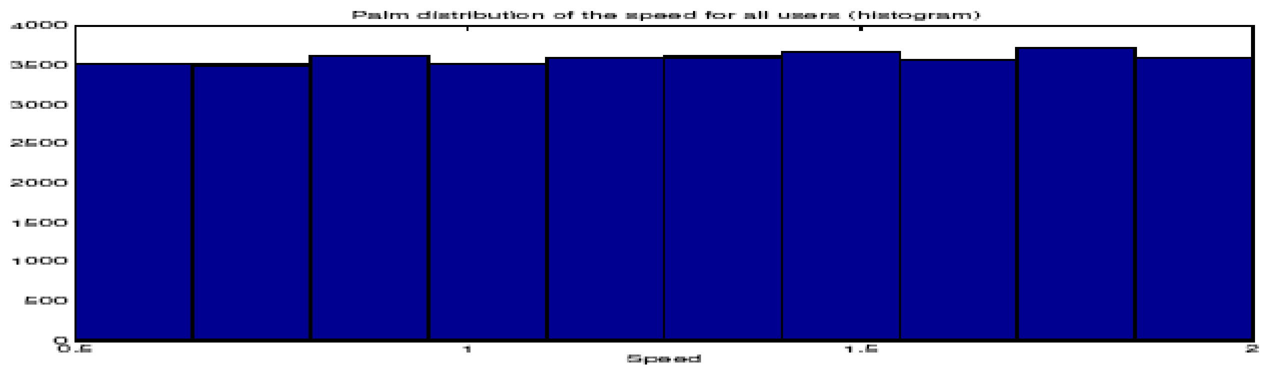
Issues Observed with These Mobility Models

- Researchers in mobile networking have used these models and observed some annoying phenomenons.

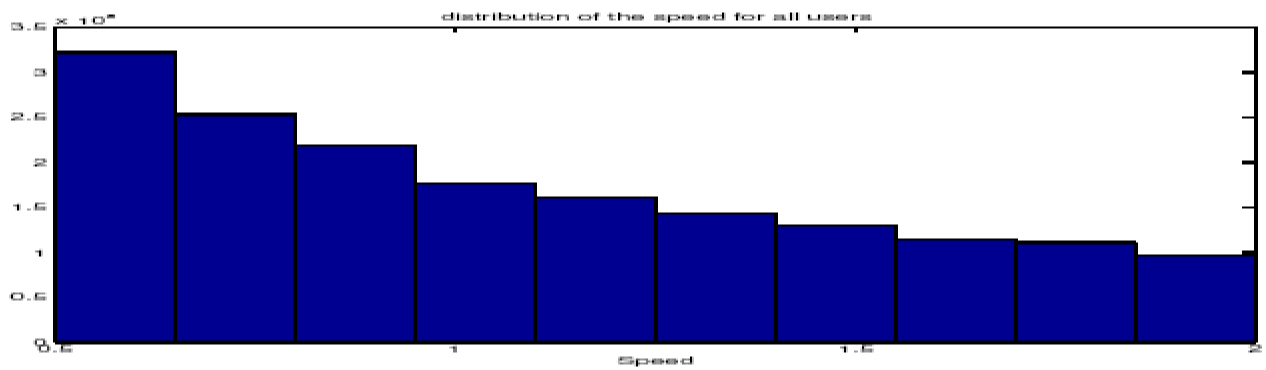
Example: Average Speed

Random Waypoint on Rectangle, without Pause:

- Speed observed at waypoints (Event average)

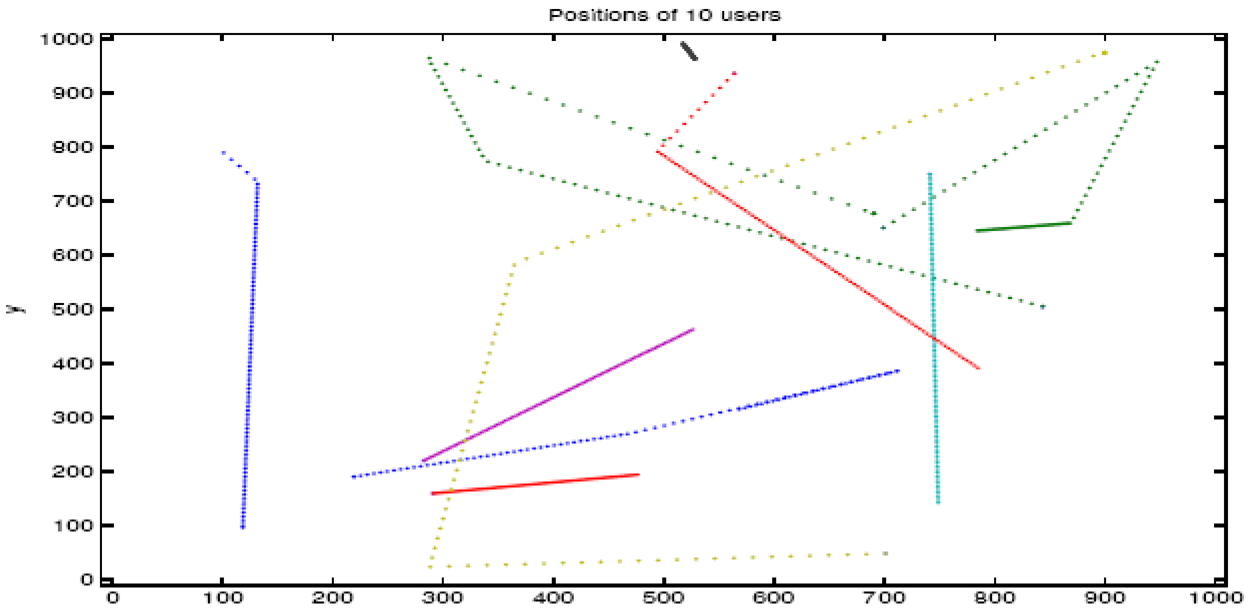


- Speed observed at an arbitrary time (Time average)



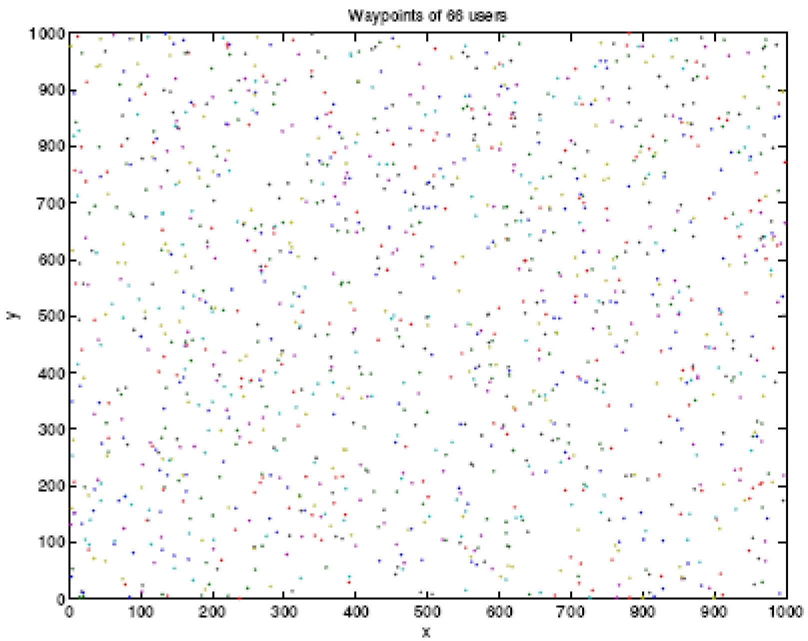
Intuitive Explanation: Difference in Sampling

- Low speed trips are more likely to be observed

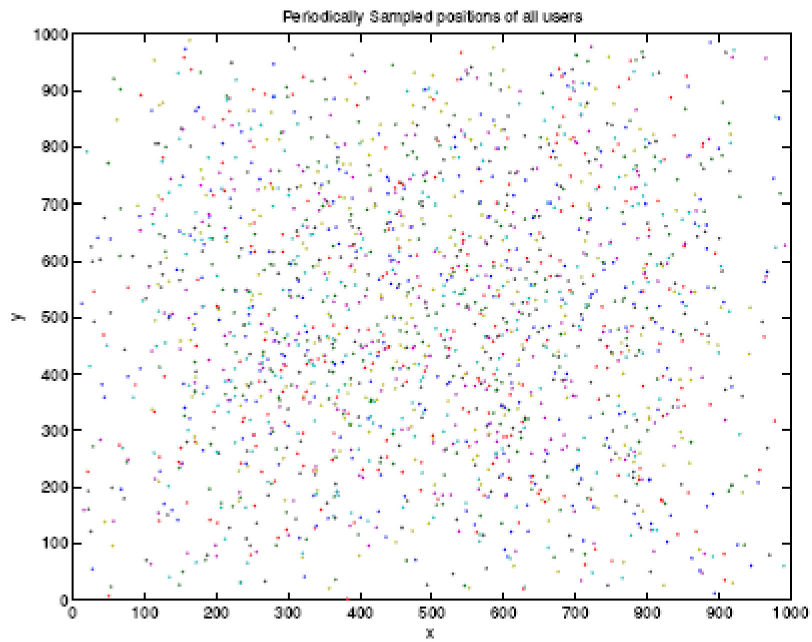


Ditribution of Node Location

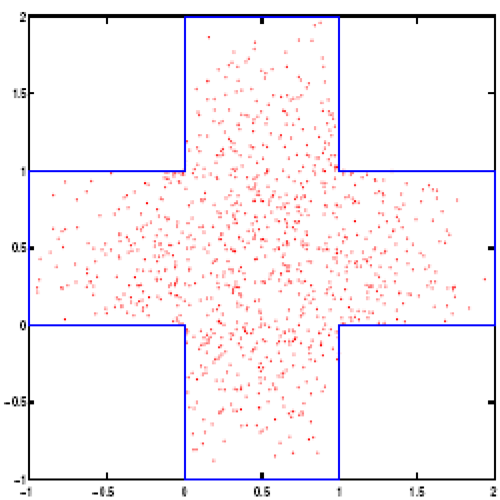
Position observed at waypoints (Event average)

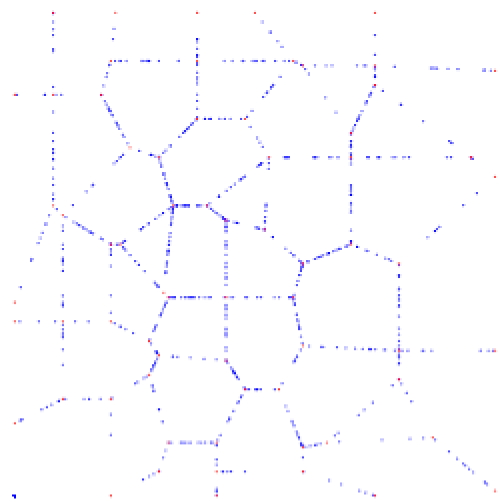


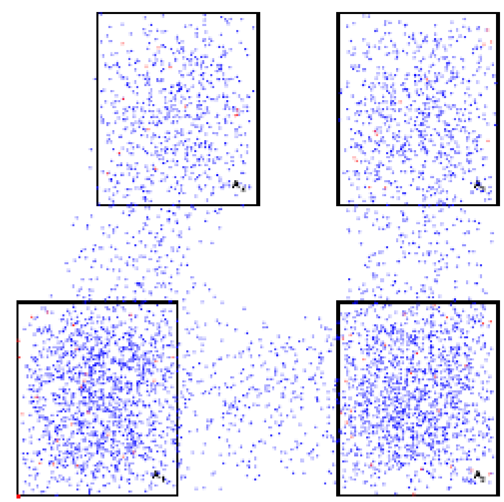
Position observed at an arbitrary time (Time average)



Node Location at Arbitrary Instant

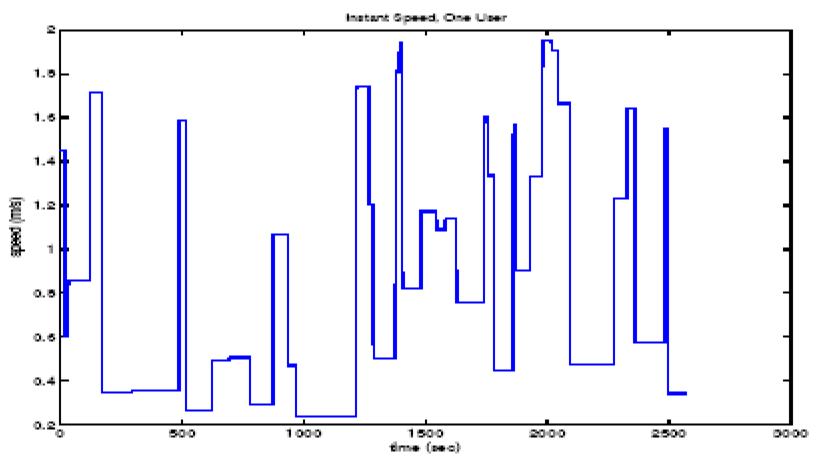




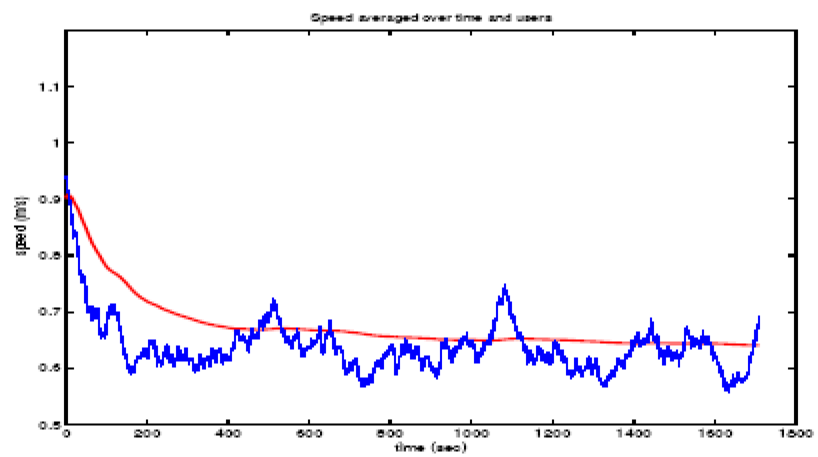


Issue: Decay in Average Speed

- “suffers from decay" “is considered harmful" [Yoon03]



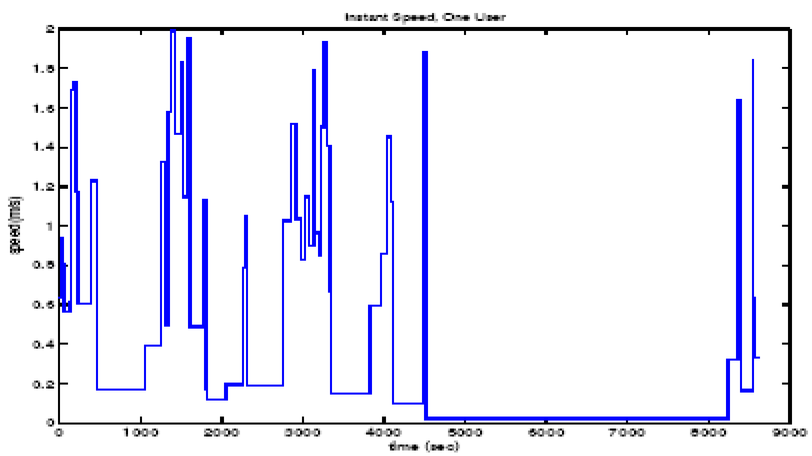
One User



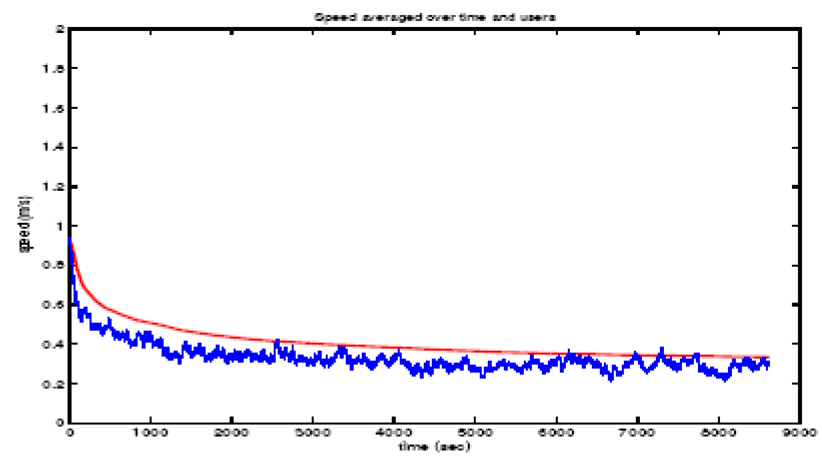
Instant Speed + Empirical speed, both averaged over users

Decay with $v_{min} = 0$

- when $v_{\min} = 0$, sample average speed decays to 0
  model freezes



One User



Instant Speed + Empirical speed, both averaged over users

2. Palm Calculus Instant Primer

Palm Calculus

- All of this has to do with time averages versus event averages
  - An old topic in queueing theory
  - Now well understood by mathematicians under the name Palm Calculus

Palm Calculus Framework

- A stationary simulation with state S_t.
- Some quantity X_t measured at time t. Assume that

  $(S_t;X_t)$ is jointly stationary

  I.e., S_t is in a stationary regime and X_t depends on the past, present and future state of the simulation in a way that is invariant by shift of time origin.
- Examples
  - S_t = current position of mobile, speed, and next waypoint
  - *Jointly stationary with* St: X_t = current speed at time t; X_t = time to be run until next waypoint
  - *Not jointly stationary with* S_t: X_t = time at which last waypoint occurred

Palm Expectation

- Consider some selected transitions of the simulation,occurring at times T_n.
  - Example: T_n = time at which n^{th} waypoint reached
- ***Definition*** : the Palm Expectation is

$$\mathbb{E}^t(X_t) = \mathbb{E}(X_t \mid \text{a selected transition occurred at time } t)$$

- By stationarity:

$$\mathbb{E}^t(X_t) = \mathbb{E}^0(X_0)$$

- Example:
  - Let T_n = time at which n^{th} waypoint reached, X_t = current speed at time t
  - $\mathbb{E}^t(X_t) = \mathbb{E}^0(X_0)$ = average speed observed at a waypoint

Event versus Time Averages

- $\mathrm{H}(X_t) = \mathrm{H}(X_0)$ expresses the time average viewpoint.
- $\mathrm{H}^t(X_t) = \mathrm{H}^0(X_0)$ expresses the event average viewpoint.

- Example:
  - Let T_n = time at which n^{th} waypoint reached, X_t = current speed at time t
  - $\mathrm{H}^t(X_t) = \mathrm{H}^0(X_0)$ = average speed observed at a waypoint
  - $\mathrm{H}(X_t)=\mathrm{H}(X_0)$ = average speed observed at an arbitrary point in time

Formal Definition

- In discrete time, we have an elementary conditional probability
  - $H^t(X_t) = H(X_t\ 1_{< n\ 5\ ]\ \text{such that } Tn=t})\ /\ S(< n\ 5\ ]\ \text{such that } T_n=t)$
- In continuous time, the definition is a little more sophisticated
  - Similar to the definition of conditional density $f_X(x|Y=y)$ for continuous random variables with joint density – see the writeup **[LeBoudec04]** for details
  - See **[BaccelliBremaud87]** for a formal treatment
- Palm probability is defined similarly
  - $S^t(\ X_t\ 5\ W) = H^t(1_{Xt\ 5\ W})$

Ergodic Interpretation

- **Assume simulation is stationary + ergodic, i.e. sample path averages converge to expectations; then we can estimate time and event averages by:**

$$\mathbb{E}(X_0) = \lim_{T \to +\infty} \frac{1}{T} \sum_{s=1}^{T} X_s$$

$$\mathbb{E}^0(X_0) = \lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} X_{T_n}$$

- **In terms of probabilities:**
  - Stationary probability: $\mathbb{P}(X_t \in W) \approx$ fraction of time that X_t is in some set W
  - Palm probability: $\mathbb{P}^t(X_t \in W) \approx$ fraction of selected transitions at which X_t is in W

Two Palm Calculus Formulas

- **Intensity of selected transitions: λ := expected number of transitions per time unit**
- **Intensity Formula:**

$$\frac{1}{\lambda} = \mathbb{E}^0(T_1 - T_0) = \mathbb{E}^0(T_1)$$

where by convention T_0 ‰ $0 < T_1$

- **Inversion Formula**

$$\mathbb{E}(X_t) = \mathbb{E}(X_0) = \lambda \mathbb{E}^0 \left(\int_0^{T_1} X_s ds \right)$$

- **The proofs are simple in discrete time – see [LeBoudec04]**

A Simple Example

- At bus stop in average λ buses per hour. Inspector measures time between all bus inter-departures. Inspector estimates $\mathbb{E}^0(T_1 - T_0) = \frac{1}{\lambda}$
- Joe arrives at time t and measures $X_t =$ (time until next bus $-$ time since last bus). Joe estimates $\mathbb{E}(X_0) = \mathbb{E}(T_1 - T_0)$
- Inversion formula:

$$\mathbb{E}(T_1 - T_0) = \lambda \mathbb{E}^0\left(\int_0^{T_1} X_t dt\right) = \lambda \mathbb{E}^0(T_1^2) = \frac{1}{\lambda} + \lambda \mathrm{var}^0(T_1 - T_0)$$

- Joe's estimate always larger than Inspector's (Feller's Paradox)

Other Palm Formulas

- **Little formula N = λ R**
- **PASTA**
- **Neveu^s exchange formula**
- **…**

See [BaccelliBremaud87,LeBoudec05] for more details

3. Application to Time Averages

Application to Average Speed

- **Assume a stationary regime exists and simulation is run long enough**
- **Apply inversion formula and obtain distribution of instantaneous speed V(t)**

$$\mathbb{E}\left(\phi(V(t))\right) = \lambda \mathbb{E}^0 \left(\int_0^{T_1} \phi(V(t))\, dt \right)$$

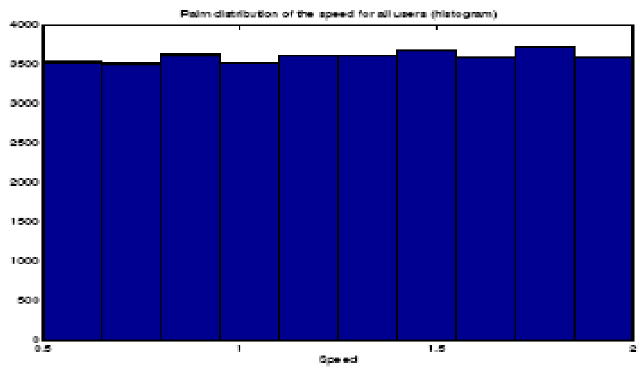
$$\begin{aligned}
&= \lambda \mathbb{E}^0 \left(\phi(V_0) T_1 \right) \\
&= \lambda \mathbb{E}^0 \left(\phi(V_0) \frac{\|M_1 - M_0\|}{V_0} \right) \\
&= \lambda \mathbb{E}^0 \left(\|M_1 - M_0\| \right) \mathbb{E}^0 \left(\frac{\phi(V_0)}{V_0} \right) \\
&= C \int_0^{v_{\max}} \frac{\phi(v)}{v} f^0_{V_0}(v)\, dv
\end{aligned}$$

Inversion Formula Gives Relation between Speed Distributions at Waypoint and at Arbitrary Point in Time

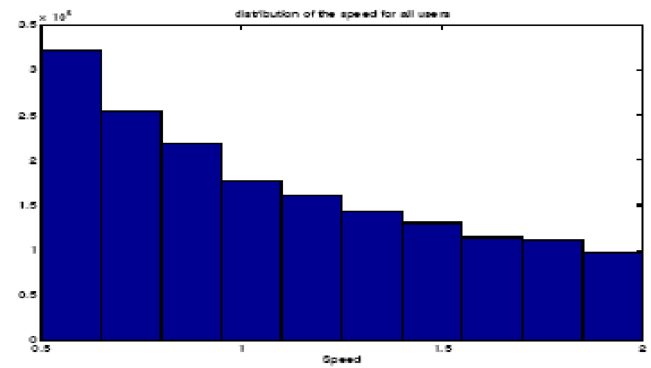
$$f_{V(t)}(v)dv = \frac{C}{v} f^0_{V_0}(v)dv$$

with: $f_{V(t)}(v)$ = stationary density of speed, $f^0_{V_0}(v)$ = Palm density of speed (i.e. uniform on $[v_{\min}, v_{\max}]$) and $C^{-1} = \mathbb{E}^0(\frac{1}{V_0})$

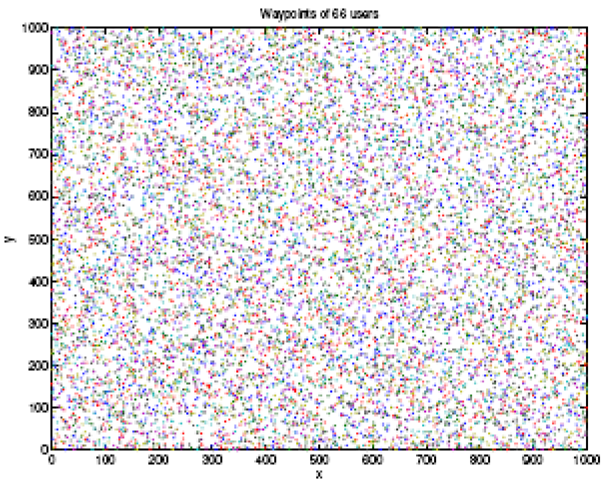
Event Average

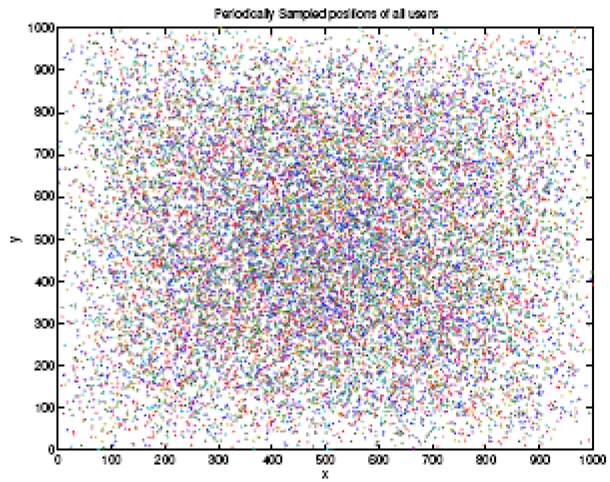


Time Average



Application to Distribution of Location





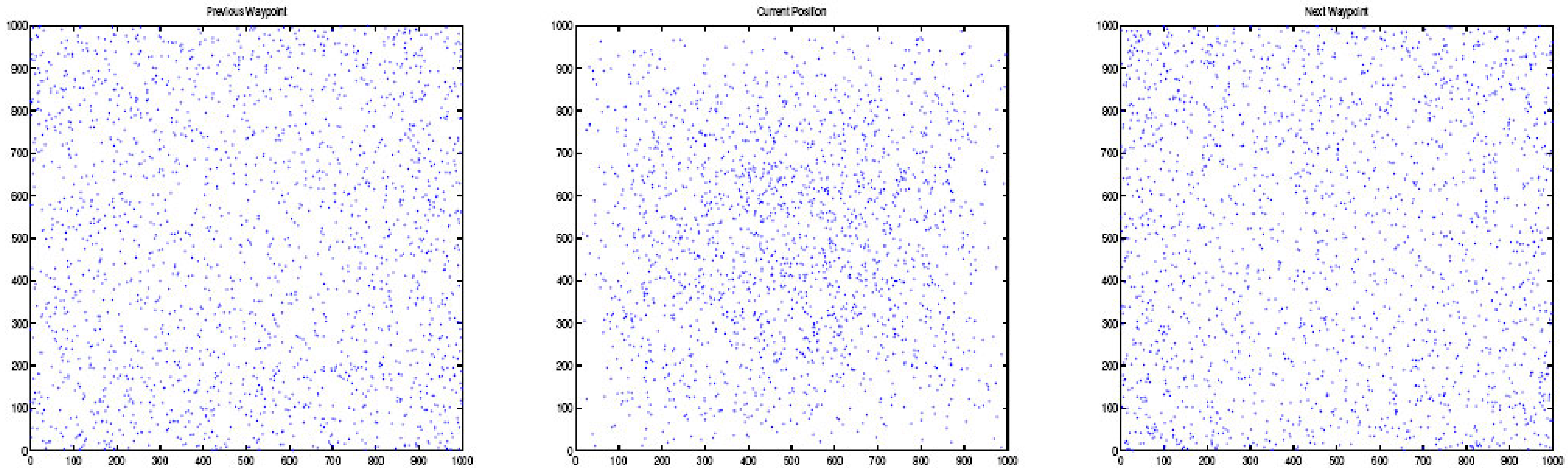
- **Conventional approaches finds that closed form expression for density is too difficult [Bettstetter04]**
- **Approximation of density in area [0; a] [0; a] [Bettstetter04]:**

$$f_{X,Y}(x,y) \approx \frac{36}{a^2}x(x-a)y(y-a)$$

Previous and Next Waypoints

- Let $M(t)$: position at time t
- Let $Prev(t), Next(t)$: previous and next waypoints

$Prev(t)$ $\qquad$ $M(t)$ $\qquad$ $Next(t)$



Q Is $Prev(t)$ [resp. $M(t)$, $Next(t)$] uniformly distributed ?

A No. But $Prev(t)$ and $Next(t)$ have same (non uniform) distribution.

Stationary Distribution of Location Is also Obtained By Inversion Formula

- Joint distribution of $(Prev(t), M(t), Next(t))$ has a simple closed form [NavidiCamp04]:

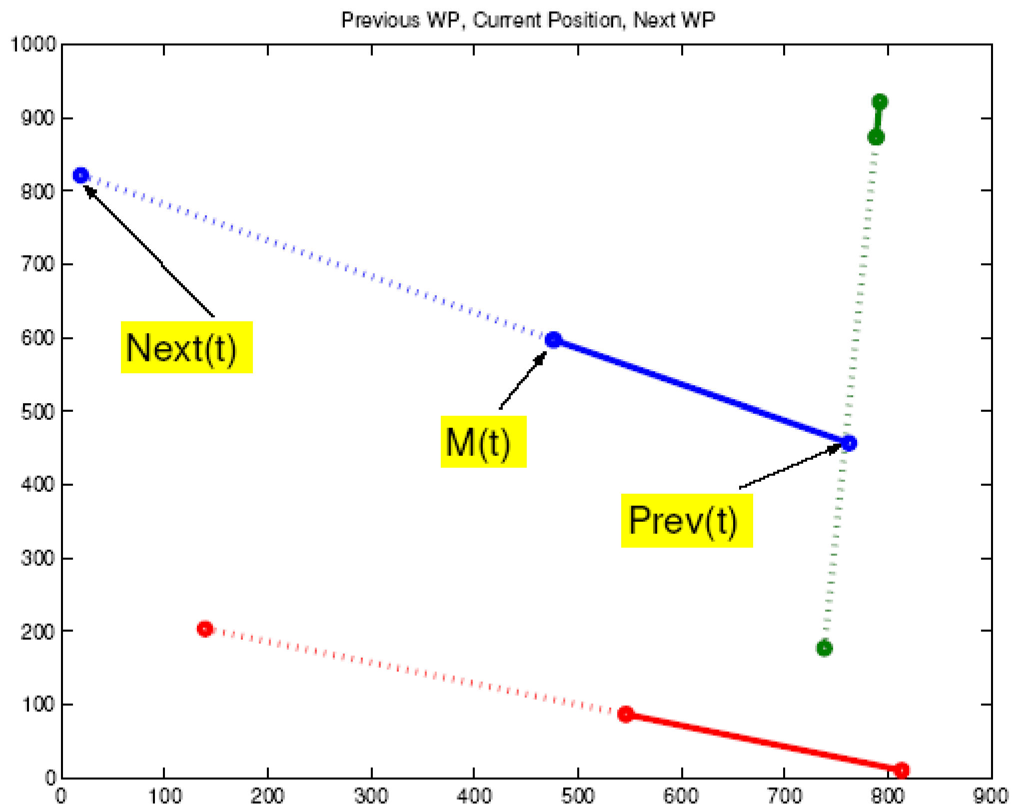
1. $((Prev(t), Next(t))$ has density over area A

$$f_{Prev(t),Next(t)}(P, N) = K \|P - N\|$$

2. Distribution of $M(t)$ given $Prev(t) = P, Next(t) = N$ is uniform on segment $[P, N]$

$K^{-1} = \text{vol}(A)^2 \bar{\Delta}(A)$, with $\bar{\Delta}(A) =$ average distance between two points in A. For $A = [0; a] \times [0; a]$, $\bar{\Delta}(A) = 0.5214a$ [Gosh1951].

Stationary Distribution of Location



- Valid for any convex area

Proof

- Apply Inversion Formula

For any bounded, non negative function ϕ:

$$\mathbb{E}(\phi(Prev(t), M(t), Next(t))) = \lambda \mathbb{E}^0 \left(\int_0^{T_1} \phi(M_0, M_0 + \frac{t}{T_1}(M_1 - M_0), M_1) dt \right)$$

By a simple change of variable in the integral, we obtain

$$\lambda \mathbb{E}^0 \left(T_1 \int_0^1 \phi(M_0, M_0 + u(M_1 - M_0), M_1) du \right)$$

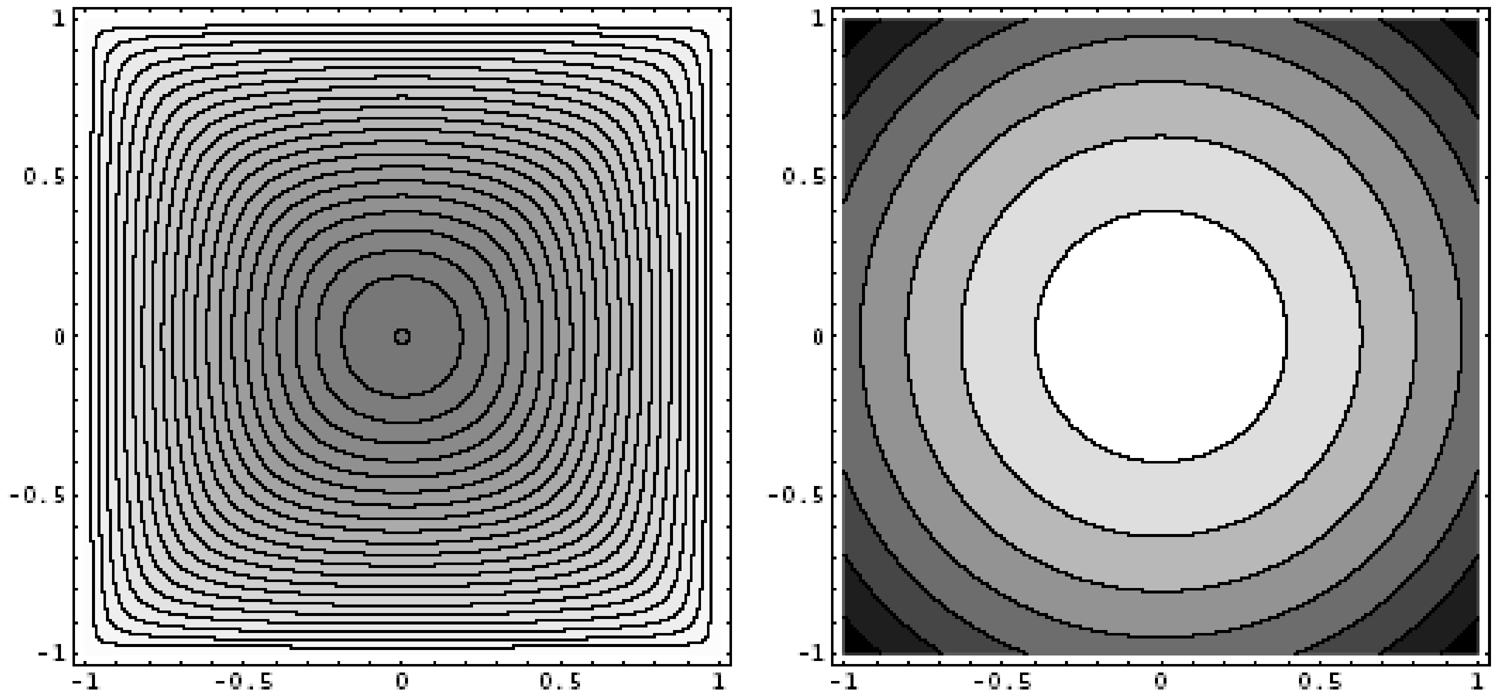
Now given that there is an arrival at time 0, $T_1 = \frac{\|M_1 - M_0\|}{V_0}$ and the speed V_0 is independent of the waypoints M_0 and M_1 thus

$$\begin{aligned} &= \lambda \mathbb{E}^0 (\frac{1}{V_0}) \mathbb{E}^0 \left(\|M_1 - M_0\| \int_0^1 \phi(M_0, M_0 + u(M_1 - M_0), M_1) du \right) \\ &= K \int_A \int_A \int_0^1 \phi(M_0, (1-u)M_0 + uM_1, M_1) \, \|M_1 - M_0\| \, du \, dM_0 \, dM_1 \end{aligned}$$

which shows the statement.

Stationary Distribution of Location

- Marginal densities $f_{M(t)}(m)$ and $f_{Next(t)}(n)$ can be computed in closed form



(a) Node Location

(b) Next Waypoint

Closed Forms

$$\begin{cases} f_{M(t)}(x,y) = f_{M(t)}(|x|,|y|) \\ \text{if } |x| < |y| \text{ then } f_{M(t)}(x,y) = f_{M(t)}(|y|,|x|) \\ \text{if } 0 \le y \le x \text{ then } f_{M(t)}(x,y) = \frac{15}{32\left(\sqrt{2}+2+5\ln(1+\sqrt{2})\right)} F(x,y) \end{cases}$$

with $F(x,y) =$

$$\begin{array}{lrlcrl}
 & (1-x)(2+x)(1-y) & \sqrt{1+\frac{(1-y)^2}{(1+x)^2}} & + & (1-x)(1-y)(2+y) & \sqrt{1+\frac{(1-x)^2}{(1+y)^2}} \\
+ & (1-x)(2+x)(1+y) & \sqrt{1+\frac{(1+y)^2}{(1+x)^2}} & + & (1-x)(1+y)(2-y) & \sqrt{1+\frac{(1-x)^2}{(1-y)^2}} \\
- & \frac{(1-x)^2(1-y)^2}{1+x} & \sqrt{1+\frac{(1+x)^2}{(1-y)^2}} & - & \frac{(1-x)^2(1-y)^2}{1+y} & \sqrt{1+\frac{(1+y)^2}{(1-x)^2}} \\
- & \frac{(1-x)^2(1+y)^2}{1+x} & \sqrt{1+\frac{(1+x)^2}{(1+y)^2}} & - & \frac{(1-x)^2(1+y)^2}{1-y} & \sqrt{1+\frac{(1-y)^2}{(1-x)^2}} \\
+ & (1-x)\left[1+x-(1-y)^2\right] & \sinh^{-1}\left(\frac{1-y}{1+x}\right) & + & (1-y)\left[1+y-(1-x)^2\right] & \sinh^{-1}\left(\frac{1-x}{1+y}\right) \\
+ & (1-x)\left[1+x-(1+y)^2\right] & \sinh^{-1}\left(\frac{1+y}{1+x}\right) & + & (1+y)\left[1-y-(1-x)^2\right] & \sinh^{-1}\left(\frac{1-x}{1-y}\right) \\
+ & (1-x)^2(1-y) & \sinh^{-1}\left(\frac{1+x}{1-y}\right) & + & (1-x)(1-y)^2 & \sinh^{-1}\left(\frac{1+y}{1-x}\right) \\
+ & (1-x)^2(1+y) & \sinh^{-1}\left(\frac{1+x}{1+y}\right) & + & (1-x)(1+y)^2 & \sinh^{-1}\left(\frac{1-y}{1-x}\right)
\end{array} \tag{14}$$

where $\sinh^{-1}(t) = \ln\left(t+\sqrt{1+t^2}\right)$ *(inverse hyperbolic sine).*

$$f_{Next(t)}(x,y) = \frac{5}{32\left(\sqrt{2}+2+5\ln(1+\sqrt{2})\right)}\left(I(x,y)+I(y,x)+I(-x,-y)+I(-y,x)\right) \quad (17)$$

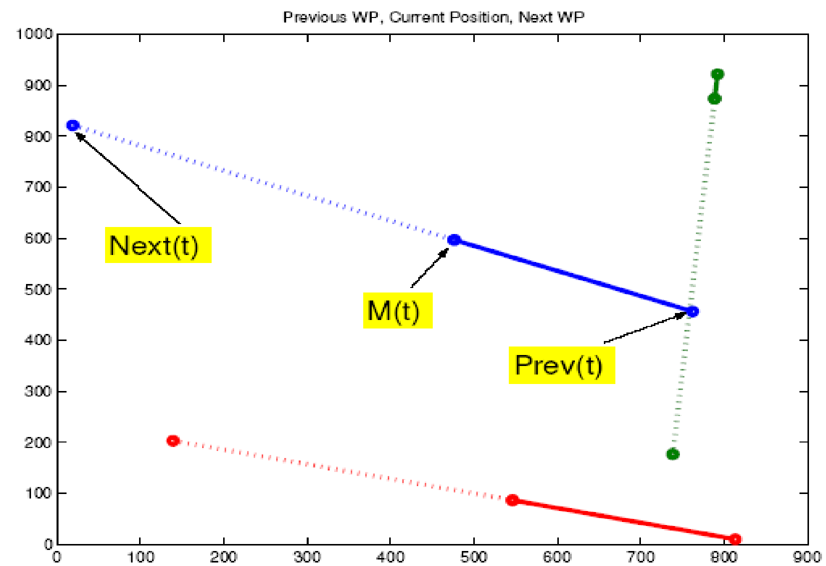
with $I(x,y) =$

$$\begin{aligned}
& \tfrac{1}{2}(1-x)\Big[-2\log(1-x)(x-1)^2 + \log\left(-y+\sqrt{(x-2)x+(y-2)y+2}+1\right) \\
+ \quad & \log\left(y+\sqrt{(x-2)x+y(y+2)+2}+1\right) \\
+ \quad & (x-2)x\log\left(\left(-y+\sqrt{(x-2)x+(y-2)y+2}+1\right)\left(y+\sqrt{(x-2)x+y(y+2)+2}+1\right)\right) \\
- \quad & y\sqrt{(x-2)x+(y-2)y+2}+\sqrt{(x-2)x+(y-2)y+2} \\
+ \quad & y\sqrt{(x-2)x+y(y+2)+2}+\sqrt{(x-2)x+y(y+2)+2}\Big]
\end{aligned}$$

4. Application to Perfect Simulation

Sampling a Stationary Location

- Assume we want to sample a location from its stationary distribution
  - We can use the closed form of the density, but it is complex and works only for some simple domains
  - Better to use the joint density of (Prev(t), M(t), Next(t))



1. $((Prev(t), Next(t))$ has density over area A

$$f_{Prev(t),Next(t)}(P,N) = K\,\|P-N\|$$

2. Distribution of $M(t)$ given $Prev(t) = P, Next(t) = N$ is uniform on segment $[P,N]$

Rejection Sampling

- We need to sample (Prev(t), Next(t)) ; it has a known, but non classical density => rejection sampling
- Density depends on some geometric constant K => need not be computed, thanks to rejection sampling
- Sampling Algorithm

1. Draw (M_0, M_1) with joint density $K\|M_1 - M_0\|$ on $A \times A$: **do**
   draw M_0, M_1 iid $\sim$ Unif(A)
   draw $V \sim$ Unif$[0, \Delta]$
   until $V < \|M_1 - M_0\|$
2. Draw $U \sim$ Unif$[0, 1]$
3. $M(t) = (1 - U)M_0 + UM_1$

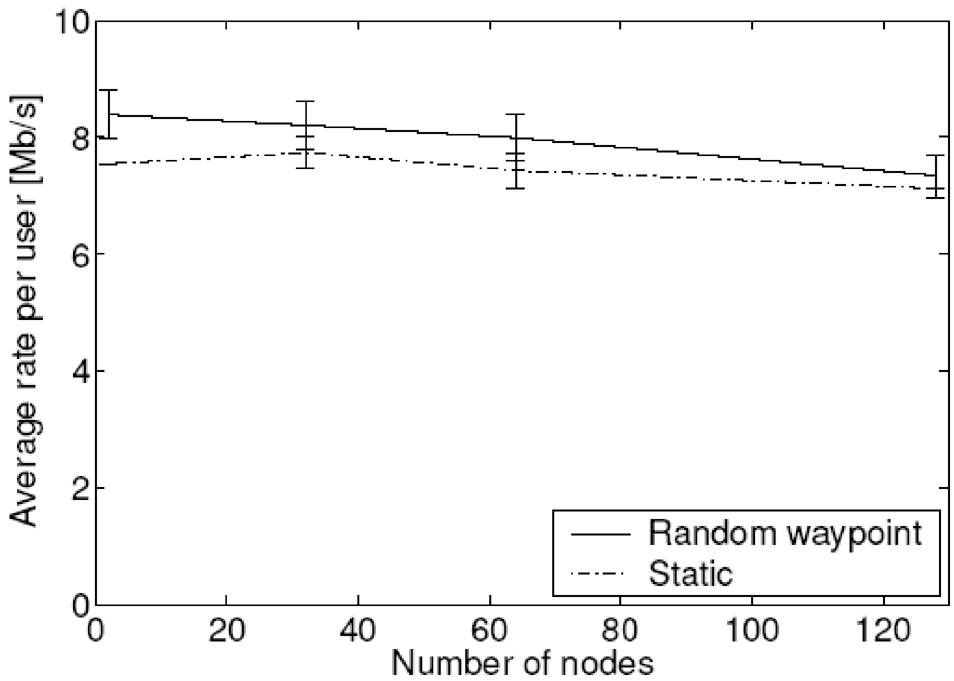
Δ: upper bound on diameter $\bar{\Delta}$ of area A

Average number of iterations to draw M_0, M_1 is $\Delta/\bar{\Delta}$ ($= 2.71$, square area, $= 2.21$, disk)

Why Does It Matter ?

From anonymous source:

- compare a MAC protocol with/without mobility
- naive user uses uniform distribution of points for the immobile case – should have used stationary distribution instead



Perfect Simulation

- We can sample complete simulation state from stationary distribution
  - This is called *perfect simulation*
  - No transient !

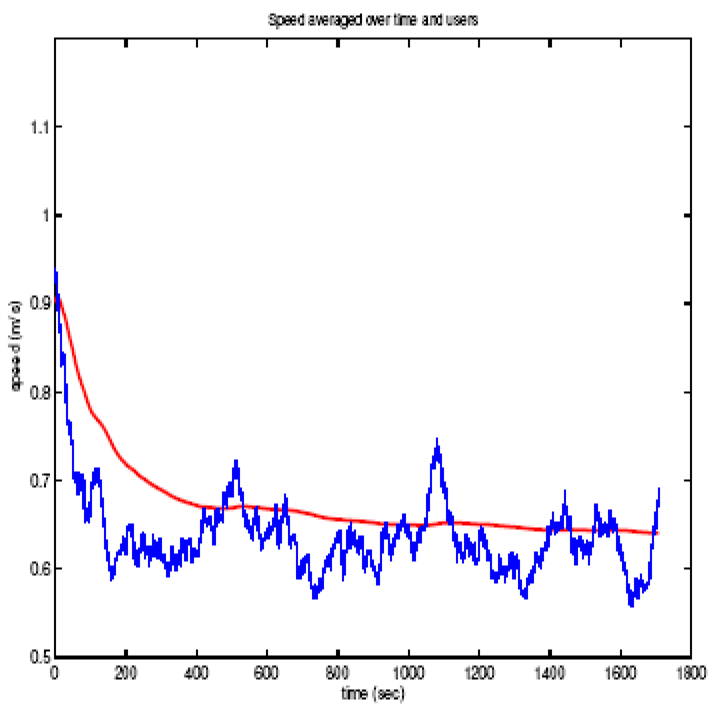
Perfect Simulation of Random Waypoint

- State $S_t =$ (position of mobile at time t, speed, next waypoint) = $(M(t), V(t), Next(t))$
- $V(t)$ is independent of $(M(t), Next(t))$
- Perfect Simulation of RWP

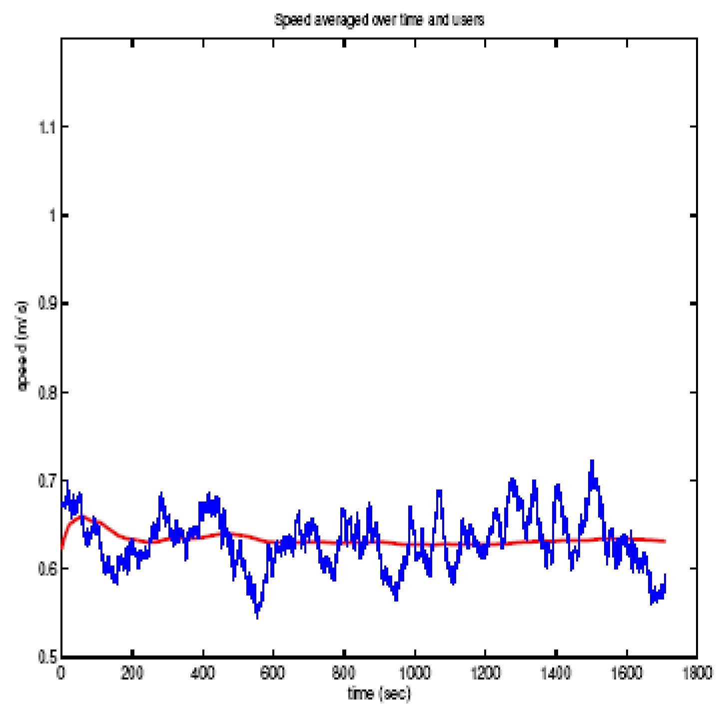
1. Sample (p, m, n) from stationary distrib of $(Prev(t), M(t), Next(t))$ (see above)
2. Sample v from stationary distrib of $V(t)$ (see above)
3. Start simulation with initial position $= m$, speed $= v$ and next waypoint $= n$

No Speed Decay

Standard Simulation



Perfect Simulation



Perfect Simulation of Random Trip

- All we saw about random waypoint also applies the same to random trip
  - Rejection sampling always applies

- In some cases, the stationary distribution of location is the same as the distribution at a trip endpoint (i.e. is uniform)
  - Random waypoint on torus or sphere
  - Random walk with reflection or wrapping

See [L-Vojnovic-Infocom05]

5. Stationarity Issues

Existence of Stationary Regime

Consider the simple Random Waypoint with (Palm) distributions of speed $f_V^0(v) = K_0 1_{\{v_{\min} \leq v \leq v_{\max}\}}$

- Model is defined by
  Sequence of waypoints $M_0, M_1, M_2, ...M_n$ and speeds $V_0, V_1, ..., V_n, ...$
- M_n, V_n are stationary with respect to sequence index n
  ... but
- Now consider simulation state $S_t =$ (position of mobile at time t, speed, next waypoint)
  Does S_t have a stationary regime ?

Application of Intensity Formula

$$\frac{1}{\lambda} = \mathbb{E}^0(T_1) = \mathbb{E}^0(\frac{D_1}{V_0}) = \mathbb{E}^0(D_1)\mathbb{E}^0(\frac{1}{V_0})$$

$\mathbb{E}^0(D_1)$ is the average distance between 2 points chosen uniformly and independently

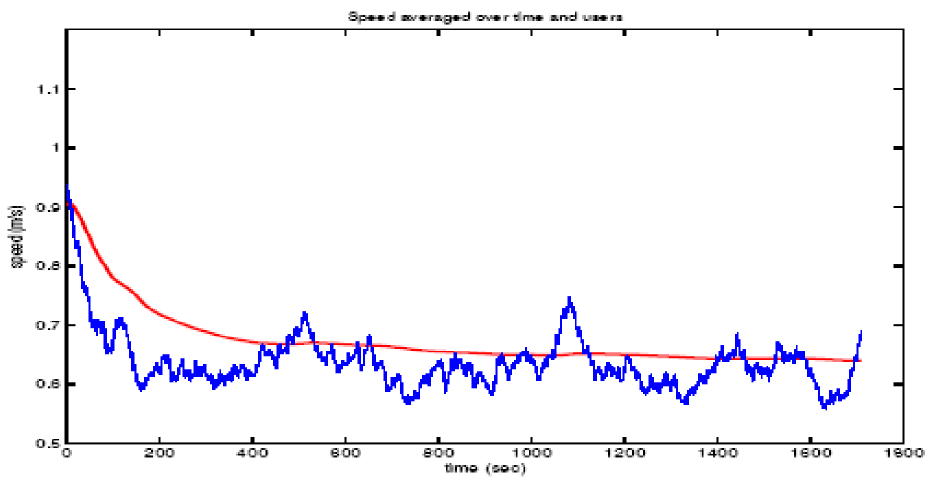
- Thus a necessary condition for stationarity is $v_{\min} > 0$
- It is also sufficient [L-Vojnovic-Infocom05]

  (Slivnyak's inverse construction [BaccelliBremaud87])

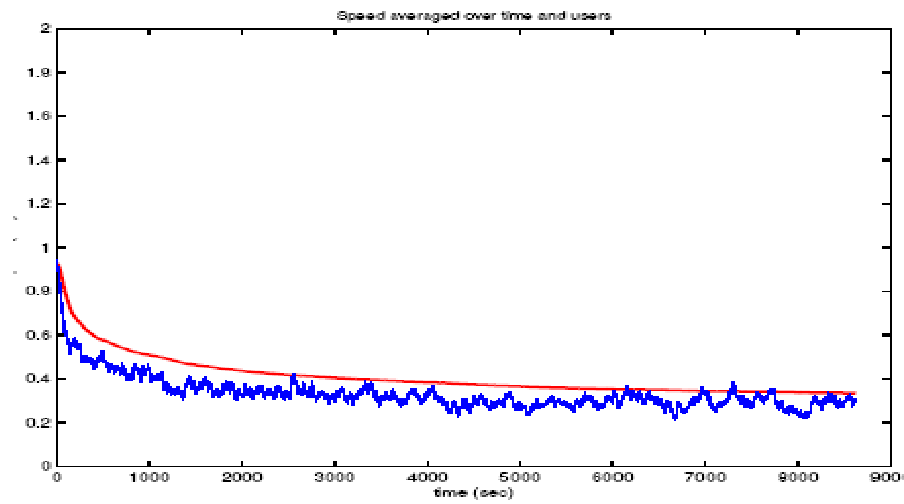
(intensity = waypoints per time unit)

Reported Issue of Decay

- Case $v_{min} > 0$
  - Decay is convergence to stationary regime



- Case $v_{min} = 0$
  - Decay is convergence to 0



What your Simulator *Really* Does in Practice

- State $S_t =$ (position of mobile at time t, speed, next waypoint)
- Finite State Space : (rescaled) state $\in \mathbb{N}^3$
- V_n is uniform in the discrete set $\{v_{\min} + \epsilon, v_{\min} + 2\epsilon, \ldots, v_{\max} - 2\epsilon, v_{\max} - \epsilon\}$
- A *realistic* model of what a simulator does

The simulation is always asymptotically stationary (finite state space)

+ ergodic (irreducible)

- **Even if vmin = 0**

- For $v_{\min} > 0$ the intensity is $\approx \frac{1}{\mathbb{E}^0(D_1)} \frac{v_{\max} - v_{\min}}{\ln v_{\max} - \ln v_{\min}}$
- For $v_{\min} = 0$ and $\epsilon \to 0$ intensity is $O\left(\frac{1}{-\ln \epsilon}\right)$

when $v_{\min} = 0$:

- Model is unstable
- but simulation is with discrete state space thus still converges to a stationary regime but
  - slow convergence;
  - stationary regime depends on accuracy term ϵ of simulator.

Condition for Random Trip Model

- **For the generic random trip model, the condition v_{min}>0 is replaced by**

  $$H^0(T_1 - T_0) < 4$$

  i.e. the mean trip duration, sampled from at trip endpoint, is finite

6. Examples with Long Range Dependence

Why Long Range Dependent Models ?

- Mobility models may exhibit some aspects of long range dependence
  - See Augustin Chaintreau, Pan Hui, Jon Crowcroft, Christophe Diot, Richard Gass, and James Scott. ***"Impact of Human Mobility on the Design of Opportunistic Forwarding Algorithms"***.
- The random trip model supports LRD

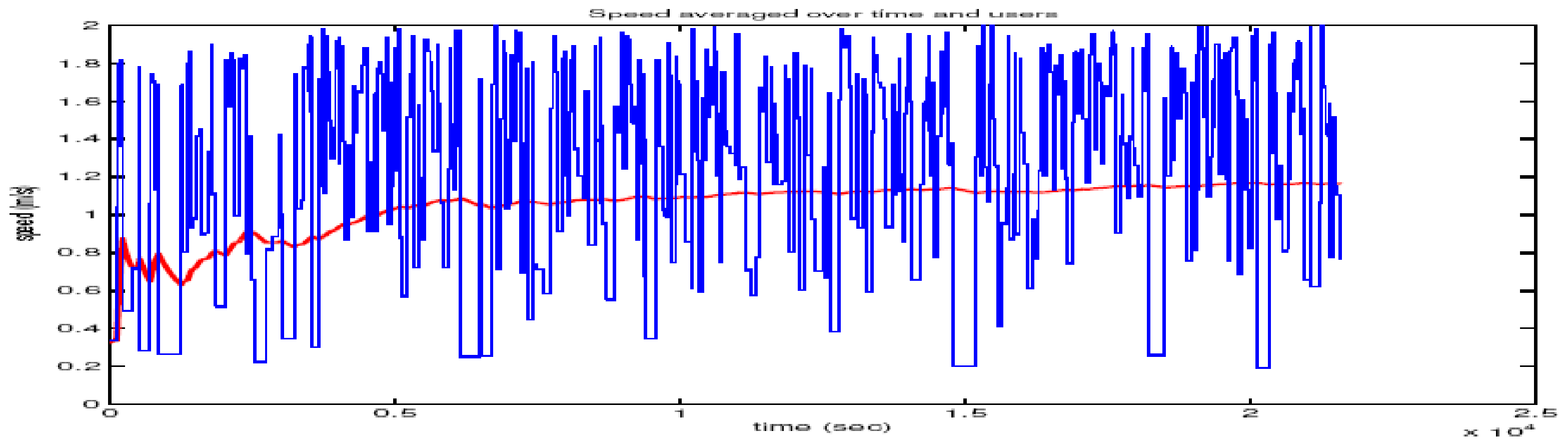
Long Range Dependent Random Waypoint

- **Consider the random waypoint without pause, like before, but change the distribution of speed:**
  - Non Stable model: $f_V^0(v) = K_0 1_{\{0 \le v \le v_{\max}\}}$
  - SRD model: $f_V^0(v) = K_2 v\sqrt{v} 1_{\{0 \le v \le v_{\max}\}}$
    $\mathbb{E}^0((T_1 - T_0)^2) < \infty$
    Model is stationary and short range dependent (SRD)
  - LRD model: $f_V^0(v) = K_1 \sqrt{v} 1_{\{0 \le v \le v_{\max}\}}$
    $\mathbb{E}^0(T_1 - T_0) < \infty$
    $\mathbb{E}^0((T_1 - T_0)^2) = \infty$
    Model is stationary but LRD with $\alpha = 0.5$ $(H = 0.75)$

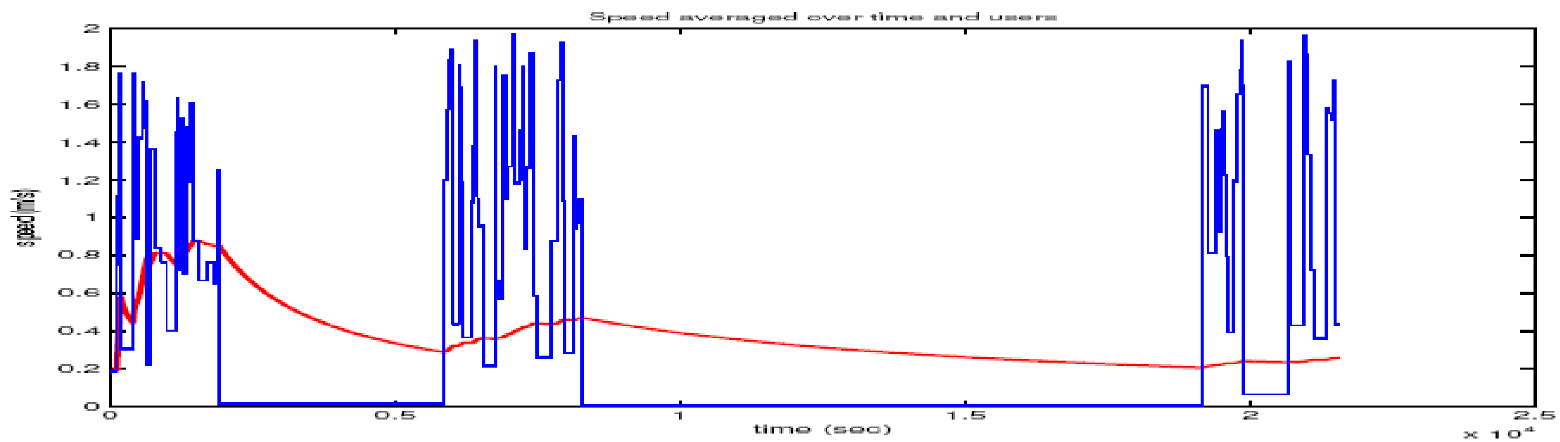
LRD means high variability

Instant Speed, One User, simulated time = 2.5 days

SRD



LRD



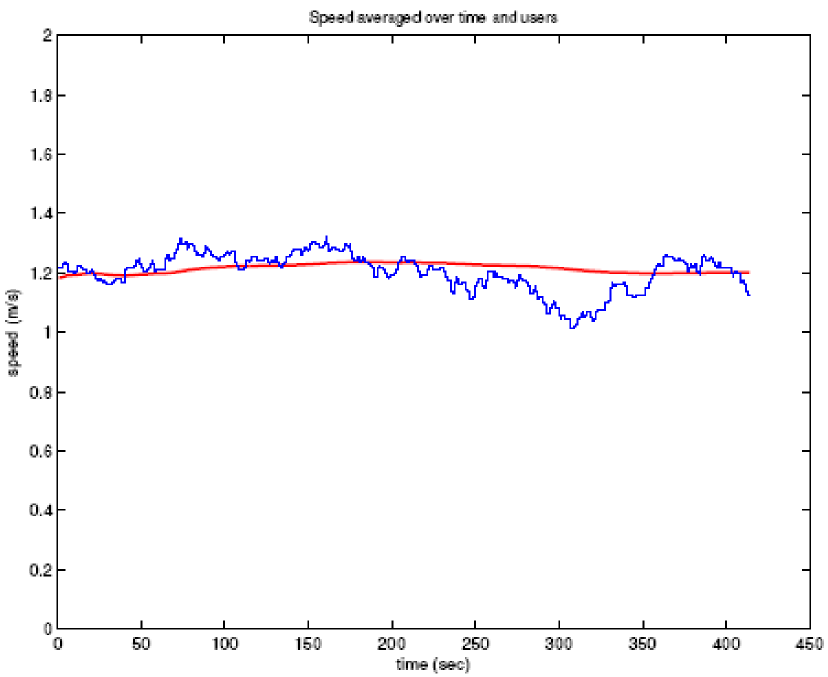
Practical Implications

- Perfect simulation is essential for LRD model (very slow convergence to averages)
- Confidence intervals should be obtained by independent replications, not by long simulation runs
  - With n runs, confidence interval for mean is order of $\frac{1}{\sqrt(n)}$

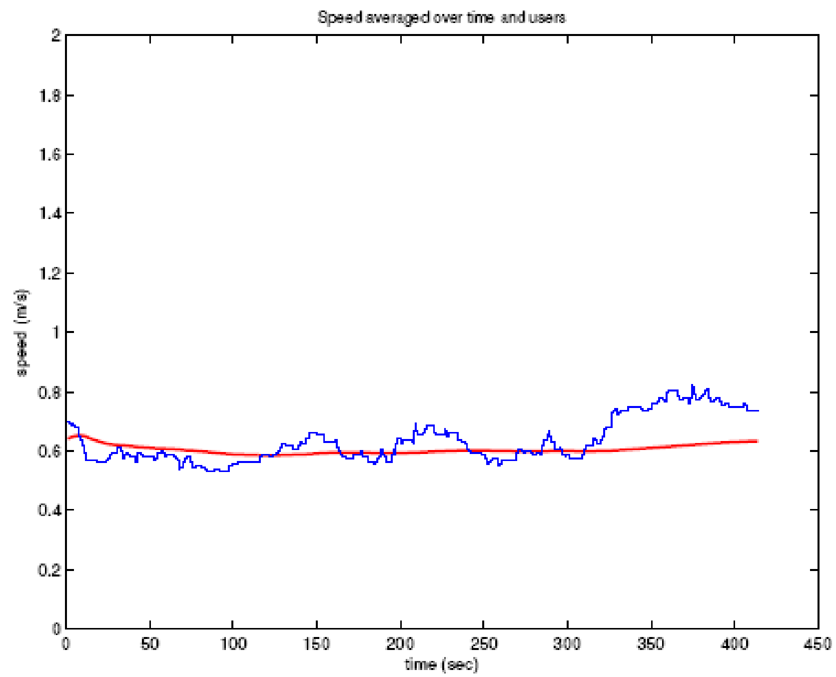
Average Over Independent Runs

Average of Instant Speed over 50 Independent Runs, Perfect Initialization, simulated time = 0.05 day

SRD



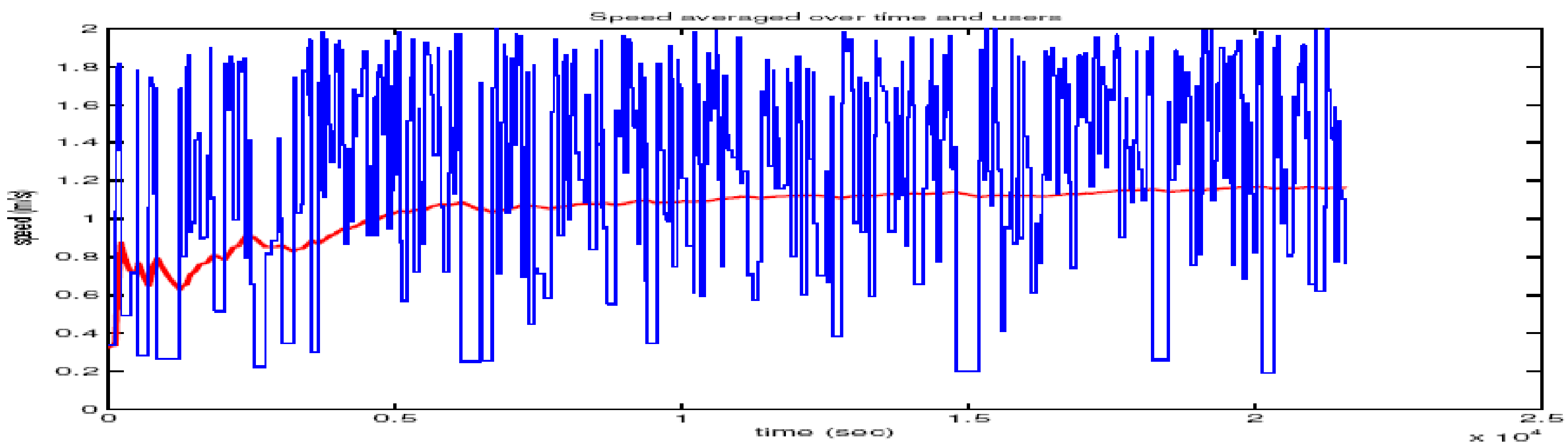
LRD



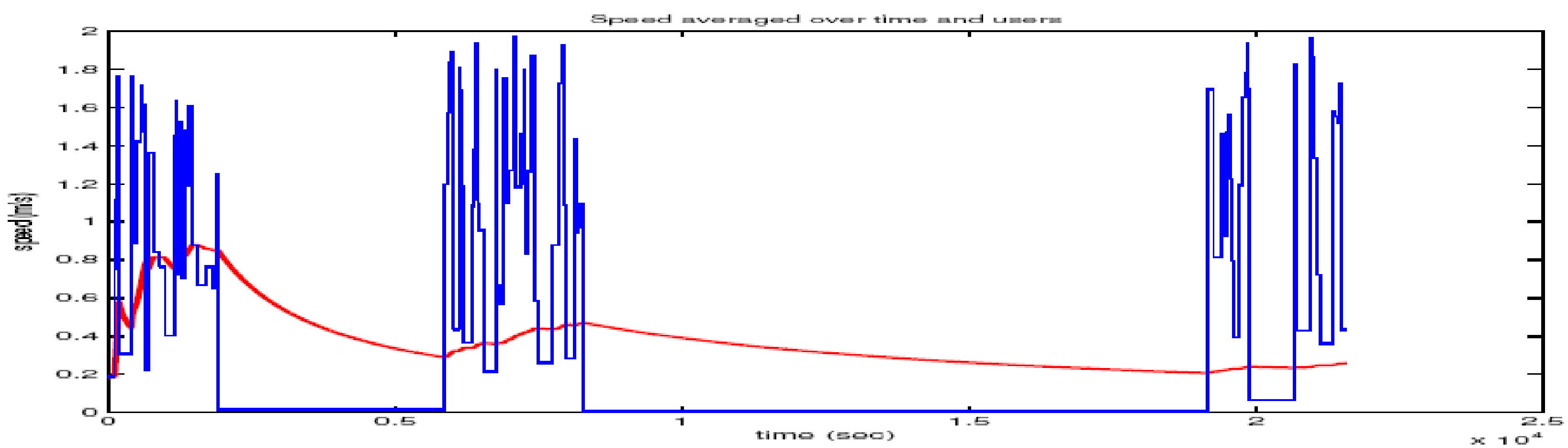
Compare to Single Long Run

Instant Speed, One User, simulated time = 2.5 days

SRD



LRD



Conclusion

- Palm Calculus with Inversion Formula essential to understand Mobility Models
- Stationary distributions are easily obtained in closed form
- Perfect Simulation is possible and simple
- Decay is simply convergence to steady state

References

Bettstetter04 Christian Bettstetter, Hannes Hartenstein and Xavier Pérez-Costa. Stochastic properties of the random waypoint mobility model. *ACM/Kluwer Wireless Networks*, Special Issue on Modeling and Analysis of Mobile Networks 2003.

Yoon03 Jungkeun Yoon, Mingyan Liu, and Brian Noble. Random waypoint considered harmful. In *Proceedings of Infocom*, 2003.

LeBoudec05 Jean-Yves Le Boudec. *Performance evaluation lecture notes* http://ica1www.epfl.ch/perfeval/

LeBoudec04 Jean-Yves Le Boudec. Understand the Simulation of Mobility Models with Palm Calculus, Technical Report EPFL IC/2004/43, June 2004 http://lcawww.epfl.ch/Publications/LeBoudec/LeBoudecV04.pdf

BaccelliBremaud87 François Baccelli and Pierre Brémaud. *Palm Probabilities and Stationary Queues*. Springer Verlag Lecture Notes in Statistics, 1987.

NavidiCamp04 W. Navidi, T. Camp, and N. Bauer, "Improving the Accuracy of Random Waypoint Simulations Through Steady-State Initialization", Proceedings of the 15th International Conference on Modeling and Simulation (MS '04), pp. 319-326, March 2004.

L-Vojnovic-Infocom05 Jean-Yves Le Boudec and Milan Vojnović, Perfect Simulation and Stationarity of a Class of Mobility Models *IEEE INFOCOM 2005*, Miami, 13-17 March 05