

Mean Field Methods for Computer and Communication Systems

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Contents

1. Mean Field Interaction Model
2. Convergence to ODE
3. Finite Horizon: Fast Simulation
and Decoupling assumption
4. Infinite Horizon: Fixed Point
Method and Decoupling
assumption

1

MEAN FIELD INTERACTION MODEL

Mean Field

- A *model* introduced in Physics
 - ▶ interaction between *particles* is via distribution of states of all particle
- An *approximation* method for a large collection of particles
 - ▶ assumes *independence* in the master equation
- Why do we care in information and communication systems ?
 - ▶ Model interaction of many objects:
 - ▶ Distributed systems, communication protocols, game theory, self-organized systems

Mean Field Interaction Model

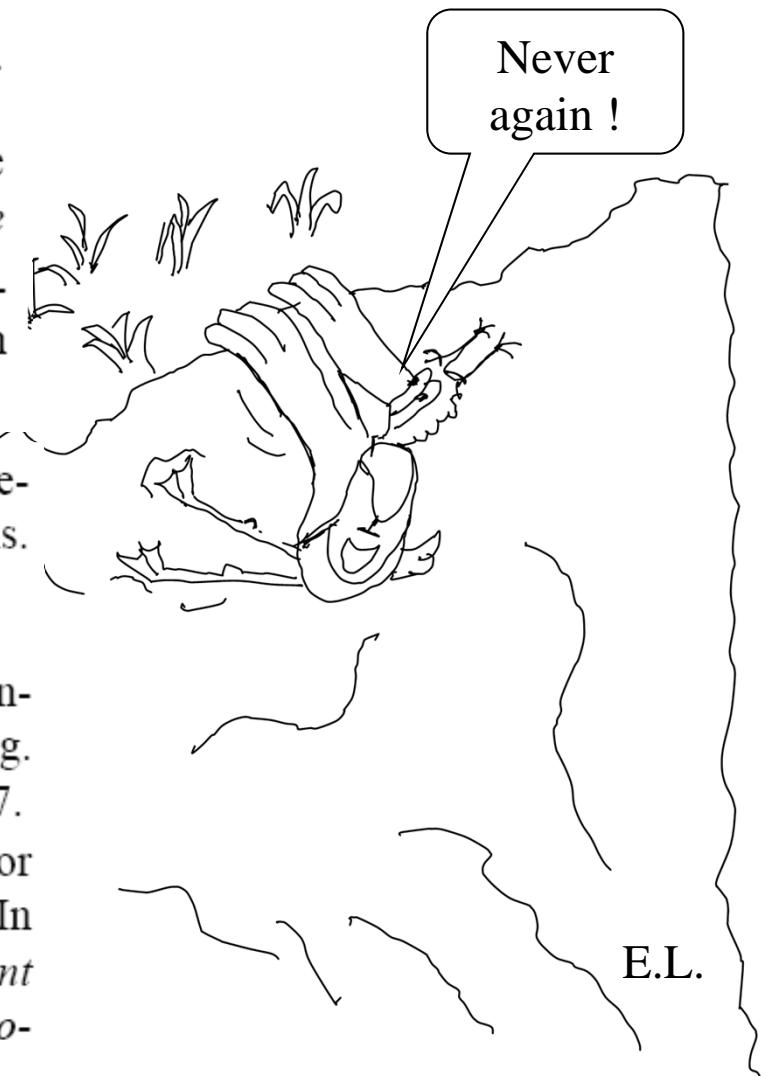
- Time is discrete
- N objects, N large
- Object n has state $X_n(t)$
- $(X^N_1(t), \dots, X^N_N(t))$ is Markov
- Objects are observable only through their state
- “Occupancy measure”
 $M^N(t)$ = distribution of object states at time t

Mean Field Interaction Model

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- Objects are observable only through their state
- “Occupancy measure”
 $M^N(t)$ = distribution of object states at time t
- **Theorem** [Gast (2011)]
 $M^N(t)$ is Markov
- Called “*Mean Field Interaction Models*” in the Performance Evaluation community
[McDonald(2007), Benaïm and Le Boudec(2008)]

A Few Examples Where Applied

- [1] L. Afanassieva, S. Popov, and G. Fayolle. Models for transportation networks. *Journal of Mathematical Sciences*, 1997 – Springer.
- [2] F. Baccelli, A. Chaintreau, D. De Vleeschauwer, and D. R. McDonald. Http turbulence, May 2004.
- [3] F. Baccelli, M. Lelarge, and D. McDonald. Metastable regimes for multiplexed tcp flows. In *Proceedings of the*
- [5] M.-D. Bordenave, Charles and A. Proutiere. A particle system in interaction with a rapidly varying environment: Mean field limits and applications. arXiv:math/0701363v2.
- [11] S. Kumar and L. Massoulié. Integrating streaming and file-transfer internet traffic: Fluid and diffusion approximations. MSR-TR-2005-160.
- [16] Y. M. Suhov and N. D. Vvedenskaya. Dobrushin's mean-field approximation for a queue with dynamic routing. *Markov Processes and Related Fields*, 3(4):493–526, 1997.
- [17] P. Tinnakornsrisuphap and A. M. Makowski. Limit behavior of ecn/red gateways under a large number of tcp flows. In *Proceedings IEEE INFOCOM 2003, The 22nd Annual Joint Conference of the IEEE Computer and Communications Societies, San Francisco, CA, USA, March 30 - April 3 2003.*



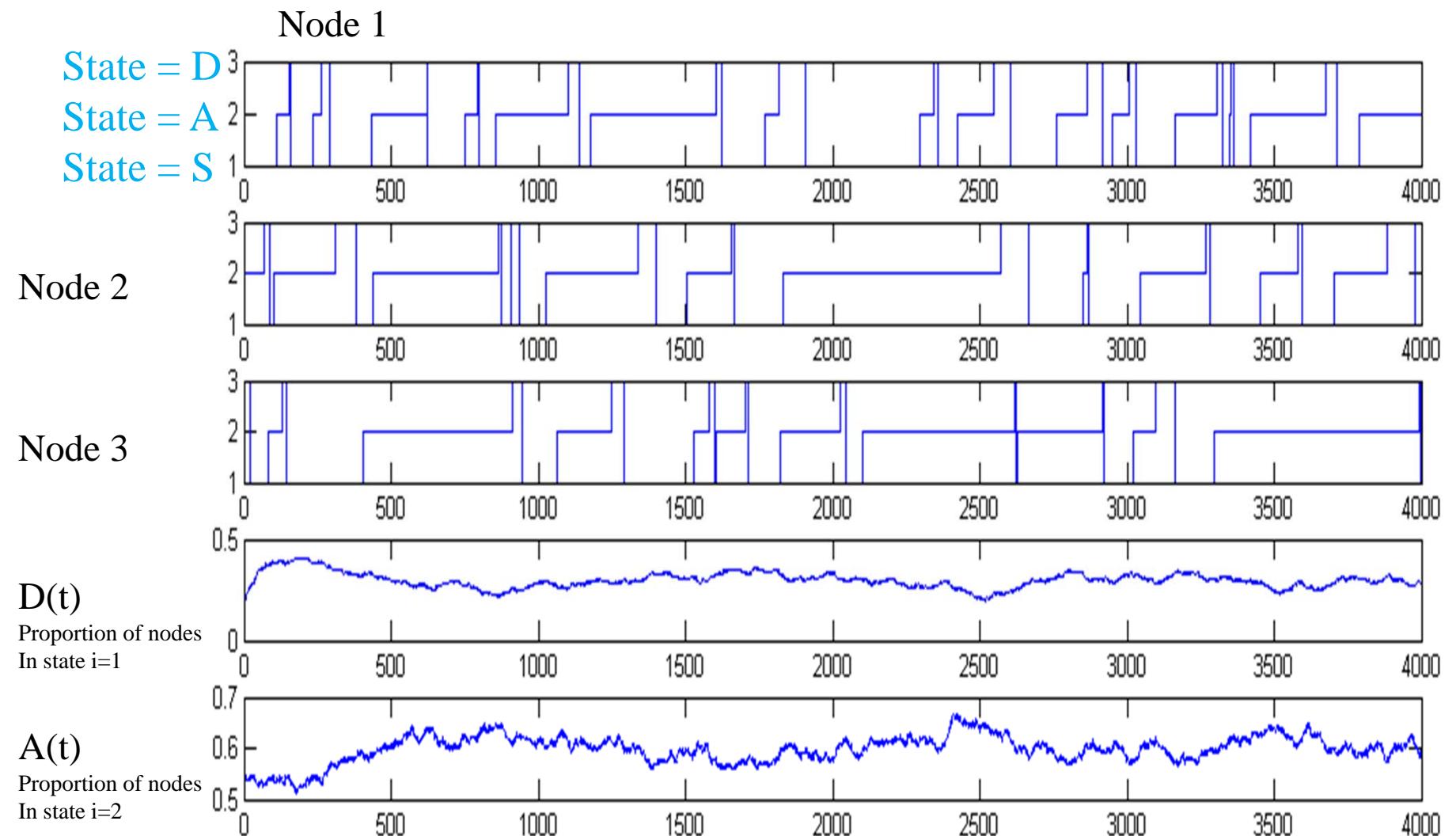
Example: 2-Step Malware

- Mobile nodes are either
 - ▶ 'S' Susceptible
 - ▶ 'D' Dormant
 - ▶ 'A' Active
 - Time is discrete
 - Nodes meet pairwise (bluetooth)
 - One interaction per time slot,
 $I(N) = 1/N$; mean field limit is an ODE
 - State space is finite
 $= \{`S', `A', `D'\}$
 - Occupancy measure is
 $M(t) = (S(t), D(t), A(t))$ with
 $S(t) + D(t) + A(t) = 1$
 $S(t)$ = proportion of nodes in state 'S'
- [Benaïm and Le Boudec(2008)]
- Possible interactions:
 1. Recovery
 - ▶ $D \rightarrow S$
 2. Mutual upgrade
 - ▶ $D + D \rightarrow A + A$
 3. Infection by active
 - ▶ $D + A \rightarrow A + A$
 4. Recovery
 - ▶ $A \rightarrow S$
 5. Recruitment by Dormant
 - ▶ $S + D \rightarrow D + D$

Direct infection

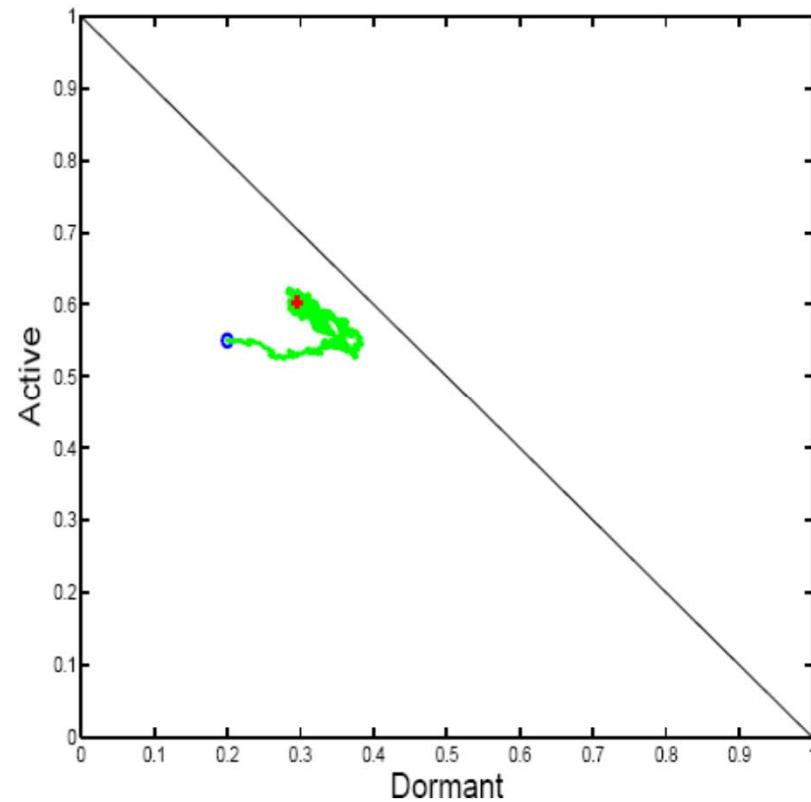
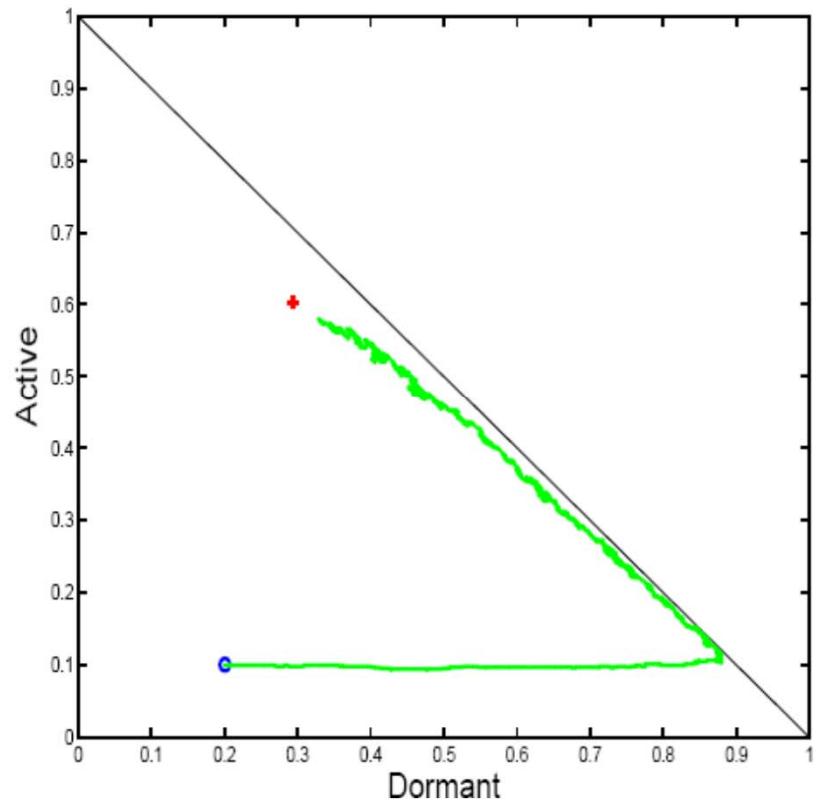
 - ▶ $S \rightarrow D$
 6. Direct infection
 - ▶ $S \rightarrow A$

Simulation Runs, N=1000 nodes



$$\beta = 0.01, \delta_A = 0.005, \delta_D = 0.0001, \alpha_0 = \alpha = 0.0001, h = 0.3, r = 0.1, \lambda = 0.0001$$

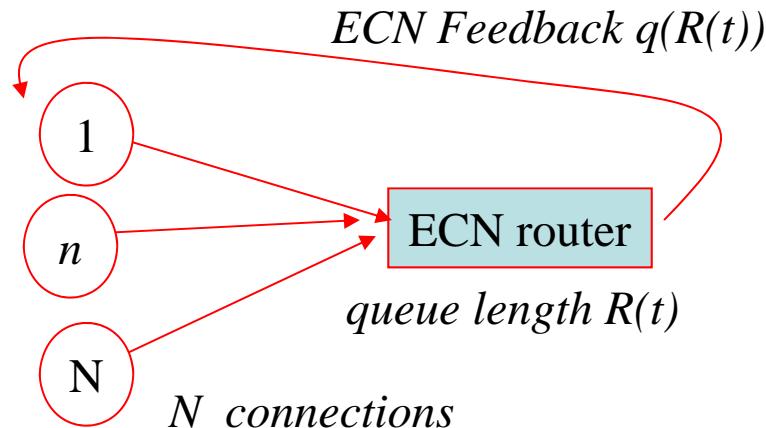
Sample Runs with N = 1000



$$\beta = 0.01, \delta_A = 0.005, \delta_D = 0.0001, \alpha_0 = \alpha = 0.0001, h = 0.3, r = 0.1, \lambda = 0.0001$$

Example: TCP and ECN

- [Tinnakornsrisuphap and Makowski(2003)]



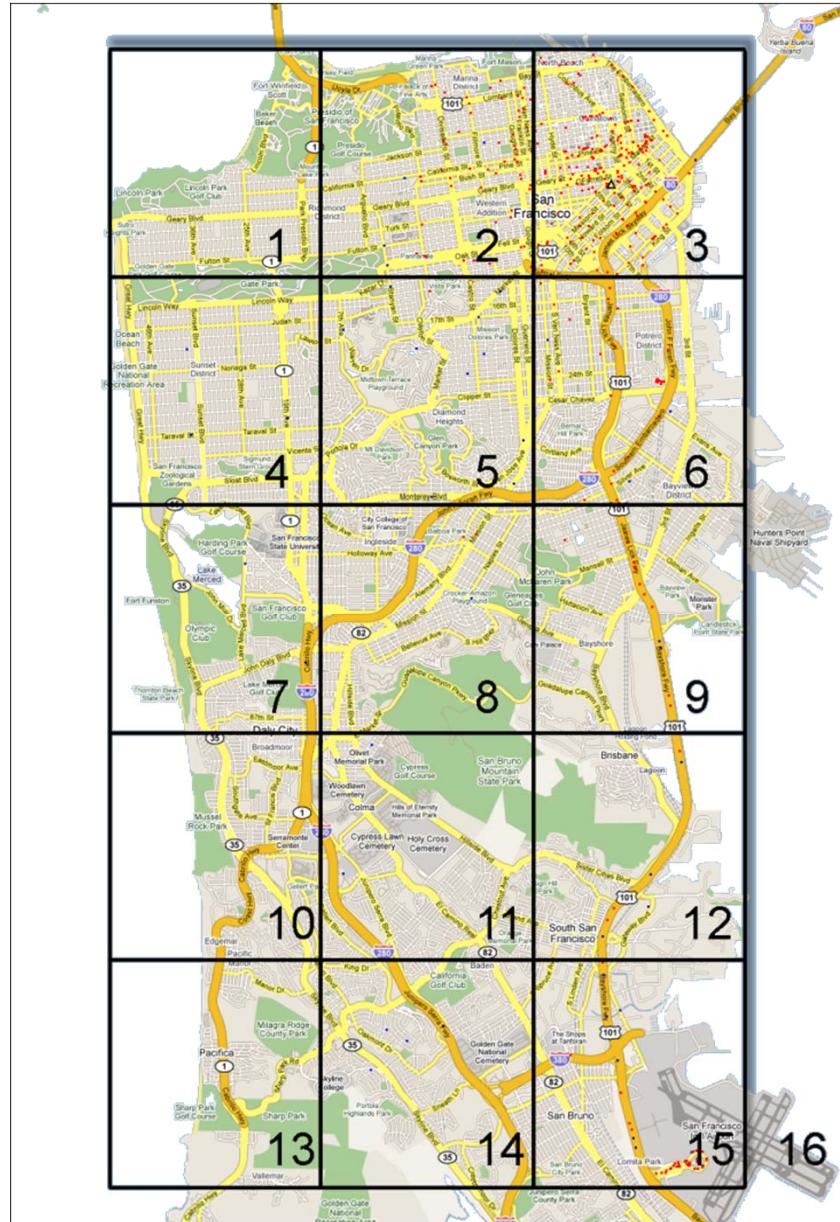
At, every time step, all connections update their state: $I(N)=1$

- Time is discrete, mean field limit is also in discrete time (iterated map)

- Similar examples:
HTTP Metastability
[Baccelli et al.(2004) Baccelli, Lelarge, and McDonald]

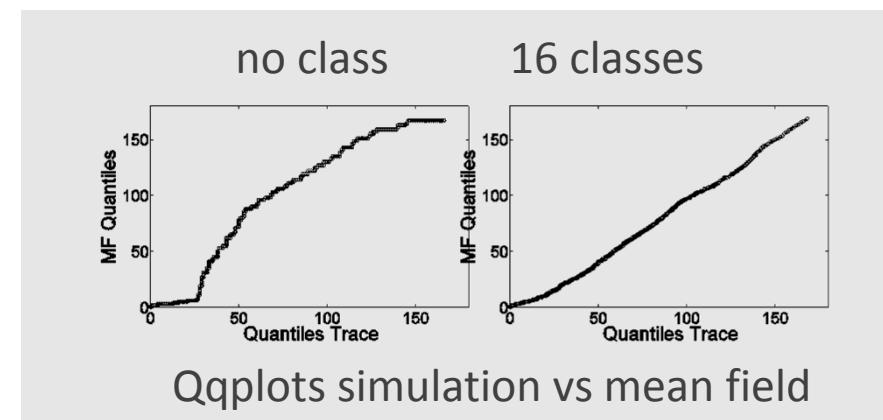
Reputation System [Le Boudec et al.(2007) Le Boudec, McDonald, and Mundinger, Gomez-Serrano et 10, 2012]

The Importance of Being Spatial



- Mobile node state = (c, t)
 $c = 1 \dots 16$ (position)
 $t \in \mathbb{R}^+$ (age of gossip)
- Time is continuous, $I(N) = 1$
- Occupancy measure is
 $F_c(z,t) = \text{proportion of nodes that at location } c \text{ and have age } \leq z$

[Age of Gossip, Chaintreau et al.(2009)]



What can we do with a Mean Field Interaction Model ?

■ Large N asymptotics, Finite Horizon

- ▶ fluid limit of occupancy measure (ODE)
- ▶ decoupling assumption (fast simulation)

■ Issues

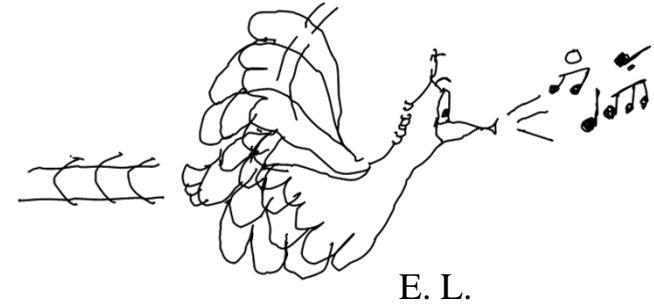
- ▶ When valid
- ▶ How to formulate the fluid limit

■ Large t asymptotic

- ▶ Stationary approximation of occupancy measure
- ▶ Decoupling assumption

■ Issues

- ▶ When valid



2.

CONVERGENCE TO ODE

Intensity $I(N)$

- $I(N)$ = expected number of transitions per object per time unit
- A mean field limit occurs when we re-scale time by $I(N)$
i.e. we consider $X^N(t/I(N))$
- $I(N) = O(1)$: mean field limit is in discrete time
[Le Boudec et al (2007)]
 $I(N) = O(1/N)$: mean field limit is in continuous time
[Benaïm and Le Boudec (2008)]

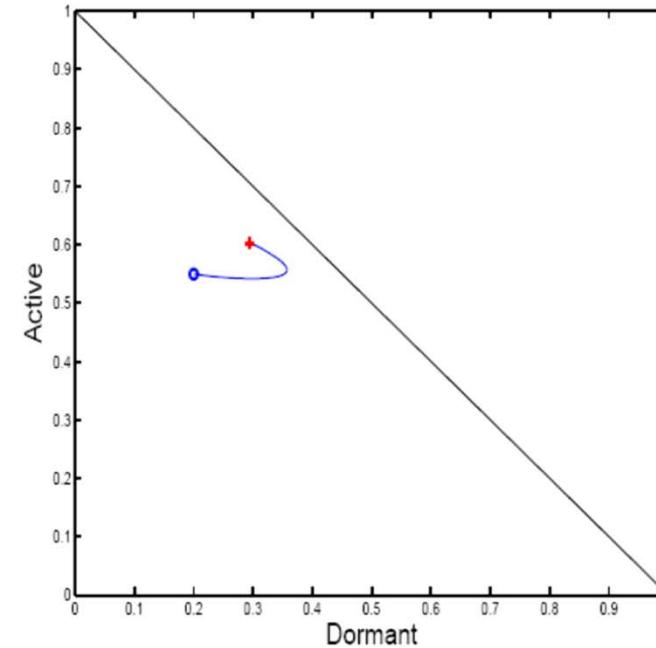
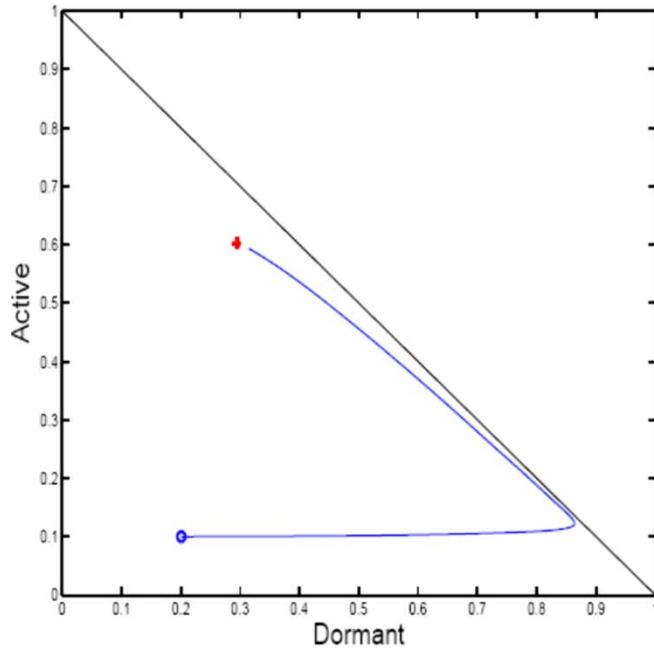
The Mean Field Limit

- Under very general conditions (given later) the occupancy measure converges, in law, to a deterministic process, $m(t)$, called the *mean field limit*

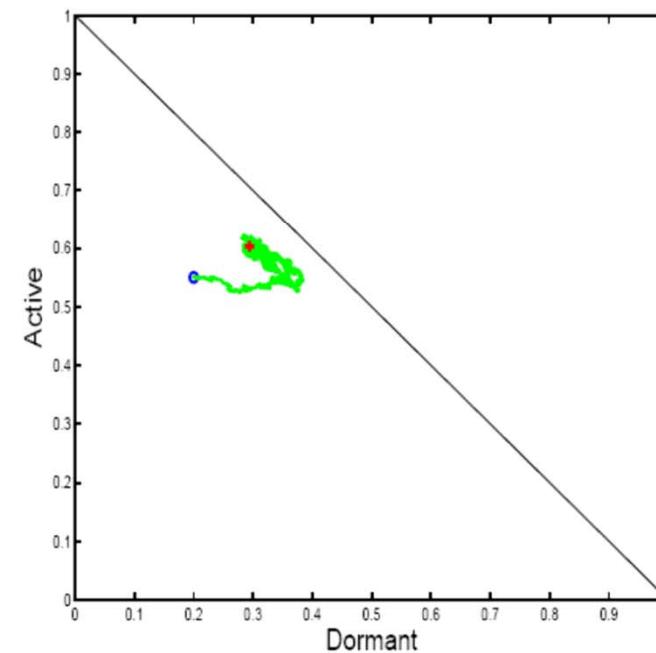
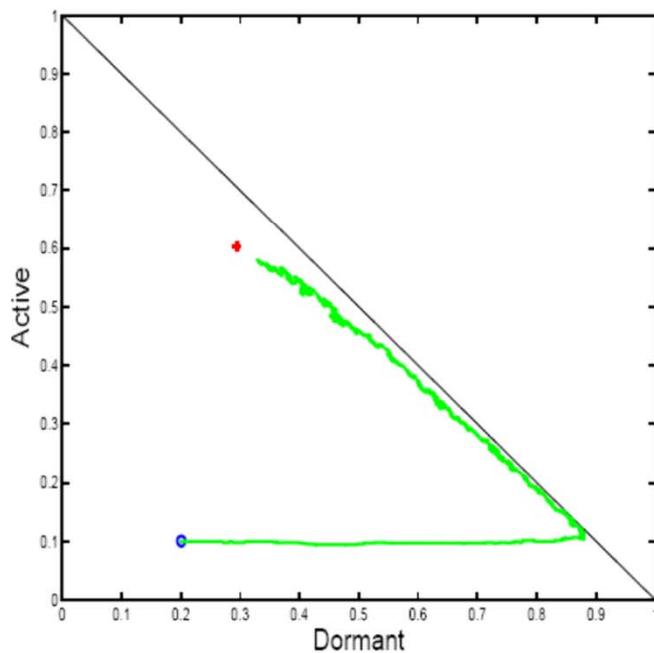
$$M^N \left(\frac{t}{I(N)} \right) \rightarrow m(t)$$

- Finite State Space => ODE

Mean Field Limit
 $N = +\infty$



Stochastic
system
 $N = 1000$



Sufficient Conditions for Convergence

- [Kurtz 1970], see also [Bordenav et al 2008], [Graham 2000]
- Sufficient condition verifiable by inspection:

[Benaïm and Le Boudec(2008), Ioannidis and Marbach(2009)]

- Let $W^N(k)$ be the number of objects that do a transition in time slot k . Note that $\mathbb{E}(W^N(k)) = NI(N)$, where $I(N) \stackrel{\text{def}}{=} \text{intensity}$. Assume

$$\mathbb{E}(W^N(k)^2) \leq \beta(N) \quad \text{with} \quad \lim_{N \rightarrow \infty} I(N)\beta(N) = 0$$

Example: $I(N) = 1/N$

Second moment of number of objects affected in one timeslot = $o(N)$

- Similar result when mean field limit is in discrete time [Le Boudec et al 2007]

Example: Convergence to Mean Field

Example: 2-Step Malware

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- One interaction per time slot,
 $I(N) = 1/N$; mean field limit is an ODE
- State space is finite
 $= \{S', A', D'\}$
- Occupancy measure is
 $M(t) = (S(t), D(t), A(t))$ with
 $S(t) + D(t) + A(t) = 1$
 $S(t)$ = proportion of nodes in state 'S'
[Benaïm and Le Boudec(2008)]

- Possible interactions:
 1. Recovery
 - ▶ $D \rightarrow S$
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 4. Recovery
 - ▶ $A \rightarrow S$
 5. Recruitment by Dormant
 - ▶ $S + D \rightarrow D + D$
 - Direct infection
 - ▶ $S \rightarrow D$
 6. Direct infection
 - ▶ $S \rightarrow A$

- Rescale time such that one time step = $1/N$
- Number of transitions per time step is bounded by 2, therefore there is convergence to mean field

$$\begin{aligned}\frac{\partial D}{\partial t} &= -\delta_D D - 2\lambda D^2 - \beta A \frac{D}{h+D} + (\alpha_0 + rD)S \\ \frac{\partial A}{\partial t} &= 2\lambda D^2 + \beta A \frac{D}{h+D} - \delta_A A + \alpha S \\ \frac{\partial S}{\partial t} &= \delta_D D + \delta_A A - (\alpha_0 + rD)S - \alpha S\end{aligned}$$

Formulating the Mean Field Limit

- **Drift** = sum over all transitions of proba of transition \times Delta to system state $M^N(t)$
- Re-scale drift by intensity
- Equation for mean field limit is $dm/dt = \lim$ of rescaled drift
- Can be automated
<http://icawww1.epfl.ch/IS/tsed>

case	prob	effect on (D, A, S)
1	$D\delta_D$	$\frac{1}{N}(-1, 0, 1)$
2	$D\lambda\frac{ND-1}{N-1}$	$\frac{1}{N}(-2, +2, 0)$
3	$A\beta\frac{D}{h+D}$	$\frac{1}{N}(-1, +1, 0)$
4	$A\delta_A$	$\frac{1}{N}(0, -1, +1)$
5	$S(\alpha_0 + rD)$	$\frac{1}{N}(+1, 0, -1)$
6	$S\alpha$	$\frac{1}{N}(0, +1, -1)$

drift =

$$\frac{1}{N} \begin{pmatrix} -D\delta_D - 2D\lambda\frac{ND-1}{N-1} - A\beta\frac{D}{h+D} + S(\alpha_0 + rD) \\ 2D\lambda\frac{ND-1}{N-1} + A\beta\frac{D}{h+D} - A\delta_A + S\alpha \\ D\delta_D + A\delta_A - S(\alpha_0 + rD) - S\alpha \end{pmatrix}$$

$$\frac{\partial D}{\partial t} = -\delta_D D - 2\lambda D^2 - \beta A \frac{D}{h+D} + (\alpha_0 + rD)S$$

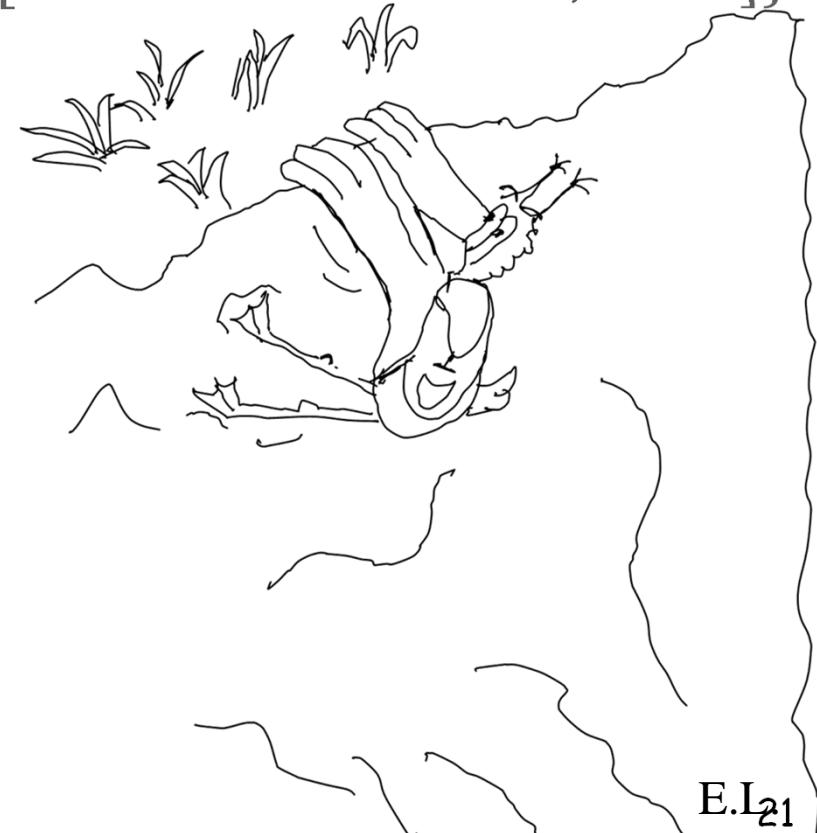
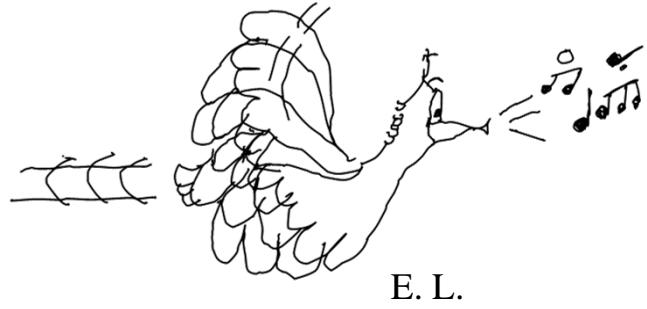
$$\frac{\partial A}{\partial t} = 2\lambda D^2 + \beta A \frac{D}{h+D} - \delta_A A + \alpha S$$

$$\frac{\partial S}{\partial t} = \delta_D D + \delta_A A - (\alpha_0 + rD)S - \alpha S$$

Convergence to Mean Field

- For the finite state space case, there are many simple results, often verifiable by inspection

For example [Kurtz 1970] or [Benaim, Le Boudec 2008]
- For the general state space, things may be more complex (fluid limit is not an ODE, e.g. [Chaintreau et al, 2009], [Gomez-Serrano et al, 2012])



E.L₂₁

3. **FINITE HORIZON : FAST SIMULATION AND DECOUPLING ASSUMPTION**

Convergence to Mean Field Limit is Equivalent to Propagation of Chaos

Definition 1.1 Let $X^N = (X_1^N, \dots, X_N^N)$ be an exchangeable sequence of processes in $\mathcal{P}(S)$ and $m \in \mathcal{P}(S)$ where S is metric complete separable. $(X^N)_N$ is m -chaotic iff for every k : $\mathcal{L}(X_1^N, \dots, X_k^N) \rightarrow m \otimes \dots \otimes m$ as $N \rightarrow \infty$.

Theorem 1.1 ([Sznitman(1991)]) $(X^N)_N$ is m -chaotic then the occupancy measure $M^N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=1}^N \delta_{X_n^N}$ converges in probability (and in law) to m .

If the occupancy measure converges in law to m then $(X^N)_N$ is m -chaotic.

Propagation of Chaos = Decoupling Assumption

■ (Propagation of Chaos)

k objects are asymptotically independent with common law equal to the mean field limit, for any fixed k

$$\mathcal{L} \left(X_1 \left(\frac{t}{I(N)} \right), \dots, X_k \left(\frac{t}{I(N)} \right) \right) \rightarrow m(t) \otimes \dots \otimes m(t)$$

■ (Decoupling Assumption)

(also called Mean Field Approximation, or Fast Simulation)

The law of one object is asymptotically as if all other objects were drawn randomly with replacement from $m(t)$

The Two Interpretations of the Mean Field Limit

- At any time t

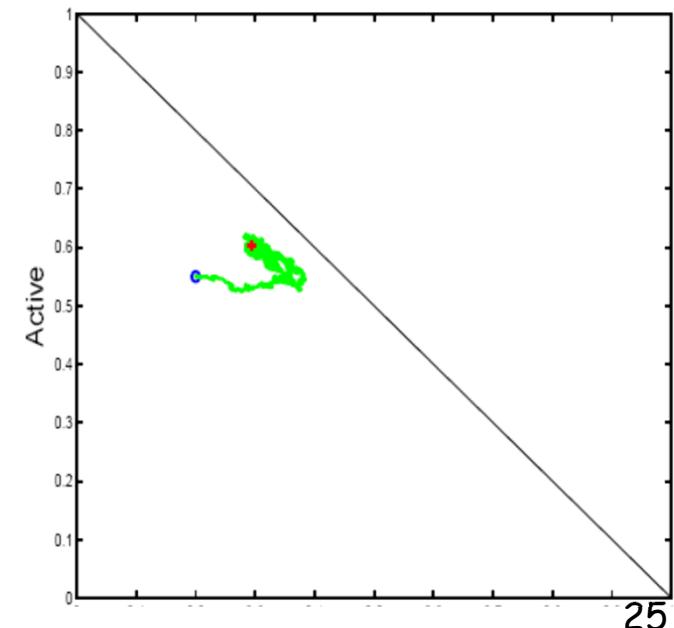
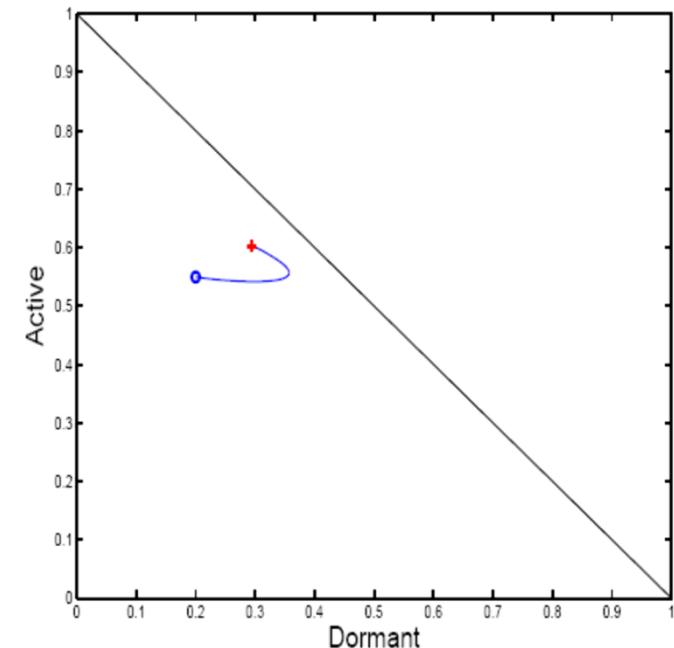
$$P(X_n(t) =' A') \approx A\left(\frac{t}{N}\right)$$

$$P(X_m(t) =' D', X_n(t) =' A') \approx D\left(\frac{t}{N}\right) A\left(\frac{t}{N}\right)$$

where (D, A, S) is solution of ODE

- Thus for large t :
 - ▶ Prob (node n is dormant) ≈ 0.3
 - ▶ Prob (node n is active) ≈ 0.6
 - ▶ Prob (node n is susceptible) ≈ 0.1

- $m(t)$ approximates both
 1. the occupancy measure $M^N(t)$
 2. the state probability for one object at time t , drawn at random among N



$$p_j^N(t|i) = \text{P}(X_n^N(t) = j | X_n^N(0) = i)$$

$$p_j^N\left(\frac{t}{N} | i\right) \approx p_j(t|i)$$

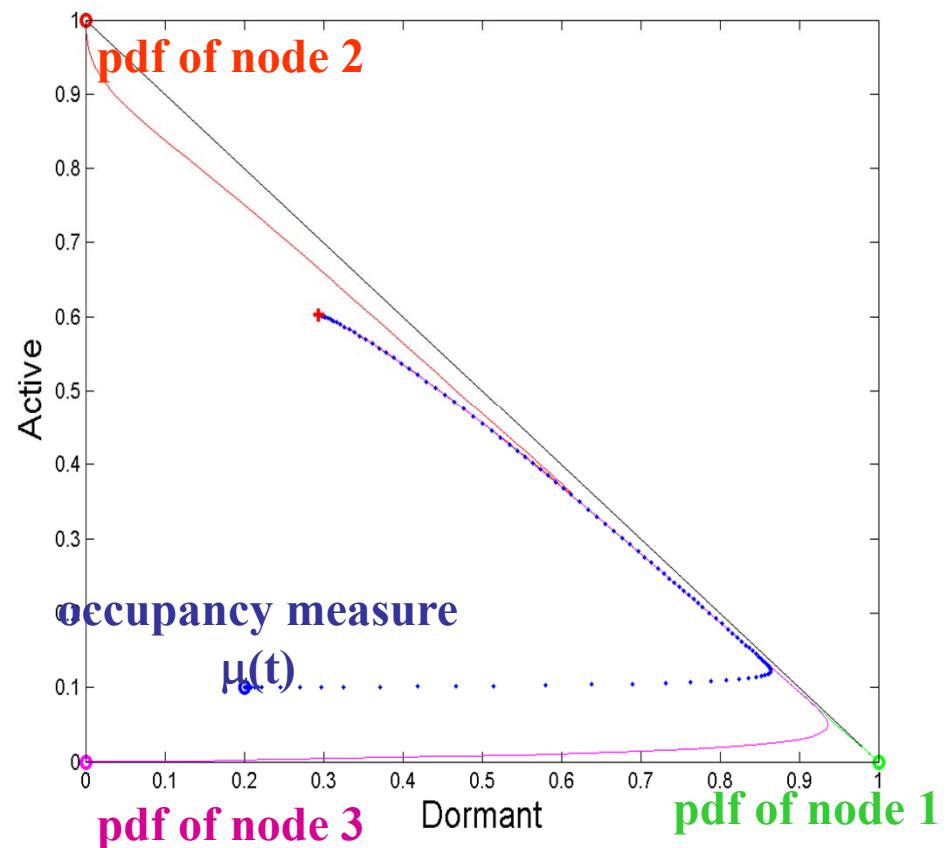
where $\vec{p}(t|i)$ is the (transient) probability of a continuous time nonhomogeneous Markov process

$$\frac{d}{dt} \vec{p}(t|i) = \vec{p}(t|i)^T A(\vec{m}(t))$$

- Same ODE as mean field limit, with different initial condition

$$\begin{aligned} \frac{d}{dt} \vec{m}(t) &= \vec{m}(t)^T A(\vec{m}(t)) \\ &= F(\vec{m}(t)) \end{aligned}$$

« Fast Simulation »



The Decoupling Assumption

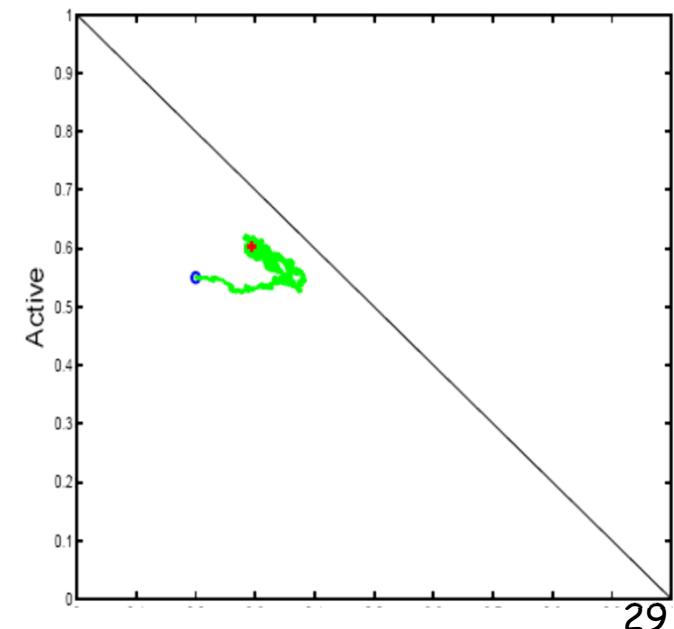
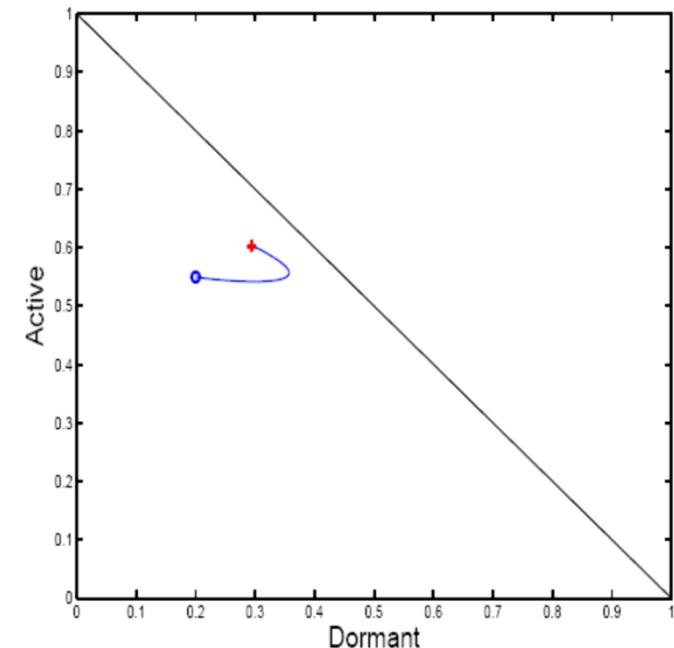
- The evolution for one object as if the other objects had a state drawn randomly and independently from the distribution $m(t)$
- Is valid *over finite horizon* whenever mean field convergence occurs
- Can be used to analyze or simulate evolution of k objects

4.

INFINITE HORIZON: FIXED POINT METHOD AND DECOUPLING ASSUMPTION

The Fixed Point Method

- Decoupling assumption says distribution of prob for state of one object is $\approx \vec{m}(t)$ with $\frac{d\vec{m}(t)}{dt} = F(\vec{m}(t))$
- We are interested in stationary regime, i.e we do $F(\vec{m}) = 0$
- This is the « Fixed Point Method »
- Example: in stationary regime:
 - ▶ Prob (node n is dormant) ≈ 0.3
 - ▶ Prob (node n is active) ≈ 0.6
 - ▶ Prob (node n is susceptible) ≈ 0.1
 - ▶ Nodes m and n are independent



Example: 802.11 Analysis, Bianchi's Formula

802.11 single cell

m_i = proba one node is in
backoff stage I

β = attempt rate

γ = collision proba

See [Benaim and Le Boudec , 2008] for this analysis

ODE for mean field limit

$$\frac{dm_0}{d\tau} = -m_0 q_0 + \beta(\vec{m}) (1 - \gamma(\vec{m})) + q_K m_K \gamma(\vec{m})$$

$$\frac{dm_i}{d\tau} = -m_i q_i + m_{i-1} q_{i-1} \gamma(\vec{m}) \quad i = 1, \dots, K$$

$$\beta(\vec{m}) = \sum_{i=0}^K q_i m_i$$

$$\gamma(\vec{m}) = 1 - e^{-\beta(\vec{m})}$$

Solve for Fixed Point:

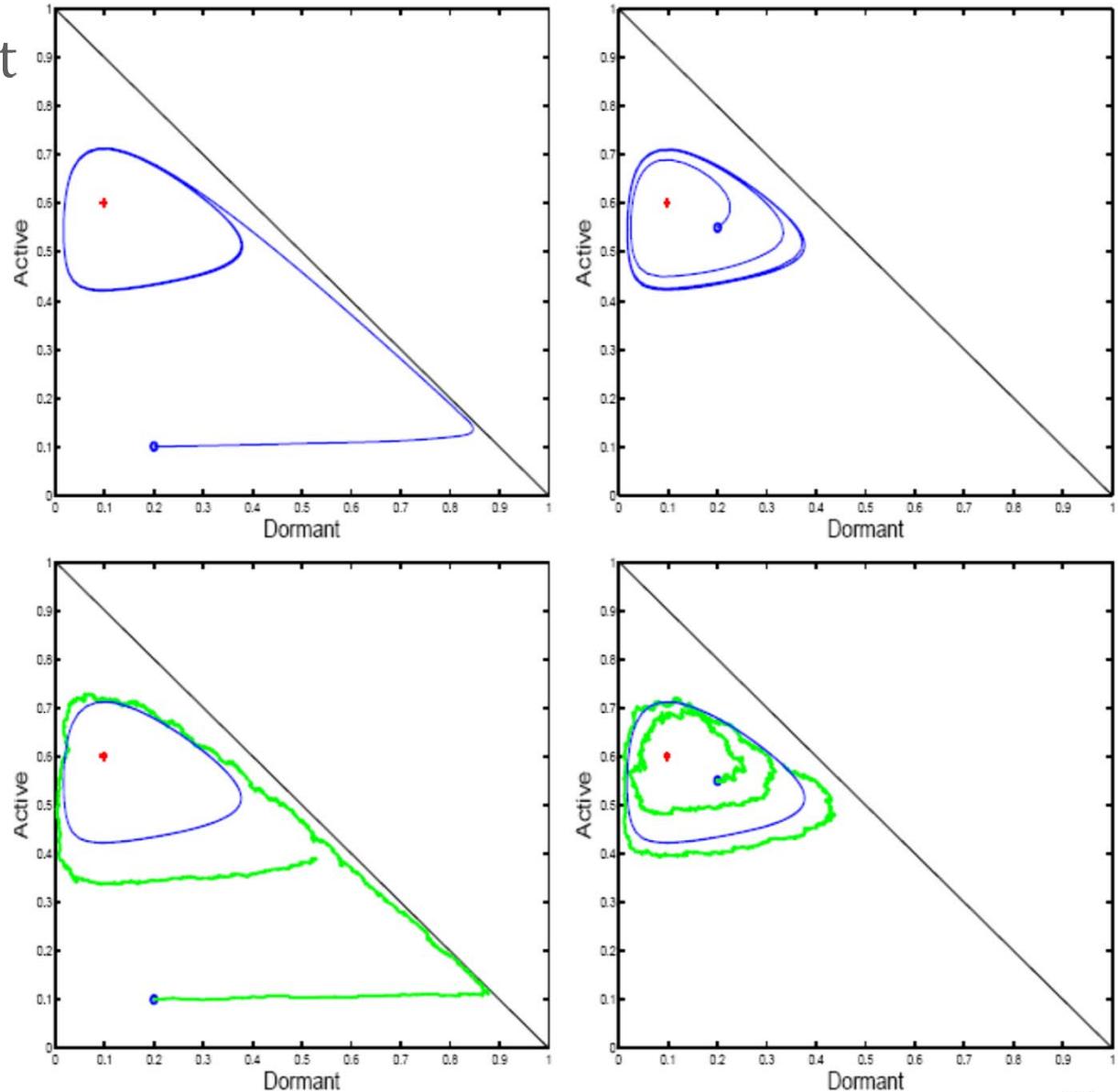
$$m_i = \frac{\gamma^i}{q_i} \frac{1}{\sum_{k=0}^K \frac{\gamma^k}{q_k}}$$

Bianchi's
Fixed
Point
Equation
[Bianchi 1998]

$$\gamma = 1 - e^{-\beta}$$
$$\beta = \frac{\sum_{k=0}^K \gamma^k}{\sum_{k=0}^K \frac{\gamma^k}{q_k}}$$

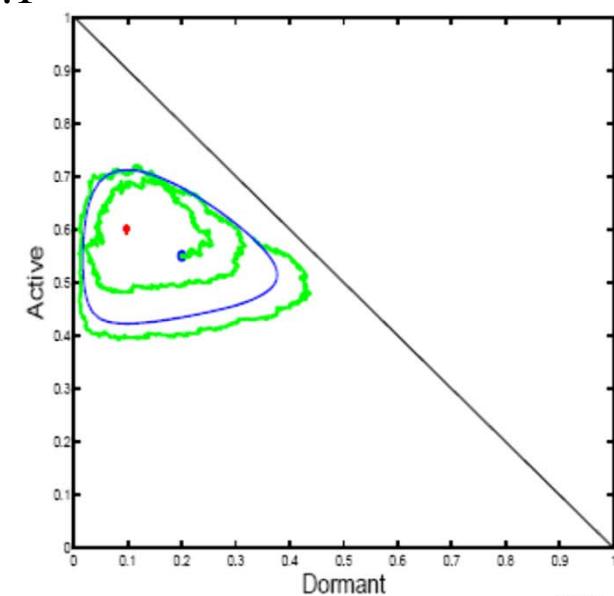
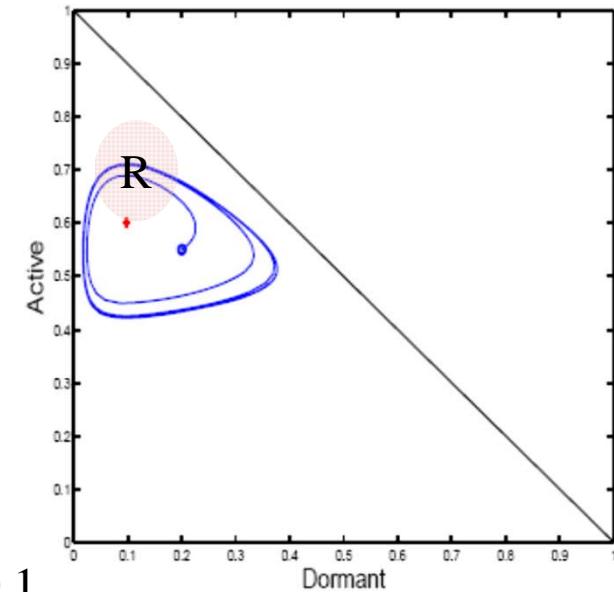
2-Step Malware, Again

- Same as before except for one parameter value :
 $h = 0.1$ instead of 0.3
- The ODE does not converge to a unique attractor (limit cycle)
- The equation $F(m) = 0$ has a **unique** solution (red cross) – but it is ***not*** the stationary regime !

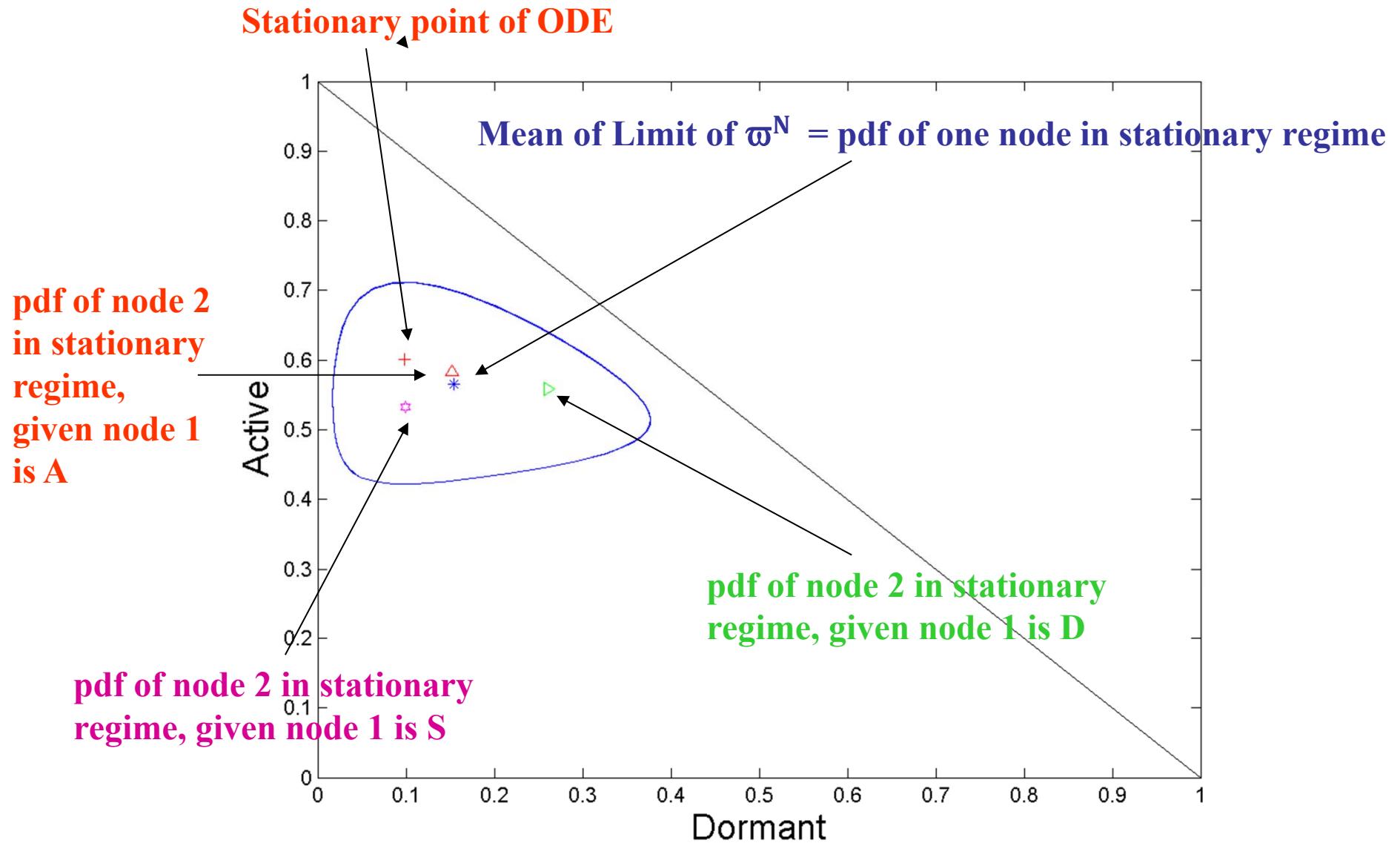


Example Where Fixed Point Method Fails

- In stationary regime,
 $\vec{m}(t) = (D(t), A(t), S(t))$ follows the limit cycle
- Assume you are in stationary regime (simulation has run for a long time) and you observe that one node, say $n = 1$, is in state 'A'
- It is more likely that $m(t)$ is in region R
- Therefore, it is more likely that some other node, say $n = 2$, is also in state 'A'
- This is synchronization



Joint PDFs of Two Nodes in Stationary Regime



Where is the Catch ?

- Decoupling assumption says that nodes m and n are asymptotically independent
- There *is* mean field convergence for this example
- But we saw that nodes may not be asymptotically independent

... is there a contradiction ?

Markov chain is ergodic

$$\mathbb{P}(X_1^N(t/N) = i \text{ and } X_1^N(t/N) = j) \xrightarrow{t \rightarrow \infty} \pi_{i,j}^N$$

$\downarrow N \rightarrow \infty$ Mean Field Convergence $\downarrow N \rightarrow \infty$

$$m_i(t) m_j(t) \neq \frac{1}{T} \int_0^T m_i(t) m_j(t) dt$$

- The *decoupling assumption may not hold in stationary regime*, even for perfectly regular models

Result 1: Fixed Point Method Holds under (H)

- Assume that

(H) ODE has a unique global stable point to which all trajectories converge

- Theorem [e.g. Benaim et al 2008] : The limit of stationary distribution of M^N is concentrated on this fixed point
- The decoupling assumption holds in stationary regime

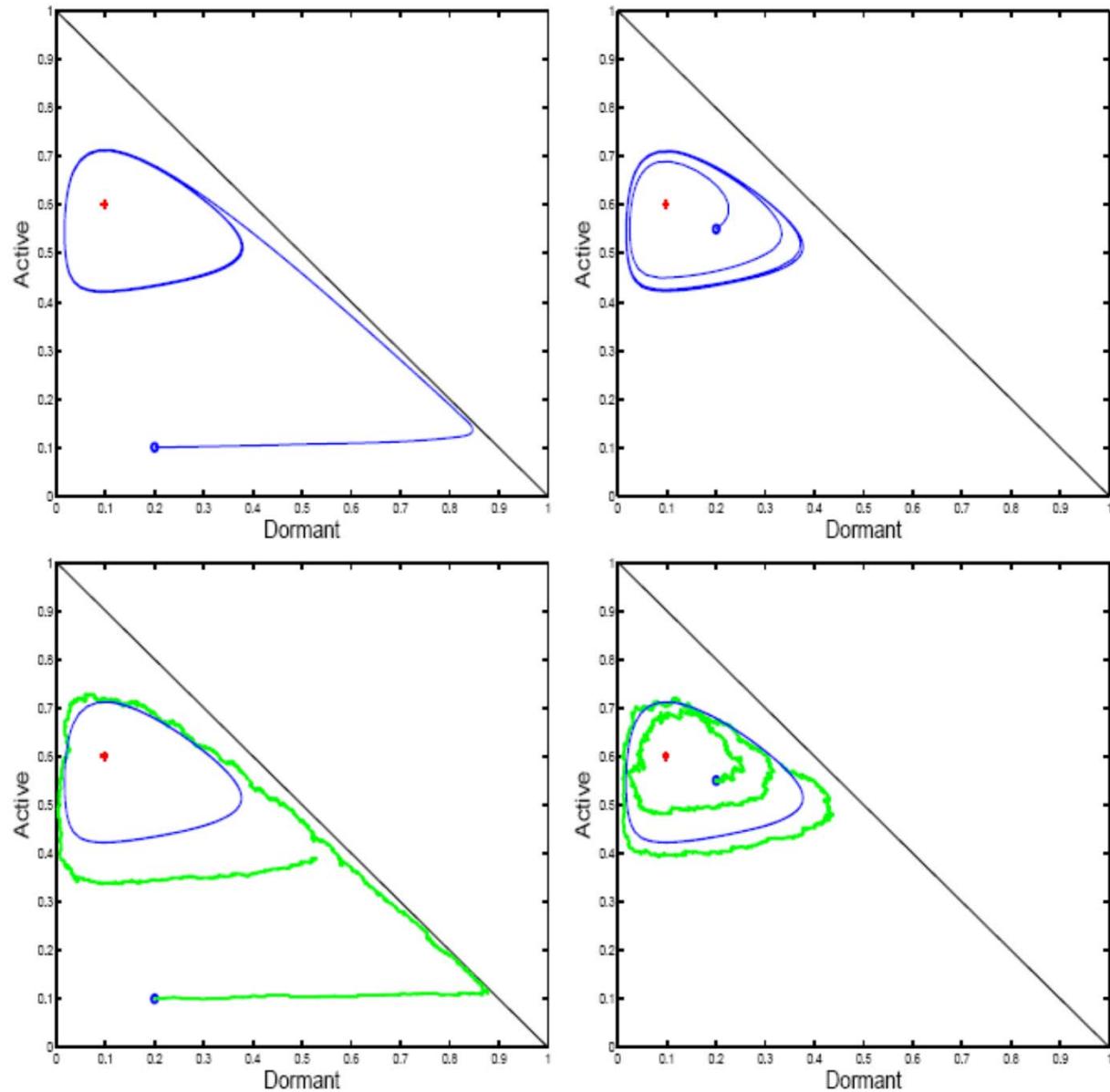
Result 2: Birkhoff Center

- Here:

Birkhoff center =
limit cycle \cup fixed
point

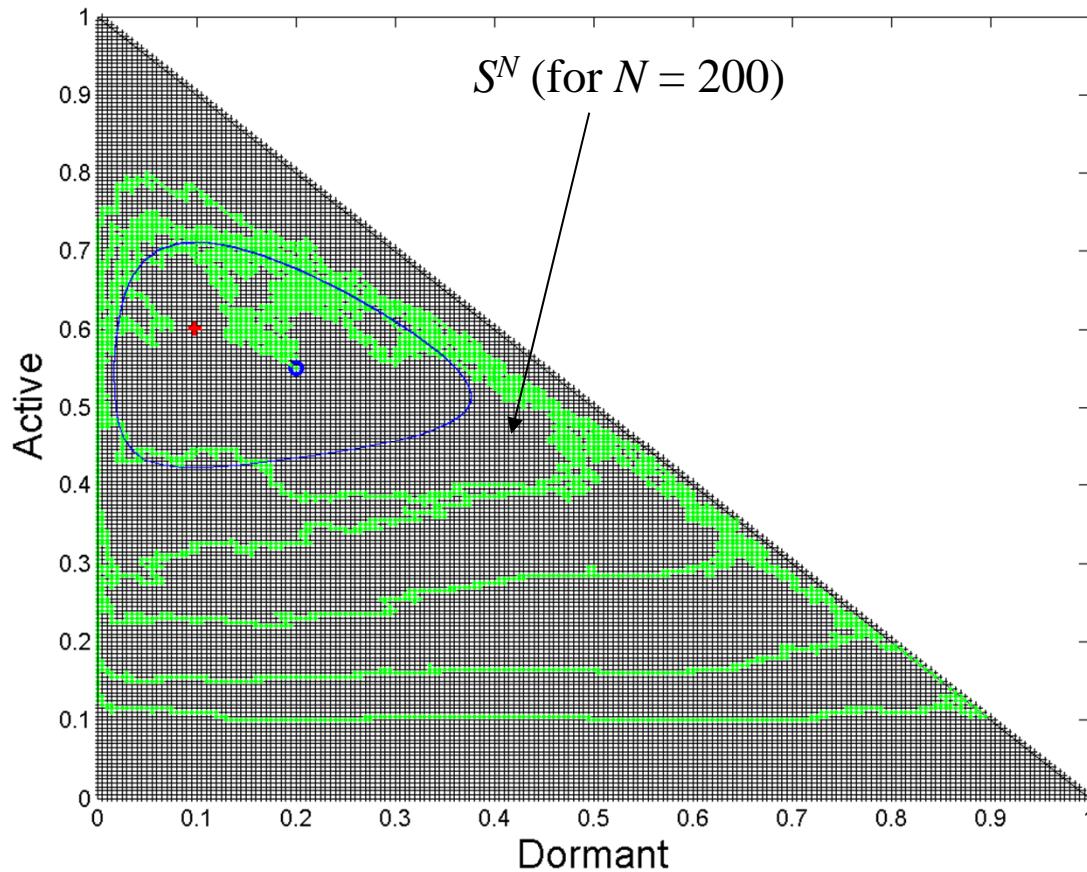
- Theorem in [Benaïm]
says that the
stochastic system for
large N is close to the
Birkhoff center,

i.e. the stationary
regime of ODE is a
good approximation
of the stationary
regime of stochastic
system

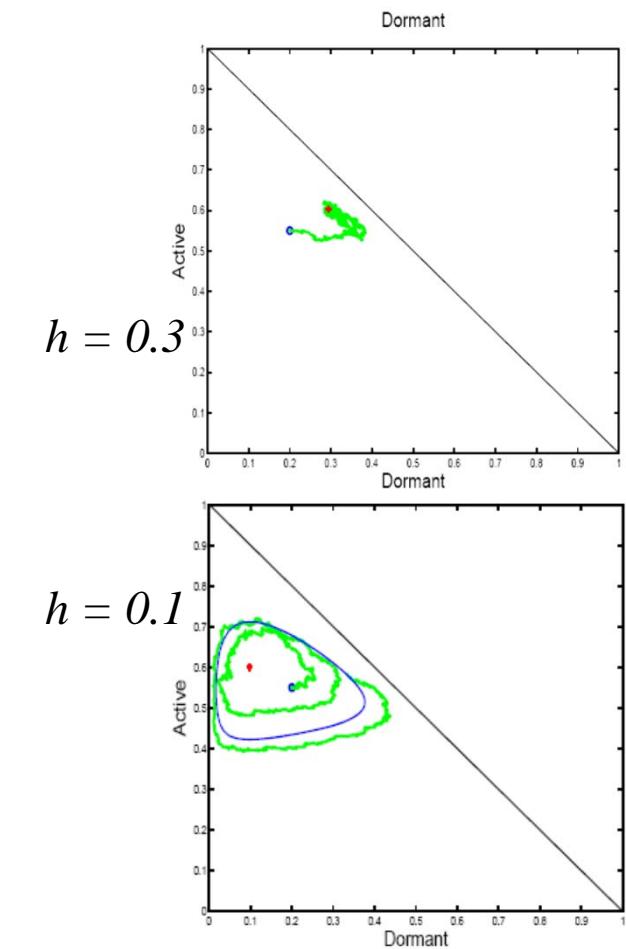


Stationary Behaviour of Mean Field Limit is not predicted by Structure of Markov Chain

- $M^N(t)$ is a Markov chain on $S^N = \{(a, b, c) \geq 0, a + b + c = 1, a, b, c \text{ multiples of } 1/N\}$
- $M^N(t)$ is ergodic and aperiodic



- Depending on parameter, there is or is not a limit cycle for $m(t)$

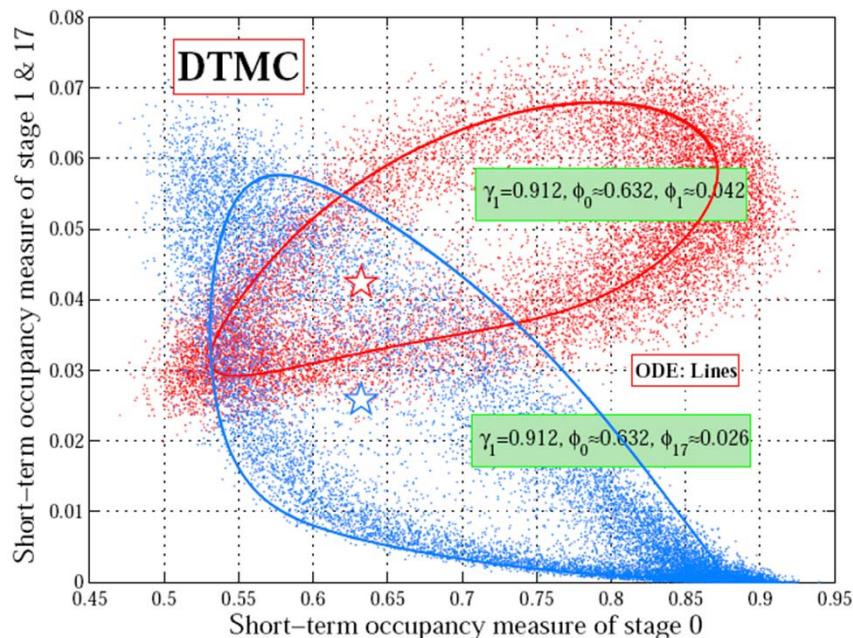


Example: 802.11 with Heterogeneous Nodes

■ [Cho et al, 2010]

Two classes of nodes with heterogeneous parameters (retransmission probability)

Fixed point equation has a unique solution, but this is not the stationary proba



There is a limit cycle

Result 3: In the Reversible Case, the Fixed Point Method Always Works

- **Definition** Markov Process $X(t)$ on enumerable state E space, with transition rates $q(i,j)$ is reversible iff
 1. it is ergodic
 2. $p(i) q(i,j) = p(j) q(j,i)$ for some p

Theorem 1.2 ([Le Boudec(2010)]) *Assume some process $Y^N(t)$ converges at any fixed t to some deterministic system $y(t)$ at any finite time. Assume the processes Y^N are reversible under some probabilities Π^N . Let $\Pi \in \mathcal{P}(E)$ be a limit point of the sequence Π^N . Π is concentrated on the set of stationary points S of the fluid limit $y(t)$*

- Stationary points = fixed points
- If process with finite N is reversible, the stationary behaviour is determined only by fixed points.

A Correct Method

- 1. Write dynamical system equations *in transient regime*
- 2. Study the *stationary regime of* dynamical system
 - ▶ if converges to unique stationary point m^* then make fixed point assumption
 - ▶ else objects are coupled in stationary regime by mean field limit $m(t)$
- Hard to predict outcome of 2 (except for reversible case)

Conclusion

- Mean field models are frequent in large scale systems
- Validity of approach is often simple by inspection
- Mean field is both
 - ▶ ODE for fluid limit
 - ▶ Fast simulation using decoupling assumption
- Decoupling assumption holds at finite horizon; may not hold in stationary regime.
- Stationary regime is more than stationary points, in general
 - (except for reversible case)

Thank You ...

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