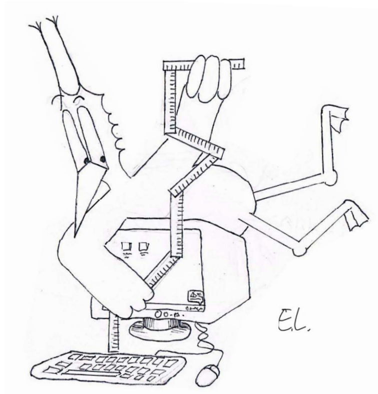
PERFORMANCE EVALUATION EXERCISES

METHODOLOGY

With Solutions Jean-Yves Le Boudec, August 23, 2022



1. A nuisance factor is . . .

   (a) ☐ An unanticipated experimental condition that corrupts the results

   (b) ☒ A condition in the system that affects the performance but that we are not interested in

   (c) ☐ An unpleasant part of the performance evaluation

2. A non-dominated metric means. . .

   (a) ☒ a metric vector for which no other vector is better

   (b) ☐ a metric value that is better than or equal to all others

   (c) ☐ a metric value that is better than all others

   (d) ☐ None of the above

3. In Borduria, there are two train companies: the incumbent operator (Sovrail) and a new company (Fastrack). The national bordurian consumer association developed an app and asked people to report whether trains are late or on time. Users also reported whether they travelled at the peak hour, off peak or during the week-end. Here are the results of the last campaign:

SOVRAIL	Late	On-time	Total (percent)
Peak hour	105397	105247	210644 (89.93)
Off peak	2805	6616	9421 (4.02)
Week-end	1369	12795	14164 (6.05)
Total (percent)	109571 (46.78)	124658 (53.22)	234229 (100.00)

FASTRACK	Late	On-Time	Total (percent)
Peak hour	182477	176468	358945 (56.68)
Off peak	64019	140990	205009 (32.37)
Week-end	7573	61794	69367 (10.95)
Total (percent)	254069 (40.12)	379252 (59.88)	633321 (100.00)

Which company performs better ? Is Simpson's paradox present here ?

Solution. If we look only at the last row of each table, we see that the proportion of users experiencing a late train is higher for Sovrail, which therefore seems to be worse than Fastrack. However, we can also observe that the period (Peak hour, Off-peak or Week-end) has an important impact, and is thus a nuisance factor. The distributions are given by the last column of every table; they are very different for both companies: more users of Sovrail took the train during the peak hour. The nuisance parameter is distributed differently for the two companies, so this may introduce a bias in the global score.

To see if there is a Simpson's inversion, let us compute the probability of being late given the period. We obtain:

$\mathbb{P}$(Late | Period) in percent		
Period	SOVRAIL	FASTRACK
Peak hour	50.04	50.84
Off-peak	29.77	31.23
Week-end	9.67	10.92
height		

The probabilities are similar, and better for Sovrail. There *is* an inversion due to Simpson's paradox. Users report Sovrail to be more often late, not because Sovrail trains are more often late but because Sovrail reporters are more often riding at the peak hour.

4. We measure the performance of a radio link as a function of the modulation rate. Day/night is a nuisance factor.

Plan A	Nb of experiments	day	night
	1 Mb/s	20	10
	11 Mb/s	30	15
	55 Mb/s	60	30

Plan B	Nb of experiments	day	night
	1 Mb/s	20	20
	11 Mb/s	20	20
	55 Mb/s	20	20

Which experimental plan is a proper randomization of the day/night factor ?

(a) ☐ A

(b) ☐ B

(c) ☒ both

(d) ☐ None

Solution. A proper randomization should be such that $\mathbb{P}(i|\text{day}) = \mathbb{P}(i|\text{night})$ for all i, which is true for both A and B.

5. The "scientific method" means . . .

(a) ☐ Carefully screen all experimental conditions

(b) ☐ Beware of hidden factors

(c) ☒ Do not draw a conclusion until you have exhausted all attempts to invalidate it

(d) ☐ None of the above

6. The random variables X and Y take integer values in $\{0, 1, ..., M\}$. Which of the following statements are equivalent to "X and Y are independent" ? (m, n are integers).

(a) ☒ $\mathbb{P}(X = m \text{ and } Y = n) = \mathbb{P}(X = m)\mathbb{P}(Y = n)$ for all m, n

(b) ☒ $\mathbb{P}(X = m|Y = n) = \mathbb{P}(X = m)$ for all m, n such that $\mathbb{P}(Y = n) \neq 0$

(c) ☒ $\mathbb{P}(Y = n|X = m) = \mathbb{P}(Y = n)$ for all m, n such that $\mathbb{P}(X = m) \neq 0$

(d) ☒ $\mathbb{P}(X = m|Y = n)$, when defined, is independent of n for all m

(e) ☒ $\mathbb{P}(Y = n|X = m)$, when defined, is independent of m for all n

(f) ☐ $\mathbb{P}(X = m|Y = n)$, when defined, is independent of m for all n

(g) ☐ $\mathbb{P}(Y = n|X = m)$, when defined, is independent of n for all m

Solution. (a) is the definition of independence.

$(a) \Rightarrow (b)$: for n such that $\mathbb{P}(Y = n) \neq 0$:

$$\mathbb{P}(X = m|Y = n) = \frac{\mathbb{P}(X = m \text{ and } Y = n)}{\mathbb{P}(Y = n)} = \frac{\mathbb{P}(X = m)\mathbb{P}(Y = n)}{\mathbb{P}(Y = n)} = \mathbb{P}(X = m) \quad (1)$$

$(b) \Rightarrow (a)$: for any m, n :

- if $\mathbb{P}(Y = n) = 0$ then $\mathbb{P}(X = m \text{ and } Y = n) \leq \mathbb{P}(Y = n) = 0$ thus (a) is true.
- else by (b) $\mathbb{P}(X = m \text{ and } Y = n) = \mathbb{P}(X = m|Y = n) \times \mathbb{P}(Y = n) = \mathbb{P}(X = m)\mathbb{P}(Y = n)$

Thus $(a) \Leftrightarrow (b)$ and similarly $(a) \Leftrightarrow (c)$

$(b) \Rightarrow (d)$ obviously.

$(d) \Rightarrow (b)$: $\mathbb{P}(X = m|Y = n)$ is independent of n, but may depend on m, let us call it $\varphi(m)$. We want to show that $\varphi(m) = \mathbb{P}(X = m)$. By the law of total probabilities:

$$\mathbb{P}(X = m) = \sum_n P(X = m|Y = n)P(Y = n) = \sum_n \varphi(m)P(Y = n) = \varphi(m)\sum_n P(Y = n) = \varphi(m)$$

Thus $(b) \Leftrightarrow (d)$ and similarly $(c) \Leftrightarrow (e)$.

(f) means that, given $Y = n$, X is uniformly distributed. Thus the distribution of X given $Y = n$ is always the same regardless of n . Thus $(f) \Rightarrow (a)$. But the converse is not true. If X and Y are independent and non uniform, then (f) is not true because then $\mathbb{P}(X = m|Y = n) = \mathbb{P}(X = m)$ which is not the same for all m because X is not uniform.

7. `genRandInt()` is a function that returns a random integer, distributed according to the Poisson distribution with mean m. Successive calls to this function produce independent results. A lazy performance analyst obtains a sequence of results as follows.

   - $X_1 \leftarrow$ `genRandInt()`
   - For $n \geq 2$, X_n is obtained as follows: flip a coin; if the result is TAIL then $X_n = X_{n-1}$ else $X_n \leftarrow$ `genRandInt()`

   Is the sequence X_n independent ?

   (a) ☒ No

   (b) ☐ Yes

   (c) ☐ It depends on m

Solution. Let us compute $\mathbb{P}(X_n = i)$ for any integer i. Let p_i be the probability that `genRandInt()` returns the value i.

Given that $X_1 = j$, what happens to X_2?

- if the coin flipping returns TAIL, X_2 is equal X_1, and is thus equal to j. This case occurs with probability 0.5
- if the coin flipping returns HEAD, X_2 is drawn by `genRandInt()`, which take any integer value and there is thus an infinite number of cases. The case where the value is i occurs with probability $0.5 \times p_i$

Thus

$$\begin{aligned}\mathbb{P}(X_2 = i | X_1 = j) &= 0.5p_i \text{ for } i \neq j \\ \mathbb{P}(X_2 = i | X_1 = i) &= 0.5 + 0.5p_i\end{aligned}$$

$\mathbb{P}(X_2 = i | X_1 = j)$ is dependent on j therefore X_1 and X_2 are not independent.

8. Is the sequence X_n in the previous question identically distributed ?

   (a) ☐ No
   (b) ☒ Yes
   (c) ☐ It depends on m

   Solution. Let us compute $\mathbb{P}(X_n = i)$ for any integer i. By construction, $\mathbb{P}(X_1 = i) = p_i$.

   From the solution of the previous question we have

$$\begin{aligned}\mathbb{P}(X_2 = i | X_1 = j) &= 0.5p_i \text{ for } i \neq j \\ \mathbb{P}(X_2 = i | X_1 = i) &= 0.5 + 0.5p_i\end{aligned}$$

   thus

$$\begin{aligned}\mathbb{P}(X_2 = i) &= \sum_{j\in\mathbb{N}} \mathbb{P}(X_2 = i | X_1 = j)\mathbb{P}(X_1 = j) && (2) \\ &= \sum_{j\in\mathbb{N}} \mathbb{P}(X_2 = i | X_1 = j)p_j && (3) \\ &= (0.5 + 0.5p_i)p_i + 0.5p_i \sum_{j\in\mathbb{N}, j\neq i} p_j && (4) \\ &= (0.5 + 0.5p_i)p_i + 0.5p_i(1 - p_i) = p_i = \mathbb{P}(X_1 = i) && (5)\end{aligned}$$

   Thus X_2 and X_1 have the same distribution. Since the construction of X_n from X_{n-1} is the same as the construction of X_2 from X_1, it follows that all X_n have the same distribution.

   Alternative proof: instead of (2) to (5) we can use the following derivation. Let F_2 be the result of the coin flipping performed to produce X_2.

$$\begin{aligned}\mathbb{P}(X_2 = i | F_2 = \text{HEAD}) &= p_i && (6) \\ \mathbb{P}(X_2 = i | F_2 = \text{TAIL}) &= \mathbb{P}(X_1 = i) = p_i && (7)\end{aligned}$$

   Thus $\mathbb{P}(X_2 = i) = p_i = \mathbb{P}(X_1 = i)$.

9. Can you find an instance of a pattern seen in class (or of another pattern) in a project that you were involved with ?