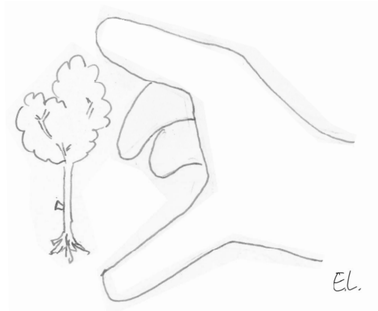
Performance Evaluation Exercises

Confidence Intervals

With Solutions Jean-Yves Le Boudec, August 23, 2022



1. **Probability Drill**

The following table gives the complementary CDF of the standard normal distribution.

z $\mathbb{P}(Z > z)$	0 0.5	1 0.1587	2 0.02275	3 0.001350	4 3.167E-05	5 2.867E-07
z $\mathbb{P}(Z > z)$	6 9.866E-10	7 1.280E-12	8 6.2216E-16	9 1.129E-19	10 7.620E-24	11 1.911E-28

(a) X is a gaussian random variable with mean μ and standard deviation σ. X measures a time and is in seconds. In which units are μ and σ ?

Solution. In seconds as well.

Which of the following random variables is standard gaussian (= standard normal) ?

i. ☐ $Z = \sigma X + \mu$

ii. ☒ $Z = \frac{X-\mu}{\sigma}$

iii. ☐ $Z = \frac{X}{\sigma} - \mu$

iv. ☐ None of these.

(b) $X_1, ...X_n$ are independent random variables with same expectation μ and same standard deviation σ. What are the expectation and the standard deviation of $\bar{X} = \frac{1}{n}(X_1 + ... + X_n)$?

Solution.

$$\mathbb{E}(\bar{X}) = \frac{1}{n}(\mathbb{E}(X_1) + ... + \mathbb{E}(X_n)) = \frac{1}{n}(\mu + ... + \mu) = \frac{1}{n}(n\mu) = \mu$$

The variance of λX is λ^2 the variance of X and the variance of the sum of independent random variables is the sum of the variances, hence

$$\text{var}(\bar{X}) = \frac{1}{n^2}(\text{var}(X_1) + ... + \text{var}(X_n)) = \frac{1}{n^2}(\sigma^2 + ... + \sigma^2) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$$

Thus the standard deviation of $\bar{X}$ is

$$\sigma_{\bar{X}} = \sqrt{\text{var}(\bar{X})} = \frac{1}{\sqrt{n}}\sigma$$

(c) Lisa models the result of an experiment that produces only positive numbers as a gaussian random variable X with mean $\mu = 10$ and standard deviation $\sigma = 1$. Bart observes that a gaussian distribution may take negative values and claims that Lisa should not use this model. Who is right ?

Solution. Bart is formally right, but practically, Lisa is also right. To see why, compute the probability that X is negative. It is given by the tables of the standard distribution. $Z = \frac{X-\mu}{\sigma}$ has a standard normal distribution. Furthermore

$$(X < 0) \Leftrightarrow \left(Z < \frac{-\mu}{\sigma}\right) \Leftrightarrow (Z < -10)$$

thus

$$\mathbb{P}\,(X < 0) = \mathbb{P}\,(Z < -10) = \mathbb{P}\,(Z > 10) = 1 - F(10) = 7.620 \cdot 10^{-24}$$

where $F()$ is the CDF of the standard normal distribution (and because the distribution of Z is symmetric around 0). This probability is astronomically small so we can accept Lisa's approach.

(d) The pdf of the gaussian distribution mean μ and standard deviation σ is $f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Give the values of

i. $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = 1$

ii. $\int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \mu$

iii. $\int_{-\infty}^{+\infty} \frac{x^2}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \mu^2 + \sigma^2$

Solution. If f_X is the pdf of a random variable X, we have: $\mathbb{E}(\varphi(X)) = \int_{-\infty}^{+\infty} \varphi(x) f_X(x)dx$ for any test function $\varphi()$. Thus expression i is $\mathbb{E}(1) = 1$. Expression ii is $\mathbb{E}(X) = \mu$. Expression iii is

$$\mathbb{E}\left((X)^2\right) = (\mathbb{E}(X))^2 + \text{var}(X) = \mu^2 + \sigma^2$$

(e) X_1 is a gaussian random variable with mean μ_1 and variance σ_1^2; X_2 is a gaussian random variable with mean μ_2 and variance σ_2^2; X_1 and X_2 are independent. What is the pdf of $X_1 + X_2$?

Solution. The sum of independent gaussian random variables is gaussian therefore $X_1 + X_2$ is gaussian. Its mean is $\mu_1 + \mu_2$; its variance is $\sigma^2 = \sigma_1^2 + \sigma_2^2$ (thus $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$). Its pdf is thus $f_{X+Y}(x) = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}e^{-\frac{(x-\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}}$.

(f) $X_1, X_2, ..., X_n$ are independent random variables with values in $\{0, 1\}$; X_i represents a success at the ith experiment, and $p = \mathbb{P}(X_i = 1)$ for all i. $N = X_1 + ... + X_n$ represents the total number of successes out of n experiments. What is the distribution of N? what is its mean and standard deviation ? For large n, can you approximate it with a continuous distribution ?

Solution. The distribution of N is (by definition) binomial with parameters n and p. The expectation and variance of X_i are

$$\begin{aligned}
\mathbb{E}(X_i) &= (1-p) \times 0 + p \times 1 = p \\
\text{var}(X_i) &= \mathbb{E}(X_i^2) - \mathbb{E}(X_i)^2 = \mathbb{E}(X_i) - \mathbb{E}(X_i)^2 = p - p^2 = p(1-p)
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}(N) &= np \\
\text{var}(N) &= np(1-p)
\end{aligned}$$

The support of N is the set of integers between 0 and n. When n is large, this can be approximated with a normal distribution (by the central limit theorem) with mean $\mu = np$ and variance $\sigma^2 = np(1-p)$.

2. Here are the results of 10 independent simulation runs (throughput in Mb/s):

   13.3 12.9 13.1 13.5 12.8 12.9 13.2 189.6 12.9 13.0

   Give a 95% confidence interval.

   (a) ☐ [12.9 ; 189.6]
   (b) ☒ [12.9 ; 13.5]
   (c) ☐ [13.0 ; 13.1]
   (d) ☐ [0 ; 200]
   (e) ☐ [12.9 ; 63.55]
   (f) ☐ [-2.11 ; 63.55]

   Solution. Method 1: CI for median : $[x_{(2)}, x_{(9)}] = [12.9; 13.5]$.

   Method 2a: CI for mean $m \pm 1.96\frac{s}{\sqrt{n}}$ using the large n asymptotic. Mean $m = 30.72$; Standard deviation $s = 52.96$ with $1/n$ formula and $\sigma = 55.83$ with $1/(n-1)$ formula. $CI = [-2.11; 63.55]$ or $[-3.88; 65.32]$ but Method 2a may be hard to justify with $n = 10$ (n is too too small).

   Method 2b: CI for mean $m \pm 2.262\frac{s}{\sqrt{n}}$ assuming the distribution is normal. Standard deviation $s = 55.83$ with $1/(n-1)$ formula. $CI = [-9.21; 70.65]$ but Method 2b may be hard to justify: the normal distribution is very unlikely to produce an outlier such as the 8th value.

3. We have tested a system for errors and found 0 error in 36 runs. Give a confidence interval for the probability of failure.

   (a) ☐ [0% ; 0%]
   (b) ☐ [0%; 1.74%]
   (c) ☒ [0% ; 9.74%]
   (d) ☐ [0% ; 33.74%]
   (e) ☐ [0 ; 100%]

   Solution. We can use the formula for a probability of success (since the successes are rare).

   CI is $[0; p_0(n)]$ with $p_0(n) = 1 - \left(\frac{1-\gamma}{2}\right)^{1/n} = 1 - \left(\frac{0{,}05}{2}\right)^{1/n} \approx 3.689/n$

   Exact formula gives $p_0(n) = 9.74\%$. Approximation gives $p_0(n) \approx 3.689/36 = 10.2\%$.

4. We expect ...

   (a) ☒ ... a 95% confidence interval to be narrower than a 99% confidence interval
   (b) ☐ ... a 95% confidence interval to be wider than a 99% confidence interval
   (c) ☐ It depends on the data
   (d) ☐ It depends on the type of confidence interval

   Solution. If we want to be more certain, we have to enlarge the confidence interval. Check this with the confidence interval for the median ($n = 37$):

   95%: $[x_{(13)}; x_{(25)}]$

   99%: $[x_{(11)}; x_{(27)}]$

   for the mean:

   95%: $m \pm 1.96s/\sqrt{n}$

   99%: $m \pm 2.58s/\sqrt{n}$

5. A data set $x_i > 0$ is such that $y_i = 1/x_i$ looks normal. A 95% confidence interval for the mean of y_i is $[L; U]$. Is it true that a confidence interval for the mean of x_i is $[1/U; 1/L]$?

   (a) ☐ Yes
   (b) ☒ No

   Solution. $Y_i = 1/X_i$ is a non linear transformation, the mean is not preserved: $m_X \neq 1/m_Y$ in general.

   $1/m_Y$ (i.e. the inverse of the mean of the inverses) is called the harmonic mean.

   Note that with probability 95%, $L \leq m_Y \leq U$ and thus with probability 95%: $1/U \leq 1/m_Y \leq 1/L$.

   $[1/U; 1/L]$ is a confidence interval for the harmonic mean of x_i.

6. A data set $x_i > 0$ is such that $y_i = 1/x_i$ looks normal. A 95% confidence interval for the median of y_i is $[L'; U']$. Is it true that a confidence interval for the median of x_i is $[1/U'; 1/L']$?

   (a) ☒ Yes
   (b) ☐ No

   Solution. $Y_i = 1/X_i$ is a monotone transformation, the order is reversed and thus the median is preserved: $\text{median}_X = 1/\text{median}_Y$

   Furthermore:

$$\left(L' \leq \text{median}_Y \leq U')\right) \quad \Leftrightarrow \quad \left(1/U' \leq 1/\text{median}_Y \leq 1/L'\right) \Leftrightarrow \left(1/U' \leq \text{median}_X \leq 1/L'\right) \tag{1}$$

   Thus $[1/U'; 1/L']$ is a confidence interval for the median of x_i.

7. We have obtained $n = 100$ independent measurements of response time. The values, in msec, sorted in increasing order, are:

   18.0062 18.7358 18.7527 19.3283 19.3893 19.6023 20.0612 20.1427 20.2037 20.4588
   ... (8 lines not shown) ...
   25.0155 25.0720 25.2769 25.2907 25.9579 26.1526 26.7172 26.7557 27.3556 27.8489

   The mean of the measurements is $m = 22.8705$ and the standard deviation is $s = 1.8840$. Give a prediction interval with confidence level 95%.

   Solution. We can use the formula based on order statistics. A 95% prediction interval is obtained a $[x_{(i)}, x_{(j)}]$ with $i = \lfloor 0.025(n+1) \rfloor = 2$ and $j = \lceil 0.975(n+1) \rceil = 99$, thus

$$I = [18.7358; 27.3556]$$

   If we could have some assurance that the data follows a normal distribution we could use the formula $m \pm 1.96s$ but since we do not have access to all the data we cannot make such an assumption.

8. The paper below is available on moodle (and elsewhere).

   Prytz, G. and Johannessen, S., 2005, September. Real-time performance measurements using UDP on Windows and Linux. In Emerging Technologies and Factory Automation, 2005. ETFA 2005. 10th IEEE Conference on (Vol. 2, pp. 8-pp). IEEE.

   Table 1 of this paper gives measurements with uncertainty bounds. For example, the first two rows give:

Time (μs)	Processor Load	Network Speed	
		100 Mbps	1 Gbps
UDP output stack time	Low	$15 \pm 2.3\mu s$	$16 \pm 1.9\mu s$
	High	$91 \pm 29\mu s$	$80 \pm 36\mu s$

Are these intervals correctly computed ? If so, what do they represent ?

Solution. The intervals are $m \pm 3\sigma$ where m is the mean of the measurement and σ the estimated standard deviation. There are $n = 25$ repeated, independent measurements. Each measurement is obtained by averaging a large number of experimental results, so it can reasonably be assumed to have a gaussian distribution. The obtained intervals are prediction intervals at confidence level 99.7%.

Compute confidence intervals at level 95% for the first two rows of table 1. Gives results in the same format as the table above.

Solution.

Confidence intervals at level 95% for the mean of the times are given by the formula for the normal case:

$$m \pm \eta \frac{\sigma}{\sqrt{25}} \tag{2}$$

with $n = 25$ and $\eta = 2.064$ ($97.5\%-$quantile of the student -24 distribution). We obtain μ as the central values in Table 1 and σ as $1/3$ the width of the given intervals. This gives finally:

This gives:

Time (μs)	Processor Load	Network Speed	
		100 Mbps	1 Gbps
UDP output stack time	Low	$15 \pm 0.32\mu s$	$16 \pm 0.26\mu s$
	High	$91 \pm 4.0\mu s$	$80 \pm 5.0\mu s$

9. We measure the round trip propagation time over a fiber infrastructure as follows. First we measure the delay Z over one fiber, then we measure the delay T over a second fiber; our result is $R = Z + T$. To obtain a good estimate of Z we perform m independent measurements and obtain $z_1, ..., z_m$; we also perform n independent measurements of T and obtain $t_1, ..., t_n$. Both m and n are very large and $m \neq n$.

   Assume that Z and T are random variables with finite mean and variance. Find a formula for a confidence interval for $\mu_R = \mu_Z + \mu_T$.

   (Hint: use inspiration from the slide "Idea of Proof" that follows the description of confidence interval for the mean, asymptotic case.)

   Solution. We model our measurements as being produced by sampling $Z_1, ..., Z_m$ and $T_1, ..., T_n$, where the Z_i [resp. T_i] are iid with mean μ_Z and variance σ_Z^2 [resp. μ_Z, σ_T^2]. We also assume that both sequences are independent. Let

$$\begin{aligned} \bar{Z} &= \frac{1}{m}\sum_{i=1}^{m} X_i,\ s_Z^2 = \frac{1}{m}\sum_{i=1}^{m}(Z_i - \bar{Z})^2 \\ \bar{T} &= \frac{1}{n}\sum_{i=1}^{n} Y_i,\ s_T^2 = \frac{1}{n}\sum_{i=1}^{m}(T_i - \bar{T})^2 \end{aligned}$$

   We estimate μ_R by $\bar{Z} + \bar{T}$. If m and n are large, $\bar{Z}$ is approximately gaussian with mean μ_Z and variance $\frac{1}{m}\sigma_Z^2$ and similarly for $\bar{T}$. Thus $\bar{Z} + \bar{T}$ is approximately gaussian with mean $\mu_Z + \mu_T$ and variance $\frac{1}{m}\sigma_Z^2 + \frac{1}{n}\sigma_T^2$ (since the Z and T measurements are independent, the variance of the sum is the sum of the variances). Thus

$$\frac{\bar{Z} + \bar{T} - (\mu_Z + \mu_T)}{\sqrt{\frac{1}{m}\sigma_Z^2 + \frac{1}{n}\sigma_T^2}} \text{ is approximately standard normal}$$

We approximate σ_Z^2 by s_Z^2 and σ_T^2 by s_T^2, thus

$$\frac{\bar{Z} + \bar{T} - (\mu_Z + \mu_T)}{\sqrt{\frac{1}{m}s_Z^2 + \frac{1}{n}s_T^2}} \text{ is approximately standard normal}$$

and with probability 95% it is in the interval $[-1.96; +1.96]$. An approximate 95% confidence interval for $\mu_Z + \mu_T$ is thus

$$\bar{Z} + \bar{T} \pm 1.96\sqrt{\frac{1}{m}s_Z^2 + \frac{1}{n}s_T^2}$$