

## Exercises : optimization problem classes

**Exercise 1** (Hyperbolic constraints as SOCP).

1. Show that, for all  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}$ ,  $z \in \mathbb{R}$ ,

$$x^\top x \leq yz, \quad y \geq 0, \quad z \geq 0$$

iff

$$\left\| \begin{pmatrix} 2x \\ y - z \end{pmatrix} \right\| \leq y + z \quad y \geq 0, \quad z \geq 0$$

2. Represent the following problem as an SOCP

$$\begin{aligned} \text{Max} \quad & \left( \sum_{i=1}^n 1/(a_i^\top x - b) \right)^{-1} \\ \text{s.t.} \quad & Ax > b \end{aligned}$$

**Exercise 2.** We consider a physical function  $\Phi$  that is approximated as the superposition of multiple simple phenomenon (e.g. waves). Each simple phenomenon  $p \in [P]$  is represented by a function  $\Phi_p : \mathbb{R}^d \rightarrow \mathbb{R}$ .

We have data points  $(x^k, y^k)_{k \in [n]}$ , and want to find the  $\Phi$  that match at best the data while being a linear combination of  $\Phi_p$ .

Propose a least-square regression that answer this question.

**Exercise 3.** Consider a chocolate manufacturing company that produces only two types of chocolate – A and B. Both the chocolates require Milk and Choco only. To manufacture each unit of A and B, the following quantities are required:

- Each unit of A requires 1 unit of Milk and 3 units of Choco
- Each unit of B requires 1 unit of Milk and 2 units of Choco

The company kitchen has a total of 5 units of Milk and 12 units of Choco. On each sale, the company makes a profit of

- 6 per unit A sold

- 5 per unit B sold.

Model this as an LP.

**Exercise 4.** A classical extension of the least-square problem, which has strong theoretical and practical interest is the LASSO problem

$$\text{Min}_{x \in \mathbb{R}^p} \quad \|Ax - y\|^2 + \lambda \|x\|_1$$

Show that this problem can be cast as a QP problem.

**Exercise 5.** Consider the following optimization problem.

$$\begin{aligned} \text{Min}_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x_i \in \{0, 1\} \quad \forall i \in I \end{aligned}$$

Write this problem as a QCQP. Is it convex ?

**Exercise 6.** Consider a facility that plan to deliver product to clients by drone (thus in direct line). Assume that you have  $N$  clients, each with position (in  $\mathbb{R}^2$ )  $x_n$ . The drone make each time a direct travel from the facility location to the client. Assume that the drone have a maximum range of  $R$ , and that you want to minimize the average travel distance while being able to serve all of your clients.

Model the problem of choosing the facility location as an SOCP.

**Exercise 7.** Consider the following robust linear program

$$\begin{aligned} \text{Min}_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & (a_i + R_i \delta_i)^\top x \leq b_i \quad \forall \|\delta_i\|_2 \leq 1, \quad \forall i \in [m] \end{aligned}$$

Write this problem as an SOCP.

**Exercise 8.** Let  $F(\theta)$  be a symmetric matrix parametrized by  $\theta \in \mathbb{R}^d$  whose coefficients are linear in  $\theta$ . Model the problem of finding the parameter  $\theta \in \Theta$ , where  $\Theta$  is a polyhedron, minimizing  $\kappa(\theta)$  as an SDP.

What happen if the coefficient of  $F(\theta)$  are affine in  $\theta$  ? Suggest a solution method ? (hard)

**Exercise 9.** Consider a finite set  $X = \{x_i\}_{i \in [n]}$ , and  $\mathcal{P}^+$  the set of probabilities on  $X$ . For  $\mathbb{P}, \mathbb{Q} \in \mathcal{P}$ , with  $\text{supp}(\mathbb{Q}) = X$ , we define the Kullback-Leibler divergence as

$$d_{KL}(\mathbb{P}|\mathbb{Q}) = \sum_{i=1}^n p_i \ln(p_i/q_i)$$

where  $p_i = \mathbb{P}(X = x_i)$  and  $q_i = \mathbb{Q}(X = x_i)$ .

Let  $X$  be 100 equidistant points spanning in  $[-1, 1]$ . Let  $\mathbb{Q}$  be uniform on  $X$ .

We are looking for the probability  $\mathbb{P}$  on  $X$  such that

- $\mathbb{E}_{\mathbb{P}}[\mathbf{X}] \in [-0.1, 0.1]$
- $\mathbb{E}_{\mathbb{P}}[\mathbf{X}^2] \in [0.5, 0.6]$
- $\mathbb{E}_{\mathbb{P}}[3\mathbf{X}^2 - 2\mathbf{X}] \in [-0.3, -0.2]$
- $\mathbb{P}(\mathbf{X} < 0) \in [0.3, 0.4]$

that minimize the Kullback-Leibler divergence from  $\mathbb{Q}$ .

Model this problem as an optimization problem. In which class does it belongs ?

**Exercise 10.** Consider that you sell a given product over  $T$  days. The demand for each day is  $d_t$ . Having a quantity  $x_t$  of items in stock have a cost (per day) of  $cx_t$ . You can order, each day, a quantity  $q_t$ , and have to satisfy the demand.

For each of the following variation : model the problem, explicit the class to which it belongs, and give the optimal solution if easily found.

1. Without any further constraint / specifications.
2. There is an "ordering cost": each time you order, you have to pay a fix cost  $\kappa$ .
3. Instead of an "ordering cost" there is a maximum number of days at which you can order a replenishment.