Robust Optimization : A tutorial

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An optimization problem

A generic optimization problem can be written

$$\begin{aligned} \min_x \quad & L(x) \\ s.t. \quad & g(x) \leq 0 \end{aligned}$$

where

- x is the decision variable
- L is the objective function
- g is the constraint function

An optimization problem with uncertainty

Adding uncertainty ξ in the mix

$$\begin{aligned} \min_x \quad & L(x, \tilde{\xi}) \\ s.t. \quad & g(x, \tilde{\xi}) \leq 0 \end{aligned}$$

Remarks:

- $\tilde{\xi}$ is unknown. Two main way of modelling it:
  - $\tilde{\xi} \in R$ with a known uncertainty set R, and a pessimistic approach. This is the robust optimization approach (RO).
  - $\tilde{\xi}$ is a random variable with known probability law. This is the Stochastic Programming approach (SP).
- Cost is not well defined.
  - RO : $\max_{\xi \in R} L(x, \xi)$.
  - SP : $\mathbb{E}\big[L(x, \boldsymbol{\xi})\big]$.
- Constraints are not well defined.
  - RO : $g(x, \xi) \leq 0, \qquad \forall \xi \in R$.
  - SP : $g(x, \boldsymbol{\xi}) \leq 0, \qquad \mathbb{P} - a.s.$.

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Requirements and limits

- Stochastic optimization :
  - requires a law of the uncertainty ξ
  - can be hard to solve (generally require discretizing the support and blowing up the dimension of the problem)
  - there exists specific methods (like Bender's decomposition)
- Robust optimization :
  - requires an uncertainty set R
  - can be overly conservative, even for reasonable R
  - complexity strongly depend on the choice of R
- Distributionally robust optimization :
  - is a mix between robust and stochastic optimization
  - consists in solving a stochastic optimization problem where the law is chosen in a robust way
  - is a fast growing fields with multiple recent results
  - but is still hard to implement than other approaches

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Some numerical tests on real-life LPs

From Ben-Tal and Nemirovski

- take LP from Netlib library
- look at non-integer coefficients, assuming that they are not known with perfect certainty
- What happens if you change them by 0.1% ?
  - constraints can be violated by up to 450%
  - $\mathbb{P}(\text{violation} > 0) = 0.5$
  - $\mathbb{P}(\text{violation} > 150\%) = 0.18$
  - $\mathbb{E}[\text{violation}] = 125\%$

What do you want from robust optimization ?

- finding a solution that is less sensible to modified data, without a great increase of price
- choosing an uncertainty set R that:
  - offer robustness guarantee
  - yield an easily solved optimization problem

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Solving a robust optimization problem

The robust optimization problem we want to solve is[1]

$$\begin{aligned} \min_x \quad & L(x) \\ s.t. \quad & g(x,\xi) \leq 0 \qquad \forall \xi \in R \end{aligned}$$

Two main approaches are possible:

Constraint generation: replace R by a finite set of ξ, that is we replace an "infinite number of contraints" by a finite number of them.

Reformulation: replace $g(x,\xi) \leq 0 \quad \forall \xi \in R$, by $\sup_{\xi \in R} g(x,\xi) \leq 0$, then explicit the sup.

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Constraint generation algorithm

Data: Problem parameters, reference uncertainty ξ_0
Result: approximate value with gap;
for $k \in \mathbb{N}$ **do**

solve $\tilde{v} = \min_{x} \left\{ L(x) \mid g(x, \xi_\kappa) \; \forall \kappa \leq k \right\} \quad \rightsquigarrow x_k$;

solve $s = \max_{\xi \in R} g(x_k, \xi) \quad \rightsquigarrow \xi_{k+1}$;

if $s \leq 0$ **then**

Robust optimization problem solved,
with value $\tilde{v}$ and optimal solution x_k

Algorithm 1: Constraint Generation Algorithm

Note that we are solving a problem similar to the deterministic problem with an increasing number of constraints.

This is easy to implement and can be numerically efficient.

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Reformulation principle

We can write the robust optimization problem as

$$\begin{array}{ll} \min_x & L(x) \\ s.t. & \sup_{\xi \in R} g(x, \xi) \leq 0 \end{array}$$

Now, there are two ways of simplifying this problem :

- we can explicitly compute $\bar{g}(x) = \sup_{\xi \in R} g(x, \xi)$;
- by duality we can write $\sup_{\xi \in R} g(x, \xi) = \min_{\eta \in Q} h(x, \eta)$

➡ $\min_{\eta \in Q} h(x, \eta) \leq 0$ is equivalent to $\exists \eta$ such that $h(x, \eta) \leq 0$, i.e. just add η as a variable in your optimization problem

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Canonization of the problem I

We consider

$$\min_{x\geq 0} \max_{(A,b,c)\in R} \quad c^\top x$$
$$s.t. \quad Ax \leq b$$

Without loss of generality we can consider a deterministic cost:

$$\begin{aligned} \min_{x\geq 0,\theta} \quad & \theta \\ s.t. \quad & Ax \leq b && \forall (A,b,c) \in R \\ & c^\top x \leq \theta && \forall (A,b,c) \in R \end{aligned}$$

That can be written as

$$\begin{aligned} \min_{x\geq 0,\theta} \quad & \theta \\ s.t. \quad & a_i{}^\top x - b_i \leq 0 && \forall (A,b,c) \in R, \forall i \in [m] \\ & c^\top x - \theta \leq 0 && \forall (A,b,c) \in R \end{aligned}$$

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That can be written as

$$\min_{x \geq 0, \theta} \quad \theta$$
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Canonization of the problem II

We now consider

$$\begin{aligned} \min_{x\geq 0} \quad & c^\top x \\ s.t. \quad & a_i^\top x - b_i \leq 0 \qquad \forall (A,b) \in R, \forall i \in [m] \end{aligned}$$

Let R_i be the projection of R onto coordinate i.
We have in particular $R \subset R_1 \times \cdots \times R_m$.
But note that, in the robust constraint, R can be replaced by $R_1 \times \cdots \times R_m$, indeed,

$$\begin{aligned} & f_i(x,\xi_i) \leq 0, \quad \forall i \in [m], \quad \forall \xi \in R \\ \iff \quad & f_i(x,\xi_i) \leq 0, \quad \forall i \in [m], \forall \xi \in R_1 \times \cdots \times R_m \\ \iff \quad & f_i(x,\xi_i) \leq 0, \quad \forall \xi_i, \in R_i \quad \forall i \in [m] \end{aligned}$$

Canonization of the problem II

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Canonization of the problem II

We now consider

$$\begin{aligned} \min_{x \geq 0} \quad & c^\top x \\ s.t. \quad & {a_i}^\top x - b_i \leq 0 \qquad\qquad \forall (A, b) \in R, \forall i \in [m] \end{aligned}$$

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To model correlation we set

$$a = \bar{a} + P\zeta \qquad b = \bar{b} + p^\top \zeta$$

where $(\bar{a}, \bar{b})$ are the nominal value, and ζ is the primitive/residual uncertainty.

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The robust constraint now reads

$$(\bar{a}^\top x - \bar{b}) + (P^\top x - p)^\top \zeta \leq 0 \qquad \forall \zeta \in \mathcal{Z}$$

Canonization of the problem IV

Example: assume that a is a random variable with mean $\bar{a}$ and covariance Σ. Then, a natural reformulation would be

$$a = \bar{a} + \Sigma^{1/2}\zeta,$$

so that ζ is centered with uncorrelated coordinates.

Finally, w.l.o.g. we assume that b is deterministic (can be obtained by adding a variable x_{n+1} constrained to be equal to 1).

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An explicit worst case value

- We consider an ellipsoidal uncertainty set

$$R = \Big\{ \xi = \{\bar{a} + P\zeta\}_i \quad | \quad \|\zeta\|_2 \leq \rho \Big\}$$

- Here we can, for a given x, explicitly compute

$$\sup_{\xi \in R} \xi^\top x = \bar{a}^\top x + \sup_{\|\zeta\|_2 \leq \rho} (P\zeta)^\top x$$
$$= \bar{a}^\top x + \rho \|P^\top x\|_2$$

- Hence, constraint

$$\sup_{\xi \in R} \xi^\top x \leq b$$

can be written

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SOCP problem

- An Second Order Cone Programming constraint is a constraint of the form

$$\|Ax + b\|_2 \leq c^\top x + d$$

- An SOCP problem is a (continuous) optimization problem with linear cost and linear and SOCP constraints
- There exists powerful software to solve SOCP (e.g. CPLEX, Gurobi, MOSEK...) with dedicated interior points methods
- There exist a duality theory akin to the LP duality theory
- If a robust optimization problem can be cast as an SOCP the formulation is deemed efficient

Contents

Linear duality : recalls

- Recall that, if finite,

$$\begin{aligned} \max_{\xi} \quad & \xi^\top x \\ s.t. \quad & D\xi \leq d \end{aligned}$$

as the same value as

$$\begin{aligned} \min_{\eta} \quad & \eta^\top d \\ s.t. \quad & \eta^\top D = x \\ & \eta \geq 0 \end{aligned}$$

- Thus,

$$\begin{aligned} \sup_{\xi : D\xi \leq d} \; \xi^\top x \leq b \quad & \iff \quad \min_{\eta \geq 0 : \eta^\top D = x} \; \eta^\top d \leq b \\ & \iff \quad \exists \eta \geq 0, \quad \eta^\top D = x, \quad \eta^\top d \leq b \end{aligned}$$

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Polyhedral uncertainty

- We consider a polyhedral uncertainty set

$$R = \Big\{\xi \quad | \quad D\xi \leq d\Big\}$$

- Then the robust optimization problem

$$\begin{aligned} \min_{x \geq 0} \quad & c^\top x \\ s.t. \quad & \sup_{\xi \in R} \xi^\top x \leq h \end{aligned}$$

reads

$$\begin{aligned} \min_{x \geq 0, \eta \geq 0} \quad & c^\top x \\ s.t. \quad & \eta^\top d \leq h \\ & \eta^\top d = x \end{aligned}$$

Contents

Soyster model

The problem

$$\min_{x} \quad c^\top x$$

$$\tilde{A}x \leq b \qquad\qquad \forall \tilde{A} \in R$$

$$\underline{x} \leq x \leq \bar{x}$$

where each coefficient $\tilde{a}_{ij} \in [\bar{a}_{ij} - \delta_{ij}, \bar{a}_{ij} + \delta_{ij}]$

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where each coefficient $\tilde{a}_{ij} \in [\bar{a}_{ij} - \delta_{ij}, \bar{a}_{ij} + \delta_{ij}]$ can be written

$$\begin{aligned} \min_{x} \quad & c^\top x \\ & \sum_j \bar{a}_{ij} x_j + \sum_j \delta_{ij} |x_j| \leq b_i \qquad \forall i \\ & \underline{x} \leq x \leq \bar{x} \end{aligned}$$

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can be written

$$\begin{aligned} \min_x \quad & c^\top x \\ & \sum_j \bar{a}_{ij} x_j + \sum_j \delta_{ij} y_j \leq b_i && \forall i \\ & \underline{x} \leq x \leq \bar{x} \\ & y_j \geq x_j, \quad y_j \geq -x_j \end{aligned}$$

Cardinality constrained LP I

Soyster's model is over conservative, we want to consider a model where only Γ_i coefficient per line have non-zero errors, leading to

$$\min_{x,y} \quad c^\top x$$

$$\sum_j \bar{a}_{ij} x_j + \max_{S_i : |S_i| = \Gamma_i} \sum_{j \in S_i} \delta_{ij} y_j \leq b_i \qquad \forall i$$

$$\underline{x} \leq x \leq \bar{x}$$

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$$
\begin{aligned}
\min_{x,y} \quad & c^\top x \\
& \sum_j \bar{a}_{ij} x_j + \beta_i \leq b_i & \forall i \\
& \max_{S_i : |S_i| = \Gamma_i} \sum_{j \in S_i} \delta_{ij} y_j \leq \beta_i \\
& \underline{x} \leq x \leq \bar{x} \\
& y_j \geq x_j, \quad y_j \geq -x_j
\end{aligned}
$$

Cardinality constrained LP II

This means that, for line i we take a margin of

$$\beta_i(x, \Gamma_i) := \max_{S_i : |S_i| = \Gamma_i} \sum_{j \in S_i} \delta_{ij} |x_j|$$

which can be obtained as

$$\begin{aligned} \beta_i(x, \Gamma_i) = \max_{z \geq 0} \quad & \sum_j \delta_{ij} |x_j| z_{ij} & \\ & \sum_j z_{ij} \leq \Gamma_i & [\lambda_i] \\ & z_{ij} \leq 1 & [\mu_{ij}] \end{aligned}$$

This LP can be then dualized to be integrated in the original LP.

Cardinality constrained LP III

$$
\begin{aligned}
\beta_i(x, \Gamma_i) = \max_{z \geq 0} \quad & \sum_j \delta_{ij} |x_j| z_{ij} & \\
& \sum_j z_{ij} \leq \Gamma_i & [\lambda_i] \\
& z_{ij} \leq 1 & [\mu_{ij}]
\end{aligned}
$$

Cardinality constrained LP III

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& \sum_j z_{ij} \leq \Gamma_i && [\lambda_i] \\
& z_{ij} \leq 1 && [\mu_{ij}]
\end{aligned}
$$

$$
\begin{aligned}
\beta_i(x,\Gamma_i) = \max_{z\geq 0} \min_{\lambda,\mu\geq 0} \quad & \sum_j \delta_{ij}|x_j| z_{ij} + \lambda_i\Big(\Gamma_i - \sum_j z_{ij}\Big) \\
& + \sum_j \mu_{ij}\Big(1 - z_{ij}\Big)
\end{aligned}
$$

Cardinality constrained LP III

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Cardinality constrained LP III

$$\beta_i(x, \Gamma_i) = \max_{z \geq 0} \quad \sum_j \delta_{ij} |x_j| z_{ij}$$

$$\sum_j z_{ij} \leq \Gamma_i \qquad [\lambda_i]$$

$$z_{ij} \leq 1 \qquad [\mu_{ij}]$$

$$\begin{aligned}
\beta_i(x, \Gamma_i) = \max_{z \geq 0} \min_{\lambda, \mu \geq 0} \quad & \sum_j \delta_{ij} |x_j| z_{ij} + \lambda_i \Big(\Gamma_i - \sum_j z_{ij} \Big) \\
& + \sum_j \mu_{ij} \Big(1 - z_{ij} \Big) \\
= \min_{\lambda, \mu \geq 0} \quad & \lambda_i \Gamma_i + \sum_j \mu_{ij} \\
\text{s.t.} \quad & \delta_{ij} |x_j| \leq \lambda_i + \mu_{ij}
\end{aligned}$$

Cardinality constrained LP IV

In the end we obtain

$$
\begin{aligned}
\min_{x,\beta,\lambda,\mu} \quad & c^\top x \\
& \sum_j \bar{a}_{ij} x_j + \beta_i \leq b_i & \forall i \\
& \lambda_i \Gamma_i + \sum_j \mu_{ij} \leq \beta_i & \forall i \\
& \delta_{ij} x_j \leq \lambda_i + \mu_{ij} & \forall i,j \\
& -\delta_{ij} x_j \leq \lambda_i + \mu_{ij} & \forall i,j \\
& \lambda \geq 0, \quad \mu \geq 0 \\
& \underline{x} \leq x \leq \bar{x}
\end{aligned}
$$

Contents

Setting

- Objective: assign facilities to satisfy demand
- horizon T, $i \in [F]$ candidate facility location, $j \in [N]$ customer demands
- η unit produce price
- At location i: c_i unit production cost, C_i unit capacity price, K_i opening cost
- $d_{i,j}$ shipping cost
- $D_{j,\tau}$ demand at location j at time τ
- $x_{i,j,\tau} \in [0,1]$ proportion of demand j satisfied by i at time τ
- $P_{i,\tau} \in \mathbb{R}^+$ amount of good produced
- $I_i \in \{0,1\}$ boolean of opening i
- Z_i capacity at i

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Nominal formulation

$$\max \quad \underbrace{\sum_{\tau\in[T],i\in[F],j\in[N]} (\eta - d_{i,j})x_{i,j,\tau}D_{j,\tau}}_{\text{income - transportation}} - \underbrace{\sum_{\tau\in[T],i\in[F]} c_i P_{i,\tau}}_{\text{prod. cost}}$$

$$- \underbrace{\sum_{i\in[F]} C_i Z_i}_{\text{capa. cost}} - \underbrace{\sum_{i\in[F]} K_i I_i}_{\text{opening cost}}$$

$$\text{s.t.} \quad \sum_{i\in[F]} x_{i,j,\tau} \leq 1 \qquad \forall j,\tau$$

$$\sum_{j\in[N]} x_{i,j,\tau}D_{j,\tau} \leq P_{i,\tau} \qquad \forall i,\tau$$

$$x_{i,j,\tau} \geq 0 \qquad \forall i,j,\tau$$

$$P_{i,\tau} \leq Z_i, \quad Z_i \leq M I_i, \qquad \forall i,\tau$$

$$I_i \in \{0,1\} \qquad \forall i$$

Uncertainty

We assume that the demand $D_{j,\tau}$ are unknown. We consider

$$R = \left\{ D \;\middle|\; \sum_{j\in[N],\tau\in[T]} \left(\frac{D_{j,\tau} - \bar{D}_{j,\tau}}{\varepsilon_\tau \bar{D}_{j,\tau}}\right)^2 \leq \rho^2 \right\}$$

where

- $\bar{D}_{j,\tau}$ is the nominal demand;
- ε_t is related to demand variability;
- ρ is a robustness parameter.

Robust formulation

First step: identifying uncertainty

$$
\begin{aligned}
\max \quad & \theta \\
\text{s.t.} \quad & \sum_{\tau\in[T],i\in[F],j\in[N]} (\eta - d_{i,j}) x_{i,j,\tau} D_{j,\tau} - \sum_{\tau\in[T],i\in[F]} c_i P_{i,\tau} \\
& \qquad - \sum_{i\in[F]} C_i Z_i - \sum_{i\in[F]} K_i I_i \geq \theta && \forall D \in R \\
& \sum_{i\in[F]} x_{i,j,\tau} \leq 1 && \forall j, \tau \\
& \sum_{j\in[N]} x_{i,j,\tau} D_{j,\tau} \leq P_{i,\tau} && \forall i, \tau, \quad \forall D \in R \\
& x_{i,j,\tau} \geq 0 && \forall i, j, \tau \\
& P_{i,\tau} \leq Z_i, \quad Z_i \leq M I_i, && \forall i, \tau \\
& I_i \in \{0,1\} && \forall i
\end{aligned}
$$

Normalization

Second step: normalize (and decorrelate) demand.

$$\zeta_{j,\tau} = \frac{D_{j,\tau} - \bar{D}_{j,\tau}}{\varepsilon_\tau \bar{D}_{j,\tau}}$$

So that $D \in R$ iff $\zeta \in \mathcal{Z} := \{\zeta \mid \|\zeta\|_2 \leq \rho\}$.

Thus, the "incomes-transportation cost" becomes

$$\sum_{\tau \in [T], i \in [F], j \in [N]} (\eta - d_{i,j}) x_{i,j,\tau} \bar{D}_{j,\tau} + \sum_{\tau \in [T], i \in [F], j \in [N]} (\eta - d_{i,j}) x_{i,j,\tau} \varepsilon_\tau \bar{D}_{j,\tau} \zeta_{j,\tau},$$

and the production capacity constraint reads

$$\sum_{j \in [N]} x_{i,j,\tau} \bar{D}_{j,\tau} + \sum_{j \in [N]} x_{i,j,\tau} \varepsilon_\tau \bar{D}_{j,\tau} \zeta_{j,\tau} \leq P_{i,\tau} \qquad \forall i, \tau.$$

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Collecting uncertainty coefficients and robust reformulation

We collect the coefficient of ζ in the cost:

$$Q_{j,\tau}(x) := -\sum_{i\in[F]} (\eta - d_{i,j}) x_{i,j,\tau} \varepsilon_\tau \bar{D}_{j,\tau}$$

so the "incomes-transportation cost" becomes

$$\sum_{\tau\in[T],i\in[F],j\in[N]} (\eta - d_{i,j}) x_{i,j,\tau} \bar{D}_{j,\tau} - \underbrace{\sup_{\zeta\in\mathcal{Z}} Q(x)^\top \zeta}_{=\rho\|Q(x)\|_2}.$$

Similarly, the production capacity constraint is reformulated as

$$\sum_{j\in[N]} x_{i,j,\tau} \bar{D}_{j,\tau} + \rho\|V_{i,\tau}\|_2 \leq P_{i,\tau} \qquad \forall i, \tau$$

where $V_{i,\tau,j} := \varepsilon_\tau x_{i,j,\tau} \bar{D}_{j,\tau}$.

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where $V_{i,\tau,j} := \varepsilon_\tau x_{i,j,\tau}\bar{D}_{j,\tau}$.

Global robust formulation as a MISOCP

$$\max_{x\geq 0, P, Z, I\in\{0,1\}} \quad \sum_{\tau\in[T], i\in[F], j\in[N]} (\eta - d_{i,j}) x_{i,j,\tau} \bar{D}_{j,\tau} - \rho \|Q(x)\|_2$$

$$\text{s.t.} \quad - \sum_{\tau\in[T], i\in[F]} c_i P_{i,\tau} - \sum_{i\in[F]} C_i Z_i - \sum_{i\in[F]} K_i I_i \geq \theta$$

$$Q_{j,\tau}(x) = - \sum_{i\in[F]} (\eta - d_{i,j}) x_{i,j,\tau} \varepsilon_\tau \bar{D}_{j,\tau} \qquad \forall j, \tau$$

$$\sum_{i\in[F]} x_{i,j,\tau} \leq 1 \qquad \forall j, \tau$$

$$\sum_{j\in[N]} x_{i,j,\tau} \bar{D}_{j,\tau} + \rho \|V_{i,\tau}(x)\|_2 \leq P_{i,\tau} \qquad \forall i, \tau$$

$$V_{i,\tau,j}(x) = \varepsilon_\tau x_{i,j,\tau} \bar{D}_{j,\tau} \forall i, \tau, j$$

$$P_{i,\tau} \leq Z_i, \quad Z_i \leq M I_i, \qquad \forall i, \tau$$

Contents

A combinatorial optimization problem with cardinality constraint

We consider a combinatorial optimization problem:

$$\min_{x \in \{0,1\}^N} \quad \max_{\tilde{c} \in R} \tilde{c}^\top x$$
$$s.t. \quad x \in X$$

where R is such that each $\tilde{c}_i \in [\bar{c}_i, \bar{c}_i + \delta_i]$, with at most Γ coefficient deviating from $\bar{c}_i$.

Thus, the problem reads

$$(P) \qquad \min_{x \in \{0,1\}^N} \quad \bar{c}^\top x + \max_{|S| \leq \Gamma} \sum_{i \in S} \delta_i x_i$$
$$s.t. \quad x \in X$$

wlog we assume that the i are ordered by decreasing cost uncertainty span : $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$.

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$$s.t. \quad x \in X$$

wlog we assume that the i are ordered by decreasing cost uncertainty span : $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$.

Solving the robust combinatorial problem I

We can write (P) as

$$\begin{aligned} \min_{x\in\{0,1\}^N} \max_{u\in[0,1]^n} \quad & \bar{c}^\top x + \sum_{i=1}^n \delta_i x_i \zeta \\ s.t. \quad & x \in X \\ & \sum_{i=1}^n \zeta \leq \Gamma \end{aligned}$$

For a given $x \in X$ we dualize the inner maximization LP problem

Solving the robust combinatorial problem II

Thus we can write (P) as

$$\begin{aligned}
\min_{x,y,\theta} \quad & \bar{c}^\top x + \Gamma\theta + \sum_{j=1}^{n} y_j \\
s.t. \quad & x \in X \\
& y_j + \theta \geq \delta_j x_j \\
& y_j, \theta \geq 0
\end{aligned}$$

Note that an optimal solution satisfies

$$y_j = (\delta_j x_j - \theta)^+ = (\delta_j - \theta)^+ x_j$$

as $x_j \in \{0,1\}$, and $\theta \geq 0$.

Solving the robust combinatorial problem II

Thus we can write (P) as

$$\begin{aligned} \min_{x,y,\theta} \quad & \bar{c}^\top x + \Gamma\theta + \sum_{j=1}^{n} y_j \\ s.t. \quad & x \in X \\ & y_j + \theta \geq \delta_j x_j \\ & y_j, \theta \geq 0 \end{aligned}$$

Note that an optimal solution satisfies

$$y_j = (\delta_j x_j - \theta)^+ = (\delta_j - \theta)^+ x_j$$

as $x_j \in \{0, 1\}$, and $\theta \geq 0$.

Solving the robust combinatorial problem III

Thus we can write (P) as

$$\min_{\theta \geq 0} \min_{x} \quad \bar{c}^\top x + \Gamma\theta + \sum_{j=1}^{n} x_j(\delta_j - \theta)^+$$
$$s.t. \quad x \in X$$

We can now decompose the problem for $\theta \in [\delta_\ell, \delta_{\ell-1}]$ where $\delta_{n+1} = 0$ and $\delta_0 = +\infty$.
Therefore, we have

$$val(P) = \min_{\ell \in [n]} Z^\ell$$

where

$$Z^\ell = \min_{x \in X, \theta \in [\delta_\ell, \delta_{\ell-1}]} \quad \bar{c}^\top x + \Gamma\theta + \sum_{j=1}^{\ell-1} x_j(\delta_j - \theta)$$

Solving the robust combinatorial problem III

Thus we can write (P) as

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Solving the robust combinatorial problem III

Thus we can write (P) as

$$\begin{aligned} \min_{\theta\geq 0}\min_{x} \quad & \bar{c}^\top x + \Gamma\theta + \sum_{j=1}^{n} x_j(\delta_j - \theta)^+ \\ s.t. \quad & x \in X \end{aligned}$$

We can now decompose the problem for $\theta \in [\delta_\ell, \delta_{\ell-1}]$ where $\delta_{n+1} = 0$ and $\delta_0 = +\infty$.
Therefore, we have

$$val(P) = \min_{\ell\in[n]} Z^\ell$$

where

$$Z^\ell = \min_{x\in X, \theta\in[\delta_\ell, \delta_{\ell-1}]} \quad \bar{c}^\top x + \Gamma\theta + \sum_{j=1}^{\ell-1} x_j(\delta_j - \theta)$$

Solving the robust combinatorial problem IV

As the problem is linear in θ we have that

$$Z^\ell = \min_{x \in X, \theta \in [\delta_\ell, \delta_{\ell-1}]} \quad \bar{c}^\top x + \Gamma\theta + \sum_{j=1}^{\ell-1} x_j(\delta_j - \theta)$$

is attained for $\theta = \delta_\ell$ or $\theta = \delta_{\ell-1}$.
So in the end, we have

$$val(P) = \min_{\ell \in [n]} G^\ell$$

where

$$G^\ell = \Gamma\delta_\ell + \min_{x \in X} \Big\{ \bar{c}^\top x + \sum_{j=1}^{\ell} \underbrace{(\delta_j - \delta_\ell)}_{\geq 0} x_j \Big\}$$

Algorithm for the robust problem

1. For $\ell \in [n]$, solve
$$G^\ell = \Gamma\delta_\ell + \min_{x\in X} \quad \Big\{\bar{c}^\top x + \sum_{i=1}^{\ell}(\delta_i - \delta_\ell)x_j\Big\}$$
with optimal solution x_ℓ
2. Set $\ell^* \in \arg\min_{\ell\in[n]} G^\ell$
3. Return $val(P) = G^{\ell^*}$ and $x^* = x_\ell$

Contents

Why do robust optimization ?

- Because you want to account for some uncertainty
- Because you want to have a solution that resists to changes in data
- Because your data is unprecise and robustness yield better out-of-sample result
- Because you do not have the law of the uncertainty
- Because you can control the robustness level
- Because your problem is "one-shot"

Which uncertainty set to choose ?

- An uncertainty set that is computationally tractable
- An uncertainty set that yields good results
- An uncertainty set that have some theoretical soundness
- An uncertainty set that take available data into account
- Select uncertainty set / level through cross-validation

Is there some theoretical results ?

- Yes: with some assumption over the randomness (e.g. bounded and symmetric around $\bar{a}$) some uncertainty set (e.g. ellipsoidal) have a probabilistic guarantee :

$$\forall \xi \in R_\varepsilon, \quad g(x,\xi) \leq 0 \qquad \Longrightarrow \qquad \mathbb{P}\Big(g(x,\boldsymbol{\xi}) \leq 0\Big) \geq 1-\varepsilon$$

- Yes: in some cases approximation scheme for nominal problem can be extended to robust problem (e.g. cardinal uncertainty in combinatorial problem)
- Yes: using relevant data we can use statistical tools to construct a robust set R that imply a probabilistic guarantee

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