Optimality conditions

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Why should I bother to learn this stuff?

- Optimality conditions enable to solve exactly some easy optimization problems (e.g. in microeconomics, some mechanical problems...)
- Optimality conditions are used to derive algorithms for complex problem
- $\Longrightarrow$ fundamental both for studying optimization as well as other science

Contents

Optimization problem: vocabulary ♡

Generically speaking, an optimization problem is

$$\underset{x \in X}{\mathrm{Min}} \qquad f(x) \qquad (P)$$

where

- $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function (a.k.a. cost function),
- X is the feasible set,
- $x \in X$ is an admissible decision variables or a solution,
- $x^\sharp \in X$ such that $val(P) = f(x^\sharp) = \inf_{x \in X} f(x)$ is an optimal solution,
- if $X = \mathbb{R}^n$ the problem is unconstrained,
- if X and f are convex, then the problem is convex,
- if X is a polyhedron and f linear then the problem is linear,
- if X is a convex cone and f linear then the problem is conic.

Optimization problem: explicit formulation ♡

The previous optimization problem is often defined explicitly in the following standard form

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & f(x) & & (P) \\ \text{s.t.} \quad & g_i(x) = 0 & & \forall i \in [n_E] \\ & h_j(x) \leq 0 & & \forall j \in [n_I] \end{aligned}$$

with

$$X := \Big\{ x \in \mathbb{R}^n \mid \forall i \in [n_E], \quad g_i(x) = 0, \quad \forall j \in [n_I], \quad h_j(x) \leq 0 \Big\}.$$

- (P) is a differentiable optimization problem if f and $\{g_i\}_{i\in[n_E]}$ and $\{h_j\}_{j\in[n_I]}$ are differentiable.
- (P) is a convex differentiable optimization problem if f, and h_j (for $j \in [n_I]$) are convex differentiable and g_i (for $i \in [n_E]$) are affine.
  ♣ Exercise: Show that in this case X is convex.

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A few remarks and tricks

- We can always write an abstract optimization problem in standard form (exercise!)
- For a given optimization problem there is an infinite number of possible standard forms (exercise!)
- We can always find an equivalent problem in dimension $\mathbb{R}^{n+1}$ with linear cost (exercise!)
- A minimization problem with $X = \emptyset$ has value $+\infty$ (by convention)
- A minimization problem has value $-\infty$ iff there exists a sequence $x_n \in X$ such that $f(x_n) \to -\infty$
- Maximizing f is just minimizing $-f$

Contents

Differentiable case

♡

Theorem

Assume that $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is differentiable at $x^\sharp$.

1. *If $x^\sharp$ is an unconstrained local minimizer of f then $\nabla f(x^\sharp) = 0$.*
2. *If in addition f is convex, then $\nabla f(x^\sharp) = 0$ iff $x^\sharp$ is a global minimizer.*

Proof:

1. Assume $\nabla f(x^\sharp) \neq 0$. DL of order 1 at $x^\sharp$ show that $f(x^\sharp - t\nabla f(x^\sharp)) < f(x^\sharp)$ for $t > 0$ small enough.
2. $f(y) \geq f(x^\sharp) + \langle \nabla f(x^\sharp), y - x^\sharp \rangle$.

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Consider $f : \mathbb{R}^n \to \bar{\mathbb{R}}$*. Then* $x^\sharp$ *is a global minimum iff*

$$0 \in \partial f(x^\sharp)$$

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Consider a proper convex function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, *and* X *a closed convex set, such that* $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(X) \neq \emptyset$.
Then $x^\sharp$ *is a minimizer of* f *on* X *iff there exists* $g \in \partial f(x^\sharp)$ *such that* $-g \in N_X(x^\sharp)$.

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Then $x^\sharp$ is a minimizer of f on X iff there exists $g \in \partial f(x^\sharp)$ such that $-g \in N_X(x^\sharp)$.

proof : The technical assumption ensures that $\partial(f + \mathbb{I}_X) = \partial f + \partial(\mathbb{I}_X)$.
As $\partial(\mathbb{I}_X) = N_X$, we have, $0 \in \partial(f + \mathbb{I}_X)(x^\sharp)$ iff there exists $g \in \partial f(x^\sharp)$ such that $-g \in N_X(x^\sharp)$.

Contents

Tangent cones

For $f : \mathbb{R}^n \to \mathbb{R}$, we consider an optimization problem of the form

$$\underset{x \in X}{\mathrm{Min}} \qquad f(x).$$

Definition

We say that $d \in \mathbb{R}^n$ is tangent to X at $x \in X$ if there exists a sequence $x_k \in X$ converging to x and a sequence $t_k \searrow 0$ such that

$$d = \lim_k \frac{x_k - x}{t_k}.$$

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Let $T_X(x)$ be the tangent cone of X at x, that is, the set of all tangent to X at x.

Equivalently,

$$T_X(x) = \{\, d \in \mathbb{R}^n \;\mid \exists t_k \searrow 0,\ \exists d_k \to d,\ x + t_k d_k \in X\}$$

Optimality conditions - differentiable case

Consider a function $f : \mathbb{R}^n \to \mathbb{R}$ and the optimization problem

$$(P) \qquad \underset{x \in X}{\text{Min}} \qquad f(x).$$

If $x^\sharp \notin \mathrm{int}(X)$ we do not necessarily need to have $\nabla f(x^\sharp) = 0$, indeed we just to have $\langle d, \nabla f(x^\sharp) \rangle \leq 0$ for all "admissible" direction d.

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Theorem

Assume that f is differentiable at $x^\sharp$.

1. *If $x^\sharp$ is a local minimizer of (P) we have*

$$\nabla f(x^\sharp) \in \left[T_X(x^\sharp) \right]^\oplus. \qquad (*)$$

2. *If f and X are both convex, and $(*)$ holds, then $x^\sharp$ is an optimal solution of (P)*

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♠ Exercise: Prove this result.

Convex case

Let $K_X^{ad}(x)$ be the cone of admissible direction

$$K_X^{ad}(x) := \left\{ t(y-x) \in \mathbb{R}^n \quad | \quad y \in X, \quad t \geq 0 \right\}$$

Lemma

If $X \subset \mathbb{R}^n$ is convex, and $x \in X$, we have

$$T_X(x) = \overline{K_X^{ad}(x)}.$$

Recall that

$$T_X(x) = \{ \ d \in \mathbb{R}^n \ \ | \, \exists t_k \searrow 0, \ \exists d_k \to d, \ x + t_k d_k \in X \}$$

♠ Exercise: Prove this lemma

Differentiable constraints

We consider the following set of admissible solution

$$X = \Big\{ x \in \mathbb{R}^n \quad | \quad g_i(x) = 0,\ i \in [n_E] \quad h_j(x) \leq 0,\ j \in [n_j] \Big\},$$

where g and h are differentiable functions.

Recall that the tangent cone is given by

$$T_X(x) = \{\ d \in \mathbb{R}^n \ | \ \exists t_k \searrow 0,\ \exists d_k \to d,\ g(x+t_k d_k) = 0,\ h(x+t_k d_k) \leq 0\}$$

We define the linearized tangent cone

$$\begin{aligned} T_X^\ell(x) := \{\ d \in \mathbb{R}^n \ | & \left\langle \nabla g_i(x)\,,d \right\rangle = 0,\ \forall i \in [n_E] \\ & \left\langle \nabla h_j(x)\,,d \right\rangle \leq 0,\ \forall j \in I_0(x)\} \end{aligned}$$

where

$$I_0(x) := \Big\{ j \in [n_I] \quad | \quad h_j(x) = 0 \Big\}.$$

Constraint qualifications

We always have

$$T_X(x) \subset T_X^\ell(x).$$

♣ Exercise: Prove it.

We say that the constraints are qualified at x if

$$T_X(x) = T_X^\ell(x).$$

Constraint qualifications

We always have
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Sufficient qualification conditions ♡

Recall that g and h are assumed differentiable.
We denote the index set of active constraints at x

$$I_0(x) := \{ i \in [n_I] \quad | \quad h_i(x) = 0 \}.$$

The following conditions are sufficient qualification conditions at x:

1. g and h_i for $i \in I_0(x)$ are locally affine;
2. (Slater) g is affine, h_j are convex, and there exists x_S such that $g(x_S) = 0$ and $h_j(x_S) < 0$;
3. (Mangasarian-Fromowitz) For all $\alpha \in \mathbb{R}^{n_E}$ and $\beta \in \mathbb{R}^{n_I}_{+}$,

$$\sum_{i \in [n_E]} \alpha_i \nabla g_i(x) + \sum_{j \in I_0(x)} \beta_j \nabla h_j(x) = 0 \qquad \Longrightarrow \qquad \alpha = 0 \text{ and } \beta = 0$$

Expliciting the optimality condition I $\diamondsuit$

Under constraint qualification, the optimality condition reads

$$\nabla f(x) \in \left[T_X^\ell(x)\right]^\oplus$$

where

$$T_X^\ell(x) = \{\, d \in \mathbb{R}^n \mid \underbrace{\left\langle \nabla g_i(x)\,, d \right\rangle = 0, i \in [n_I] \quad \left\langle \nabla h_j(x)\,, d \right\rangle \leq 0, j \in I_0(x)}_{= A_x d \in C} \}.$$

with $A_x = \begin{pmatrix} ((\nabla g_i(x))^\top)_{i \in [n_I]} \\ ((\nabla h_j(x))^\top)_{j \in I_0(x)} \end{pmatrix}$ and $C = \{0\}^{n_E} \times (\mathbb{R}_-)^{n_I}$.

♣ Exercise: Show that $C^\oplus = \mathbb{R}^{n_E} \times (\mathbb{R}_-)^{n_I}$

Expliciting the optimality condition II ♢

Recall that the positive dual cone of a set K is

$$K^{\oplus} := \{ d \in \mathbb{R}^n \quad | \quad \langle d, x \rangle \geq 0, \forall x \in K \}.$$

Let C be a closed convex set. Consider

$$K = A^{-1}C := \{ x \in \mathbb{R}^n \mid Ax \in C \},$$

then

$$K^{\oplus} = \{ A^{\top}\lambda \mid \lambda \in C^{\oplus} \}.$$

♣ Exercise: prove it.

Hence,

$$\nabla f(x) \in \Big[\underbrace{T_X^{\ell}(x)}_{A_x^{-1}C} \Big]^{\oplus}$$

$$\iff \qquad \exists \lambda \in C^{\oplus}, \quad \nabla f(x) = A_x^{\top}\lambda$$

$$\iff \quad \exists \lambda \in \mathbb{R}^{n_E},\ \exists \mu \in \mathbb{R}_+^{I_0(x)} \quad \nabla f(x) + \sum_{i=1}^{n_E} \lambda_i \nabla g_i(x) + \sum_{j \in I_0(x)} \mu_j \nabla h_j(x) = 0.$$

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Karush Kuhn Tucker condition

Theorem (KKT)

Assume that the objective function f and the constraint function g_i and h_j are differentiable. Assume that the constraints are qualified at x.

Then if x is a local minimum of

$$\min_{\tilde{x}\in\mathbb{R}^n} \left\{ f(\tilde{x}) \quad | \quad g_i(\tilde{x}) = 0,\ \forall i \in [n_E] \quad h_j(\tilde{x}) \leq 0,\ \forall j \in [n_I] \right\}$$

then there exists dual variables λ, μ such that

$$\begin{cases} \nabla f(x) + \sum_{i=1}^{n_E} \lambda_i \nabla g_i(x) + \sum_{j=1}^{n_I} \mu_j \nabla h_j(x) = 0 & \nabla_x \mathcal{L} = 0 \\ g(x) = 0, \quad h(x) \leq 0 & \textit{Primal feasibility} \\ \lambda \in \mathbb{R}^{n_E}, \quad \mu \in \mathbb{R}_+^{n_I} & \textit{dual feasibility} \\ \mu_j h_j(x) = 0 \qquad \forall j \in [n_I] & \textit{complementarity constraint} \end{cases}$$

Exercise

Solve the following optimization problem

$$\begin{aligned} \underset{\tilde{x}, y \in \mathbb{R}^2}{\text{Min}} \quad & (x-1)^2 + (y-2)^2 \\ & x \leq y \\ & x + 2y \leq 2 \end{aligned}$$

Contents

What you have to know

- Basic vocabulary: objective, constraint, admissible solution, differentiable optimization problem
- First order necessary KKT conditions

What you really should know

- What is a tangent cone
- Sufficient qualification conditions (linear and Slater's)
- That KKT conditions are sufficient in the convex case

What you have to be able to do

- Write the KKT condition for a given explicit problem and use them to solve said problem

What you should be able to do

- Check that constraints are qualified