

Exercises: optimization problem classes

Exercise 1 (Hyperbolic constraints as SOCP).

1. Show that, for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $z \in \mathbb{R}$,

$$x^\top x \leq yz, \quad y \geq 0, \quad z \geq 0$$

iff

$$\left\| \begin{pmatrix} 2x \\ y - z \end{pmatrix} \right\| \leq y + z \quad y \geq 0, \quad z \geq 0$$

2. Represent the following problem as an SOCP

$$(P) \quad \begin{aligned} & \text{Max} \quad \left(\sum_{i=1}^n 1/(a_i^\top x - b) \right)^{-1} \\ & \text{s.t.} \quad Ax > b \end{aligned}$$

Exercise 2. We consider a physical function Φ that is approximated as the superposition of multiple simple phenomenon (e.g. waves). Each simple phenomenon $p \in [P]$ is represented by a function $\Phi_p : \mathbb{R}^d \rightarrow \mathbb{R}$.

We have data points $(x^k, y^k)_{k \in [n]}$, and want to find the Φ that match at best the data while being a linear combination of Φ_p .

Propose a least-square regression that answer this question.

Exercise 3. Consider a chocolate manufacturing company that produces only two types of chocolate – A and B. Both the chocolates require Milk and Choco only. To manufacture each unit of A and B, the following quantities are required:

- Each unit of A requires 1 unit of Milk and 3 units of Choco
- Each unit of B requires 1 unit of Milk and 2 units of Choco

The company kitchen has a total of 5 units of Milk and 12 units of Choco. On each sale, the company makes a profit of

- 6 per unit A sold

- 5 per unit B sold.

Model this as an LP.

Exercise 4. A classical extension of the least-square problem, which has strong theoretical and practical interest is the LASSO problem

$$\text{Min}_{x \in \mathbb{R}^p} \quad \|Ax - b\|^2 + \lambda \|x\|_1$$

Show that this problem can be cast as a QP problem.

Exercise 5. Consider the following optimization problem.

$$\begin{aligned} & \text{Min}_{x \in \mathbb{R}^n} \quad c^\top x \\ & \text{s.t.} \quad Ax = b \\ & \quad \quad x_i \in \{0, 1\} \quad \forall i \in I \end{aligned}$$

Write this problem as a QCQP. Is it convex ?

Exercise 6. Consider a facility that plan to deliver product to clients by drone (thus in direct line). Assume that you have N clients, each with position (in \mathbb{R}^2) x_n . The drone make each time a direct travel from the facility location to the client. Assume that the drone have a maximum range of R , and that you want to minimize the average travel distance while being able to serve all of your clients.

Model the problem of choosing the facility location as an SOCP.

Exercise 7. Consider the following robust linear program

$$\begin{aligned} & \text{Min}_{x \in \mathbb{R}^n} \quad c^\top x \\ & \text{s.t.} \quad (a_i + R_i \delta_i)^\top x \leq b_i \quad \forall \|\delta_i\|_2 \leq 1, \quad \forall i \in [m] \end{aligned}$$

where R_i are positive real numbers. Write this problem as an SOCP.

Exercise 8. Let $F(\theta)$ be a symmetric matrix parametrized by $\theta \in \mathbb{R}^d$ whose coefficients are linear in θ . Model the problem of finding the parameter $\theta \in \Theta$, where Θ is a polyhedron, minimizing $\kappa(\theta)$ as an SDP.

What happen if the coefficient of $F(\theta)$ are affine in θ ? Suggest a solution method ? (hard)

Exercise 9. Consider a finite set $X = \{x_i\}_{i \in [n]}$, and \mathcal{P}^+ the set of probabilities on X . For $\mathbb{P}, \mathbb{Q} \in \mathcal{P}$, with $\text{supp}(\mathbb{Q}) = X$, we define the Kullback-Leibler divergence as

$$d_{KL}(\mathbb{P}|\mathbb{Q}) = \sum_{i=1}^n p_i \ln(p_i/q_i)$$

where $p_i = \mathbb{P}(X = x_i)$ and $q_i = \mathbb{Q}(X = x_i)$.

Let X be 100 equidistant points spanning in $[-1, 1]$. Let \mathbb{Q} be uniform on X .

We are looking for the probability \mathbb{P} on X such that

- $\mathbb{E}_{\mathbb{P}}[\mathbf{X}] \in [-0.1, 0.1]$
- $\mathbb{E}_{\mathbb{P}}[\mathbf{X}^2] \in [0.5, 0.6]$
- $\mathbb{E}_{\mathbb{P}}[3\mathbf{X}^2 - 2\mathbf{X}] \in [-0.3, -0.2]$
- $\mathbb{P}(\mathbf{X} < 0) \in [0.3, 0.4]$

that minimize the Kullback-Leibler divergence from \mathbb{Q} .

Model this problem as an optimization problem. In which class does it belongs ?

Exercise 10. Consider that you sell a given product over T days. The demand for each day is d_t . Having a quantity x_t of items in stock have a cost (per day) of cx_t . You can order, each day, a quantity q_t , and have to satisfy the demand.

For each of the following variation : model the problem, explicit the class to which it belongs, and give the optimal solution if easily found.

1. Without any further constraint / specifications.
2. There is an "ordering cost": each time you order, you have to pay a fix cost κ .
3. Instead of an "ordering cost" there is a maximum number of days at which you can order a replenishment.