

Wardrop Equilibrium

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Convex set

- A set $C \subset \mathbb{R}^n$ is *convex* iff

$$\forall x, y \in C, \quad \forall t \in [0, 1], \quad tx + (1 - t)y \in C.$$

- Intersection of convex sets is convex.
- A closed convex set C is equal to the intersection of all half-spaces containing it.

Convex function

- The *epigraph* of a function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is

$$\text{epi}(f) := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid t \geq f(x)\}.$$

- The *domain* of a function f is

$$\text{dom}(f) := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$$

- The function f is said to be *convex* iff its epigraph is convex, in other words iff

$$\forall t \in [0, 1], \quad f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

- The function f is said to be *strictly convex* iff

$$\forall t \in (0, 1), \quad f(tx + (1 - t)y) < tf(x) + (1 - t)f(y).$$

Convexity and differentiable

We assume sufficient regularity for the written object to exist.

- If $f : \mathbb{R} \rightarrow \mathbb{R}$.
 - f is convex iff f' non-decreasing.
 - If $f' > 0$ then f is strictly convex.
 - f is convex iff $f'' \geq 0$.
 - If $f'' > 0$ then f is strictly convex.
- If $f : \mathbb{R}^n \rightarrow \mathbb{R}$
 - f is convex iff ∇f non-decreasing (i.e. $(\nabla f(y) - \nabla f(x)) \cdot (y - x) \geq 0$).
 - f is convex iff $\nabla^2 f(x) \succeq 0$ for all x .
 - If $\nabla^2 f(x) \succ 0$ for all x then f is strictly convex.

Video explanation

<https://www.youtube.com/watch?v=qF0aDJfEa4Y>

Convex differentiable optimization problem

Consider the following optimization problem.

$$\begin{array}{ll}
 \min_{x \in \mathbb{R}^n} & f(x) \\
 \text{s.t.} & g_i(x) = 0 \quad \forall i \in [n_E] \\
 & h_j(x) \leq 0 \quad \forall j \in [n_I]
 \end{array} \quad (P)$$

with

$$X := \{x \in \mathbb{R}^n \mid \forall i \in [n_E], \quad g_i(x) = 0, \quad \forall j \in [n_I], \quad h_j(x) \leq 0\}.$$

- (P) is a *convex optimization problem* if f and X are convex.
- (P) is a *convex differentiable optimization problem* if f , and h_j (for $j \in [n_I]$) are convex differentiable and g_i (for $i \in [n_E]$) are affine

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KKT conditions

Theorem (KKT)

Let x^\sharp be an optimal solution to a differentiable optimization problem (P) . If the constraints are qualified at x^\sharp then there exists optimal multipliers $\lambda^\sharp \in \mathbb{R}^{n_E}$ and $\mu^\sharp \in \mathbb{R}^{n_I}$ satisfying

$$\left\{ \begin{array}{ll} \nabla f(x^\sharp) + \sum_{i=1}^n \lambda_i^\sharp \nabla g_i(x^\sharp) + \sum_{j=1}^{n_I} \mu_j^\sharp \nabla h_j(x^\sharp) = 0 & \text{first order condition} \\ g(x^\sharp) = 0 & \text{primal admissibility} \\ h(x^\sharp) \leq 0 & \\ \mu \geq 0 & \text{dual admissibility} \\ \mu_i g_i(x^\sharp) = 0, \quad \forall i \in \llbracket 1, n_I \rrbracket & \text{complementarity} \end{array} \right.$$

The three last conditions are sometimes compactly written

$$0 \leq g(x^\sharp) \perp \mu \geq 0.$$

Video explanation

Intro to constrained optimization

<https://www.youtube.com/watch?v=vwUV2IDLp8Q>

Explaining tangeancy of multipliers

<https://www.youtube.com/watch?v=yuqB-d5MjZA>

Marginal interpretation of multipliers

<https://www.youtube.com/watch?v=m-G3K2GPmEQ>

Slater condition

A convex optimization problem (P) satisfies the *Slater* condition if there exists a strictly admissible $x_0 \in \mathbb{R}^n$ that is

$$\forall i \in [n_E], \quad g_i(x_0) = 0, \quad \forall j \in [n_I], \quad h_j(x_0) < 0.$$

If the Slater condition is satisfied, then the constraints are qualified at any $x \in X$.

Another optimality condition (convex case)

Theorem

If (P) is a convex differentiable optimization problem, then $x^\# \in X$ is an optimal solution iff

$$\forall y \in X, \quad \nabla f(x) \cdot (y - x) \geq 0.$$

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The set-up

- $G = (V, E)$ is a directed graph
- x_e for $e \in E$ represent the flux (number of people per hour) taking edge e
- $\ell_e : \mathbb{R} \rightarrow \mathbb{R}^+$ the cost incurred by a given user to take edge e
- We consider K origin-destination vertex pair $\{o^k, d^k\}_{k \in \llbracket 1, K \rrbracket}$, such that there exists at least one path from o^k to d^k .
- r_k is the rate of people going from o^k to d^k
- \mathcal{P}_k the set of all simple (i.e. without cycle) path form o^k to d^k
- We denote f_p the flux of people taking path $p \in \mathcal{P}_k$

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Some physical relations

People going from o^k to d^k have to choose a path

$$r^k = \sum_{p \in \mathcal{P}^k} f_p.$$

People going through an edge are on a simple path taking this edge

$$x_e = \sum_{p \ni e} f_p.$$

The flux are non-negative

$$\forall p \in \mathcal{P}, \quad f_p \geq 0, \quad \text{and} \quad \forall e \in E, \quad x_e \geq 0$$

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System optimum problem

The system optimum consists in minimizing the sum of all costs over the admissible flux $x = (x_e)_{e \in E}$

- Given x , the cost of taking edge e for one person is $\ell_e(x_e)$.
- The cost for the system for edge e is thus $x_e \ell_e(x_e)$.
- Thus minimizing the system costs consists in solving

$$\min_{x, f} \quad \sum_{e \in E} x_e \ell_e(x_e) \quad (SO)$$

$$s.t. \quad r_k = \sum_{p \in \mathcal{P}_k} f_p \quad k \in \llbracket 1, K \rrbracket$$

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Path intensity formulation

- We can reformulate the (SO) problem only using path-intensity $f = (f_p)_{p \in \mathcal{P}}$.
- Define $x_e(f) := \sum_{p \ni e} f_p$, and $x = (x_e)_{e \in E}$.
- Define the loss along a path $\ell_p(f) = \sum_{e \in p} \ell_e(\underbrace{\sum_{p' \ni e} f_{p'}}_{x_e(f)})$.
- The total cost is thus

$$C(f) = \sum_{p \in \mathcal{P}} f_p \ell_p(f) = \sum_{e \in E} x_e \ell_e(x_e(f)) = C(x(f)).$$

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Path intensity problem

$$\begin{aligned}
 \min_f \quad & \sum_{p \in \mathcal{P}} f_p \ell_p(f) && (SO) \\
 s.t. \quad & r_k = \sum_{p \in \mathcal{P}_k} f_p && k \in \llbracket 1, K \rrbracket \\
 & f_p \geq 0 && p \in \mathcal{P}
 \end{aligned}$$

Equilibrium definition

John Wardrop defined a traffic equilibrium as follows. "Under equilibrium conditions traffic arranges itself in congested networks such that all used routes between an O-D pair have equal and minimum costs, while all unused routes have greater or equal costs."

A mathematical definition reads as follows.

Definition

A user flow f is a User Equilibrium if

$$\forall k \in \llbracket 1, K \rrbracket, \quad \forall (p, p') \in \mathcal{P}_k^2, \quad f_p > 0 \implies \ell_p(f) \leq \ell_{p'}(f).$$

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A new cost function

We are going to show that a user-equilibrium f is defined as a vector satisfying the KKT conditions of a certain optimization problem.

Let define a new edge-loss function by

$$L_e(x_e) := \int_0^{x_e} \ell_e(u) du.$$

The Wardrop potential is defined (for edge intensity) as

$$W(f) = W(x(f)) = \sum_{e \in E} L_e(x_e(f)).$$

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User optimum problem

Theorem

A flow f is a user equilibrium if and only if it satisfies the first order KKT conditions of the following optimization problem

$$\begin{array}{ll}
 \min_{x,f} & W(x) \\
 \text{s.t.} & r_k = \sum_{p \in \mathcal{P}_k} f_p \quad k \in \llbracket 1, K \rrbracket \\
 & x_e = \sum_{p \ni e} f_p \quad e \in E \\
 & f_p \geq 0 \quad p \in \mathcal{P}
 \end{array}$$

Proof

I

In path intensity formulation

$$\begin{aligned}
 \min_f \quad & \sum_{e \in E} L_e \left(\sum_{p \ni e} f_p \right) \\
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 & f_p \geq 0 & p \in \mathcal{P}
 \end{aligned}$$

with Lagrangian

$$L(f, \lambda, \mu) := W(f) + \sum_{k=1}^K \lambda_k \left(r_k - \sum_{p \in \mathcal{P}_k} f_p \right) + \sum_{p \in \mathcal{P}} \mu_p f_p.$$

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$$\begin{aligned}
 \min_f \quad & \sum_{e \in E} L_e \left(\sum_{p \ni e} f_p \right) \\
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Proof



Now note that we have

$$\begin{aligned}
 \frac{\partial W}{\partial f_p}(f) &= \frac{\partial}{\partial f_p} \left(\sum_{e \in E} L_e \left(\sum_{p' \ni e} f_{p'} \right) \right) \\
 &= \sum_{e \in p} \frac{\partial}{\partial x_e} L_e(x_e(f)) \\
 &= \sum_{e \in p} \ell_e(x_e(f)) = \ell_p(f),
 \end{aligned}$$

Recall that $L_e(x_e) := \int_0^{x_e} \ell_e(u) du$.

Proof



The constraints of (UE) are qualified. Consequently its first-order KKT conditions reads

$$\begin{cases} \frac{\partial L(f, \lambda, \mu)}{\partial f_p} = \ell_p(f) - \lambda_k + \mu_p = 0 & \forall p \in \mathcal{P}_k, \forall k \in \llbracket 1, K \rrbracket \\ \frac{\partial L(f, \lambda, \mu)}{\partial \lambda_k} = r_k - \sum_{p \in \mathcal{P}_k} f_p = 0 & \forall k \in \llbracket 1, K \rrbracket \\ \mu_p = 0 \text{ or } f_p = 0 & \forall p \in \mathcal{P} \\ \mu_p \leq 0, f_p \geq 0 & \forall p \in \mathcal{P} \end{cases}$$

f satisfies the KKT conditions iff for all origin-destination pair $k \in \llbracket 1, K \rrbracket$, and all path $p \in \mathcal{P}_k$ we have

$$\begin{cases} \ell_p(f) = \lambda_k & \text{if } f_p > 0 \\ \ell_p(f) \geq \lambda_k & \text{if } f_p = 0 \end{cases}$$

In other words, if the path $p \in \mathcal{P}_k$ is used, then its cost is λ_k , and all other path $p' \in \mathcal{P}_i$ have a greater or equal cost, which is the definition of

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Convex case : equivalence

If the loss functions (in edge-intensity) are non-decreasing then the Wardrop potential W is convex.

Theorem

Assume that the loss function ℓ_e are non-decreasing for all $e \in E$. Then there exists at least one user equilibrium, and a flow f is a user equilibrium if and only if it solves (UE)

Proof : the cost is convex as composition of convex and affine functions, thus KKT is a necessary and sufficient condition for optimality.

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Convex case : characterization

define the system cost of a flow f for a given flow f' , as

$$C^f(f) := \sum_{e \in E} x_e(f) \ell_e(x_e(f')).$$

Theorem

Assume that the cost functions ℓ_e are continuous and non-decreasing. Then, f^{UE} is a user equilibrium iff

$$\forall f \in F^{ad}, \quad C^{f^{UE}}(f^{UE}) \leq C^{f^{UE}}(f),$$

where F^{ad} is the set of admissible flows.

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where F^{ad} is the set of admissible flows.

Proof

By convexity (f^{UE}) is an optimal solution to (UE) iff

$$\nabla W(f^{UE}) \cdot (f - f^{UE}) \geq 0, \quad \forall f \in F^{ad}$$

which is equivalent to

$$\sum_{p \in \mathcal{P}} \underbrace{\frac{\partial W}{\partial f_p}(f^{UE})}_{\ell_p(f^{UE})} f_p \geq \sum_{p \in \mathcal{P}} \underbrace{\frac{\partial W}{\partial f_p}(f^{UE})}_{\ell_p(f^{UE})} f_p^{UE}, \quad \forall f \in F^{ad}$$

which can be written

$$C^{f^{UE}}(f^{UE}) \leq C^{f^{UE}}(f), \quad \forall f \in F^{ad}.$$

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Definition

Definition

Consider increasing loss functions ℓ_e . Let f^{UE} be a user equilibrium, and f^{SO} be a system optimum. Then the price of anarchy of our network is given by

$$PoA := \frac{C(f^{UE})}{C(f^{SO})} \geq 1.$$

Theorem

Let ℓ_e be the affine function $x_e \mapsto b_e x_e + c_e$, with $b_e, c_e \geq 0$. Then the price of anarchy is lower than $4/3$, and the bound is tight.

Proof

Let f be a feasible flow, and f^{UE} be the user equilibrium. For ease of notation we fix $x^{UE} = x(f^{UE})$, and $x = x(f)$.

By Theorem we have

$$\begin{aligned}
 C(f^{UE}) &\leq C^{f^{UE}}(f) \\
 &= \sum_{e \in E} (b_e x_e^{UE} + c_e) x_e \\
 &\leq \sum_{e \in E} \left[(b_e x_e + c_e) x_e + \frac{1}{4} b_e (x_e^{UE})^2 \right] \quad \text{as } (x_e - x_e^{UE}/2)^2 \geq 0 \\
 &\leq C(f) + \frac{1}{4} \sum_{e \in E} (b_e x_e^{UE} + c_e) x_e^{UE} \quad \text{as } c_e x_e^{UE} \geq 0 \\
 &= C(f) + \frac{1}{4} C^{f^{UE}}(f^{UE})
 \end{aligned}$$

Hence we have $3/4 C(f^{UE}) \leq C(f)$.

Minimizing over admissible flow f ends the proof.

Proof

Let f be a feasible flow, and f^{UE} be the user equilibrium. For ease of notation we fix $x^{UE} = x(f^{UE})$, and $x = x(f)$.

By Theorem we have

$$\begin{aligned}
 C(f^{UE}) &\leq C^{f^{UE}}(f) \\
 &= \sum_{e \in E} (b_e x_e^{UE} + c_e) x_e \\
 &\leq \sum_{e \in E} \left[(b_e x_e + c_e) x_e + \frac{1}{4} b_e (x_e^{UE})^2 \right] \quad \text{as } (x_e - x_e^{UE}/2)^2 \geq 0 \\
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Pigou's Example

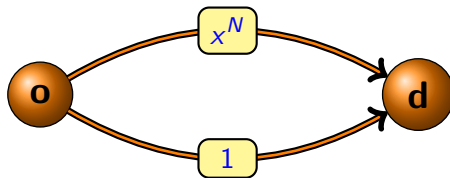


Figure: Pigou example

On a graph with two nodes: one origin, one destination, a total flow of 1, a fixed cost of 1 on one edge, and a cost of x^N on the other, where $N \in \mathbb{N}$ and x is the intensity of the flow using this edge (see Figure 1).

- 1 Compute the system optimum for a given N .
- 2 Compute the user equilibrium for a given N .
- 3 Compute the price of anarchy on this network when $N \rightarrow \infty$.

Exercise for next week (3.2)

Consider a (finite) directed, strongly connected, graph $G = (V, E)$. We consider K origin-destination vertex pair $\{o^k, d^k\}_{k \in \llbracket 1, K \rrbracket}$, such that there exists at least one path from o^k to d^k .

We want to find bounds on the price of anarchy, assuming that, for each arc e , $\ell_e : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is non-decreasing, and that we have

$$x\ell_e(x) \leq \gamma L_e(x), \quad \forall x \in \mathbb{R}^+$$

- ① Recall which optimization problems solves the social optimum x^{SO} and the user equilibrium x^{UE} .
- ② Let x be a feasible vector of arc-intensity. Show that $W(x) \leq C(x) \leq \gamma W(x)$.
- ③ Show that the price of anarchy $C(x^{UE})/C(x^{SO})$ is lower than γ .
- ④ If the cost per arc ℓ_e are polynomial of order at most p with non-negative coefficient, find a bound on the price of anarchy. Is this bound sharp ?

Further video content

This is a research seminar by one of the expert in the domain. The first half is very interesting to get a better intuition of the concepts. The second half is more dedicated to the proof of the result presented in the talk.

https://www.youtube.com/watch?v=e30_tMsN2t8