Robust Optimization: Static Case

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An optimization problem

A generic optimization problem can be written

$$\begin{aligned} \min_{x} \quad & L(x) \\ s.t. \quad & g(x) \leq 0 \end{aligned}$$

where

- x is the decision variable
- L is the objective function
- g is the constraint function

An optimization problem with uncertainty

Adding uncertainty ξ in the mix

$$
\begin{aligned}
\min_{x} \quad & L(x, \tilde{\xi}) \\
s.t. \quad & g(x, \tilde{\xi}) \leq 0
\end{aligned}
$$

Remarks:

- $\tilde{\xi}$ is unknown. Two main way of modelling it:
  - $\tilde{\xi} \in R$ with a known uncertainty set R, and a pessimistic approach. This is the robust optimization approach (RO).
  - $\tilde{\xi}$ is a random variable with known probability law. This is the Stochastic Programming approach (SP).
- Cost is not well defined.
  - RO: $\max_{\xi \in R} L(x, \xi)$.
  - SP: $\mathbb{E}\big[L(x, \xi)\big]$.
- Constraints are not well defined.
  - RO: $g(x, \xi) \leq 0, \qquad \forall \xi \in R$.
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Requirements and limits

- Stochastic optimization:
  - requires a law of the uncertainty ξ
  - can be hard to solve (generally require discretizing the support and blowing up the dimension of the problem)
  - there exists specific methods (like Bender's decomposition)
- Robust optimization:
  - requires an uncertainty set R
  - can be overly conservative, even for reasonable R
  - complexity strongly depend on the choice of R
- Distributionally robust optimization:
  - is a mix between robust and stochastic optimization
  - consists in solving a stochastic optimization problem where the law is chosen in a robust way
  - is a fast growing fields with multiple recent results
  - but is still hard to implement than other approaches

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Some numerical tests on real-life LPs

From Ben-Tal and Nemirovski

- take LP from Netlib library
- look at non-integer coefficients, assuming that they are not known with perfect certainty
- What happens if you change them by 0.1% ?
  - constraints can be violated by up to 450%
  - $\mathbb{P}(\text{violation} > 0) = 0.5$
  - $\mathbb{P}(\text{violation} > 150\%) = 0.18$
  - $\mathbb{E}[\text{violation}] = 125\%$

What do you want from robust optimization ?

- finding a solution that is less sensible to modified data, without a great increase of price
- choosing an uncertainty set R that:
  - offer robustness guarantee
  - yield an easily solved optimization problem

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Solving a robust optimization problem

The robust optimization problem we want to solve is[1]

$$\begin{aligned} \min_x \quad & L(x) \\ s.t. \quad & g(x,\xi) \leq 0 \qquad \forall \xi \in R \end{aligned}$$

Two main approaches are possible:

Constraint generation: replace R by a finite set of ξ, that is we replace an "infinite number of contraints" by a finite number of them.

Reformulation: replace $g(x,\xi) \leq 0 \quad \forall \xi \in R$,
by $\sup_{\xi \in R} g(x,\xi) \leq 0$,
then explicit the sup.

[1]For simplicity reason we dropped w.l.o.g. the uncertainty in the objective.

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Constraint generation algorithm

Data: Problem parameters, reference uncertainty ξ_0
Result: approximate value with gap;
for $k \in \mathbb{N}$ **do**
  solve $\tilde{v} = \min_{x} \{L(x) \mid g(x, \xi_\kappa) \; \forall \kappa \leq k\} \quad \rightsquigarrow x_k$;
  solve $s = \max_{\xi \in R} g(x_k, \xi) \quad \rightsquigarrow \xi_{k+1}$;
  if $s \leq 0$ **then**
    Robust optimization problem solved,
    with value $\tilde{v}$ and optimal solution x_k

Algorithm 1: Constraint Generation Algorithm

Note that we are solving a problem similar to the deterministic problem with an increasing number of constraints.

This is easy to implement and can be numerically efficient.

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Reformulation principle

We can write the robust optimization problem as

$$\begin{aligned} \min_{x} \quad & L(x) \\ s.t. \quad & \sup_{\xi \in R} g(x,\xi) \leq 0 \end{aligned}$$

Now, there are two ways of simplifying this problem:

- we can explicitly compute $\bar{g}(x) = \sup_{\xi \in R} g(x,\xi)$;
- by duality we can write $\sup_{\xi \in R} g(x,\xi) = \min_{\eta \in Q} h(x,\eta)$

➥ $\min_{\eta \in Q} h(x,\eta) \leq 0$ is equivalent to $\exists \eta$ such that $h(x,\eta) \leq 0$, i.e. just add η as a variable in your optimization problem

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Canonization of the problem

We consider

$$\min_{x\geq 0} \max_{(A,b,c)\in R} \quad c^\top x$$
$$s.t. \quad Ax \leq b$$

Without loss of generality we can consider a deterministic cost:

$$\min_{x\geq 0,\theta} \quad \theta$$
$$s.t. \quad Ax \leq b \qquad \forall (A,b,c) \in R$$
$$c^\top x \leq \theta \qquad \forall (A,b,c) \in R$$

That can be written as

$$\min_{x\geq 0,\theta} \quad \theta$$
$$s.t. \quad {a_i}^\top x - b_i \leq 0 \qquad \forall (A,b,c) \in R, \forall i \in [m]$$

Canonization of the problem I

We consider

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$$\begin{aligned}\min_{x\geq 0,\theta} \quad & \theta \\ s.t. \quad & Ax \leq b & \forall (A,b,c) \in R \\ & c^\top x \leq \theta & \forall (A,b,c) \in R\end{aligned}$$

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Canonization of the problem II

We now consider

$$\min_{x\geq 0} \quad c^\top x$$
$$s.t. \quad {a_i}^\top x - b_i \leq 0 \qquad \forall (A,b) \in R, \forall i \in [m]$$

Let R_i be the projection of R onto coordinate i.
We have in particular $R \subset R_1 \times \cdots \times R_m$.
But note that, in the robust constraint, R can be replaced by $R_1 \times \cdots \times R_m$, indeed,

$$f_i(x,\xi_i) \leq 0, \quad \forall i \in [m], \quad \forall \xi \in R$$
$$\iff \quad f_i(x,\xi_i) \leq 0, \quad \forall i \in [m], \forall \xi \in R_1 \times \cdots \times R_m$$
$$\iff \quad f_i(x,\xi_i) \leq 0, \quad \forall \xi_i, \in R_i \quad \forall i \in [m]$$

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Canonization of the problem III

We now consider

$$\begin{aligned} \min_{x\geq 0} \quad & c^\top x \\ s.t. \quad & a_i^\top x - b_i \leq 0 \qquad \forall (a_i, b_i) \in R_i, \forall i \in [m] \end{aligned}$$

Canonization of the problem III

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$$\begin{aligned} \min_{x \geq 0} \quad & c^\top x \\ s.t. \quad & a^\top x - b \leq 0 \qquad \forall (a, b) \in R, \end{aligned}$$

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$$\begin{aligned} \min_{x \geq 0} \quad & c^\top x \\ s.t. \quad & a^\top x - b \leq 0 \qquad \forall (a, b) \in R, \end{aligned}$$

To model correlation we set

$$a = \bar{a} + P\zeta \qquad b = \bar{b} + p^\top \zeta$$

where $(\bar{a}, \bar{b})$ are the nominal value, and ζ is the primitive/residual uncertainty.

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The robust constraint now reads

$$(\bar{a}^\top x - \bar{b}) + (P^\top x - p)^\top \zeta \leq 0 \qquad \forall \zeta \in \mathcal{Z}$$

Canonization of the problem IV

Example: assume that a is a random variable with mean $\bar{a}$ and covariance Σ. Then, a natural reformulation would be

$$a = \bar{a} + \Sigma^{1/2}\zeta,$$

so that ζ is centered with uncorrelated coordinates.

Finally, w.l.o.g. we assume that b is deterministic (can be obtained by adding a variable x_{n+1} constrained to be equal to 1).

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An explicit worst case value

- We consider an ellipsoidal uncertainty set

$$R = \Big\{ \xi = \{ \bar{a} + P\zeta \}_i \quad | \quad \|\zeta\|_2 \leq \rho \Big\}$$

- Here we can, for a given x, explicitly compute

$$\begin{aligned} \sup_{\xi \in R} \xi^\top x &= \bar{a}^\top x + \sup_{\|\zeta\|_2 \leq \rho} (P\zeta)^\top x \\ &= \bar{a}^\top x + \rho \|P^\top x\|_2 \end{aligned}$$

- Hence, constraint

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SOCP problem

- An Second Order Cone Programming constraint is a constraint of the form

$$\|Ax + b\|_2 \leq c^\top x + d$$

- An SOCP problem is a (continuous) optimization problem with linear cost and linear and SOCP constraints
- There exists powerful software to solve SOCP (e.g. CPLEX, Gurobi, MOSEK...) with dedicated interior points methods
- There exist a duality theory akin to the LP duality theory
- If a robust optimization problem can be cast as an SOCP the formulation is deemed efficient

Contents

Linear duality: recalls

- Recall that, if finite,

$$\begin{aligned} \max_{\xi} \quad & \xi^\top x \\ s.t. \quad & D\xi \leq d \end{aligned}$$

as the same value as

$$\begin{aligned} \min_{\eta} \quad & \eta^\top d \\ s.t. \quad & \eta^\top D = x \\ & \eta \geq 0 \end{aligned}$$

- Thus,

$$\begin{aligned} \sup_{\xi : D\xi \leq d} \xi^\top x \leq b \quad & \Longleftrightarrow \quad \min_{\eta \geq 0 : \eta^\top D = x} \eta^\top d \leq b \\ & \Longleftrightarrow \quad \exists \eta \geq 0, \quad \eta^\top D = x, \quad \eta^\top d \leq b \end{aligned}$$

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Polyhedral uncertainty

- We consider a polyhedral uncertainty set

$$R = \left\{ \xi \quad \middle| \quad D\xi \leq d \right\}$$

- Then the robust optimization problem

$$\begin{aligned} \min_{x \geq 0} \quad & c^\top x \\ s.t. \quad & \sup_{\xi \in R} \xi^\top x \leq h \end{aligned}$$

reads

$$\begin{aligned} \min_{x \geq 0, \eta \geq 0} \quad & c^\top x \\ s.t. \quad & \eta^\top d \leq h \\ & \eta^\top d = x \end{aligned}$$

Contents

Soyster model

The problem

$$\min_{x} \quad c^\top x$$

$$\tilde{A}x \leq b \qquad \forall \tilde{A} \in R$$

$$\underline{x} \leq x \leq \bar{x}$$

where each coefficient $\tilde{a}_{ij} \in [\bar{a}_{ij} - \delta_{ij}, \bar{a}_{ij} + \delta_{ij}]$

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where each coefficient $\tilde{a}_{ij} \in [\bar{a}_{ij} - \delta_{ij}, \bar{a}_{ij} + \delta_{ij}]$ can be written

$$\begin{aligned} \min_{x} \quad & c^\top x \\ & \sum_j \bar{a}_{ij} x_j + \sum_j \delta_{ij} |x_j| \leq b_i & \forall i \\ & \underline{x} \leq x \leq \bar{x} \end{aligned}$$

Soyster model

The problem

$$
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\begin{aligned}
\min_{x} \quad & c^\top x \\
& \sum_j \bar{a}_{ij} x_j + \sum_j \delta_{ij} y_j \leq b_i & \forall i \\
& \underline{x} \leq x \leq \bar{x} \\
& y_j \geq x_j, \quad y_j \geq -x_j
\end{aligned}
$$

Cardinality constrained LP

Soyster's model is over conservative, we want to consider a model where only Γ_i coefficient per line have non-zero errors, leading to

$$\min_{x,y} \quad c^\top x$$

$$\sum_j \bar{a}_{ij} x_j + \max_{S_i : |S_i| = \Gamma_i} \sum_{j \in S_i} \delta_{ij} y_j \leq b_i \qquad \forall i$$

$$\underline{x} \leq x \leq \bar{x}$$

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$$
\begin{aligned}
\min_{x,y} \quad & c^\top x \\
& \sum_j \bar{a}_{ij} x_j + \beta_i \leq b_i && \forall i \\
& \max_{S_i : |S_i| = \Gamma_i} \sum_{j \in S_i} \delta_{ij} y_j \leq \beta_i \\
& \underline{x} \leq x \leq \bar{x} \\
& y_j \geq x_j, \quad y_j \geq -x_j
\end{aligned}
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Cardinality constrained LP II

This means that, for line i we take a margin of

$$\beta_i(x, \Gamma_i) := \max_{S_i : |S_i| = \Gamma_i} \sum_{j \in S_i} \delta_{ij} |x_j|$$

which can be obtained as

$$\begin{aligned} \beta_i(x, \Gamma_i) = \max_{z \geq 0} \quad & \sum_j \delta_{ij} |x_j| z_{ij} & \\ & \sum_j z_{ij} \leq \Gamma_i & [\lambda_i] \\ & z_{ij} \leq 1 & [\mu_{ij}] \end{aligned}$$

This LP can be then dualized to be integrated in the original LP.

Cardinality constrained LP III

$$\beta_i(x,\Gamma_i) = \max_{z \geq 0} \quad \sum_j \delta_{ij}|x_j| z_{ij}$$

$$\sum_j z_{ij} \leq \Gamma_i \qquad [\lambda_i]$$

$$z_{ij} \leq 1 \qquad [\mu_{ij}]$$

Cardinality constrained LP III

$$
\begin{aligned}
\beta_i(x,\Gamma_i) = \max_{z \geq 0} \quad & \sum_j \delta_{ij}|x_j| z_{ij} & \\
& \sum_j z_{ij} \leq \Gamma_i & [\lambda_i] \\
& z_{ij} \leq 1 & [\mu_{ij}]
\end{aligned}
$$

$$
\beta_i(x,\Gamma_i) = \max_{z \geq 0} \min_{\lambda,\mu \geq 0} \quad \sum_j \delta_{ij}|x_j| z_{ij} + \lambda_i\Big(\Gamma_i - \sum_j z_{ij}\Big) \sum_j \mu_{ij}\Big(1 - z_{ij}\Big)
$$

Cardinality constrained LP III

$$
\begin{aligned}
\beta_i(x,\Gamma_i) = \max_{z \geq 0} \quad & \sum_j \delta_{ij}|x_j| z_{ij} & \\
& \sum_j z_{ij} \leq \Gamma_i & [\lambda_i] \\
& z_{ij} \leq 1 & [\mu_{ij}]
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$$

$$
\begin{aligned}
\beta_i(x,\Gamma_i) &= \max_{z\geq 0} \min_{\lambda,\mu \geq 0} \quad \sum_j \delta_{ij}|x_j| z_{ij} + \lambda_i \Big(\Gamma_i - \sum_j z_{ij}\Big) \sum_j \mu_{ij} \Big(1 - z_{ij}\Big) \\
&= \min_{\lambda,\mu\geq 0} \max_{z \geq 0} \quad \lambda_i \Gamma_i + \sum_j \mu_{ij} \qquad + \sum_j z_{ij} \Big(\delta_{ij}|x_j| - \lambda_i - \mu_{ij}\Big)
\end{aligned}
$$

Cardinality constrained LP III

$$\beta_i(x,\Gamma_i) = \max_{z\geq 0} \quad \sum_j \delta_{ij}|x_j| z_{ij}$$

$$\sum_j z_{ij} \leq \Gamma_i \qquad [\lambda_i]$$

$$z_{ij} \leq 1 \qquad [\mu_{ij}]$$

$$\beta_i(x,\Gamma_i) = \max_{z\geq 0} \min_{\lambda,\mu\geq 0} \quad \sum_j \delta_{ij}|x_j| z_{ij} + \lambda_i\Big(\Gamma_i - \sum_j z_{ij}\Big)\sum_j \mu_{ij}\Big(1 - z_{ij}\Big)$$

$$= \min_{\lambda,\mu\geq 0} \quad \lambda_i\Gamma_i + \sum_j \mu_{ij}$$

$$\text{s.t.} \quad \delta_{ij}|x_j| \leq \lambda_i + \mu_{ij}$$

Cardinality constrained LP IV

In the end we obtain

$$
\begin{aligned}
\min_{x,\beta,\lambda,\mu} \quad & c^\top x \\
& \sum_j \bar{a}_{ij} x_j + \beta_i \leq b_i && \forall i \\
& \lambda_i \Gamma_i + \sum_j \mu_{ij} \leq \beta_i && \forall i \\
& \delta_{ij} x_j \leq \lambda_i + \mu_{ij} && \forall i,j \\
& -\delta_{ij} x_j \leq \lambda_i + \mu_{ij} && \forall i,j \\
& \lambda \geq 0, \quad \mu \geq 0 \\
& \underline{x} \leq x \leq \bar{x}
\end{aligned}
$$

Contents

Robust constraint implying a probabilistic guarantee

Definition

We say that, for a given set of probability measures $\mathbb{P} \in \mathcal{P}$, the constraint

$$g(x, \xi) \leq 0, \qquad \forall \xi \in R,$$

implies a probabilistic guarantee of level ε if, for all $\mathbb{P} \in \mathcal{P}$,

$$\mathbb{P}\Big(g(x, \xi) \leq 0\Big) \geq 1 - \varepsilon.$$

Probability guarantee for ellipsoidal uncertainty

- We consider a linear constraint

$$\sum_j \tilde{\boldsymbol{a}}_{ij} x_j \leq b_i, \qquad \forall i \in [m]$$

- We assume that $\tilde{\boldsymbol{a}}_{ij} = \bar{a}_{ij}(1 + \varepsilon \boldsymbol{\xi}_{ij})$ where $\boldsymbol{\xi}_{ij}$ is a random variable with mean 0, contained in $[-1, 1]$, and independent in j.
- Then the robust constraint

$$\sum_j \bar{a}_{ij} x_j + \varepsilon \Omega \sqrt{\sum_j \bar{a}_{ij}^2 x_j^2} \leq b_i^+, \qquad \forall i \in [m]$$

implies a probabilistic guarantee of level $1 - e^{-\Omega^2/2}$.

Contents

A combinatorial optimization problem with cardinality constraint

We consider a combinatorial optimization problem:

$$\min_{x\in\{0,1\}^N} \max_{\tilde{c}\in R} \tilde{c}^\top x$$
$$s.t. \quad x \in X$$

where R is such that each $\tilde{c}_i \in [\bar{c}_i, \bar{c}_i + \delta_i]$, with at most Γ coefficient deviating from $\bar{c}_i$.
Thus, the problem reads

$$(P) \qquad \min_{x\in\{0,1\}^N} \bar{c}^\top x + \max_{|S|\leq\Gamma} \sum_{i\in S} \delta_i x_i$$
$$s.t. \quad x \in X$$

wlog we assume that the i are ordered by decreasing cost uncertainty span: $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$.

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wlog we assume that the i are ordered by decreasing cost uncertainty span: $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$.

Solving the robust combinatorial problem I

We can write (P) as

$$\begin{aligned} \min_{x \in \{0,1\}^N} \max_{\zeta \in [0,1]^n} \quad & \bar{c}^\top x + \sum_{i=1}^n \delta_i x_i \zeta \\ s.t. \quad & x \in X \\ & \sum_{i=1}^n \zeta \leq \Gamma \end{aligned}$$

For a given $x \in X$ we dualize the inner maximization LP problem

Solving the robust combinatorial problem II

Thus we can write (P) as

$$\begin{aligned} \min_{x,y,\theta} \quad & \bar{c}^\top x + \Gamma\theta + \sum_{j=1}^n y_j \\ s.t. \quad & x \in X \\ & y_j + \theta \geq \delta_j x_j \\ & y_j, \theta \geq 0 \end{aligned}$$

Note that an optimal solution satisfies

$$y_j = (\delta_j x_j - \theta)^+ = (\delta_j - \theta)^+ x_j$$

as $x_j \in \{0,1\}$, and $\theta \geq 0$.

Solving the robust combinatorial problem II

Thus we can write (P) as

$$\begin{aligned} \min_{x,y,\theta} \quad & \bar{c}^\top x + \Gamma\theta + \sum_{j=1}^n y_j \\ s.t. \quad & x \in X \\ & y_j + \theta \geq \delta_j x_j \\ & y_j, \theta \geq 0 \end{aligned}$$

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Solving the robust combinatorial problem III

Thus we can write (P) as

$$\begin{aligned} \min_{\theta \geq 0} \min_{x} \quad & \bar{c}^\top x + \Gamma\theta + \sum_{j=1}^{n} x_j(\delta_j - \theta)^+ \\ s.t. \quad & x \in X \end{aligned}$$

We can now decompose the problem for $\theta \in [\delta_\ell, \delta_{\ell-1}]$ where $\delta_{n+1} = 0$ and $\delta_0 = +\infty$. Therefore, we have

$$val(P) = \min_{\ell \in [n]} Z^\ell$$

where

$$Z^\ell = \min_{x \in X, \theta \in [\delta_\ell, \delta_{\ell-1}]} \quad \bar{c}^\top x + \Gamma\theta + \sum_{j=1}^{\ell-1} x_j(\delta_j - \theta)$$

Solving the robust combinatorial problem III

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Solving the robust combinatorial problem III

Thus we can write (P) as

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We can now decompose the problem for $\theta \in [\delta_\ell, \delta_{\ell-1}]$ where $\delta_{n+1} = 0$ and $\delta_0 = +\infty$. Therefore, we have

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Solving the robust combinatorial problem IV

As the problem is linear in θ we have that

$$Z^\ell = \min_{x \in X, \theta \in [\delta_\ell, \delta_{\ell-1}]} \quad \bar{c}^\top x + \Gamma\theta + \sum_{j=1}^{\ell-1} x_j(\delta_j - \theta)$$

is attained for $\theta = \delta_\ell$ or $\theta = \delta_{\ell-1}$.
So in the end, we have

$$val(P) = \min_{\ell \in [n]} G^\ell$$

where

$$G^\ell = \Gamma\delta_\ell + \min_{x \in X} \Big\{ \bar{c}^\top x + \sum_{j=1}^{\ell} \underbrace{(\delta_j - \delta_\ell)}_{\geq 0} x_j \Big\}$$

Algorithm for the robust problem

1. For $\ell \in [n]$, solve

$$G^\ell = \Gamma\delta_\ell + \min_{x \in X} \quad \Big\{\bar{c}^\top x + \sum_{i=1}^{\ell} (\delta_i - \delta_\ell) x_j \Big\}$$

   with optimal solution x_ℓ

2. Set $\ell^* \in \arg\min_{\ell \in [n]} G^\ell$
3. Return $val(P) = G^{\ell^*}$ and $x^* = x_\ell$

Contents

Why do robust optimization ?

- Because you want to account for some uncertainty
- Because you want to have a solution that resists to changes in data
- Because your data is unprecise and robustness yield better out-of-sample result
- Because you do not have the law of the uncertainty
- Because you can control the robustness level
- Because your problem is "one-shot"

Which uncertainty set to choose ?

- An uncertainty set that is computationally tractable
- An uncertainty set that yields good results
- An uncertainty set that have some theoretical soundness
- An uncertainty set that take available data into account
- Select uncertainty set / level through cross-validation

Is there some theoretical results ?

- Yes: with some assumption over the randomness (e.g. bounded and symmetric around $\bar{a}$) some uncertainty set (e.g. ellipsoidal) have a probabilistic guarantee:

$$\forall \xi \in R_\varepsilon, \quad g(x,\xi) \leq 0 \qquad \Longrightarrow \qquad \mathbb{P}\Big(g(x,\xi) \leq 0\Big) \geq 1-\varepsilon$$

- Yes: in some cases approximation scheme for nominal problem can be extended to robust problem (e.g. cardinal uncertainty in combinatorial problem)
- Yes: using relevant data we can use statistical tools to construct a robust set R that imply a probabilistic guarantee

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