

# Exercises : Duality

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**Exercise 1** (Dual formulation). Let  $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . Show that

$$1. \mathbb{I}_{g(x)=0} = \sup_{\lambda \in \mathbb{R}^m} \lambda^\top g(x)$$

$$2. \mathbb{I}_{g(x) \leq 0} = \sup_{\lambda \in \mathbb{R}_+^m} \lambda^\top g(x)$$

$$3. \mathbb{I}_{g(x) \in C} = \sup_{\lambda \in -C^\oplus} \lambda^\top g(x) \text{ where } C \text{ is a closed convex cone, and } C^\oplus := \{\lambda \in \mathbb{R}^m \mid \lambda^\top c \geq 0, \forall c \in C\}.$$

**Answers:**

1. If  $g(x) \neq 0$  there is  $i \in [m]$  such that  $g_i(x) \neq 0$ , and we choose  $\lambda_i$  accordingly.

2. Same reasoning.

3. If  $g(x) \in C$ ,  $\lambda^\top g(x) \leq 0$ , and  $0 \in -C^\oplus$ . If  $g(x) \notin C$ , by separation of the convex compact  $\{g(x)\}$  from the closed convex set  $C$  there exists  $\lambda \in \mathbb{R}^n$  such that  $\lambda^\top g(x) > b > \lambda^\top c$  for all  $c \in C$ . As  $C$  is a cone,  $tc \in C$  for all  $t > 0$ , and thus  $\lambda \in -C^\oplus$ . Further  $b \geq 0$ , thus  $t\lambda^\top g(x) \rightarrow +\infty$  when  $t \rightarrow \infty$ .

**Exercise 2** (Linear Programming). Consider the following linear problem (LP)

$$(P) \quad \begin{aligned} & \text{Min}_{x \geq 0} \quad c^\top x \\ & \text{s.t.} \quad Ax = b \end{aligned}$$

1. Show that the dual of (P) is an LP.

2. Show that the dual of the dual of (P) is equivalent to (P).

**Answers:**

1. The dual of (P) is

$$(D) \quad \begin{aligned} & \text{Max}_{\lambda} \quad -b^\top \lambda \\ & \text{s.t.} \quad A^\top \lambda + c \leq 0 \end{aligned}$$

2. Direct by computing the dual of (D).

**Exercise 3** (Quadratically Constrained Quadratic Programming). Consider the problem

$$(QCQP) \quad \begin{aligned} & \text{Min}_{x \in \mathbb{R}^n} \quad \frac{1}{2} x^\top P_0 x + q_0^\top x + r_0 \\ & \quad \frac{1}{2} x^\top P_i x + q_i^\top x + r_i \leq 0 \quad \forall i \in [m] \end{aligned}$$

where  $P_0 \in S_{++}^n$  and  $P_i \in S_+^n$ .

1. Show by duality that, for  $\mu \in \mathbb{R}_+^m$ , there exists  $P_\mu, q_\mu$  and  $r_\mu$ , such that  $g(\mu) = -\frac{1}{2} q_\mu P_\mu^{-1} q_\mu + r_\mu \leq \text{val}(P)$ .

2. Give an easy condition under which  $\text{val}(P) = \sup_{\mu \geq 0} g(\mu)$ .

**Answers:**

1. Simply write the dual function we get

$$P_\mu = P_0 + \sum_i \mu_i P_i, \quad q_\mu = q_0 + \sum_i \mu_i q_i$$

$$r_\mu = r_0 + \sum_i \mu_i r_i$$

2. The problem is convex, Slater's condition ensure constraint qualification, thus a condition would be the existence of a strictly satisfying all constraints.

**Exercise 4** (Conic Programming). Let  $K \subset \mathbb{R}^n$  be a closed convex pointed cone, and denote  $x \preceq_K y$  iff  $y \in x + K$ . Consider the following program, with  $A \in M_{m,n}$  and  $b \in \mathbb{R}^m$ .

$$(P) \quad \begin{aligned} \text{Min}_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \preceq_K 0 \end{aligned}$$

1. Show that (P) is a convex optimization problem.
2. Denote  $\mathcal{L}(x, \lambda, \mu) = c^\top x + \lambda^\top (Ax - b) + \mu^\top x$ . Show that  $\text{val}(P) = \text{Min}_{x \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^m, \mu \in K^\oplus} \mathcal{L}(x, \lambda, \mu)$ .
3. Give a dual problem to (P).

**Answers:**

1.  $x \preceq_K 0$  iff  $x \in -K$ , and  $-K$  is convex.
2. If  $x \in -K$ , for any  $\mu \in K^\oplus$ ,  $\mu^\top x \leq 0$ , thus  $\sup_{\mu \in K^\oplus} \mu^\top x = 0$ . If  $-x \notin K = K^{\oplus\oplus}$ , there exists  $\lambda \in K^\oplus$ , such that  $-x^\top \lambda < 0$ , hence  $\sup_{\mu \in K^\oplus} \mu^\top x = +\infty$ . (or see ex 1).
3. By min-max duality we consider

$$\text{Max}_{\lambda \in \mathbb{R}^m, \mu \in K^\oplus} -b^\top \lambda + \inf_{x \in \mathbb{R}^n} (A^\top \lambda + c + \mu)^\top x +$$

which yields

$$(D) \quad \begin{aligned} \text{Max} \quad & -b^\top \lambda \\ & A^\top \lambda + c + \mu = 0 \quad \mu \in K^\oplus \end{aligned}$$

**Exercise 5** (Duality gap). Consider the following problem

$$\begin{aligned} \text{Min}_{x \in \mathbb{R}, y \in \mathbb{R}_*^+} \quad & e^{-x} \\ \text{s.t.} \quad & x^2/y \leq 0 \end{aligned}$$

1. Find the optimal solution of this problem.
2. Write and solve the (Lagrangian) dual problem. Is there a duality gap?

**Answers:**

$$\mathcal{L}(x, y; \mu) = e^{-x} + \mu x^2/y$$

$$g(\mu) = \inf_{x \in \mathbb{R}, y > 0} e^{-x} + \mu x^2/y = 0$$

as the term inside the inf is positive, and choosing  $x = t$ ,  $y = t^3$  goes to 0 for all  $\mu$ .

**Exercise 6** (Two-way partitioning). Let  $W \in S_n$  be a symmetric matrix, consider the following problem.

$$(P) \quad \begin{aligned} \text{Min}_{x \in \mathbb{R}^n} \quad & x^\top W x \\ \text{s.t.} \quad & x_i^2 = 1 \quad \forall i \in [n] \end{aligned}$$

1. Consider a set of  $n$  element that you want to partition in 2 subsets, with a cost  $c_{i,j}$  if  $i$  and  $j$  are in the same set, and a cost  $-c_{i,j}$  if they are in a different set. Justify that it can be solved by solving (P).

2. Is (P) a convex problem?

3. Show that, for any  $\lambda \in \mathbb{R}^n$  such that  $W + \text{diag}(\lambda) \succeq 0$ , we have  $\text{val}(P) \geq -\sum \lambda_i$ . Deduce a lower bound on  $\text{val}(P)$ .

**Answers:**

1. The constraint ensures that  $x_i \in \{-1, 1\}$ , each value representing one subset. We set  $W_{i,j} = c_{i,j}$ .
2. No, because the set of admissible points is  $\{-1, 1\}^n$ .
3. The Lagrangian of (P) is

$$\begin{aligned} \mathcal{L}(x, \lambda) &= x^\top W x + \sum_{i=1}^n \lambda_i (x_i^2 - 1) \\ &= x^\top (W + \text{diag}(\lambda)) x - \sum_{i=1}^n \lambda_i \end{aligned}$$

And we have,

$$\begin{aligned} g(\lambda) &= \inf_x x^\top (W + \text{diag}(\lambda)) x - \sum_{i=1}^n \lambda_i \\ &= -\sum_{i=1}^n \lambda_i - \mathbb{I}_{W + \text{diag}(\lambda) \succeq 0} \leq \text{val}(P) \end{aligned}$$

Thus, if  $\lambda_{\min}$  is the small eigenvalue of  $W$  we have  $W + \text{diag}(\lambda) \geq 0$ , and  $\text{val}(P) \geq n\lambda_{\min}$ .

**Exercise 7** (Linear SVM : duality). Consider the following problem (see : <https://www.youtube.com/watch?v=IOetFPgsMUc> for background)

$$\begin{aligned} \min_{w \in \mathbb{R}^d, b \in \mathbb{R}} \quad & \frac{1}{2} \|w\|^2 \\ \text{s.t.} \quad & y_i(w^\top x_i + b) \geq 1 \quad \forall i \in [n] \\ & \eta_i \geq 0 \quad \forall i \in [n] \end{aligned}$$

1. In which case can we guarantee strong duality ?
2. Write the dual of this optimization problem and express the optimal primal solution  $(w^\#, b^\#)$  in terms of the optimal dual solution.

**Exercise 8.** We consider the following problem.

$$\begin{aligned} \text{Min}_{x_1, x_2} \quad & x_1^2 + x_2^2 \quad (1) \\ \text{s.t.} \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 \quad (2) \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1 \quad (3) \end{aligned}$$

1. Classify this problem (After 5th course)
2. Find the optimal solution and value of this problem.
3. Write and solve the KKT equation for this problem.
4. Derive and solve the Lagrangian dual of this problem.
5. Do we have strong duality ? If yes, could we have known it from the start ? If not, can you comment on why ?

**Answers:**

1. This is a convex QCQP
2. The only admissible point, and hence the optimal solution is  $(1, 0)$ , with value 1.

3. The Lagrangian is

$$\begin{aligned} \mathcal{L}(x, \lambda) = & x_1^2 + x_2^2 + \lambda_1((x_1 - 1)^2 + (x_2 - 1)^2 - 1) \\ & + \lambda_2((x_1 - 1)^2 + (x_2 + 1)^2 - 1) \end{aligned}$$

KKT condition are

- Gradient of Lagrangian is null :

$$\begin{aligned} 2x_1 + 2\lambda_1(x_1 - 1) + 2\lambda_2(x_1 - 1) &= 0 \\ 2x_2 + 2\lambda_1(x_2 - 1) + 2\lambda_2(x_2 + 1) &= 0 \end{aligned}$$

- $x$  is primal feasible :  $(x_1 - 1)^2 + (x_2 - 1)^2 \leq 1$  and  $(x_1 - 1)^2 + (x_2 + 1)^2 \leq 1$
- $\lambda$  is dual feasible  $\lambda_1 \geq 0, \lambda_2 \geq 0$ .
- Complementary slackness:

$$\begin{aligned} \lambda_1 = 0 \quad \text{or} \quad (x_1 - 1)^2 + (x_2 - 1)^2 &= 1 \\ \lambda_2 = 0 \quad \text{or} \quad (x_1 - 1)^2 + (x_2 + 1)^2 &= 1 \end{aligned}$$

$x$  feasible is  $x = (1, 0)$ , which imply  $2 = 0$  which is impossible. Thus there is no pair  $(x, \lambda)$  satisfying the KKT equations. The KKT equations fails to give the optimal solution because the constraints are not qualified.

4. The Lagrange dual function is

$$\begin{aligned} g(\lambda_1, \lambda_2) &= \inf_{x_1, x_2} \mathcal{L}(x, \lambda) \\ &= \inf_{x_1, x_2} (1 + \lambda_1 + \lambda_2)(x_1^2 + x_2^2) \\ &\quad - 2(\lambda_1 + \lambda_2)x_1 - 2(\lambda_1 - \lambda_2)x_2 + \lambda_1 + \lambda_2 \\ &= \lambda_1 + \lambda_2 - \frac{(\lambda_1 + \lambda_2)^2 + (\lambda_1 - \lambda_2)^2}{1 + \lambda_1 + \lambda_2} \\ &\quad \text{if } 1 + \lambda_1 + \lambda_2 > 0 \end{aligned}$$

The dual problem reads

$$\begin{aligned} \text{Max}_{\lambda} \quad & \frac{\lambda_1 + \lambda_2 - (\lambda_1 - \lambda_2)^2}{1 + \lambda_1 + \lambda_2} \\ \text{s.t.} \quad & \lambda_1 \geq 0, \lambda_2 \geq 0 \end{aligned}$$

By symmetry the optimum is attained at  $\lambda_1 = \lambda_2$ , thus the dual reads

$$\text{Max}_{\lambda_1 \geq 0} \quad \frac{2\lambda_1}{2\lambda_1 + 1}$$

Which has value 1 and no solution.

5. The dual problem have the same value as the primal problem, thus we have strong duality.

However there does not exist a dual multiplier, which is why there is no solution to the KKT equations.

We could not guarantee the existence of a primal-dual optimal solution through KKT as the constraints were not qualified.