

Exercises : Optimality conditions

March 27, 2023

Exercise 1. Solve the following optimization problem 2.

$$\begin{aligned} \text{Min}_{x,y \in \mathbb{R}^2} \quad & (x-1)^2 + (y-2)^2 \\ \text{s.t.} \quad & x \leq y \\ & x + 2y \leq 2 \end{aligned}$$

$$\begin{aligned} \min_{x_1, x_2} \quad & 4x_1^2 - x_1x_2 + x_2^2 - 12x_1 \\ \text{s.t.} \quad & x_1 - 2x_2 + x_3 = 5 \\ & x_1^2 + 3x_2^2 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Exercise 2 (First order optimality condition). 3.
Consider, for f differentiable,

$$\begin{aligned} (P) \quad \text{Min}_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

$$\begin{aligned} \min_{x_1, x_2, x_3} \quad & e_1^x - x_1x_2 + x_3^3 \\ \text{s.t.} \quad & \ln(e^{x_1-4x_2} + e^{x_1+x_3}) \leq 2x_1 + 3 \\ & 2x_1^2 + x_2^2 \leq 2 \end{aligned}$$

Recall that

$$\begin{aligned} T_X(x_0) = \{ d \in \mathbb{R}^n \mid \exists d_k \rightarrow d, \exists t_k \searrow 0, \\ \text{s.t. } x_0 + t_k d_k \in X \} \end{aligned} \quad 4.$$

and $K^\oplus = \{ \lambda \mid \lambda^\top x \geq 0, \forall x \in K \}$.

Show that

$$\begin{aligned} \min_{x_1, x_2} \quad & -x_1 \\ \text{s.t.} \quad & -x_2 - (x_1 - 1)^3 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

1. If x_0 is an optimal solution to (P), then $\nabla f(x_0) \in [T_X(x_0)]^+$. 5.

2. If f is convex, X is closed convex, and $\nabla f(x_0) \in [T_X(x_0)]^+$, then x_0 is an optimal solution to (P).

$$\begin{aligned} \min_{x_1, x_2} \quad & -x_1 \\ \text{s.t.} \quad & x_2 - (x_1 - 1)^3 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Exercise 3. In the following cases, are the KKT conditions necessary / sufficient ?

1.

$$\begin{aligned} \min_{x_1, x_2, x_3} \quad & 12x_1 - 5x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 - x_3 = 5 \\ & x_1 - x_2 \geq -2 \\ & 2x_1 - 4x_2 \leq 12 \end{aligned}$$

Exercise 4. Solve the following problem using first order optimality conditions

$$\begin{aligned} \min_{x_1, x_2} \quad & -2(x_1 - 2)^2 - x_2^2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 25 \\ & x_1 \geq 0 \end{aligned}$$