Constrained optimization

V. Leclère (ENPC)

May 24th, 2024

Why should I bother to learn this stuff?

- Most real problems have constraints that you have to deal with.
- This course give a snapshot of the tools available to you.
- $\Longrightarrow$ useful for
  - having an idea of what can be done when you have constraints

Constrained optimization problem

- In the previous courses we have developed algorithms for unconstrained optimization problem.
- We now want to sketch some methods to deal with the constrained problem

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

- We are going to discuss multiple types of constraints set X:
  - X is a ball : $\{x \mid \|x - x_0\|_2 \leq r\}$
  - X is a box : $\{x \mid \underline{x_i} \leq x_i \leq \bar{x}_i \quad \forall i \in [n]\}$
  - X is a polyhedron: $\{x \mid Ax \leq b\}$
  - X is given through explicit constraints $\{x \mid g(x) = 0, \quad h(x) \leq 0\}$

Contents

Contents

Admissible descent direction

- Recall that a descent direction d at point $x^{(k)} \in \mathbb{R}^n$ is a vector such that $\nabla f(x^{(k)})^\top d < 0$.
- An admissible descent direction at point $x^{(k)} \in X$ is a descent direction $d \in \mathbb{R}^n$ such that,

$$\exists \varepsilon > 0, \quad \forall t \leq \varepsilon, \qquad x^{(k)} + td \in X.$$

- In other words, an admissible descent direction, is a direction that locally decreases the objective while staying in the constraint set.
- An admissible descent direction algorithm is naturally defined by:
  - A choice of admissible descent direction $d^{(k)}$
  - A choice of (sufficiently small) step $t^{(k)}$
  - $x^{(k+1)} = x^{(k)} + t^{(k)} d^{(k)} \in X$
- Warning: this does not necessarily converge. We can construct examples where the step size gets increasingly small because of the constraints.

A counter example

Consider

$$\min_{x\in\mathbb{R}^3} \quad f(x) := \frac{4}{3}(x_1^2 - x_1x_2 + x_2^2)^{3/4} - x_3$$
$$\text{s.t.} \quad x \geq 0$$

We set $x^{(0)} = (0, 2^{-3/2}, 0)$, and $d^{(k)}$ such that $d_i^{(k)} = -g_i^{(k)}\mathbb{1}_{x_i^{(k)}>0}$, with $g_i^{(k)} = \nabla f(x^{(k)})$, and choose $t^{(k)}$ as the optimal step.

- This is an admissible direction descent with optimal step.
- f is strictly convex.
- $x^{(k)}$ converges toward a non-optimal point.

A counter example

Consider

$$\begin{aligned} \min_{x\in\mathbb{R}^3} \quad & f(x) := \frac{4}{3}(x_1^2 - x_1x_2 + x_2^2)^{3/4} - x_3 \\ \text{s.t.} \quad & x \geq 0 \end{aligned}$$

We set $x^{(0)} = (0, 2^{-3/2}, 0)$, and $d^{(k)}$ such that $d_i^{(k)} = -g_i^{(k)} \mathbb{1}_{x_i^{(k)}>0}$, with $g_i^{(k)} = \nabla f(x^{(k)})$, and choose $t^{(k)}$ as the optimal step.

- This is an admissible direction descent with optimal step.
- f is strictly convex.
- $x^{(k)}$ converges toward a non-optimal point.

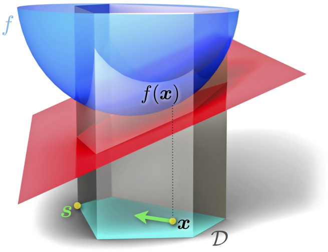
Conditional gradient algorithm

We address an optimization problem with a convex objective function f and compact polyhedral constraint set X, i.e.

$$\min_{x \in X \subset \mathbb{R}^n} \qquad f(x)$$

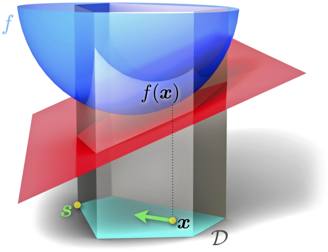
where

$$X = \left\{ x \in \mathbb{R}^n \quad \middle| \quad Ax \leq b, \quad \tilde{A}x = \tilde{b} \right\}$$



Conditional gradient algorithm

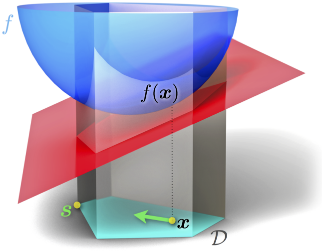
It is a descent algorithm, where we first look for an admissible descent direction $d^{(k)}$, and then look for the optimal step.



Conditional gradient algorithm

It is a descent algorithm, where we first look for an admissible descent direction $d^{(k)}$, and then look for the optimal step.
As f is convex, we know that for any point $x^{(k)}$,

$$f(y) \geq f(x^{(k)}) + \nabla f(x^{(k)}) \cdot (y - x^{(k)})$$

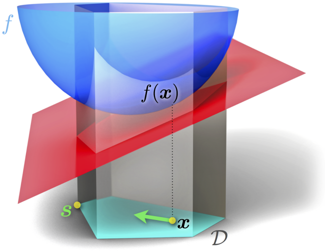


Conditional gradient algorithm

It is a descent algorithm, where we first look for an admissible descent direction $d^{(k)}$, and then look for the optimal step.
As f is convex, we know that for any point $x^{(k)}$,

$$f(y) \geq f(x^{(k)}) + \nabla f(x^{(k)}) \cdot (y - x^{(k)})$$

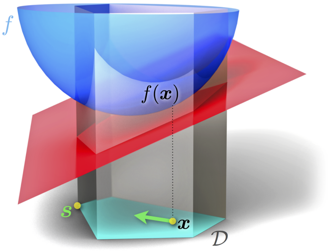
The conditional gradient method consists in choosing the descent direction that minimizes the linearization of f over X.



Conditional gradient algorithm

The conditional gradient method consists in choosing the descent direction that minimizes the linearization of f over X. More precisely, at step k we solve

$$y^{(k)} \in \underset{y \in X}{\arg\min} \quad f(x^{(k)}) + \nabla f(x^{(k)}) \cdot (y - x^{(k)}).$$



Remarks on conditional gradient

$$y^{(k)} \in \arg\min_{y \in X} \quad f(x^{(k)}) + \nabla f(x^{(k)}) \cdot (y - x^{(k)}).$$

- This problem is linear, hence easy to solve.
- By the convexity inequality, the value of the linearized Problem is a lower bound to the true problem.
- As $y^{(k)} \in X$, $d^{(k)} = y^{(k)} - x^{(k)}$ is a *feasable direction*, in the sense that for all $t \in [0, 1]$, $x^{(k)} + td^{(k)} \in X$.
- If $y^{(k)}$ is obtained through the simplex method it is an extreme point of X, which means that, for $t > 1$, $x^{(k)} + td^{(k)} \notin X$.
- If $y^{(k)} = x^{(k)}$ then we have found an optimal solution.
- We also have $y^{(k)} \in \arg\min_{y \in X} \nabla f(x^{(k)}) \cdot y$, the lower-bound being obtained easily.

Remarks on conditional gradient

$$y^{(k)} \in \arg\min_{y \in X} \quad f(x^{(k)}) + \nabla f(x^{(k)}) \cdot (y - x^{(k)}).$$

- This problem is linear, hence easy to solve.
- By the convexity inequality, the value of the linearized Problem is a lower bound to the true problem.
- As $y^{(k)} \in X$, $d^{(k)} = y^{(k)} - x^{(k)}$ is a *feasable direction*, in the sense that for all $t \in [0,1]$, $x^{(k)} + td^{(k)} \in X$.
- If $y^{(k)}$ is obtained through the simplex method it is an extreme point of X, which means that, for $t > 1$, $x^{(k)} + td^{(k)} \notin X$.
- If $y^{(k)} = x^{(k)}$ then we have found an optimal solution.
- We also have $y^{(k)} \in \arg\min_{y \in X} \nabla f(x^{(k)}) \cdot y$, the lower-bound being obtained easily.

Remarks on conditional gradient

$$y^{(k)} \in \arg\min_{y \in X} \quad f(x^{(k)}) + \nabla f(x^{(k)}) \cdot (y - x^{(k)}).$$

- This problem is linear, hence easy to solve.
- By the convexity inequality, the value of the linearized Problem is a lower bound to the true problem.
- As $y^{(k)} \in X$, $d^{(k)} = y^{(k)} - x^{(k)}$ is a *feasable direction*, in the sense that for all $t \in [0, 1]$, $x^{(k)} + td^{(k)} \in X$.
- If $y^{(k)}$ is obtained through the simplex method it is an extreme point of X, which means that, for $t > 1$, $x^{(k)} + td^{(k)} \notin X$.
- If $y^{(k)} = x^{(k)}$ then we have found an optimal solution.
- We also have $y^{(k)} \in \arg\min_{y \in X} \nabla f(x^{(k)}) \cdot y$, the lower-bound being obtained easily.

Remarks on conditional gradient

$$y^{(k)} \in \arg\min_{y \in X} \quad f(x^{(k)}) + \nabla f(x^{(k)}) \cdot (y - x^{(k)}).$$

- This problem is linear, hence easy to solve.
- By the convexity inequality, the value of the linearized Problem is a lower bound to the true problem.
- As $y^{(k)} \in X$, $d^{(k)} = y^{(k)} - x^{(k)}$ is a *feasable direction*, in the sense that for all $t \in [0,1]$, $x^{(k)} + td^{(k)} \in X$.
- If $y^{(k)}$ is obtained through the simplex method it is an extreme point of X, which means that, for $t > 1$, $x^{(k)} + td^{(k)} \notin X$.
- If $y^{(k)} = x^{(k)}$ then we have found an optimal solution.
- We also have $y^{(k)} \in \arg\min_{y \in X} \nabla f(x^{(k)}) \cdot y$, the lower-bound being obtained easily.

Remarks on conditional gradient

$$y^{(k)} \in \arg\min_{y \in X} \quad f(x^{(k)}) + \nabla f(x^{(k)}) \cdot (y - x^{(k)}).$$

- This problem is linear, hence easy to solve.
- By the convexity inequality, the value of the linearized Problem is a lower bound to the true problem.
- As $y^{(k)} \in X$, $d^{(k)} = y^{(k)} - x^{(k)}$ is a *feasable direction*, in the sense that for all $t \in [0,1]$, $x^{(k)} + td^{(k)} \in X$.
- If $y^{(k)}$ is obtained through the simplex method it is an extreme point of X, which means that, for $t > 1$, $x^{(k)} + td^{(k)} \notin X$.
- If $y^{(k)} = x^{(k)}$ then we have found an optimal solution.
- We also have $y^{(k)} \in \arg\min_{y \in X} \nabla f(x^{(k)}) \cdot y$, the lower-bound being obtained easily.

Remarks on conditional gradient

$$y^{(k)} \in \arg\min_{y \in X} \quad f(x^{(k)}) + \nabla f(x^{(k)}) \cdot (y - x^{(k)}).$$

- This problem is linear, hence easy to solve.
- By the convexity inequality, the value of the linearized Problem is a lower bound to the true problem.
- As $y^{(k)} \in X$, $d^{(k)} = y^{(k)} - x^{(k)}$ is a *feasable direction*, in the sense that for all $t \in [0,1]$, $x^{(k)} + td^{(k)} \in X$.
- If $y^{(k)}$ is obtained through the simplex method it is an extreme point of X, which means that, for $t > 1$, $x^{(k)} + td^{(k)} \notin X$.
- If $y^{(k)} = x^{(k)}$ then we have found an optimal solution.
- We also have $y^{(k)} \in \arg\min_{y \in X} \nabla f(x^{(k)}) \cdot y$, the lower-bound being obtained easily.

Contents

Projection on a convex set ♡

Let $X \subset \mathbb{R}^n$ be a nonempty closed convex set. We call $P_X : \mathbb{R}^n \to \mathbb{R}^n$ the projection on X the fonction such that

$$P_X(x) = \underset{x' \in X}{\arg\min} \|x' - x\|_2^2$$

We have

- $\bar{x} = P_X(x)$ iff $(x - \bar{x}) \in N_X(\bar{x})$ (i.e. $\langle x - \bar{x}, x' - \bar{x} \rangle \leq 0, \quad \forall x' \in X$)
- $\langle P_X(y) - P_X(x), y - x \rangle \geq 0$ (P_X is *non-decreasing*)
- $\|P_X(y) - P_X(x)\|_2 \leq \|y - x\|$ (P_X is a *contraction*)

♠ Exercise: Prove these results

Projection on a convex set

♡

Let $X \subset \mathbb{R}^n$ be a nonempty closed convex set. We call $P_X : \mathbb{R}^n \to \mathbb{R}^n$ the projection on X the fonction such that

$$P_X(x) = \underset{x' \in X}{\arg\min} \, \|x' - x\|_2^2$$

We have

- $\bar{x} = P_X(x)$ iff $(x - \bar{x}) \in N_X(\bar{x})$ (i.e. $\langle x - \bar{x} \, , x' - \bar{x} \rangle \leq 0, \quad \forall x' \in X$)
- $\langle P_X(y) - P_X(x) \, , y - x \rangle \geq 0$ (P_X is *non-decreasing*)
- $\|P_X(y) - P_X(x)\|_2 \leq \|y - x\|$ (P_X is a *contraction*)

♠ Exercise: Prove these results

Projected gradient

Consider

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{Min}} \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

where f is differentiable and X convex.
The projected gradient algorithm generates the following sequence

$$x^{(k+1)} = P_X\left[x^{(k)} - t^{(k)} g^{(k)}\right]$$

Projected gradient

Theorem

Assume that $X \neq \emptyset$ is a closed convex set. $x^\sharp \in X$ is a critical point if and only if for one (or all) $t > 0$,

$$x^\sharp = P_X\big[x^\sharp - t\nabla f(x^\sharp)\big].$$

Projected gradient II $\diamondsuit$

Theorem

Assume that $X \neq \emptyset$ is a closed convex set. $x^\sharp \in X$ is a critical point if and only if for one (or all) $t > 0$,

$$x^\sharp = P_X\left[x^\sharp - t\nabla f(x^\sharp)\right].$$

Theorem

If f is lower bounded on X, and with L-Lipschitz gradients, and X closed convex (nonempty) set. Then the projected gradient algorithm with step staying in $[a,b] \subset]0, 2/L[$, then $\|x^{(k+1)} - x^{(k)}\| \to 0$, and any adherence point of $\{x^{(k)}\}_{k\in\mathbb{N}}$ is a critical point.

Corollary: if f convex differentiable with L-Lipschitz gradient, X compact convex nonempty, the projected gradient algorithm with step $1/L$ is converging toward the optimal solution.

When to use ? ♡

- Projected gradient is useful only if the projection is simple, as projecting over a convex set consists in solving a constrained optimization problem.
- Projection is simple for balls and boxes.
- Finding an admissible direction is doable if the constraint set is polyhedral, or more generally conic-representable.

Contents

Contents

Idea of penalization

♡

We consider the constrained optimization problem

$$(\mathcal{P}) \quad \underset{x \in \mathbb{R}^n}{\text{Min}} \quad f(x) \qquad \text{s.t.} \quad x \in X$$

and the following penalized version

$$(\mathcal{P}_r) \quad \underset{x \in \mathbb{R}^n}{\text{Min}} \quad f(x) + r p(x)$$

Thus, a (constrained) problem is replaced by a sequence of (unconstrained) problems.

♣ Exercise: What is happening if $p = \mathbb{I}_X$?

Idea of penalization ♡

We consider the constrained optimization problem

$$(\mathcal{P}) \quad \underset{x \in \mathbb{R}^n}{\mathrm{Min}} \quad f(x) \qquad \text{s.t.} \quad x \in X$$

and the following penalized version

$$(\mathcal{P}_r) \quad \underset{x \in \mathbb{R}^n}{\mathrm{Min}} \quad f(x) + r p(x)$$

Thus, a (constrained) problem is replaced by a sequence of (unconstrained) problems.

♣ Exercise: What is happening if $p = \mathbb{I}_X$?

Some monotonicity results

$$(\mathcal{P}_r) \quad \underset{x \in \mathbb{R}^n}{\text{Min}} \qquad f(x) + r p(x)$$

The idea is that, with higher r, the penalization has more impact on the problem.
More precisely, let $0 < r_1 < r_2$, and x_{r_i} be an optimal solution of $(\mathcal{P}_{r_i})$.
We have:

- $p(x_{r_1}) \geq p(x_{r_2})$
- $f(x_{r_1}) \leq f(x_{r_2})$

♣ Exercise: prove these results.

Outer penalization

A first idea for choosing a penalization function p consists in choosing a function p such that:

- $p(x) = 0$ for $x \in X$
- $p(x) > 0$ for $x \notin X$

intuitively the idea is that p is the fine to pay for not respecting the constraint. Heuristically, it should be increasing with the distance to X.

Outer penalization - theoretical results

Assume that

- p is l.s.c on $\mathbb{R}^n$
- $p \geq 0$
- $p(x) = 0$ iff $x \in X$

Further assume that f is l.s.c and there exists $r_0 > 0$ such that $x \mapsto f(x) + r_0 p(x)$ is coercive (i.e. $\to \infty$ if $\|x\| \to \infty$).
Then,

1. for $r > r_0$, $(\mathcal{P}_r)$ admit at least one optimal solution
2. $(x_r)_{r \to +\infty}$ is bounded
3. any adherence point of $(x_r)_{r \to +\infty}$ is an optimal solution of $\mathcal{P}$.

Outer penalization - quadratic case

Assume that

$$X = \left\{ x \in \mathbb{R}^n \quad \middle| \quad g(x) = 0, \quad h(x) \leq 0 \right\}$$

then the quadratic penalization consists in choosing

$$p : x \mapsto \|g(x)\|^2 + \|(h(x))^+\|^2$$

This choice is interesting as (for affinely lower-bounded f):

- $x \mapsto f(x) + rp(x)$ is differentiable if f is differentiable
- $x_r \to x^\sharp$ if $r \to \infty$

However, generally speaking, if the constraints are impactful (e.g. have non-zero optimal multipliers), then

$$x_r \notin X$$

Outer penalization - quadratic case

Assume that

$$X = \left\{x \in \mathbb{R}^n \quad | \quad g(x) = 0, \quad h(x) \leq 0\right\}$$

then the quadratic penalization consists in choosing

$$p : x \mapsto \|g(x)\|^2 + \|(h(x))^+\|^2$$

This choice is interesting as (for affinely lower-bounded f):

- $x \mapsto f(x) + rp(x)$ is differentiable if f is differentiable
- $x_r \to x^\sharp$ if $r \to \infty$

However, generally speaking, if the constraints are impactful (e.g. have non-zero optimal multipliers), then

$$x_r \notin X$$

Outer penalization - quadratic case

Assume that

$$X = \{x \in \mathbb{R}^n \quad | \quad g(x) = 0, \quad h(x) \leq 0\}$$

then the quadratic penalization consists in choosing

$$p : x \mapsto \|g(x)\|^2 + \|(h(x))^+\|^2$$

This choice is interesting as (for affinely lower-bounded f):

- $x \mapsto f(x) + rp(x)$ is differentiable if f is differentiable
- $x_r \to x^\sharp$ if $r \to \infty$

However, generally speaking, if the constraints are impactful (e.g. have non-zero optimal multipliers), then

$$x_r \notin X$$

Outer penalization - L^1 case

Assume that

$$X = \left\{ x \in \mathbb{R}^n \quad | \quad g(x) = 0, \quad h(x) \leq 0 \right\}$$

another natural penalization consists in choosing

$$p : x \mapsto \|g(x)\|_1 + \|(h(x))^+\|_1$$

The interest of this approach is that, if the problem is convex and the constraints are qualified at optimality, then, for r large enough, an optimal solution to the penalized problem $(\mathcal{P}_r)$ is an optimal solution to the original problem $(\mathcal{P})$. Thus, we speak of exact penalization.

Unfortunately, this comes to the price of non-differentiability.

Outer penalization - L^1 case

Assume that

$$X = \left\{ x \in \mathbb{R}^n \quad | \quad g(x) = 0, \quad h(x) \leq 0 \right\}$$

another natural penalization consists in choosing

$$p : x \mapsto \|g(x)\|_1 + \|(h(x))^+\|_1$$

The interest of this approach is that, if the problem is convex and the constraints are qualified at optimality, then, for r large enough, an optimal solution to the penalized problem $(\mathcal{P}_r)$ is an optimal solution to the original problem $(\mathcal{P})$. Thus, we speak of exact penalization.

Unfortunately, this comes to the price of non-differentiability.

Outer penalization - L^1 case

Assume that

$$X = \{x \in \mathbb{R}^n \quad | \quad g(x) = 0, \quad h(x) \leq 0\}$$

another natural penalization consists in choosing

$$p : x \mapsto \|g(x)\|_1 + \|(h(x))^+\|_1$$

The interest of this approach is that, if the problem is convex and the constraints are qualified at optimality, then, for r large enough, an optimal solution to the penalized problem $(\mathcal{P}_r)$ is an optimal solution to the original problem $(\mathcal{P})$. Thus, we speak of exact penalization.

Unfortunately, this comes to the price of non-differentiability.

Inner penalization

Another approach consists in choosing a penalization function p that takes value $+\infty$ outside of X.

The idea here is to add a potential that repulses the optimal solution from the boundary.

This is typically done in a way to keep $f + \frac{1}{s}p$ smooth, and if possible convex.

Note that, for the inner penalization, we need the coefficient $\dfrac{1}{s} \to 0$, (hence $s \to +\infty$) for the penalized problem to converges toward the original one.

More on that in the next course.

Inner penalization

Another approach consists in choosing a penalization function p that takes value $+\infty$ outside of X.

The idea here is to add a potential that repulses the optimal solution from the boundary.

This is typically done in a way to keep $f + \frac{1}{s}p$ smooth, and if possible convex.

Note that, for the inner penalization, we need the coefficient $\frac{1}{s} \to 0$, (hence $s \to +\infty$) for the penalized problem to converges toward the original one.

More on that in the next course.

Inner penalization

Another approach consists in choosing a penalization function p that takes value $+\infty$ outside of X.

The idea here is to add a potential that repulses the optimal solution from the boundary.

This is typically done in a way to keep $f + \frac{1}{s}p$ smooth, and if possible convex.

Note that, for the inner penalization, we need the coefficient $\dfrac{1}{s} \to 0$, (hence $s \to +\infty$) for the penalized problem to converges toward the original one.

More on that in the next course.

Inner penalization

Another approach consists in choosing a penalization function p that takes value $+\infty$ outside of X.

The idea here is to add a potential that repulses the optimal solution from the boundary.

This is typically done in a way to keep $f + \frac{1}{s}p$ smooth, and if possible convex.

Note that, for the inner penalization, we need the coefficient $\dfrac{1}{s} \to 0$, (hence $s \to +\infty$) for the penalized problem to converges toward the original one.

More on that in the next course.

Contents

Duality, here we go again

♡

Recall that to a primal problem

$$
\begin{aligned}
(\mathcal{P}) \quad \underset{x \in \mathbb{R}^n}{\mathrm{Min}} \quad & f(x) & (1)\\
\text{s.t.} \quad & g(x) = 0 & (2)\\
& h(x) \leq 0 & (3)
\end{aligned}
$$

we associate the dual problem

$$
(\mathcal{D}) \quad \underset{\lambda, \mu \geq 0}{\mathrm{Max}} \quad \underbrace{\underset{x}{\mathrm{Min}} \quad f(x) + \lambda^\top g(x) + \mu^\top h(x)}_{\Phi(\lambda,\mu)}
$$

♣ Exercise: Under which sufficient conditions are these problems equivalent ?

Duality, here we go again ♡

Recall that to a primal problem

$$
\begin{aligned}
(\mathcal{P}) \quad \min_{x \in \mathbb{R}^n} \quad & f(x) && (1) \\
\text{s.t.} \quad & g(x) = 0 && (2) \\
& h(x) \leq 0 && (3)
\end{aligned}
$$

we associate the dual problem

$$
(\mathcal{D}) \quad \max_{\lambda, \mu \geq 0} \quad \underbrace{\min_{x} \quad f(x) + \lambda^\top g(x) + \mu^\top h(x)}_{\Phi(\lambda,\mu)}
$$

♣ Exercise: Under which sufficient conditions are these problems equivalent ?

Duality seen as exact penalization ♡

If $(\mathcal{P})$ is convex differentiable and the constraints are qualified, then for any optimal multiplier $\overline{\lambda}, \overline{\mu}$ the unconstrained problem

$$\underset{x}{\text{Min}} \qquad f(x) + \overline{\lambda}^\top g(x) + \overline{\mu}^\top h(x)$$

have the same optimal solution as the original problem $(\mathcal{P})$.

Projected gradient in the dual

Consider the dual problem

$$(\mathcal{D}) \quad \underset{\lambda,\mu\geq 0}{\text{Max}} \qquad \Phi(\lambda,\mu)$$

Recall that, under technical conditions,

$$\nabla\Phi(\lambda,\mu) = \begin{pmatrix} g(x^\sharp(\lambda,\mu)) \\ h(x^\sharp(\lambda,\mu)) \end{pmatrix}$$

where $x^\sharp(\lambda,\mu)$ is an optimal solution of the inner minimization problem for given λ,μ.

We suggest solving this problem through projected gradient with step t:

$$\lambda^{(k+1)} = \lambda^{(k)} + t g(x^\sharp(\lambda^{(k)},\mu^{(k)}))$$

$$\mu^{(k+1)} = [\mu^{(k)} + t h(x^\sharp(\lambda^{(k)},\mu^{(k)}))]^+$$

Projected gradient in the dual

Consider the dual problem

$$(\mathcal{D}) \quad \underset{\lambda,\mu\geq 0}{\text{Max}} \qquad \Phi(\lambda,\mu)$$

Recall that, under technical conditions,

$$\nabla\Phi(\lambda,\mu) = \begin{pmatrix} g(x^\sharp(\lambda,\mu)) \\ h(x^\sharp(\lambda,\mu)) \end{pmatrix}$$

where $x^\sharp(\lambda,\mu)$ is an optimal solution of the inner minimization problem for given λ,μ.
We suggest solving this problem through projected gradient with step t:

$$\lambda^{(k+1)} = \lambda^{(k)} + t g(x^\sharp(\lambda^{(k)},\mu^{(k)}))$$
$$\mu^{(k+1)} = [\mu^{(k)} + t h(x^\sharp(\lambda^{(k)},\mu^{(k)}))]^+$$

Uzawa's algorithm

Data: Initial primal point $x^{(0)}$, Initial dual points $\lambda^{(0)}, \mu^{(0)}$, unconstrained optimization method, dual step $t > 0$.

while $\|g(x^{(k)})\|_2 + \|(h(x^{(k)}))^+\|_2 \geq \varepsilon$ **do**

Solve for $x^{(k+1)}$

$$\underset{x}{\text{Min}} \qquad f(x) + \lambda^{(k)\top} g(x) + \mu^{(k)\top} h(x)$$

Update the multipliers

$$\lambda^{(k+1)} = \lambda^{(k)} + t g(x^{(k+1)})$$
$$\mu^{(k+1)} = [\mu^{(k)} + t h(x^{(k+1)})]^+$$

Algorithm 1: Uzawa algorithm

Convergence requires strong convexity and constraint qualifications.

Exercise: decomposition by prices

We consider the following energy problem:

- you are an energy producer with N production unit
- you have to satisfy a given demand planning for the next 24h (i.e. the total output at time t should be equal to d_t)
- the time step is the hour, and each unit has a production cost for each planning given as a convex quadratic function of the planning

1. Model this problem as an optimization problem. In which class does it belong? How many variables?
2. Apply Uzawa's algorithm to this problem. Why could this be an interesting idea?
3. Give an economic interpretation of this method.
4. What would happen if each unit had production constraints?

What you have to know

- There is three main ways of dealing with constraints:
  - choosing an admissible direction
  - projection of the next iterate
  - penalizing the constraints

What you really should know

- admissible direction methods are mainly usefull for polyhedral constraint set
- projection is usefull only if the admissible set is simple (ball or bound constraints)
- penalization can be inner or outer, differentiable or not.

What you have to be able to do

- Implement a penalization approach.

What you should be able to do

- Implement Uzawa's algorithm.