MATH/COMP 61: Discrete Mathematics
Tufts University

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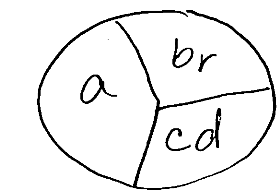
Name: SOLUTIONS **Exam 2 (11/2/18)** **Tufts ID:**________________

50 minutes. No notes, books or devices allowed. Please SHOW YOUR WORK and/or explain answers for full credit!

10:05

1. (10 pts) Let $L = \{a, b, c, d, r\}$.

   - Decide whether or not each of the following relations on L is an equivalence relation. Prove your answer.
   - If it is an equivalence relation, describe and/or draw the partition.



a = 5 times
b=2
c=1
d=1
r=2

(a) $R = \{(x, y) : x \text{ and } y \text{ appear the same number of times in the word “abracadabra.”}\}$

Claim R is an equivalence relation

Proof
REFL xRx for all x since a letter appears the same # of times as itself.

SYMM If xRy then x and y appear the same number of times so yRx also.

TRANS If xRy and yRz then x, y, and z all appear the same # of times, so xRz.

(b) $R = \{(x, y) : x \text{ and } y \text{ appear next to each other in the word “abracadabra.”}\}$

Claim R is not an equivalence relation.
R is not transitive; bRa and aRc but $b \not R c$.
(R is also not reflexive; no letter appears twice in a row so $a \not R a$, etc.)

10:10

10:30

2. SHORT ANSWER (5 pts each, 30 pts total)

   (a) Compute $\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \ldots + \binom{10}{10}$.
   Hint: There are fast and slow ways to do this.

   $= 2^{10} = \boxed{1024}$

   (b) Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7\}$. For each description below, choose a final ordered pair to obtain a relation from A to B which satisfies the description.

   i. $f = \{(1, 5), (2, 5), (3, 6), (\underline{1}, \underline{5})\}$ where the relation f is not a function.

   ii. $f = \{(1, 5), (2, 5), (3, 6), (\underline{4}, \underline{6})\}$ where the relation f is a function from A to B but is not onto B.

   iii. $f = \{(1, 5), (2, 5), (3, 6), (\underline{4}, \underline{7})\}$ where the relation f is a function from A to B and is onto B.

(c) You are visiting an ice cream store which has 7 different flavors. How many ways can you choose a quadruple cone with 4 different flavors (order matters)?

Simplify your answer.

$$7 \cdot 6 \cdot 5 \cdot 4 = 42 \cdot 20 = \boxed{840}$$

(d) You are visiting an ice cream store which has 7 different flavors. How many ways can you choose a quadruple cone with 4 scoops (order matters and repetition is allowed)?

You do not need to simplify your answer.

$$\boxed{7^4}$$

(e) You are visiting an ice cream store which has 7 different flavors. How many ways can you choose a sampler bowl with 4 different flavors (order **doesn't** matter)?

Simplify your answer.

$$\binom{7}{4} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = \boxed{35}$$

(f) Why does $\binom{10}{4}$ tell you the coefficient of x^4y^6 in $(x+y)^{10}$? (Don't just cite a theorem; explain in a sentence or two the justification behind this theorem.)

When we expand $(x+y)^{10} = (x+y)(x+y)(x+y)\cdots(x+y)$, we create terms by choosing one option (x or y) from each parenthesis. To get a term with four x's and six y's we choose x's from four of the ten parentheses. the remaining ones will automatically be y's. Thus there are $\binom{10}{4}$ ways to do this, and this will give us the coefficient of x^4y^6.

PROOFS: You must choose THREE of the following FOUR proofs. If you attempt all four, the highest three scores will count.

10:40

3. (20 pts) Prove using contradiction that the sum of a rational number and an irrational number is irrational. *Recall: A number is rational if it can be written as $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. If this is not possible, the number is called irrational.*

Proof Let r be a rational # and i be an irrational #.
For sake of contradiction, assume $r+i$ is rational.
- Since r is rational, $r = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$.
- Since $r+i$ is rational, $r+i = \frac{c}{d}$ where $c, d \in \mathbb{Z}$ and $d \neq 0$.

Then $i = \frac{c}{d} - r = \frac{c}{d} - \frac{a}{b} = \frac{cb-ad}{bd}$ where $(cb-ad)$ and $bd \in \mathbb{Z}$, and $bd \neq 0$.

But then i is rational.
Contradiction $\rightarrow\leftarrow$.
Thus $r+i$ must be irrational.

10:45

4. (20 pts) Prove using some form of induction that for all $n \in \mathbb{N}$, $1 + 3 + 5 + \ldots + (2n-1) = n^2$.

Proof Base case $n=0$. Last term of summation is $(2 \cdot 0 - 1) = -1$ which means empty sum; $0 = 0^2$. But that's weird so let's also do
$n=1$ Last term of summation is $(2 \cdot 1 - 1) = 1$ and $1 = 1^2$ so ✓.

Inductive step Assume true for k.

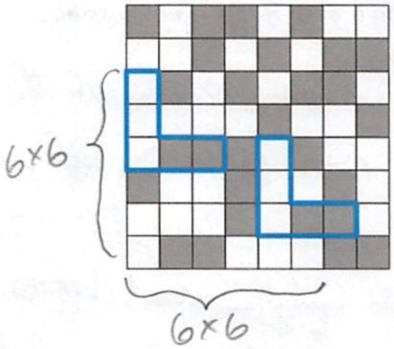
Thus $1+3+5+ \ldots +(2k-1) = k^2$
Adding $(2(k+1)-1)$ to both sides gives us:

$$1+3+5+\ldots+(2k-1)+(2(k+1)-1) = k^2 + (2(k+1)-1)$$
$$= k^2 + 2k+2-1$$
$$= k^2+2k+1$$
$$= (k+1)^2$$

which proves the claim for $k+1$.

Conclusion The claim holds for all $n \geq 1$. (really, $n \geq 0$).

10:56 5. (20 pts) Suppose the squares of an 8×8 board are colored black or white at random. For this problem, an L-region is a collection of 5 squares in the shape of a capital L (namely, a corner square with two squares above and two squares to the right; see two examples in the picture below). Use the Pigeonhole Principle to prove that no matter how we color the board, there must be two L-regions that are colored identically (as illustrated by the two L-regions in the picture.)



How many L-regions? Every L can be identified with its corner square, which can be anywhere in the 6x6 board which occupies the lower left of the 8x8 board. So there are 36 such L-regions.

How many L-colorings? There are 5 squares in an L, and each square can be independently colored black or white, so there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$ ways to color an L.

Conclusion Since there are 36 L-regions and only 32 colorings, by Pigeonhole there must be 2 L-regions with the same coloring.

11:01 6. (20 pts) Use a combinatorial proof to prove that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

We will count the # of K-element subsets of the set $\{1, 2, 3, \dots n\}$.

LHS $\binom{n}{k}$ gives us this directly by definition.

RHS We can split up the k-element subsets into

(subsets not containing the number 1) + (subsets containing the number 1)

Need to choose all k elements from $\{2, 3, \dots n\}$, which is $\binom{n-1}{k}$

Need to choose the remaining k-1 elements from $\{2, 3, \dots n\}$, which is $\binom{n-1}{k-1}$

Thus $\binom{n-1}{k} + \binom{n-1}{k-1}$ also counts the # of k-elt subsets.

Conclusion $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$