Karen Edwards

Name:________________________ **Exam 2 (3/29/19)** **Tufts ID:**__________________

60 minutes. No notes, books or devices allowed.
Please SHOW YOUR WORK and/or explain answers for full credit!

1. (10 pts) Let $A = \{1, 2, 3\}$. Recall that $P(A)$ denotes the power set of A, i.e., the set of all subsets of A.

   - Decide whether or not each of the following relations on $P(A)$ is an equivalence relation. Prove your answer.
   - If it is an equivalence relation, describe and/or draw the partition.

   (a) $R = \{(S, T) : S \subseteq T\}$

   (b) $R = \{(S, T) : |S| = |T|\}$

2. (10 pts) Let $P = \{a, b, c\}$ and $Q = \{d, e\}$.

   (a) Give an example of a function from P to Q. (You may use any sufficiently clear method of defining the function: a verbal description, list of ordered pairs, blob picture, etc.). How many different functions are possible?

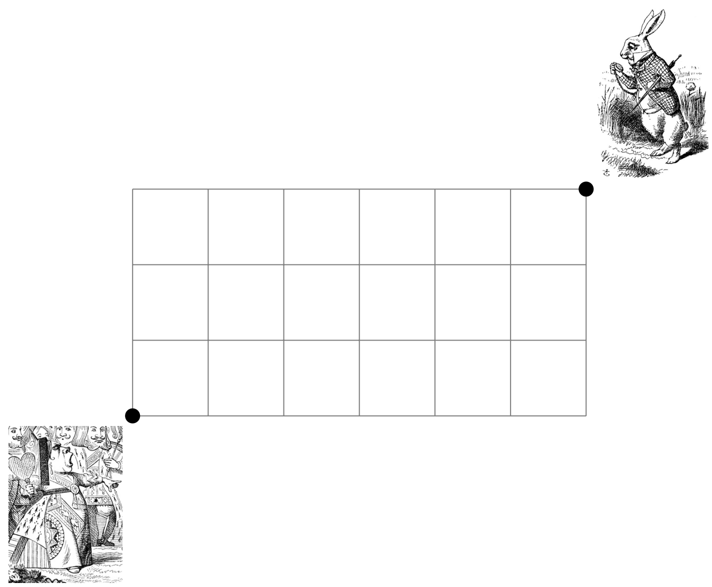
   (b) If possible, give an example of a one-to-one function from P to Q. How many different one-to-one functions are possible?

   (c) If possible, give an example of an onto function from P to Q. How many different onto functions are possible?

3. SHORT ANSWER (5 pts each, 20 pts total)

   (a) How many different ways can the letters of the word GOOGOLPLEX be arranged such that the result contains the letters XL next to each other in that order? You do not need to simplify your answer.

   (b) The White Rabbit is late for a very important date! How many ways can he run to the Queen of Hearts along gridlines if he can only travel down and to the left? Simplify your answer (your final answer should be an integer.)



   (c) At a CS conference, each participant selects 4 special-interest groups out of 10 possible groups (e.g., ”First-Time Participants”, ”Robotics”, ”Undergrads”, etc.) How many participants are needed to guarantee that somewhere among them there is a set of 3 participants with the same set of choices? Your answer should be an integer.

   (d) A robot needs to compute $\binom{13}{8}$. In its memory it already has stored $\binom{n}{k}$ for $0 \leq n \leq 12$. How can it quickly compute $\binom{13}{8}$ from those smaller results? (You do not need to give an actual integer answer, just an expression.)

PROOFS: You must choose THREE of the following FOUR proofs. If you attempt all four, the highest three scores will count.

4. (20 pts) Prove using contradiction or contrapositive:
   If n is divisible by 6 then $n + 10$ is not divisible by 6.

5. (20 pts) Union Pacific is assembling freight trains. There are two types of train cars: length 1 and length 2. A train is made by a assembling an ordered sequence of cars; for example, a train of length 3 can be made as $1 + 2, 2 + 1$ or $1 + 1 + 1$. Prove using some form of induction that $F_n =$ the number of ways to make a train of length n, where F_n is the nth Fibonacci number.
   Hint: Consider the options for what the first car in the train could be.

6. (20 pts) According to the North American Numbering Plan (NANP), neither the 1st nor 4th digit of a 10-digit phone number can be a 0. Prove that among the current 745 CS majors at Tufts there must be two whose phone numbers "match zeros," i.e., have zeros in exactly the same positions. (Assume every student has exactly one phone number.)

7. (20 pts) Use a combinatorial proof to prove that

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

BONUS questions (5 pts each, max of 10 pts added to your score)

(A) How many different ways can the letters of the word GOOGOLPLEX be arranged such that the result contains the letters GO next to each other in that order? This may be harder than you think.

(B) How many lattice paths are there in R^3 (three-dimensional space) from $(0, 0, 0)$ to $(2, 3, 5)$ if you always travel along edges in a positive direction?

(C) Consider the first-order recurrence $a_n = 1 + 5a_{n-1}$ where $a_0 = 0$.

  (a) Compute $a_0, a_1, \ldots a_4$.

  (b) Find a formula for a_n in terms of n (i.e., not a recursive formula.)

(D) Let $f(n) = 5n^2 + 100n$ be a function on $\mathbb{N}$. Show that $f(n) = O(n^2)$.

(E) Describe at least one bijection between two of the following sets, thus proving the two infinite sets are the same size: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$