MATH/COMP 61: Discrete Mathematics
Tufts University

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Name: SOLUTIONS **Exam 2 (3/29/19)** **Tufts ID:**____________

60 minutes. No notes, books or devices allowed.
Please SHOW YOUR WORK and/or explain answers for full credit!

10:37

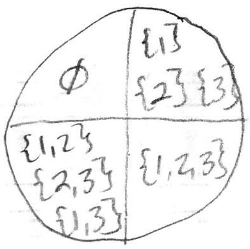
1. (10 pts) Let $A = \{1, 2, 3\}$. Recall that $P(A)$ denotes the power set of A, i.e., the set of all subsets of A.

   - Decide whether or not each of the following relations on $P(A)$ is an equivalence relation. Prove your answer.
   - If it is an equivalence relation, describe and/or draw the partition.

   (a) $R = \{(S, T) : S \subseteq T\}$

   No, R is <u>not</u> an equivalence relation because it is not symmetric. For example, $(\{1\}, \{1,2\}) \in R$ but $(\{1,2\}, \{1\}) \notin R$. I.e., $\{1\} \subseteq \{1,2\}$ but $\{1,2\} \not\subseteq \{1\}$.

   (b) $R = \{(S, T) : |S| = |T|\}$

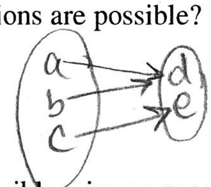
   

   Yes, R <u>is</u> an equivalence relation.
   - It is reflexive because $|S| = |S|$ is true for any set
   - It is symmetric because if S is the same size as T, then T is the same size as S.
   - It is transitive because if S is the same size as T and T is the same size as U then S is the same size as U.

10:42

2. (10 pts) Let $P = \{a, b, c\}$ and $Q = \{d, e\}$.

   (a) Give an example of a function from P to Q. (You may use any sufficiently clear method of defining the function: a verbal description, list of ordered pairs, blob picture, etc.). How many different functions are possible?

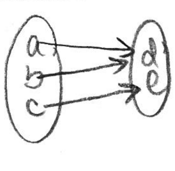
   

   For each elt of P, we choose which elt of Q to send it to. So there are $2 \cdot 2 \cdot 2 = 2^3 = 8$ functions.

   (b) If possible, give an example of a one-to-one function from P to Q. How many different one-to-one functions are possible?

   It is not possible. a, b and c would have to go to 3 distinct elements of Q but Q doesn't have 3 distinct elts.

   (c) If possible, give an example of an onto function from P to Q. How many different onto functions are possible?

   Of the 8 possible functions, there are only 2 that are <u>not</u> onto (either send everything to d or everything to e.) The other 6 must be onto.

10:46

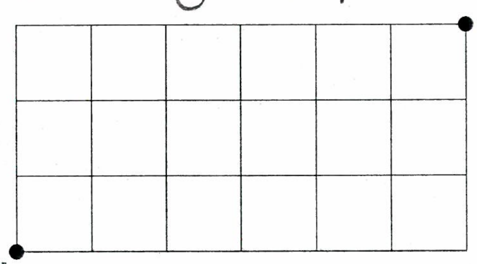
3. SHORT ANSWER (5 pts each, 20 pts total)

(a) How many different ways can the letters of the word GOOGOLPLEX be arranged such that the result contains the letters XL next to each other in that order? You do not need to simplify your answer.

Treat XL as a single letter. Then there are 9 letters total, including three O's and two G's. So the # of arrangements = $\frac{\#\text{ permutations}}{\#\text{duplicates}} = \frac{9!}{3!\,2!}$

(b) The White Rabbit is late for a very important date! How many ways can he run to the Queen of Hearts along gridlines if he can only travel down and to the left? Simplify your answer (your final answer should be an integer.)

His path can be denoted by 3 D's (for "down") and 6 L's (for "left"), in some order. Thus out of the 9 steps in his path, 3 must be D's. There are $\binom{9}{3}$ ways to do this.



$\binom{9}{3} = \frac{\cancel{9}^{3}\cdot\cancel{8}^{4}\cdot 7}{\cancel{3}\cdot\cancel{2}\cdot 1} = 12\cdot 7$
$= 84$

(c) At a CS conference, each participant selects 4 special-interest groups out of 10 possible groups (e.g., "First-Time Participants", "Robotics", "Undergrads", etc.) How many participants are needed to guarantee that somewhere among them there is a set of 3 participants with the same set of choices? Your answer should be an integer.

There are $\binom{10}{4}$ ways to select special-interest groups. $\binom{10}{4} = \frac{10\cdot\cancel{9}^{3}\cdot\cancel{8}\cdot 7}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 10\cdot 21 = 210$. If we want to guarantee 3 of the same, our avg must be above 2, so $\lceil \frac{n}{210} \rceil = 3$. This works for $n = 210\cdot 2+1 = 421$ (or higher)

(d) A robot needs to compute $\binom{13}{8}$. In its memory it already has stored $\binom{n}{k}$ for $0 \leq n \leq 12$. How can it quickly compute $\binom{13}{8}$ from those smaller results? (You do not need to give an actual integer answer, just an expression.)

$\binom{13}{8} = \binom{12}{7} + \binom{12}{8}$

PROOFS: You must choose THREE of the following FOUR proofs. If you attempt all four, the highest three scores will count.

10:56

4. (20 pts) Prove using contradiction or contrapositive:

Claim If n is divisible by 6 then $n + 10$ is not divisible by 6.

Proof: Assume n is divisible by 6.
For sake of contradiction, assume $n+10$ is divisible by 6.
If $6 \mid n+10$ and $6 \mid n$, then $6 \mid (n+10)-(n)$ so $6 \mid 10$.
But this is false. $\Rightarrow\Leftarrow$ (contradiction)
So we conclude $n+10$ is not divisible by 6.

10:59 5. (20 pts) Union Pacific is assembling freight trains. There are two types of train cars: length 1 and length 2. A train is made by a assembling an ordered sequence of cars; for example, a train of length 3 can be made as $1 + 2, 2 + 1$ or $1 + 1 + 1$. Prove using some form of induction that F_n = the number of ways to make a train of length n, where F_n is the nth Fibonacci number.
Hint: Consider the options for what the first car in the train could be.

Claim F_n = # ways to make a train of length n

by strong induction

Proof Base cases $F_0 = 1$, and there is 1 way to make a train of length 0 (empty train). ✓

$F_1 = 1$, and there is is 1 way to make a train of length 1 (namely, "1"). ✓

Inductive Step Assume F_K = # ways to make a train of length k, for $0, 1, \dots K$, where $K \geq 1$.

Now suppose we have a train of length $K+1$.
The first car is either length 1 or length 2.

- If length 1, then the rest of the train is length K so there are F_K ways to complete the train
- If length 2, then the rest of the train is length $K-1$ and our ind. hypothesis still applies so there are F_{K-1} ways to complete the train

In total there are $F_K + F_{K-1} = F_{K+1}$ ways to make the train.

Conclusion True for all $n \geq 0$.

11:11

6. (20 pts) According to the North American Numbering Plan (NANP), neither the 1st nor 4th digit of a 10-digit phone number can be a 0. Prove that among the current 745 CS majors at Tufts there must be two whose phone numbers "match zeros," i.e., have zeros in exactly the same positions. (Assume every student has exactly one phone number.)

In a phone number there are 8 positions which could have zeros, or not. Thus there are $2^8 = 256$ possible configurations of zeros, Since 745 is more than 256, at least two CS majors must have zeros in the same positions by the Pigeonhole Principle. (Actually, by generalized Pigeonhole there are at least 3 CS majors with zeros in the same position, since $\lceil \frac{745}{256} \rceil = 3$.)

11:14

11:17

7. (20 pts) Use a combinatorial proof to prove that

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

Let us count the # of subsets of an n element set.

LHS: Every one of the n elements could be in or out of the subset, so there are $\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n \text{ times}} = 2^n$ choices.

RHS: If we group the subsets by size, we have

$$\binom{\#\text{ subsets}}{\text{of size } 0} + \binom{\#\text{ subsets}}{\text{of size } 1} + \binom{\#\text{ subsets}}{\text{of size } 2} + \ldots + \binom{\#\text{ subsets}}{\text{of size } n}$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots \binom{n}{n}$$

Since both sides count the same thing, they are equal.

11:20

BONUS questions (5 pts each, max of 10 pts added to your score)

(A) How many different ways can the letters of the word GOOGOLPLEX be arranged such that the result contains the letters GO next to each other in that order? This may be harder than you think.

(B) How many lattice paths are there in R^3 (three-dimensional space) from $(0, 0, 0)$ to $(2, 3, 5)$ if you always travel along edges in a positive direction?

(C) Consider the first-order recurrence $a_n = 1 + 5a_{n-1}$ where $a_0 = 0$.

(a) Compute $a_0, a_1, \ldots a_4$.

(b) Find a formula for a_n in terms of n (i.e., not a recursive formula.)

(D) Let $f(n) = 5n^2 + 100n$ be a function on $\mathbb{N}$. Show that $f(n) = O(n^2)$.

(E) Describe at least one bijection between two of the following sets, thus proving the two infinite sets are the same size: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$