Lecture Notes

MEK4420

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The Wave-Green function

To solve the Laplace equation in a wave field, we will need to construct a Green function. We say *a priori* that it ought to be of the form

$$\text{Ц}(\boldsymbol{x}, t) = \operatorname{Re}\big(\operatorname{G}(\boldsymbol{x}) \exp{(i\omega t)}\big),$$

where $\operatorname{G}(\boldsymbol{x}) = \ln r$ in the absence of waves. For waves present the Green function must satisfy the following conditions. It must obviously satisfy the Laplace equation,

$$\nabla^2 \operatorname{G}(\boldsymbol{x}) = 0 \qquad \text{in } \Omega \backslash \{\boldsymbol{\text{ж}}\}.$$

It must also satisfy the dynamic boundary condition at the surface,

$$\omega^2 \operatorname{G} = \mathrm{g}\partial_y \operatorname{G} \qquad \text{at } y = 0.$$

We will be looking for radiating solutions, emanating from some body at the origin, such that

$$\operatorname{G}(\boldsymbol{x}) \sim \exp{(\mp ikx)} \qquad \text{as } x \to \pm\infty.$$

We furthermore assume infinite depth, so that

$$|\nabla \operatorname{G}| \to 0 \qquad \text{as } y \to -\infty.$$

We allow ourselves, for brevity and so as to not exhaust our glyphic arsenal so to speak, to write $\boldsymbol{x}, \boldsymbol{\text{ж}} \in \mathbb{C}^1$, the vector space over the complex numbers. That is, we write for the time being that $\boldsymbol{x} = x + iy$ and $\boldsymbol{\text{ж}} = \text{ж} + i\text{ч}$. The complex conjugate is denoted $\boldsymbol{x}^* = x - iy$. It is found that[1]

$$\operatorname{G}(\boldsymbol{x}) = \ln {}^{r}\!/_{r_1} + \operatorname{Re}(f_1) + i \operatorname{Re}(f_2),$$

where $r = |\boldsymbol{x} - \boldsymbol{\text{ж}}|$ and $r_1 = |\boldsymbol{x} - \boldsymbol{\text{ж}}^*|$. For the wave in question, $\omega = \sqrt{\kappa \mathrm{g}}$, where κ is the wave number. We introduce $\text{з} = -\kappa i(\boldsymbol{x} - \boldsymbol{\text{ж}}^*)$. We have that

$$\begin{aligned} f_1(\text{з}) &= 2 \operatorname{pv} \int_0^\infty \frac{\exp{(\text{з}^{k}/_{\kappa})}}{\kappa - k} \,\mathrm{d}k \\ &= -2\big(\operatorname{E}_1(\text{з}) + \ln{(\text{з})} - \ln{(-\text{з})}\big) \exp{(\text{з})}, \end{aligned}$$

and

$$f_2(\text{з}) = 2\pi \exp{(\text{з})}.$$

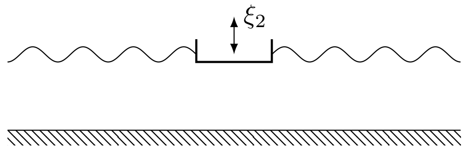
The asymptotic behavior of f_1 is as follows.[2]

$$f_1 = \pm 2\pi i \exp{(\text{з})} \qquad \text{for } x - \text{ж} \to \pm\infty.$$

[1] [2] Wehausen & Laitone, sect.17

The Radiation Problem

It is perhaps appropriate to provide a concrete example before divulging in the general position of the radiation problem. We imagine some geometry described by the boundary S_{B} bobbing in water with a heave motion $\xi_2(t) = \operatorname{Re}\big(\hat{\xi}_2 \exp{(i\omega t)}\big)$. The body of water may be bounded by some bottom bathymetry. It is the effect of radiation we are interested in—the waves generated by the motion of the geometry.



The kinematic boundary condition between the geometry and the water is given by

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{u}_{\mathrm{B}} = \hat{\boldsymbol{n}} \cdot \nabla \Phi_{\mathrm{R}} \qquad \text{on } S_{\mathrm{B}},$$

where $\boldsymbol{u}_{\mathrm{B}} = \partial_t \xi_2 \hat{\boldsymbol{\jmath}}$ is the velocity of the geometry. As an ansatz, we say that the radiation potential is given by

$$\Phi_{\mathrm{R}}(\boldsymbol{x}, t) = \operatorname{Re}\big(i\omega\hat{\xi}_2 \exp{(i\omega t)} \phi_2(\boldsymbol{x})\big).$$

Given $\hat{\boldsymbol{n}} \cdot \boldsymbol{u}_{\mathrm{B}} = \omega \hat{n}_y \xi_2$, we get the reformulated kinematic boundary condition

$$\operatorname{Re}\Big(i\omega\hat{\xi}_2 \exp{(i\omega t)} \big(\hat{n}_y - \partial_{\hat{\boldsymbol{n}}} \phi_2\big)\Big) = 0 \qquad \text{on } S_{\mathrm{B}}.$$

We recall the linearized dynamic boundary condition for surface waves,

$$\partial_t^2 \Phi_{\mathrm{R}} = -\mathrm{g}\partial_y \Phi_{\mathrm{R}} \qquad \text{at } y = 0.$$

We have that $\partial_t^2 \Phi_{\mathrm{R}} = -\omega^2 \Phi_{\mathrm{R}}$, yielding the dynamic boundary condition

$$\operatorname{Re}\Big(i\omega\hat{\xi}_2 \exp{(i\omega t)} \big(\mathrm{g}\partial_y \phi - \omega^2 \phi\big)\Big) = 0 \quad \text{at } y = 0.$$

[2] [1] Abramowitz & Stegun, ch.5

With this specific example in mind, we say that the general radiation potential is given as a superposition of the potential modes,

$$\Phi_{\mathrm{R}}(\boldsymbol{x}, t) = \mathrm{Re} \sum_j i\omega\xi_j \phi_j(\boldsymbol{x}) \exp{(i\omega t)}.$$

The kinematic boundary condition at the body is

$$\partial_{\hat{\boldsymbol{n}}} \phi_j = \hat{n}_j \qquad \text{on } S_{\mathrm{B}}.$$

When $\boldsymbol{\varkappa}$ is on S_{B}, the integral equation is given by

$$\pi\phi(\boldsymbol{\varkappa}) = \int_{S_{\mathrm{B}}} \phi_j \partial_{\hat{\boldsymbol{n}}} \mathrm{G} - \mathrm{G}\, \partial_{\hat{\boldsymbol{n}}} \phi_j \,\mathrm{d}S,$$

and when $\boldsymbol{\varkappa}$ is in the fluid, the left hand side of the equation is of twice the magnitude.

Damping

We notice now that the potentials ϕ_j may indeed be complex, indicating that the added mass may only represent the real part of the added mass integral. Recall that $\Phi_i = i\omega\hat{\xi}_i\phi_i e^{i\omega t}$. Omitting the temporal exponential, we find that the pressure is given by BERNOULLI's equation $p_i = i\varrho\omega\hat{\xi}_i\phi_i$. We write that the displacement amplitude velocity and acceleration are given by $i\omega\hat{\xi}_i$ and $-\omega^2\hat{\xi}_i$, repsectively. We then have that

$$\begin{aligned} F_{ij}(t) &= -\varrho\omega^2\hat{\xi}_i \int_{S_{\mathrm{B}}} \phi_i \hat{n}_j \,\mathrm{d}S \\ &= -\omega^2\hat{\xi}_i m_{ij} + i\omega\hat{\xi}_i r_{ij}, \end{aligned}$$

where the last equality is an ansatz due to us expecting an imaginary part, extrapolating from NEWTON's second law of motion. We then say that the *added mass* $\mathfrak{m}$, and *radiation damping* $\mathfrak{r}$ tensors are defined as follows.

$$\mathfrak{m} = \varrho\, \mathrm{Re} \int_{S_{\mathrm{B}}} \phi_i \hat{n}_j \,\mathrm{d}S, \quad \mathfrak{r} = -\omega\varrho\, \mathrm{Im} \int_{S_{\mathrm{B}}} \phi_i \hat{n}_j \,\mathrm{d}S$$

Example: *The work done by a geometry*

We want to find the rate of work for a motor that sustains the heave in the illustration above. In general the average rate of work is given by $\overline{\partial_t W} = -\overline{F_2 U_2}$, where the overline is the time average, introducing a factor of one half. We have that $F_2 = -\omega^2\hat{\xi}_2 m_{22} + i\omega\hat{\xi}_2 r_{22}$, and $U_2 = i\omega\hat{\xi}_2$. Since the rate of work is a real value, we have that

$$\overline{\partial_t W} = \frac{\omega^2 \hat{\xi}_2^{\,2} r_{22}}{2}.$$

We recall the following approximations for the kinetic and potential energies from the *Hydrodynamic Wave Theory* course, being

$$\frac{\varrho}{2}\overline{\int_{-h}^{\eta} \boldsymbol{u}\cdot\boldsymbol{u} \,\mathrm{d}y} \sim \frac{\varrho \mathrm{g}|A|^2}{4}, \quad \frac{\varrho}{2}\overline{\int_{-h}^{\eta} \varrho \mathrm{g} y \,\mathrm{d}y} \sim \frac{\varrho \mathrm{g}|A|^2}{4},$$

respectively. We also recall the approximation for the energy flux,

$$\overline{\int_{-h}^{\eta} pu \,\mathrm{d}y} \sim Ec_g, \qquad c_g = \frac{\mathrm{g}}{2\omega}$$

where E is the sum of potential and kinetic energies, and c_g is the group velocity for deep water. We have that

$$|A|^2 = \hat{\xi}_2^{\,2} |\hat{\eta}_2^{\pm\infty}|^2,$$

requiring a superposition when considering the total energy flux. The actual work done is the characteristic time multiplied by the temporal average, being equivalent to dividing by ω in this case. The work put into the system must equal the energy flux in the waves, so we now get that

$$\frac{r_{22}}{\varrho\omega} = \frac{1}{2}\left(|A_j^{\infty}|^2 + |A_j^{-\infty}|^2\right).$$

Example: *Far field amplitudes of a symmetrical geometry*

In the far field, when $\varkappa \to \pm\infty$, we have that $\mathrm{G} = \mathrm{Re}\,(f_1) + i\,\mathrm{Re}\,(f_2)$. Carrying out the calculation, we find that

$$\mathrm{G}(\boldsymbol{x}; \boldsymbol{\varkappa}) = 2\pi i \exp{(\mp\mathfrak{z})} \quad \text{as } \varkappa \to \pm\infty,$$

where $\mathfrak{z} = -i\kappa(\boldsymbol{x} - \boldsymbol{\varkappa}^*)$. It follows that

$$\partial_x \mathrm{G} = \pm i\kappa\, \mathrm{G}, \quad \partial_y \mathrm{G} = \pm\kappa\, \mathrm{G} \quad \text{as } \varkappa \to \pm\infty,$$

yielding

$$2\pi\phi_j(\boldsymbol{\varkappa}) = \pm \int_{S_{\mathrm{B}}} (i\kappa\phi_j \hat{\boldsymbol{n}}^* - \hat{n}_j)\, \mathrm{G} \,\mathrm{d}S.$$

We say

$$A_j^{\pm\infty} = \mp \int_{S_{\mathrm{B}}} (\kappa\phi_j \hat{\boldsymbol{n}}^* + i\hat{n}_j)\, e^{\kappa(y \pm ix)} \,\mathrm{d}S,$$

so that

$$\phi_j(\boldsymbol{\varkappa}) = A_j^{\pm\infty} e^{-\kappa(\text{ч} \pm i\varkappa)}, \qquad \text{as } \varkappa \to \pm\infty.$$

We recall the kinematic boundary condition, $\partial_t\eta = \partial_z\Phi_{\mathrm{R}}$, and the dynamic boundary condition, $\partial_t^2\Phi_{\mathrm{R}} = -\mathrm{g}\partial_z\Phi_{\mathrm{R}}$. Having already assumed a FOURIER decomposition of the potential, we find that the far field surface elevation may be expressed as a superposition

$$\eta(\varkappa, t) = \sum_j \hat{\xi}_j \hat{\eta}_j^{\pm\infty} \exp\left(i(\omega t \mp \kappa\varkappa)\right),$$

where $\hat{\eta}_j^{\pm\infty} = \kappa A_j^{\pm\infty}$. Supposing a combined sway and heave motion of the geometry, the outgoing waves on the respective sides of the geometry, is given by

$$\eta(\text{эж}, t) = \kappa \left(\hat{\xi}_1 A_1^{\infty} + \hat{\xi}_2 A_2^{\infty} \right) e^{i(\omega t - \kappa \text{эж})},$$

$$\eta(\text{эж}, t) = \kappa \left(\hat{\xi}_1 A_1^{-\infty} + \hat{\xi}_2 A_2^{-\infty} \right) e^{i(\omega t + \kappa \text{эж})},$$

for $\text{эж} \to \infty$ and $\text{эж} \to -\infty$, respectively.

We see that we are now free to choose a combination of $\hat{\xi}_1$ and $\hat{\xi}_2$ such that the far field on the left-hand side disappears. The normal vectors of a geometry symmetrical about the y-axis has the property that $\hat{n}_x(x,y) = -\hat{n}_x(-x,y)$, and $\hat{n}_y(x,y) = \hat{n}_y(-x,y)$. Furthermore, $\phi_1(x,y) = -\phi_1(-x,y)$ and $\phi_2(x,y) = \phi_2(-x,y)$ on the geometry. This yields the properties that $A_1^{\infty} = -A_1^{-\infty}$ and $A_2^{\infty} = A_2^{-\infty}$. It follows that

$$\eta(\text{эж}, t) = 2\kappa \hat{\xi}_2 A_2^{\infty} e^{i(\omega t - \kappa \text{эж})},$$

whilst the far field in the negative direction is negligible.

References

[1] Abramowitz, Milton and Stegun, Irene A. *Handbook of Mathematical Functions.* 3rd ed. National Bureau of Standards, 1965.

[2] Wehausen, John V. and Laitone, Edmund V. "Surface Waves". In: *Encyclopedia of Physics.* Ed. by Siegfried Flügge. Vol. IX. Springer-Verlag, 1960.