The Forces and Response of a Heaving Rectangle

MEKSP100

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The Heave Problem

We consider a box floating on the ocean surface, whose motion is described by the radiation potential

$$\Phi_{\mathrm{R}}(\boldsymbol{x};t) = \mathrm{Re}\left(i\omega\phi_2\hat{\xi}_2\exp\left(i\omega t\right)\right),$$

for which the heave potential ϕ_2 is determined the conditions of continuity in velocity, incompressibility, and evanescence, and the kinematic bounary condition

$$\mathrm{g}\partial_y\phi_2 = -\omega^2\phi_2.$$

The first of the above conditions may be expressed as $\partial_{\hat{\boldsymbol{n}}}\Phi_{\mathrm{R}} = \boldsymbol{u}_{\mathrm{B}}\cdot\hat{\boldsymbol{n}}$, or

$$\partial_{\hat{\boldsymbol{n}}}\phi_2 = \hat{n}_y \qquad \text{on } S_{\mathrm{B}}.$$

The incompressibility of the fluid yields the Laplace equation

$$\nabla^2\phi_2 = 0 \qquad \text{in } \Omega.$$

The condition of evanescence states that the gradient of the velocity potential ought to disappear at infinity, namely

$$|\nabla\phi_2| \to 0 \qquad \text{as } y \to \infty.$$

As for the accompanying Green function,

$$\text{Ц}(\boldsymbol{x},t) = \mathrm{Re}\left(\mathrm{G}\left(\boldsymbol{x}\right)\exp\left(i\omega t\right)\right),$$

it must also satisfy the Laplace equation, kinematic boundary condition, and evanescence conditions. We are looking for radiating solutions, emanating from the body at the origin such that

$$\mathrm{G}\left(\boldsymbol{x}\right) \sim \exp\left(\mp ikx\right) \qquad \text{as } x \to \pm\infty.$$

The derivation of the integral equation follows from that found in lecture notes from January 21^{st}, yielding

$$-\pi\phi_2(\boldsymbol{\text{ж}}) + \int_{S_{\mathrm{B}}}\phi_2\partial_{\hat{\boldsymbol{n}}}\,\mathrm{G}\;\mathrm{d}S = \int_{S_{\mathrm{B}}}\mathrm{G}\,\partial_{\hat{\boldsymbol{n}}}\phi_2\,\mathrm{d}S.$$

Discretization of the Integral Equation and Boundary

The logarithm terms are integrated in much the same way they were in the first mandatory assignment. The boundary element method assumes the potential is constant, set to the value of the midpoint between nodes. Integration of the logarithm terms in the Green function is outlined in the lecture notes from January 28^{th}. The gradient of the other terms are given to be

$$\partial_x\,\mathrm{G} = \kappa\big(\mathrm{Im}\left(f_1(\mathfrak{z})\right) + i\,\mathrm{Im}\left(f_2(\mathfrak{z})\right)\big),$$

$$\partial_y\,\mathrm{G} = \kappa\big(\mathrm{Re}\left(f_1(\mathfrak{z})\right) + i\,\mathrm{Re}\left(f_2(\mathfrak{z})\right)\big),$$

and are treated with a midpoint rule, setting the complex variable

$$\mathfrak{z} = \kappa\big(\text{ч}_m + \text{ч}_n - i(\text{ж}_m - \text{ж}_n)\big).$$

The boundary we discretize with a Chebyshov distribution in two coordinates $\boldsymbol{x}_{\mathtt{p}}$ and $\boldsymbol{x}_{\mathtt{m}}$. Constructing a partition of the interval whose midpoints forms Chebyshov distribution seems to be more effort than it is worth, so we concede that Chebyshov distributions in $\boldsymbol{x}_{\mathtt{p}}$ and $\boldsymbol{x}_{\mathtt{m}}$ suffice to get accurate enough results near the corners of the box.

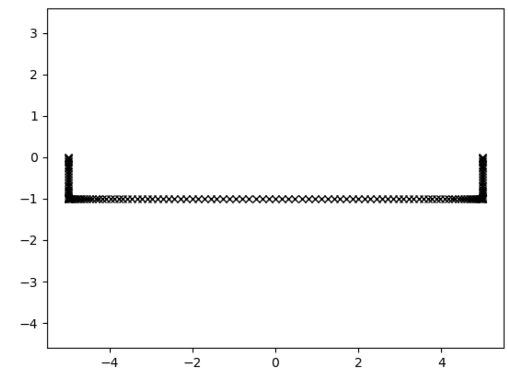


Figure 1: Rectangle with $L/D = 10$, $N_x = 100$, $N_y = 25$.

We may test the numerical scheme by checking that the left-hand side equals the right-hand side of the matrix equation for the real and imaginary parts, using a known function. We consider

$$\phi_0 = e^{\kappa(y-ix)}, \qquad \partial_{\hat{\boldsymbol{n}}}\phi_0 = \kappa(\hat{n}_y - i\hat{n}_x)\phi_0.$$

Plotting the left-hand side against the right-hand side,

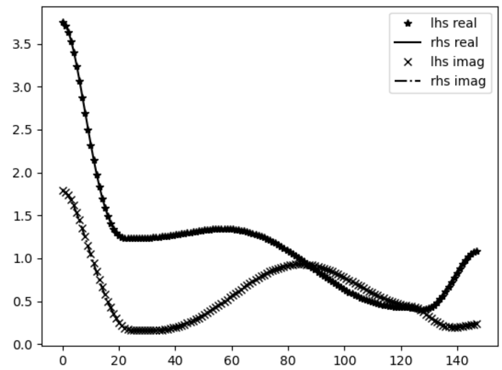
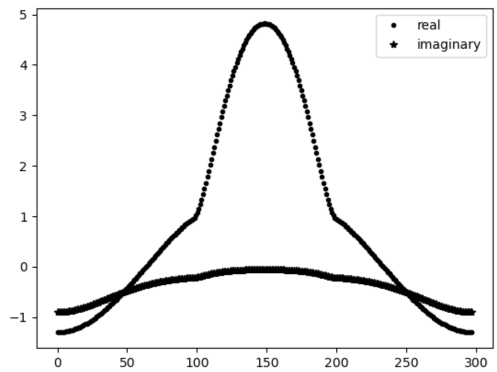


Figure 2: Left-hand and right-hand side of integral equation with ϕ_0. Rectangle $L/D = 2$.

We see that the implementation seems to work, so we may solve for the heave potential.



Wave Interactions

The far field amplitudes is shown to be

$$A_j^{\pm\infty} = \mp \int_{S_\mathrm{B}} \left(\kappa\phi_j \hat{\boldsymbol{n}}^* - \hat{n}_j\right) \mathrm{d}S.$$

The added mass and damping coefficients are found by taking the real and imaginary parts of the integral

$$\int_{S_\mathrm{B}} \phi_j \hat{n} \,\mathrm{d}S.$$

The Diffraction Problem

The Body Response