Lecture Notes

MEK4420

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The Discrete Integral Equation

We want to numerically solve the integral equation

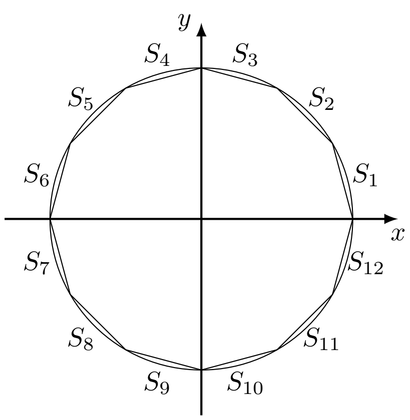
$$\int_{\partial\Omega} \left(\phi\partial_{\hat{\boldsymbol{n}}} \ln r - \ln r \partial_{\hat{\boldsymbol{n}}}\phi\right) \mathrm{d}S = \pi\phi(\boldsymbol{\varkappa}).$$

The Boundary Element Method

We discretize the boundary with N collocation points, and connect them with straight line segments S_k such that

$$S = \bigcup_{m=1}^{N} S_m.$$

For convex Ω, the discrete boundary S will enclose a smaller area.



We can then set the potential function and its derivatives constant across these line segments as an approximation so that

$$\int_{\partial\Omega} \phi\partial_{\hat{\boldsymbol{n}}} \ln r \,\mathrm{d}S \approx \sum_{m=1}^{N} \phi_m \int_{S_m} \partial_{\hat{\boldsymbol{n}}} \ln r \,\mathrm{d}S,$$

$$\int_{\partial\Omega} \ln r \partial_{\hat{\boldsymbol{n}}}\phi \,\mathrm{d}S \approx \sum_{m=1}^{N} \partial_{\hat{\boldsymbol{n}}}\phi_m \int_{S_m} \ln r \,\mathrm{d}S,$$

where $\phi_m \equiv \phi(\boldsymbol{\varkappa}_m)$, and $\boldsymbol{\varkappa}_m = {}^1\!/_2(\boldsymbol{x}_m + \boldsymbol{x}_{m-1})$.

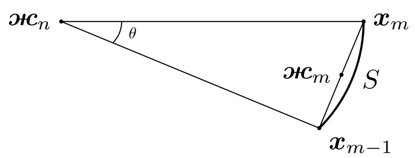
[1] [1] Abramowitz & Stegun, p.67, eq.4.1.2

Logarithmic Flux Integral

We consider the flux integral of the natural logarithm over some segment S of the geometry boundary,

$$\int_S \hat{\boldsymbol{n}} \cdot \nabla \ln r \,\mathrm{d}S.$$

On this segment, the differential element $\mathrm{d}S$ can be decomposed into differentials along the abscissal and ordinal components, and the logarithm may be expressed in terms of the real part of its complex counterpart. That is, $\hat{\boldsymbol{n}} \cdot \nabla = n_x\partial_x + n_y\partial_y$, and $n_x\mathrm{d}S = -\mathrm{d}y$ and $n_y\mathrm{d}S = \mathrm{d}x$.



Letting $\mathrm{Re}(\star)$ denote the real part of a complex number, and noting that $r_m = |\boldsymbol{x}_m - \boldsymbol{\varkappa}_n|$, we have that[1]

$$\ln r = \mathrm{Re}\left(\ln\left(\mathfrak{z} - \boldsymbol{\varkappa}_n\right)\right), \quad \mathfrak{z} = x + iy.$$

Now,

$$\int_S \hat{\boldsymbol{n}} \cdot \nabla \ln r \,\mathrm{d}S = \mathrm{Re} \int_S \frac{i}{\mathfrak{z} - \boldsymbol{\varkappa}_n} \,\mathrm{d}\mathfrak{z}.$$

Evaluating the integral at the points $\boldsymbol{x}_m$ and $\boldsymbol{x}_{m-1}$, and using the fact that $\ln\mathfrak{z} = \ln|\mathfrak{z}| + i\arg\mathfrak{z}$, we get that

$$\int_S \hat{\boldsymbol{n}} \cdot \nabla \ln r \,\mathrm{d}S = \Theta_{n,m-1} - \Theta_{n,m} \equiv -\boldsymbol{\theta},$$

where $\Theta_{n,m} = \arg\left(\boldsymbol{x}_m - \boldsymbol{\varkappa}_n\right)$. We note that this contribution to the discrete integral equation is exact, and it is the assumption that ϕ is constant along the line segment S_m that causes inaccuracy.

Quadrature Methods

To integrate the logarithm, we employ a so-called GAUSS quadrature method of the second order. This method has us map the domain of integration to the one dimensional unit circle,[1]

$$\int_a^b y(x)\,\mathrm{d}x \mapsto \int_{-1}^{1} \eta(\xi)\,\mathrm{d}\xi,$$

where

$$x = \frac{b-a}{2}\xi + \frac{a+b}{2}, \qquad \eta(\xi) \equiv \frac{b-a}{2} y(x).$$

This latter integral we approximate by

$$\sum_{k=1}^{N} w_k \eta(\xi_k), \quad w_k = \frac{2}{\left(1-{\xi_k}^2\right)\left(\mathrm{P}'_N(\xi_k)\right)^2},$$

Where P_N is the N^{th}-degree LEGENDRE polynomial, and ξ_k are its N zeros. Setting $N = 2$, and recalling the three first LEGENDRE polynomials,

$$\mathrm{P}_0(\xi) = 1, \quad \mathrm{P}_1(\xi) = \xi, \quad \mathrm{P}_2(\xi) = \frac{3\xi^2-1}{2},$$

We find that $w_1 = 1$ and $w_2 = 1$, and $\xi_k = \pm 1/\sqrt{3}$. Recalling the logarithm rule $\ln(x^\alpha) = \alpha \ln x$, we have that

$$\int_{S_m} \ln r \,\mathrm{d}S \approx \frac{1}{2}\sum_{k=1}^{2} \frac{|\boldsymbol{x}_m - \boldsymbol{x}_{m-1}|}{2} \ln |\boldsymbol{x}'_m - \boldsymbol{\varkappa}_n|^2,$$

where

$$\boldsymbol{x}'_m = \frac{\boldsymbol{x}_m + \boldsymbol{x}_{m-1}}{2} + \frac{(-1)^k(\boldsymbol{x}_m - \boldsymbol{x}_{m-1})}{2\sqrt{3}}.$$

We usually label the tensor collection of such integrals $\mathtt{h}$.

The Discrete Integral Equation

We note that $\partial_{\hat{\boldsymbol{n}}}\phi_m$ has three modes corresponding to each of the Cartesian unit vectors, and that the subscript m denotes evaluation at that indexed node, and likewise with the rotational normal vectors. Since we consider here a Galilean coordinate system—moving with the geometry—the ϕ here represent velocity potentials due to geometry motion with unit velocity some mode.[2] It is meant here, then, that $\partial_{\hat{\boldsymbol{n}}}\phi_m = n_i$, where n_i is whichever normal vector component. The integral equation is then given as follows.

$$-\pi\phi_n - \sum_{m=1}^{N} \phi_m \boldsymbol{\theta}_{n,m} = \sum_{m=1}^{N} \partial_{\hat{\boldsymbol{n}}}\phi_m \mathtt{h}_{n,m}.$$

[1] [1] ABRAMOWITZ & STEGUN, pp.779 & 887

Added Mass

The added mass tensor $\mathfrak{m}$ may be similarly approximated.

$$\mathfrak{m} = \varrho \int_{\partial\Omega} \phi_j n_i \,\mathrm{d}S = \varrho \sum_{m=1}^{N} |\boldsymbol{x}_m - \boldsymbol{x}_{m-1}| {\phi_j}_m n_{im}$$

More on the Added Mass Tensor

Symmetry

The added mass tensor is symmetric. We may prove this by showing that $\mathfrak{m} - \mathfrak{m}^\dagger \equiv 0$. That is,

$$\mathfrak{m} - \mathfrak{m}^\dagger = \varrho \int_{\partial\Omega} \left(\phi_i \partial_{\hat{\boldsymbol{n}}}\phi_j - \phi_j \partial_{\hat{\boldsymbol{n}}}\phi_i\right) \mathrm{d}S \equiv 0.$$

Kinetic Energy

The kinetic energy of the fluid may be expressed as the integral

$$\frac{\varrho}{2}\int_\Omega \nabla\Phi \cdot \nabla\Phi \,\mathrm{d}V.$$

We recall the identity $\nabla\Phi \cdot \nabla\Phi + \Phi\nabla \cdot \nabla\Phi = \nabla \cdot (\Phi\nabla\Phi)$, and using GAUSS' divergence theorem, we may decompose the potentials, yielding the following expression of the kinetic energy in terms of the added mass tensor, $1/2 \sum_{i,j} U_i U_j m_{ij}$.

Example: *Added mass of sphere*

We wish to calculate the added mass of a sphere of radius R_0 moving according to the potential

$$\phi_1 = -\frac{{R_0}^3}{2}\partial_x(-1/r) = -\frac{x{R_0}^3}{2r^3}.$$

We use the definition of the added mass,

$$m_{11} = \varrho \int_{\partial\Omega} \phi_1 n_1 \,\mathrm{d}S = \frac{\varrho R_0}{2}\int_{\partial\Omega} {n_1}^2 \,\mathrm{d}S.$$

Using the fact that the three components of the Cartesian unit normal vector are the same length, the integrand must be a third of that of the unit normal vector. Integrating that across the surface yields the surface area of the sphere, so that indeed $m_{11} = \rho/2 \maltese$, where $\maltese = 4/3\pi {R_0}^3$ is the displaced fluid.

References

[1] ABRAMOWITZ, MILTON and STEGUN, IRENE A. *Handbook of Mathematical Functions.* 3rd ed. National Bureau of Standards, 1965.

[2] NEWMAN, JOHN NICHOLAS. *Marine Hydrodynamics.* The MIT press, 2018.

[2] [2] NEWMAN, pp.143–144