Lecture Notes

MEK4420

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Fixed Bodies and Moving Bodies

The class of Green functions is central to this course on marine hydrodynamics. From the theory of partial differential equations, we recall that for the Poisson equation

$$\nabla^2\phi(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{on } \Omega,$$

the solution is given by the convolution

$$\phi(\boldsymbol{x}) = \mathrm{G} * f = \int_\Omega \mathrm{G}\,(\boldsymbol{x},\boldsymbol{\xi}) f(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi},$$

where

$$\nabla^2\,\mathrm{G}(\boldsymbol{x},\boldsymbol{\xi}) = \delta(\boldsymbol{x}-\boldsymbol{\xi}).$$

For the purposes of this part of the course, the Green function for the Laplace operator on the d-sphere is the Newton kernel

$$\mathrm{G}(\boldsymbol{x}) = \begin{cases} \ln(\boldsymbol{x}), & d = 2 \\ \boldsymbol{x}^{2-d}, & d \neq 2 \end{cases}.$$

Example: *Circle*

Consider $r^2 = (x-\text{ж})^2 + (y-\text{ч})^2$, and the volume flux through the circle centered at the origin of radius R_0 of the source function $\phi(r) = \ln r$. The volume flux is given by

$$\int_{\partial\Omega} \boldsymbol{u}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S, \qquad \hat{\boldsymbol{n}} = \frac{\boldsymbol{x}}{r}.$$

We have that $\nabla\phi = \boldsymbol{u}$, meaning we have that

$$\boldsymbol{u} = \frac{x-\text{ж}}{r^2}\hat{\boldsymbol{\imath}} + \frac{y-\text{ч}}{r^2}\hat{\boldsymbol{\jmath}}.$$

Since the circle is centered at the origin, $\text{ж} = 0$ and $\text{ч} = 0$, meaning $\boldsymbol{u} = r^{-2}\boldsymbol{x}$. We parametrize the circle through the differential element $\mathrm{d}S = R_0\mathrm{d}\theta$, and evaluate the integrand at $r = R_0$,

$$\int_0^{2\pi} \boldsymbol{x}\cdot\boldsymbol{x}r^{-3}R_0\,\mathrm{d}\theta = \int_0^{2\pi}\mathrm{d}\theta = 2\pi.$$

Example: *Sphere*

Consider now $r^2 = (x-\text{ж})^2 + (y-\text{ч})^2 + (z-\text{з})^2$ and the sphere of radius R_0 centered at the origin, and the volume flux through its surface by the source function $\phi(r) = 1/r$. We have that

$$\boldsymbol{u} = \nabla\phi = -\frac{\boldsymbol{x}}{r}, \qquad \hat{\boldsymbol{n}} = \frac{\boldsymbol{x}}{r}.$$

The differential surface element on the sphere is $\mathrm{d}S = {R_0}^2\sin\theta\,\mathrm{d}\varphi\mathrm{d}\theta$, where θ and φ are the polar and azimuthal variables, respectively. Now the volume flux is given by

$$-\frac{1}{{R_0}^2}\int_{\partial\Omega}\mathrm{d}S = -\int_0^\pi\int_0^{2\pi}\sin\theta\,\mathrm{d}\varphi\mathrm{d}\theta = -4\pi.$$

Green's theorem and Distributions of Singularities

Because of incompressibility, an irrotational flow can be described simply by the Laplace equation

$$\nabla^2\phi = 0 \quad \text{on } \Omega,$$

such that $\nabla\phi = \boldsymbol{u}$. We consider the following expression equality.

$$\nabla\cdot(\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1) = 0.$$

By the product rule, we have that the left-hand side of the equality is equal to

$$\nabla\phi_1\cdot\nabla\phi_2 + \phi_1\nabla^2\phi_2 - \nabla\phi_2\cdot\nabla\phi_1 + \phi_2\nabla^2\phi_1,$$

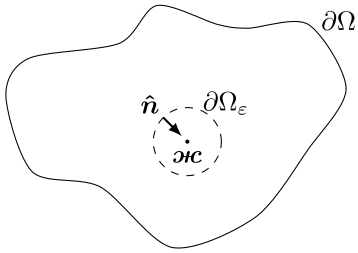
which is indeed equal to zero by virtue of the potentials being harmonic. Considering now the case $\phi_2 = \mathrm{G}$, we may integrate the above equation

$$\int_\Omega \nabla\cdot(\phi\nabla\,\mathrm{G} - \mathrm{G}\,\nabla\phi)\,\mathrm{d}V = 0,$$

which by Gauss' divergence theorem leads to

$$\int_{\partial\Omega}(\phi\partial_n\,\mathrm{G} - \mathrm{G}\,\partial_n\phi)\,\mathrm{d}S = 0.$$

As will be shown later on in the course, we want to integrate over surfaces where the potential is singular. The divergence theorem does not hold in such cases, so the singularity must be removed. This can be remedied by considering a small circle around the singularity situated at $\boldsymbol{\varkappa}$ of radius ε.



We may then integrate about this contour to account for the singularity, and ensuring the divergence theorem holds.

$$\int_{\partial\Omega_\varepsilon} (\phi\partial_n \mathrm{G} - \mathrm{G}\,\partial_n\phi)\,\mathrm{d}S, \quad \hat{\boldsymbol{n}} = -\frac{\boldsymbol{x}}{r}.$$

In two dimensions, the GREEN function is the logarithm, and we calculate that $\partial_n \ln r = -\varepsilon^{-1}$. The differential line element may be discretized as $\mathrm{d}S = \varepsilon \mathrm{d}\theta$, and for the limiting case of an evanescent ε, we may approximate the potential by its TAYLOR expansion as follows.

$$\begin{aligned}\phi(\boldsymbol{x}) &= \phi(\boldsymbol{\varkappa}) + (\boldsymbol{x} - \boldsymbol{\varkappa})\cdot\nabla\phi(\boldsymbol{\varkappa}) + \cdots \\ &= \phi(\boldsymbol{\varkappa}) + O(\varepsilon),\end{aligned}$$

since we know $\boldsymbol{x} - \boldsymbol{\varkappa} \leq \varepsilon$. The above integral is then approximated to the first order by

$$-\int_0^{2\pi} (\phi(\boldsymbol{\varkappa}) + \varepsilon\ln\varepsilon\partial_n\phi)\,\mathrm{d}\theta,$$

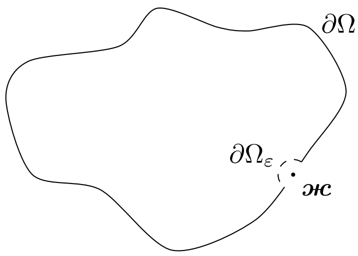
for which we may consider the limiting case, yielding

$$\int_{\partial\Omega_\varepsilon} (\phi\partial_n \mathrm{G} - \mathrm{G}\,\partial_n\phi)\,\mathrm{d}S \approx -2\pi\phi(\boldsymbol{\varkappa}).$$

Then to the first order, we have in general that for areas of integration enclosing a singularity,

$$\int_{\partial\Omega} (\phi\partial_n \mathrm{G} - \mathrm{G}\,\partial_n\phi)\,\mathrm{d}S = 2\pi\phi(\boldsymbol{\varkappa}).$$

Should the singularity lie on the boundary, the integration path is a semicircle.

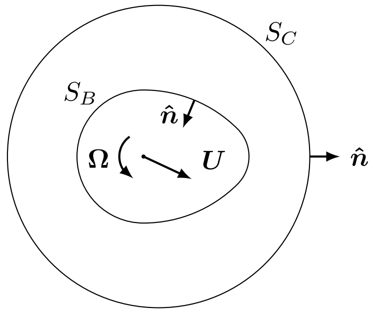


When the singularity is at the boundary like this, the resulting contribution to the integral is consequently halved:

$$\int_{\partial\Omega} (\phi\partial_n \mathrm{G} - \mathrm{G}\,\partial_n\phi)\,\mathrm{d}S = \pi\phi(\boldsymbol{\varkappa}).$$

When the singularity is outside the domain of integration, we have seen that the integral is equal to zero.

Hydrodynamic Pressure Forces



We consider a submerged body in an unbounded fluid moving with velocity $\boldsymbol{U} = U(t)\hat{\boldsymbol{\imath}}$. The fluid is described by a potential $\Phi(\boldsymbol{x}, t)$, whose gradient is evanescent at large r. Physically, this means that the motion of the body through the fluid does not influence the fluid in the far field. We impose the kinematic boundary condition on the body of slip, meaning $\partial_n\Phi = \boldsymbol{U}\cdot\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}} = n_x\hat{\boldsymbol{\imath}} + n_y\hat{\boldsymbol{\jmath}}$. As an ansatz, we say that

$$\Phi(\boldsymbol{x}, t) = U(t)\phi(\boldsymbol{x}),$$

where $\phi(\boldsymbol{x})$ satisfies the LAPLACE equation

$$\nabla^2\phi = 0, \quad \text{in } \Omega,$$

$$\partial_n\phi = n_x, \quad \text{on } \partial\Omega,$$

$$\lim_{r\to\infty} |\nabla\phi| = 0,$$

The very essence of this course is to predict the forces and moments acting on a body past which fluid motion flows. By BERNOULLI's equation, we have that

$$\begin{aligned}\boldsymbol{F} &= \int_{S_B} p\hat{\boldsymbol{n}}\,\mathrm{d}S \\ &= -\varrho\int_{S_B}\left(\partial_t\phi + {}^1\!/\!{}_2|\nabla\phi|^2\right)\hat{\boldsymbol{n}}\,\mathrm{d}S,\end{aligned}$$

$$\begin{aligned}\boldsymbol{M} &= \int_{S_B} p\boldsymbol{x}\times\hat{\boldsymbol{n}}\,\mathrm{d}S \\ &= -\varrho\int_{S_B}\left(\partial_t\phi + {}^1\!/\!{}_2|\nabla\phi|^2\right)\boldsymbol{x}\times\hat{\boldsymbol{n}}\,\mathrm{d}S.\end{aligned}$$

We now need the REYNOLDS transport theorem, which states that for a function of the form

$$I(t) = \int_\Omega f(\boldsymbol{x}, t)\,\mathrm{d}V,$$

its derivative is given by

$$I'(t) = \int_\Omega \partial_t f\,\mathrm{d}V + \int_{\partial\Omega} f\boldsymbol{U}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S, \tag{3.11}$$

and the generalized STOKES theorem,

$$\int_\Omega \nabla f\,\mathrm{d}V = \int_{\partial\Omega} f\hat{\boldsymbol{n}}\,\mathrm{d}S. \tag{4.85}$$

The rate of change of the momentum of the volume enclosed by S_C is given by

$$\varrho\frac{\mathrm{d}}{\mathrm{d}t}\int_V \nabla\phi\,\mathrm{d}V = \varrho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_B+S_C}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S.$$

Using the REYNOLDS transport theorem on the left-hand side, and interchanging the gradient and temporal derivative, we have that it is equal to

$$\varrho\int_\Omega \nabla\partial_t\phi\,\mathrm{d}V + \varrho\int_{\partial\Omega}\nabla\phi(\boldsymbol{U}\cdot\hat{\boldsymbol{n}})\,\mathrm{d}S,$$

Now using the genrealized STOKES theorem, we have that it further simplifies to

$$\varrho\int_{\partial\Omega}\partial_t\phi\hat{\boldsymbol{n}}\,\mathrm{d}S + \varrho\int_{\partial\Omega}\nabla\phi(\boldsymbol{U}\cdot\hat{\boldsymbol{n}})\,\mathrm{d}S.$$

Since the contour S_B is steady, we may interchange the temporal differential operator with the integral for that contribution to $\partial\Omega$. Due to impermeability, we have that $\boldsymbol{U}\cdot\hat{\boldsymbol{n}}$ on that contour. Furthermore, $\partial_n\phi = \boldsymbol{U}\cdot\hat{\boldsymbol{n}}$, so

$$\varrho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_B}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S = \varrho\int_{S_B}\left(\partial_t\phi\hat{\boldsymbol{n}} + \partial_n\phi\nabla\phi\right)\mathrm{d}S.$$

We may rewrite the rate of change of the momentum, or the force, in terms of the above, we have that

$$\begin{aligned}\boldsymbol{F} = &-\varrho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_B}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S\\ &-\varrho\int_{S_C}\left(\partial_n\phi\nabla\phi - {}^1\!/\!_2|\nabla\phi|^2\hat{\boldsymbol{n}}\right)\mathrm{d}S,\end{aligned}$$

where we have used that

$$\int_{\partial\Omega}\left(\partial_n\phi\nabla\phi - {}^1\!/\!_2|\nabla\phi|^2\hat{\boldsymbol{n}}\right)\mathrm{d}S = \int_\Omega\nabla\phi\nabla^2\phi\,\mathrm{d}V \equiv 0.$$

In a similar manner, one may use

$$\int_\Omega\nabla\times\boldsymbol{Q}\,\mathrm{d}V = \int_{\partial\Omega}\hat{\boldsymbol{n}}\times\boldsymbol{Q}\,\mathrm{d}S$$

to show that

$$\begin{aligned}\boldsymbol{M} = &-\varrho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_B}\phi(\boldsymbol{x}\times\hat{\boldsymbol{n}})\,\mathrm{d}S\\ &-\varrho\int_{S_C}\boldsymbol{x}\times\left(\partial_n\phi\nabla\phi - {}^1\!/\!_2|\nabla\phi|^2\right)\mathrm{d}S.\end{aligned}$$

Force on a Moving Body in an Unbounded Fluid

Linearizing the pressure initially instead, such that $p \approx -\varrho\partial_t\phi = -\varrho\phi\partial_t U$, we have that the force is simply

$$\boldsymbol{F} = -\varrho\partial_t U\int_{\partial\Omega}\phi\hat{\boldsymbol{n}}\,\mathrm{d}S,$$

and the moment

$$\boldsymbol{M} = -\varrho\partial_t U\int_{\partial\Omega}\phi\boldsymbol{x}\times\hat{\boldsymbol{n}}\,\mathrm{d}S.$$

Using NEWTON's second law of motion, we express the force in terms of mass and acceleration,

$$F_i = -m_{1i}\partial_t U, \qquad m_{1i} = \varrho\int_{\partial\Omega}\phi n_i\,\mathrm{d}S.$$

A body moving with velocity $\boldsymbol{U} = U_1\hat{\boldsymbol{\imath}} + U_2\hat{\boldsymbol{\jmath}}$ and angular velocity $\boldsymbol{\Omega} = U_6\hat{\boldsymbol{k}}$ may have its potential expressed as a superposition

$$\Phi = U_1\phi_1 + U_2\phi_2 + U_6\phi_6$$

Similarly to the linearization above, we find that in general, the force may be expressed

$$F_i = -\sum_{j\in\{1,2,6\}} m_{ji}\partial_t U_j, \qquad m_{ji} = \varrho\int_{\partial\Omega}\phi_j n_i\,\mathrm{d}S,$$

where $n_i = \partial_n\phi_i$, and m_{ji} is the added mass tensor.

Example: ***Added mass of circular cross-section***

We wish to calculate the added mass coefficient of a circle of radius R_0 moving according to the potential

$$\phi_1 = -{R_0}^2\partial_x\ln r = -\frac{x{R_0}^2}{r^2}.$$

We parametrize the circle with $\mathrm{d}S = R_0\,\mathrm{d}\theta$, and consider the normal vector corresponding to translation along $\hat{\boldsymbol{\imath}}$, $n_1 = \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\imath}} = -{}^x\!/\!_r$, and find that

$$m_{11} = \varrho R_0\int_0^{2\pi}\left(-\frac{x{R_0}^2}{r^2}\right)\left(-\frac{x}{r}\right)\mathrm{d}\theta = \varrho\pi{R_0}^2.$$

We see that the added mass corresponds to the fluid displaced by the surface meeting the fluid.