The Forces and Response of a Heaving Rectangle

MEKSP100

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The Heave Problem

We consider a box floating on the ocean surface with draft D and length L, whose motion is described by the radiation potential

$$\Phi_{\mathrm{R}}(\boldsymbol{x};t) = \operatorname{Re}\big(i\omega\phi_2\hat{\xi}_2 \exp{(i\omega t)}\big),$$

for which the heave potential ϕ_2 is determined the conditions of continuity in velocity, incompressibility, and evanescence, and the kinematic bounary condition

$$\mathrm{g}\partial_y\phi_2 = -\omega^2\phi_2.$$

The first of the above conditions may be expressed as $\partial_{\hat{\boldsymbol{n}}}\Phi_{\mathrm{R}} = \boldsymbol{u}_{\mathrm{B}}\cdot\hat{\boldsymbol{n}}$, or

$$\partial_{\hat{\boldsymbol{n}}}\phi_2 = \hat{n}_y \qquad \text{on } S_{\mathrm{B}}.$$

The incompressibility of the fluid yields the Laplace equation

$$\nabla^2\phi_2 = 0 \qquad \text{in } \Omega.$$

The condition of evanescence states that the gradient of the velocity potential ought to disappear at infinity, namely

$$|\nabla\phi_2| \to 0 \qquad \text{as } y\to\infty.$$

As for the accompanying Green function,

$$\text{Ц}(\boldsymbol{x},t) = \operatorname{Re}\big(\,\mathrm{G}\,(\boldsymbol{x})\exp{(i\omega t)}\big),$$

it must also satisfy the Laplace equation, kinematic boundary condition, and evanescence conditions. We are looking for radiating solutions, emanating from the body at the origin such that

$$\mathrm{G}\,(\boldsymbol{x}) \sim \exp{(\mp ikx)} \qquad \text{as } x\to\pm\infty.$$

The derivation of the integral equation follows from that found in lecture notes from January 21st, yielding

$$-\pi\phi_2(\boldsymbol{\text{жс}}) + \int_{S_{\mathrm{B}}}\phi_2\partial_{\hat{\boldsymbol{n}}}\,\mathrm{G}\;\mathrm{d}S = \int_{S_{\mathrm{B}}}\mathrm{G}\,\partial_{\hat{\boldsymbol{n}}}\phi_2\,\mathrm{d}S.$$

Discretization of the Integral Equation and Boundary

The logarithm terms are integrated in much the same way they were in the first mandatory assignment. The boundary element method assumes the potential is constant, set to the value of the midpoint between nodes. Integration of the logarithm terms in the Green function is outlined in the lecture notes from January 28th. The gradient of the other terms are given to be

$$\partial_x\,\mathrm{G} = \kappa\big(\operatorname{Im}{(f_1(\text{ʒ}))} + i\operatorname{Im}{(f_2(\text{ʒ}))}\big),$$

$$\partial_y\,\mathrm{G} = \kappa\big(\operatorname{Re}{(f_1(\text{ʒ}))} + i\operatorname{Re}{(f_2(\text{ʒ}))}\big),$$

and are treated with a midtpoint rule, setting the complex variable

$$\text{ʒ} = \kappa\big(\text{ч}_m + \text{ч}_n - i(\text{жс}_m - \text{жс}_n)\big).$$

The boundary we discretize with Chebyshov distributions along the three line segments in two coordinates $\boldsymbol{x}_{\mathtt{p}}$ and $\boldsymbol{x}_{\mathtt{m}}$. Constructing a partition of the interval whose midpoints forms Chebyshov distribution seems to be more effort than it is worth, so we concede that Chebyshov distributions in $\boldsymbol{x}_{\mathtt{p}}$ and $\boldsymbol{x}_{\mathtt{m}}$ suffice to get accurate enough results near the corners of the box.

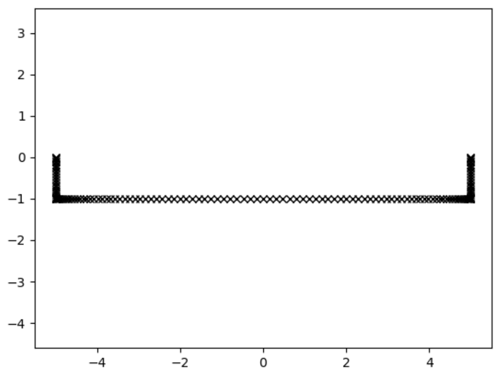

Figure 1: Rectangle with $L/D = 10$, $N_x = 100$, $N_y = 25$.

We may test the numerical scheme by checking that the left-hand side equals the right-hand side of the matrix equation for the real and imaginary parts, using a known function. We consider

$$\phi_0 = \frac{ig e^{\kappa(y-ix)}}{\omega}, \qquad \partial_{\hat{\boldsymbol{n}}}\phi_0 = \kappa(\hat{n}_y - i\hat{n}_x)\phi_0.$$

We have that

$$\pi\phi_0 + \int_{S_{\mathrm{B}}} \phi_0 \partial_{\hat{\boldsymbol{n}}}\, \mathrm{G}\, \mathrm{d}S = \int_{S_{\mathrm{B}}} \mathrm{G}\, \partial_{\hat{\boldsymbol{n}}}\phi_0 \,\mathrm{d}S,$$

meaning we may benchmark the implementation of the integral equation by comparison. Plotting the left-hand side against the right-hand side,

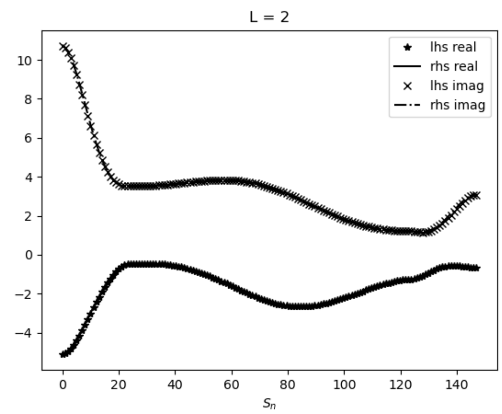

Figure 2: Left-hand and right-hand side of integral equation with ϕ_0. Rectangle $L/D = 2$.

We see that the implementation seems to work, so we may solve for the heave potential.

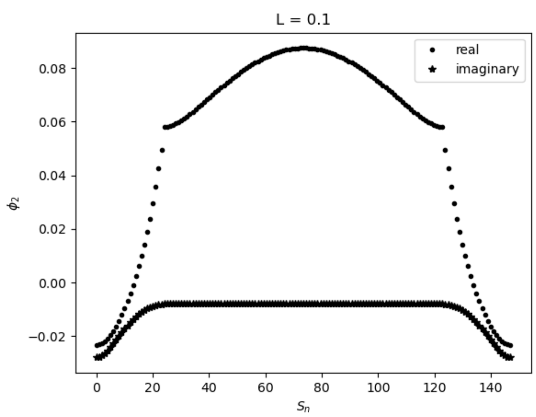

Figure 3: Heave potential for $L/D = 0.1$, and $\kappa D = 1.2$.

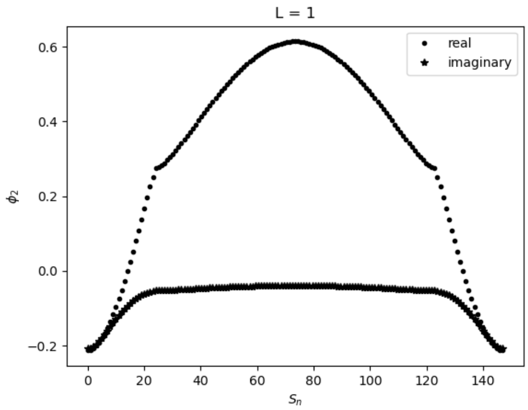

Figure 4: Heave potential for $L/D = 1$, and $\kappa D = 1.2$.

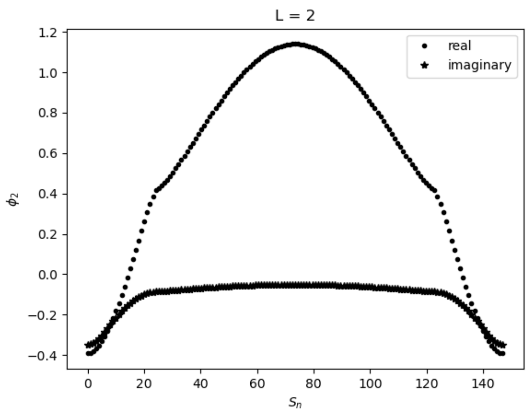

Figure 5: Heave potential for $L/D = 2$, and $\kappa D = 1.2$.

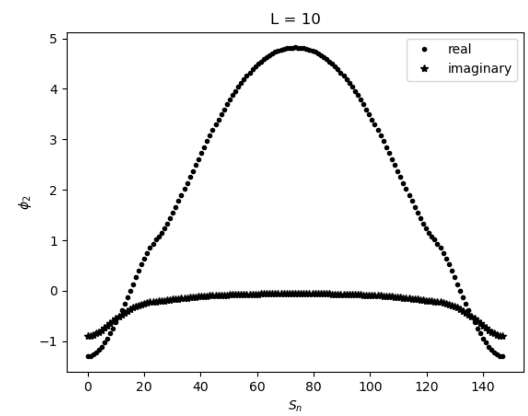

Figure 6: Heave potential for $L/D = 10$, and $\kappa D = 1.2$.

Wave Interactions

The far field amplitudes is shown to be

$$A_j^{\pm\infty} = \mp \int_{S_{\mathrm{B}}} \left(\kappa\phi_j \hat{\boldsymbol{n}}^* + i\hat{n}_j\right) e^{\kappa(y\pm ix)}\,\mathrm{d}S.$$

The added mass and damping coefficients are found by taking the real and imaginary parts of the integral

$$\int_{S_\mathrm{B}} \phi_j \hat{n}_i \,\mathrm{d}S.$$

Consulting the course notes, we furthermore have that the damping may be approximated by

$$r_{22} = \frac{\varrho\omega}{2}\left(|A_2^{\infty}|^2 + |A_2^{-\infty}|^2\right).$$

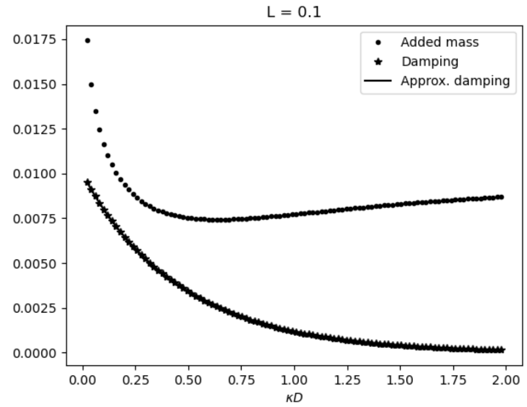

Figure 7: Added mass for $L/D = 0.1$.

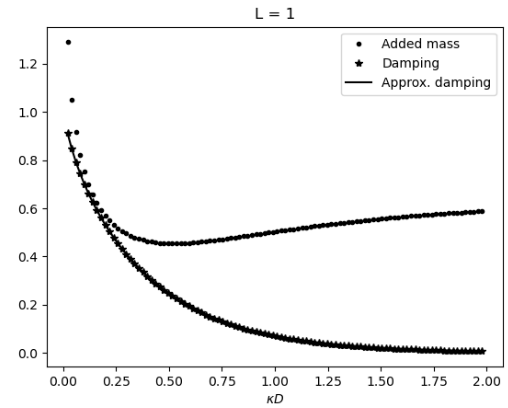

Figure 8: Added mass for $L/D = 1$.

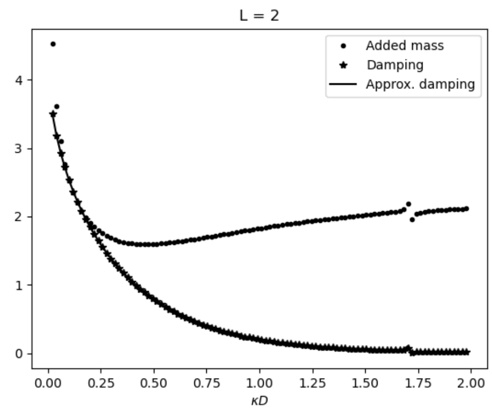

Figure 9: Added mass for $L/D = 2$

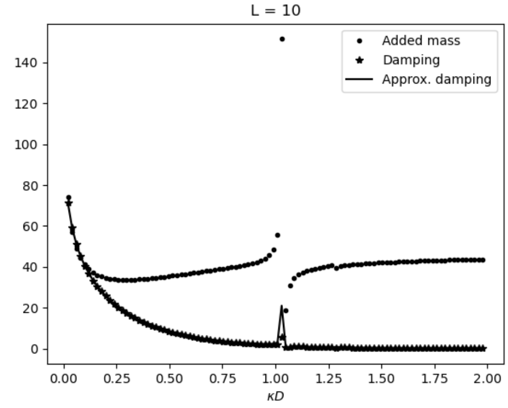

Figure 10: Added mass for $L/D = 10$.

We have plotted the added mass for $\kappa D \in (0, 2)$, and the approximated damping is scaled with respect to ωD^2.

The Diffraction Problem

We now instead imagine the geometry fixed, with the incoming wave ϕ_0 interacting with the geometry. The diffraction potential is described by

$$\Phi_\mathrm{D}(\boldsymbol{x};t) = \mathrm{Re}\left(A\phi_\mathrm{D} e^{i\omega t}\right),$$

where $\phi_\mathrm{D} = \phi_0 + \phi_7$. We may find ϕ_7 by solving the integral equation

$$-\pi\phi_\mathrm{D}(\boldsymbol{\varkappa}) + \int_{S_\mathrm{B}} \phi_\mathrm{D}\partial_{\hat{\boldsymbol{n}}}\,\mathrm{G}\;\mathrm{d}S = -2\pi\phi_0(\boldsymbol{\varkappa}).$$

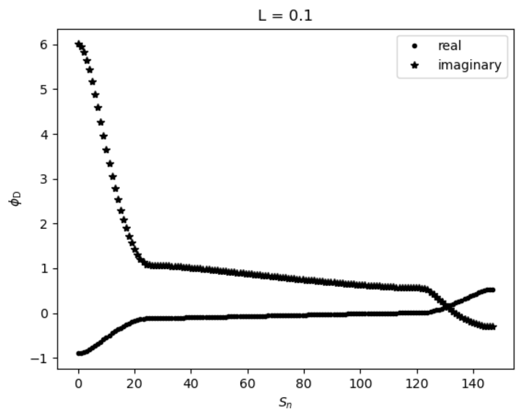

Figure 11: ϕ_D for $L/D = 0.1$.

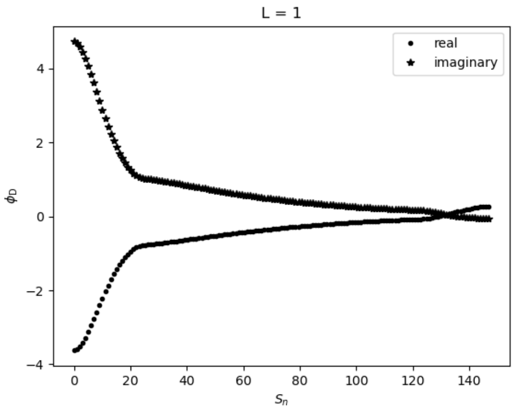

Figure 12: ϕ_{D} for $L/D = 1$.

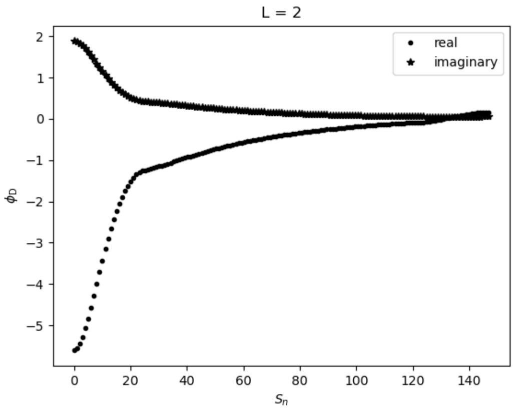

Figure 13: ϕ_{D} for $L/D = 2$.

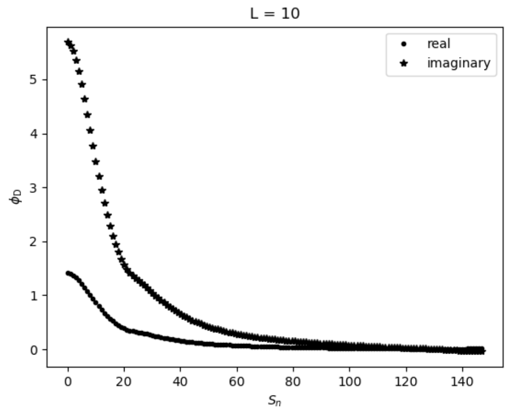

Figure 14: ϕ_{D} for $L/D = 10$.

The exciting force may then be obtained through

$$X_2 = -i\omega\varrho \int_{S_{\mathrm{B}}} \phi_{\mathrm{D}} \hat{n}_2 \,\mathrm{d}S.$$

We also utilize the Haskind relations

$$X_2^{\mathrm{H1}} = -i\omega\varrho \int_{S_{\mathrm{B}}} \left(\phi_0 \partial_{\hat{\boldsymbol{n}}} \phi_2 - \phi_2 \partial_{\hat{\boldsymbol{n}}} \phi_0\right) \mathrm{d}S,$$

$$X_2^{\mathrm{H2}} = i\varrho g A_2^{-\infty},$$

and the Froude–Krylov approximation

$$X_2^{\mathrm{FK}} = \varrho g \sin\left(\frac{\kappa L}{2}\right) \frac{e^{-\kappa D}}{\kappa}.$$

The Haskind relations and Froude–Krylov approximation will be derived in the lecture notes.

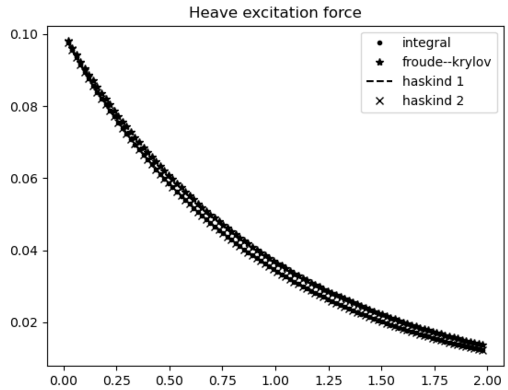

Figure 15: Excitation force for $L/D = 0.1$.

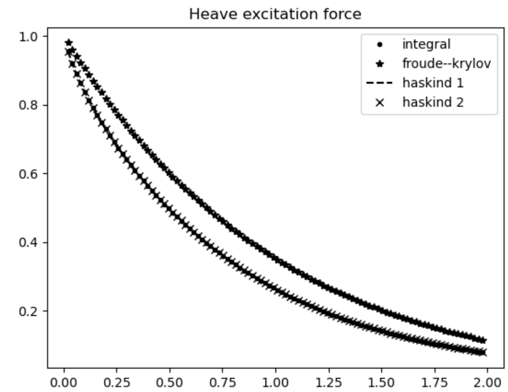

Figure 16: Excitation force for $L/D = 1$.

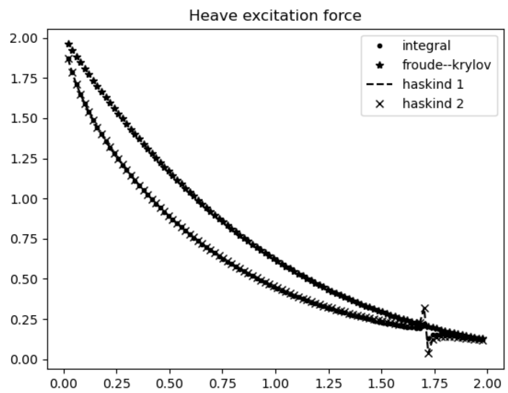

Figure 17: Excitation force for $L/D = 2$.

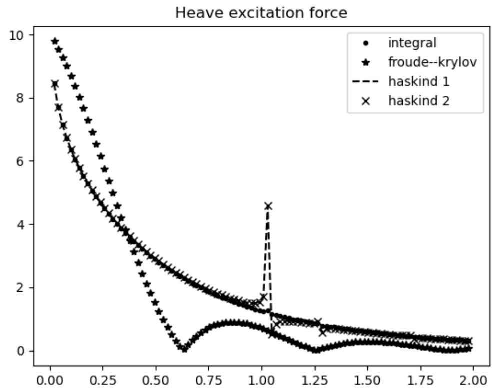

Figure 18: Excitation force for $L/D = 10$.

The Body Response

Problem 6.17

A vertical spar buoy of circular cylindrical form, draft T, and diameter d is freely floating. Compute the hydrostatic restoring forces and moments. Estimate the natural frequency in heave, assuming that the buoy is sufficiently slender that the added mass and dmaping coefficients can be neglected by comparison to the mass of the buoy. Estimate the ecxiting force force from the FROUDE–KRYLOV approximation, the damping coefficient from the HASKIND relations, and compute the heave response.

Cylindrical buoy with bouyant center y_B and center of gravity y_G, diameter d, draft T. Displaced volume is $\forall = S \times T$, $S = {}^{\pi d^2}/_4$ waterline area. Hydrostatic restoring forces

$$c_{22} = \varrho \mathrm{g} S, \qquad c_{44} = \varrho \mathrm{g} S_{33} + \varrho \mathrm{g} \forall (y_\mathrm{B} - y_\mathrm{G}), \qquad c_{66} = \varrho \mathrm{g} S_{11} + \varrho \mathrm{g} \forall (y_\mathrm{B} - y_\mathrm{G})$$

Cylinder, $S = {}^{\pi d^2}/_4$,

$$S_{11} = \int_{S_\mathrm{B}} x^2 \,\mathrm{d}S = \int_0^{d/2} \int_0^{2\pi} r^3 \cos^2\theta \,\mathrm{d}\theta \mathrm{d}r, \quad S_{33} = \int_{S_\mathrm{B}} z^2 \,\mathrm{d}S = \int_0^{d/2} \int_0^{2\pi} r^3 \sin^2\theta \,\mathrm{d}\theta \mathrm{d}r.$$

As expected, $S_{11} = S_{33}$. We get that

$$c_{22} = \frac{\pi \varrho \mathrm{g} d^2}{4}, \qquad c_{44} = c_{66} = \frac{\pi \varrho \mathrm{g} d^4}{64} + \frac{\pi \varrho \mathrm{g} T d^2 (y_\mathrm{B} - y_\mathrm{G})}{4}$$

Units check out, force $\mathrm{kg}\,\mathrm{s}^{-2}$, moment of inertia $\mathrm{kg}\,\mathrm{m}^2\,\mathrm{s}^{-2}$. Natural frequency given by the singularity of repsonse amplitude operator,

$$\frac{\hat{\xi}_2}{A} = \frac{|X_2|}{c_{22} - \omega^2 (m + m_{22}) + i\omega r_{22}},$$

where the $m = \varrho \forall$ is the mass. That is, $\omega_n = \sqrt{{}^1/_T}$. FROUDE–KRYLOV force on submerged cylinder:

$$X_2^{\mathrm{FK}} = -i\omega\varrho \int_{S_\mathrm{B}} \phi_0 \hat{n}_2 \,\mathrm{d}S, \qquad \phi_0 = \frac{i\mathrm{g} e^{\kappa(y - ix)}}{\omega}.$$

Here S_B is composed of two distinct surface integrals—one over the bottom of the cylinder, and one over the cylinder walls. The component of the normal vector coinciding with the heave motion is zero on the cylinder walls, so there is no contribution, yielding

$$X_2^{\mathrm{FK}} = -i\omega\varrho e^{-\kappa T} \int_{|\mathfrak{z}| \le d/2} e^{-i\kappa x} \,\mathrm{d}\mathfrak{z} = \frac{\pi \varrho \mathrm{g} d^2 e^{-\kappa T}}{4},$$

where $\mathfrak{z} = x + iz$. Consulting the lecture notes, we know $A_2^{\infty} = A_2^{-\infty}$, yielding from the second HASKIND relation and the FROUDE–KRYLOV approximation

$$r_{22} = \frac{\omega |X_2^{\mathrm{FK}}|^2}{\varrho \mathrm{g}} = \frac{\pi^2 \omega \varrho d^4 e^{-2\kappa T}}{16}.$$