

Lecture Notes

MEK4420

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The Discrete Integral Equation

We want to numerically solve the integral equation

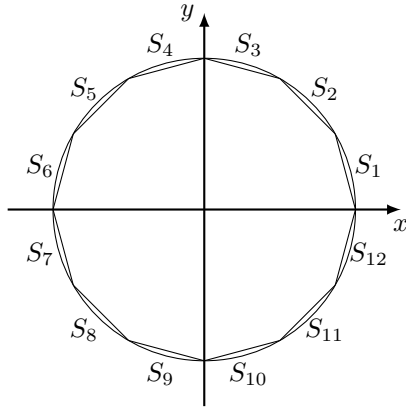
$$\int_{\partial\Omega} (\phi \partial_{\hat{n}} \ln r - \ln r \partial_{\hat{n}} \phi) dS = \pi \phi(\mathbf{x}).$$

The Boundary Element Method

We discretize the boundary with N collocation points, and connect them with straight line segments S_k such that

$$S = \bigcup_{m=1}^N S_m.$$

For convex Ω , the discrete boundary S will enclose a smaller area.



We can then set the potential function and its derivatives constant across these line segments as an approximation so that

$$\int_{\partial\Omega} \phi \partial_{\hat{n}} \ln r dS \approx \sum_{m=1}^N \phi_m \int_{S_m} \partial_{\hat{n}} \ln r dS,$$

$$\int_{\partial\Omega} \ln r \partial_{\hat{n}} \phi dS \approx \sum_{m=1}^N \partial_{\hat{n}} \phi_m \int_{S_m} \ln r dS,$$

where $\phi_m \equiv \phi(\mathbf{x}_m)$, and $\mathbf{x}_m = 1/2(\mathbf{x}_m + \mathbf{x}_{m-1})$.

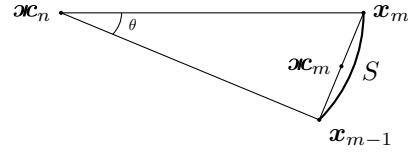
¹[1] ABRAMOWITZ & STEGUN, p.67, eq.4.1.2

Logarithmic Flux Integral

We consider the flux integral of the natural logarithm over some segment S of the geometry boundary,

$$\int_S \hat{n} \cdot \nabla \ln r dS.$$

On this segment, the differential element dS can be decomposed into differentials along the abscissal and ordinal components, and the logarithm may be expressed in terms of the real part of its complex counterpart. That is, $\hat{n} \cdot \nabla = n_x \partial_x + n_y \partial_y$, and $n_x dS = -dy$ and $n_y dS = dx$.



Letting $\text{Re}(\star)$ denote the real part of a complex number, and noting that $r_m = |\mathbf{x}_m - \mathbf{x}_n|$, we have that¹

$$\ln r = \text{Re}(\ln(\mathcal{z} - \mathbf{x}_n)), \quad \mathcal{z} = x + iy.$$

Now,

$$\int_S \hat{n} \cdot \nabla \ln r dS = \text{Re} \int_S \frac{i}{\mathcal{z} - \mathbf{x}_n} d\mathcal{z}.$$

Evaluating the integral at the points \mathbf{x}_m and \mathbf{x}_{m-1} , and using the fact that $\ln \mathcal{z} = \ln |\mathcal{z}| + i \arg \mathcal{z}$, we get that

$$\int_S \hat{n} \cdot \nabla \ln r dS = \Theta_{n,m-1} - \Theta_{n,m} \equiv -\theta,$$

where $\Theta_{n,m} = \arg(\mathbf{x}_m - \mathbf{x}_n)$. We note that this contribution to the discrete integral equation is exact, and it is the assumption that ϕ is constant along the line segment S_m that causes inaccuracy.

Quadrature Methods

To integrate the logarithm, we employ a so-called GAUSS quadrature method of the second order. This method has us map the domain of integration to the one dimensional unit circle,¹

$$\int_a^b y(x) dx \mapsto \int_{-1}^1 \eta(\xi) d\xi,$$

where

$$x = \frac{b-a}{2}\xi + \frac{a+b}{2}, \quad \eta(\xi) \equiv \frac{b-a}{2}y(x).$$

This latter integral we approximate by

$$\sum_{k=1}^N w_k \eta(\xi_k), \quad w_k = \frac{2}{(1 - \xi_k^2)(P'_N(\xi_k))^2},$$

Where P_N is the N^{th} -degree LEGENDRE polynomial, and ξ_k are its N zeros. Setting $N = 2$, and recalling the three first LEGENDRE polynomials,

$$P_0(\xi) = 1, \quad P_1(\xi) = \xi, \quad P_2(\xi) = \frac{3\xi^2 - 1}{2},$$

We find that $w_1 = 1$ and $w_2 = 1$, and $\xi_k = \pm 1/\sqrt{3}$. Recalling the logarithm rule $\ln(x^\alpha) = \alpha \ln x$, we have that

$$\int_{S_m} \ln r dS \approx \frac{1}{2} \sum_{k=1}^2 \frac{|\mathbf{x}_m - \mathbf{x}_{m-1}|}{2} \ln |\mathbf{x}'_m - \mathbf{x}_n|^2,$$

where

$$\mathbf{x}'_m = \frac{\mathbf{x}_m + \mathbf{x}_{m-1}}{2} + \frac{(-1)^k(\mathbf{x}_m - \mathbf{x}_{m-1})}{2\sqrt{3}}.$$

We usually label the tensor collection of such integrals \mathbf{h} .

The Discrete Integral Equation

We note that $\partial_{\hat{\mathbf{n}}} \phi_m$ has three modes corresponding to each of the Cartesian unit vectors, and that the subscript m denotes evaluation at that indexed node, and likewise with the rotational normal vectors. Since we consider here a Galilean coordinate system—moving with the geometry—the ϕ here represent velocity potentials due to geometry motion with unit velocity some mode.² It is meant here, then, that $\partial_{\hat{\mathbf{n}}} \phi_m = n_i$, where n_i is whichever normal vector component. The integral equation is then given as follows.

$$-\pi \phi_n - \sum_{m=1}^N \phi_m \theta_{n,m} = \sum_{m=1}^N \partial_{\hat{\mathbf{n}}} \phi_m \mathbf{h}_{n,m}.$$

¹[1] ABRAMOWITZ & STEGUN, pp.779 & 887

Added Mass

The added mass tensor \mathbf{m} may be similarly approximated.

$$\mathbf{m} = \varrho \int_{\partial\Omega} \phi_j n_i dS = \varrho \sum_{m=1}^N |\mathbf{x}_m - \mathbf{x}_{m-1}| \phi_{j_m} n_{im}$$

More on the Added Mass Tensor

Symmetry

The added mass tensor is symmetric. We may prove this by showing that $\mathbf{m} - \mathbf{m}^\dagger \equiv 0$. That is,

$$\mathbf{m} - \mathbf{m}^\dagger = \varrho \int_{\partial\Omega} (\phi_i \partial_{\hat{\mathbf{n}}} \phi_j - \phi_j \partial_{\hat{\mathbf{n}}} \phi_i) dS \equiv 0.$$

Kinetic Energy

The kinetic energy of the fluid may be expressed as the integral

$$\frac{\varrho}{2} \int_{\Omega} \nabla \Phi \cdot \nabla \Phi dV.$$

We recall the identity $\nabla \Phi \cdot \nabla \Phi + \Phi \nabla \cdot \nabla \Phi = \nabla \cdot (\Phi \nabla \Phi)$, and using GAUSS' divergence theorem, we may decompose the potentials, yielding the following expression of the kinetic energy in terms of the added mass tensor, $1/2 \sum_{i,j} U_i U_j m_{ij}$.

Example: Added mass of sphere

We wish to calculate the added mass of a sphere of radius R_0 moving according to the potential

$$\phi_1 = -\frac{R_0^3}{2} \partial_x (-1/r) = -\frac{x R_0^3}{2r^3}.$$

We use the definition of the added mass,

$$m_{11} = \varrho \int_{\partial\Omega} \phi_1 n_1 dS = \frac{\varrho R_0}{2} \int_{\partial\Omega} n_1^2 dS.$$

Using the fact that the three components of the Cartesian unit normal vector are the same length, the integrand must be a third of that of the unit normal vector. Integrating that across the surface yields the surface area of the sphere, so that indeed $m_{11} = \rho/2 V$, where $V = 4/3 \pi R_0^3$ is the displaced fluid.

References

- [1] ABRAMOWITZ, MILTON and STEGUN, IRENE A. *Handbook of Mathematical Functions*. 3rd ed. National Bureau of Standards, 1965.
- [2] NEWMAN, JOHN NICHOLAS. *Marine Hydrodynamics*. The MIT press, 2018.

²[2] NEWMAN, pp.143–144