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We want to implement the fast multipole method, as presented in the article *Acceleration of Unsteady Vortex Lattice Method via Dipole Panel Fast Multipole Method* by Deng *et al.*.[1]

[1] [5] Deng *et al.* (2020)

2 Vector fields

Throughout this text, we shall draw from the results of the theory of differential geometry, the most important of which is *the fundamental theorem of multivariate calculus*. The conditions under which this theorem holds is of such importance to the validity of this work that some of the underlying theory will be discussed. Since a thorough understanding of the theory requires such attention to detail in order to discuss in any more meaningful detail than merely superficial mention, only the minimal required vocabulary and ideas to grasp will be introduced. For a more thorough derivation and explanation of the formalities of the theory and the background thereof, the reader is therefore refered to a dedicated text on the topic. The work at hand will utilize the conventions of Shigeyuki MORITA's *Geometry of Differential Forms* for this purpose, and any rigorous proofs or additional definitions to that end are referred thither. Admittedly, not including proofs of what is stated to be such foundational results preceding this work will not serve to enhance intuition *per se*, and so the reason for its inclusion is in part due to completeness, and in part so as to insure ourselves that there is indeed work from which we can draw already proven rules of arithmetic. Subsequently, the theory of functions of a complex variable will be discussed. For although the work at hand is concerned with fluid flow of three dimensions, results from the study of fluid flows of two dimensions will turn out to be useful, which we shall find can be viewed through the lens of complex analysis.

It ultimately is results pertaining to the fluid flow of air around and in the wake of a wind turbine rotor we wish to model. Of pertinence, then, is to outline the mathematical description of fluid flow altogether. For the purposes of this text, it is sufficient we establish for the moment a fluid to be a volume of a specified density and viscosity, both of which govern its dynamics, or motion. We suppose this motion may be represented in terms of a velocity vector of scalar functions in the infinite Euclidean space it occupies. In practice, such a vector is found through the integration of a differential equation governed by the applicable physical principles. We shall justify this notion of fluids for application to our problem of wind turbines, as it will in time become clear that the assumptions made are in no way general to the physics of fluid flow, or even correct, for that matter.

The aforementioned fundamental theorem of multivariate calculus may then be employed to transform such an integrated differential equation evaluated throughout the entire fluid domain to one over its boundary. In the case of a wind turbine blade, this entails integrating over its surface. There are two special cases of this fundamental theorem of particular interest, namely the GAUSS–OSTROGRADSKY theorem, and the THOMSON–STOKES theorem. The names are inconsequential, as they are indeed only merely offshoots of this more fundamental theorem, and are as such simply often labeled the divergence theorem and curl theorem. Unsurprisingly, they respectively relate the divergence and curl in the fluid to some flux of fluid at the boundary, and are given by

$$\int_\Omega \operatorname{div} \boldsymbol{u} \, \mathrm{d}V = \int_{\partial\Omega} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S}, \qquad \int_\Omega \operatorname{curl} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} = \int_{\partial\Omega} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{s}. \tag{2.1a, b}$$

As is stated, the derivation of this fundamental theorem is beyond the scope of this work, so it will simply be presented once the foundation has been laid. This foundation will also reveal the theory of harmonic analysis, upon which we may build complex analysis. Once discussed, the topic of fluid mechanics will at last be addressed in full, and we shall discuss its dynamics and kinematics. Finally, we shall derive the theory with which this thesis seeks to address its problems—namely the decomposition of fluid processes in solenoidal and vortical porcesses.

2.1 Exterior Calculus

The majority of the theoretical background in this work will hinge almost in its entirety on the validity of the equations (2.1), and theory sufficient for understanding their applications will be discussed in this section.

> **Definition 2.1** (Pseudo-RIEMANN manifold)**.** An n-dimensional Ω is a topological space locally similar to Euclidean space $\mathbb{E}^n$, in a smooth sense. It is said to be of the pseudo-RIEMANN kind if it is equipped with the non-degenerate RIEMANN metric $\mathfrak{g}$.

The RIEMANN metric is left purposefully unspecified, as it depends on the representation of the manifold chosen. Throughout this work, we shall work the pesudo-RIEMANN manifold of dimension 3, but we shall for the moment consider one of dimension n so as to speak of the details in general. The intrinsic smoothness of the manifold reveals something about its differentiability, which is defined by the tangent space of some point in the manifold, $\mathrm{T}_p\,\Omega$. A tangent space $\mathrm{T}_p\,\Omega$ is technically a local space containing all tangent vectors at some point $p \in \Omega$,[1] so we shall introduce the *tangent bundle* as the disjoint union of such spaces for all points on the manifold, $\mathrm{T}\,\Omega = \sqcup_p \mathrm{T}_p\,\Omega$,[2] equipped with a natural projection $\Pi\colon\ \mathrm{T}\,\Omega \to \Omega$, such that vector fields $\boldsymbol{u} \in \mathrm{T}_p\,\Omega$ are mapped to $\Pi(\boldsymbol{u}) = p$.[3] Defining an inner product on this space, $\langle \boldsymbol{a}, \boldsymbol{b}\rangle = [\mathfrak{g}]_{ij} a^i b^j$, where EINSTEIN summation is implied, induces the metric.

In order to construct a meaningful notion of operations on the tangent bundle, we endow its dual, the cotangent bundle $\mathrm{T}^*\,\Omega$, the $\wedge$, forming the exterior algebra $\Lambda\Omega$. The exterior product may appear obtuse when first presented in the abstract manner we will shortly do, but rest assured it does indeed have a geometric interpretation. For, in an effort to unite the exterior algebra of Hermann GRASSMANN[4] and the quaternion algebra of William Rowan HAMILTON,[5] William Kingdon CLIFFORD laid the foundation on which the theory of the nominal CLIFFORD algebras were built.[6] On such algebras, the exterior product of k covectors is understood to be the oriented volume of the k-paralellotope bound by them,[7] and is just one part of the so-called *geometric product*, which provides a coherent definition of a vector product in general. We shall only draw from this intuition at a later point, as we shall focus on the properties of the exterior product with respect to objects on sections of the cotangent bundle.

Labeling the projection of a covector field $\mathfrak{u} \in \mathrm{T}^*\,\Omega$ in the same manner as a vector field, $\Pi^*(\mathfrak{u}) = p$, we define the section to be its inverse operation, uniquely returning the point to the covector field. The repeated application of the exterior product however admits higher grade covector fields. We will therefore use for a grade k monomial section of the exterior algebra $\mathfrak{a} \in \Lambda^k\Omega$ a k-form, and avoid the terminology of covectors altogether, as forms serve a much broader purpose to our end. The exterior product is alternating in the sense that for $\mathfrak{a}_i \in \Lambda\Omega$, given a permutation σ of the positive integers up to k, we have $\mathfrak{a}_{\sigma(1)} \wedge \cdots \wedge \mathfrak{a}_{\sigma(k)} = \operatorname{sgn}(\sigma)\left(\mathfrak{a}_1 \wedge \cdots \wedge \mathfrak{a}_k\right)$. The sign of a permutation $\operatorname{sgn}(\sigma)$ is 1 if the number of transpositions from the identity permutation is even, and -1 if the number of transpositions is odd. This alternating property further reduces to the two properties that

$$\mathfrak{a} \wedge \mathfrak{a} \equiv 0, \qquad \mathfrak{a} \wedge \mathfrak{b} \equiv -\mathfrak{b} \wedge \mathfrak{a}, \tag{2.2a, b}$$

for any forms $\mathfrak{a}, \mathfrak{b}$, of the same or different grade. To summarize, then:

Definition 2.2 (Exterior algebra)**.** Let $\mathrm{T}\,\Omega$ be a tangent bundle with basis ∂_i, and let $\wedge$ denote the product of the algebra $\Lambda\Omega = \mathrm{T}^*\,\Omega$. Then the members of the algebra

$$\mathrm{d}x^i = \left([\mathfrak{g}]^{ij}\partial_j\right)^\flat$$

form a basis for the cotangent bundle. When $\wedge : \Lambda^k\Omega \times \Lambda^l\Omega \to \Lambda^{k+l}\Omega$ satisfies the conditions of bilinearity and associativity, in addition to the modified alternating condition of equation (2.2b)

$$\mathfrak{a} \wedge \mathfrak{b} = (-1)^{kl}\,\mathfrak{b} \wedge \mathfrak{a}, \tag{2.3}$$

it is an exterior product, and $\Lambda^{k+l}\Omega$ is an exterior algebra.

Vector fields, then, as far as this work is concerned, are sections of the tangent space, though we will not notationally distinguish them. The notion of a tangent space is inherently connected to some sense of differentiability. Thus, given a differentiable curve on the manifold, the tangent vector at a point is described by this differentiability, and the partial derivative operators form a basis for the tangent space with respect to local coordinates on the manifold. We may choose for it some local basis $\{\boldsymbol{e}_j\}$, such that a vector field may be expressed $\boldsymbol{u} = u^j \boldsymbol{e}_j$. For $\mathbb{E}^3$, with coordinates x, y, z specifically, we shall always write its basis as $\{\partial_x, \partial_y, \partial_z\} \simeq \{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$. This *dual space* we discussed earlier may be regarded as the space of all linear functionals from the tangent space onto the field over which the vector field lies. Consider still the general basis vectors $\boldsymbol{e}_j$, along with some dual basis $\{\boldsymbol{e}^i\}$. As a result of the above discussion, we must have that $\boldsymbol{e}^i(\boldsymbol{e}_j) = \delta^i_j$. Clearly,

$$\boldsymbol{e}^i(\boldsymbol{u}) = \boldsymbol{e}^i(u^j \boldsymbol{e}_j) = u^j \boldsymbol{e}^i \boldsymbol{e}_j = u^j \delta^i_j = u^i.$$

In fact, any k-form is by construction a functional over the space of vector fields on the manifold. It is clear that there exists some isomorphism between the vectors and covectors. If $\mathfrak{u} \in \Lambda\Omega$ and $\boldsymbol{u} \in \mathrm{T}\,\Omega$

[1] [6] FLANDERS, p.54
[2] [12] JOST, p.12
[3] [13] MORITA, pp.169–171
[4] [7] GRASSMANN, §55
[5] [9] HAMILTON (1844)
[6] [4] CLIFFORD (1878)
[7] [7] GRASSMANN, pp.50–51

are isomorphic, these isomorphisms are usually denoted by the sharp $\mathfrak{u}^\sharp = \boldsymbol{u}$ and the flat $\boldsymbol{u}^\flat = \mathfrak{u}$, respectively raising and lowering the indices of the coefficients. Stated explicitly, with these operators, we may describe any vector field $\boldsymbol{u} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}}$ in terms of differential forms as $\boldsymbol{u}^\flat = u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$, so that $u^i\partial_i \simeq u_i\,\mathrm{d}x^i$. This notation however tends to clutter expressions and is cumbersome to read, and it will only be used when strictly necessary. As the astute reader may have noticed, we already readily distinguish between elements of the cotangent and tangent spaces by denoting the former with a fraktur typeface, and the latter with an italic bold. Thus it should be clear from context whichever of the two is meant at any given time.

We note the above expression implies the basis covectors, and indeed reiterate the fact that k-forms themselves are functions of vectors fields. In fact, when $\mathfrak{a} \in \Lambda\Omega$, we have that $\mathfrak{a}(\boldsymbol{u}) = a_i u^j$, which is a zero-form, or scalar. This generalizes to higher grade forms, with the introduction of the interior product, acting as a contraction in the usual sense of the term.

Definition 2.3 (Interior product)**.** Let $\boldsymbol{u} \in \mathrm{T}\,\Omega$, and $\mathfrak{a} \wedge \mathfrak{b} \in \Lambda^{k+l}\Omega$, noting the order of the addition of exponents in the exterior algebra. The interior product $\lrcorner : \mathrm{T}\,\Omega \times \Lambda^{k+l}\Omega \to \Lambda^{k+l-1}$, is given by

$$\boldsymbol{u} \lrcorner (\mathfrak{a} \wedge \mathfrak{b}) = (\boldsymbol{u} \lrcorner \mathfrak{a}) \wedge \mathfrak{b} + (-1)^k \mathfrak{a} \wedge \boldsymbol{u} \lrcorner \mathfrak{b}.$$

For unit $\mathfrak{b}$, we have $\mathfrak{a} \wedge 1 = \mathfrak{a}$, and so we say $\boldsymbol{u} \lrcorner \mathfrak{a} = \mathfrak{a}(\boldsymbol{u})$.

Extrapolating on this idea that the tangent space is inherently dependent on some notion of differentiability forming a basis for the tangent space, the cotangent space must then have a corresponding operation, which we will label the exterior derivative.

Definition 2.4 (Exterior derivative)**.** Given a manifold Ω and basis $\{\partial_i\}$, the exterior derivative $\mathrm{d}\colon \Lambda^k\Omega \to \Lambda^{k+1}\Omega$ of a k-form $\mathfrak{a} = a_{\imath}\,\mathrm{d}x^{\imath}$ is given by $\mathrm{d}\mathfrak{a} = \partial_i a_{\imath}\,\mathrm{d}x^i \wedge \mathrm{d}x^{\imath}$, where $\imath$ is a multi-index of i. The product rule is graded, namely $\mathrm{d}(\mathfrak{a} \wedge \mathfrak{b}) = \mathrm{d}\mathfrak{a} \wedge \mathfrak{b} + (-1)^k \mathfrak{a} \wedge \mathrm{d}\mathfrak{b}$. Furthermore, $\mathrm{d}\mathrm{d}\mathfrak{a} \equiv 0$.

In the manipulation of expressions of forms, it shall be necessary to convert forms of a higher grade to one of lower grade, and *vice versa*, for example before or after computing the exterior derivative. To that end, we introduce the Hodge star-operator.

Definition 2.5 (Hodge operator)**.** The Hodge operator $\star : \Lambda^k\Omega \to \Lambda^{n-k}\Omega$, where n is the dimension of Ω, such that $\mathfrak{a} \wedge \star\mathfrak{b} = \langle \mathfrak{a}, \mathfrak{b} \rangle\,\mathrm{d}V$. It holds that $\star\star\mathfrak{a} = (-1)^{k(n-k)}\mathfrak{a}$.

With the exterior derivative, we may describe the volume of the space with respect to local coordinates. In particular, for a manifold Ω of dimension n, $\mathrm{d}V \equiv \sqrt{|\mathfrak{g}|}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, where $|\mathfrak{g}|$ is the determinant of the local metric. We find from the definition of the Hodge star above that $\star\mathrm{d}V = 1$.

In order to study the exterior derivative under the Hodge operator, we define the *codifferential*, given as follows.[1]

Definition 2.6 (Coderivative)**.** The codifferential of a k-form is adjoint to the exterior derivative, and is given by

$$\delta = (-1)^k \star^{-1} \mathrm{d}\star = (-1)^{n(k+1)+1} \star \mathrm{d} \star .$$

For illustrative purposes, the relationship between the exterior derivative, coderivative, and Hodge star is given by the following commutative diagram.

$$\begin{tikzcd} \Lambda^k\Omega \arrow[r, "\star"] \arrow[d, "\delta"'] & \Lambda^{n-k}\Omega \arrow[d, "\mathrm{d}"] \\ \Lambda^{k-1}\Omega \arrow[r, "(-1)^k\star"'] & \Lambda^{n-k+1}\Omega \end{tikzcd}$$

[1] [13] Morita, §4.2

As we have noted, we are strictly speakning not performing these operations on the space $\Lambda^k\Omega$, but rather sections of it. On these sections we may define an inner product that should be familiar,

$$(\mathfrak{a},\mathfrak{b}) = \int_\Omega \langle \mathfrak{a},\mathfrak{b}\rangle \,\mathrm{d}V = \int_\Omega \mathfrak{a}\wedge\star\mathfrak{b}. \tag{2.4}$$

We have of course not defined what the integral of a differential form actually is. Still, assuming inheritance from our familiar notion of the integral as a linear operator, we find that it does indeed constitute an integral analogous to that of the L^2 inner product. This inner product is symmetric, and is the inner product with respect to which the codifferential is adjoint to the exterior derivative, in the sense that $(\mathrm{d}\mathfrak{a},\mathfrak{b}) = (\mathfrak{a},\delta\mathfrak{b})$. For proofs and further discussion on the codifferential operator, the reader is again refered to MORITA's monograph.[1]

We are now nearing the revelation of the fundamental theorem of multivariate calculus, and with it the end of this section. Before this, however, we ought to finally apply the operators to these differential forms to get a sense of the reason for which they were introduced to begin with. Consider the Euclidean vector field $\boldsymbol{u} = u\,\mathrm{d}x + v\,\mathrm{d}y + w\mathrm{d}z$. Upon calculating its exterior derivative $\mathrm{d}\boldsymbol{u}$, one finds that

$$\mathrm{d}\boldsymbol{u} = (\partial_x v - \partial_y u)\,\mathrm{d}x\wedge\mathrm{d}y + (\partial_z u - \partial_x w)\,\mathrm{d}z\wedge\mathrm{d}x + (\partial_y w - \partial_z v)\,\mathrm{d}y\wedge\mathrm{d}z.$$

This is nothing less than the two-form corresponding to the curl. Furthermore, the coderivative of a vector field is given by $\delta\boldsymbol{u} = -\star\mathrm{d}\star\boldsymbol{u}$, which upon calculation one finds to be

$$\delta\boldsymbol{u} = -(\partial_x u + \partial_y v + \partial_z w).$$

This is of course the divergence with an opposite sign. And so, we define the operators to be such.

Definition 2.7 (Curl and divergence)**.** Let $\mathfrak{u}\in\Lambda^k\Omega$, with $\dim\Omega = 3$. The curl and divergence of $\mathfrak{u}$ are respectively given by

$$\operatorname{curl}\mathfrak{u} = \star\mathrm{d}\mathfrak{u}, \qquad \operatorname{div}\mathfrak{u} = \star\mathrm{d}\star\mathfrak{u}.$$

From these definitions, it is trivial to prove identities of great utility. For example, $\operatorname{div}\operatorname{curl}\mathfrak{u}$ and $\operatorname{curl}\operatorname{grad}\mathfrak{u}$ both being identically equal to zero. Another way to express the divergence is to consider the contraction of the gradient. We understand the trace of a matrix to be a form of contraction,[2] and the divergence of a vector field is simply the trace of the gradient.[3] For brevity, we take an *ad hoc* approach. Consider the exterior derivative of the interior product $\boldsymbol{u}\lrcorner\,\mathrm{d}V$. We calculate that

$$\begin{aligned}\mathrm{d}(\boldsymbol{u}\lrcorner\,\mathrm{d}V) &= \mathrm{d}\sum_i (-1)^{i+1} u_i \mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^n \\ &= \partial_i u_i \mathrm{d}V \\ &= \operatorname{div}\boldsymbol{u}\,\mathrm{d}V,\end{aligned} \tag{2.5}$$

where $\widehat{\mathrm{d}x^i}$ in the first equality in equation (2.5) means the term is omitted.

Having an idea of what the divergence and curl actually are, we are now ready for the progenitor of their named theorems—the fundamental theorem of multivariate calculus. The proof for this theorem is beyond the scope of the present work, so it will only be stated.

Theorem 2.8 (Fundamental Theorem of Multivariate Calculus)**.** *Let $\mathfrak{u}\in\Lambda^{n-1}\Omega$, and $\partial\Omega$ be the sufficiently regular boundary of Ω. Then,*

$$\int_{\partial\Omega}\mathfrak{u} = \int_\Omega \mathrm{d}\mathfrak{u}. \tag{2.6}$$

The convergence of integrals of the form (2.6) only makes sense for an unbounded fluid domain should the differential form $\mathfrak{u}$ evanesce sufficiently fast. This is clear from the condition that $\mathfrak{u}$ shall have compact support, meaning it would need to be a distribution. Such liberties are not ones we may always take, so we will make a concession in our definition of the manifold Ω. Suppose $\partial\Omega = S\cup\Sigma$, such that Σ encloses Ω, whilst S excises it, as is shown in figure 1. We may now extend Σ in such a way that Ω remains compact, and the general properties of $\mathfrak{u}$ are conserved, now only requiring sufficiently rapid evanescence.

[1][13] MORITA, pp.153–155
[2]expand on this.
[3]confirm this.

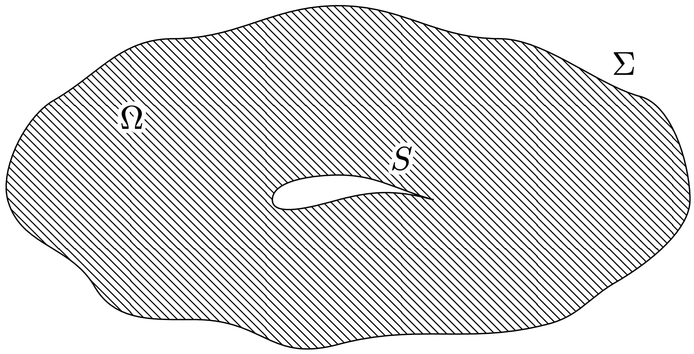

Figure 1: A cross section of Ω, illustrating the boundaries $S \cup \Sigma = \partial\Omega$.

An interesting inquiry to propose is of course how a punctured manifold behaves differently than one which is not. The answer is simple—problems will arise concerning the behavior of closed differential forms. We consider the following.

Lemma 2.9 (POINCARÉ). *Let $\mathfrak{a} \in \Lambda^k\mathbb{R}^n$ be an arbitrary closed form, meaning* $\mathrm{d}\mathfrak{a} = 0$. *There then exists a form $\mathfrak{b} \in \Lambda^{k-1}\mathbb{R}^n$ such that $\mathfrak{a} = \mathrm{d}\mathfrak{b}$.*

The terminology for a form being the exterior derivative of some other form is *exact*. The POINCARÉ lemma states the manifold of figure 1, we can always find some local representation such that any closed form is exact.

As was noted above, the GAUSS–OSTROGRADSKY theorem of equation (2.1a) is a special case of equation (2.6). To show this, we recall the result of equation (2.5), and consider the following equality.

$$\int_\Omega \mathrm{d}(\boldsymbol{u} \lrcorner \mathrm{d}V) = \int_{\partial\Omega} \boldsymbol{u} \lrcorner \mathrm{d}V.$$

Indeed, $\boldsymbol{u} \cdot \hat{\boldsymbol{n}}\mathrm{d}S = u\,\mathrm{d}y \wedge \mathrm{d}z - v\,\mathrm{d}z \wedge \mathrm{d}x + w\,\mathrm{d}x \wedge \mathrm{d}y = \boldsymbol{u} \lrcorner \mathrm{d}V$. And with that, the major result of the boundary element method is proved—namely its progenitor.

Definition 2.10 (HODGE–LAPLACE operator). Let $\mathfrak{u} \in \Lambda^k\Omega$. We define the HODGE Laplacian to be

$$\nabla^2\mathfrak{u} = \delta\mathrm{d}\mathfrak{u} + \mathrm{d}\delta\mathfrak{u}. \tag{2.7}$$

Should $\nabla^2\mathfrak{u} = 0$, then $\mathfrak{u}$ is said to be harmonic.

Theorem 2.11 (HODGE decomposition). *Any differential form $\mathfrak{u}$ may be decomposed as* $\mathrm{d}\mathfrak{a} + \delta\mathfrak{b} + \mathfrak{c}$, *where* $\nabla^2\mathfrak{c} = 0$.

2.2 Complex Analysis

An unbounded fluid, whose flow is invariant in a plane, will then be perfectly described by considering only the directions in which the fluid does vary. Examples of such flow are the cross section of an infinitely long cylinder, or any plane bisection a finite axisymmetric shape aligned with the flow.

2.3 Fluid Mechanics

In our effort to mathematically describe fluid flow, our work shall be inspired by the principles of nature. The discussion that follows will elucidate to which extent our modelling actually reflects real fluids. Studying the physics of fluids is in large part shared by the consideration of the forces applied—dynamics—and the resulting motion—kinematics. It is the fluid kinematics we shall study first. Regarding the motion of the fluid, we should like to distinguish two frames of reference, namely the material and the spatial. The material frame of reference follows some specific fluid particle, whilst the spatial frame observes the fluid passing through some fixed point.[1] Consider some particle $\boldsymbol{x}_0 \in \Omega_0$, which is carried by some flow onto the point $\boldsymbol{x} \in \Omega$ at time t. Distinguishing Ω_0 is a formality—for our purposes it is the very same manifold as we otherwise consider, and the flow is an automorphism. This configuration

[1] [14] TRUESDELL, §14, pp.29–35

decribes nothing less than the dynamical system $\boldsymbol{x} = \boldsymbol{c}(\boldsymbol{x}_0, t)$, $\boldsymbol{x}_0 = \boldsymbol{c}(\boldsymbol{x}_0, 0)$. Clearly the velocity of the fluid at such a fixed point $\boldsymbol{x}$ must depend on the motion of the fluid particle. The nature of his dependency is however not immediately apparent, as the chain rule of differentiation is not something we have established with regards to differentiation with respect to vector fields. To this end, we consider the integral curves of the fluid velocity vector field $\boldsymbol{u}$ charted by $\boldsymbol{x}$. Such an integral curve we may in general label $\boldsymbol{c}(t)$, and are defined by its derivative coinciding with the velocity field at each point. These are not necessarily constrained by an initial condition, but do correspond with the flow when they are. Should an explicit formula for temporal differentiation be established, equations of motion may be derived from Newtonian principles and deliberation of dynamics, as we would possess the constitutive acceleration.

We shall, however, explore another avenue through Hamiltonian mechanics. Considerable amount of such work can be done from very minimal physical considerations, as is shown in Vladimir ARNOLD's works.[1,2] Although the results of these works are magnificent, the processes by which they are derived are beyond the scope of this work, and indeed the author at the present time. We shall therefore draw from Jürgen JOST's monograph *Riemannian Geometry and Geometric Analysis* as a source on connexions and symplectic forms to present these results.[3] The geometric formulation of hydrodynamics developed in the periphery of the works of ARNOLD build upon the Hamiltonian formalism using the theory of groups.

We get the following theorem.[4]

Theorem 2.12 (DARBOUX)**.** *Let $\mathfrak{h}$ be a closed, nondegenerate bivector on some manifold of even dimension. Then every point $\boldsymbol{x}$ has some neighborhood on which we can find a coordinate system $(x^1, \dots, p_1, \dots)$, with*

$$\mathfrak{h} = \mathrm{d}p_i \wedge \mathrm{d}x^i. \tag{2.8}$$

The meaning of this theorem is that we can not only guarantee that $\mathfrak{h}$ is closed, but also that it is exact, since we can write $\mathrm{d}p_i \wedge \mathrm{d}x^i = \mathrm{d}\mathfrak{p}$. Consider the projection $\Pi : \mathrm{T}^* \Omega \to \Omega$ such that the pullback is defined by $\mathrm{d}\Pi = \Pi_* : \mathrm{T}\,\mathrm{T}^* \Omega \to \mathrm{T}\,\Omega$. Then some tangent vector $\boldsymbol{a} \in \mathrm{T}_{(\boldsymbol{x},\mathfrak{p})} \Omega$ is given as $\boldsymbol{a} = a^i \partial_{x^i} + b_j \partial_{p_j}$, and $\Pi_* \boldsymbol{a} = a^i \partial_{x^i} \in \mathrm{T}_x \Omega$.

Definition 2.13 (Symplectic manifold)**.** A symplectic structure on a differentiable manifold of even dimension is a closed, nondegenerate bivector. The pair is a symplectic manifold.

$$\boldsymbol{u} \lrcorner \mathfrak{h} = -\mathrm{d}\mathscr{H}.$$

The symplectic form is preserved along the flow.

$$\frac{\partial x^i}{\partial t} = \frac{\partial \mathscr{H}}{\partial p_i}, \qquad \frac{\partial p_i}{\partial t} = -\frac{\partial \mathscr{H}}{\partial x^i}$$

Definition 2.14 (LIE derivative)**.** Let $\boldsymbol{u}$ be a vector field on the manifold Ω, and ϕ the flow. For an arbitrary $\mathfrak{a} \in \Lambda^k\Omega$, the LIE derivative of $\mathfrak{a}$ by $\boldsymbol{u}$ is

$$\mathscr{L}_{\boldsymbol{u}} \mathfrak{a} = \lim_{t \to 0} \frac{\phi^*(t, \mathfrak{a}) - \mathfrak{a}}{t} = \mathrm{d}(\boldsymbol{u} \lrcorner \mathfrak{a}) + \boldsymbol{u} \lrcorner \mathrm{d}\mathfrak{a}.$$

The vector field $\boldsymbol{u}$ is symplectic if $\mathscr{L}_{\boldsymbol{u}} \mathfrak{a} = 0$ for a symplectic form $\mathfrak{a}$.

Definition 2.15 (LEVI-CIVITA connexion)**.** Let $\boldsymbol{u} \in \mathrm{T}\,\Omega$ and $\mathfrak{a}$ be a section of some vector bundle. A connexion ∇ is a bilinear form. Should it preserve the metric in the sense that $\nabla \mathfrak{g} = 0$, and be torsion free, then it is a LEVI-CIVITA connexion. These are determined by the CHRISTOFFEL symbols

$$\nabla_i \partial_j = \Gamma^k_{ij} \partial_k, \qquad \Gamma^k_{ij} = {}^{\mathrm{g}^{k\ell}}\!/_2 (\partial_j \mathrm{g}_{i\ell} + \partial_i \mathrm{g}_{j\ell} - \partial_\ell \mathrm{g}_{ij}). \tag{2.9}$$

Theorem 2.16 (Fundamental thoerem of pseudo-Riemannian geometry)**.** *Every pseudo-Riemannian manifold has a unique LEVI-CIVITA connexion.*

A curve is called geodesic if $\nabla_{\dot{c}} \dot{c} = 0$.

$$\varrho \mathrm{D}_t \boldsymbol{u} = \nabla \cdot \mathfrak{P} + \mathbf{g}, \qquad \mathrm{D}_t \varrho = -\varrho \nabla \cdot \boldsymbol{u} \tag{2.10a, b}$$

[1] [3] ARNOLD & HESIN, ch.X
[2] [2] ARNOLD (1966)
[3] [12] JOST, §2.4, 4.3
[4] [12] JOST, p.79

2.4 Potential Fluid Flow

The generation of lift is undoubtedly a viscous phenomenon, fully realized by the well known NAVIER–STOKES and continuity equations.

Upon assuming conservative body forces, the NAVIER–STOKES equation then reduces to the EULER and incompressibility equations[1]

$$\varrho \mathrm{D}_t \boldsymbol{u} = -\nabla p, \qquad \nabla \cdot \boldsymbol{u} = 0. \tag{2.11a, b}$$

A vector field satisfying (2.11b) is called solenoidal. It is apparent that the curl of the EULER equation (2.11) is zero, such that the vector field describing the fluid flow is harmonic.

$$\nabla^2 \Phi = 0, \qquad \boldsymbol{x} \in \Omega \tag{2.12}$$

This representation of the fluid is manageable and allows for powerful tools with which to investigate it. It is known that the effects of viscosity are negligible for flow at large scale should the fluid not interact with an object.

We consider two harmonic functions ϕ, ψ over the fluid domain Ω, and find that $\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = 0$, which upon integration yields by the GAUSS–OSTROGRADSKY theorem that[2]

$$\int_{\partial\Omega} (\phi \nabla \psi - \psi \nabla \phi) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = 0. \tag{2.13}$$

This result is often referred to as GREEN's second identity. Constructing now a GREEN function defined to be harmonic everywhere, with the exception at some point $\boldsymbol{\varkappa} \in \mathrm{Q}$, we may recast equation (2.13) in terms of the CAUCHY principal value.[3] For the case of three dimensional flow, the GREEN function is given by

$$\mathrm{G}(\boldsymbol{x}; \boldsymbol{\varkappa}) = r^{-1}, \qquad r = |\boldsymbol{x} - \boldsymbol{\varkappa}|. \tag{2.14a, b}$$

We confirm that $\nabla^2 \mathrm{G}(\boldsymbol{x}; \boldsymbol{\varkappa}) = \delta(\boldsymbol{x} - \boldsymbol{\varkappa})$, where δ is the DIRAC delta function, meaning the GREEN function of equation (2.14a) is harmonic almost everywhere. We may insert this function for ψ in equation (2.13), with the appropriate adjustments to the right-hand side. By integrating around the singularity at $\boldsymbol{\varkappa}$, we find that

$$\mathrm{pv} \int_{\partial\Omega} (\Phi \partial_{\hat{\boldsymbol{n}}} \mathrm{G} - \mathrm{G}\, \partial_{\hat{\boldsymbol{n}}} \Phi) \, \mathrm{d}S = k\pi\Phi(\boldsymbol{\varkappa}), \qquad k = \begin{cases} 2, & \boldsymbol{\varkappa} \in \partial\Omega \\ 4, & \boldsymbol{\varkappa} \in \Omega \\ 0, & \text{otherwise} \end{cases}. \tag{2.15}$$

For the directional derivative, we mean $\partial_{\hat{\boldsymbol{n}}} = \hat{\boldsymbol{n}} \cdot \nabla$.

2.5 Vortical Fluid Flow

In order to study the effects of vortices, we define the vorticity $\mathfrak{w} = \star \mathrm{d}\mathfrak{u}$, with the relation $\mathfrak{w}^\sharp = \boldsymbol{w}$. From the NAVIER–STOKES equation, we find that the vorticity must also satisfy a similar set of equations to that of the velocity, namely

$$\mathrm{D}_t \boldsymbol{w} = 0, \qquad \nabla \cdot \boldsymbol{w} = 0.$$

The question at hand is whether the vorticity is useful in the representation of the fluid flow we are interested in modeling. For any structure interacting with a fluid inducing a perturbed velocity, this velocity ought to not propagate indefinitely. In other words, it should be evanescent. Such vector fields may be described by decomposition into constituent HELMHOLTZ components, wherein

$$\boldsymbol{u} = \nabla \Phi + \nabla \times \boldsymbol{\Psi}, \qquad \boldsymbol{\Psi} = \psi \nabla A. \tag{2.16a, b}$$

We understand from the above discussion that this is just a special case of the HODGE decomposition, where Φ is a sharped zero-form, and $\boldsymbol{\Psi}$ is a sharped one-form. Indeed, as the curl operator is simply $\star\mathrm{d}$, any choice of bivector $\mathfrak{v}$ such that $\star\mathfrak{v} = \Psi_i \, \mathrm{d}x^i$ will be approporiate, revealing the familiar HODGE decomposition of theorem 2.11. Such a choice of representation of the vector potential is called a gauge, and the gauge chosen in equation (2.16b) is that of GUMEROV & DURAISWAMI,[4] where we impose the

[1] discuss MACH number, incompressibility is very relevant here.

[2] ad hoc. motivate.

[3] explain

[4] [8] GUMEROV & DURAISWAMI (2006)

condition that $\operatorname{div}\boldsymbol{\Psi} = 0$. In other words, the bivector $\mathfrak{v}$ is exact. It ought to be noted that the condition of evanescence is not one which applies to the Helmholtz decomposition, but rather that of the convergence of integrals which will be discussed promptly.

Should the fluid happen to interact with a structure, it is only near the surface the effects of viscosity are apparent. A formulation, then, wherein the vorticity is nonzero only on the boundary of the object, we may be able to proceed with a potential flow formulation in the rest of the fluid domain. Nevertheless, the laplacian term in the Navier–Stokes equation may be expanded in terms of equation (2.7) so as to reveal that

$$\mu\nabla^2\boldsymbol{u} = \mu\nabla(\nabla\cdot\boldsymbol{u}) - \mu\nabla\times\boldsymbol{w}. \tag{2.17}$$

This immediately yields the biharmonic equation for the potential function Φ when the vorticity is zero, which has been discussed by Gumerov and Duraiswami in terms of implementation through fast multipole methods.[1] For we know the divergence of the velocity field to be zero, so that the Laplacian is equal to the curl of the vorticity. The connection is immediately apparent between that of viscous effects encoded in the Laplacian term, and the vorticity of the fluid. Thus our motivation is plainly laid out before us—we wish to imitate the viscous effects that the fluid exhibits through a potential formulation. Employing the above Helmholtz decomposition of equation (2.16), we find that by taking the laplacian thereof and substituting in the value of the vorticity of equation (2.17),

$$-\nabla\times\boldsymbol{w} = \nabla\times\nabla^2\boldsymbol{\Psi}.$$

We are free to equate the two arguments of the curl so that the vorticity equals the negative laplacian of the vector field $\boldsymbol{\Psi}$, or in terms of differential forms, $\nabla^2\star\mathfrak{v} = -\mathfrak{w}$. This is a Poisson equation, which is solved by the convolution $\star\mathfrak{v} = -\mathfrak{w} * \mathrm{G}$,[2] where G is the familiar Green function, expressed in local coordinates as

$$\Psi_i(\boldsymbol{x}) = -\int_{\Omega(\boldsymbol{\varkappa})} w_i\,\mathrm{G}\;\mathrm{d}V, \qquad \mathrm{G}(\boldsymbol{x},\boldsymbol{\varkappa}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{\varkappa}|}.$$

Since it is the velocity $\mathfrak{u} = \operatorname{div}\mathfrak{v}$ we wish to find, we may simply calculate it from the above expression by taking the curl. Note that integration is done with respect to the variable $\boldsymbol{\varkappa}$, whilst the exterior derivative will be taken with respect to $\boldsymbol{x}$. To that end, we have $\partial_i\mathfrak{w}(\boldsymbol{\varkappa})\mathrm{d}x^i = 0$, so that by the graded product rule, we only differentiate the Green function. We then find the velocity vector field in terms of the vorticity, such that

$$\mathfrak{u}(\boldsymbol{x}) = -\int_\Omega \star\mathrm{d}(\mathfrak{w}\,\mathrm{G}) = -\int_\Omega \star(\mathfrak{w}\wedge\mathrm{d}\,\mathrm{G}\,).$$

We notice the relation between the Hodge star of the exterior product of two 1-forms and the cross product of two vectors; $\star(\mathfrak{a}\wedge\mathfrak{b}) = (\boldsymbol{a}\times\boldsymbol{b})$, up to musical isomorphism. Expressing this in terms of vectors, we have the generalized Biot–Savart equation,

$$\boldsymbol{u}(\boldsymbol{x}) = -\frac{1}{4\pi}\int_\Omega \frac{\boldsymbol{w}\times(\boldsymbol{x}-\boldsymbol{\varkappa})}{|\boldsymbol{x}-\boldsymbol{\varkappa}|^3}\,\mathrm{d}V. \tag{2.18}$$

[1] *ibid.*

[2] cite

Introduce circulation, such that the vorticity is given along vortex filaments, $\mathfrak{w} = \Gamma \mathfrak{t}$. Gives compact support, etc. so integrals converge nicely. Provide definition of circulation.

$$\Gamma = \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{s} = \int_S \boldsymbol{w} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S.$$

We can derive the circulation theorem.

$$\mathrm{D}_t \Gamma = 0.$$

We will derive how boundary layers may be approximated by vortex sheets.

The BIOT–SAVART formulation of the disturbance velocity field at any point by the filamtent C is then given by[1]

$$\boldsymbol{u}(\boldsymbol{\varkappa}) = \frac{1}{4\pi} \int_C \Gamma(\boldsymbol{x}) \frac{\boldsymbol{x} - \boldsymbol{\varkappa}}{r^3} \times \mathrm{d}\boldsymbol{s}. \tag{2.19}$$

Multiple authors[2] have cited John L. HESS and A.M.O. SMITH's monumental work *Calculation of Potential Flow about Arbitrary Bodies*[3] to the equivalence between a dipole panel and a vortex panel. Such a statement was not found. Rather, it was shown by HESS[4] that the induced velocity by a distribution of dipoles is equivalent to the induced velocity by a certain vortex distribution:

3 Boundary Element Method

The boundary element method, or panel method, takes the integral equation (2.15) and discretizes the boundary of the fluid into quadrilateral panels. We often set the boundary of the fluid to be the surface of some object, and impose the NEUMANN boundary condition of impermeability,

$$\partial_{\hat{\boldsymbol{n}}} \Phi(\boldsymbol{x}) = \hat{\boldsymbol{n}} \cdot \boldsymbol{U}, \qquad \boldsymbol{x} \in \partial\Omega. \tag{3.1}$$

This boundary condition can be modified to model non-rigidity as well. Some approximation of the potential at the panel is then set, and the integral equation can be posed as a set of algebraic equations.

Equation (2.15) is in fact a FREDHOLM integral equation of the second kind for $\phi(\boldsymbol{\varkappa})$. We can now solve for the potential function Φ of equation (2.12) by the means we see fit. The alternative construction is to consider distributions of simple and double sources over the boundary. The potential at the point $\boldsymbol{x}$ due to simple or double sources are given respectively by

$$\Phi(\boldsymbol{\varkappa}) = \int_S \frac{\sigma(\boldsymbol{x})}{r} \, \mathrm{d}S, \qquad \Phi(\boldsymbol{\varkappa}) = \int_S \frac{\mu(\boldsymbol{x}) \cos\alpha}{r^2} \, \mathrm{d}S. \tag{3.2a ,b}$$

Imposing the condition of impermeability of equation (3.1) on the simple pole distribution, we find an integral equation for the source distribution. It can be shown that equation (3.1) can be recast in terms of an integral equation for te simple source distribution,

$$-2\pi\sigma + \int_S \sigma \partial_{\hat{\boldsymbol{n}}} r^{-1} \, \mathrm{d}S = \hat{\boldsymbol{n}} \cdot \boldsymbol{U}, \qquad \boldsymbol{x} \in \partial\Omega.$$

This distribution can now be inserted into equation (3.2a) to find the potential. The potential

3.1 The Vortex Lattice Method

The vortex lattice method exchanges the source panels of the boundary element method for vortex filaments C_i along their boundaries. Some form of interpolation is done for the circulation on each filament, for which we choose a constant value. Thus,

$$\boldsymbol{u}(\boldsymbol{\varkappa}) \approx \frac{1}{4\pi} \sum_{i=1}^{4} \Gamma_i \int_{C_i} \frac{\boldsymbol{x} - \boldsymbol{\varkappa}}{r^3} \times \mathrm{d}\boldsymbol{s}.$$

For each filament, we can calculate the induced velocity. In order to reduce clutter, we drop the index indicating filament.

$$\boldsymbol{u}(\boldsymbol{\varkappa}) = \frac{\Gamma}{4\pi} \frac{\boldsymbol{r}^{\mathrm{I}} \times \boldsymbol{r}^{\mathrm{II}}}{r^{\mathrm{I}} r^{\mathrm{II}} + \boldsymbol{r}^{\mathrm{I}} \cdot \boldsymbol{r}^{\mathrm{II}}} \left(\frac{1}{r^{\mathrm{I}}} + \frac{1}{r^{\mathrm{II}}} \right), \qquad \boldsymbol{r}_i = \boldsymbol{\varkappa} - \boldsymbol{x}_i, \qquad \boldsymbol{x}_i \in \partial C_i. \tag{3.3}$$

1 [5] DENG *et al.* (2020), eq.6
2 cite
3 [11] HESS & SMITH (1967)
4 [10] HESS (1972)

3.2 Multipole Expansions

We consider the points $\boldsymbol{x}$ and $\boldsymbol{\varkappa}$ in $\mathbb{R}^3$, whose angle between them, of the triangle in the plane described by them and the origin, is denoted α. Although these points lie in three dimensional space, they form a unique plane, upon which the angle α may be defined. The points themselves may be given in terms of spherical coordinates, so that the distance between the points $\boldsymbol{x}$ and $\boldsymbol{\varkappa}$ may be written in terms of the law of cosines,

$$r = |\boldsymbol{x} - \boldsymbol{\varkappa}| = \sqrt{|\boldsymbol{x}|^2 - 2|\boldsymbol{x}||\boldsymbol{\varkappa}|\cos\alpha + |\boldsymbol{\varkappa}|^2}.$$

Given the above expression of the value r, we seek to approximate the GREEN function so that integrating it will be more efficient. It is clear from the above that we may write

$$\mathrm{G}(\boldsymbol{x}; \boldsymbol{\varkappa}) = \frac{1}{|\boldsymbol{\varkappa}|\sqrt{1 + \varepsilon^2 - 2\varepsilon\cos\alpha}}, \quad \varepsilon = \frac{|\boldsymbol{x}|}{|\boldsymbol{\varkappa}|}.$$

Through a binomial expansion in $\varepsilon^2 - 2\varepsilon\cos\alpha$, and rearranging so that we have a power series in ε, we have that

$$\mathrm{G}(\boldsymbol{x}; \boldsymbol{\varkappa}) = \frac{1}{|\boldsymbol{\varkappa}|}\sum_{k=0}^{\infty}\mathrm{P}_k(\cos\alpha)\varepsilon^k, \quad \varepsilon < 1,$$

where P_k is the k^{th} LEGENDRE polynomial.[1]

3.3 Octree Data Structure

An *octree* is a data structure which partitions the cube.

3.4 The Fast Multipole Method

The fast multipole method calculates the velocity at some point $\boldsymbol{\varkappa}$ using a sum of values using the methods of equation (?) for near field $\boldsymbol{x}$ and values using the multipole method. A local coordinate system is adopted, requiring translation and rotation of the origin, at which point a multipole expansion can be made. We suppose a dipole panel

$$\Phi(\boldsymbol{\varkappa}) = -\frac{1}{4\pi}\int_S \frac{\boldsymbol{\mu}\cdot\boldsymbol{r}}{r^3}\,\mathrm{d}S,$$

where $\boldsymbol{\mu}$ is the dipole strength vector.

4 Numerical Implementation

5 Conclusion

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