First Mandatory Assignment

MAT-MEK4270

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Problem 1

Consider the POISSON equation

$$\nabla^2 u = f, \qquad \boldsymbol{x} \in \text{Q}, \tag{1.1}$$

where $\Omega = [0, L] \times [0, L]$. Use a uniform mesh with equal length in each direction, so that the mesh may be described by $\boldsymbol{x} \times \boldsymbol{y}$:

$$\boldsymbol{x} = [i\boldsymbol{h}]_{i=0}^{N}, \qquad \boldsymbol{y} = [i\boldsymbol{h}]_{i=0}^{N},$$

where $\boldsymbol{h} = {}^{L}/{}_{N}$. An outline for a script is found in texttt Poisson2D.py.

a.

Modify the `Poisson2D` class such that `test_convergence_poisson2D` and `test_interpolation` pass.

Implementing the finite difference scheme for the second derivative at the inner points,

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{\boldsymbol{u}_{i+1,j} - 2\boldsymbol{u}_{i,j} + \boldsymbol{u}_{i-1,j}}{\Delta \boldsymbol{x}^2}, \qquad , \qquad \boldsymbol{x} \in \Omega,$$

in both $\boldsymbol{x}$ and $\boldsymbol{y}$, and a forward and backward EULER scheme at the boundaries, viz.

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{\boldsymbol{u}_{0,j} - 5\boldsymbol{u}_{1,j} + 4\boldsymbol{u}_{2,j} - \boldsymbol{u}_{3,j}}{\Delta \boldsymbol{x}^2}, \qquad \boldsymbol{x} \in \partial\Omega,$$

on the left-most boundary, and similarly for the right-most boundary, only backwards. This can then be written in terms of a matrix, whose inverse acting on $f(\boldsymbol{x})$ should yield $u(\boldsymbol{x})$.

$$\texttt{D2} = \begin{pmatrix} 1 & -5 & 4 & -1 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & -1 & 4 & -5 & 1 \end{pmatrix}$$

The meshpoints themselves are made up of *linspaces* which are weaved into the ordinals and abscissae of a *meshgrid*.

Problem 2

Consider the wave equation

$$\partial_t^2 \eta = c^2 \nabla^2 \eta, \qquad \boldsymbol{x} \in \text{Q}, \quad t \in \Theta, \tag{1.2}$$

where Q is the unit square and $\Theta = [0, T]$. Consider also the COURANT number with the spatial and temporal discretizations $\boldsymbol{h} = {}^{L}/{}_{N_x}$ and ${}^{t_m}/{}_{m} = \Delta \mathtt{t} = {}^{T}/{}_{N_t}$, respectively,

$$\text{Co} = \frac{c\Delta\mathtt{t}}{\boldsymbol{h}}.$$

The wave equation (1.2) admits two real staitonary waves as solutions. For the DIRICHLET and NEUMANN problems we have the imaginary and real parts respectively of

$$\eta(\boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \cos(\omega t),$$

where we restrict $\boldsymbol{k} = \boldsymbol{m}\pi$, where $\boldsymbol{m} = m_x\hat{\boldsymbol{\imath}} + m_y\hat{\boldsymbol{\jmath}}$ for integer m_x, m_y. The numerical scheme for the wave equation is guven by

$$\frac{\eta_{n,m}^{m+1} - 2\eta_{n,m}^{m} + \eta_{n,m}^{m-1}}{\Delta\mathtt{t}^2} = c^2 \left(\frac{\eta_{n+1,m}^{m} - 2\eta_{n,m}^{m} + \eta_{n-1,m}^{m}}{\boldsymbol{h}^2} + \frac{\eta_{n,m+1}^{m} - 2\eta_{n,m}^{m} + \eta_{n,m-1}^{m}}{\boldsymbol{h}^2} \right) \tag{1.3}$$

a.

Implement the `Wave2D.py` class fully for a homogeneous Dirichlet boundary condition. Compute also the l^2 error as follows.

$$E = \sqrt{h^2 \sum_{n,m} {e_{nm}^{m}}^2},$$

where e_{nm}^{m} is the difference between the numerical and analytical solutions.

b.

Implement the `Wave2D_Neumann` class for Neumann boundary conditions.

c.

Show that

$$\eta(\boldsymbol{x}, t) = e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \tag{1.6}$$

satisfies the wave equation (1.2)

> We set $|\boldsymbol{k}| = k$. We then have that
>
> $$\nabla^2 \eta = -k^2 \eta, \qquad \partial_t^2 \eta = -\omega^2 \eta.$$
>
> Using the standard definition of wave celerity, $c = {}^{\omega}/_{k}$, what we wanted to show is evident.

d.

Assuming isotropic k, we have that the discretization of (1.6) is

$$\eta_{nm}^{m} = e^{i(kh(m+n) - \varpi n \Delta t)}, \tag{1.7}$$

where ϖ is the numerical approximation of ω. Show that this also solves the wave equation (1.2), and that for Courant number $\mathrm{Co} = \sqrt{2}/2$, we have that $\varpi = \omega$.

> Evidently, $\eta_{m+1,n}^{m} = \eta_{m,n+1}^{m}$. Cross-multiplying equation (1.3) by Δt, the right-hand side coefficient reduces to unity, given $\mathrm{Co} = \sqrt{2}/2$. Furthermore, we may neglect the common factors $\exp(ikh(m+n))$ and $\exp(-i\varpi m \Delta t)$. We must then have that
>
> $$e^{i\varpi\Delta t} + e^{-i\varpi\Delta t} = e^{ikh} + e^{-ikh}.$$
>
> Clearly,
>
> $$\varpi = \frac{kh}{\Delta t} = \frac{\omega h}{c\Delta t\sqrt{2}}.$$
>
> We have used the fact that this system dictates a dispersion relation $\omega = ck\sqrt{2}$. What we wanted to show follows.