Vietnam National University - Ho Chi Minh City, University of Science, Faculty of Mathematics and Computer Science

FDM: Practical Assignment 1

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Problem

Let $\Omega = (0,1) \subset \mathbb{R}$ and $f \in \mathrm{L}^2(\Omega)$.

$$-u_{xx} = f(x) \quad \text{in } \Omega \tag{1}$$

1. Dirichlet boundary condition
   a. Solve equation (1) with uniform mesh subject to a Dirichlet boundary condition:

$$u(0) = 1, \quad u(1) = 2.$$

   b. Solve equation (1) subject to a Dirichlet boundary condition:

$$u(0) = 0, u(1) = 0$$

with non uniform mesh $x_i = 1 - \cos\frac{\pi i}{2N}$ for $i = 0, \cdots, N$.

2. Dirichlet - Neumann boundary condition
   Solve equation (1) with uniform mesh subject to a Dirichlet Neumann boundary condition:

$$u'(0) = 0, \quad u(1) = 3.$$

3. Neumann boundary condition
   Solve equation (1) with uniform mesh subject to a Neumann boundary condition:

$$u'(0) = 0, \quad u'(1) = 0.$$

with condition $\displaystyle\int_0^1 f(x)\,\mathrm{d}x = 0$.

Solution

1. Discrete similar problem for the case of the homogeneous dirichlet boundary, we have:

$$\begin{cases} -\dfrac{u_{i+1} - 2u_i + u_{i-1}}{\Delta^2 x} = f_i \quad \forall i \in \overline{1, N-1} \\ u(0) = 1, u(1) = 2 \end{cases}$$

We have linear system for the scheme

$$\begin{cases} i = 1, \frac{2u_1 - u_2}{\Delta^2 x} & = f_1 + \frac{1}{\Delta^2 x} \\ i = 2, \frac{-u_1 + 2u_2 - u_3}{\Delta^2 x} & = f_2 \\ i = 3, \quad \frac{-u_2 + 2u_3 - u_4}{\Delta^2 x} & = f_3 \\ \qquad \cdots & \\ i = N-2, \quad \frac{-u_{N-3} + 2u_{N-2} - u_{N-1}}{\Delta^2 x} & = f_{N-2} \\ i = N-1, \quad \frac{-u_{N-2} + 2u_{N-1}}{\Delta^2 x} & = f_{N-1} + \frac{2}{\Delta^2 x} \end{cases}$$

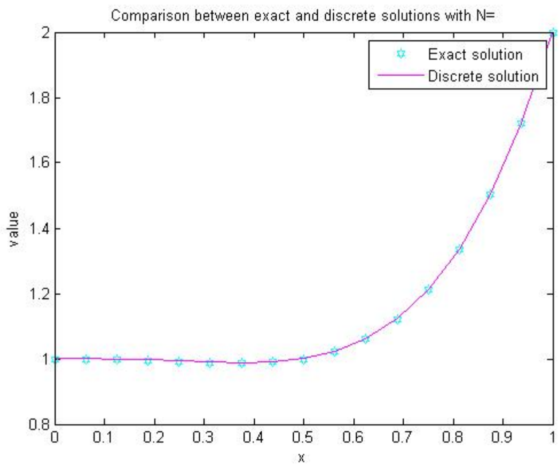
Matrix form $\color{red}{AU = F}$, $A \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$, $U, F \in \mathbb{R}^{N-1}$,

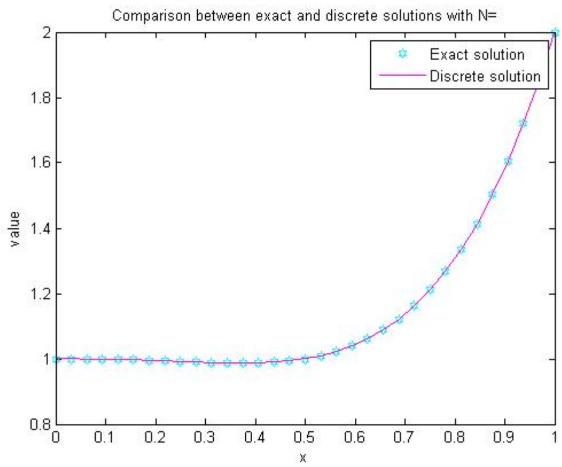
$$A = \frac{1}{\Delta^2 x}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

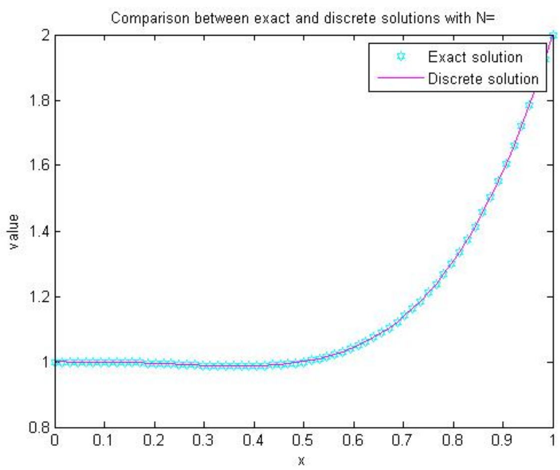
$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix} \qquad F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-2} \\ f_{N-1} \end{bmatrix}$$

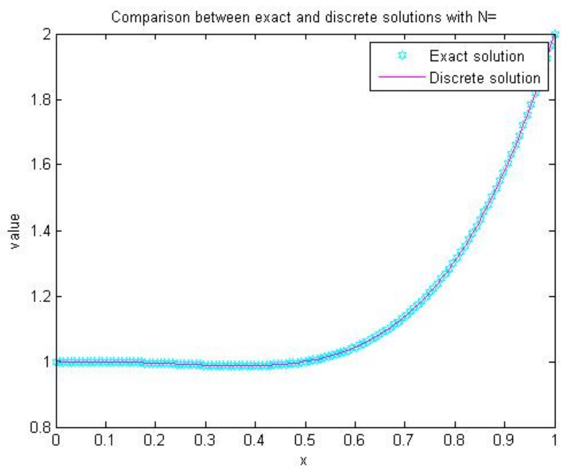
a. We set up with the following exact solution u and function f

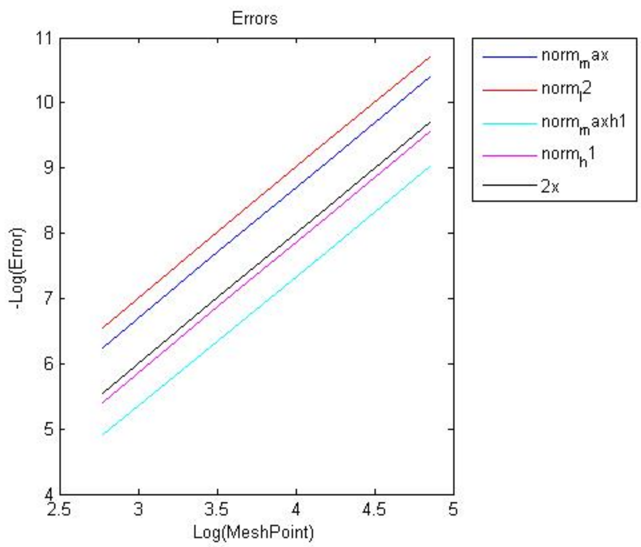
$$\begin{cases} u\left(x\right) = 2x^4 - x^3 + 1 \\ f\left(x\right) = 6x - 24x^2 \\ u\left(0\right) = 1, u\left(1\right) = 2 \end{cases}$$





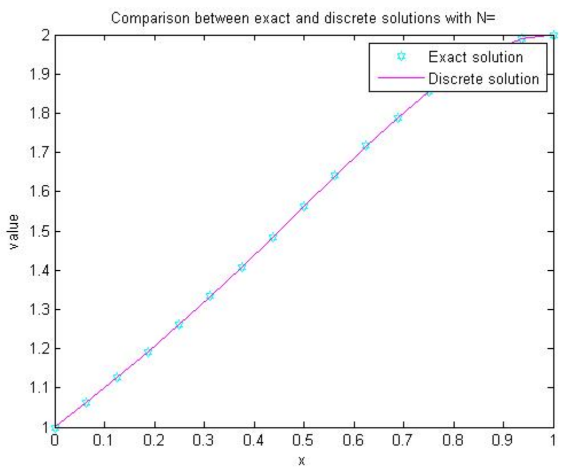


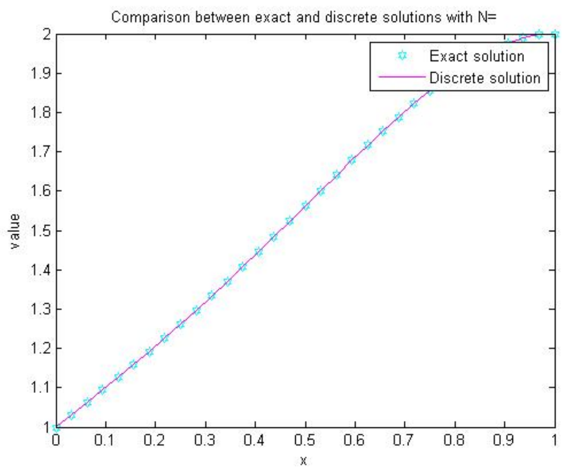


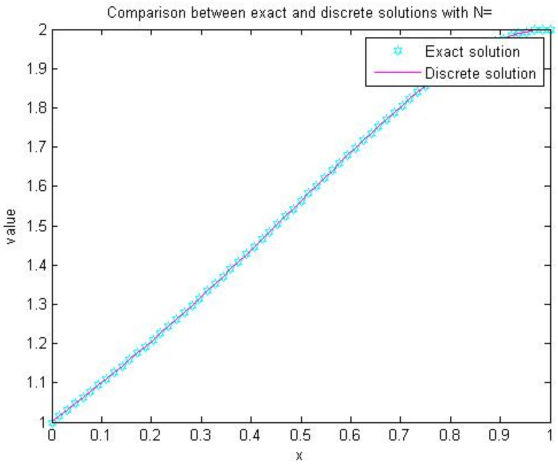


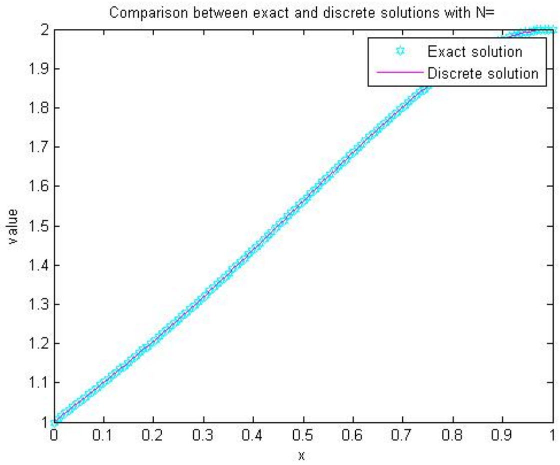
With the following exact solution u and function f

$$\begin{cases} u\left(x\right) = -x^4 + x^3 + x + 1 \\ f\left(x\right) = -6x + 12x^2 \\ u\left(0\right) = 1, u\left(1\right) = 2 \end{cases}$$






Comparison between exact and discrete solutions with N=
Exact solution
Discrete solution
value
x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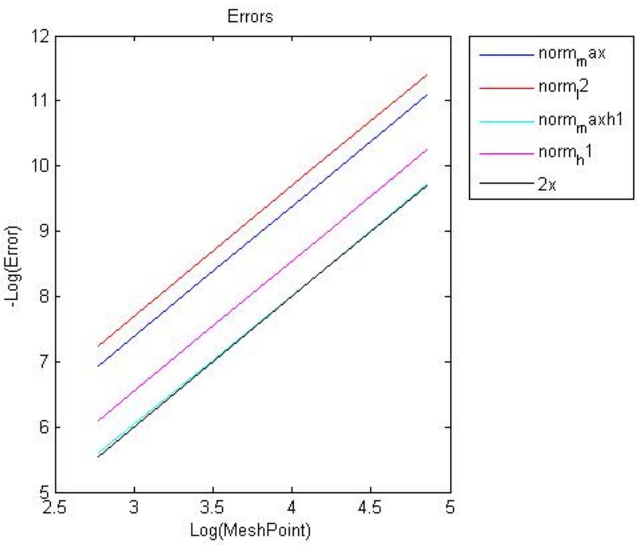

Errors
norm_max
norm_l2
norm_maxh1
norm_h1
2x
-Log(Error)
Log(MeshPoint)

b. We use the Taylor series expansion, there holds

$$u_{i+1} = u_i + \frac{\partial u}{\partial x}(x_i)(x_{i+1} - x_i) + \frac{1}{2!}\frac{\partial^2 u}{\partial x^2}(x_i)(x_{i+1} - u_i)^2 + O\left[(x_{i+1} - x_i)^3\right]$$

and

$$u_{i-1} = u_i + \frac{\partial u}{\partial x}(x_i)(x_{i-1} - x_i) + \frac{1}{2!}\frac{\partial^2 u}{\partial x^2}(x_i)(x_{i-1} - u_i)^2 + O\left[(x_{i-1} - x_i)^3\right].$$

Let $\Delta_i^{(1)}x = x_{i+1} - x_i$ and $\Delta_i^{(2)}x = x_i - x_{i-1}$, we have

$$u_{i+1} = u_i + \frac{\partial u}{\partial x}(x_i)\,\Delta_i^{(1)}x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_i)\left(\Delta_i^{(1)}x\right)^2 + O\left[\left(\Delta_i^{(1)}x\right)^3\right]$$

and

$$u_{i-1} = u_i - \frac{\partial u}{\partial x}(x_i)\,\Delta_i^{(2)}x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_i)\left(\Delta_i^{(2)}x\right)^2 + O\left[\left(\Delta_i^{(2)}x\right)^3\right]. \tag{2}$$

We have:

$$\frac{\Delta_i^{(2)}x}{\Delta_i^{(1)}x}u_{i+1} = \frac{\Delta_i^{(2)}x}{\Delta_i^{(1)}x}u_i + \frac{\partial u}{\partial x}(x_i)\,\Delta_i^{(2)}x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_i)\,\Delta_i^{(1)}x\Delta_i^{(2)}x \tag{3}$$

Let (2) + (3), we have:

$$\frac{\Delta_i^{(2)}x}{\Delta_i^{(1)}x}u_{i+1} + u_{i-1} = \left(\frac{\Delta_i^{(2)}x}{\Delta_i^{(1)}x} + 1\right)u_i + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_i)\left[\Delta_i^{(1)}x\Delta_i^{(2)}x + \left(\Delta_i^{(1)}x\right)^2\right]$$

So,

$$\frac{\partial^2 u}{\partial x^2}(x_i) = 2\frac{\frac{\Delta_i^{(2)}x}{\Delta_i^{(1)}x}u_{i+1} + u_{i-1} - \left(\frac{\Delta_i^{(2)}x}{\Delta_i^{(1)}x} + 1\right)u_i}{\Delta_i^{(1)}x\Delta_i^{(2)}x + \left(\Delta_i^{(2)}x\right)^2}$$

Combined with (1), we obtain:

$$-\frac{2}{\Delta_i^{(1)}x\Delta_i^{(2)}x + \left(\Delta_i^{(2)}x\right)^2}\left[\frac{\Delta_i^{(2)}x}{\Delta_i^{(1)}x}u_{i+1} + u_{i-1} - \left(\frac{\Delta_i^{(2)}x}{\Delta_i^{(1)}x} + 1\right)u_i\right] = f_i \quad \forall i \in \overline{1, N-1}$$

Let

$$\alpha = \frac{2}{\Delta_i^{(1)}x\Delta_i^{(2)}x + \left(\Delta_i^{(2)}x\right)^2},$$
$$\beta = \frac{\Delta_i^{(2)}x}{\Delta_i^{(1)}x},$$
$$\gamma = \frac{\Delta_i^{(2)}x}{\Delta_i^{(1)}x} + 1.$$

Thus,

$$-\alpha\left(\beta u_{i+1} - \gamma u_i + u_{i-1}\right) = f_i \quad \forall i \in \overline{1, N-1}$$

We have linear system for the schemes

$$\begin{cases} i=1, -\alpha\left(u_0-\gamma u_1+\beta u_2\right) & =f_1 \\ i=2, \qquad -\alpha\left(u_1-\gamma u_2+\beta u_3\right) & =f_2 \\ i=3, \qquad\qquad -\alpha\left(u_2-\gamma u_3+\beta u_4\right) & =f_3 \\ \qquad\qquad \cdots & \\ i=N-2, \qquad\qquad -\alpha\left(u_{N-3}-\gamma u_{N-2}+\beta u_{N-1}\right) & =f_{N-2} \\ i=N-1, \qquad\qquad\qquad -\alpha\left(u_{N-2}-\gamma u_{N-1}+\beta u_N\right) & =f_{N-1} \end{cases}$$

Using the Dirichlet boundary condition, we obtain:

$$\begin{cases} i=1, -\alpha\left(-\gamma u_1+\beta u_2\right) & =f_1 \\ i=2, \qquad -\alpha\left(u_1-\gamma u_2+\beta u_3\right) & =f_2 \\ i=3, \qquad\qquad -\alpha\left(u_2-\gamma u_3+\beta u_4\right) & =f_3 \\ \qquad\qquad \cdots & \\ i=N-2, \qquad\qquad -\alpha\left(u_{N-3}-\gamma u_{N-2}+\beta u_{N-1}\right) & =f_{N-2} \\ i=N-1, \qquad\qquad\qquad -\alpha\left(u_{N-2}-\gamma u_{N-1}\right) & =f_{N-1} \end{cases}$$

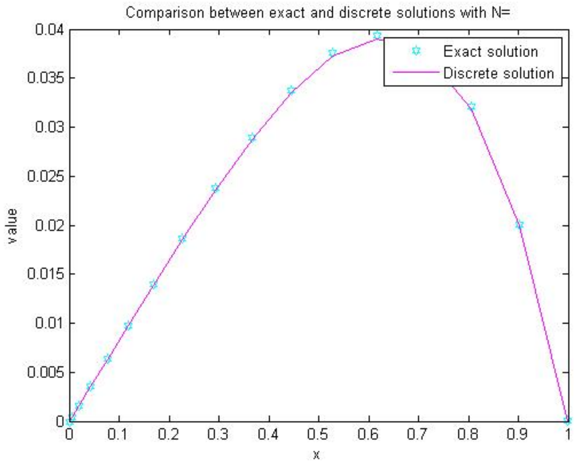
Matrix form $AU=F$, $A\in\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}$, $U,F\in\mathbb{R}^{N-1}$,

$$A=-\alpha\begin{bmatrix} -\gamma & \beta & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -\gamma & \beta & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\gamma & \beta & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\gamma & \beta \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\gamma \end{bmatrix}$$

$$U=\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix} \qquad F=\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-2} \\ f_{N-1} \end{bmatrix}$$

We set up with the following exact solution u and function f

$$\begin{cases} u\left(x\right)=-\frac{1}{12}x^4+\frac{1}{12}x \\ f\left(x\right)=x^2 \\ u\left(0\right)=u\left(1\right)=0 \end{cases}$$



Comparison between exact and discrete solutions with N=
Exact solution
Discrete solution
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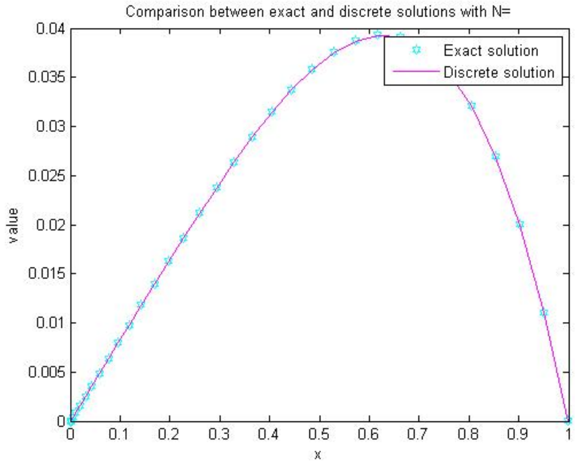

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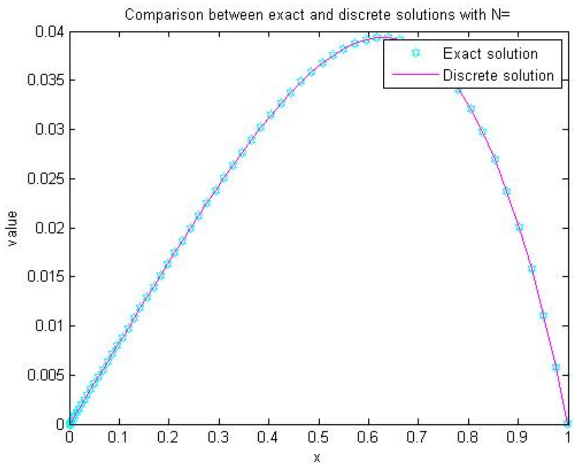

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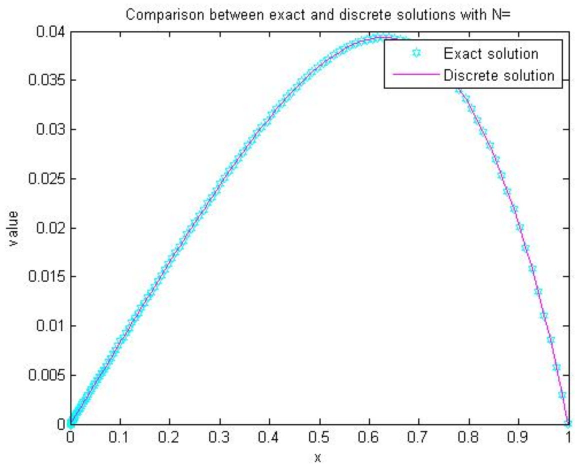

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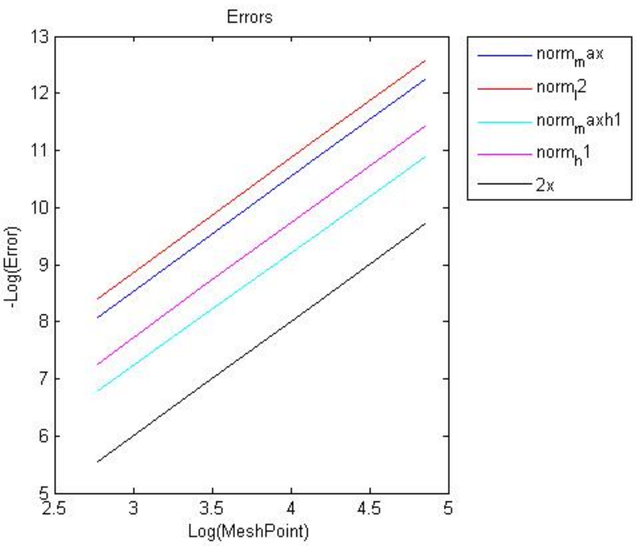

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With the following exact solution u and function f

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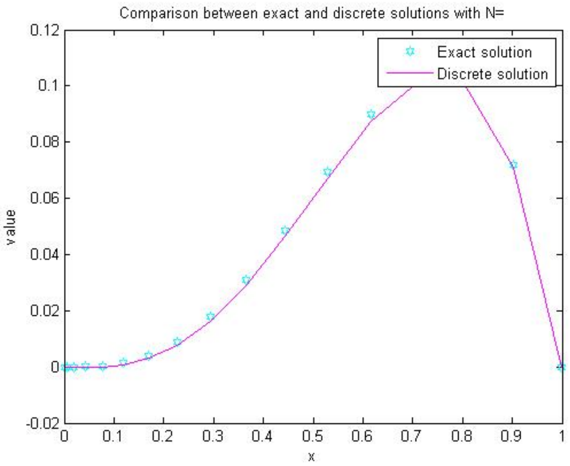
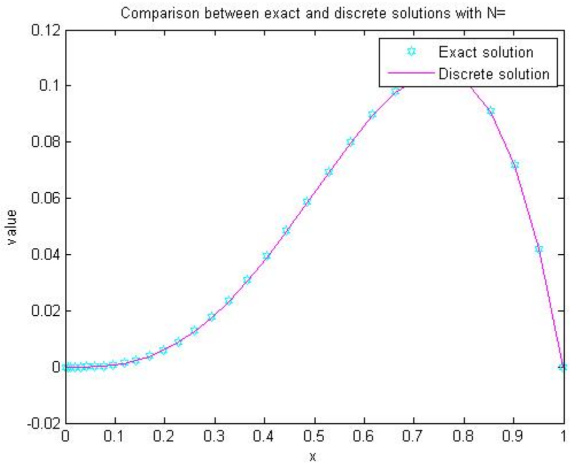
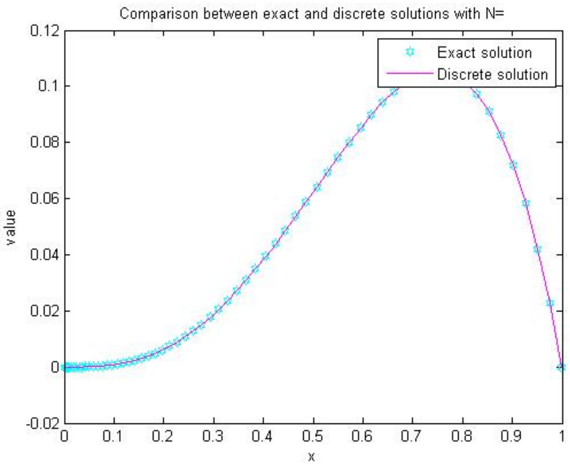
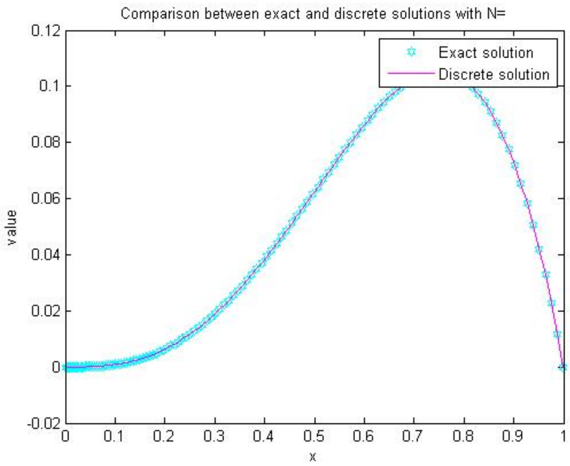
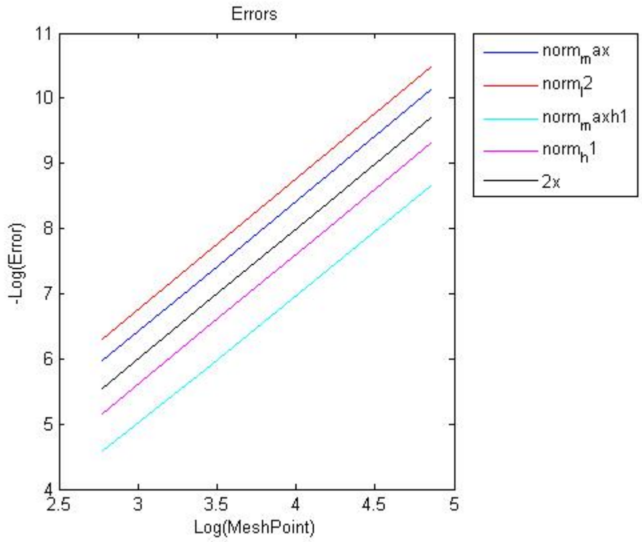










2. Similar to the Problem 1, we have:

$$\begin{cases} -\dfrac{u_{i+1} - 2u_i + u_{i-1}}{\Delta^2 x} = f_i \quad \forall i \in \overline{1, N-1} \\ u'(0) = 0, u(1) = 3 \end{cases}$$

We have linear system for the scheme

$$\begin{cases} i = 1, \frac{-u_0+2u_1-u_2}{\Delta^2 x} & = f_1 \\ i = 2, \quad \frac{-u_1+2u_2-u_3}{\Delta^2 x} & = f_2 \\ i = 3, \quad \frac{-u_2+2u_3-u_4}{\Delta^2 x} & = f_3 \\ \quad \dots & \\ i = N-2, \quad \frac{-u_{N-3}+2u_{N-2}-u_{N-1}}{\Delta^2 x} & = f_{N-2} \\ i = N-1, \quad \frac{-u_{N-2}+2u_{N-1}-u_N}{\Delta^2 x} & = f_{N-1} \end{cases} \tag{4}$$

Using the forward difference at 0, it means that:

$$\frac{\partial u}{\partial x}(0) = \frac{u_1 - u_0}{\Delta x} = 0$$
$$\Rightarrow u_1 = u_0 \tag{5}$$

Moreover, we have:

$$u(1) = 3$$
$$\Rightarrow u_N = 3 \tag{6}$$

From (4), (5) and (6), we have:

$$\begin{cases} i = 1, \frac{u_1-u_2}{\Delta^2 x} & = f_1 \\ i = 2, \frac{-u_1+2u_2-u_3}{\Delta^2 x} & = f_2 \\ i = 3, \quad \frac{-u_2+2u_3-u_4}{\Delta^2 x} & = f_3 \\ \quad \dots & \\ i = N-2, \quad \frac{-u_{N-3}+2u_{N-2}-u_{N-1}}{\Delta^2 x} & = f_{N-2} \\ i = N-1, \quad \frac{-u_{N-2}+2u_{N-1}}{\Delta^2 x} & = f_{N-1} + \frac{3}{\Delta^2 x} \end{cases}$$

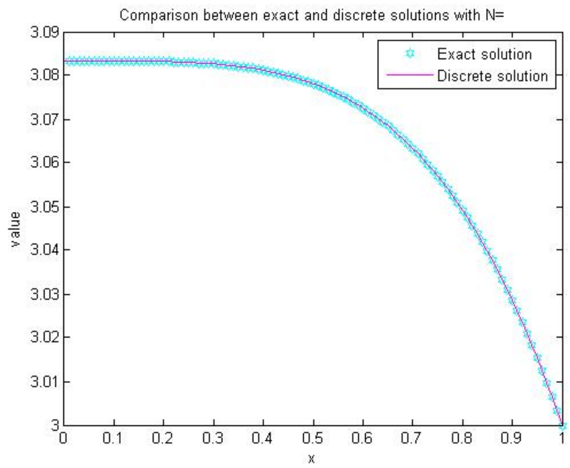
Matrix form $AU = F$, $A \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$, $U, F \in \mathbb{R}^{N-1}$,

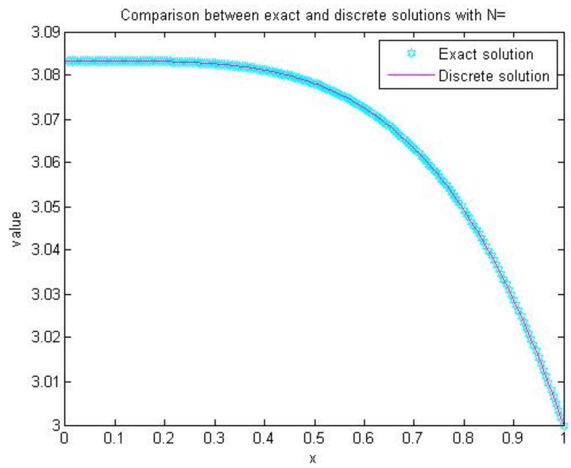
$$A = \frac{1}{\Delta^2 x}\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

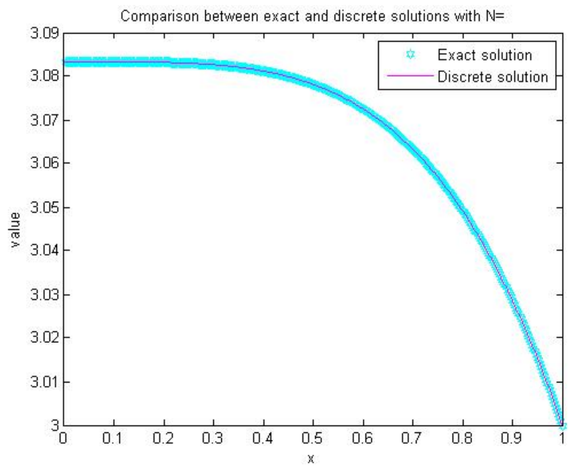
$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix} \qquad F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-2} \\ f_{N-1} \end{bmatrix}$$

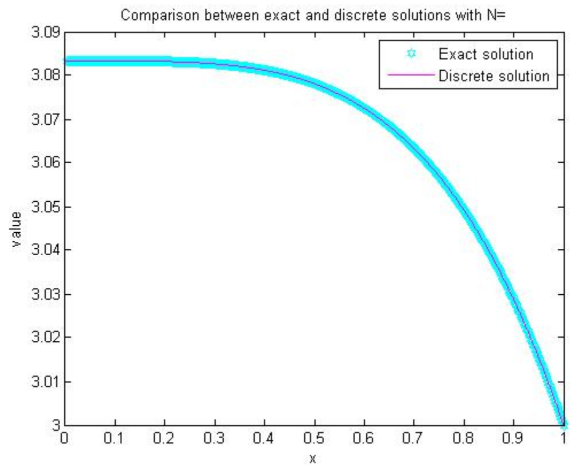
We set up with the following exact solution u and function f

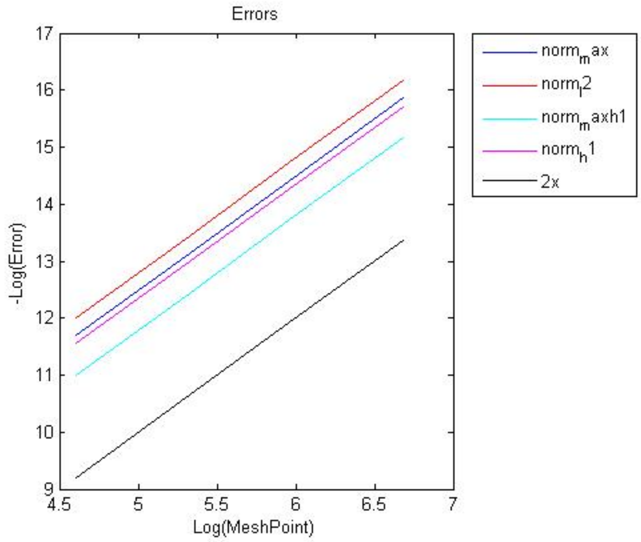
$$\begin{cases} u(x) = -\frac{1}{12}x^4 + \frac{37}{12} \\ f(x) = x^2 \\ u'(0) = 0, u(1) = 3 \end{cases}$$





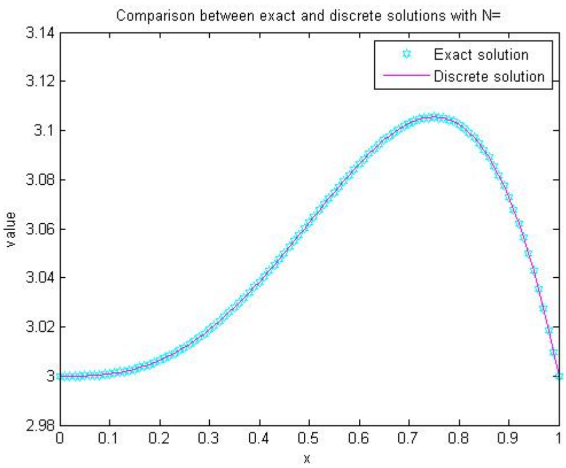


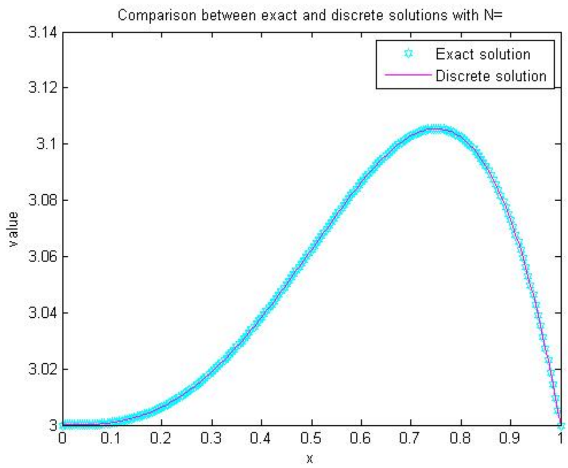


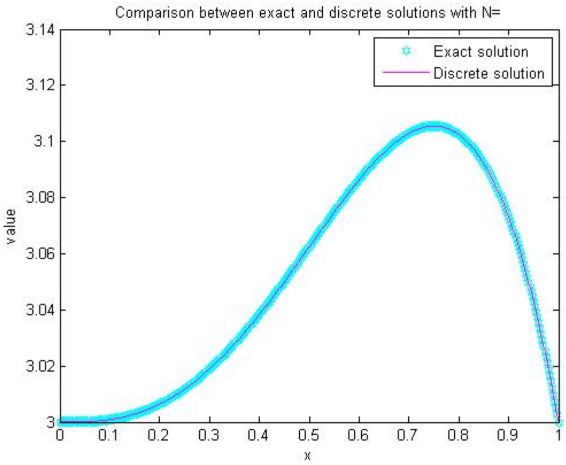


With the following exact solution u and function f

$$\begin{cases} u\left(x\right)=-x^{4}+x^{3}+3 \\ f\left(x\right)=-6x+12x^{2} \\ u'\left(0\right)=0, u\left(1\right)=3 \end{cases}$$






Comparison between exact and discrete solutions with N=
Exact solution
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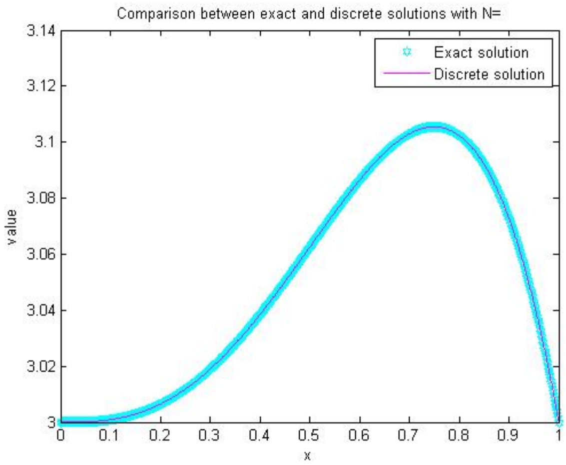
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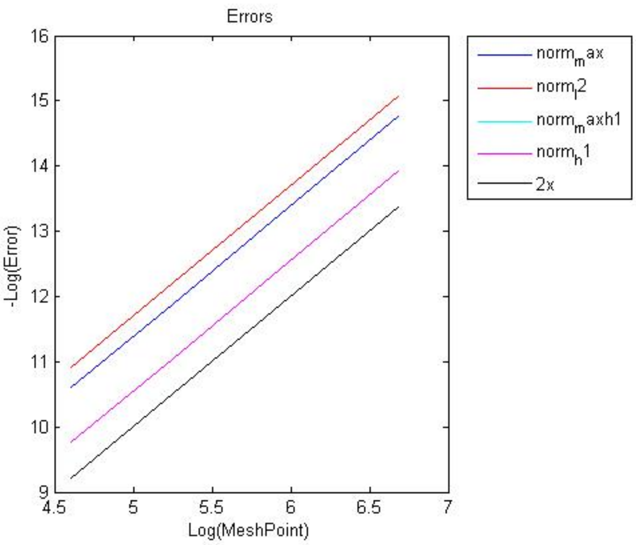
Errors
norm_max
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3. We see that if we solve this problem analytically by integrating twice and trying to determine the constants of integration from the boundary conditions, there are infinitely many solutions.