Vietnam National University - Ho Chi Minh City, University of Science, Faculty of Mathematics and Computer Science

FDM: Practical Assignment 2

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November 18, 2017

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Problem

Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ and $f \in \mathrm{L}^2(\Omega)$.

$$-\Delta u(x,y) = f(x,y) \quad \text{in } \Omega. \tag{1}$$

1. Dirichlet boundary condition

   Solve equation (1) with uniform mesh subject to a Dirichlet boundary condition:

$$u(x,y) = 0 \qquad \forall (x,y) \in \partial\Omega.$$

   b. Solve equation (1) with uniform mesh subject to a Dirichlet boundary condition:

$$u(x,y) = g(x,y) \qquad \forall (x,y) \in \partial\Omega.$$

   c. Solve equation (1) subject to a Dirichlet boundary condition:

$$u(x,y) = g(x,y) \quad \forall (x,y) \in \partial\Omega$$

with following mesh: $\{x_i\}_{i\in[0,N_x]}$ with $x_i = ih$, $h = \frac{1}{N_x}$ and $\{y_j\}_{j\in[0,N_y]}$ with $y_i = jk$, $k = \frac{1}{N_y}$.

Noting that there exit positive constants α, β such that

$$\alpha \leq \frac{h}{k} \leq \beta$$

2. Dirichlet-Neumann boundary condition

   Solve equtation (1) with uniform mesh subject to a Dirichlet Neumann boundary condition:

$$u(x,0) = g_1(x), \ u(0,y) = g_2(y), \ u(x,1) = g_3(x), \ \frac{\partial u}{\partial x}(1,y) = g_4(y).$$

3. Neumann boundary condition

   Solve equation (1) with conditon $\int_\Omega f(x,y)\,\mathrm{d}x\mathrm{d}y = 0$ and with uniform mesh subject to a Neumann boundary condition:

$$\nabla u \cdot \vec{\mathbf{n}}(x,y) = 0 \quad \forall (x,y) \in \partial\Omega,$$

where $\vec{\mathbf{n}}$ is unit normal vector to boundary $\partial\Omega$.

Solution

1. We have the scheme for finite difference discretization:

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j} \tag{2}$$

and

$$u_{0,j} = u_{N,j} = u_{i,0} = u_{i,N} = 0, \quad \forall i, j = 0, \cdots, N.$$

- If $j = 1$, and $i = 1$ then $u_{i,j-1} = u_{i-1,j} = 0$, the equation in (2) becomes:

$$\frac{4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

- If $j = 1$, and $i = N-1$ then $u_{i,j-1} = u_{i+1,j} = 0$, the equation in (2) becomes

$$\frac{-u_{i-1,j} + 4u_{i,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

  If $j = 1$ and $i \notin \{1, N-1\}$ then $u_{i,j-1} = 0$, the equation in (2) becomes:

$$\frac{-u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

- If $j = N-1$ and $i = 1$ then $u_{i,j+1} = u_{i-1,j} = 0$, the equation in (2) becomes:

$$\frac{-u_{i,j-1} + 4u_{i,j} - u_{i+1,j}}{h^2} = f_{i,j}.$$

- If $j = N-1$ and $i = N-1$ then $u_{i,j+1} = u_{i+1,j} = 0$, the equation in (2) becomes:

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j}}{h^2} = f_{i,j}.$$

- If $j = N-1$ and $i \notin \{1, N-1\}$ then $u_{i,j+1} = 0$, the equation in (2) becomes:

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j}}{h^2} = f_{i,j}.$$

- If $j \notin \{1, N-1\}$ and $i = 1$ then $u_{i-1,j} = 0$, the equation in (2) becomes

$$\frac{-u_{i,j-1} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

- If $j \notin \{1, N-1\}$ and $i = N-1$ then $u_{i+1,j} = 0$, the equation in (2) becomes

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

- If $j \notin \{1, N-1\}$ and $i \notin \{1, N-1\}$ then the equation in (2) becomes

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

We get a linear system

$$Au = F, \qquad A \in \mathbb{R}^{(N-1)^2} \times \mathbb{R}^{(N-1)^2},$$

$$u = \left[u_{1,1}, \cdots, u_{N-1,1}, u_{1,2}, \cdots, u_{N-1,2}, \cdots, u_{1,N-1}, \cdots, u_{N-1,N-1}\right]^T,$$
$$F = \left[f_{1,1}, \cdots, f_{N-1,1}, f_{1,2}, \cdots, f_{N-1,2}, \cdots, f_{1,N-1}, \cdots, f_{N-1,N-1}\right]^T,$$

$$A = \frac{1}{h^2}\begin{bmatrix} B & -I & & & & \\ -I & B & & & & \\ & -I & B & -I & & \\ & & & \cdots & & \\ & & & -I & B & -I \\ & & & & -I & B \end{bmatrix}$$

where

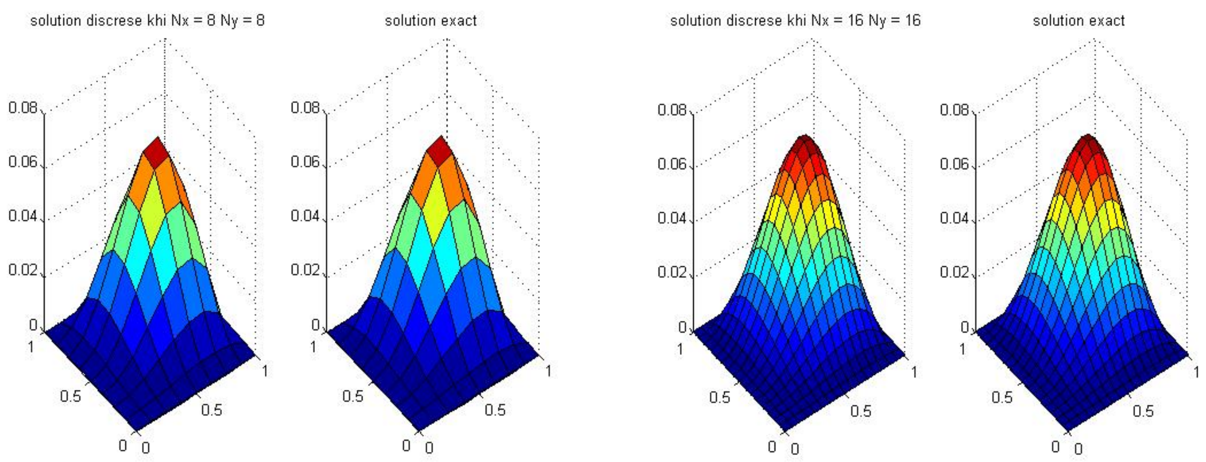
$$B = \begin{bmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & & \cdots & & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{bmatrix}$$

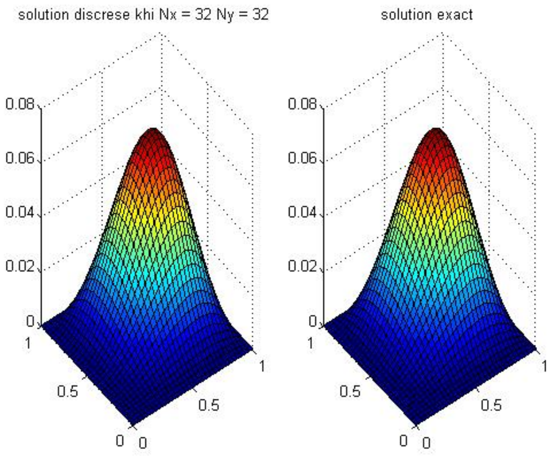
and

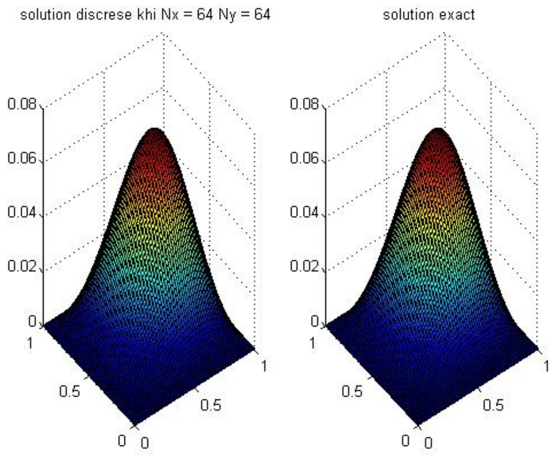
$$I = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \cdots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}.$$

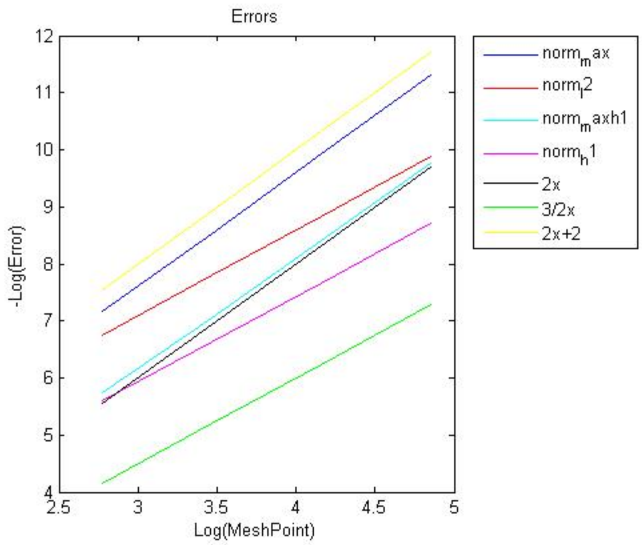
a. We set up with the following exact solution u and function f:

$$\begin{cases} u(x,y) = x^2\left(1-x^2\right)y^2\left(1-y^2\right) \\ f(x,y) = 12x^2y^2\left(2-x^2-y^2\right) - 2x^2\left(1-x^2\right) - 2y^2\left(1-y^2\right) \\ u(x,y) = 0 \quad \forall (x,y) \in \partial\Omega \end{cases}$$



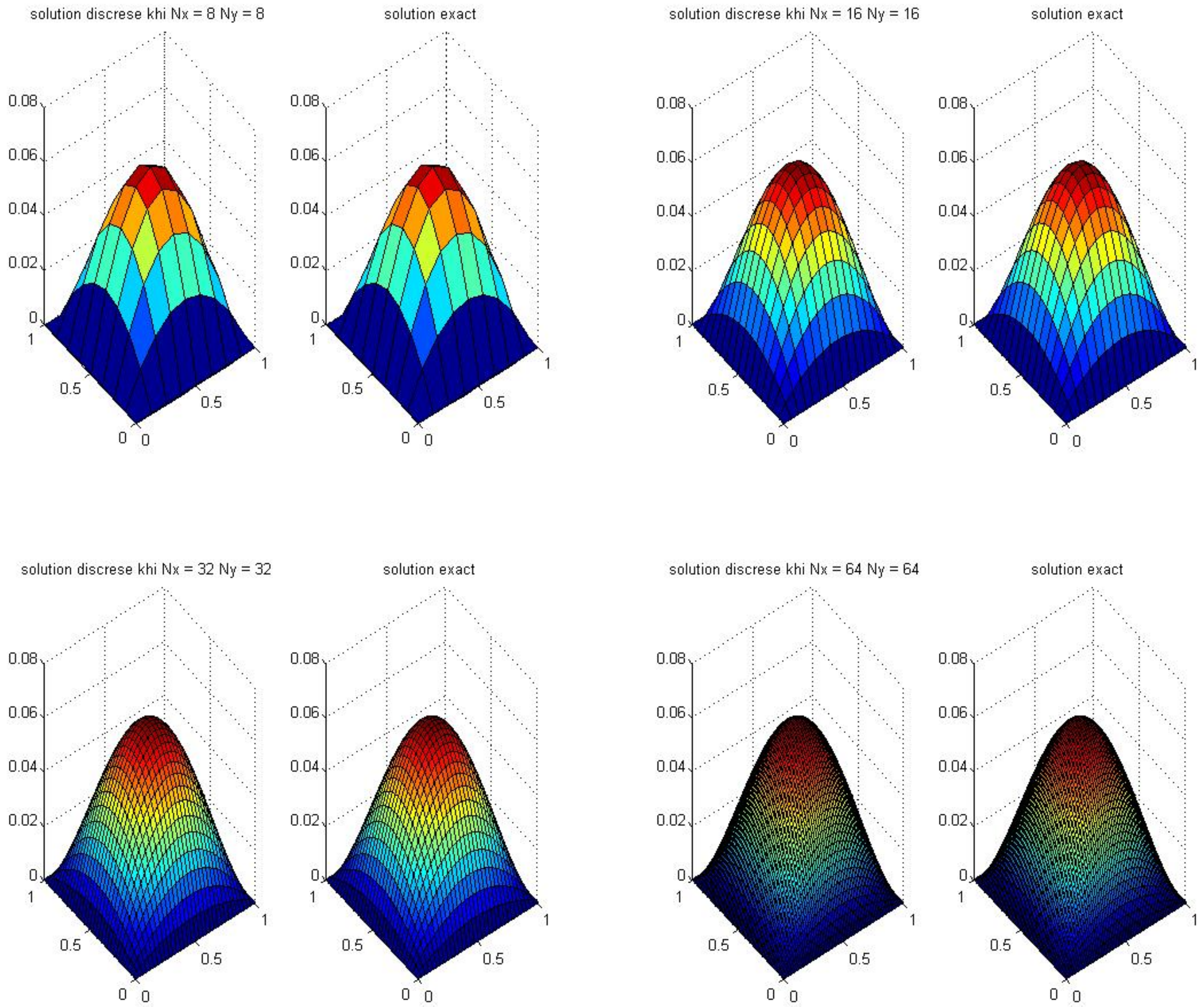

solution discrese khi Nx = 32 Ny = 32
solution exact

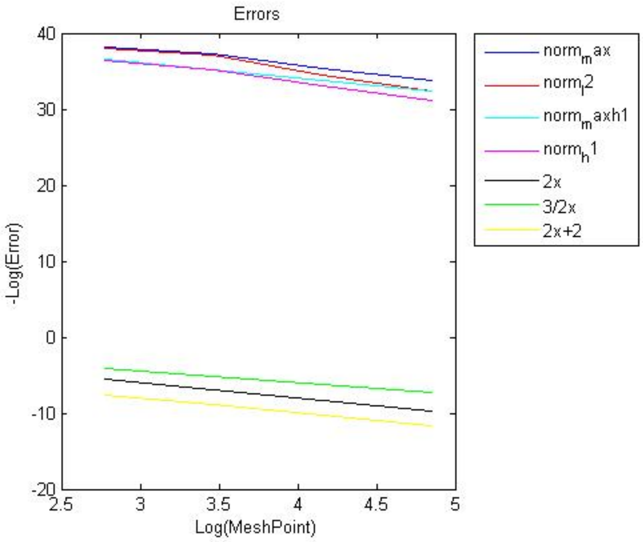

solution discrese khi Nx = 64 Ny = 64
solution exact


Errors
-Log(Error)
Log(MeshPoint)
norm_max
norm_l2
norm_maxh1
norm_h1
2x
3/2x
2x+2

With the following exact solution u and function f

$$\begin{cases} u(x) = x(1-x)\,y(1-y) \\ f(x) = -2x(x-1) - 2y(y-1) \\ u(x,y) = 0 \quad \forall (x,y) \in \partial\Omega \end{cases}$$





b. We have the scheme for finite difference discretization:

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j} \tag{3}$$

and

$$\begin{aligned} u_{0,j} &= g\left(0, y_j\right), \quad && \forall j = 0, \cdots, N, \\ u_{N,j} &= g\left(1, y_j\right), \quad && \forall j = 0, \cdots, N, \\ u_{i,0} &= g\left(x_i, 0\right), \quad && \forall i = 0, \cdots, N, \\ u_{i,N} &= g\left(x_i, 1\right), \quad && \forall i = 0, \cdots, N. \end{aligned}$$

- If $j = 1$, and $i = 1$ then $u_{i,j-1} = g\left(x_i, 0\right)$ and $u_{i-1,j} = g\left(0, y_j\right)$, the equation in (3) becomes:

$$\frac{4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j} + \frac{g\left(x_i, 0\right)}{h^2} + \frac{g\left(0, y_j\right)}{h^2}.$$

- If $j = 1$, and $i = N - 1$ then $u_{i,j-1} = g\left(x_i, 0\right)$ and $u_{i+1,j} = g\left(1, y_j\right)$, the equation in (3) becomes

$$\frac{-u_{i-1,j} + 4u_{i,j} - u_{i,j+1}}{h^2} = f_{i,j} + \frac{g\left(x_i, 0\right)}{h^2} + \frac{g\left(1, y_j\right)}{h^2}.$$

  If $j = 1$ and $i \notin \{1, N-1\}$ then $u_{i,j-1} = g\left(x_i, 0\right)$, the equation in (3) becomes:

$$\frac{-u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j} + \frac{g\left(x_i, 0\right)}{h^2}.$$

- If $j = N - 1$ and $i = 1$ then $u_{i,j+1} = g\left(x_i, 1\right)$ and $u_{i-1,j} = g\left(0, y_j\right)$, the equation in (3) becomes:

$$\frac{-u_{i,j-1} + 4u_{i,j} - u_{i+1,j}}{h^2} = f_{i,j} + \frac{g\left(x_i, 1\right)}{h^2} + \frac{g\left(0, y_j\right)}{h^2}.$$

- If $j = N - 1$ and $i = N - 1$ then $u_{i,j+1} = g\left(x_i, 1\right)$ and $u_{i+1,j} = g\left(1, y_j\right)$, the equation in (3) becomes:

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j}}{h^2} = f_{i,j} + \frac{g\left(x_i, 1\right)}{h^2} + \frac{g\left(1, y_j\right)}{h^2}.$$

- If $j = N-1$ and $i \notin \{1, N-1\}$ then $u_{i,j+1} = g(x_i, 1)$, the equation in (3) becomes:

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j}}{h^2} = f_{i,j} + \frac{g(x_i, 1)}{h^2}.$$

- If $j \notin \{1, N-1\}$ and $i = 1$ then $u_{i-1,j} = g(0, y_j)$, the equation in (3) becomes:

$$\frac{-u_{i,j-1} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j} + \frac{g(0, y_j)}{h^2}.$$

- If $j \notin \{1, N-1\}$ and $i = N-1$ then $u_{i+1,j} = g(1, y_j)$, the equation in (3) becomes:

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i,j+1}}{h^2} = f_{i,j} + \frac{g(1, y_j)}{h^2}.$$

- If $j \notin \{1, N-1\}$ and $i \notin \{1, N-1\}$ then the equation in (3) holds:

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

We get a linear system

$$Au = F, \qquad A \in \mathbb{R}^{(N-1)^2} \times \mathbb{R}^{(N-1)^2},$$

$$u = \begin{bmatrix} u_{1,1} \\ \vdots \\ u_{N-1,1} \\ u_{1,2} \\ \vdots \\ u_{N-1,2} \\ \vdots \\ u_{1,N-1} \\ \vdots \\ u_{N-1,N-1} \end{bmatrix} \qquad F = \begin{bmatrix} f_{1,1} + \frac{g(x_1,0)}{h^2} + \frac{g(0,y_1)}{h^2} \\ \vdots \\ f_{N-1,1} + \frac{g(x_{N-1},0)}{h^2} + \frac{g(1,y_1)}{h^2} \\ f_{1,2} + \frac{g(0,y_2)}{h^2} \\ \vdots \\ f_{N-1,2} + \frac{g(1,y_2)}{h^2} \\ \vdots \\ f_{1,N-1} + \frac{g(x_1,1)}{h^2} + \frac{g(0,y_{N-1})}{h^2} \\ \cdots \\ f_{N-1,N-1} + \frac{g(x_{N-1},1)}{h^2} + \frac{g(1,y_{N-1})}{h^2} \end{bmatrix}$$

$$A = \frac{1}{h^2} \begin{bmatrix} B & -I & & & & \\ -I & B & & & & \\ & -I & B & -I & & \\ & & & \cdots & & \\ & & & -I & B & -I \\ & & & & -I & B \end{bmatrix}$$

where

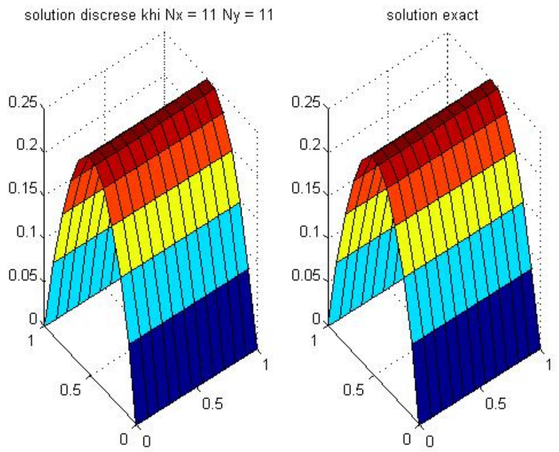
$$B = \begin{bmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & & \cdots & & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{bmatrix}$$

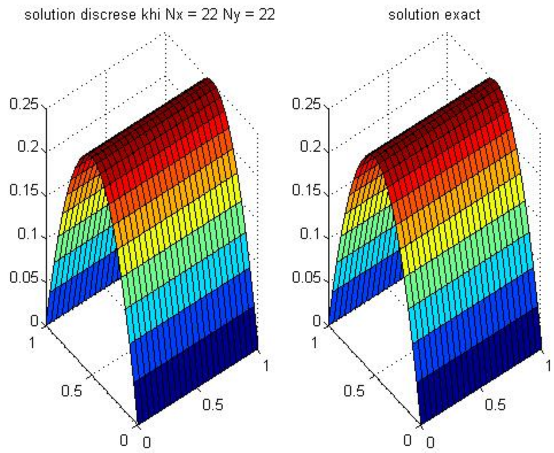
and

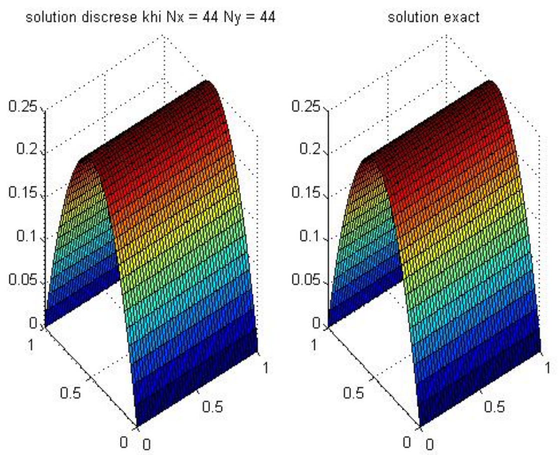
$$I = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \dots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}.$$

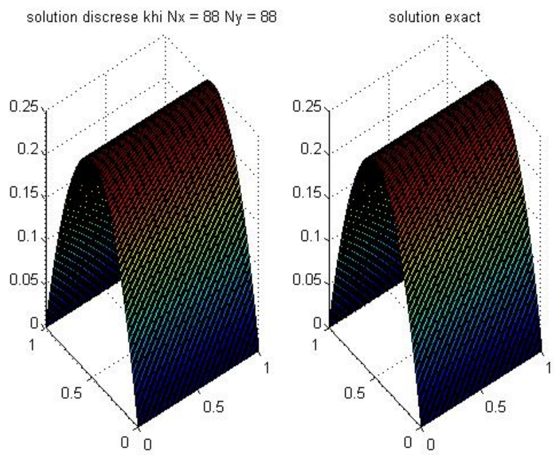
We set up with the following exact solution u and function f:

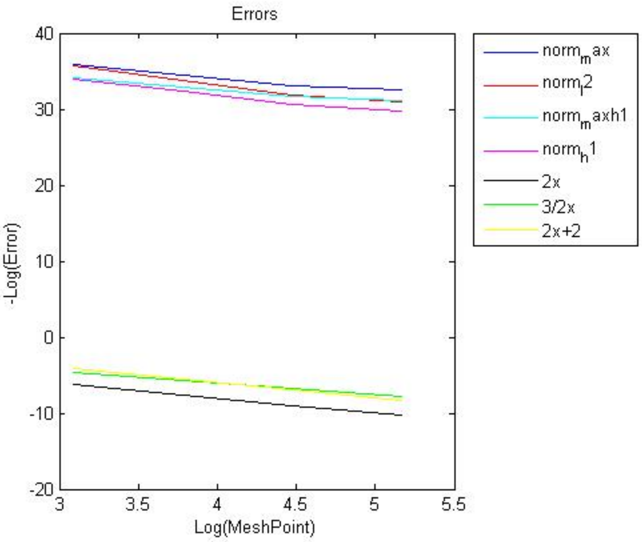
$$\begin{cases} u\,(x,y) = x\,(1-x) \\ f\,(x,y) = 2 \\ u\,(x,y) = x\,(1-x) \qquad \forall\,(x,y) \in \partial\Omega \end{cases}$$





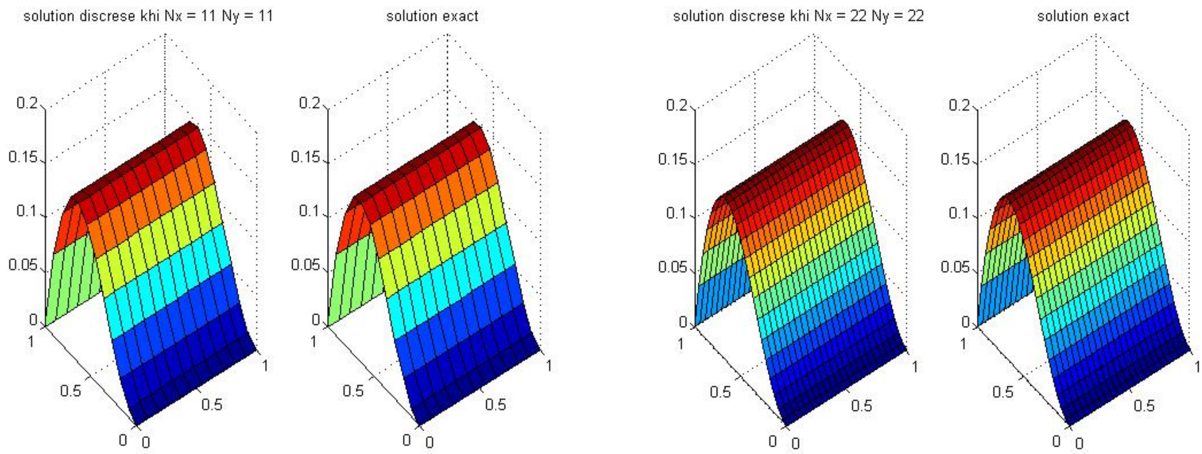


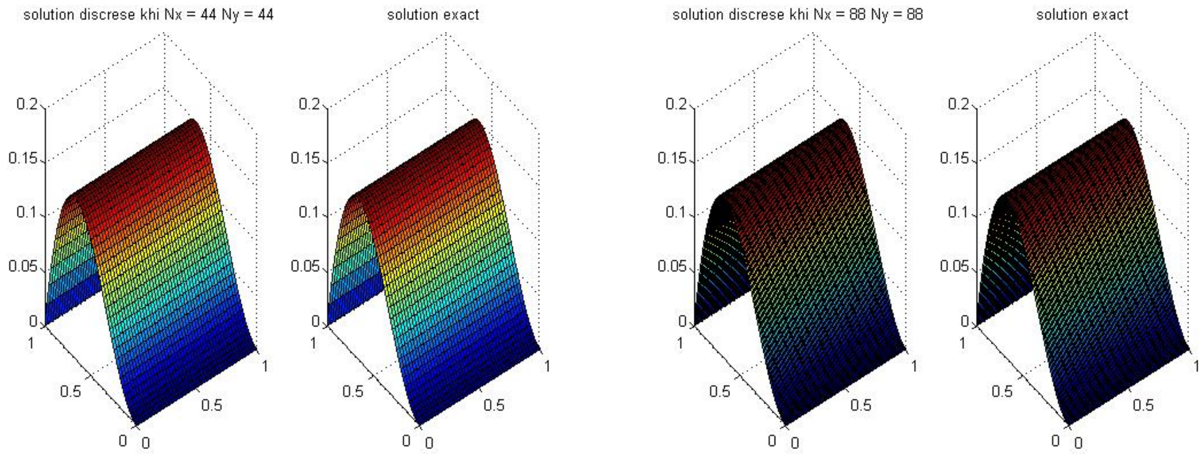


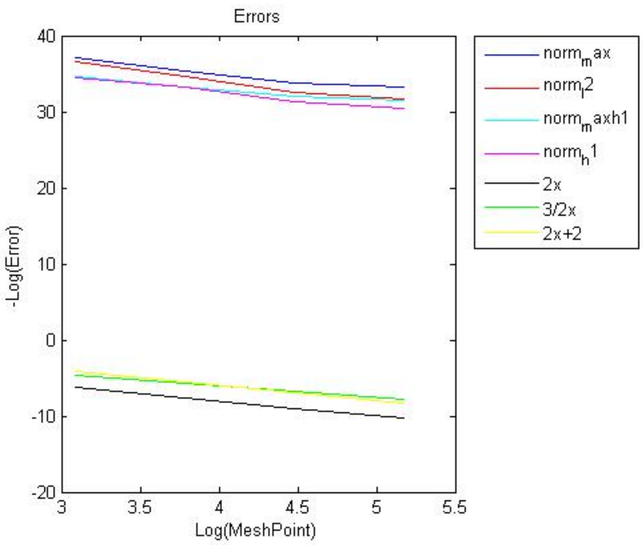


With the following exact solution u and function f

$$\begin{cases} u\left(x,y\right)=x^{2}\left(1-x\right) \\ f\left(x,y\right)=6x-2 \\ u\left(x,y\right)=x^{2}\left(1-x\right) \qquad \forall\left(x,y\right)\in\partial\Omega \end{cases}$$




solution discrese khi Nx = 44 Ny = 44
solution exact
solution discrese khi Nx = 88 Ny = 88
solution exact


Errors
-Log(Error)
Log(MeshPoint)
norm_max
norm_l2
norm_maxh1
norm_h1
2x
3/2x
2x+2

c. With $x_i = ih$, $h = \frac{1}{N_x}$, using the approximation of the second order derivative respect x, we have:

$$-\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \frac{-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}}{h^2}.$$

It's similar, with $y_i = jk$, $k = \frac{1}{N_y}$, we have

$$-\frac{\partial^2 u}{\partial y^2}(x_i, y_j) = \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{k^2}.$$

Then, we get the scheme for finite difference discretization:

$$\frac{-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}}{h^2} + \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{k^2} = f_{i,j}. \tag{4}$$

and

$$\begin{aligned} u_{0,j} &= g(0, y_j), & \forall j = 0, \cdots, N_y, \\ u_{N_x,j} &= g(1, y_j), & \forall j = 0, \cdots, N_y, \\ u_{i,0} &= g(x_i, 0), & \forall i = 0, \cdots, N_x, \\ u_{i,N_y} &= g(x_i, 1), & \forall i = 0, \cdots, N_x. \end{aligned}$$

- If $j = 1$, and $i = 1$ then $u_{i,j-1} = g(x_i, 0)$ and $u_{i-1,j} = g(0, y_j)$, the equation in (4) becomes:

$$\frac{2u_{i,j} - u_{i+1,j}}{h^2} + \frac{2u_{i,j} - u_{i,j+1}}{k^2} = f_{i,j} + \frac{g(x_i, 0)}{k^2} + \frac{g(0, y_j)}{h^2}.$$

- If $j = 1$, and $i = N_x - 1$ then $u_{i,j-1} = g(x_i, 0)$ and $u_{i+1,j} = g(1, y_j)$, the equation in (4) becomes

$$\frac{-u_{i-1,j} + 2u_{i,j}}{h^2} + \frac{2u_{i,j} - u_{i,j+1}}{k^2} = f_{i,j} + \frac{g(x_i, 0)}{k^2} + \frac{g(1, y_j)}{h^2}.$$

  If $j = 1$ and $i \notin \{1, N_x - 1\}$ then $u_{i,j-1} = g(x_i, 0)$, the equation in (4) becomes:

$$\frac{-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}}{h^2} + \frac{2u_{i,j} - u_{i,j+1}}{k^2} = f_{i,j} + \frac{g(x_i, 0)}{k^2}.$$

- If $j = N_y - 1$ and $i = 1$ then $u_{i,j+1} = g(x_i, 1)$ and $u_{i-1,j} = g(0, y_j)$, the equation in (4) becomes:

$$\frac{2u_{i,j} - u_{i+1,j}}{h^2} + \frac{-u_{i,j-1} + 2u_{i,j}}{k^2} = f_{i,j} + \frac{g(x_i, 1)}{k^2} + \frac{g(0, y_j)}{h^2}.$$

- If $j = N_y - 1$ and $i = N_x - 1$ then $u_{i,j+1} = g(x_i, 1)$ and $u_{i+1,j} = g(1, y_j)$, the equation in (4) becomes:

$$\frac{-u_{i-1,j} + 2u_{i,j}}{h^2} + \frac{-u_{i,j-1} + 2u_{i,j}}{k^2} = f_{i,j} + \frac{g(x_i, 1)}{k^2} + \frac{g(1, y_j)}{h^2}.$$

- If $j = N_y - 1$ and $i \notin \{1, N_x - 1\}$ then $u_{i,j+1} = g(x_i, 1)$, the equation in (4) becomes:

$$\frac{-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}}{h^2} + \frac{-u_{i,j-1} + 2u_{i,j}}{k^2} = f_{i,j} + \frac{g(x_i, 1)}{k^2}.$$

- If $j \notin \{1, N_y - 1\}$ and $i = 1$ then $u_{i-1,j} = g(0, y_j)$, the equation in (4) becomes:

$$\frac{2u_{i,j} - u_{i+1,j}}{h^2} + \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{k^2} = f_{i,j} + \frac{g(0, y_j)}{h^2}.$$

- If $j \notin \{1, N_y - 1\}$ and $i = N_x - 1$ then $u_{i+1,j} = g(1, y_j)$, the equation in (4) becomes:

$$\frac{-u_{i-1,j} + 2u_{i,j}}{h^2} + \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{k^2} = f_{i,j} + \frac{g(1, y_j)}{h^2}.$$

- If $j \notin \{1, N_y - 1\}$ and $i \notin \{1, N_x - 1\}$ then the equation (4) holds:

$$\frac{-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}}{h^2} + \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{k^2} = f_{i,j}.$$

We get a linear system

$$Au = F, \quad A \in \mathbb{R}^{(N_x-1)(N_y-1)} \times \mathbb{R}^{(N_x-1)(N_y-1)},$$

$$u = \begin{bmatrix} u_{1,1} \\ \vdots \\ u_{N_x-1,1} \\ u_{1,2} \\ \vdots \\ u_{N_x-1,2} \\ \vdots \\ u_{1,N_y-1} \\ \vdots \\ u_{N_x-1,N_y-1} \end{bmatrix} \qquad F = \begin{bmatrix} f_{1,1} + \frac{g(x_1,0)}{k^2} + \frac{g(0,y_1)}{h^2} \\ \vdots \\ f_{N_x-1,1} + \frac{g(x_{N_x-1},0)}{k^2} + \frac{g(1,y_1)}{h^2} \\ f_{1,2} + \frac{g(0,y_2)}{h^2} \\ \vdots \\ f_{N_x-1,2} + \frac{g(1,y_2)}{h^2} \\ \vdots \\ f_{1,N_y-1} + \frac{g(x_1,1)}{k^2} + \frac{g\left(0,y_{N_y-1}\right)}{h^2} \\ \cdots \\ f_{N_x-1,N_y-1} + \frac{g(x_{N_x-1},1)}{k^2} + \frac{g\left(1,y_{N_y-1}\right)}{h^2} \end{bmatrix}$$

$$A = \frac{1}{h^2k^2} \begin{bmatrix} B & -I & & & & \\ -I & B & & & & \\ & -I & B & -I & & \\ & & & \cdots & & \\ & & & -I & B & -I \\ & & & & -I & B \end{bmatrix}$$

where

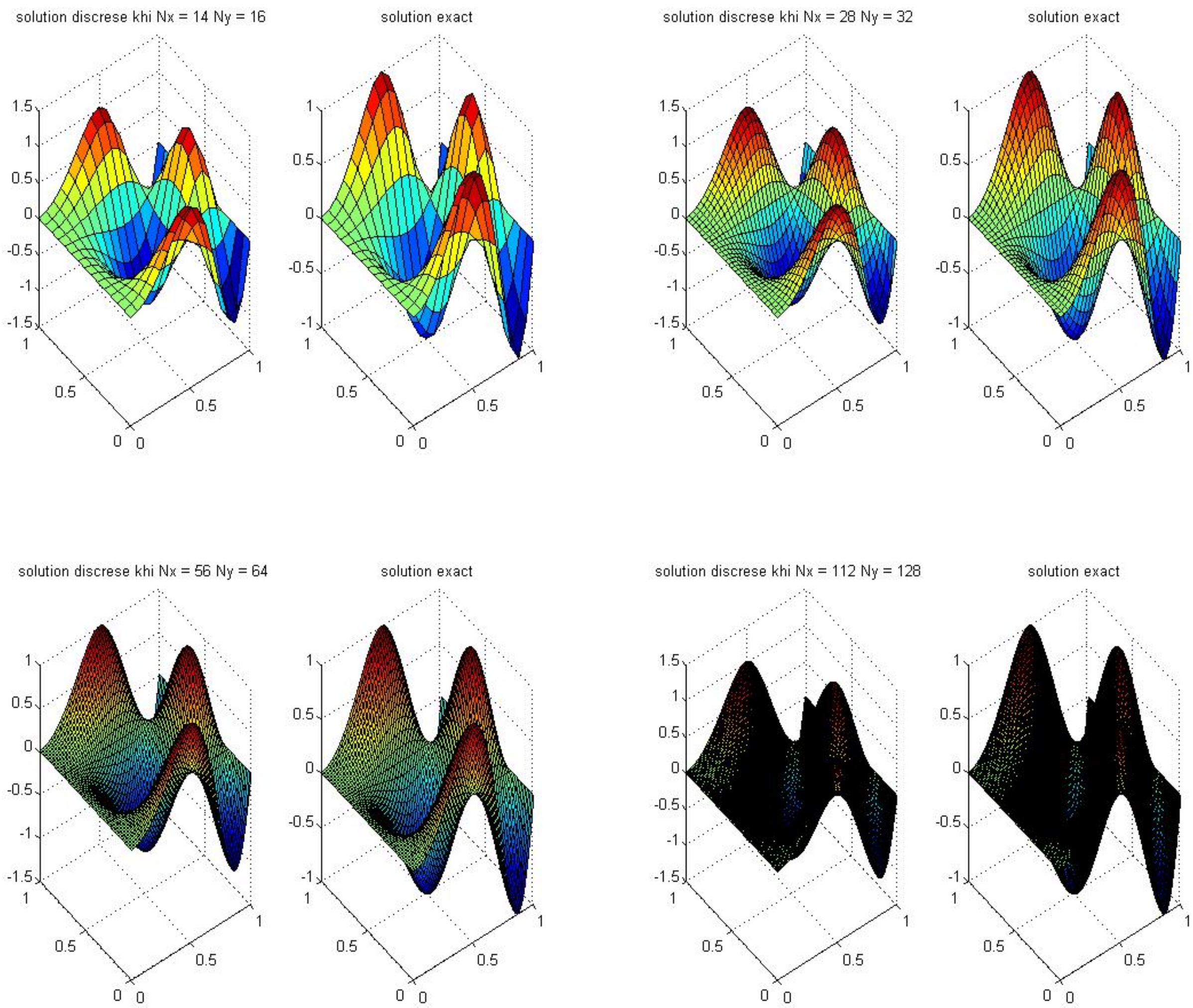
$$B = \begin{bmatrix} 2\left(h^2 + k^2\right) & -k^2 & & & & \\ -k^2 & 2\left(h^2 + k^2\right) & -k^2 & & & \\ & -k^2 & 2\left(h^2 + k^2\right) & -k^2 & & \\ & & & \cdots & & \\ & & & -k^2 & 2\left(h^2 + k^2\right) & -k^2 \\ & & & & -k^2 & 2\left(h^2 + k^2\right) \end{bmatrix}$$

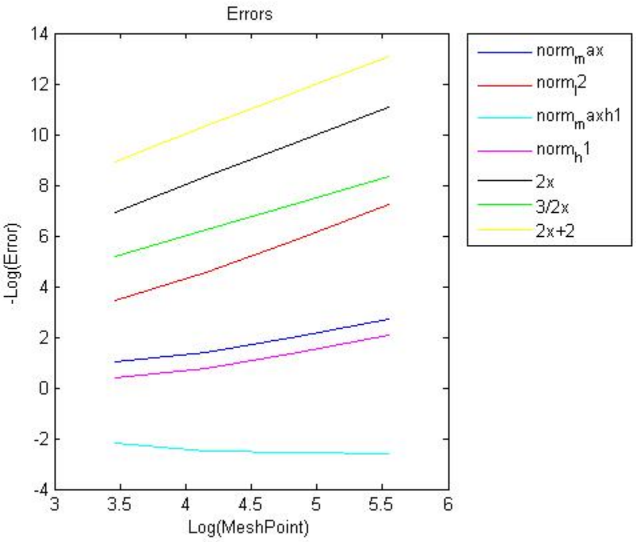
and

$$I = \begin{bmatrix} h^2 & & & & & \\ & h^2 & & & & \\ & & h^2 & & & \\ & & & \cdots & & \\ & & & & h^2 & \\ & & & & & h^2 \end{bmatrix}.$$

We set up with the following exact solution u and function f:

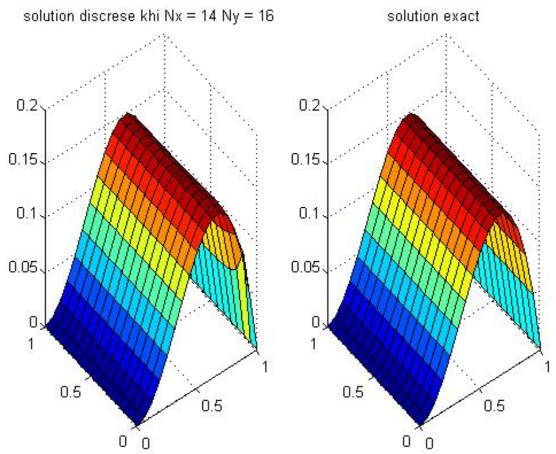
$$\begin{cases} u\left(x,y\right) = \cos\left(2\pi x\right)\sin\left(2\pi y^2\right) \\ f\left(x,y\right) = 4\pi\cos\left(2\pi x\right)\left[\pi\sin\left(2\pi y^2\right)\left(1+4y^2\right) - \cos\left(2\pi y^2\right)\right] \\ u\left(x,y\right) = \cos\left(2\pi x\right)\sin\left(2\pi y^2\right) \quad \forall\left(x,y\right)\in\partial\Omega \end{cases}$$

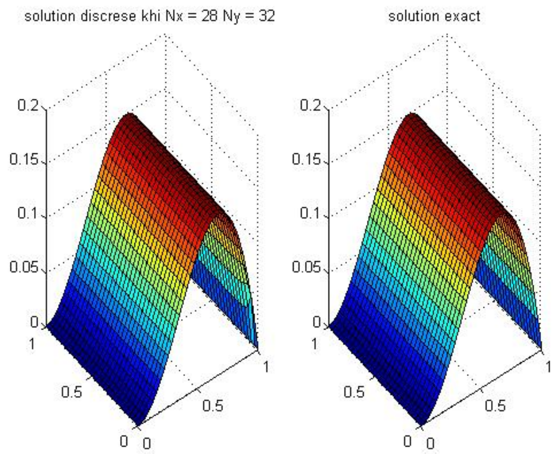


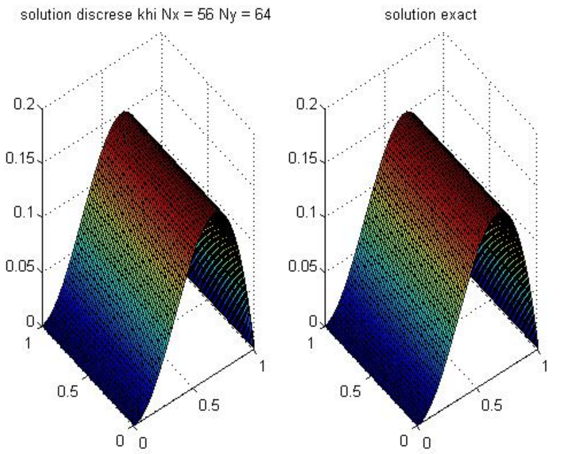


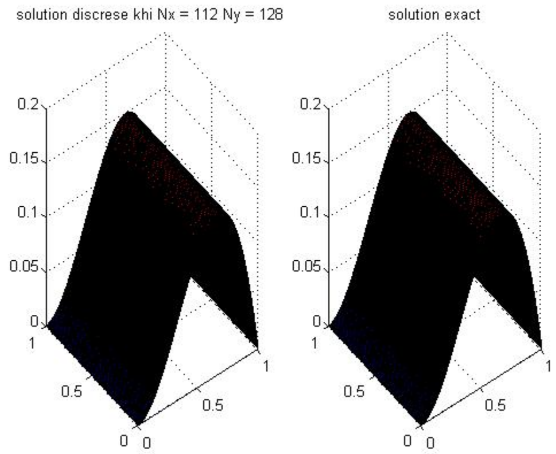
With the following exact solution u and function f

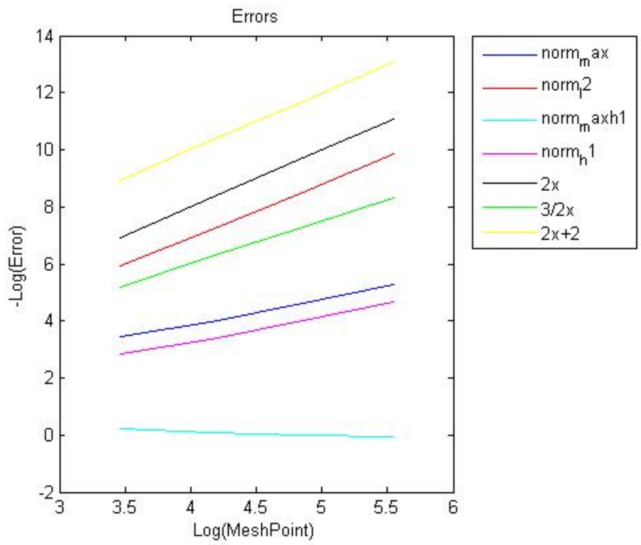
$$\begin{cases} u(x,y) = y^2(1-y) \\ f(x,y) = 6y-2 \\ u(x,y) = y^2(1-y) \quad \forall (x,y) \in \partial\Omega \end{cases}$$






solution discrese khi Nx = 56 Ny = 64
solution exact


solution discrese khi Nx = 112 Ny = 128
solution exact


Errors
-Log(Error)
Log(MeshPoint)
norm_max
norm_l2
norm_maxh1
norm_h1
2x
3/2x
2x+2

2. Dirichlet-Neumann boundary condition
We have:

$$u(x,0)=g_1(x),\ u(0,y)=g_2(y),\ u(x,1)=g_3(x),\ \frac{\partial u}{\partial x}(1,y)=g_4(y).$$

It's mean that:

$$\begin{aligned} u_{i,0}&=g_1(x_i), && \forall i=0,\cdots N,\\ u_{0,j}&=g_2(y_j), && \forall j=0,\cdots N,\\ u_{i,N}&=g_3(x_i), && \forall i=0,\cdots N,\\ \frac{\partial u}{\partial x}(1,y_j)&=g_4(y_j), && \forall j=0,\cdots N. \end{aligned}$$

Thus,

$$\frac{\partial u}{\partial x}(1,y_i)=\frac{u_{N,j}-u_{N-1,j}}{h}=g_4(y_j) \quad \forall j=0,\cdots N.$$

We have the scheme for finite difference discretization:

$$\frac{-u_{i,j-1}-u_{i-1,j}+4u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2}=f_{i,j} \tag{5}$$

- If $j=1$, and $i=1$ then $u_{i,j-1}=g_1(x_i)$ and $u_{i-1,j}=g_2(y_j)$, the equation in (5) becomes:

$$\frac{4u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2}=f_{i,j}+\frac{g_1(x_i)}{h^2}+\frac{g_2(y_j)}{h^2}.$$

- If $j=1$, and $i=N-1$ then $u_{i,j-1}=g_1(x_i)$ and $\frac{u_{N,j}-u_{N-1,j}}{h^2}=\frac{g_4(yj)}{h}$, the equation in (5) becomes

$$\frac{-u_{i-1,j}+3u_{i,j}-u_{i,j+1}}{h^2}=f_{i,j}+\frac{g_1(x_i)}{h^2}+\frac{g_4(y_j)}{h}.$$

If $j=1$ and $i\notin\{1,N-1\}$ then $u_{i,j-1}=g_1(x_i)$, the equation in (5) becomes:

$$\frac{-u_{i-1,j}+4u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2}=f_{i,j}+\frac{g_1(x_i)}{h^2}.$$

- If $j=N-1$ and $i=1$ then $u_{i,j+1}=g_3(x_i)$ and $u_{i-1,j}=g_2(y_j)$, the equation in (5) becomes:

$$\frac{-u_{i,j-1}+4u_{i,j}-u_{i+1,j}}{h^2}=f_{i,j}+\frac{g_3(x_i)}{h^2}+\frac{g_2(y_j)}{h^2}.$$

- If $j=N-1$ and $i=N-1$ then $u_{i,j+1}=g_3(x_i)$ and $\frac{u_{N,j}-u_{N-1,j}}{h^2}=\frac{g_4(y_j)}{h}$, the equation in (5) becomes:

$$\frac{-u_{i,j-1}-u_{i-1,j}+3u_{i,j}}{h^2}=f_{i,j}+\frac{g_3(x_i)}{h^2}+\frac{g_4(y_j)}{h}.$$

- If $j=N-1$ and $i\notin\{1,N-1\}$ then $u_{i,j+1}=g_3(x_i)$, the equation in (5) becomes:

$$\frac{-u_{i,j-1}-u_{i-1,j}+4u_{i,j}-u_{i+1,j}}{h^2}=f_{i,j}+\frac{g_3(x_i)}{h^2}.$$

- If $j \notin \{1, N-1\}$ and $i = 1$ then $u_{i-1,j} = g_2(y_j)$, the equation in (5) becomes:

$$\frac{-u_{i,j-1} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j} + \frac{g_2(y_j)}{h^2}.$$

- If $j \notin \{1, N-1\}$ and $i = N-1$ then $\frac{u_{N,j}-u_{N-1,j}}{h^2} = \frac{g_4(y_j)}{h}$, the equation in (5) becomes:

$$\frac{-u_{i,j-1} - u_{i-1,j} + 3u_{i,j} - u_{i,j+1}}{h^2} = f_{i,j} + \frac{g_4(y_j)}{h}.$$

- If $j \notin \{1, N-1\}$ and $i \notin \{1, N-1\}$ then the equation in (5) holds:

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

We get a linear system

$$Au = F, \quad A \in \mathbb{R}^{(N-1)^2} \times \mathbb{R}^{(N-1)^2},$$

$$u = \begin{bmatrix} u_{1,1} \\ \vdots \\ u_{N-1,1} \\ u_{1,2} \\ \vdots \\ u_{N-1,2} \\ \vdots \\ u_{1,N-1} \\ \vdots \\ u_{N-1,N-1} \end{bmatrix} \qquad F = \begin{bmatrix} f_{1,1} + \frac{g_1(x_1)}{h^2} + \frac{g_2(y_1)}{h^2} \\ \vdots \\ f_{N-1,1} + \frac{g_1(x_{N-1})}{h^2} + \frac{g_4(y_1)}{h} \\ f_{1,2} + \frac{g_2(y_2)}{h^2} \\ \vdots \\ f_{N-1,2} + \frac{g_4(y_2)}{h} \\ \vdots \\ f_{1,N-1} + \frac{g_3(x_1)}{h^2} + \frac{g_2(y_{N-1})}{h^2} \\ \cdots \\ f_{N-1,N-1} + \frac{g_3(x_{N-1})}{h^2} + \frac{g_4(y_{N-1})}{h} \end{bmatrix}$$

$$A = \frac{1}{h^2} \begin{bmatrix} B & -I & & & & \\ -I & B & & & & \\ & -I & B & -I & & \\ & & & \cdots & & \\ & & & -I & B & -I \\ & & & & -I & C \end{bmatrix}$$

where

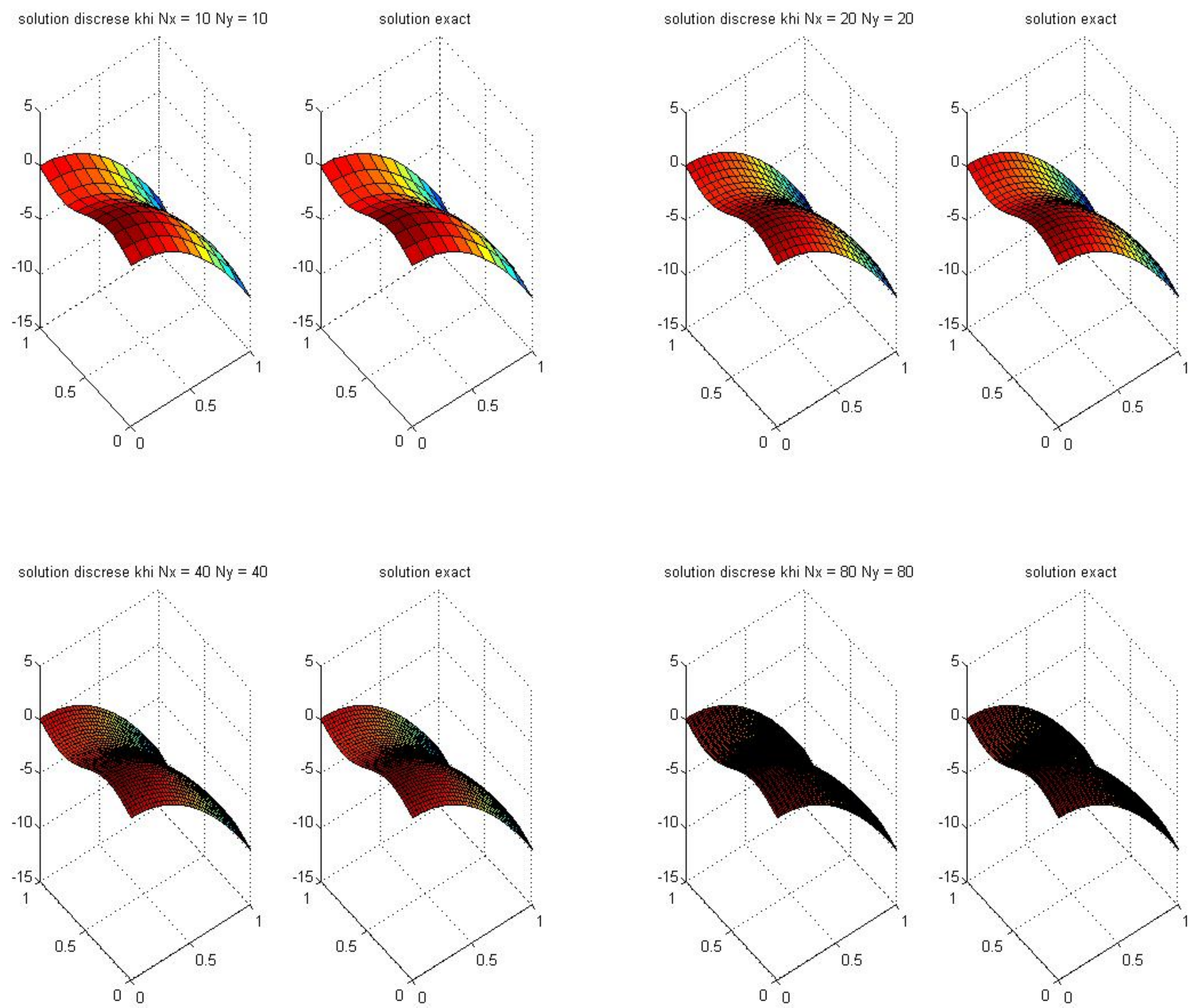
$$B = \begin{bmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & & \cdots & & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & -1 & & & & \\ -1 & 3 & -1 & & & \\ & -1 & 3 & -1 & & \\ & & & \cdots & & \\ & & & -1 & 3 & -1 \\ & & & & -1 & 3 \end{bmatrix}$$

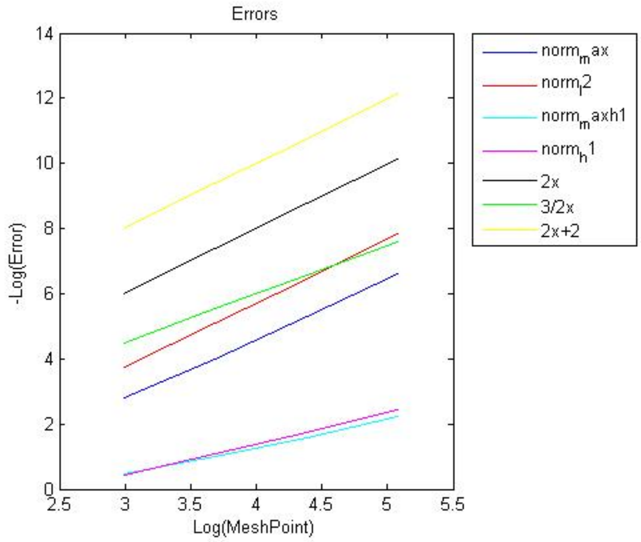
and

$$I = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \cdots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}.$$

We set up with the following exact solution u and function f:

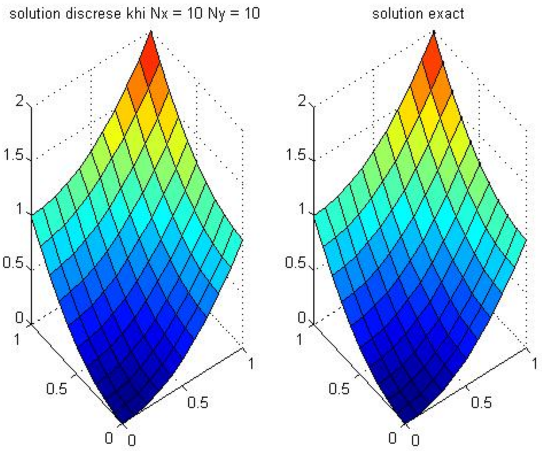
$$\begin{cases} u\left(x,y\right) & = \sin\left(2\pi\left(x-1\right)\right] - 10y^2 \\ f\left(x,y\right) & = 4\pi^2 \sin\left[2\pi\left(x-1\right)\right] + 20 \\ u\left(x,0\right) & = \sin\left(2\pi\left(x-1\right)\right] \\ u\left(0,y\right) & = -10y^2 \\ u\left(x,1\right) & = \sin\left(2\pi\left(x-1\right)\right] - 10 \\ \frac{\partial u}{\partial x}\left(1,y\right) & = 1 \end{cases}$$

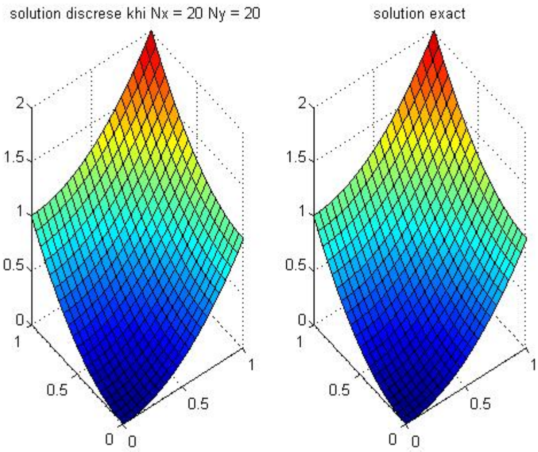


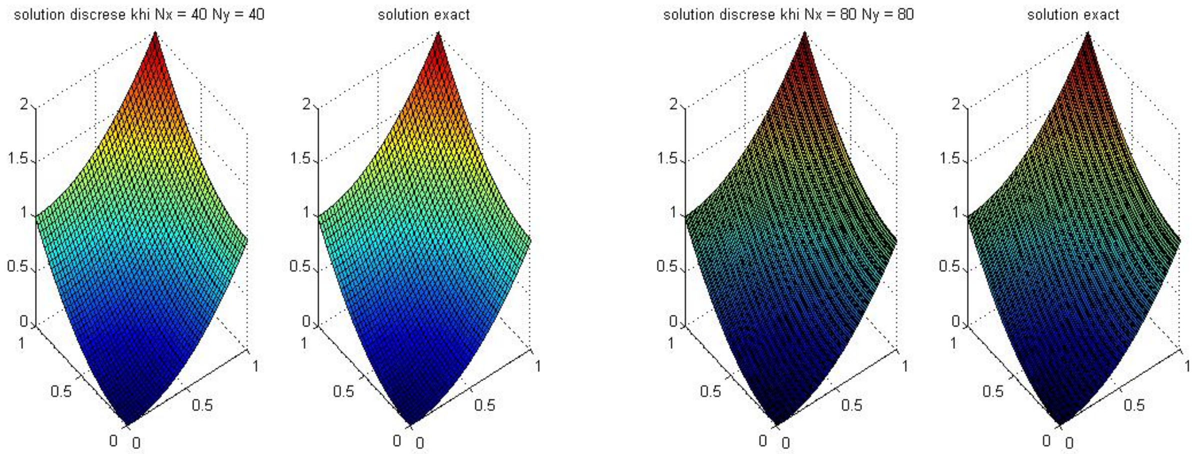


With the following exact solution u and function f

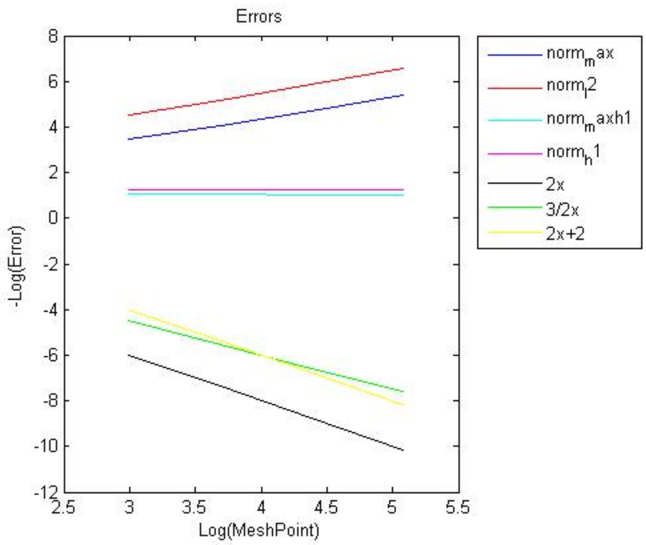
$$\begin{cases} u(x,y) & = x^2 + y^2 \\ f(x,y) & = -4 \\ u(x,0) & = x^2 \\ u(0,y) & = y^2 \\ u(x,1) & = x^2 + 1 \\ \frac{\partial u}{\partial x}(1,y) & = 2 \end{cases}$$







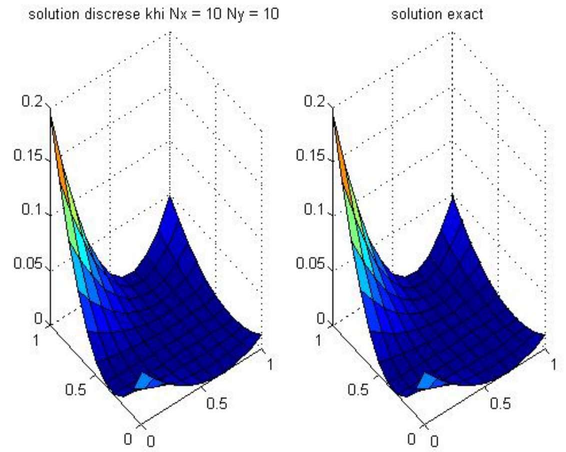
solution discrese khi Nx = 40 Ny = 40
solution exact
solution discrese khi Nx = 80 Ny = 80
solution exact

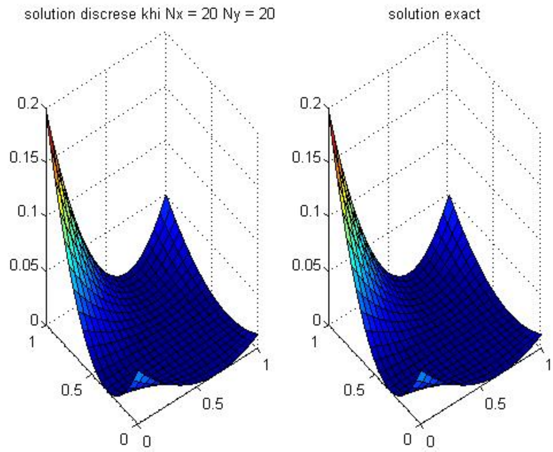


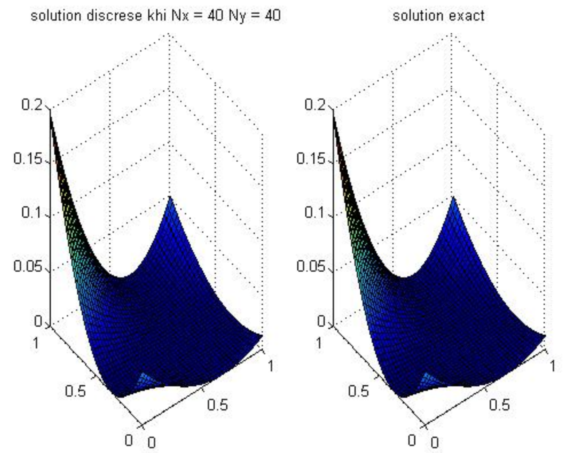
Errors
-Log(Error)
Log(MeshPoint)
norm_max
norm_l2
norm_maxh1
norm_h1
2x
3/2x
2x+2

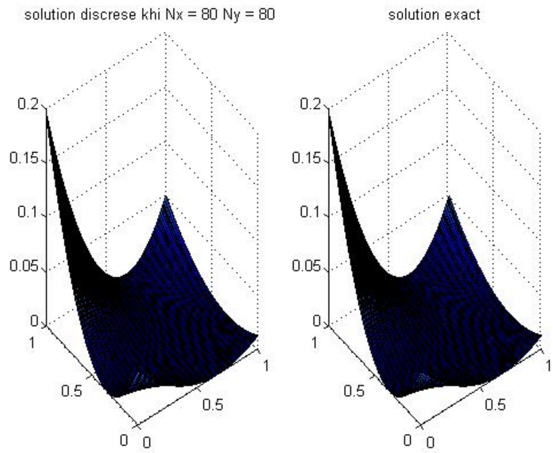
With the following exact solution u and function f

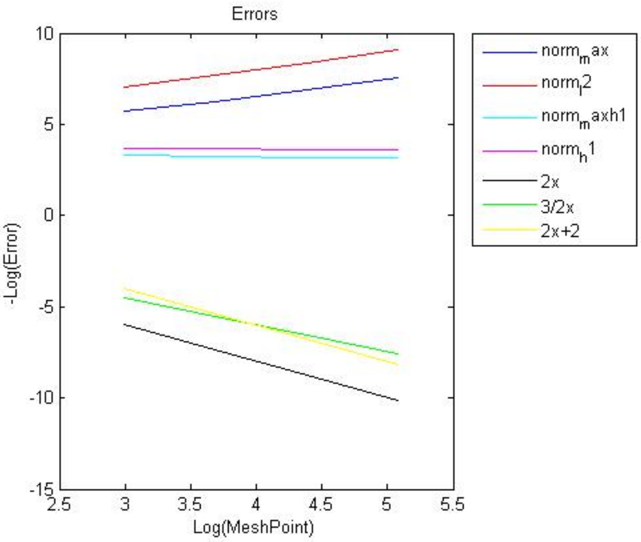
$$\begin{cases} u(x,y) & = \left(x-\frac{1}{3}\right)^2\left(y-\frac{2}{3}\right)^2 \\ f(x,y) & = -2\left(x-\frac{1}{3}\right)^2 - 2\left(y-\frac{2}{3}\right)^2 \\ u(x,0) & = \frac{4}{9}\left(x-\frac{1}{3}\right)^2 \\ u(0,y) & = \frac{1}{9}\left(y-\frac{2}{3}\right)^2 \\ u(x,1) & = \frac{1}{9}\left(x-\frac{1}{3}\right)^2 \\ \frac{\partial u}{\partial x}(1,y) & = \frac{4}{3}\left(y-\frac{2}{3}\right)^2 \end{cases}$$











3. Neumann boundary condition
We have:

$$\nabla u \cdot \vec{\mathbf{n}}\,(x,y) = 0 \quad \forall\,(x,y) \in \partial\Omega.$$

Because $\vec{\mathbf{n}}$ is unit normal vector to boundary $\partial\Omega$, we have:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \quad \forall\,(x,y) \in \partial\Omega.$$

We have:

$$\begin{aligned}
\frac{\partial u}{\partial x}(0,y_j) &= \frac{u_{1,j}-u_{0,j}}{h} = 0 \Rightarrow u_{1,j} = u_{0,j},\\
\frac{\partial u}{\partial x}(1,y_j) &= \frac{u_{N,j}-u_{N-1,j}}{h} = 0 \Rightarrow u_{N,j} = u_{N-1,j},\\
\frac{\partial u}{\partial y}(x_i,0) &= \frac{u_{i,1}-u_{i,0}}{h} = 0 \Rightarrow u_{i,1} = u_{i,0},\\
\frac{\partial u}{\partial y}(x_i,1) &= \frac{u_{i,N}-u_{i,N-1}}{h} = 0 \Rightarrow u_{i,N} = u_{i,N-1}.
\end{aligned}$$

We have the scheme for finite difference discretization:

$$\frac{-u_{i,j-1}-u_{i-1,j}+4u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2} = f_{i,j} \tag{6}$$

- If $j=1$, and $i=1$ then $u_{i,j-1}=u_{i,j}$ and $u_{i-1,j}=u_{i,j}$, the equation in (6) becomes:

$$\frac{2u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2} = f_{i,j}.$$

- If $j=1$, and $i=N-1$ then $u_{i,j-1}=u_{i,j}$ and $u_{i+1,j}=u_{i,j}$, the equation in (6) becomes

$$\frac{-u_{i-1,j}+2u_{i,j}-u_{i,j+1}}{h^2} = f_{i,j}.$$

  If $j=1$ and $i \notin \{1,N-1\}$ then $u_{i,j-1}=u_{i,j}$, the equation in (6) becomes:

$$\frac{-u_{i-1,j}+3u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2} = f_{i,j}.$$

- If $j = N-1$ and $i = 1$ then $u_{i,j+1} = u_{i,j}$ and $u_{i-1,j} = u_{i,j}$, the equation in (6) becomes:
$$\frac{-u_{i,j-1} + 2u_{i,j} - u_{i+1,j}}{h^2} = f_{i,j}.$$
- If $j = N-1$ and $i = N-1$ then $u_{i,j+1} = u_{i,j}$ and $u_{i+1,j} = u_{i,j}$, the equation in (6) becomes:
$$\frac{-u_{i,j-1} - u_{i-1,j} + 2u_{i,j}}{h^2} = f_{i,j}.$$
- If $j = N-1$ and $i \notin \{1, N-1\}$ then $u_{i,j+1} = u_{i,j}$, the equation in (6) becomes:
$$\frac{-u_{i,j-1} - u_{i-1,j} + 3u_{i,j} - u_{i+1,j}}{h^2} = f_{i,j}.$$
- If $j \notin \{1, N-1\}$ and $i = 1$ then $u_{i-1,j} = u_{i,j}$, the equation in (6) becomes:
$$\frac{-u_{i,j-1} + 3u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$
- If $j \notin \{1, N-1\}$ and $i = N-1$ then $u_{i+1,j} = u_{i,j}$, the equation in (6) becomes:
$$\frac{-u_{i,j-1} - u_{i-1,j} + 3u_{i,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$
- If $j \notin \{1, N-1\}$ and $i \notin \{1, N-1\}$ then the equation in (6) holds:
$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

We get a linear system
$$Au = F, \qquad A \in \mathbb{R}^{(N-1)^2} \times \mathbb{R}^{(N-1)^2},$$
$$u = \begin{bmatrix} u_{1,1} \\ \vdots \\ u_{N-1,1} \\ u_{1,2} \\ \vdots \\ u_{N-1,2} \\ \vdots \\ u_{1,N-1} \\ \vdots \\ u_{N-1,N-1} \end{bmatrix} \qquad F = \begin{bmatrix} f_{1,1} \\ \vdots \\ f_{N-1,1} \\ f_{1,2} \\ \vdots \\ f_{N-1,2} \\ \vdots \\ f_{1,N-1} \\ \cdots \\ f_{N-1,N-1} \end{bmatrix}$$
$$A = \frac{1}{h^2} \begin{bmatrix} C & -I & & & & \\ -I & B & & & & \\ & -I & B & -I & & \\ & & & \cdots & & \\ & & & -I & B & -I \\ & & & & -I & C \end{bmatrix}$$

where

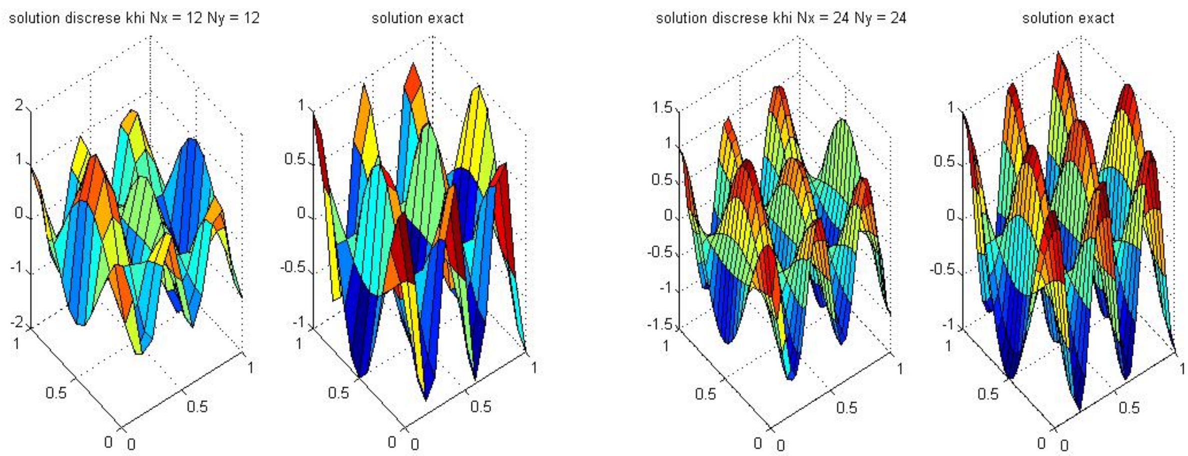
$$B = \begin{bmatrix} 3 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & & \cdots & & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 3 & -1 & & & \\ & -1 & 3 & -1 & & \\ & & & \cdots & & \\ & & & -1 & 3 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

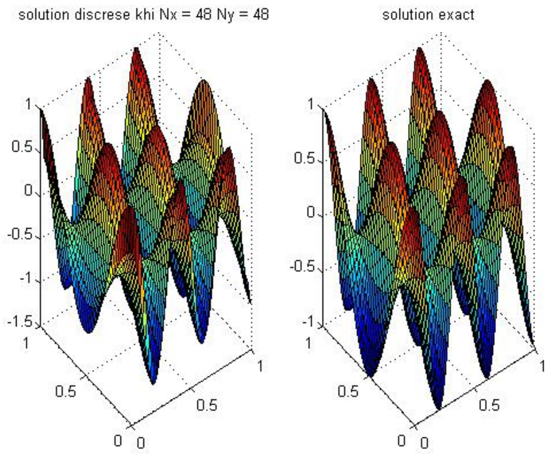
and

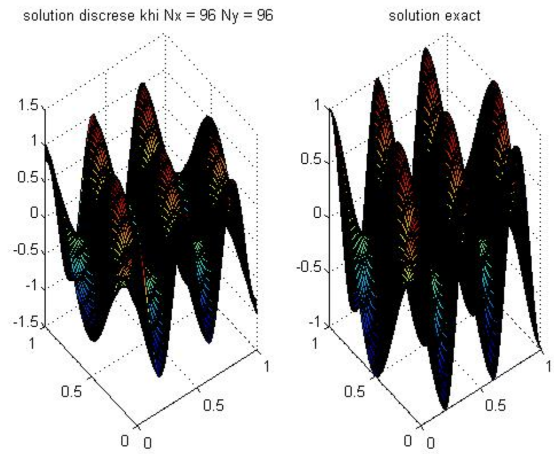
$$I = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \cdots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}.$$

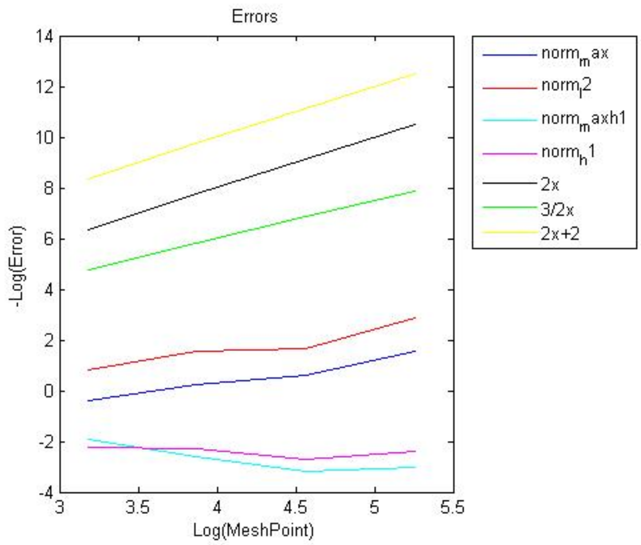
We set up with the following exact solution u and function f:

$$\begin{cases} u\left(x,y\right) & = \cos\left(2\pi x\right)\cos\left(5\pi y\right) \\ f\left(x,y\right) & = 29\pi^2\cos\left(2\pi x\right)\cos\left(5\pi y\right) \\ \nabla u\cdot\vec{\mathbf{n}}\left(x,y\right) & = 0 \quad \forall\left(x,y\right)\in\partial\Omega \end{cases}$$



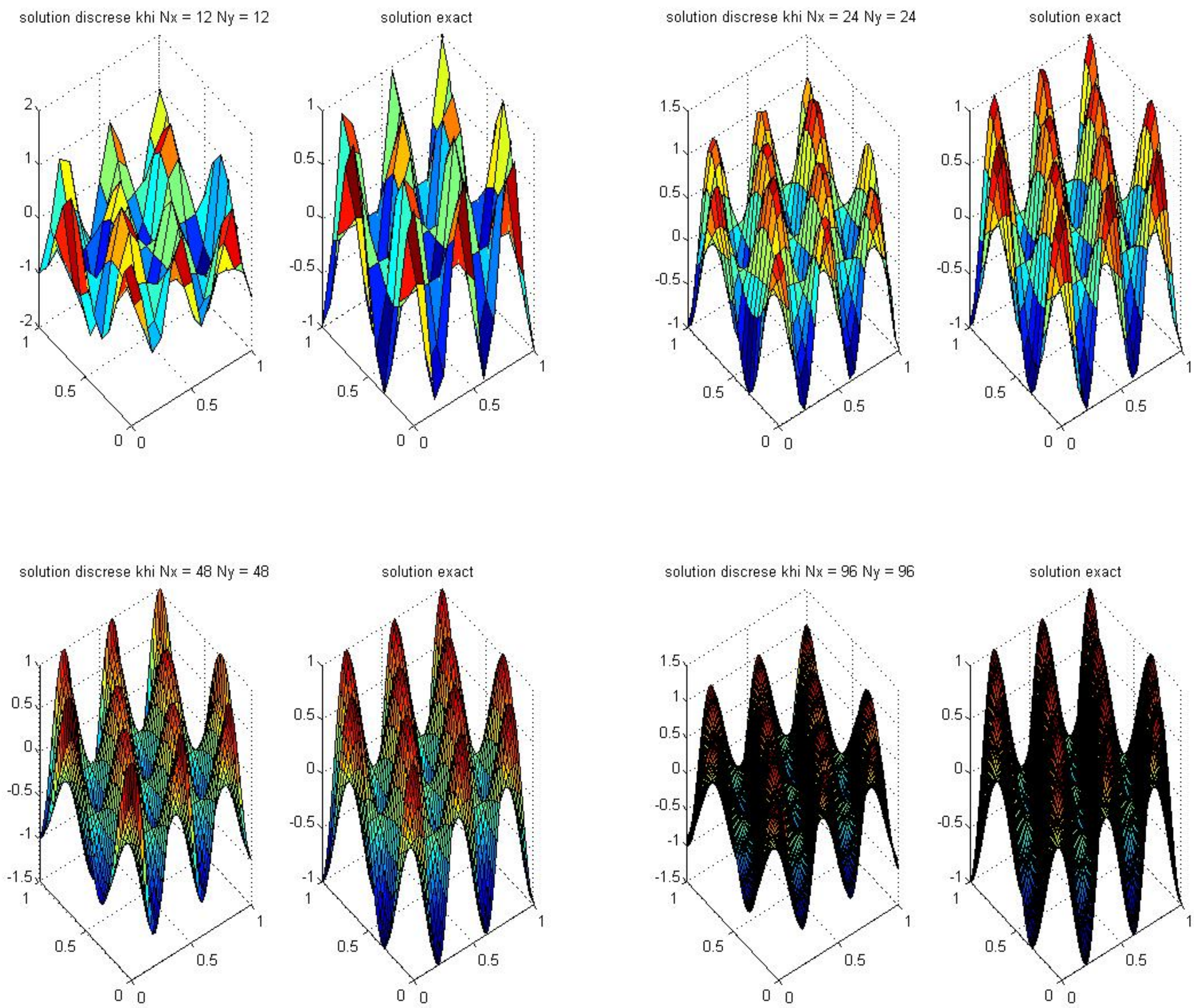

solution discrese khi Nx = 48 Ny = 48
solution exact

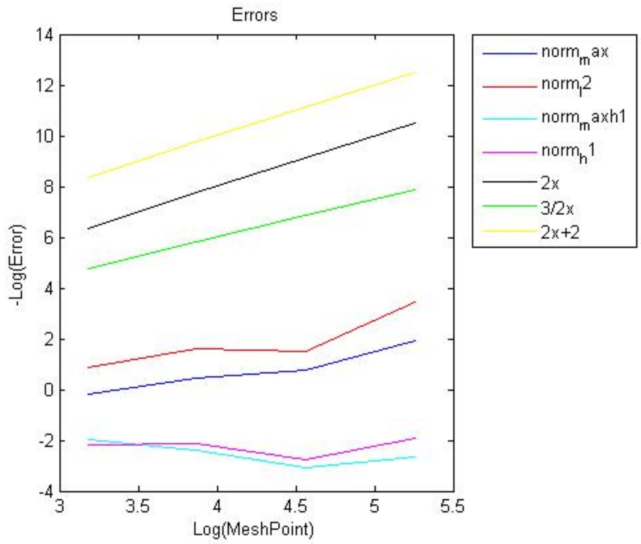

solution discrese khi Nx = 96 Ny = 96
solution exact


Errors
-Log(Error)
Log(MeshPoint)
norm$_m$ax
norm$_l$2
norm$_m$axh1
norm$_h$1
2x
3/2x
2x+2

With the following exact solution u and function f

$$\begin{cases} u\left(x,y\right) & =\cos\left(3\pi x\right)\cos\left(5\pi y\right) \\ f\left(x,y\right) & =34\pi^{2}\cos\left(3\pi x\right)\cos\left(5\pi y\right) \\ \nabla u\cdot\vec{\mathbf{n}}\left(x,y\right) & =0 \quad \forall\left(x,y\right)\in\partial\Omega \end{cases}$$




Errors
-Log(Error)
Log(MeshPoint)
norm_max
norm_l2
norm_maxh1
norm_h1
2x
3/2x
2x+2