Vietnam National University - Ho Chi Minh City, University of Science, Faculty of Mathematics and Computer Science

FDM: Practical Assignment 3

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Problem

For $f \in \mathrm{L}^2\left([0,1]\times[0,T]\right)$ and we consider heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f\left(x,t\right) \qquad \forall\left(x,t\right) \in [0,1]\times[0,T]\,. \tag{1}$$

Along with this equation we need initial conditions at time 0

$$u\left(x,0\right) = u_0\left(x\right). \tag{2}$$

and also boundary conditions if we are working on a bounded domain, e.g., the Dirichlet conditions

$$\begin{aligned} u\left(t,0\right) &= 0 \\ u\left(t,1\right) &= 0 \end{aligned} \tag{3}$$

1. Find the discrete solution using finite difference scheme with three methods (Forward, Backward Euler and Crank-Nicolson) and consider them when

$$\begin{aligned} u_{ex}\left(x,t\right) &= x\left(1-x\right)^2 e^{-2t}, \\ f\left(x,t\right) &= \left(2-8x+4x^2-2x^3\right)e^{-2t}. \end{aligned}$$

2. Compute the error for three methods on space with discrete H_0^2 and L^2 norms when $k=0.01$, $k=0.005$, $k=0.0001$ and $k=h^2$. Do you consider about these errors.

Solution

1. Let us consider a uniform partion with $N_x + 1$ points x_i for all $i = 0, 1, 2, \cdot, N_x$, we have space step is $h = \frac{1}{N_x}$. We divide the interval $[0, T]$ into $N_t - 1$ sub-intervals of constant length $k = \frac{T}{N_t}$. Then

$$x_i = ih \text{ and } t_n = nk.$$

We have the natural discretization of (1)

$$\frac{U_i^{n+1} - U_i^n}{k} = \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{h^2} + f_i^n$$

where $U_i^n = u(x_i, t_n)$ and $f_i^n = f(x_i, t_n)$.

Apply "θ method", we have

$$\frac{U_i^{n+1} - U_i^n}{k} = \left[(1-\theta) D_2 U_i^n + \theta D_2 U_i^{n+1}\right] + f_i^n \tag{4}$$

where $D_2 U_i^n = \dfrac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{h^2}$ and $D_2 U_i^{n+1} = \dfrac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{h^2}$ $\forall i = \overline{1, N_x - 1}$.

We can rewrite that:

$$\begin{aligned} -r\theta U_{i-1}^{n+1} + (1 + 2r\theta) U_i^{n+1} - r\theta U_{i+1}^{n+1} &= r(1-\theta) U_{i-1}^n \\ &+ [1 - 2r(1-\theta)] U_i^n + r(1-\theta) U_{i+1}^n + kf_i^n \end{aligned} \tag{5}$$

where $r = \frac{k}{h^2}$

We put

$$A = \begin{bmatrix} -2r & r & 0 & 0 & 0 & 0 \\ r & -2r & r & 0 & 0 & 0 \\ 0 & r & -2r & r & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & r & -2r & r \\ 0 & 0 & 0 & 0 & r & -2r \end{bmatrix} \qquad F = \begin{bmatrix} kf_1^n \\ kf_2^n \\ \vdots \\ kf_{N_x-2}^n \\ kf_{N_x-1}^n \end{bmatrix}$$

Thus, (5) can rewrite that:

$$(I - \theta A) U^{n+1} = [I + (1-\theta) A] U^n + F \tag{6}$$

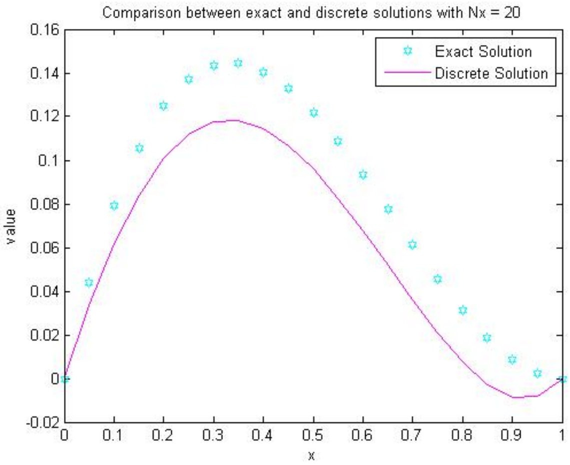
- Forward Euler method

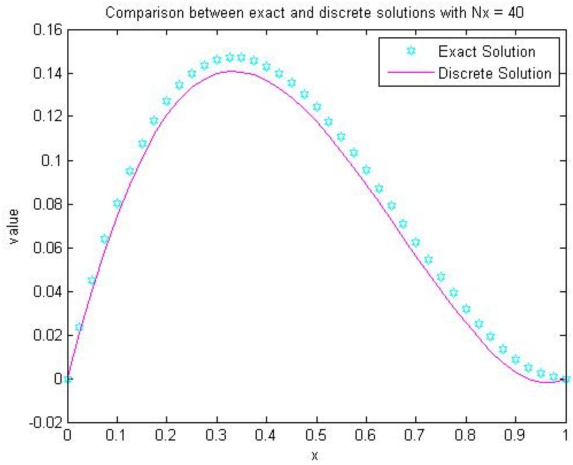
  Let $\theta = 0$, we have (6) rewrite that:

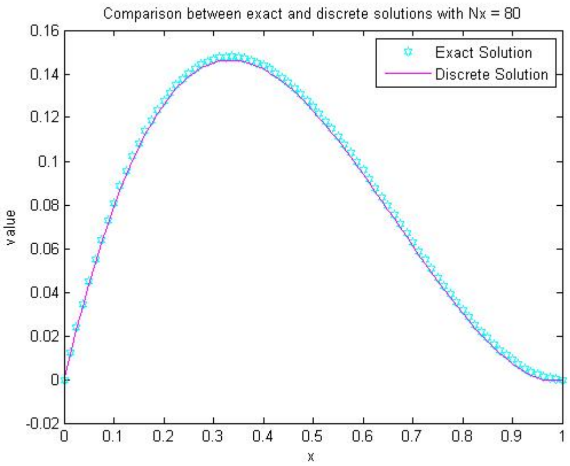
$$U^{n+1} = (I + A) U^n + F \tag{7}$$

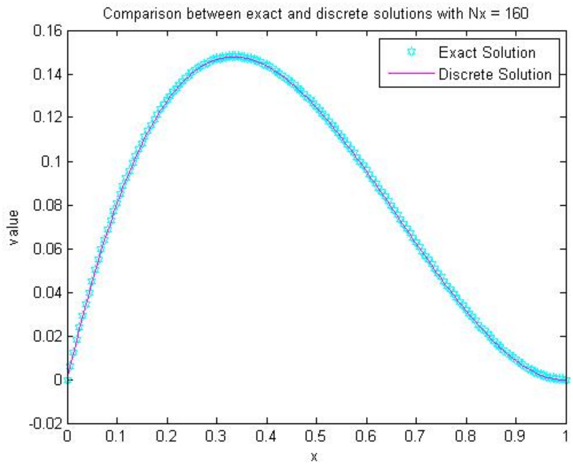
  We set up with the following exact solution u and function f:

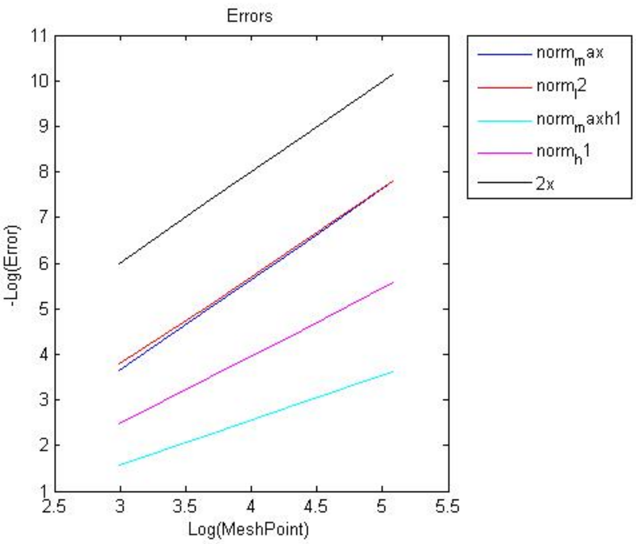
$$\begin{cases} u_{ex}(x,t) &= x(1-x)^2 e^{-2t}, \\ f(x,t) &= (2 - 8x + 4x^2 - 2x^3) e^{-2t} \end{cases}$$


Comparison between exact and discrete solutions with Nx = 20
Exact Solution
Discrete Solution
value
x


Comparison between exact and discrete solutions with Nx = 40
Exact Solution
Discrete Solution
value
x


Comparison between exact and discrete solutions with Nx = 80
Exact Solution
Discrete Solution
value
x


Comparison between exact and discrete solutions with Nx = 160
Exact Solution
Discrete Solution
value
x


Errors
norm_max
norm_l2
norm_maxh1
norm_h1
2x
-Log(Error)
Log(MeshPoint)

- Backward Euler method

  Let $\theta = 1$, we have (6) rewrite that:

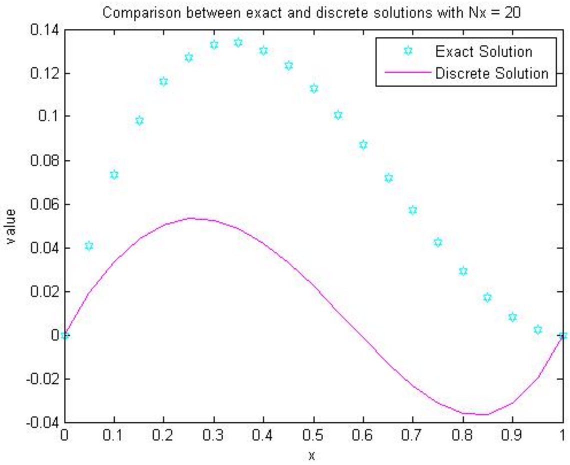
$$(I - A)\, U^{n+1} = U^n + F \tag{8}$$

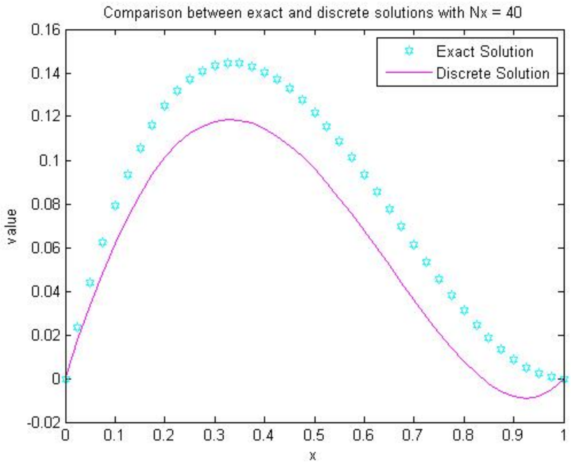
Thus,

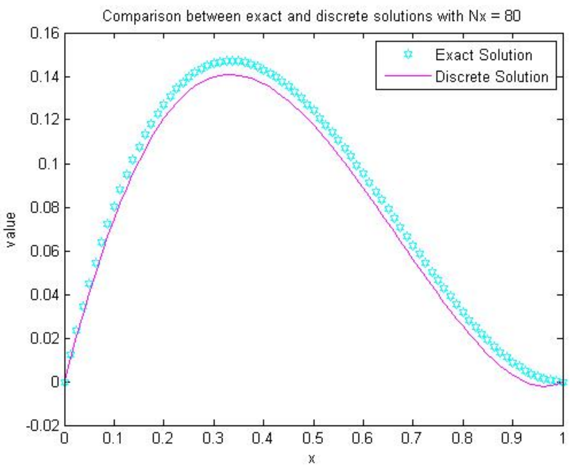
$$U^{n+1} = (I - A)^{-1} \left(U^n + F \right) \tag{9}$$

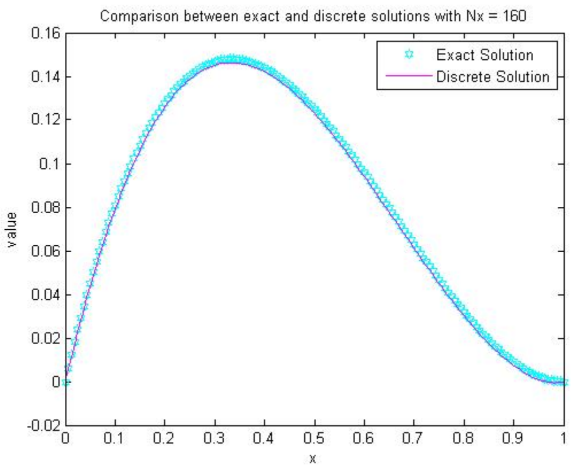
We set up with the following exact solution u and function f:

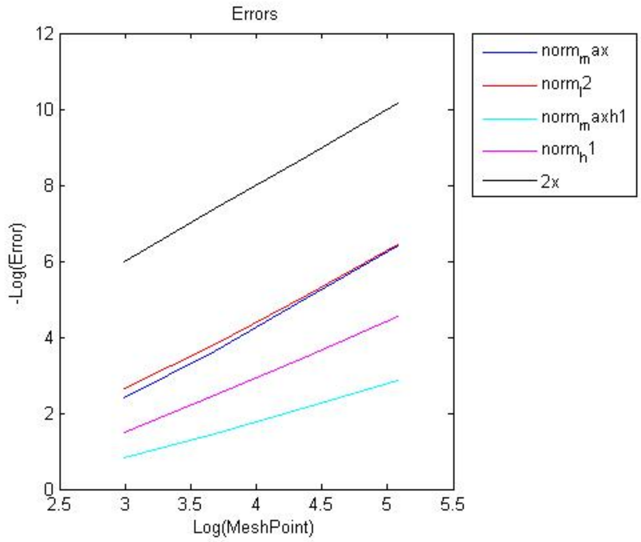
$$\begin{cases} u_{ex}(x,t) &= x(1-x)^2 e^{-2t}, \\ f(x,t) &= \left(2 - 8x + 4x^2 - 2x^3\right) e^{-2t} \end{cases}$$











- Crank-Nicolson

  Let $\theta = \frac{1}{2}$, we have (6) rewrite that:

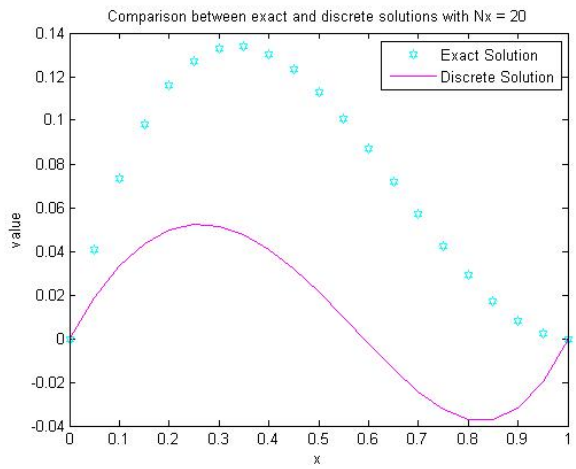
$$\left(I - \frac{1}{2}A\right)U^{n+1} = \left(I + \frac{1}{2}A\right)U^n + F \tag{10}$$

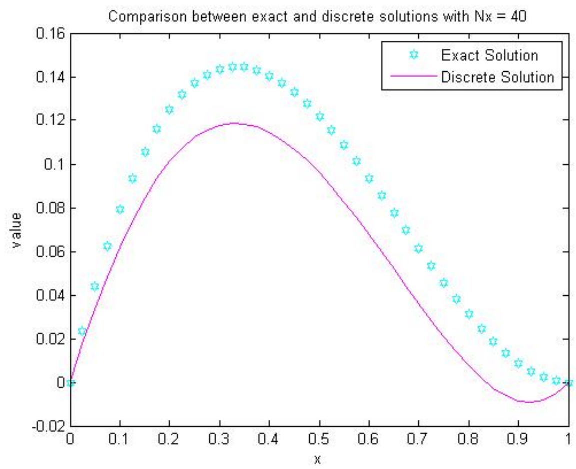
  Thus,

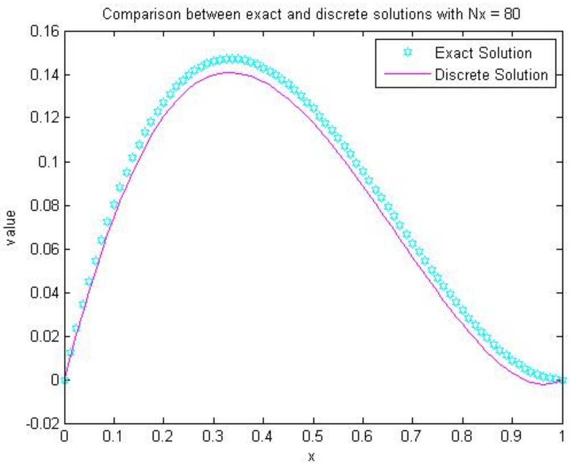
$$U^{n+1} = \left(I - \frac{1}{2}A\right)^{-1}\left[\left(I + \frac{1}{2}A\right)U^n + F\right] \tag{11}$$

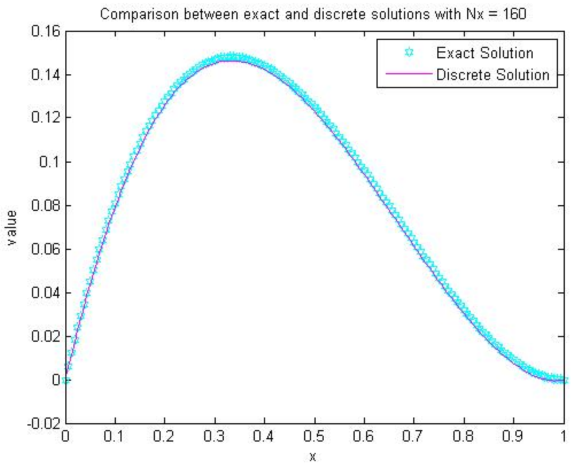
  We set up with the following exact solution u and function f:

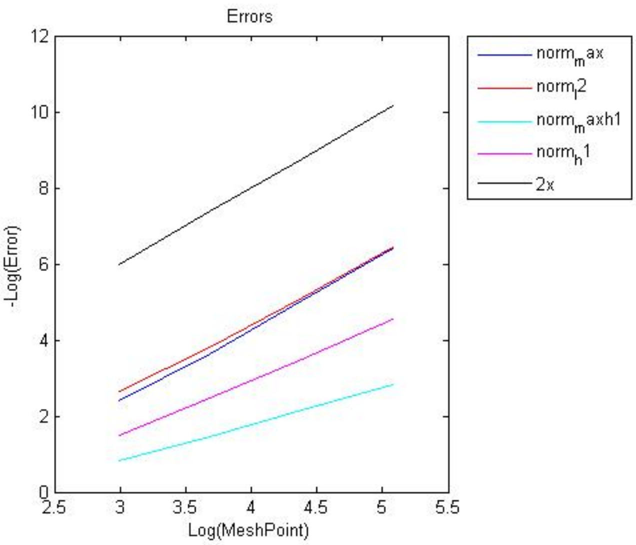
$$\begin{cases} u_{ex}(x,t) & = x(1-x)^2 e^{-2t}, \\ f(x,t) & = (2 - 8x + 4x^2 - 2x^3)\, e^{-2t} \end{cases}$$






Comparison between exact and discrete solutions with Nx = 80
Exact Solution
Discrete Solution
value
x


Comparison between exact and discrete solutions with Nx = 160
Exact Solution
Discrete Solution
value
x


Errors
norm_max
norm_l2
norm_maxh1
norm_h1
2x
-Log(Error)
Log(MeshPoint)