Vietnam National University - Ho Chi Minh City, University of Science, Faculty of Mathematics and Computer Science

# FVM: Practical Assignment 1

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## Problem

Given a 1D Poisson problem on $\Omega = (0,1)$

$$-u''(x) = f(x), x \in \Omega \,(*)$$

1. Dirichlet boundary condition

a) Solve equation $(*)$ subject to homogeneous Dirichlet boundary condition

$$u(0) = a, u(1) = b$$

by finite volume method on a regular grid and the control point is the midpoint of each control volume $x_i = \left(x_{i-1/2} + x_{i+1/2}\right)/2$

b) Solve equation $(*)$ with regular grid and the control point is 1/3 from the left of each control volume $\left(x_i = 2/3x_{i-1/2} + 1/3x_{i+1/2}\right)$.

c) How to approximate the mean-value of f over $T_i$ and compare some ways approximation.

d) Solve equation $(*)$ with singular grid (not uniform grid).

2.Neumann boundary condition

Solve equation $(*)$ subject to homogeneous Neumann boundary condition

$$u'(0) = 0,\ u'(1) = 0 \text{ with } \int_0^1 f(x)\,dx = 0 \text{ and } \int_0^1 u(x)\,dx = 0$$

by finite volume method on a regular grid and singular grid with the control point be the midpoint of each control volume $x_i = \left(x_{i-1/2} + x_{i+1/2}\right)/2$.

## Solution

1. Discrete similar problem for the case of homogeneous dirichlet boundary, we have:

$$\begin{cases} \alpha_i u_{i-2} + \beta_i u_i + \gamma_i u_{i+1} = f_i & \forall i \in \overline{1,N} \\ u(0) = a, u(1) = b \end{cases}$$

where,

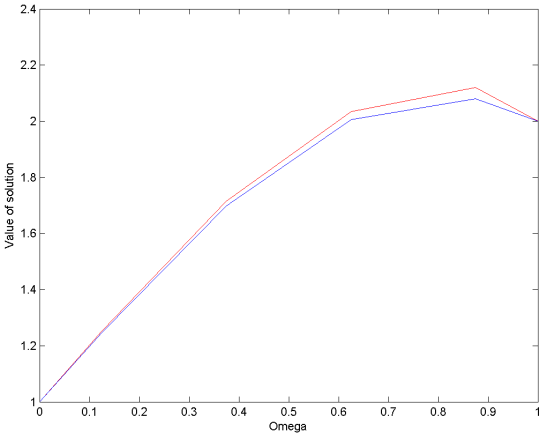
$$\begin{aligned} \alpha_i &= \frac{-1}{|D_{i-1/2}||T_i|} \\ \beta_i &= \frac{1}{|D_{i+1/2}||T_i|} + \frac{1}{|D_{i-1/2}||T_i|} \\ \gamma_i &= \frac{-1}{|D_{i+1/2}||T_i|} \end{aligned}$$

We have linear system for the scheme

$$\begin{cases} i = 1, \beta_1 u_1 + \gamma_1 u_2 & = f_1 - a\alpha_1 \\ i = 2, \alpha_2 u_1 + \beta_2 u_2 + \gamma_2 u_3 & = f_2 \\ i = 3, \qquad \alpha_3 u_2 + \beta_3 u_3 + \gamma_3 u_4 & = f_4 \\ \qquad \qquad \ldots & \\ i = N-1, \qquad \alpha_{N-1} u_{N-2} + \beta_{N-1} u_{N-1} + \gamma_{N-1} u_N & = f_{N-1} \\ i = N, \qquad \alpha_N u_{N-1} + \beta_N u_N & = f_N - b\gamma_N \end{cases}$$

a. We set up with the following exact solution u and function f

$$\begin{cases} u(x) = -x^3 + 2x + 1 \\ f(x) = 6x \\ u(0) = 1, u(1) = 2 \end{cases}$$


2.4
2.2
2
1.8
1.6
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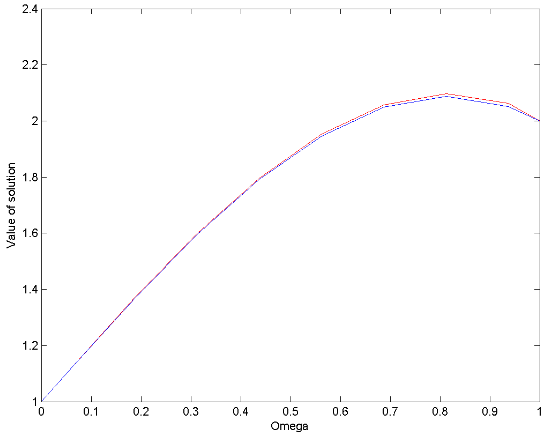
Value of solution
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Omega


2.4
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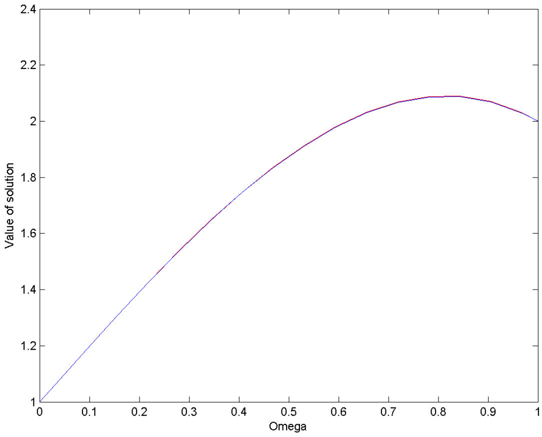
Value of solution
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Omega


2.4
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1.4
1.2
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Value of solution
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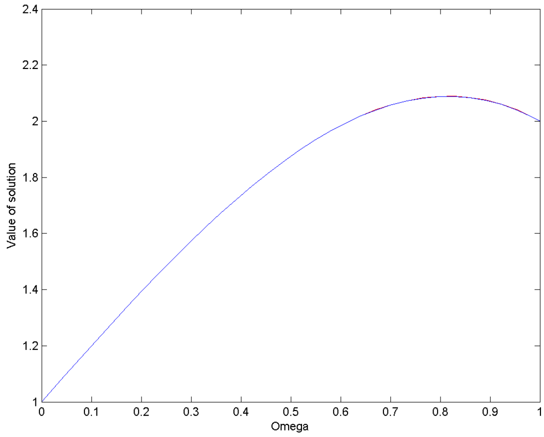
Omega


2.4
2.2
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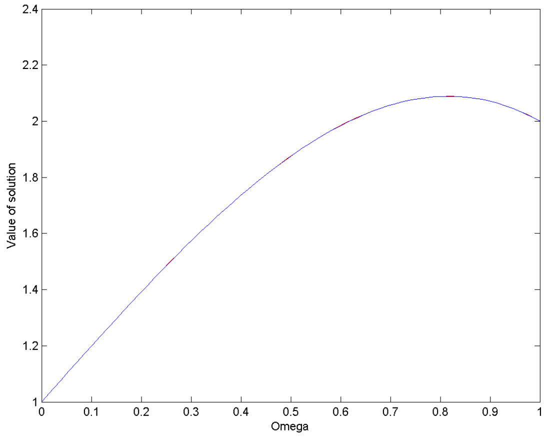
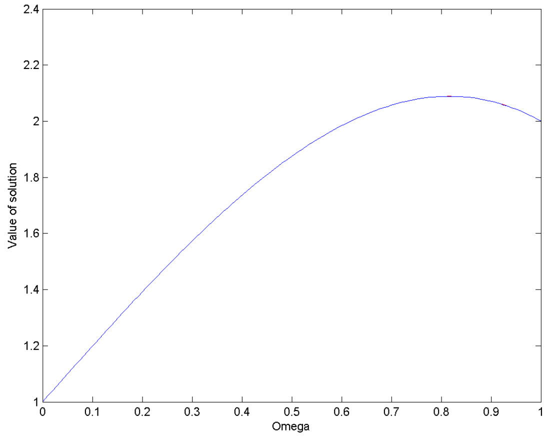
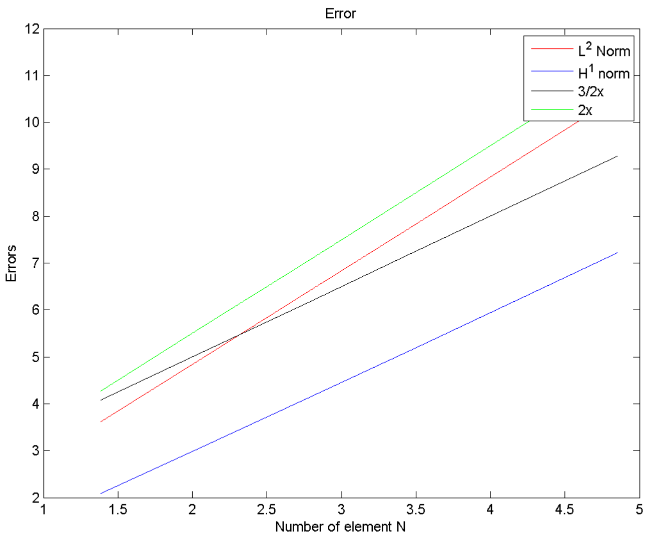
Value of solution
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Omega

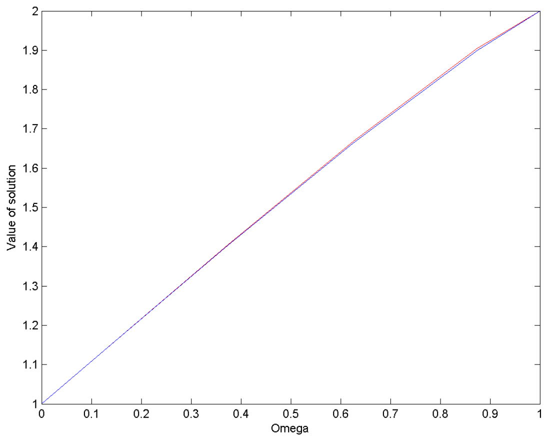

Value of solution
Omega

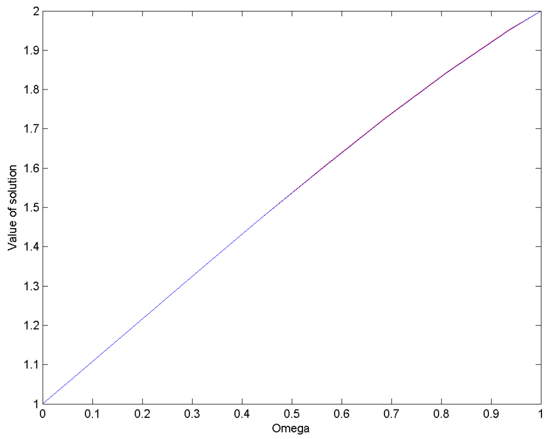

Value of solution
Omega

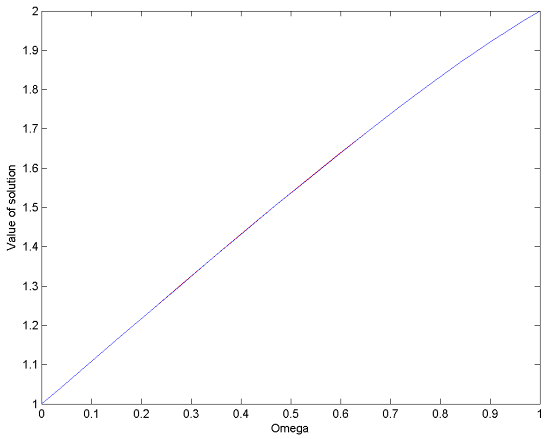

Error
$L^2$ Norm
$H^1$ norm
3/2x
2x
Errors
Number of element N

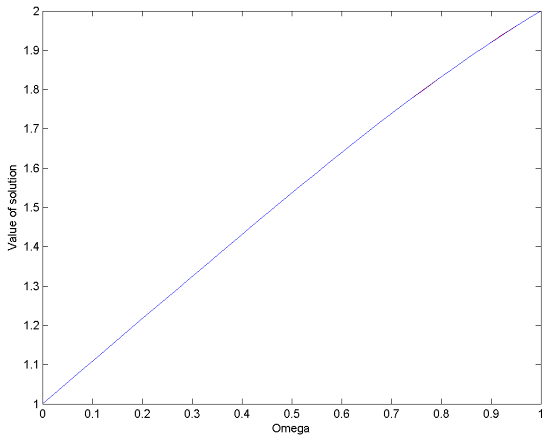
With the following exact solution u and function f

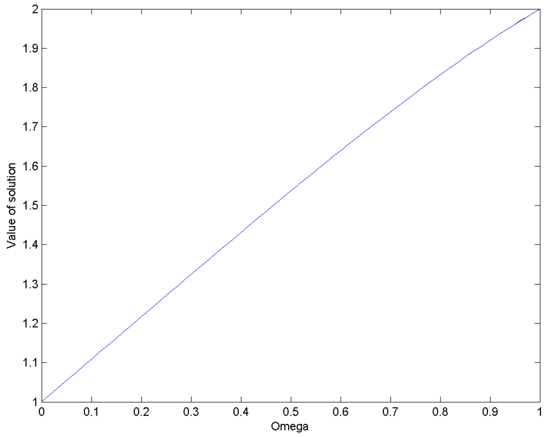
$$\begin{cases} u\left(x\right) = -\dfrac{1}{12}x^4 + \dfrac{13}{12}x + 1 \\ f\left(x\right) = x^2 \\ u(0) = 1, u(1) = 2 \end{cases}$$

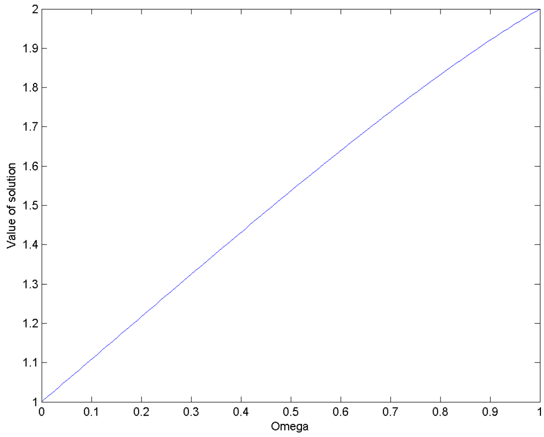


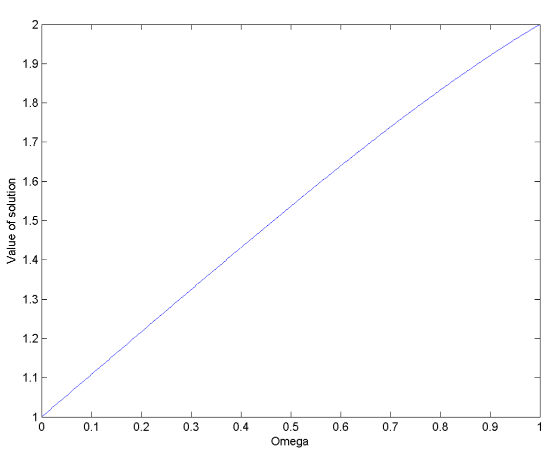



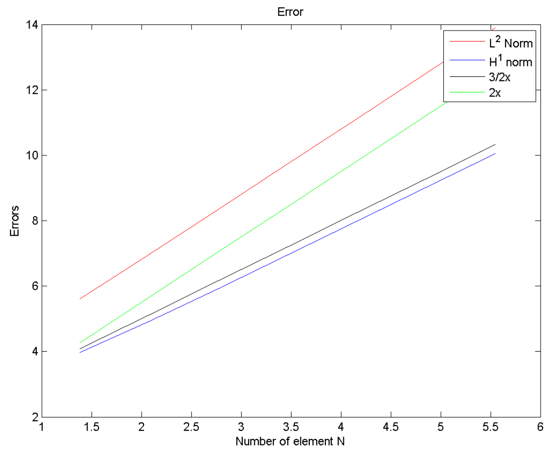




Value of solution
Omega

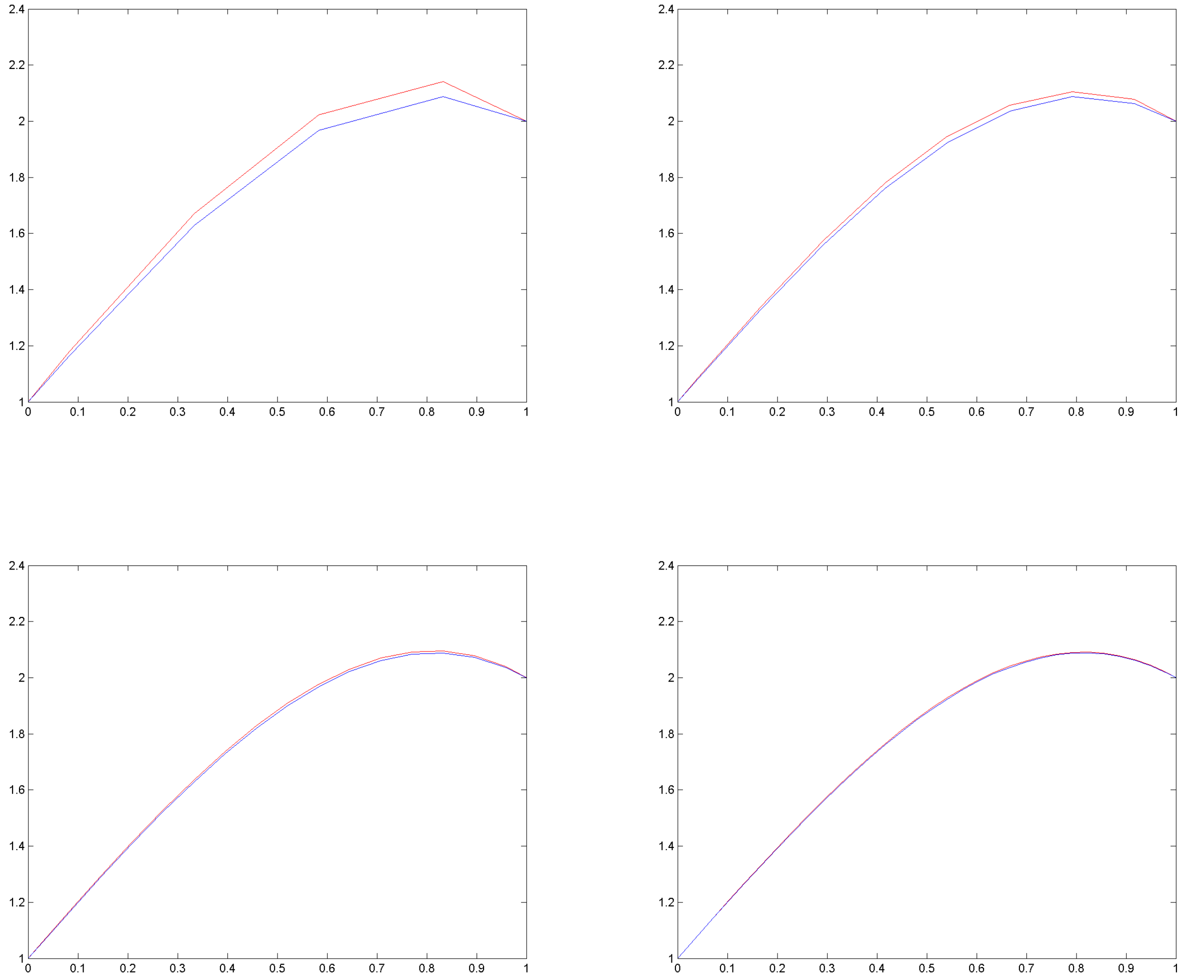

Value of solution
Omega

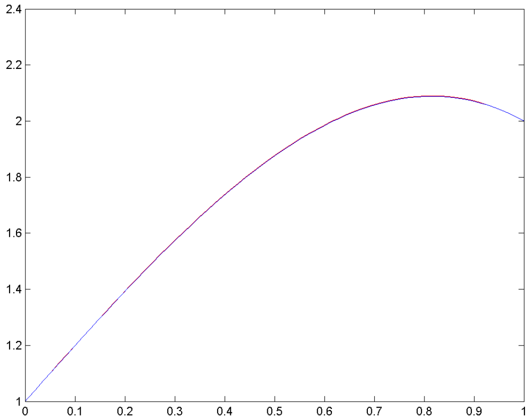

Value of solution
Omega


Error
$L^2$ Norm
$H^1$ norm
3/2x
2x
Errors
Number of element N

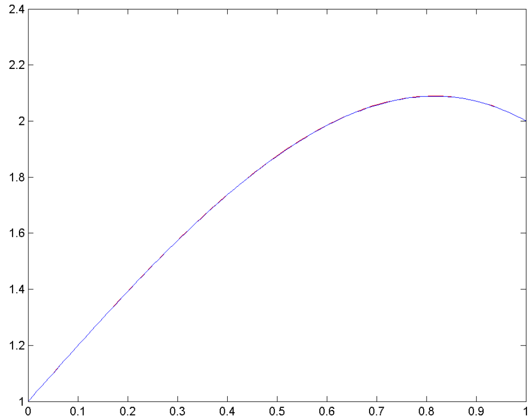
b. We set up with the following exact solution u and function f

$$\begin{cases} u\left(x\right) = -x^3 + 2x + 1 \\ f\left(x\right) = 6x \\ u(0) = 1, u(1) = 2 \end{cases}$$





2.4
2.2
2
1.8
1.6
1.4
1.2
1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

2.4
2.2
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1.4
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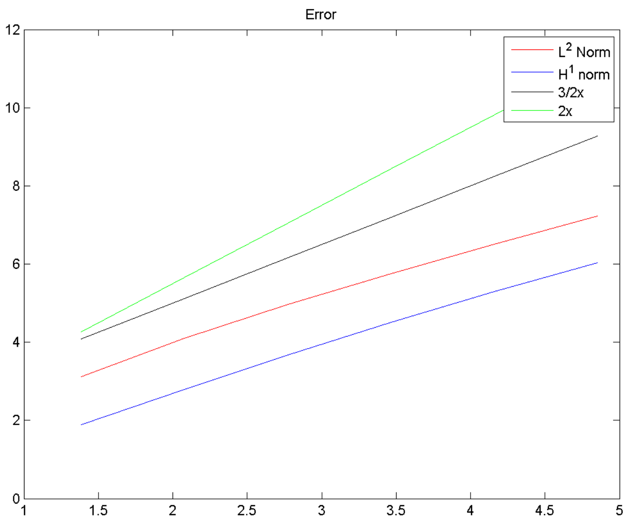
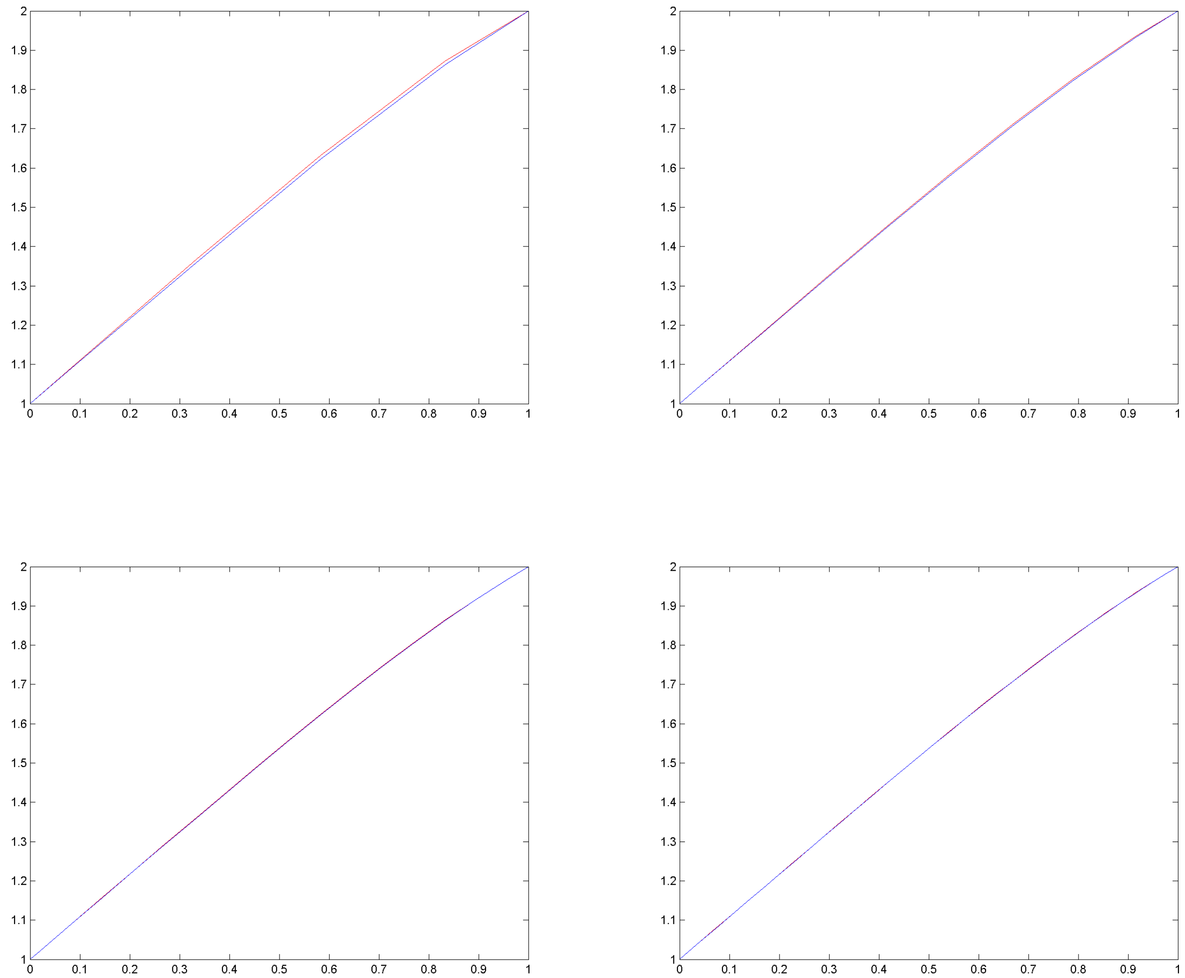
1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

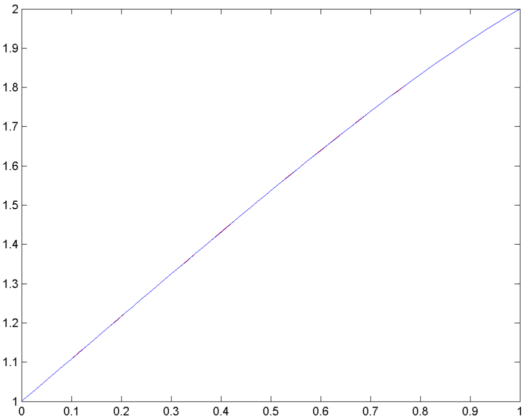



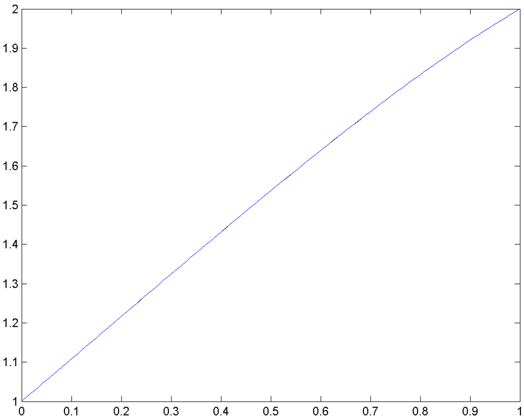
Error
L2 Norm
H1 norm
3/2x
2x
12
10
8
6
4
2
0
1 1.5 2 2.5 3 3.5 4 4.5 5

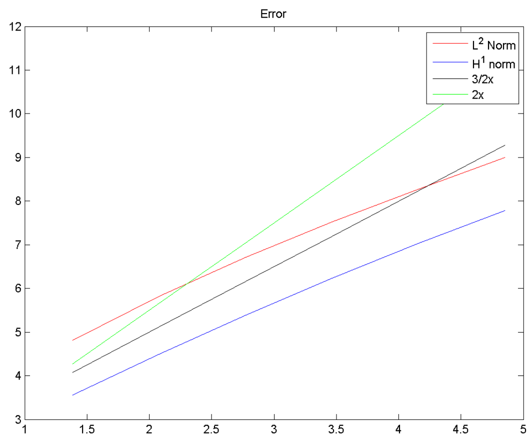
With the following exact solution u and function f

$$\begin{cases} u\left(x\right) = -\dfrac{1}{12}x^4 + \dfrac{13}{12}x + 1 \\ f\left(x\right) = x^2 \\ u(0) = 1, u(1) = 2 \end{cases}$$








Error
$L^2$ Norm
$H^1$ norm
3/2x
2x

c. To approximate the mean-value of f over $T_i$, it means that $f_{ave} = \dfrac{1}{T_i}\displaystyle\int_{T_i} f(x)\,dx$. We have to approximate $\displaystyle\int_{T_i} f(x)\,dx$. We can use the methods like midpoint, trapezoidal, etc.

Midpoint rule:

$$\int_{T_i} f(x)\,dx = \int_{x_{i-1/2}}^{x_{i+1/2}} f\left(\frac{x_{i-1/2}+x_{i+1/2}}{2}\right)dx = \left(x_{i+1/2}-x_{i-1/2}\right) f\left(\frac{x_{i-1/2}+x_{i+1/2}}{2}\right)$$

If $f \in C^2(T_i)$ then

$$\left|\int_{T_i} f(x)\,dx - \left(x_{i+1/2}-x_{i-1/2}\right) f\left(\frac{x_{i-1/2}+x_{i+1/2}}{2}\right)\right| \leqslant \frac{||f^{(2)}||}{24}\left(x_{i+1/2}-x_{i-1/2}\right)^3$$

Trapezoidal rule:

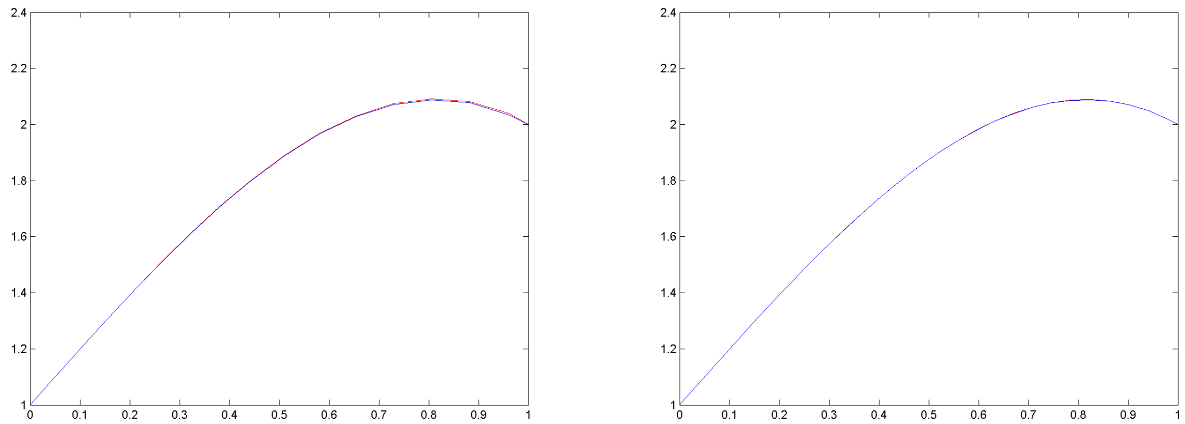
$$\int_{T_i} f(x)\,dx = \frac{x_{i+1/2}-x_{i-1/2}}{2} f\left(x_{i-1/2}\right) + \frac{x_{i+1/2}-x_{i-1/2}}{2} f\left(x_{i+1/2}\right)$$

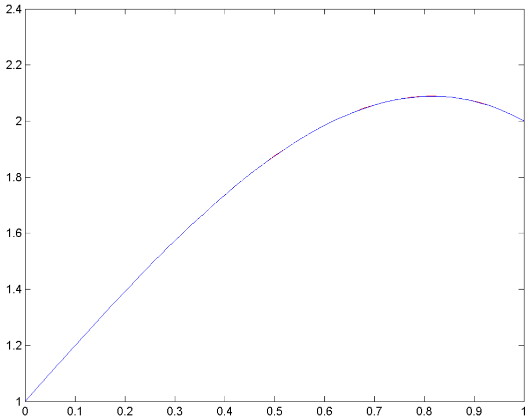
We have

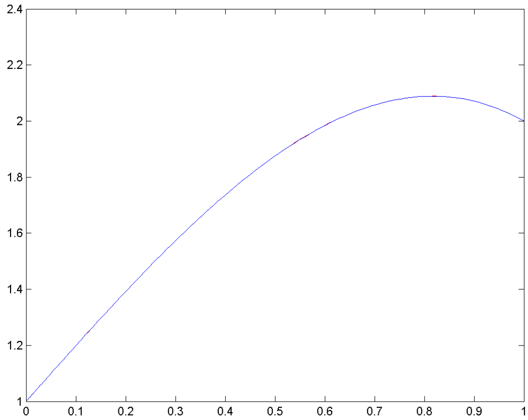
$$\left|\int_{T_i} f(x)\,dx - \frac{x_{i+1/2}-x_{i-1/2}}{2} f\left(x_{i-1/2}\right) - \frac{x_{i+1/2}-x_{i-1/2}}{2} f\left(x_{i+1/2}\right)\right| \leqslant \frac{\left(x_{i+1/2}-x_{i-1/2}\right)^2}{2}||f^{(2)}||$$

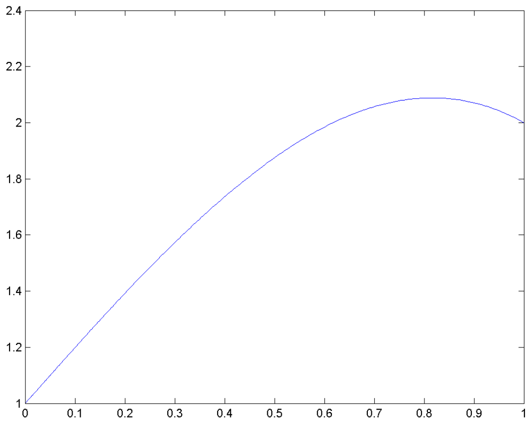
d. We set up with the following exact solution u and function f

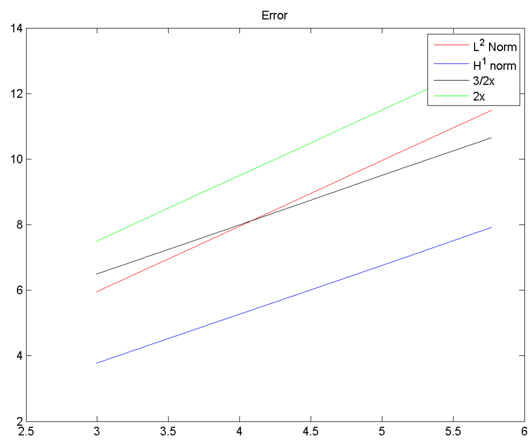
$$\begin{cases} u(x) = -x^3 + 2x + 1 \\ f(x) = 6x \\ u(0) = 1, u(1) = 2 \end{cases}$$





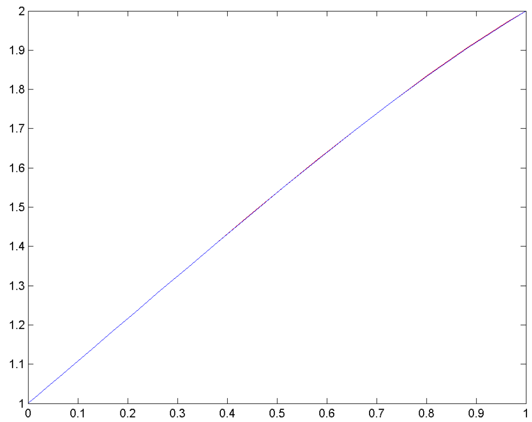


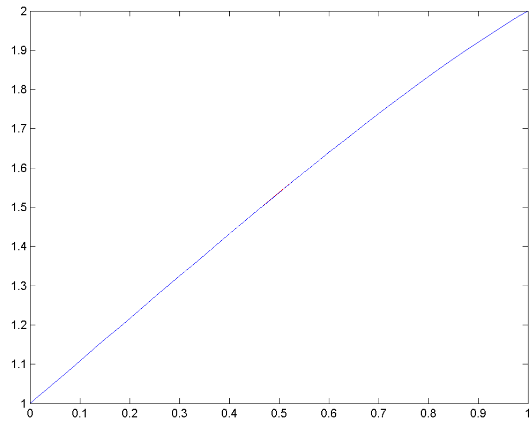


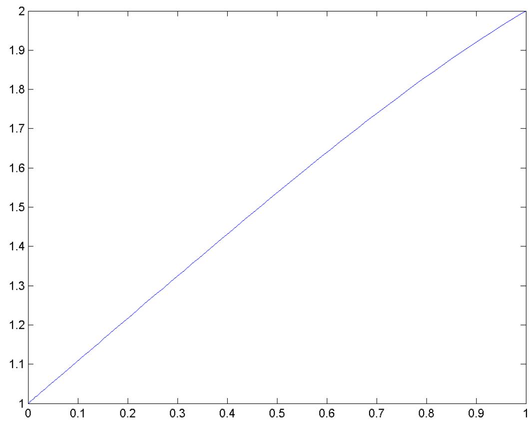

Error
L2 Norm
H1 norm
3/2x
2x

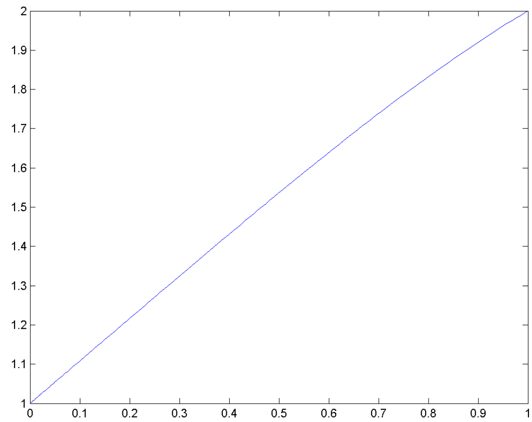
With the following exact solution u and function f

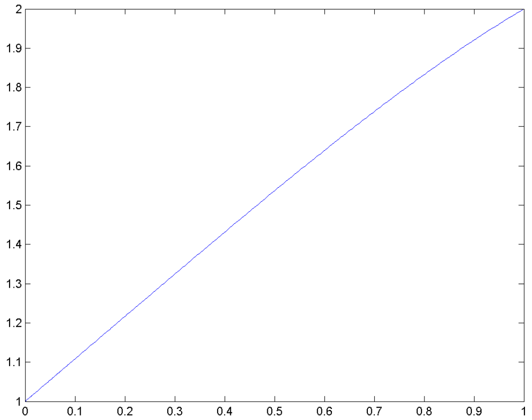
$$\begin{cases} u\left(x\right)=-\dfrac{1}{12}x^{4}+\dfrac{13}{12}x+1 \\ f\left(x\right)=x^{2} \\ u(0)=1,u(1)=2 \end{cases}$$

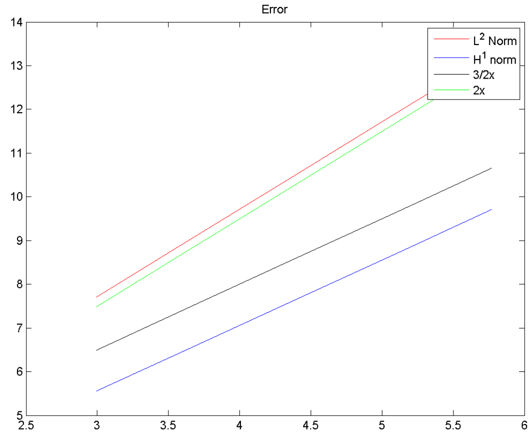







2. We consider the equation with Neumann boundary condition

$$\begin{cases} -u_{xx}(x) = f(x) \, in \Omega \\ u'(0) = u'(1) = 0 \end{cases}$$

We have $N+3$ equations:

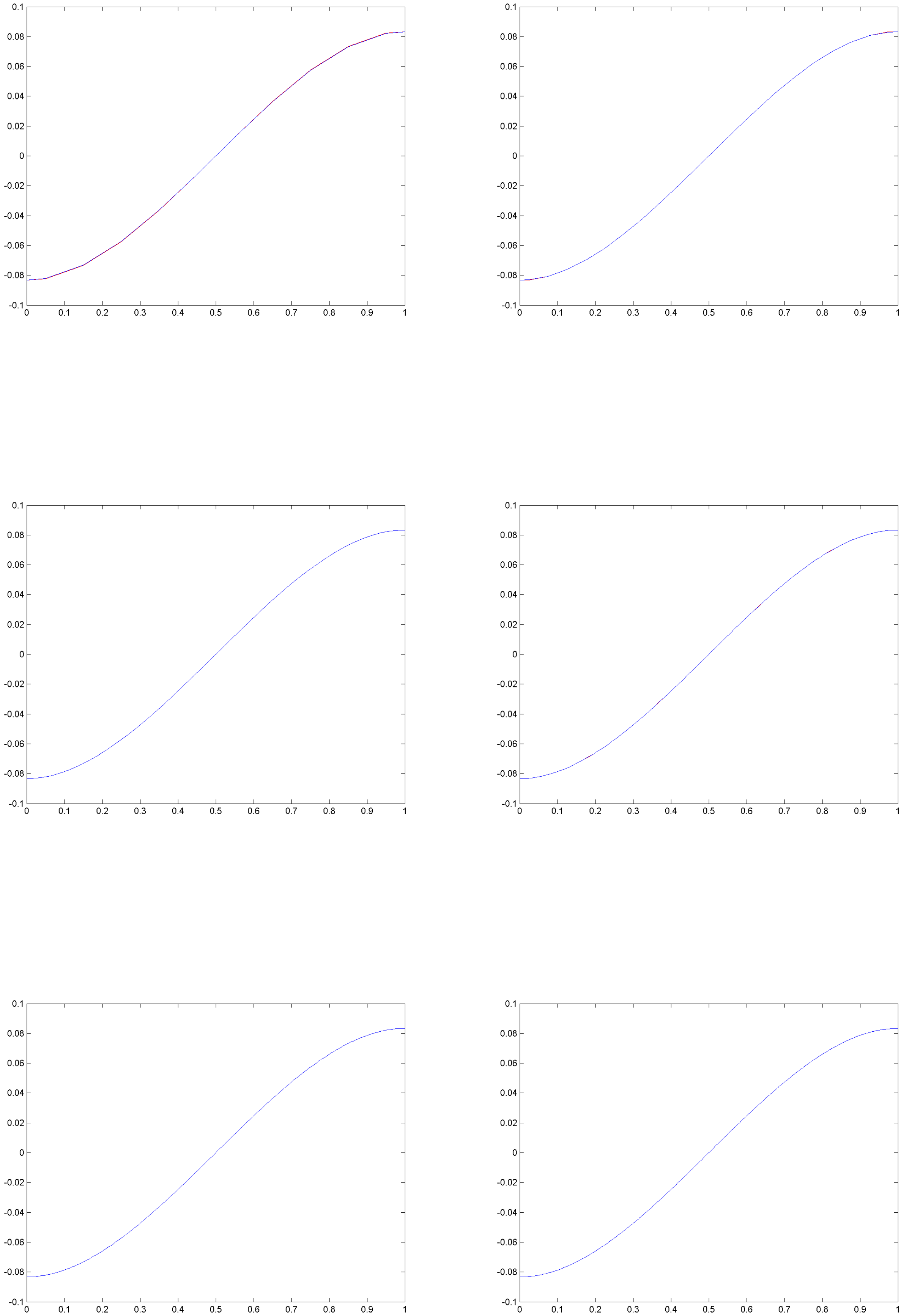
$$\begin{cases} i=1, \alpha_1 u_0 + \beta_1 u_1 + \gamma_1 u_2 & = f_1 \\ i=2, \qquad \alpha_2 u_1 + \beta_2 u_2 + \gamma_2 u_3 & = f_2 \\ i=3, \qquad \alpha_3 u_2 + \beta_3 u_3 + \gamma_3 u_4 & = f_4 \\ \qquad \dots & \\ i=N-1, \qquad \alpha_{N-1} u_{N-2} + \beta_{N-1} u_{N-1} + \gamma_{N-1} u_N & = f_{N-1} \\ i=N, \qquad \alpha_N u_{N-1} + \beta_N u_N + \gamma_N u_{N+1} & = f_N \\ \qquad u_0 & = u_1 \\ \qquad u_N & = u_{N+1} \\ \qquad \sum_{i=1}^{N} |T_i| u_i & = 0 \end{cases}$$

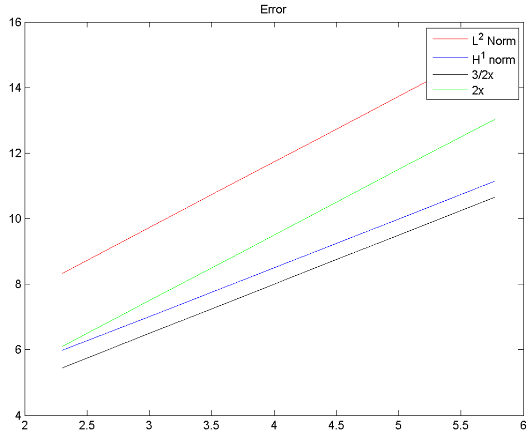
However, the set of equations $u_0 = u_1$ and $u_N = u_{N+1}$ not independent comparing to the other equations of the system. Thus, we have:

$$\begin{cases} i=1, (\alpha_1 + \beta_1) u_1 + \gamma_1 u_2 & = f_1 \\ i=2, \alpha_2 u_1 + \beta_2 u_2 + \gamma_2 u_3 & = f_2 \\ i=3, \qquad \alpha_3 u_2 + \beta_3 u_3 + \gamma_3 u_4 & = f_4 \\ \qquad \dots & \\ i=N-1, \qquad \alpha_{N-1} u_{N-2} + \beta_{N-1} u_{N-1} + \gamma_{N-1} u_N & = f_{N-1} \\ i=N, \qquad \alpha_N u_{N-1} + (\beta_N + \gamma_N) u_N & = f_N \\ \qquad \sum_{i=1}^{N} |T_i| u_i & = 0 \end{cases}$$

With regular grid, we set up with the following exact solution u and function f

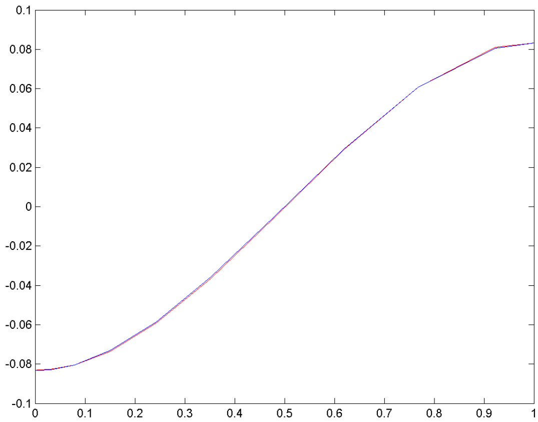
$$\begin{cases} u(x) = -\frac{1}{3} * x^3 + \frac{1}{2} * x^2 - \frac{1}{12} \\ f(x) = 2x - 1 \\ u'(0) = u'(1) = 0 \end{cases}$$

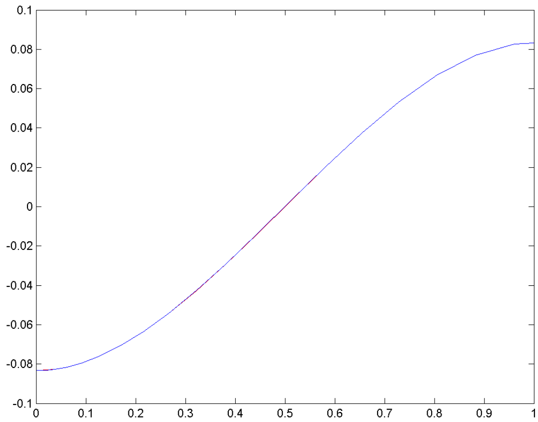

0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

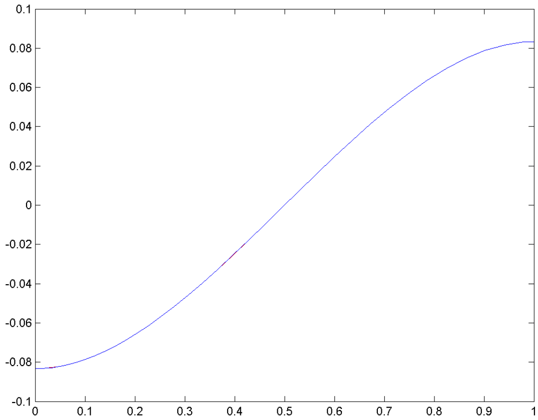


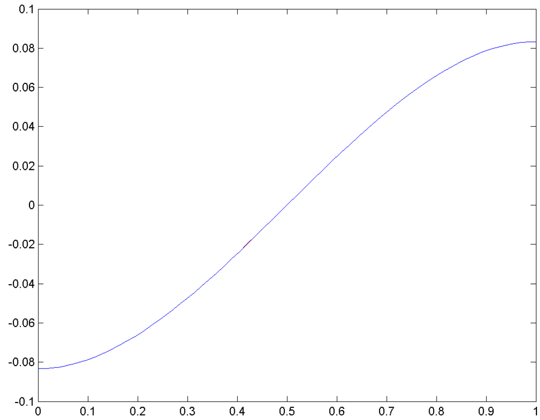

With singular grid, we set up with the following exact solution u and function f

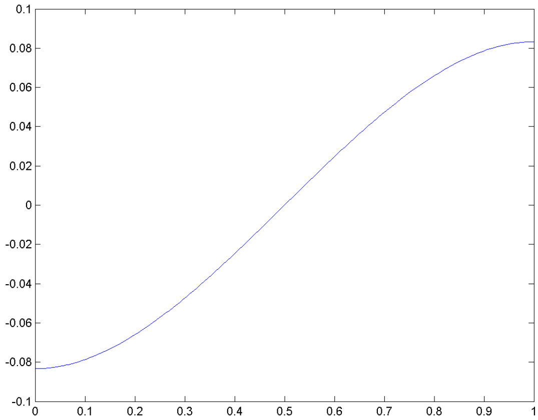
$$\begin{cases} u\left(x\right) = -\dfrac{1}{3} * x^3 + \dfrac{1}{2} * x^2 - \dfrac{1}{12} \\ f\left(x\right) = 2x - 1 \\ u'(0) = u'(1) = 0 \end{cases}$$

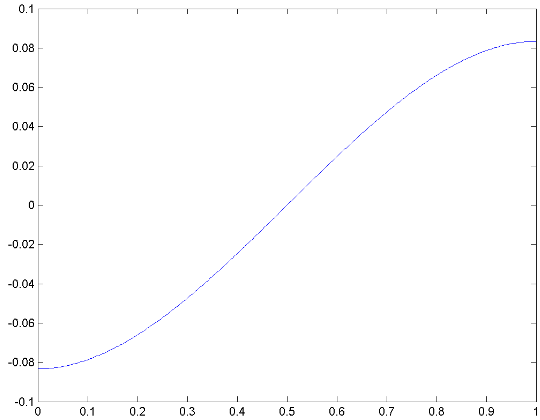


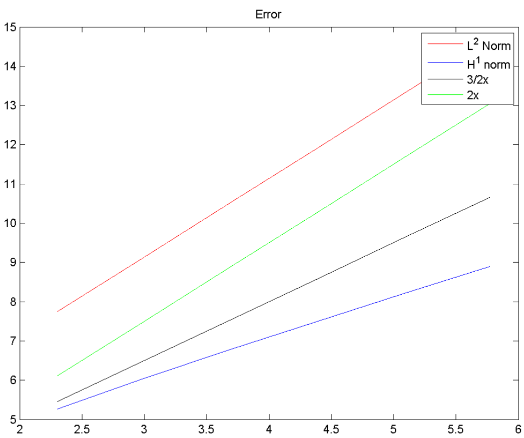











Error
$L^2$ Norm
$H^1$ norm
3/2x
2x