Vietnam National University - Ho Chi Minh City, University of Science, Faculty of Mathematics and Computer Science

FVM: Practical Assignment 3

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0.1 Dirichlet Boundary Condition

Give a $2D$ Poisson problem on $\Omega = (0,1) \times (0,1)$

$$-\Delta u\left(x,y\right) = f\left(x,y\right), \left(x,y\right) \in \Omega \tag{1}$$

Solve equation (1) subject to homogeneous Dirichlet boundary condition

$$u\left(x,y\right) = u_d \quad on \quad \Gamma = \partial\Omega$$

by finite volume method.

a. Case 1: $u_d = 0$.

b. Case 2: $u_d \neq 0$.

c. You can compare if the mesh are retangular and square.

d. You can compare some approximate kinds of f_{ij} on T_{ij} and some positions of control point.

f. You can explain some methods to approximate discrete solution on boundary to get higher convergent order.

0.2 Neumann Boundary Condition

Give a $2D$ Poisson problem on $\Omega = (0,1) \times (0,1)$

$$-\Delta u\left(x,y\right) = f\left(x,y\right), \left(x,y\right) \in \Omega \tag{2}$$

Solve equation (2) subject to homogeneous Dirichlet boundary condition

$$\nabla u \cdot n = 0 \qquad \Gamma = \partial\Omega$$

and

$$\int_\Omega f\left(x,y\right) dxdy = 0$$

by finite volume method.

Solution

Dirichlet Boundary Condition

At cell(i,j) for $i \in [1, N_1]$ and $j \in [1, N_2]$ the discrete equation is written as

$$-a_i u_{i-1,j} - b_i u_{i+1,j} - c_j u_{i,j-1} - d_j u_{i,j+1} + s_{i,j} u_{i,j} = f_{ij} \tag{3}$$

where,

$$\begin{aligned} a_i &= \frac{1}{h_i h_{i-\frac{1}{2}}}, b_i = \frac{1}{h_i h_{i+\frac{1}{2}}} \\ c_j &= \frac{1}{k_j k_{j-\frac{1}{2}}}, d_j = \frac{1}{k_j k_{j+\frac{1}{2}}} \\ s_{i,j} &= a_i + b_i + c_j + d_j \end{aligned}$$

The system is closed with boundary conditions

$$
\begin{aligned}
u_{0,j} = u_{N_1+1,j} = 0, &\quad j \in \overline{1, N_2} \\
u_{i,0} = u_{i,N_2+1} = 0, &\quad i \in \overline{1, N_1}
\end{aligned}
$$

We arrange the discrete unknowns
$(u_{i,j})$, $i = 1, \ldots, N_1$, $j = 1, \ldots, N_2$ in the following form
$u = (u_{1,1}, u_{1,2}, \ldots, u_{1,N_2}; u_{2,1}, u_{2,2}, \ldots, u_{2,N_2}; \ldots; u_{N_1,1}, u_{N_1,2}, \ldots, u_{N_1,N_2})^T$
and
$f = (f_{1,1}, f_{1,2}, \ldots, f_{1,N_2}; f_{2,1}, f_{2,2}, \ldots, f_{2,N_2}; \ldots; f_{N_1,1}, f_{N_1,2}, \ldots, f_{N_1,N_2})^T$
Then the discrete equation is written in the matrix form $Au = f$
where

$$
A = \begin{bmatrix}
A_1 & D_2 & 0 & \ldots & 0 & 0 \\
C_1 & A_2 & D_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & & \vdots & \vdots \\
\vdots & \vdots & \ddots & & \vdots & \vdots \\
0 & 0 & 0 & \ldots & A_{N_1-1} & D_{N_1-1} \\
0 & 0 & 0 & \ldots & C_{N_1} & A_{N_1}
\end{bmatrix}
$$

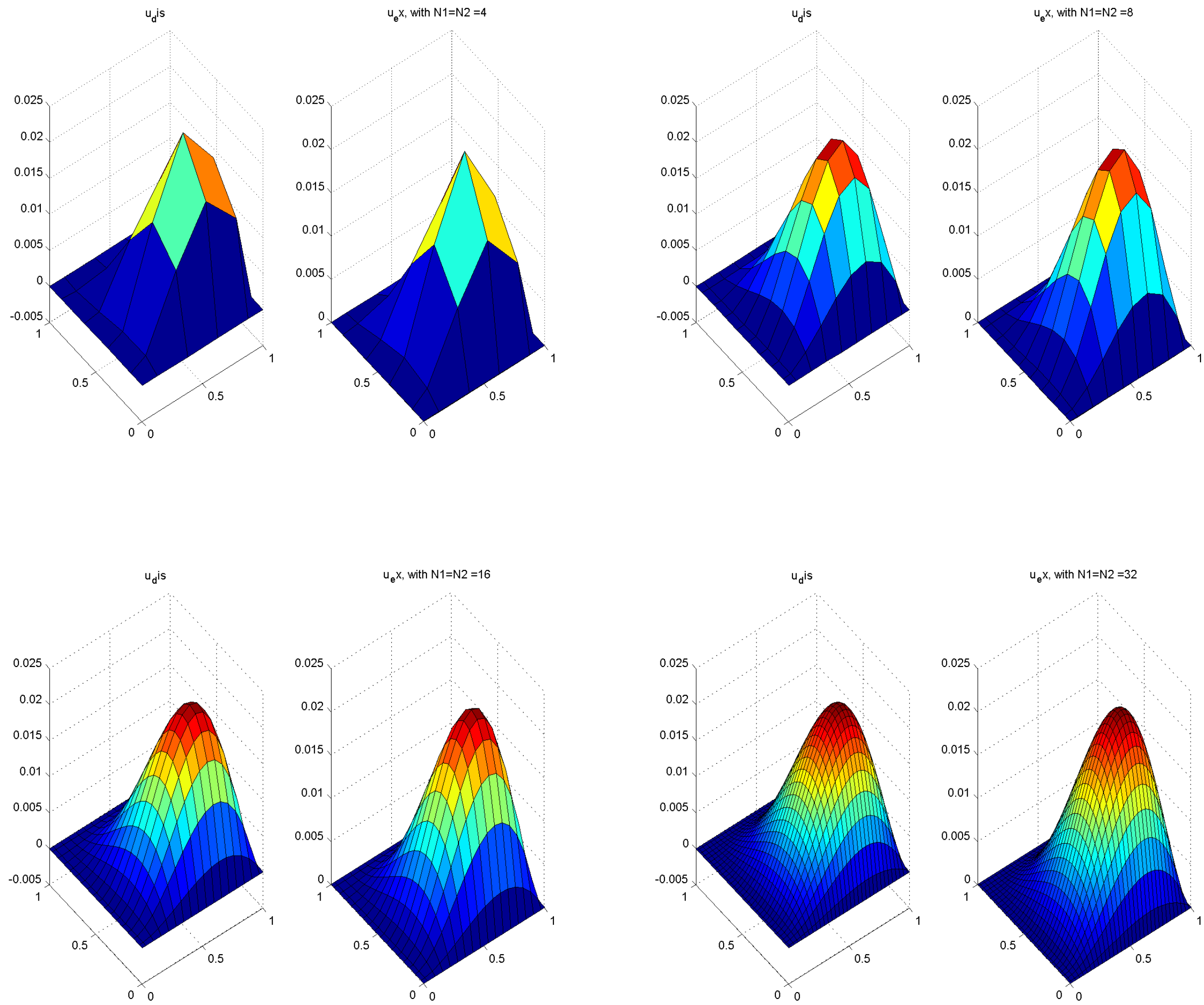
where

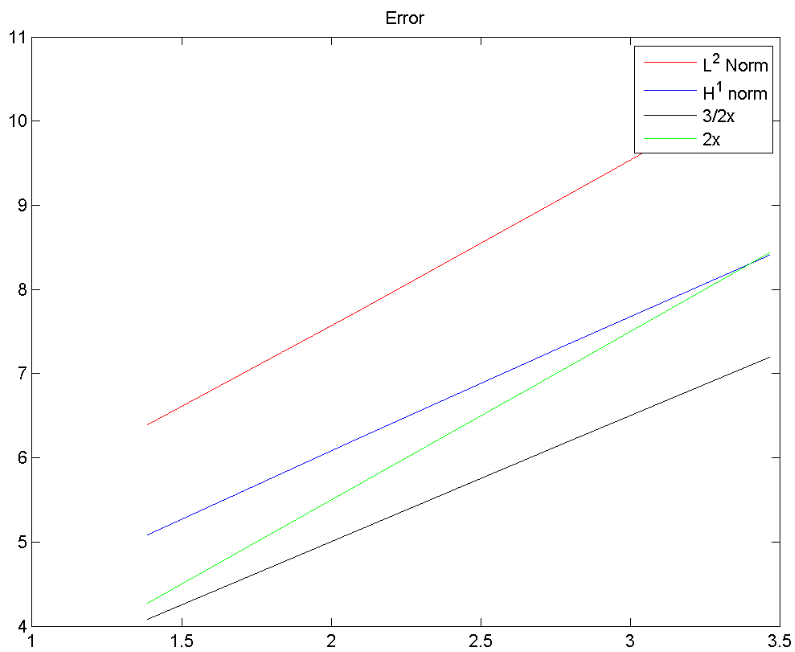
$$
A_i = \begin{bmatrix}
s_{i,1} & -d_1 & 0 & \ldots & 0 & 0 \\
-c_2 & s_{i,2} & -d_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & & \vdots & \vdots \\
\vdots & \vdots & \ddots & & \vdots & \vdots \\
0 & 0 & 0 & \ldots & s_{i,N_2-1} & -d_{N_1-1} \\
0 & 0 & 0 & \ldots & -c_{N_1} & s_{i,N2}
\end{bmatrix}
$$

$$
C_i = \begin{bmatrix}
-a_i & 0 & \ldots & 0 \\
0 & -a_i & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & -a_i
\end{bmatrix}, \qquad
D_i = \begin{bmatrix}
-b_i & 0 & \ldots & 0 \\
0 & -b_i & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & -b_i
\end{bmatrix}
$$

a. We set up with the following exact solution u and function f

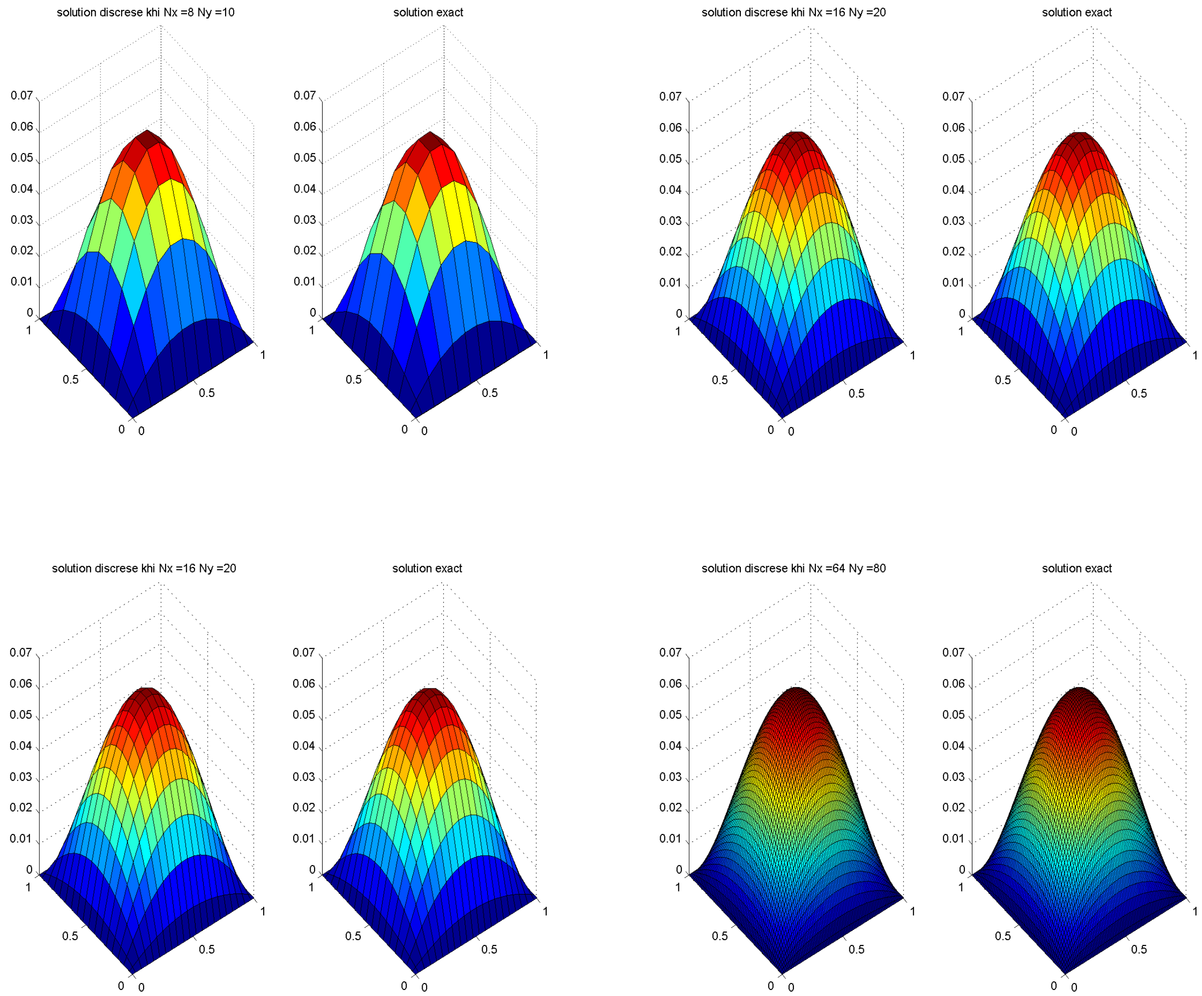
$$
\begin{cases}
u\,(x, y) = x^2(1-x)y(1-y)^2 \\
f\,(x, y) = (6y-4)x^2(x-1) + 2y(1-y)^2(3x-1)
\end{cases}
$$

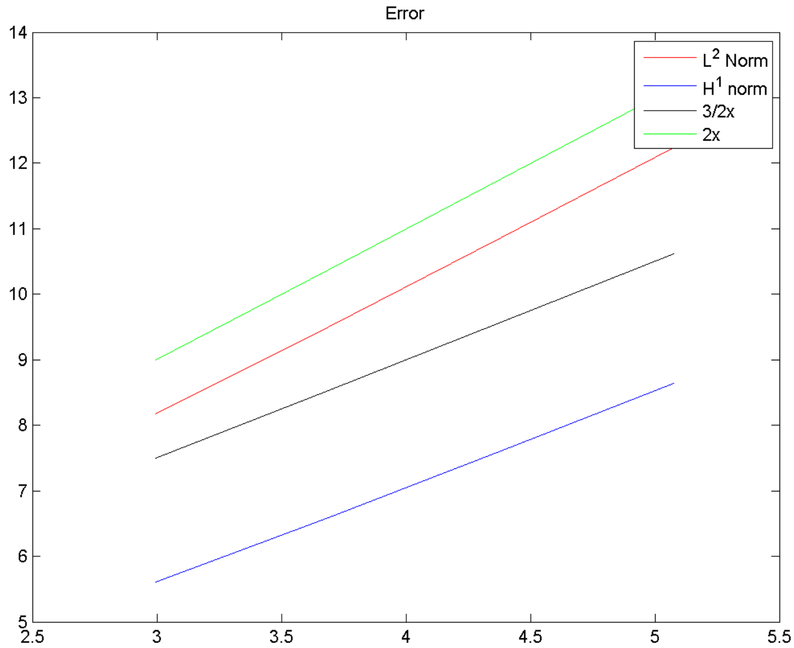

u_dis
u_ex, with N1=N2 =4
u_dis
u_ex, with N1=N2 =8
u_dis
u_ex, with N1=N2 =16
u_dis
u_ex, with N1=N2 =32


Error
L^2 Norm
H^1 norm
3/2x
2x

We set up with the following exact solution u and function f

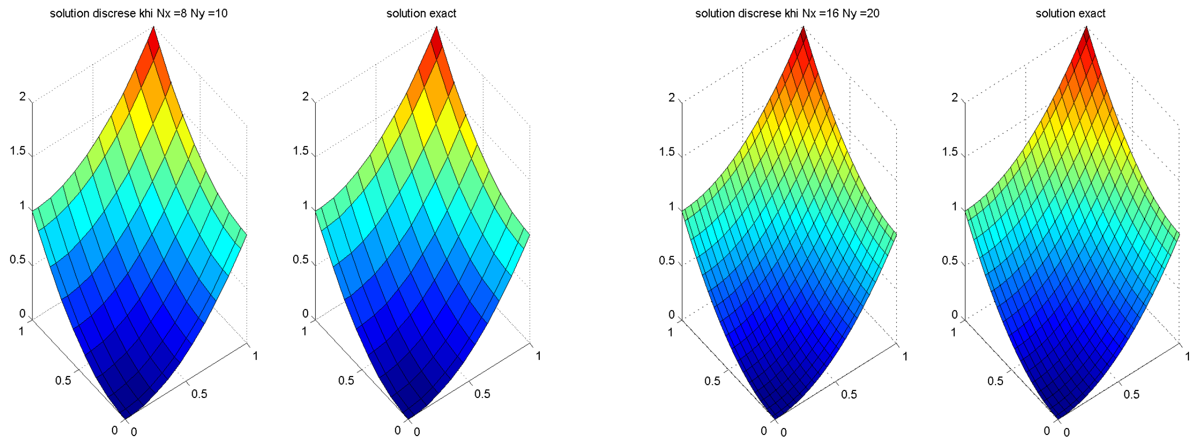
$$\begin{cases} u\,(x,y) = x(1-x)y(1-y) \\ f\,(x,y) = -2x(x-1) - 2y(y-1) \end{cases}$$

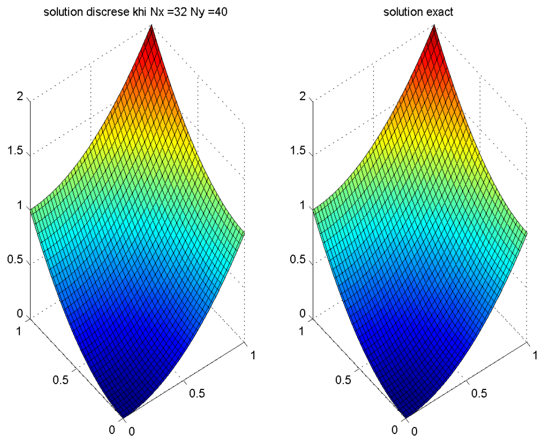


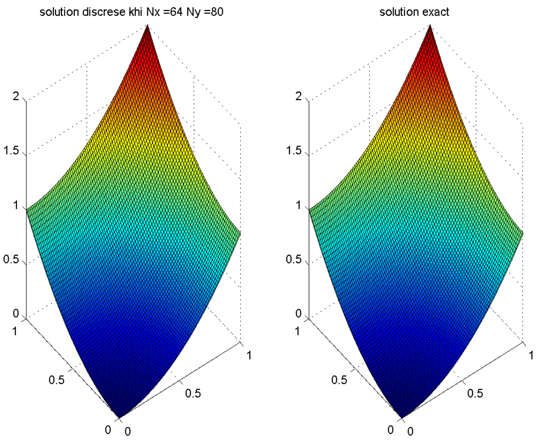


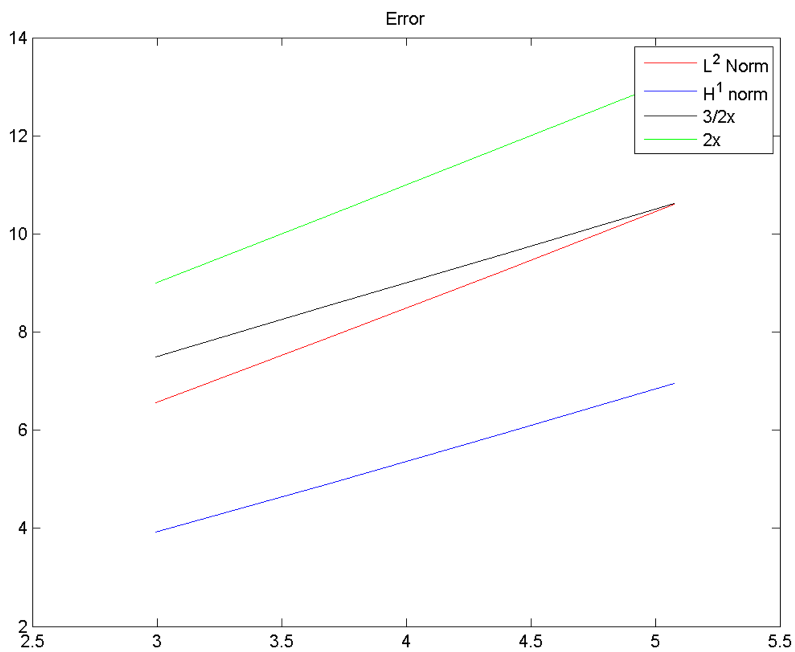
b. We set up with the following exact solution u and function f

$$\begin{cases} u(x,y) = x^2 + y^2 \\ f(x,y) = -4 \end{cases}$$



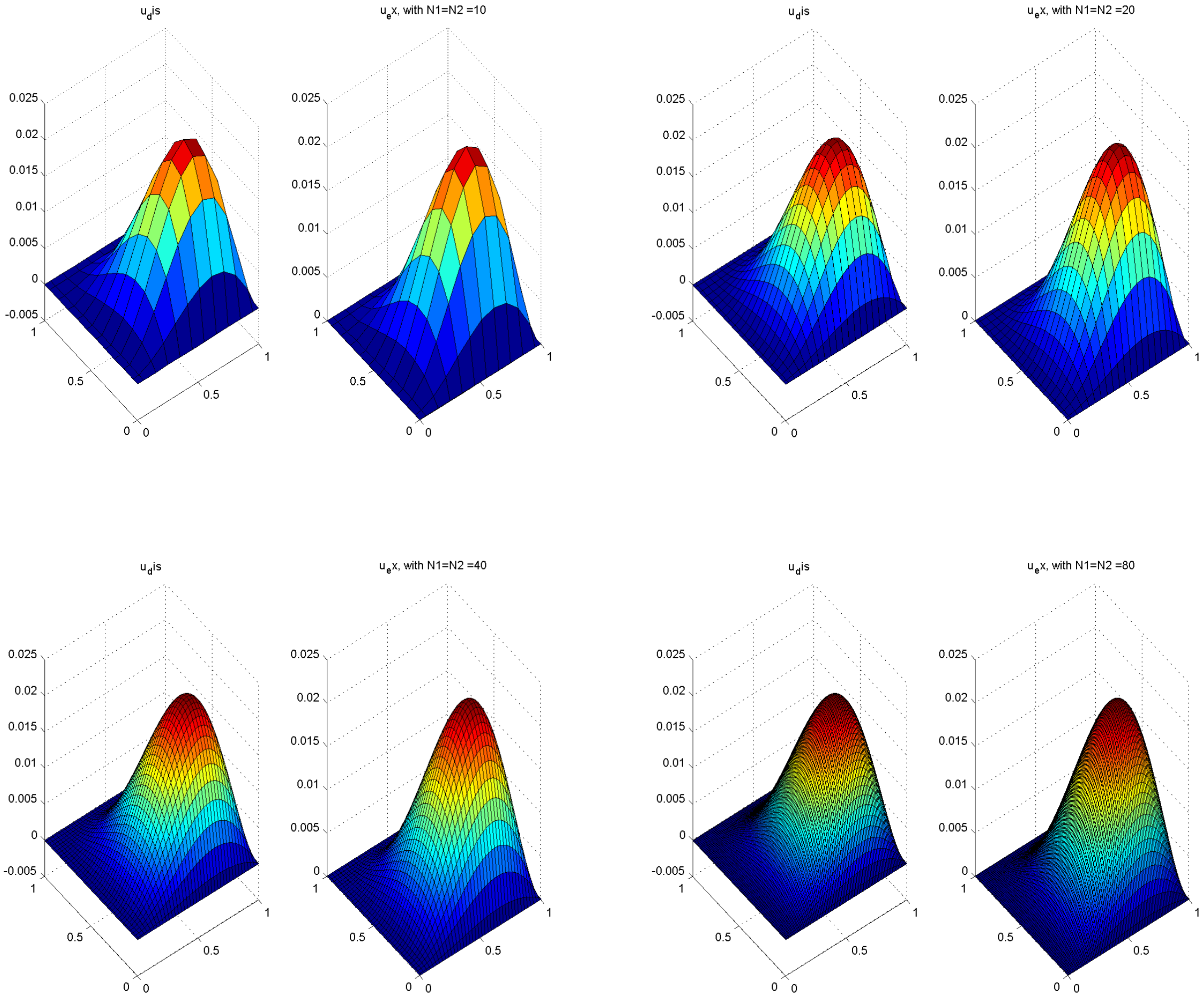

solution discrese khi Nx =32 Ny =40
solution exact

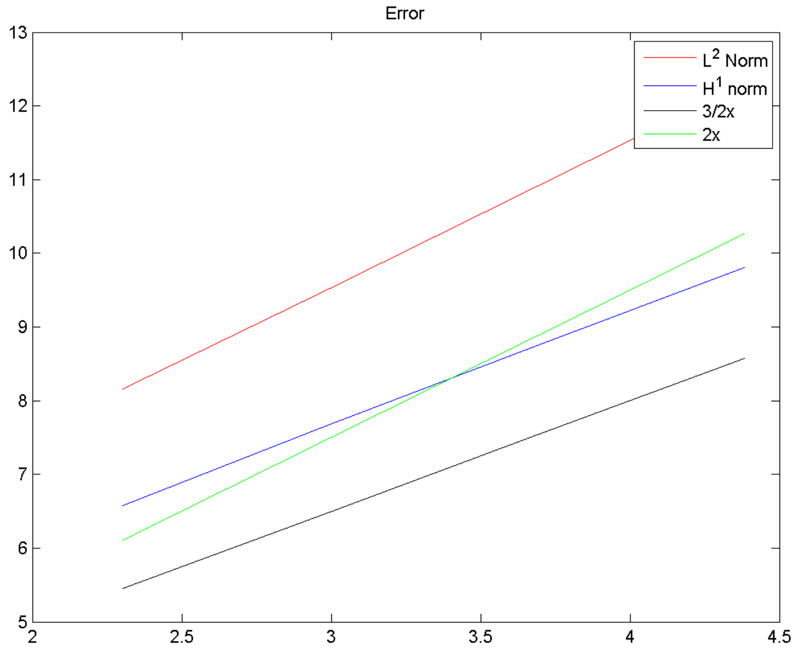

solution discrese khi Nx =64 Ny =80
solution exact


Error
L2 Norm
H1 norm
3/2x
2x

c. With the mesh is square:

$$\begin{cases} u\,(x, y) = x^2(1 - x)y(1 - y)^2 \\ f\,(x, y) = (6y - 4)x^2(x - 1) + 2y(1 - y)^2(3x - 1) \end{cases}$$





d. With greater N, the solution will be more exact, or the solutions are found by finite volume method convergence the exactly solution.

Neumann Boundary Condition

Discrete similar problem for the case of homogeneous dirichlet boundary, we have:

At cell(i,j) for $i \in [1, N_1]$ and $j \in [1, N_2]$ the discrete equation is written as

$$-a_i u_{i-1,j} - b_i u_{i+1,j} - c_j u_{i,j-1} - d_j u_{i,j+1} + s_{i,j} u_{i,j} = f_{ij} \tag{4}$$

where,

$$\begin{aligned} a_i &= \frac{1}{h_i h_{i-\frac{1}{2}}}, b_i = \frac{1}{h_i h_{i+\frac{1}{2}}} \\ c_j &= \frac{1}{k_j k_{j-\frac{1}{2}}}, d_j = \frac{1}{k_j k_{j+\frac{1}{2}}} \\ s_{i,j} &= a_i + b_i + c_j + d_j \end{aligned}$$

The system is closed with boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial x}(0, y) &= \frac{\partial u}{\partial x}(1, y) = 0 \\ \frac{\partial u}{\partial y}(x, 0) &= \frac{\partial u}{\partial y}(x, 1) = 0 \end{aligned}$$

We arrange the discrete unknowns

$(u_{i,j})$, $i = 1, \ldots, N_1$, $j = 1, \ldots, N_2$ in the following form

$u = (u_{1,1}, u_{1,2}, \ldots, u_{1,N_2}; u_{2,1}, u_{2,2}, \ldots, u_{2,N_2}; \ldots; u_{N_1,1}, u_{N_1,2}, \ldots, u_{N_1,N_2})^T$

and

$f = (f_{1,1}, f_{1,2}, \ldots, f_{1,N_2}; f_{2,1}, f_{2,2}, \ldots, f_{2,N_2}; \ldots; f_{N_1,1}, f_{N_1,2}, \ldots, f_{N_1,N_2})^T$

Then the discrete equation is written in the matrix form $Au = f$ where

$$A = \begin{bmatrix} A_1 & D_2 & 0 & \dots & 0 & 0 \\ C_1 & A_2 & D_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A_{N_1-1} & D_{N_1-1} \\ 0 & 0 & 0 & \dots & C_{N_1} & A_{N_1} \end{bmatrix}$$

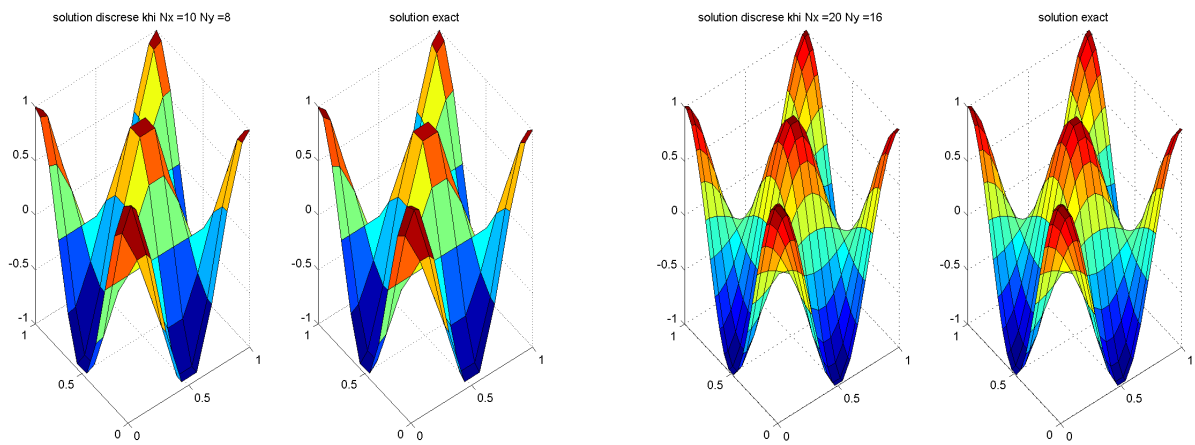
where

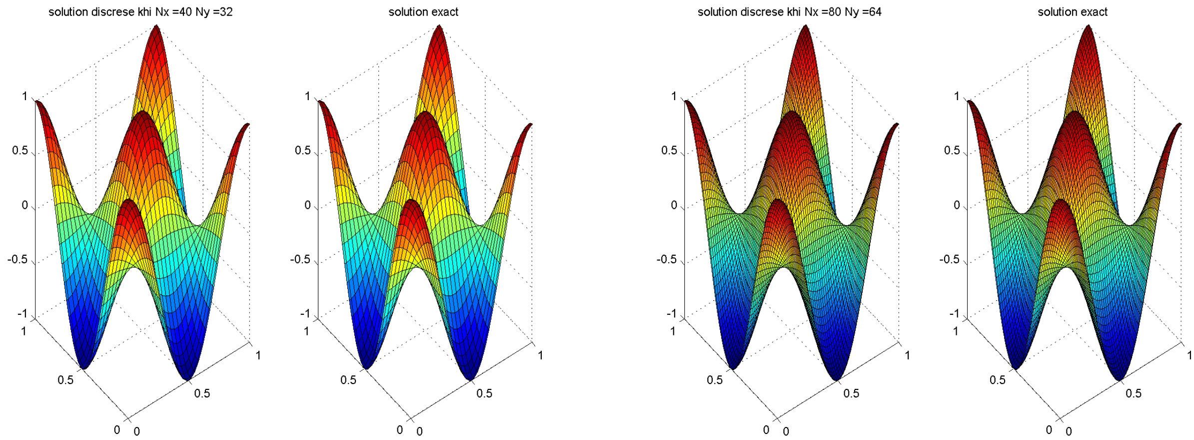
$$A_i = \begin{bmatrix} b_1 + d_1 & -d_1 & 0 & \dots & 0 & 0 \\ -c_2 & s_{i,2} & -d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & s_{i,N_2-1} & -d_{N_1-1} \\ 0 & 0 & 0 & \dots & -c_{N_1} & a_{N_1} + c_{N_2} \end{bmatrix}$$

$$C_i = \begin{bmatrix} -a_i & 0 & \dots & 0 \\ 0 & -a_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -a_i \end{bmatrix}, \qquad D_i = \begin{bmatrix} -b_i & 0 & \dots & 0 \\ 0 & -b_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -b_i \end{bmatrix}$$

We set up with the following exact solution u and function f

$$\begin{cases} u\,(x,y) = cos(2\pi x)cos(2\pi y) \\ f\,(x,y) = 4\pi^2 cos(2\pi x)cos(2\pi y) + 4\pi^2 cos(2\pi x)cos(2\pi y) \end{cases}$$

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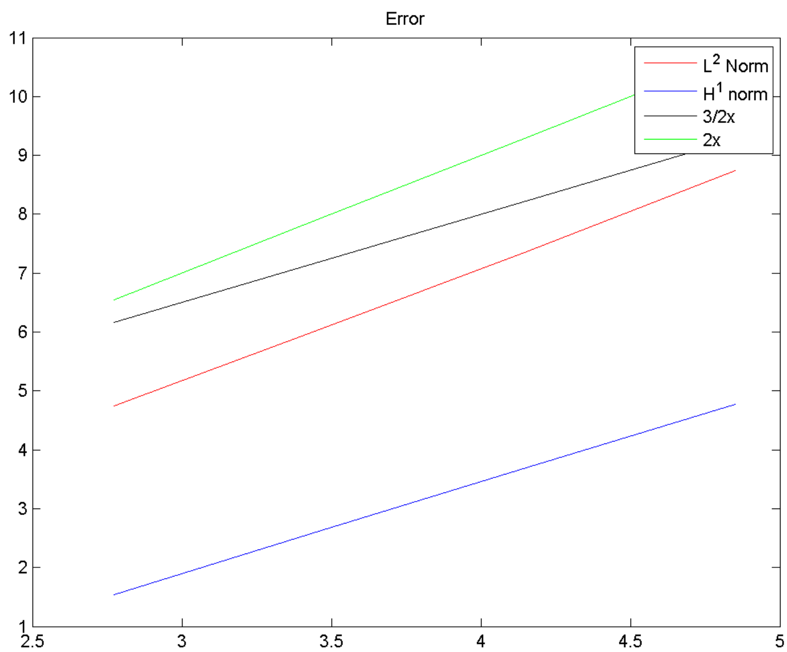
solution exact
solution discrese khi Nx =80 Ny =64
solution exact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3/2x
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