Vietnam National University - Ho Chi Minh City, University of Science, Faculty of Mathematics and Computer Science

FVM: Practical Assignment 4: Heat equations

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Parabolic equation 1D

Consider the diffusion equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad a < x < b$$

where α is the diffusion coefficient. The intial condition is

$$u(x, 0) = u_0(x)$$

The boundary condition is

$$u(a, t) = \phi_a(t), \quad u(b, t) = \phi_b(t)$$

or

$$\frac{\partial u}{\partial x}(a, t) = \psi_a(t), \quad \frac{\partial u}{\partial x}(b, t) = \psi_b(t)$$

Example: $u_t = \frac{1}{16} u_{xx}$, $x \in (0, 1), t \in (0, T)$
Initial condition: $u_0(x) = \sin(2\pi x)$
Boundary condition: $u(0, t) = u(1, t) = 0$
Exact solution: $u(x, t) = e^{-\frac{1}{4}\pi^2 t} \sin(2\pi x)$
We implement and compare the following methods
1.1 Explicit method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}$$

Accuracy: $O\left(\Delta t, h^2\right)$.
Fourier condition: $0 < \frac{\alpha \Delta t}{h^2} \leq \frac{1}{2}$
1.2 Explicit method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}$$

Unconditionally stable.
1.3 Crank-Nicolson method (1947)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2} \left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} \right)$$

Accuracy: $O\left((\Delta t)^2, h^2\right)$.
1.4 Generalization

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \left(\theta \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + (1 - \theta) \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} \right)$$

1.5 **Question:** Compare previous method? (accuracy, stability, computing time).

Solution

Discrete problem are similar in the slides, we have,

$$\frac{du_i(t)}{dt} - \frac{u_{i-1}(t)}{(x_i - x_{i-1})|T_i|} + \left(\frac{1}{(x_{i+1} - x_i)|T_i|} + \frac{1}{(x_i - x_{i-1})|T_i|}\right) u_i(t) \\ - \frac{u_{i+1}(t)}{(x_{i+1} - x_i)|T_i|} = f_i(t) \quad \forall i \in \overline{1, N}$$

We set, for all $i \in \overline{1, N}$,

$$\begin{aligned} \alpha_i &= \frac{-1}{(x_i - x_{i-1})|T_i|} \\ \beta_i &= \frac{1}{(x_{i+1} - x_i)|T_i|} + \frac{1}{(x_i - x_{i-1})|T_i|} \\ \gamma_i &= \frac{-1}{(x_{i+1} - x_i)|T_i|} \end{aligned}$$

Thus, we get

$$\frac{du_i(t)}{dt} + \alpha_i u_{i-1}(t) + \beta_i u_i(t) + \gamma_i u_{i+1}(t) = f_i(t) \quad \forall i \in \overline{1, N}$$

Linear system for the scheme

$$\begin{cases} \frac{du_1(t)}{dt} + \beta_1 u_1(t) + \gamma_1 u_2(t) & = f_1(t) \\ \frac{du_2(t)}{dt} + \alpha_2 u_1(t) + \beta_2 u_2(t) + \gamma_2 u_3(t) & = f_2(t) \\ \frac{du_3(t)}{dt} + \quad \alpha_3 u_2(t) + \beta_3 u_3(t) + \gamma_3 u_4(t) & = f_3(t) \\ \qquad \cdots\cdots\cdots & \\ \frac{du_{N-1}(t)}{dt} + \quad \alpha_{N-1} u_{N-2}(t) + \beta_{N-1} u_{N-1}(t) + \gamma_{N-1} u_N(t) & = f_{N-1}(t) \\ \frac{du_N(t)}{dt} + \quad \alpha_N u_{N-1}(t) + \beta_N u_N(t) & = f_N(t) \end{cases}$$

If the spacing T_i is uniform, for each $i \in 1, \cdot, N$ there holds

$$\frac{du_i(t)}{dt} = ru_{i+1}^n - 2ru_i^n + ru_{i-1}^n, \quad u_m^n \approx u(x_m, n_k)$$

where $r = k/h^2$. Then we get the linear ODE system

$$\frac{dU(t)}{dt} AU(t) + F(t)$$

where A is a discrete approximation of the differential operator ∂_{xx}^2.

$$A = \begin{bmatrix} r & -2r & 0 & 0 & 0 & 0 \\ r & -2r & r & 0 & 0 & 0 \\ 0 & r & -2r & r & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & r & -2r & r \\ 0 & 0 & 0 & 0 & r & -2r \end{bmatrix}, \quad F = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_{N-1}(t) \\ f_N(t) \end{bmatrix}.$$

The matrix A is tridiagonal and ssymmetric positive definite.

Now, we use the "θ-method"

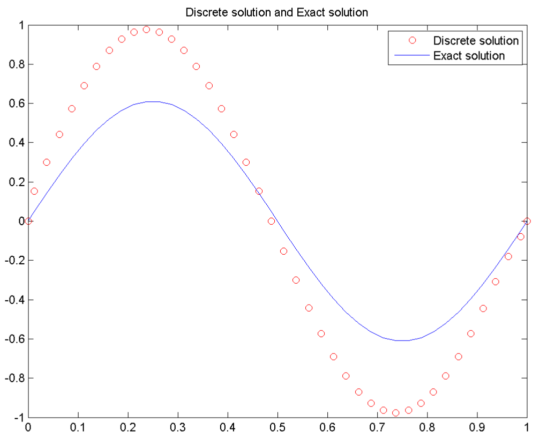
$$U^{n+1} = U^n + kA\left[\theta U^{n+1} + (1-\theta)\,U^n\right] + k\underbrace{\left[\theta F^{n+1} + (1-\theta)\,F^n\right]}_{F_\theta^n}$$

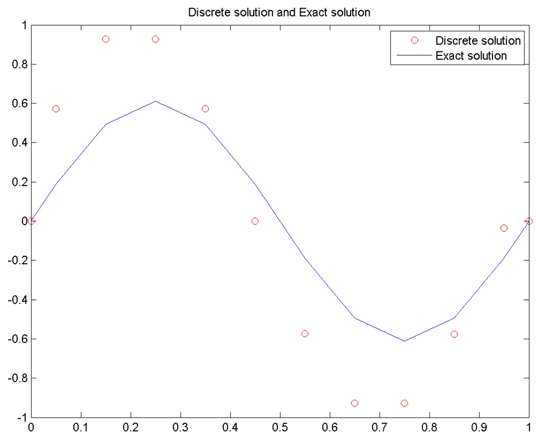
or

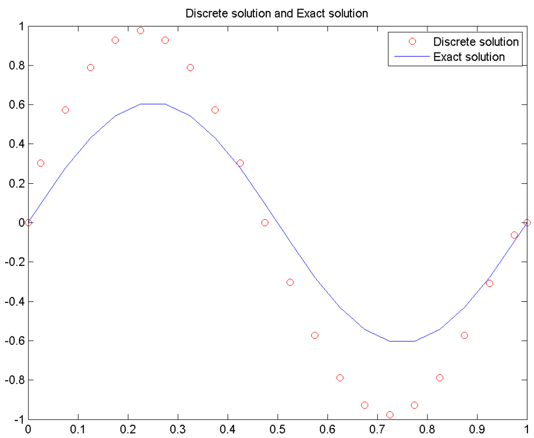
$$(I - \theta k A)\,U^{n+1} = \big(I + (1-\theta)\,kA\big)\,U^n + kF_\theta^n$$

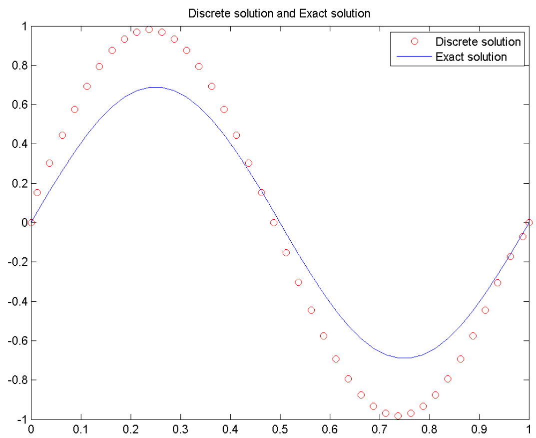
1.1 We set up with the following exact solution u and $\theta = 0$

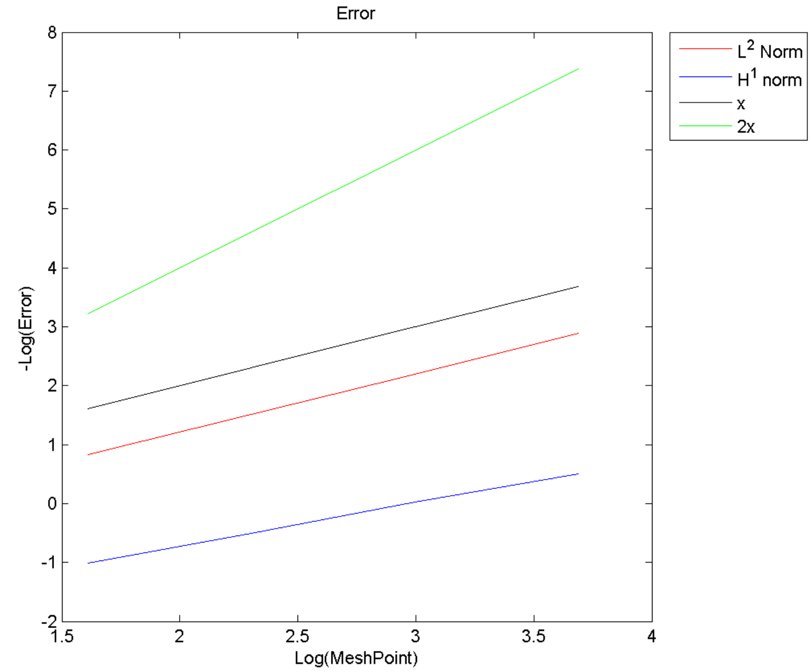
$$\begin{cases} u\,(x) = \sin(2\pi x)e^{-\frac{1}{4}\pi^2 t} \\ \theta = 0(\text{Forward Euler}) \end{cases}$$





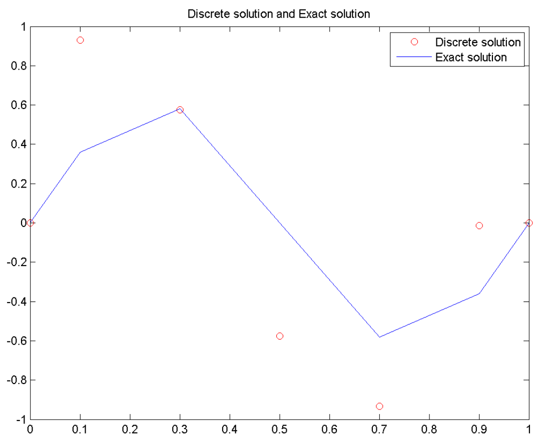


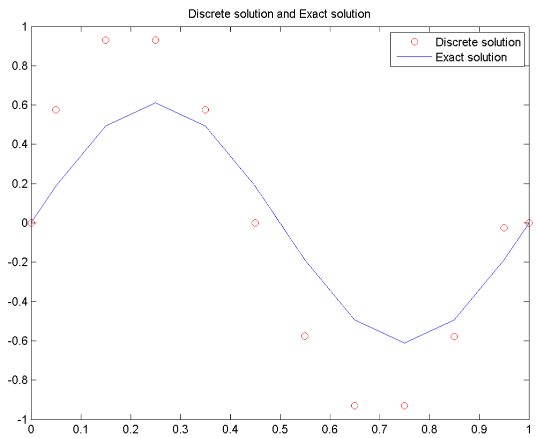


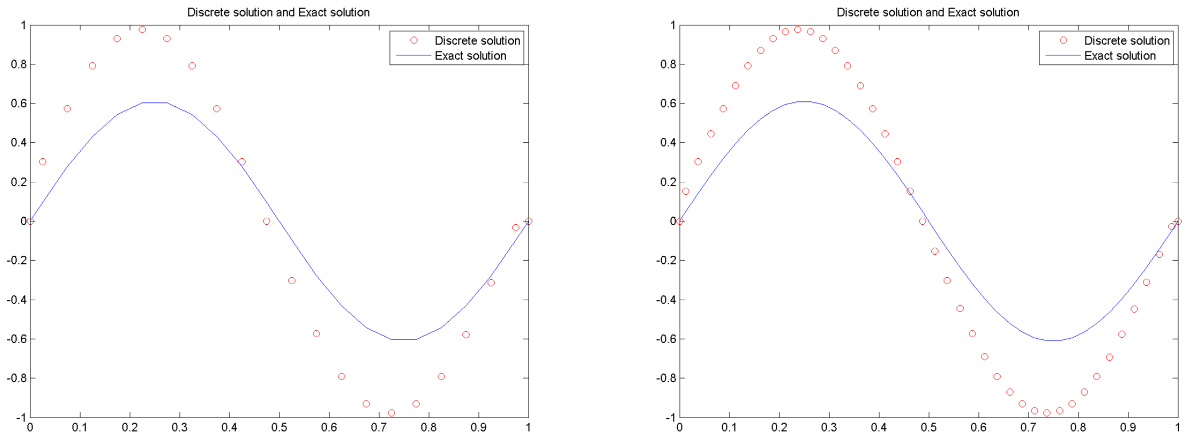


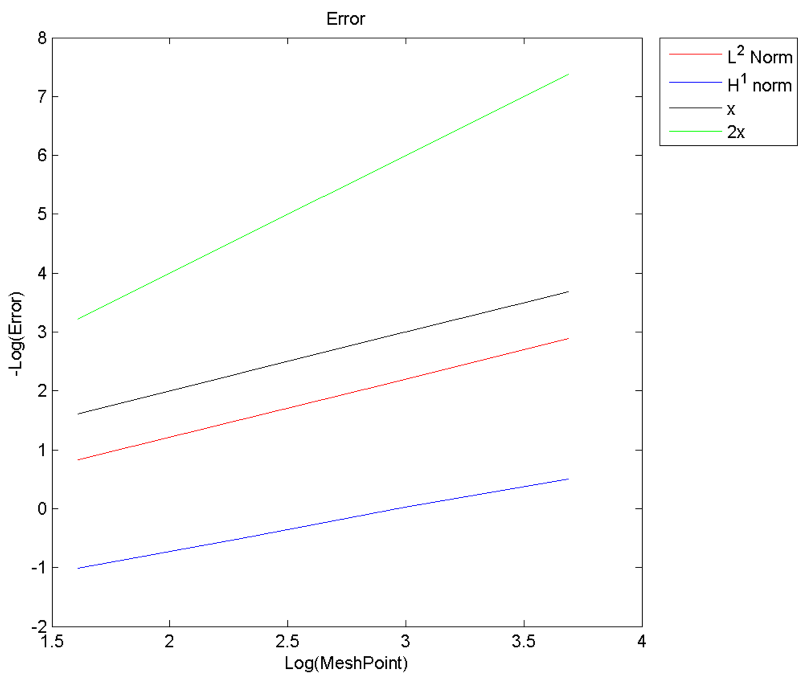
1.2 We set up with the following exact solution u and $\theta = 1$

$$\begin{cases} u\left(x\right) = \sin(2\pi x)e^{-\frac{1}{4}\pi^2 t} \\ \theta = 1(\text{Backward Euler}) \end{cases}$$



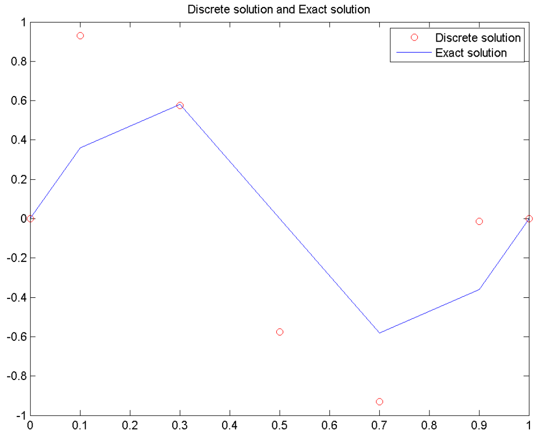


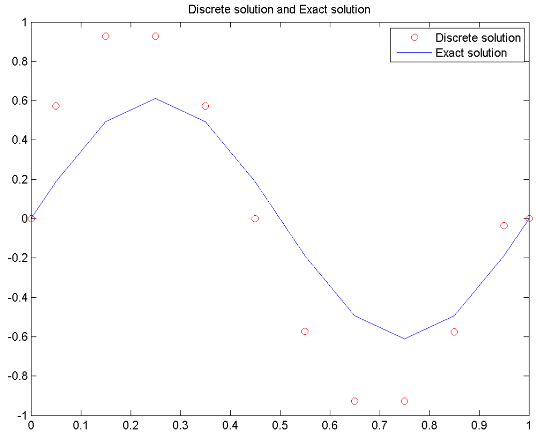

Discrete solution and Exact solution
Discrete solution
Exact solution
Discrete solution and Exact solution
Discrete solution
Exact solution

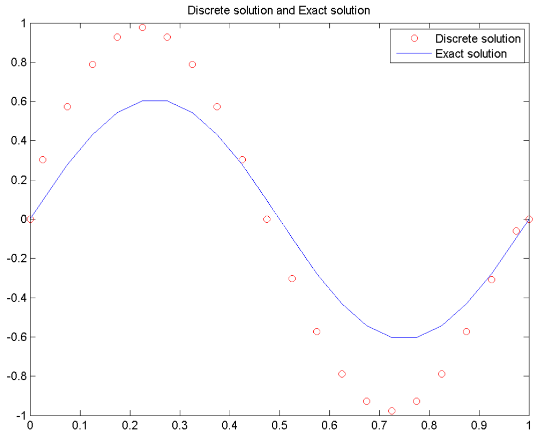

Error
L2 Norm
H1 norm
x
2x
-Log(Error)
Log(MeshPoint)

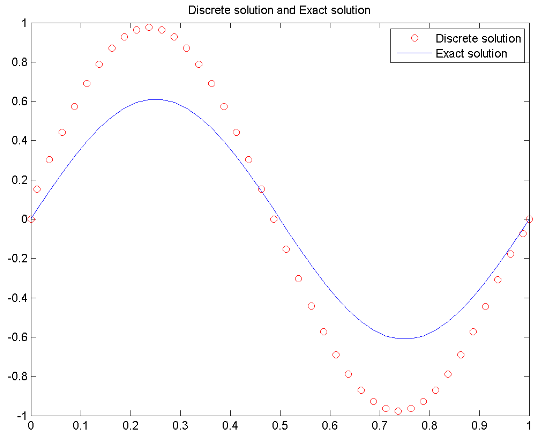
1.3 We set up with the following exact solution u and $\theta = \frac{1}{2}$

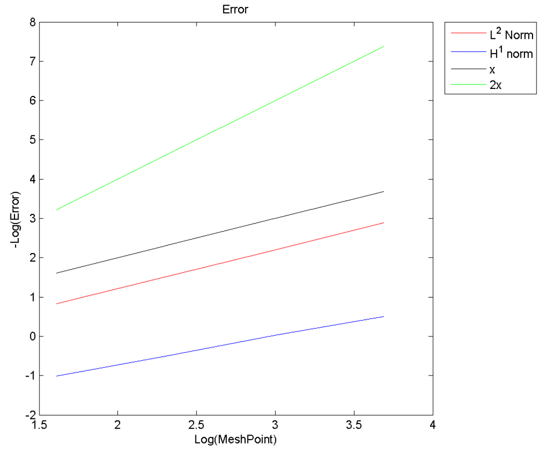
$$\begin{cases} u\,(x) = \sin(2\pi x)e^{-\frac{1}{4}\pi^2 t} \\ \theta = \frac{1}{2}(\text{Crank-Nicolson}) \end{cases}$$





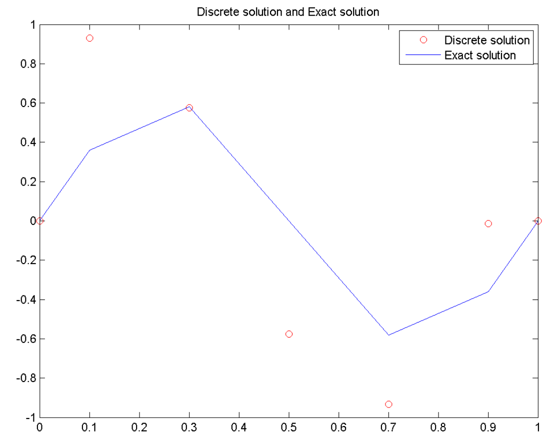


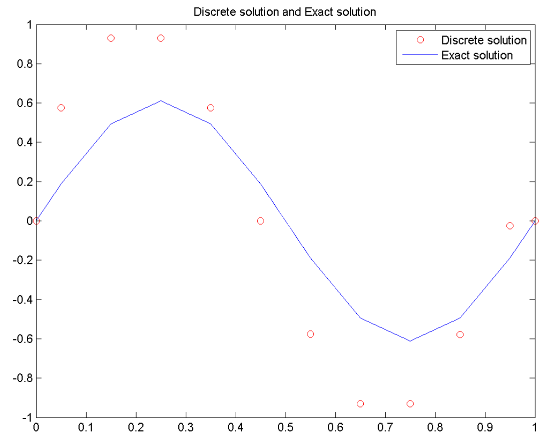


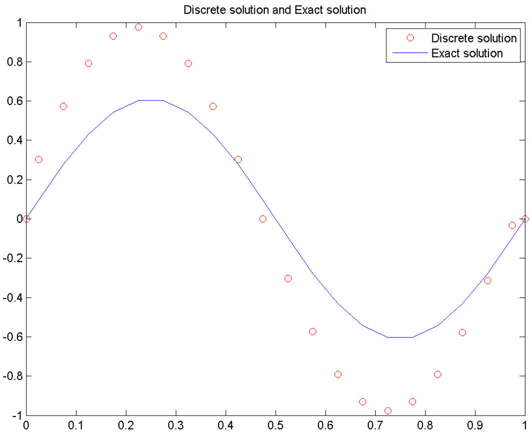


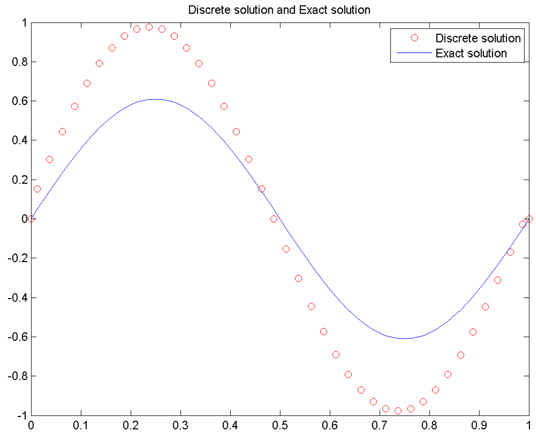
1.4 We set up with the following exact solution u and $\theta = 10$

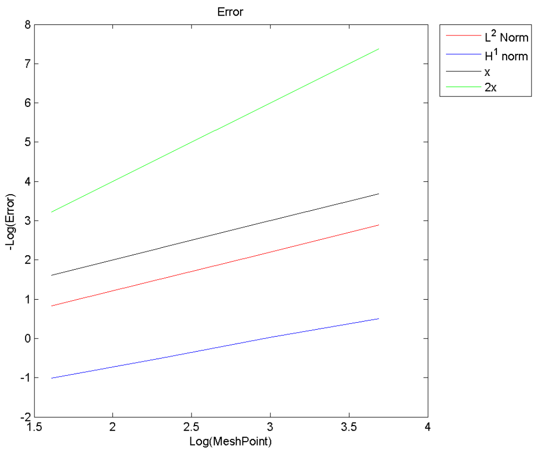
$$\begin{cases} u\left(x\right) = \sin(2\pi x)e^{-\frac{1}{4}\pi^2 t} \\ \theta = 10(\text{Generalization}) \end{cases}$$






Discrete solution and Exact solution
Discrete solution
Exact solution


Discrete solution and Exact solution
Discrete solution
Exact solution


Error
-Log(Error)
Log(MeshPoint)
L2 Norm
H1 norm
x
2x

1.5 The previous methods have the same accuracy, stability and computing time.