To prove this property, we use the definition of the Fourier transform in (4.4.1) and differentiate the series term by term with respect to ω. Thus we obtain

$$\begin{aligned}\frac{dX(\omega)}{d\omega} &= \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right] \\ &= \sum_{n=-\infty}^{\infty} x(n)\frac{d}{d\omega}e^{-j\omega n} \\ &= -j\sum_{n=-\infty}^{\infty} nx(n)e^{-j\omega n}\end{aligned}$$

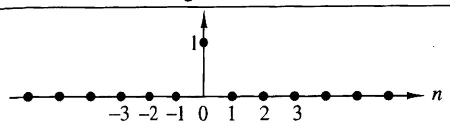
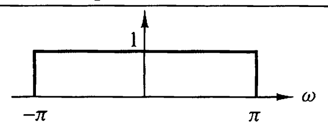
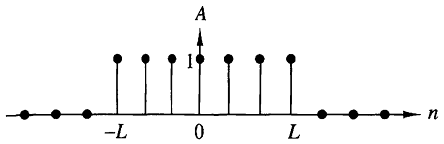
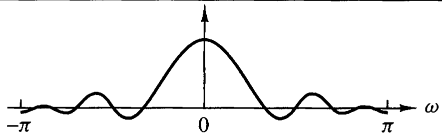
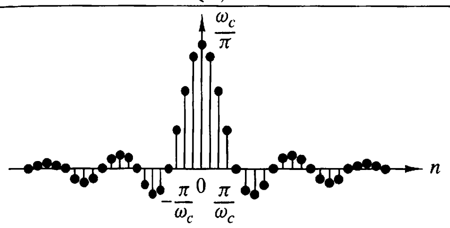
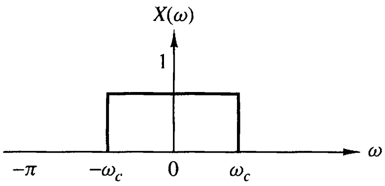
Now we multiply both sides of the equation by j to obtain the desired result in (4.4.58).

The properties derived in this section are summarized in Table 4.5, which serves as a convenient reference. Table 4.6 illustrates some useful Fourier transform pairs that will be encountered in later chapters.

TABLE 4.5 Properties of the Fourier Transform for Discrete-Time Signals

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$ $= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ is real]
Wiener–Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in the frequency domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	

TABLE 4.6 Some Useful Fourier Transform Pairs for Discrete-Time Aperiodic Signals

Signal $x(n)$	Spectrum $X(\omega)$
 $x(n) = \delta(n)$	 $X(\pi) = 1$
 $x(n) = \begin{cases} A, & \lvert n\rvert \le L \\ 0, & \lvert n\rvert > L \end{cases}$	 $X(\omega) = A\dfrac{\sin\left(L+\frac{1}{2}\right)\omega}{\sin\frac{\omega}{2}}$
 $x(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\sin \omega_c n}{\pi n}, & n \ne 0 \end{cases}$	 $X(\omega) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c \\ 0, & \omega_c \le \lvert\omega\rvert \le \pi \end{cases}$
$x(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n > 0 \end{cases}$	$X(\omega) = \frac{1}{1-ae^{-j\omega}}$

4.5 Summary and References

The Fourier series and the Fourier transform are the mathematical tools for analyzing the characteristics of signals in the frequency domain. The Fourier series is appropriate for representing a periodic signal as a weighted sum of harmonically related sinusoidal components, where the weighting coefficients represent the strengths of each of the harmonics, and the magnitude squared of each weighting coefficient represents the power of the corresponding harmonic. As we have indicated, the Fourier series is one of many possible orthogonal series expansions for a periodic signal. Its importance stems from the characteristic behavior of LTI systems, as we shall see in Chapter 5.

The Fourier transform is appropriate for representing the spectral characteristics of aperiodic signals with finite energy. The important properties of the Fourier transform were also presented in this chapter.

There are many excellent texts on Fourier series and Fourier transforms. For reference, we include the texts by Bracewell (1978), Davis (1963), Dym and McKean (1972), and Papoulis (1962).

Problems

4.1 Consider the full-wave rectified sinusoid in Fig. P4.1.

(a) Determine its spectrum $X_a(F)$.

(b) Compute the power of the signal.

(c) Plot the power spectral density.

(d) Check the validity of Parseval's relation for this signal.

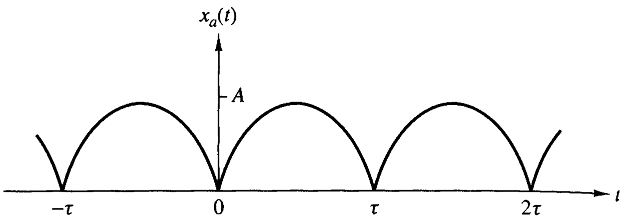


Figure P4.1

4.2 Compute and sketch the magnitude and phase spectra for the following signals $(a > 0)$.

(a) $x_a(t) = \begin{cases} Ae^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

(b) $x_a(t) = Ae^{-a|t|}$

4.3 Consider the signal

$$x(t) = \begin{cases} 1 - |t|/\tau, & |t| \leq \tau \\ 0, & \text{elsewhere} \end{cases}$$

(a) Determine and sketch its magnitude and phase spectra, $|X_a(F)|$ and $\measuredangle X_a(F)$, respectively.

(b) Create a periodic signal $x_p(t)$ with fundamental period $T_p \geq 2\tau$, so that $x(t) = x_p(t)$ for $|t| < T_p/2$. What are the Fourier coefficients c_k for the signal $x_p(t)$?

(c) Using the results in parts (a) and (b), show that $c_k = (1/T_p)X_a(k/T_p)$.

4.4 Consider the following periodic signal:

$$x(n) = \{\ldots, 1, 0, 1, 2, \underset{\uparrow}{3}, 2, 1, 0, 1, \ldots\}$$

(a) Sketch the signal $x(n)$ and its magnitude and phase spectra.

(b) Using the results in part (a), verify Parseval's relation by computing the power in the time and frequency domains.

4.5 Consider the signal

$$x(n) = 2 + 2\cos\frac{\pi n}{4} + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi n}{4}$$

(a) Determine and sketch its power density spectrum.

(b) Evaluate the power of the signal.

Determine and sketch the magnitude and phase spectra of the following periodic signals.

(a) $x(n) = 4 \sin \frac{\pi(n-2)}{3}$

(b) $x(n) = \cos \frac{2\pi}{3} n + \sin \frac{2\pi}{5} n$

(c) $x(n) = \cos \frac{2\pi}{3} n \sin \frac{2\pi}{5} n$

(d) $x(n) = \{\ldots, -2, -1, \underset{\uparrow}{0}, 1, 2, -2, -1, 0, 1, 2, \ldots\}$

(e) $x(n) = \{\ldots, -1, 2, \underset{\uparrow}{1}, 2, -1, 0, -1, 2, 1, 2, \ldots\}$

(f) $x(n) = \{\ldots, 0, 0, \underset{\uparrow}{1}, 1, 0, 0, 0, 1, 1, 0, 0, \ldots\}$

(g) $x(n) = 1, -\infty < n < \infty$

(h) $x(n) = (-1)^n, -\infty < n < \infty$

Determine the periodic signals $x(n)$, with fundamental period $N = 8$, if their Fourier coefficients are given by:

(a) $c_k = \cos \frac{k\pi}{4} + \sin \frac{3k\pi}{4}$

(b) $c_k = \begin{cases} \sin \frac{k\pi}{3}, & 0 \le k \le 6 \\ 0, & k = 7 \end{cases}$

(c) $\{c_k\} = \{\ldots, 0, \frac{1}{4}, \frac{1}{2}, 1, \underset{\uparrow}{2}, 1, \frac{1}{2}, \frac{1}{4}, 0 \ldots\}$

Two DT signals, $s_k(n)$ and $s_l(n)$, are said to be orthogonal over an interval $[N_1, N_2]$ if

$$\sum_{n=N_1}^{N_2} s_k(n) s_l^*(n) = \begin{cases} A_k, & k = l \\ 0, & k \neq l \end{cases}$$

If $A_k = 1$, the signals are called orthonormal.

(a) Prove the relation

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$$

(b) Illustrate the validity of the relation in part (a) by plotting for every value of $k = 1, 2, \ldots, 6$, the signals $s_k(n) = e^{j(2\pi/6)kn}$, $n = 0, 1, \ldots, 5$. [*Note*: For a given k, n the signal $s_k(n)$ can be represented as a vector in the complex plane.]

(c) Show that the harmonically related signals

$$s_k(n) = e^{j(2\pi/N)kn}$$

are orthogonal over any interval of length N.

4.9 Compute the Fourier transform of the following signals.

(a) $x(n) = u(n) - u(n-6)$

(b) $x(n) = 2^n u(-n)$

(c) $x(n) = (\frac{1}{4})^n u(n+4)$

(d) $x(n) = (\alpha^n \sin \omega_0 n) u(n), \qquad |\alpha| < 1$

(e) $x(n) = |\alpha|^n \sin \omega_0 n, \qquad |\alpha| < 1$

(f) $x(n) = \begin{cases} 2 - (\frac{1}{2})n, & |n| \le 4 \\ 0, & \text{elsewhere} \end{cases}$

(g) $x(n) = \{-2, -1, \underset{\uparrow}{0}, 1, 2\}$

(h) $x(n) = \begin{cases} A(2M + 1 - |n|), & |n| \le M \\ 0, & |n| > M \end{cases}$

Sketch the magnitude and phase spectra for parts (a), (f), and (g).

4.10 Determine the signals having the following Fourier transforms.

(a) $X(\omega) = \begin{cases} 0, & 0 \le |\omega| \le \omega_0 \\ 1, & \omega_0 < |\omega| \le \pi \end{cases}$

(b) $X(\omega) = \cos^2 \omega$

(c) $X(\omega) = \begin{cases} 1, & \omega_0 - \delta\omega/2 \le |\omega| \le \omega_0 + \delta\omega/2 \\ 0, & \text{elsewhere} \end{cases}$

(d) The signal shown in Fig. P4.10.

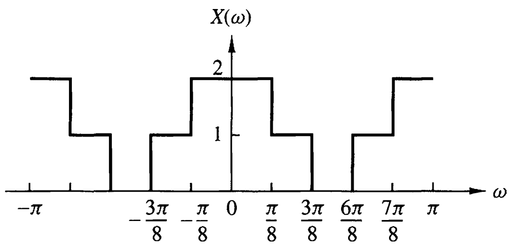


Figure P4.10

4.11 Consider the signal

$$x(n) = \{1, 0, -1, \underset{\uparrow}{2}, 3\}$$

with Fourier transform $X(\omega) = X_R(\omega) + j(X_I(\omega))$. Determine and sketch the signal $y(n)$ with Fourier transform

$$Y(\omega) = X_I(\omega) + X_R(\omega)e^{j2\omega}$$