VIETNAM NATIONAL UNIVERSITY-HO CHII MINH CITY UNIVERSITY OF SCIENCE
FACULTY PHYSICS AND ENGINEERING-DEPARTMENT OF THEORETICAL PHYSICS

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ANALYTICAL CALCULATION OF ONE-LOOP THREE-POINT SCALAR INTEGRALS

ADVISOR: **Dr. LE DUC NINH**

STUDENT: **NGUYEN THANH TIEN DAT**

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Contents

1 One loop integrals

Consider one loop diagram with n legs and n propagators, if k is loop momentum, the propagators are $q_i = k + p_i$ where $p_i = \sum_0^{n-1} r_i$ as is showed in the figure (1). Momentum conservation implies $\sum_i^n r_i = 0$ hence $r_n = 0$ Using Feynman rules, we have the integral

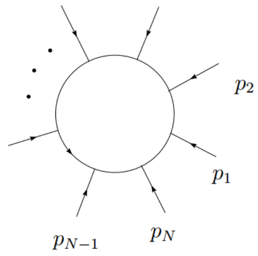

Figure 1: One loop N point integral diagram

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$$I^S_{\mu_1...\mu_p}(p_0, ..., p_{n-1}, m_0, ..., m_{n-1}) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q_{\mu_1}...q_{\mu_P}}{N_0...N_{n-1}} \tag{1}$$

with $S = \{0, 1, ...N-1\}$ and denominators factors

$$N_i = (q + p_i)^2 - m_i^2 + i\delta, \qquad n = 0, ...N-1. \tag{2}$$

We will firstly consider the scalar integral only, i.e the case where the numerator is equal to one. We follow the usual convention to denote N-point integrals with $N = 1, 2, ..$ as

$$T^{(1)} = A, \quad T^{(2)} = B, T^{(3)} = C, \quad T^{(4)} = D, \quad T^{(5)} = E, ... \tag{3}$$

2 Feynman parameters

To combine products of denominators of the type $N_i = (q + p_i)^2 - m_i^2 + i\delta$ into one single denominator, we can use this identity

$$\frac{1}{N_0 N_1 ... N_{n-1}} = \Gamma(n) \int_0^1 \left(\prod_{i=0}^{n-1} dx_i \right) \frac{\delta(1 - \sum_{j=0}^{n-1} x_i)}{(N_0 x_0 + N_1 x_1 + ... + N_{n-1} x_n)^n}. \tag{4}$$

The integral above with parameters z_i are called Feynman parameters. Using(4) to the scalar integral version of equation (1)

$$\begin{aligned} I^S(p_0, ...p_{N-1}, m_0, .., m_{n-1}) &= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \Gamma(N) \left(\int_0^1 \prod_{i=0}^{n-1} dx_i \right) \delta \left(1 - \sum_{j=0}^{n-1} x_i \right) \int \frac{d^D q}{(N_0 x_0 + N_1 x_1 + ... + N_{n-1} x_{n-1})^n} \\ &= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \Gamma(N) \left(\int_0^1 \prod_{i=0}^{n-1} dx_i \right) \delta \left(1 - \sum_{j=0}^{n-1} x_i \right) \int \frac{d^D q}{(q^2 + 2q.Q + \sum_{i=0}^{n-1} x_i (p_i^2 - m_i^2) + i\delta)^n} \end{aligned} \tag{5}$$

where $Q = \sum_{i=0}^{n-1} x_i q_i^\mu$. Now we perform the shift $l = q + Q$

$$I^S(p_0, ...p_{N-1}, m_0, .., m_{n-1}) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \Gamma(N) \left(\int_0^1 \prod_{i=0}^{n-1} dx_i \right) \delta \left(1 - \sum_{j=0}^{n-1} x_i \right) \int \frac{d^D l}{(l^2 - M^2 + i\delta)^n} \tag{6}$$

where

$$\begin{aligned} M^2 &= Q^2 - \sum_{i=0}^{n-1} x_i (p_i^2 - m_i^2) = \sum_{i,j=0}^{n-1} x_i x_j (p_i . p_j) - \frac{1}{2} \sum_{i=0}^{n-1} x_i (p_i^2 - m_i^2) \sum_{j=0}^{n-1} x_j - \frac{1}{2} \sum_{j=0}^{n-1} x_j (p_j^2 - m_j^2) \sum_{i=0}^{n-1} x_i \\ &= -\frac{1}{2} \sum_{i=0}^{n-1} x_i x_j \left(p_i^2 + p_j^2 - 2 p_i p_j - m_i^2 - m_j^2 \right) \\ &= -\frac{1}{2} \sum_{i,j=0}^{n-1} x_i x_j S_{ij}, \end{aligned} \tag{7}$$

$$S_{ij} = p_i^2 + p_j^2 - 2 x_i x_j - m_i^2 - m_j^2. \tag{8}$$

S_{ij} is called Cayley matrix. We have

$$\int_{-\infty}^{\infty} \frac{d^D l}{(l^2 - M^2 + i\delta)^n} = (-1)^n i \pi^{D/2} \frac{\Gamma(n - D/2)}{\Gamma(n)} (M^2 - i\delta)^{D/2 - n}. \tag{9}$$

To be more detailed, see Appendix A_2. Combining all results that were derived above, the integral becomes

$$I^S(p_0, ..p_{n-1}, m_0, .., m_{n-1}) = (-1)^n \frac{(2\pi\mu)^{4-D}}{i\pi^2} \left[\Gamma(n) \left(\int_0^1 \prod_{i=0}^{n-1} dx_i \right) \delta \left(1 - \sum_{j=0}^{n-1} x_i \right) \right] \left[i\pi^{D/2} \frac{\Gamma(n - D/2)}{\Gamma(n)} (M^2 - i\delta)^{D/2 - n} \right]. \tag{10}$$

3 Dimensional regularization

The idea of dimensional regularization is to work in $D = 4 - 2\varepsilon$ space time dimensions. Divergences for $D \to 4$ will thus appear as a poles in $1/\varepsilon$. Applying the dimension $D = 4 - 2\varepsilon$, the prefactors in the integral (10)

$$\frac{(2\pi\mu)^{4-D}}{i\pi^2} \times (-1)^n i \pi^{D/2} \to \frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2} \times (-1)^n i \pi^{2-\varepsilon} = (-1)^n (4\pi\mu^2)^\varepsilon. \tag{11}$$

Now, our integral will become

$$I^S(p_0, .., p_{n-1}, m_0, .., m_{n-1}) = (-1)^n (4\pi\mu^2)^\varepsilon \Gamma(n - 2 + \varepsilon) \left(\int_0^1 \prod_{i=0}^{n-1} dx_i \right) \delta \left(1 - \sum_{j=0}^{n-1} x_i \right) \left(M^2 - i\delta \right)^{2 - \varepsilon - n} \tag{12}$$

and this is our starting point to calculate one loop scalar 3 points and one loop scalar 4 points integrals.

4 Singularities

In $D = 4$ dimensions, the loop integrals (1) may be divergent either for $q \to \infty$ (ultraviolet divergences) or for $q_i^2 - m_i^2 \to 0$ (infrared divergences) and therefore need a regulator. A convenient regularization method is dimensional regularization.

An important feature of dimensional regularization is that it regulates infrared (IR) singularities, i.e soft/ or collinear divergence due to massless particles, as well. Because the divergence of UV singularities happens if the loop integral $q \to \infty$, so in general, UV behaviour becomes better if $\varepsilon > 0$ while IR singularities becomes better if $\varepsilon < 0$.
In this note, we only focus on infrared singularities which can be classified [3] as.

Soft singularities: A massless particle is exchanged between two on shell particles, i.e there is an n with

$$m_n \to 0, \quad (p_{n-1} - p_n)^2 - m_{n-1}^2 \to 0, (p_{n-1} - p_n)^2 - m_{n+1}^2 \to 0. \tag{13}$$

Collinear singularities: An external line with a light-like momentum (e.g. a massless external on-shell particle) is attached to two massless propagators, i.e there is an n with

$$m_n \to 0, \quad m_{n+1} \to 0 \quad (p_{n-1} - p_n)^2 \to 0. \tag{14}$$

5 One loop scalar 3 point integral

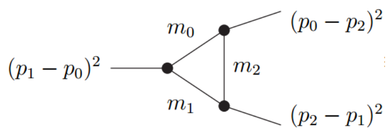
In the case of one loop three points integral, our integral becomes

$$\begin{aligned} C_0(p_0, .., p_{n-1}, m_0, .., m_{n-1}) &= -(4\pi\mu^2)^\varepsilon \Gamma(1+\varepsilon) \left(\int_0^1 \prod_{i=0}^{2} dx_i \right) \delta \left(1 - \sum_{j=0}^{2} x_i \right) \left(M^2 - i\delta \right)^{-1-\varepsilon} \\ &= -(4\pi\mu^2)^\varepsilon \Gamma(1+\varepsilon) \int_0^1 dx \int_0^{1-x} dy \left(M^2 - i\delta \right)^{-1-\varepsilon} \end{aligned} \tag{15}$$

where

$$M^2 = \sum_{i,j=0}^{2} x_i x_j (q_i.q_j) - \sum_{i=0}^{2} x_i (p_i^2 - m_i^2) = (x_0 q_0 + x_1 q_1 + x_2 q_2)^2 - \left[x_0(p_0^2 - m_0^2) + x_1(p_1^2 - m_1^2) + x_2(p_2^2 - m_2^2) \right]. \tag{16}$$

For convinience, a graphical notation is used as in [[3]] .Overlined variables are understood to receive an infinitesimally imaginary part $\bar{s} = s + i0$, etc. The function $Li_2(x) =$



$-\int_0^1 \ln(1 - xt)/x dt$ denotes the usual dilogarithm. In this note, I put $p_0 = 0$ for computational convinience. Moreover, whenever the mass parameter λ appears, it is understood as infinitesimal.

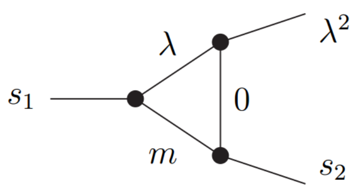
The list of all possibilities of 3 point diagrams have been published in [3]. In [3], there are two regularization schemes, dimensional and mass regularization. Dimensional regularization is used for diagrams which have external or internal massless lines, the mass regularization is used for diagram involving the external or internal lines which are very small but it has a non-zero value mass.

About the mass regularization scheme, the equation(16) will become (it happens when $\varepsilon \to 0$ and the massless propagators are replaced by fictitious mass λ)

$$C = -\int_0^1 dx \int_0^{1-x} dy \left\{ M^2(\lambda) - i\delta \right\}^{-1} \tag{17}$$

5.1 Collinear singularities

5.1.1 B2



Using the symmetry of C_0, we have $C_0(p_1, p_2 - p_1, p_2, m_0, m_1, m_2) = C_0(p_2 - p_1, p_1, p_2, m_2, m_1, m_0)$ so [1]

$$\begin{aligned} C_0 &= -\int_0^1 dx \int_0^{1-x} dy \left\{ = s_1 y^2 + \lambda^2 (1-x)^2 + (1-x) y (s_2 - s_1 - \lambda^2) - y(s_2 - m^2) - i\delta \right\}^{-1} \\ &\simeq -\int_0^1 dx \int_0^{1-x} dy \left\{ \bar{s}_1 y^2 + \lambda^2 (1-x)^2 + (1-x) y (\bar{s}_2 - \bar{s}_1 - \lambda^2) - (\bar{s}_2 - m^2) y - i\delta(y^2 - y) \right\}^{-1} \\ &= -\int_0^1 dx \int_0^{1-x} dy \left\{ \bar{s}_1 y^2 + \lambda^2 (1-x)^2 + (1-x) y (\bar{s}_2 - \bar{s}_1 - \lambda^2) - (\bar{s}_2 - m^2) y \right\}^{-1}. \end{aligned} \tag{18}$$

[1]It means that the M^2 will change from

$$M^2 = s_2 y^2 + x^2 \lambda^2 + xy(\lambda^2 + s_2 - s_1) - y(s_2 - m^2)$$

into

$$M^2 = s_1 y^2 + \lambda^2 (1-x)^2 + (1-x) y (s_2 - s_1 - \lambda^2) - y(s_2 - m^2).$$

Use the transformation, $y = \omega\eta$, $\omega = 1-x$, our integral now reads

$$
\begin{aligned}
C_0 &= -\int_0^1 \omega d\omega \int_0^1 d\eta \left\{\bar{s}_1\omega^2\eta^2 + \lambda^2\omega^2 + \omega^2\eta(\bar{s}_2-\bar{s}_1-\lambda^2) - (\bar{s}_2-m^2)\omega\eta\right\}^{-1} \\
&= -\int_0^1 d\omega \int_0^1 d\eta \left\{\bar{s}_1\omega\eta^2 + \lambda^2\omega + \omega\eta(\bar{s}_2-\bar{s}_1-\lambda^2) - (\bar{s}_2-m^2)\eta\right\}^{-1} \\
&= -\int_0^1 d\omega \int_0^1 d\eta \left\{\omega\left(\bar{s}_1\eta^2 + \eta(\bar{s}_2-\bar{s}_1-\lambda^2) + \lambda^2\right) - (\bar{s}_2-m^2)\eta\right\}^{-1} \\
&= -\int_0^1 \frac{d\eta}{\bar{s}_1\eta^2 + (\bar{s}_2-\bar{s}_1-\lambda^2)\eta + \lambda^2}\left\{\ln\left(\bar{s}_1\eta^2 + \eta(m^2-\bar{s}_1-\lambda^2) + \lambda^2\right) - \ln(m^2-\bar{s}_2)\eta\right\} \\
&= -\frac{1}{\bar{s}_1(\eta_+-\eta_-)}\int_0^1 d\eta\left(\frac{1}{\eta-\eta_+} - \frac{1}{\eta-\eta_-}\right)\left\{\ln\left(\bar{s}_1(\eta-\eta_{+1})(\eta-\eta_{-1})\right) - \ln(m^2-\bar{s}_2)\eta\right\} \\
&= -\frac{1}{\bar{s}_1(\eta_+-\eta_-)}\int_0^1 d\eta\left(\frac{1}{\eta-\eta_+} - \frac{1}{\eta-\eta_-}\right)\left\{\ln(\eta-\eta_{+1}) + \ln(\eta-\eta_{-1}) - \ln\frac{(m^2-\bar{s}_2)}{\bar{s}_1} - \ln\eta\right\} \\
&= -\frac{1}{\bar{s}_1(\eta_+-\eta_-)}\int_0^1 d\eta (I+J)
\end{aligned} \tag{19}
$$

where $\eta_\pm$ are the root solutions of equation $\bar{s}_1\eta^2 + \eta(\bar{s}_1-\bar{s}_1-\lambda^2)\eta + \lambda^2 = 0$, i.e

$$
\eta_\pm = \frac{\bar{s}_1-\bar{s}_2-\lambda^2}{2\bar{s}_1}\left(-1 \pm \sqrt{1-\frac{4\lambda^2\bar{s}_1}{(\bar{s}_1-\bar{s}_2-\lambda^2)^2}}\right) = \begin{cases}\eta_+ \simeq \dfrac{\bar{s}_1-\bar{s}_2}{\bar{s}_1} - \dfrac{\lambda^2}{\bar{s}_1-\bar{s}_2} \\ \eta_- \simeq \dfrac{\lambda^2}{\bar{s}_1-\bar{s}_2}\end{cases} \tag{20}
$$

and $\eta_{1,\pm}$ are the root solutions of the equation $\bar{s}_1\eta^2 - \eta(\bar{s}_1+\lambda^2-m^2) + \lambda^2 = 0$, i.e

$$
\eta_{1\pm} = \frac{\bar{s}_1+\lambda^2-m^2}{2\bar{s}_1}\left(1 \pm \sqrt{1-\frac{4\lambda^2\bar{s}_1}{(\bar{s}_1+\lambda^2-m^2)^2}}\right) = \begin{cases}\eta_{1+} \simeq \dfrac{\bar{s}_1-m^2}{\bar{s}_1} - \dfrac{\lambda^2}{\bar{s}_1-m^2} \\ \eta_{1-} \simeq \dfrac{\lambda^2}{\bar{s}_1-m^2}\end{cases}. \tag{21}
$$

About I

$$
\begin{aligned}
I &= \int_0^1 \frac{d\eta}{\eta-\eta_+}\left\{\ln(\eta-\eta_{+1}) + \ln(\eta-\eta_{-1}) - \left(\ln\frac{m^2-\bar{s}_2}{\bar{s}_1} + \ln\eta\right)\right\} \\
&= \left\{Li_2\left(\frac{\eta_{+1}-\eta_+}{\eta-\eta_+}\right) + Li_2\left(\frac{\eta_{-1}-\eta_+}{\eta-\eta_+}\right) + \ln^2(\eta-\eta_+) - \left[\ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln(\eta-\eta_+) + Li_2\left(\frac{-\eta_+}{\eta-\eta_+}\right) + \frac{1}{2}\ln^2(\eta-\eta_+)\right]\right\}\Big|_0^1 \\
&\simeq Li_2\left(\frac{(\bar{s}_2-m^2)/\bar{s}_1}{\bar{s}_2/\bar{s}_1}\right) - Li_2\left(1-\frac{\bar{s}_1-m^2}{\bar{s}_1-\bar{s}_2}\right) + Li_2\left(\frac{-(\bar{s}_1-\bar{s}_2)/\bar{s}_1}{\bar{s}_2/\bar{s}_1}\right) - Li_2\left(\frac{-(\bar{s}_1-\bar{s}_2)/\bar{s}_1}{-(\bar{s}_1-\bar{s}_2)/\bar{s}_1}\right) \\
&+ \frac{1}{2}\ln^2\frac{\bar{s}_2}{\bar{s}_1} - \frac{1}{2}\ln^2\left(-\frac{\bar{s}_1-\bar{s}_2}{\bar{s}_1}\right) - \ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\left(1-\frac{\bar{s}_1}{\bar{s}_1-s_2}\right) - Li_2\left(\frac{-(\bar{s}_1-\bar{s}_2)/\bar{s}_1}{\bar{s}_2/\bar{s}_1}\right) + Li_2(1) \\
&= Li_2\left(\frac{\bar{s}_2-m^2}{\bar{s}_2}\right) - Li_2\left(\frac{m^2-\bar{s}_2}{\bar{s}_1-\bar{s}_2}\right) + \frac{1}{2}\ln^2\frac{\bar{s}_2}{\bar{s}_1} - \frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_1} - \ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\frac{\bar{s}_2}{\bar{s}_2-\bar{s}_1} \\
&= Li_2\left(\frac{\bar{s}_2}{m^2}\right) + Li_2\left(\frac{\bar{s}_1-\bar{s}_2}{m^2-\bar{s}_2}\right) + \frac{1}{2}\ln^2\frac{\bar{s}_2}{m^2} + \ln\frac{\bar{s}_2}{m^2}\ln\frac{m^2-\bar{s}_2}{\bar{s}_2} + \frac{1}{2}\ln^2\left(\frac{\bar{s}_2-\bar{s}_1}{m^2-\bar{s}_2}\right) \\
&+ \frac{1}{2}\ln^2\frac{\bar{s}_2}{\bar{s}_1} - \frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_1} - \ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\frac{\bar{s}_2}{\bar{s}_2-\bar{s}_1}
\end{aligned} \tag{22}
$$

and J

$$
\begin{aligned}
J &= \int_0^1 \frac{d\eta}{\eta-\eta_-}\left\{\ln(\eta-\eta_{+1}) + \ln(\eta-\eta_{-1}) - \left(\ln\frac{m^2-\bar{s}_2}{\bar{s}_1} + \ln\eta\right)\right\} \\
&= \left\{Li_2\left(\frac{\eta_{+1}-\eta_-}{\eta-\eta_-}\right) + Li_2\left(\frac{\eta_{-1}-\eta_-}{\eta-\eta_-}\right) + \ln^2(\eta-\eta_-) - \left[\ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln(\eta-\eta_-) + Li_2\left(\frac{-\eta_-}{\eta-\eta_-}\right) + \frac{1}{2}\ln^2(\eta-\eta_-)\right]\right\}\Big|_0^1 \\
&= Li_2\left(\frac{\bar{s}_1-m^2}{\bar{s}_1}\right) - Li_2\left(1-\frac{(\bar{s}_1-m^2)(\bar{s}_1-\bar{s}_2)}{\bar{s}_1\lambda^2}\right) + Li_2\left(\frac{(m^2-\bar{s}_2)\lambda^2}{(\bar{s}_1-m^2)(\bar{s}_1-\bar{s}_2)}\right) - Li_2\left(1-\frac{\bar{s}_1-\bar{s}_2}{\bar{s}_1-m^2}\right) + \frac{1}{2}\ln^2\left(1-\frac{\lambda^2}{\bar{s}_1-\bar{s}_2}\right) - \frac{1}{2}\ln^2\left(-\frac{\lambda^2}{\bar{s}_1-\bar{s}_2}\right) \\
&- \ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\left(1-\frac{\bar{s}_1-\bar{s}_2}{\lambda^2}\right) - Li_2\left(-\frac{\lambda^2}{\bar{s}_1-\bar{s}_2}\right) + Li_2(1) \\
&= Li_2\left(\frac{\bar{s}_1-m^2}{\bar{s}_1}\right) - Li_2\left(1-\frac{(\bar{s}_1-m^2)(\bar{s}_1-\bar{s}_2)}{\bar{s}_1\lambda^2}\right) - Li_2\left(1-\frac{\bar{s}_1-\bar{s}_2}{\bar{s}_1-m^2}\right) - \frac{1}{2}\ln^2\left(-\frac{\lambda^2}{\bar{s}_1-\bar{s}_2}\right) - \ln\left(\frac{m^2-\bar{s}_2}{\bar{s}_1}\right)\ln\left(1-\frac{\bar{s}_1-\bar{s}_2}{\lambda^2}\right) + \frac{\pi^2}{6} \\
&= Li_2\left(\frac{\bar{s}_1-m^2}{\bar{s}_1}\right) + Li_2\left(1-\frac{\lambda^2\bar{s}_1}{(\bar{s}_1-m^2)(\bar{s}_1-\bar{s}_2)}\right) + \frac{1}{2}\ln^2\frac{\lambda^2\bar{s}_1}{(\bar{s}_1-m^2)(\bar{s}_1-\bar{s}_2)} + Li_2\left(1-\frac{\bar{s}_1-m^2}{\bar{s}_1-\bar{s}_2}\right) + \frac{1}{2}\ln^2\left(\frac{\bar{s}_1-\bar{s}_2}{\bar{s}_1-m^2}\right) - \frac{1}{2}\ln^2\left(-\frac{\lambda^2}{\bar{s}_1-\bar{s}_2}\right) - \ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\left(1-\frac{\bar{s}_1-\bar{s}_2}{\lambda^2}\right) + \frac{\pi^2}{6} \\
&= Li_2\left(\frac{\bar{s}_1-m^2}{\bar{s}_1}\right) + Li_2\left(\frac{m^2-\bar{s}_2}{\bar{s}_1-\bar{s}_2}\right) + \frac{\pi^2}{3} + \frac{1}{2}\ln^2\frac{\lambda^2\bar{s}_1}{(\bar{s}_1-m^2)(\bar{s}_1-\bar{s}_2)} + \frac{1}{2}\ln^2\left(\frac{\bar{s}_1-\bar{s}_2}{\bar{s}_1-m^2}\right) - \frac{1}{2}\ln^2\left(-\frac{\lambda^2}{\bar{s}_1-\bar{s}_2}\right) - \ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\left(1-\frac{\bar{s}_1-\bar{s}_2}{\lambda^2}\right) \\
&= Li_2\left(\frac{\bar{s}_1}{m^2}\right) - Li_2\left(\frac{\bar{s}_1-\bar{s}_2}{m^2-\bar{s}_2}\right) - \frac{\pi^2}{3} + \frac{1}{2}\ln^2\frac{\bar{s}_1}{m^2} + \ln\frac{\bar{s}_1}{m^2}\ln\frac{m^2-\bar{s}_1}{\bar{s}_1} - \frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{m^2-\bar{s}_2} + \frac{\pi^2}{3} + \frac{1}{2}\ln^2\frac{\lambda^2\bar{s}_1}{(\bar{s}_1-m^2)(\bar{s}_1-\bar{s}_2)} + \frac{1}{2}\ln^2\left(\frac{\bar{s}_1-\bar{s}_2}{\bar{s}_1-m^2}\right) \\
&- \frac{1}{2}\ln^2\left(-\frac{\lambda^2}{\bar{s}_1-\bar{s}_2}\right) - \ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\left(1-\frac{\bar{s}_1-\bar{s}_2}{\lambda^2}\right).
\end{aligned} \tag{23}
$$

Note that $s_1 \neq s_2$ and $s_1 \neq m^2$, $s_2 \neq m^2$. To sum up

$$
\begin{aligned}
C_0 &= -\frac{1}{\bar{s}_1-\bar{s}_2}\left\{Li_2\left(\frac{\bar{s}_2}{m^2}\right) - Li_2\left(\frac{\bar{s}_1}{m^2}\right) + 2Li_2\left(\frac{\bar{s}_1-\bar{s}_2}{m^2-\bar{s}_2}\right) + \frac{1}{2}\ln^2\frac{\bar{s}_2}{m^2} + \right. \\
&+ \ln\frac{\bar{s}_2}{m^2}\ln\frac{m^2-\bar{s}_2}{\bar{s}_2} + \frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{m^2-\bar{s}_1} + \frac{1}{2}\ln^2\frac{\bar{s}_2}{\bar{s}_1} - \frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_1} - \ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\frac{\bar{s}_2}{\bar{s}_2-\bar{s}_1} - \frac{1}{2}\ln^2\frac{\bar{s}_1}{m^2} - \ln\frac{\bar{s}_1}{m^2}\ln\frac{m^2-\bar{s}_1}{\bar{s}_1} \\
&\left. + \frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{m^2-\bar{s}_2} - \frac{1}{2}\ln^2\frac{\lambda^2\bar{s}_1}{(m^2-\bar{s}_1)(\bar{s}_2-\bar{s}_1)} - \frac{1}{2}\ln^2\frac{\bar{s}_1-\bar{s}_2}{\bar{s}_1-m^2} + \frac{1}{2}\ln^2\frac{\lambda^2}{\bar{s}_2-\bar{s}_1} + \ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\frac{\bar{s}_2-\bar{s}_1}{\lambda^2}\right\} \\
&= \frac{1}{\bar{s}_1-\bar{s}_2}\left\{\ln\frac{m^2-\bar{s}_1}{\lambda^2}\ln\frac{m^2-\bar{s}_1}{m^2} - \ln\frac{m^2-\bar{s}_2}{\lambda^2}\ln\frac{m^2-\bar{s}_2}{m^2} - 2Li_2\frac{\bar{s}_1-\bar{s}_2}{m^2-\bar{s}_2} + Li_2\left(\frac{\bar{s}_1}{m^2}\right) - Li_2\left(\frac{\bar{s}_2}{m^2}\right)\right\}.
\end{aligned} \tag{24}
$$

Note that

$$
\begin{aligned}
&-\frac{1}{2}\ln^2\frac{\lambda^2\bar{s}_1}{(m^2-\bar{s}_1)(\bar{s}_2-\bar{s}_1)}+\frac{1}{2}\ln^2\frac{\lambda^2}{\bar{s}_2-\bar{s}_1}+\frac{1}{2}\ln^2\frac{m^2-\bar{s}_1}{\bar{s}_1}+\frac{1}{2}\ln^2\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\frac{\bar{s}_2-\bar{s}_1}{\lambda^2}\\
&=-\ln\frac{\lambda^2}{\bar{s}_2-\bar{s}_1}\ln\frac{\bar{s}_1}{m^2-\bar{s}_1}+\ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\frac{\bar{s}_2-\bar{s}_1}{\lambda^2}+\frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{m^2-\bar{s}_2}-\frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{m^2-\bar{s}_1}\\
&=\ln\frac{\bar{s}_2-\bar{s}_1}{\lambda^2}\left(\ln\frac{m^2-\bar{s}_2}{\bar{s}_1}-\ln\frac{m^2-\bar{s}_1}{\bar{s}_1}\right)-\frac{1}{2}\left(\ln\frac{m^2-\bar{s}_2}{m^2-\bar{s}_1}\ln\frac{\bar{s}_2-\bar{s}_1}{m^2-\bar{s}_1}+\ln\frac{m^2-\bar{s}_2}{m^2-\bar{s}_1}\ln\frac{\bar{s}_2-\bar{s}_1}{m^2-\bar{s}_2}\right)\\
&=\ln\frac{\bar{s}_2-\bar{s}_1}{\lambda^2}\ln\frac{m^2-\bar{s}_2}{m^2-\bar{s}_1}-\frac{1}{2}\left(\ln\frac{m^2-\bar{s}_2}{m^2-\bar{s}_1}\ln\frac{\bar{s}_2-\bar{s}_1}{m^2-\bar{s}_1}+\ln\frac{m^2-\bar{s}_2}{m^2-\bar{s}_1}\ln\frac{\bar{s}_2-\bar{s}_1}{m^2-\bar{s}_2}\right)\\
&=\frac{1}{2}\ln\frac{m^2-\bar{s}_2}{m^2-\bar{s}_1}\ln\frac{m^2-\bar{s}_1}{\lambda^2}+\frac{1}{2}\ln\frac{m^2-\bar{s}_2}{m^2-\bar{s}_1}\ln\frac{m^2-\bar{s}_2}{\lambda^2}\\
&=\frac{1}{2}\left(\ln\frac{m^2-\bar{s}_2}{m^2}\ln\frac{m^2-\bar{s}_2}{\lambda^2}-\ln\frac{m^2-\bar{s}_1}{\lambda^2}\ln\frac{m^2-\bar{s}_1}{m^2}\right)+\frac{1}{2}\left(\ln\frac{m^2-\bar{s}_2}{m^2}\ln\frac{m^2-\bar{s}_1}{\lambda^2}-\ln\frac{m^2-\bar{s}_1}{m^2}\ln\frac{m^2-\bar{s}_2}{\lambda^2}\right)
\end{aligned}
\tag{25}
$$

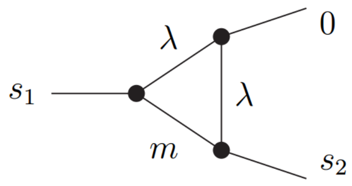
and

$$
\begin{aligned}
&\frac{1}{2}\left(\ln\frac{m^2-\bar{s}_2}{m^2}\ln\frac{m^2-\bar{s}_1}{\lambda^2}-\ln\frac{m^2-\bar{s}_1}{m^2}\ln\frac{m^2-\bar{s}_2}{\lambda^2}\right)+\frac{1}{2}\ln^2\frac{m^2-\bar{s}_2}{m^2}-\frac{1}{2}\ln^2\frac{m^2-\bar{s}_1}{m^2}\\
&=\frac{1}{2}\ln\frac{m^2-\bar{s}_2}{m^2-\bar{s}_1}\left(\ln\frac{m^2-\bar{s}_2}{m^2}-\ln\frac{m^2-\bar{s}_1}{m^2}\right)+\frac{1}{2}\left(\ln\frac{m^2-\bar{s}_2}{m^2}\ln\frac{m^2-\bar{s}_1}{\lambda^2}-\ln\frac{m^2-\bar{s}_1}{m^2}\ln\frac{m^2-\bar{s}_2}{\lambda^2}\right)\\
&=\ln\frac{m^2-\bar{s}_2}{m^2}\ln\frac{m^2-\bar{s}_2}{\lambda^2}-\ln\frac{m^2-\bar{s}_1}{m^2}\ln\frac{m^2-\bar{s}_1}{\lambda^2}
\end{aligned}
\tag{26}
$$

and finally,

$$
\begin{aligned}
&\frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{m^2-\bar{s}_2}-\frac{1}{2}\ln^2\frac{m^2-\bar{s}_2}{\bar{s}_2}+\frac{1}{2}\ln^2\frac{\bar{s}_2}{\bar{s}_1}-\frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_1}-\ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\frac{\bar{s}_2}{\bar{s}_2-\bar{s}_1}\\
&=\frac{1}{2}\left(\ln^2\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_2}+\ln^2\frac{m^2-\bar{s}_2}{\bar{s}_2}-2\ln\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_2}\ln\frac{m^2-\bar{s}_2}{\bar{s}_2}\right)-\frac{1}{2}\ln^2\frac{m^2-\bar{s}_2}{\bar{s}_2}+\frac{1}{2}\ln^2\frac{\bar{s}_2}{\bar{s}_1}-\frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_1}-\ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\frac{\bar{s}_2}{\bar{s}_2-\bar{s}_1}\\
&=\frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_2}-\frac{1}{2}\ln^2\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_1}-\ln\frac{m^2-\bar{s}_2}{\bar{s}_1}\ln\frac{\bar{s}_2}{\bar{s}_2-\bar{s}_1}-\ln\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_2}\ln\frac{m^2-\bar{s}_2}{\bar{s}_2}+\frac{1}{2}\ln^2\frac{\bar{s}_2}{\bar{s}_1}\\
&=\frac{1}{2}\ln\frac{\bar{s}_1}{\bar{s}_2}\left(\ln\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_2}-\ln\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_1}\right)+\frac{1}{2}\ln^2\frac{\bar{s}_2}{\bar{s}_1}+\ln\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_2}\left(\ln\frac{m^2-\bar{s}_2}{\bar{s}_1}-\ln\frac{m^2-\bar{s}_2}{\bar{s}_2}\right)\\
&=\frac{1}{2}\ln\frac{\bar{s}_1}{\bar{s}_2}\left(\ln\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_2}+\ln\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_1}\right)+\frac{1}{2}\ln^2\frac{\bar{s}_2}{\bar{s}_1}-\ln\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_2}\ln\frac{\bar{s}_1}{\bar{s}_2}\\
&=\frac{1}{2}\ln\frac{\bar{s}_1}{\bar{s}_2}\left(\ln\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_1}-\ln\frac{\bar{s}_2-\bar{s}_1}{\bar{s}_2}\right)+\frac{1}{2}\ln^2\frac{\bar{s}_1}{\bar{s}_2}\\
&=-\frac{1}{2}\ln^2\frac{\bar{s}_1}{\bar{s}_2}+\frac{1}{2}\ln^2\frac{\bar{s}_1}{\bar{s}_2}=0.
\end{aligned}
\tag{27}
$$

5.1.2 B3



where $p_2^2 = 0$, $2p_1p_2 = s_1 - s_2$, $m_0 = m_2 = \lambda$, $m_1 = m$ and

$$M^2 = s_2y^2 + (1-x)y(s_1 - s_2) + \lambda^2(1-y) - y(s_1 - m^2) \tag{28}$$

so

$$\begin{aligned} C_0 &= -\int_0^1 dx \int_0^{1-x} dy \left\{s_2y^2 + (1-x)y(s_1 - s_2) + \lambda^2(1-y) - y(s_1 - m^2) - i\delta\right\}^{-1} \\ &= -\int_0^1 dy \int_0^{1-y} dx \left\{\bar{s}_2y^2 + (1-x)y(\bar{s}_1 - \bar{s}_2) + \lambda^2(1-y) - y(\bar{s}_1 - m^2) - i\delta(1-y+y^2)\right\}^{-1} \\ &\simeq \frac{1}{\bar{s}_1 - \bar{s}_2}\int_0^1 \frac{dy}{y}\left\{\ln\left(\bar{s}_1y^2 + y(m^2 - \bar{s}_1 - \lambda^2) + \lambda^2\right) - \ln\left(\bar{s}_2y^2 + y(m^2 - \bar{s}_2 - \lambda^2) + \lambda^2\right)\right\} \\ &= \frac{1}{\bar{s}_1 - \bar{s}_2}(I_1 + I_2). \end{aligned} \tag{29}$$

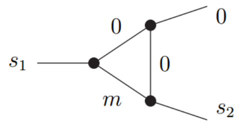
We have

$$\begin{aligned} I_1 &= \int_0^1 \frac{dy}{y} \ln\left(\bar{s}_1y^2 + y(m^2 - \bar{s}_1 - \lambda^2) + \lambda^2\right) \\ &= \int_0^1 \frac{dy}{y}\left\{\ln\bar{s}_1 + \ln\left(\frac{\lambda^2}{\bar{s}}\right) + \ln(y - y_+)(y - y_-) - \ln\left(\frac{\lambda}{\bar{s}_1}\right)\right\} \\ &= \int_0^1 \frac{dy}{y}\left\{\ln\lambda^2 + \ln(y_+ - y)(y_- - y) - \ln(y_+y_-)\right\} \\ &= \int_0^1 \frac{dy}{y}\left\{\ln\lambda^2 + \ln(y_+ - y) - \ln y_+ + \ln(y_- - y) - \ln y_-\right\} \\ &= \int_0^1 \frac{dy}{y}\left\{\ln\lambda^2 + \ln\left(1 - \frac{y}{y_+}\right) + \ln\left(1 - \frac{y}{y_-}\right)\right\} \\ &= \ln\lambda^2\int_0^1\frac{dy}{y} - Li_2\left(\frac{1}{y_+}\right) - Li_2\left(\frac{1}{y_-}\right) \\ &= -Li_2\left(\frac{\bar{s}_1}{\bar{s}_1 - m^2}\right) - Li_2\left(\frac{\bar{s}_1 - m^2}{\lambda^2}\right) + \ln\lambda^2\int_0^1\frac{dy}{y} = \ln\lambda^2\int_0^1\frac{dy}{y} + Li_2\left(\frac{\bar{s}_1 - m^2}{\bar{s}_1}\right) + \frac{1}{2}\ln^2\left(\frac{m^2 - \bar{s}_1}{\bar{s}_1}\right) + Li_2\left(\frac{\lambda^2}{\bar{s}_1 - m^2}\right) + \frac{1}{2}\ln^2\left(\frac{\lambda^2}{m^2 - \bar{s}_1}\right) \\ &= \ln\lambda^2\int_0^1\frac{dy}{y} + Li_2\left(\frac{\bar{s}_1}{m^2}\right) - \frac{\pi^2}{6} + \frac{1}{2}\ln^2\left(\frac{m^2 - \bar{s}_1}{\bar{s}_1}\right) + \frac{1}{2}\ln^2\left(\frac{\lambda^2}{m^2 - \bar{s}_1}\right) - \ln^2\left(\frac{\lambda}{m}\right) + \ln\left(\frac{\bar{s}_1}{m}\right)\ln\left(\frac{m^2 - \bar{s}_1}{m^2}\right) - \frac{1}{2}\ln^2\left(\frac{\bar{s}_1}{m^2}\right) + \ln^2\left(\frac{\lambda}{m}\right) \\ &= \ln\lambda^2\int_0^1\frac{dy}{y} + Li_2\left(\frac{\bar{s}_1}{m^2}\right) - \frac{\pi^2}{6} + \ln^2\left(\frac{m^2 - \bar{s}_1}{m\lambda}\right) + \ln^2\left(\frac{\lambda}{m}\right) \end{aligned} \tag{30}$$

where (82) and (83) are used. To sum up, the final result is

$$\begin{aligned} C_0 &= \frac{1}{\bar{s}_1 - \bar{s}_2}\left\{\ln\lambda^2\int_0^1\frac{dy}{y} + Li_2\left(\frac{\bar{s}_1}{m^2}\right) - \frac{\pi^2}{6} + \ln^2\left(\frac{m^2 - \bar{s}_1}{m\lambda}\right) + \ln^2\left(\frac{\lambda}{m}\right) - \ln\lambda^2\int_0^1\frac{dy}{y} - Li_2\left(\frac{\bar{s}_2}{m^2}\right) + \frac{\pi^2}{6} - \ln^2\left(\frac{m^2 - \bar{s}_2}{m\lambda}\right) - \ln^2\left(\frac{\lambda}{m}\right)\right\} \\ &= \frac{1}{\bar{s}_1 - \bar{s}_2}\left\{Li_2\left(\frac{\bar{s}_1}{m^2}\right) - Li_2\left(\frac{\bar{s}_2}{m^2}\right) + \ln^2\left(\frac{m^2 - \bar{s}_1}{m\lambda}\right) - \ln^2\left(\frac{m^2 - \bar{s}_2}{m\lambda}\right)\right\}. \end{aligned} \tag{31}$$

5.1.3 B4



We have $p_0 = 0$, $p_1^2 = s_1$, $p_2^2 = 0$, $2p_1p_2 = s_1 - s_2$ and

$$M^2 = s_2y^2 + y(1-x)(s_1 - s_2) - (s_1 - m^2)y \tag{32}$$

Applying the same procedure of B12 calculation, the integral now becomes

$$\begin{aligned}
C_0 &= -(4\pi\mu)^\varepsilon \Gamma(1+\varepsilon) \int_0^1 dx \int_0^{1-x} dy \left\{s_2y^2 + y(1-x)(s_1-s_2) - (s_1-m^2)y - i\delta\right\}^{-1-\varepsilon} \\
&= -(4\pi\mu)^\varepsilon \Gamma(1+\varepsilon) \int_0^1 dx \int_0^{x} dy \left\{s_2y^2 + xy(s_1-s_2) - (s_1-m^2)y - i\delta\right\}^{-1-\varepsilon} \\
&= -(4\pi\mu)^\varepsilon \Gamma(1+\varepsilon) \int_0^1 dy \int_y^{1} dx \left\{s_2y^2 + xy(s_1-s_2) - (s_1-m^2)y - i\delta\right\}^{-1-\varepsilon} \\
&= (4\pi\mu)^\varepsilon \frac{\Gamma(1+\varepsilon)}{\varepsilon(s_1-s_2)} \int_0^1 \frac{dy}{y} \left\{\left[s_2y^2 - y(s_2-m^2) - i\delta\right]^{-\varepsilon} - \left[s_1y^2 - y(s_1-m^2) - i\delta\right]^{-\varepsilon}\right\} \\
&= (4\pi\mu)^\varepsilon \frac{\Gamma(1+\varepsilon)}{\varepsilon(s_1-s_2)} \int_0^1 \frac{dy}{y} \left\{\left[(s_2+i\delta)y^2 - y(s_2-m^2+i\delta) + i\delta(-1+y-y^2)\right]^{-\varepsilon} - \left[(s_1+i\delta)y^2 - y(s_1-m^2+i\delta) + i\delta(-1+y-y^2)\right]^{-\varepsilon}\right\} \\
&\simeq (4\pi\mu)^\varepsilon \frac{\Gamma(1+\varepsilon)}{\varepsilon(s_1-s_2)} \int_0^1 \frac{dy}{y} \left\{\left[\bar{s}_2y^2 - y(\bar{s}_2-m^2)\right]^{-\varepsilon} - \left[\bar{s}_1y^2 - y(\bar{s}_1-m^2)\right]^{-\varepsilon}\right\}
\end{aligned} \tag{33}$$

where $\bar{s}_i = s_i + i\delta$, $i = 1,2$. We now focus on the first term in bracelets because the result of second term can be derived by changing $s_1 \to s_2$. Calling the first integral is I, we have

$$\begin{aligned}
I &= \int_0^1 dy y^{-1-\varepsilon} \left(\bar{s}_2y - (\bar{s}_2 - m^2)\right)^{-\varepsilon} \\
&= \int_0^1 dy y^{-1-\varepsilon} \left\{\left(\bar{s}_2y - (\bar{s}_2 - m^2)\right)^{-\varepsilon} - (-\bar{s}_2 + m^2)^{-\varepsilon}\right\} + (-\bar{s}_2 + m^2))^{-\varepsilon} \int_0^1 \frac{dy}{y^{-1-\varepsilon}}.
\end{aligned} \tag{34}$$

The term in bracelets can be expanded into $O(\varepsilon)$.

$$\begin{aligned}
\left(\bar{s}_2y - (\bar{s}_2 - m^2)\right)^{-\varepsilon} - (-\bar{s}_2 + m^2)^{-\varepsilon} &= -\varepsilon\left(\ln\left(\bar{s}_2y - (\bar{s}_2 - m^2)\right) - \ln\left(-\bar{s}_2 + m^2\right)\right) + O(\varepsilon^2) \\
&= -\varepsilon \ln\left(\frac{\bar{s}_2y - (\bar{s}_2 - m^2)}{-\bar{s}_2 + m^2}\right) + O(\varepsilon^2) \\
&= -\varepsilon \ln\left(1 - \frac{\bar{s}_2}{\bar{s}_2 - m^2}y\right) + O(\varepsilon^2).
\end{aligned} \tag{35}$$

The second integral evaluates

$$(-\bar{s}_2 + m^2))^{-\varepsilon} \int_0^1 \frac{dy}{y^{-1-\varepsilon}} = \frac{(-\bar{s}_2 + m^2))^{-\varepsilon}}{-\varepsilon} \tag{36}$$

and

$$\begin{aligned}
I &= -\varepsilon \int_0^1 \frac{dy}{y}(1 - \varepsilon \ln y) \ln\left(1 - \frac{\bar{s}_2}{\bar{s}_2 - m^2}y\right) + \frac{(-\bar{s}_2 + m^2)^{-\varepsilon}}{-\varepsilon} + O(\varepsilon^2) \\
&= -\varepsilon \int_0^1 \frac{dy}{y} \ln\left(1 - \frac{\bar{s}_2}{\bar{s}_2 - m^2}y\right) + O(\varepsilon^2) + \frac{(-\bar{s}_2 + m^2)^{-\varepsilon}}{-\varepsilon} \\
&= -\varepsilon Li_2\left(\frac{\bar{s}_2}{\bar{s}_2 - m^2}\right) + \frac{(-\bar{s}_2 + m^2)^{-\varepsilon}}{-\varepsilon} + O(\varepsilon^2) \\
&= -\varepsilon Li\left(\frac{s_2}{m^2}\right) - \frac{\varepsilon}{2}\ln^2\left(\frac{m^2 - \bar{s}_2}{\bar{s}_2}\right) - \frac{(-\bar{s}_2 + m^2)^{\varepsilon}}{-\varepsilon} + O(\varepsilon^2).
\end{aligned} \tag{37}$$

Because

$$\begin{aligned}
Li\left(\frac{\bar{s}_2}{\bar{s}_2 - m^2}\right) &= -Li\left(\frac{\bar{s}_2 - m^2}{\bar{s}_2}\right) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2\left(-\frac{\bar{s}_2}{\bar{s}_2 - m^2}\right) \\
&= Li\left(1 - \frac{\bar{s}_2}{m^2}\right) + \frac{1}{2}\ln^2\left(\frac{\bar{s}_2}{m^2}\right) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2\left(-\frac{\bar{s}_2}{\bar{s}_2 - m^2}\right) \\
&= -Li\left(\frac{s_2}{m^2}\right) + \frac{\pi^2}{6} - \ln\left(\frac{\bar{s}_2}{m^2}\right)\ln\left(\frac{m^2 - \bar{s}_2}{m^2}\right) + \frac{1}{2}\ln^2\left(\frac{\bar{s}_2}{m^2}\right) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2\left(-\frac{\bar{s}_2}{\bar{s}_2 - m^2}\right) \\
&= -Li\left(\frac{\bar{s}_2}{m^2}\right) + \frac{1}{2}\left(\ln\left(\frac{\bar{s}_2}{m^2}\right) - \ln\left(\frac{m^2 - \bar{s}_2}{m^2}\right)\right)^2 - \ln^2\left(\frac{\bar{s}_2}{m^2 - \bar{s}_2}\right) \\
&= -Li\left(\frac{\bar{s}_2}{m^2}\right) + \frac{1}{2}\ln^2\left(\frac{\bar{s}_2}{m^2 - \bar{s}_2}\right) - \ln^2\left(\frac{\bar{s}_2}{m^2 - \bar{s}_2}\right) \\
&= -Li\left(\frac{\bar{s}_2}{m^2}\right) - \frac{1}{2}\ln^2\left(\frac{\bar{s}_2}{m^2 - \bar{s}_2}\right) \\
&= -Li\left(\frac{\bar{s}_2}{m^2}\right) - \frac{1}{2}\ln^2\left(\frac{m^2 - \bar{s}_2}{\bar{s}_2}\right).
\end{aligned} \tag{38}$$

Similarity for the second term in bracelets of equation (37) we change $s_2 \to s_1$.

$$-\varepsilon Li\left(\frac{\bar{s}_1}{m^2}\right) - \frac{\varepsilon}{2}\ln^2\left(\frac{m^2 - \bar{s}_1}{\bar{s}_1}\right) - \frac{(-\bar{s}_1 + m^2))^{-\varepsilon}}{-\varepsilon} + O(\varepsilon^2) \tag{39}$$

so the first integral now becomes

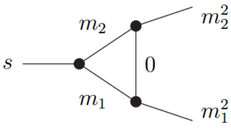
$$
\begin{aligned}
&\varepsilon\left\{-Li\left(\frac{\bar{s}_2}{m^2}\right)+Li\left(\frac{\bar{s}_1}{m^2}\right)-\frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}_2}{\bar{s}_2}\right)+\frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}_1}{\bar{s}_1}\right)\right\}-\frac{(-\bar{s}_2+m^2)^{-\varepsilon}}{-\varepsilon}+\frac{(-\bar{s}_1+m^2)^{-\varepsilon}}{-\varepsilon}\\
=&\varepsilon\left\{-Li\left(\frac{\bar{s}_2}{m^2}\right)+Li\left(\frac{\bar{s}_1}{m^2}\right)-\frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}_2}{\bar{s}_2}\right)+\frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}_1}{\bar{s}_1}\right)\right\}-\frac{m^{-\varepsilon}}{\varepsilon}\left(\left(1-\frac{\bar{s}_1}{m^2}\right)^{-\varepsilon}-\left(1-\frac{\bar{s}_2}{m^2}\right)^{-\varepsilon}\right)+O(\varepsilon^2)\\
&=\varepsilon\left\{-Li\left(\frac{\bar{s}_2}{m^2}\right)+Li\left(\frac{\bar{s}_1}{m^2}\right)-\frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}_2}{\bar{s}_2}\right)+\frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}_1}{\bar{s}_1}\right)\right\}-\\
&-\frac{m^{-\varepsilon}}{\varepsilon}\left\{-\varepsilon\ln\left(\frac{\bar{s}_2-m^2}{\bar{s}_1-m^2}\right)+\frac{\varepsilon^2}{2}\left(\ln^2\left(\frac{m^2-\bar{s}_1}{m^2}\right)-\ln^2\left(\frac{m^2-\bar{s}_2}{m^2}\right)\right)\right\}+O(\varepsilon^2)\\
&=\varepsilon\left\{-Li\left(\frac{\bar{s}_2}{m^2}\right)+Li\left(\frac{\bar{s}_1}{m^2}\right)-\ln^2\left(\frac{m^2-\bar{s}_2}{\bar{s}_2}\right)+\ln^2\left(\frac{m^2-\bar{s}_1}{\bar{s}_1}\right)\right\}+m^{-\varepsilon}\ln\left(\frac{\bar{s}_2-m^2}{\bar{s}_1-m^2}\right)+O(\varepsilon^2).
\end{aligned}
\tag{40}
$$

Then the final result is

$$
\begin{aligned}
C_0=&\frac{1}{s_1-s_2}(4\pi\mu)^{\varepsilon}\frac{\Gamma(1+\varepsilon)}{\varepsilon}\left\{-\varepsilon Li\left(\frac{\bar{s}_2}{m^2}\right)+\varepsilon Li\left(\frac{\bar{s}_1}{m^2}\right)-\varepsilon\ln^2\left(\frac{m^2-\bar{s}_2}{\bar{s}_2}\right)+\varepsilon\ln^2\left(\frac{m^2-\bar{s}_1}{\bar{s}_1}\right)\right.\\
&\left.+m^{-\varepsilon}\ln\left(\frac{\bar{s}_2-m^2}{\bar{s}_1-m^2}\right)+O(\varepsilon^2)\right\}\\
=&\frac{1}{s_1-s_2}\left\{\left(\frac{4\pi\mu}{m^2}\right)^{\varepsilon}\frac{\Gamma(1+\varepsilon)}{\varepsilon}\ln\left(\frac{\bar{s}_2-m^2}{\bar{s}_1-m^2}\right)+\left(1-\gamma_E\varepsilon+O(\varepsilon^2)\right)\left(1+\varepsilon\ln 4\pi\mu+O(\varepsilon^2)\right)\right.\\
&\left.\times\left(-Li\left(\frac{\bar{s}_2}{m^2}\right)+Li\left(\frac{\bar{s}_1}{m^2}\right)-\ln^2\left(\frac{m^2-\bar{s}_2}{\bar{s}_2}\right)+\ln^2\left(\frac{m^2-\bar{s}_1}{\bar{s}_1}\right)\right)+O(\varepsilon)\right\}\\
=&\frac{1}{s_1-s_2}\left\{\left(\frac{4\pi\mu}{m^2}\right)^{\varepsilon}\frac{\Gamma(1+\varepsilon)}{\varepsilon}\ln\left(\frac{\bar{s}_2-m^2}{\bar{s}_1-m^2}\right)-Li\left(\frac{\bar{s}_2}{m^2}\right)+Li\left(\frac{\bar{s}_1}{m^2}\right)\right.\\
&\left.-\ln^2\left(\frac{m^2-\bar{s}_2}{\bar{s}_2}\right)+\ln^2\left(\frac{m^2-\bar{s}_1}{\bar{s}_1}\right)+O(\varepsilon)\right\}.
\end{aligned}
\tag{41}
$$

5.2 Soft singularities

5.2.1 B6



Here we have $m_1 = m_1$, $m_0 = m_2$, $m_2 = 0$, $p_1^2 = s$, $p_2^2 = m_2^2$ and $2p_1p_2 = s + m_2^2 - m_1^2$

$$
\begin{aligned}
M^2 - i\delta &= (yp_1 + (1-x-y)p_2)^2 - \left(-xm_2^2 + (s-m_1^2)y + m_2^2(1-x-y)\right) - i\delta \\
&= y^2(p_1-p_2)^2 + (1-x)^2p_2^2 + 2y(1-x)(p_1-p_2)p_2 + xm_2^2 - (s-m_1^2)y - m_2^2(1-x-y) - i\delta \\
&= m_1^2y^2 + m_2^2(1-x)^2 + y(1-x)(s+m_2^2-m_1^2-2m_2^2) + xm_2^2 - (s-m_1^2)y - m_2^2(1-x-y) - i\delta \\
&= m_1^2y^2 + m_2^2x'^2 + yx'(s-m_2^2-m_1^2) + (1-x')m_2^2 - (s-m_1^2)y - m_2^2(x'-y) - i\delta \\
&= m_1^2y^2 + m_2^2x'^2 + yx'(s-m_2^2-m_1^2) - 2x'm_2^2 - (s-m_1^2-m_2^2)y + m_2^2 - i\delta \\
&= m_1^2y^2 + m^2(x'-1)^2 + y(x'-1)(s-m_1^2-m_2^2) - i\delta \\
&= (m_1y + m_2(1-x'))^2 - y(1-x')(s-(m_1-m_2)^2) - i\delta \\
&= (m_1y + m_2(1-x'))^2 - y(1-x')(s+i\delta-(m_1-m_2)^2) + i\delta(-1+y(1-x')) \\
&= (m_1y + m_2(1-x'))^2 - y(1-x')(\bar{s}-(m_1-m_2)^2) + i\delta(-1+y(1-x')) \\
&= (m_1y + m_2x)^2 - xy(\bar{s}-(m_1-m_2)^2) + i\delta(-1+yx).
\end{aligned} \tag{42}
$$

Where $x' = 1 - x$. So

$$
C_0 \simeq -(4\pi\mu)^\varepsilon\,\Gamma(1+\varepsilon)\int_0^1 dx\int_0^{1-x}\left\{m_1^2y^2 + m_2^2x^2 - xy\left(\bar{s}-m_1^2-m_2^2\right)\right\}^{-1-\varepsilon} dy \tag{43}
$$

with the transformation

$$
u = \frac{m_2x}{m_1y}, \qquad v = y \tag{44}
$$

or

$$
x = \frac{m_1}{m_2}vu, \quad y = v. \tag{45}
$$

Then the integration area which was bounded by $x=0$, $y=0$, and $y=1-x$ now is bounded by $uv=0$, $v=0$ and $u=(m_1/m_2)(-1+1/v)$, Moreover, the Jacobian of transformation is

$$
J = \frac{m_1}{m_2}v \tag{46}
$$

then

$$
\begin{aligned}
C_0 &= (4\pi\mu)^\varepsilon\Gamma(1+\varepsilon)\int_1^0\int_0^{(m_1/m_2)(-1+1/v)} v\frac{m_1}{m_2}m_1^{-2-2\varepsilon}v^{-2-2\varepsilon}\left[1+u^2-u\frac{(\bar{s}-m_1^2-m_2^2)}{m_1m_2}+i\delta\left(1+i\frac{\delta}{m_1m_2}\right)\right]^{-1-\varepsilon} dudv \\
&= -(4\pi\mu)^\varepsilon\Gamma(1+\varepsilon)\int_0^1 m_1^{-1-2\varepsilon}m_2^{-1}v^{-1-2\varepsilon}\int_0^{(m_1/m_2)(-1+1/v)}\left[1+u^2-u\frac{(\bar{s}-m_1^2-m_2^2)}{m_1m_2}\right]^{-1-\varepsilon} dudv \\
&= -(4\pi\mu)^\varepsilon\Gamma(1+\varepsilon)\int_0^1 m_1^{-1-2\varepsilon}m_2^{-1}v^{-1-2\varepsilon}\int_0^{(m_1/m_2)(-1+1/v)}\left((u-u_-)(u-u_+)\right)^{-1-\varepsilon} dudv \\
&= -(4\pi\mu)^\varepsilon\Gamma(1+\varepsilon)\int_0^1 m_1^{-1-2\varepsilon}m_2^{-1}v^{-1-2\varepsilon}\int_0^{(m_1/m_2)(-1+1/v)}\frac{1}{u_+-u_-}\left(\frac{1}{u-u_+}-\frac{1}{u-u_-}\right)\left\{1-\varepsilon\left(\ln(u-u_+)+\ln(u-u_-)\right)\right\} dudv \\
&= -(4\pi\mu)^\varepsilon\frac{\Gamma(1+\varepsilon)}{u_+-u_-}\int_0^1 m_1^{-1-2\varepsilon}m_2^{-1}v^{-1-2\varepsilon}\left\{\ln\left(\frac{m_1}{m_2}\frac{1-v}{v}-u_+\right)-\ln\left(\frac{m_1}{m_2}\frac{1-v}{v}-u_-\right)-\ln(-u_+)+\ln(-u_-)+\right. \\
&\left.+\varepsilon\int_0^{(m_1/m_2)(-1+1/v)}\left(\frac{1}{u-u_+}-\frac{1}{u-u_-}\right)\left[\ln(u_--u_+)-\ln(u-u_+)+\ln(u_+-u_-)-\ln(u-u_-)\right]du-\left(\ln(u_--u_+)+\ln(u_+-u_-)\right)\right\} dv \\
&= -m_1^{-1-2\varepsilon}m_2^{-1}(4\pi\mu)^\varepsilon\frac{\Gamma(1+\varepsilon)}{u_+-u_-}\int_0^1 v^{-1-2\varepsilon}\left\{\left[1-\varepsilon\left(\ln(u_--u_+)+\ln(u_+-u_-)\right)\right]\left(\ln\left(\frac{m_1}{m_2}(1-v)-vu_+\right)-\ln\left(\frac{m_1}{m_2}(1-v)-vu_-\right)-\ln(-u_+)+\ln(-u_-)\right)+\right. \\
&\left.+\varepsilon\int_0^{(m_1/m_2)(-1+1/v)} du\left(\frac{1}{u-u_+}-\frac{1}{u-u_-}\right)\left[\ln(u_--u_+)-\ln(u-u_+)+\ln(u_+-u_-)-\ln(u-u_-)\right]\right\} dv \\
&= -m_1^{-1-2\varepsilon}m_2^{-1}(4\pi\mu)^\varepsilon\frac{\Gamma(1+\varepsilon)}{u_+-u_-}\int_0^1 v^{-1-2\varepsilon}\left\{\left[1-\varepsilon\left(\ln(u_--u_+)+\ln(u_+-u_-)\right)\right]\left(\ln\left(\frac{m_1}{m_2}-v\left(\frac{m_1}{m_2}+u_+\right)\right)-\ln\left(\frac{m_1}{m_2}-v\left(\frac{m_1}{m_2}+u_-\right)\right)-\ln(-u_+)+\ln(-u_-)\right)+\right. \\
&\left.+\varepsilon\int_0^{(m_1/m_2)(-1+1/v)} du\left(\frac{1}{u-u_+}-\frac{1}{u-u_-}\right)\left[\ln(u_--u_+)-\ln(u-u_+)+\ln(u_+-u_-)-\ln(u-u_-)\right]\right\} dv \\
&= -m_1^{-1-2\varepsilon}m_2^{-1}(4\pi\mu)^\varepsilon\frac{\Gamma(1+\varepsilon)}{u_+-u_-}\int_0^1 v^{-1-2\varepsilon}\left\{\left[1-\varepsilon\left(\ln(u_--u_+)+\ln(u_+-u_-)\right)\right]\left(\ln\left(1-v\left(1+u_+\frac{m_2}{m_1}\right)\right)-\ln\left(1-v\left(1+u_-\frac{m_2}{m_1}\right)\right)-\ln(-u_+)+\ln(-u_-)\right)+\right. \\
&\left.+\varepsilon\int_0^{(m_1/m_2)(-1+1/v)} du\left(\frac{1}{u-u_+}-\frac{1}{u-u_-}\right)\left[\ln(u_--u_+)-\ln(u-u_+)+\ln(u_+-u_-)-\ln(u-u_-)\right]\right\} dv.
\end{aligned} \tag{47}
$$

Note that

$$
\begin{aligned}
\frac{\partial}{\partial z}Li_2\left(\frac{z-a}{z-b}\right) &= -\ln\left(1-\frac{z-a}{z-b}\right)\frac{z-b}{z-a}\left(\frac{1}{z-b}-\frac{z-a}{(z-b)^2}\right) = -\ln\left(\frac{a-b}{z-b}\right)\frac{z-b}{z-a}\frac{a-b}{(z-b)^2} = -\frac{a-b}{(z-a)(z-b)}\ln\left(\frac{a-b}{z-b}\right) \\
&= -\left(\frac{1}{z-a}-\frac{1}{z-b}\right)\ln\left(\frac{a-b}{z-b}\right)
\end{aligned} \tag{48}
$$

so

$$
-Li_2\left(\frac{A-a}{A-b}\right)+Li_2\left(\frac{a}{b}\right) = \int_0^A\left(\frac{1}{z-a}-\frac{1}{z-b}\right)\left(\ln(a-b)-\ln(z-b)\right). \tag{49}
$$

In our case, a and b always have the opposite sign in imaginary part. So

$$
Sign\left(Im(a)\right) = -Sign(Im(b)), \rightarrow Sign(Im(a-b)) = Sign(Im(z-b)), \quad \text{z is a real number.} \tag{50}
$$

then

$$
\begin{aligned}
&\varepsilon\int_0^{(m_1/m_2)(-1+1/v)} dv\left(\frac{1}{u-u_+}-\frac{1}{u-u_-}\right)\left[\ln(u_--u_+)-\ln(u-u_+)+\ln(u_+-u_-)-\ln(u-u_-)\right]=\\
&=\varepsilon\left\{Li_2\left(\frac{u_+}{u_-}\right)-Li_2\left(\frac{(m_1/m_2)(1-v)-vu_+}{(m_1/m_2)(1-v)-u_-v}\right)-Li_2\left(\frac{u_-}{u_+}\right)+Li_2\left(\frac{(m_1/m_2)(1-v)-vu_-}{(m_1/m_2)(1-v)-vu_+}\right)\right\}\\
&=\varepsilon\left\{Li_2\left(\frac{u_+}{u_-}\right)-Li_2\left(\frac{u_-}{u_+}\right)+Li_2\left(\frac{1-v\left(1+\frac{m_1}{m_2}u_-\right)}{1-v\left(1+\frac{m_1}{m_2}u_+\right)}\right)-Li_2\left(\frac{1-v\left(1+\frac{m_1}{m_2}u_+\right)}{1-v\left(1+\frac{m_1}{m_2}u_-\right)}\right)\right\}
\end{aligned}
\tag{51}
$$

so

$$
\begin{aligned}
C_0&=-m_1^{-1-2\varepsilon}m_2^{-1}(4\pi\mu)^\varepsilon\frac{\Gamma(1+\varepsilon)}{u_+-u_-}\int_0^1 v^{-1-2\varepsilon}\left\{\left[1-\varepsilon\left(\ln(u_-+u_+)+\ln(u_+-u_-)\right)\right]\left(\ln\left(1-v\left(1+u_+\frac{m_1}{m_2}\right)\right)-\ln\left(1-v\left(1+u_-\frac{m_1}{m_2}\right)\right)-\ln(-u_+)+\ln(-u_-)\right)+\right.\\
&\left.+\varepsilon\left[Li_2\left(\frac{u_+}{u_-}\right)-Li_2\left(\frac{u_-}{u_+}\right)+Li_2\left(\frac{1-v\left(1+\frac{m_1}{m_2}u_-\right)}{1-v\left(1+\frac{m_1}{m_2}u_+\right)}\right)-Li_2\left(\frac{1-v\left(1+\frac{m_1}{m_2}u_+\right)}{1-v\left(1+\frac{m_1}{m_2}u_-\right)}\right)\right]\right\}dv\\
&=-m_1^{-1-2\varepsilon}m_2^{-1}(4\pi\mu)^\varepsilon\frac{\Gamma(1+\varepsilon)}{u_+-u_-}\left\{\left[1-\varepsilon\left(\ln(u_--u_+)+\ln(u_+-u_-)\right)\right]\int_0^1\frac{dv}{v^{1+2\varepsilon}}\left[\ln\left[1-v\left(1+u_+\frac{m_1}{m_2}\right)\right]-\ln\left[1-v\left(1+u_-\frac{m_1}{m_2}\right)\right]\right]+\right.\\
&+\varepsilon\int_0^1\frac{dv}{v^{1+2\varepsilon}}\left[Li_2\left(\frac{1-v\left(1+\frac{m_1}{m_2}u_-\right)}{1-v\left(1+\frac{m_1}{m_2}u_+\right)}\right)-Li_2\left(\frac{1-v\left(1+\frac{m_1}{m_2}u_+\right)}{1-v\left(1+\frac{m_1}{m_2}u_-\right)}\right)\right]+\left[\left(1-\varepsilon\left(\ln(u_--u_+)+\ln(u_+-u_-)\right)\right)\left(-\ln(-u_+)+\ln(-u_-)\right)+\varepsilon\left(Li_2\left(\frac{u_+}{u_-}\right)-Li_2\left(\frac{u_-}{u_+}\right)\right)\right]\left.\int_0^1\frac{dv}{v^{1+2\varepsilon}}\right\}\\
&=-m_1^{-1-2\varepsilon}m_2^{-1}(4\pi\mu)^\varepsilon\frac{\Gamma(1+\varepsilon)}{u_+-u_-}\left\{\left[1-\varepsilon\left(\ln(u_--u_+)+\ln(u_+-u_-)\right)\right]\int_0^1\frac{dv}{v^{1+2\varepsilon}}\left[\ln\left[1-v\left(1+u_+\frac{m_1}{m_2}\right)\right]-\ln\left[1-v\left(1+u_-\frac{m_1}{m_2}\right)\right]\right]+\right.\\
&+\varepsilon\int_0^1\frac{dv}{v^{1+2\varepsilon}}\left[Li_2\left(\frac{1-v\left(1+\frac{m_1}{m_2}u_-\right)}{1-v\left(1+\frac{m_1}{m_2}u_+\right)}\right)-Li_2\left(\frac{1-v\left(1+\frac{m_1}{m_2}u_+\right)}{1-v\left(1+\frac{m_1}{m_2}u_-\right)}\right)\right]-\frac{1}{2}\left[\left(\frac{1}{\varepsilon}-\left(\ln(u_--u_+)+\ln(u_+-u_-)\right)\right)\left(-\ln(-u_+)+\ln(-u_-)\right)+\left(Li_2\left(\frac{u_+}{u_-}\right)-Li_2\left(\frac{u_-}{u_+}\right)\right)\right]\left.\right\}\\
&=-m_1^{-1-2\varepsilon}m_2^{-1}(4\pi\mu)^\varepsilon\frac{\Gamma(1+\varepsilon)}{u_+-u_-}\left(\left[1-\varepsilon\left(\ln(u_--u_+)+\ln(u_+-u_-)\right)\right]I+\varepsilon J-\frac{1}{2}\left[\left(\frac{1}{\varepsilon}-\left(\ln(u_--u_+)+\ln(u_+-u_-)\right)\right)\left(-\ln(-u_+)+\ln(-u_-)\right)+\left(Li_2\left(\frac{u_+}{u_-}\right)-Li_2\left(\frac{u_-}{u_+}\right)\right)\right]\right)\\
&=-m_1^{-1-2\varepsilon}m_2^{-1}(4\pi\mu)^\varepsilon\frac{\Gamma(1+\varepsilon)}{u_+-u_-}\left\{-\frac{-\ln(-u_+)+\ln(-u_-))}{2\varepsilon}+I+\frac{1}{2}(\ln(u_--u_+)+\ln(u_+-u_-))(\ln(-u_-)-\ln(-u_+))-\frac{1}{2}\left(Li_2\left(\frac{u_+}{u_-}\right)-Li_2\left(\frac{u_-}{u_+}\right)\right)+\varepsilon\left(J-(\ln(u_--u_+)+\ln(u_+-\right.\right.
\end{aligned}
\tag{52}
$$

About I

$$
I=\int_0^1\frac{dv}{v^{1+2\varepsilon}}\left(\ln\left[1-v\left(1+u_+\frac{m_1}{m_2}\right)\right]-\ln\left[1-v\left(1+u_-\frac{m_1}{m_2}\right)\right]\right)\simeq\int_0^1\frac{dv}{v}\left(\ln\left[1-v\left(1+u_+\frac{m_1}{m_2}\right)\right]-\ln\left[1-v\left(1+u_-\frac{m_1}{m_2}\right)\right]\right)=Li_2\left(1+u_-\frac{m_1}{m_2}\right)-Li_2\left(1+u_+\frac{m_1}{m_2}\right).
\tag{53}
$$

In the second step, since the integrand has the form 0/0 when v tends to zero, the integral will give finite value. Because of that, this integral defined in 4 dimensions so we can take ε equal to zero. About J

$$
J=\int_0^1\frac{dv}{v^{1+2\varepsilon}}\left[Li_2\left(\frac{1-v\left(1+\frac{m_1}{m_2}u_-\right)}{1-v\left(1+\frac{m_1}{m_2}u_+\right)}\right)-Li_2\left(\frac{1-v\left(1+\frac{m_1}{m_2}u_+\right)}{1-v\left(1+\frac{m_1}{m_2}u_-\right)}\right)\right]
\tag{54}
$$

when v tends to zero, the integrand has the form 0/0 which makes this integral defined at $v=0$. Because of that, the integral gives a finite result after doing integration. And because I and J are both finite so it will be safely to let ε to be zero in equation (52) and remains the pole $1/\varepsilon$ in the result.

$$
\begin{aligned}
C_0&=-m_1^{-1-2\varepsilon}m_2^{-1}(4\pi\mu)^\varepsilon\frac{\Gamma(1+\varepsilon)}{u_+-u_-}\left\{-\frac{\ln(-u_-)-\ln(-u_+))}{2\varepsilon}+I-\frac{1}{2}(\ln(u_--u_+)+\ln(u_+-u_-))(\ln(-u_-)-\ln(-u_+))-\frac{1}{2}\left(Li_2\left(\frac{u_+}{u_-}\right)-Li_2\left(\frac{u_-}{u_+}\right)\right)+O(\varepsilon)\right\}\\
&=m_1^{-1-\varepsilon}m_2^{-1+\varepsilon}\left(\frac{4\pi\mu^2}{m_1m_2}\right)^\varepsilon\frac{\Gamma(1+\varepsilon)}{u_--u_+}\left\{-\frac{\ln(-u_-)-\ln(-u_+))}{2\varepsilon}+Li_2\left(1+u_-\frac{m_1}{m_2}\right)-Li_2\left(1+u_+\frac{m_1}{m_2}\right)+\frac{1}{2}(\ln(u_--u_+)+\ln(u_+-u_-))(\ln(-u_-)-\ln(-u_+))-\frac{1}{2}\left(Li_2\left(\frac{u_+}{u_-}\right)-Li_2\left(\frac{u}{u}\right.\right.\right.
\end{aligned}
\tag{55}
$$

with $u_\pm$ is one of two solution of quadratic equation $u^2-u\frac{\bar{s}-m_1^2-m_2^2}{m_1m_2}+1=0$

$$
u_\pm=\frac{1}{2m_1m_2}\left(\bar{s}-m_1^2-m_2^2\pm\sqrt{(\bar{s}-m_1^2-m_2^2)^2-4m_1^2m_2^2}\right)=\frac{1}{2m_1m_2}\left(\bar{s}-m_1^2-m_2^2-\sqrt{\left((\bar{s}-(m_1+m_2)^2\right)\left(\bar{s}-(m_1-m_2)^2\right)}\right).
$$

Moreover,

$$
x_s=\frac{\sqrt{1-4m_1m_2/(\bar{s}-(m_1-m_2)^2)}-1}{\sqrt{1-4m_1m_2/(\bar{s}-(m_1-m_2)^2)}+1}=\frac{A-1}{A+1}=\frac{(A-1)^2}{A^2-1}
\tag{56}
$$

and

$$
\begin{aligned}
A^2-1&=1-\frac{4m_1m_2}{\bar{s}-(m_1-m_2)^2}-1=-\frac{4m_1m_2}{\bar{s}-(m_1-m_2)^2}\\
(A-1)^2=A^2-2A+1&=2-\frac{4m_1m_2}{\bar{s}-(m_1-m_2)^2}-2\sqrt{1-\frac{4m_1m_2}{\bar{s}-(m_1-m_2)^2}}=\frac{2\left(\bar{s}-m_1^2-m_2^2-\sqrt{\left((\bar{s}-(m_1+m_2)^2\right)\left(\bar{s}-(m_1-m_2)^2\right)}\right)}{\bar{s}-(m_1-m_2)^2}.
\end{aligned}
\tag{57}
$$

With this, we have

$$
x_s=-\frac{1}{2m_1m_2}\left(\bar{s}-m_1^2-m_2^2-\sqrt{\left((\bar{s}-(m_1+m_2)^2\right)\left(\bar{s}-(m_1-m_2)^2\right)}\right)=-u_-
\tag{58}
$$

and because $x_+x_-=1$ so

$$
x_+=-\frac{1}{x_s}
\tag{59}
$$

and

$$
\begin{aligned}
u_- - u_+ &= \left(\frac{1}{x_s} - x_s\right) = \frac{1-x_s^2}{x_s}, \\
\ln(-u_-) - \ln(-u_+) &= \ln(x_s) - \ln\left(\frac{1}{x_s}\right) = 2\ln(x_s), \\
Li_2\left(1+u_-\frac{m_1}{m_2}\right) - Li_2\left(1+u_+\frac{m_1}{m_2}\right) &= Li_2\left(1-x_s\frac{m_1}{m_2}\right) + Li_2\left(1-x_s\frac{m_2}{m_1}\right) + \frac{1}{2}\ln^2\left(\frac{m_2}{m_1}x_s\right), \\
Li_2\left(\frac{u_+}{u_-}\right) - Li_2\left(\frac{u_-}{u_+}\right) &= -2Li_2\left(x_s^2\right) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2(-x_s^2), \\
(\ln(u_- - u_+) + \ln(u_+ - u_-))(\ln(-u_-) - \ln(-u_+)) &= 2\ln(x_s)\ln\left(-\frac{(1-x_s^2)^2}{x_s^2}\right) = 2\ln(x_s)\left(\ln(1-x_s^2)^2 - \ln(-x_s^2)\right).
\end{aligned} \tag{60}
$$

so

$$
\begin{aligned}
C_0 &= \left(\frac{4\pi\mu^2}{m_1m_2}\right)^{\varepsilon}\Gamma(1+\varepsilon)\frac{x_s}{m_1m_2(1-x_s^2)}\left(\frac{m_2}{m_1}\right)^{\varepsilon}\left\{-\frac{\ln x_s}{\varepsilon} + Li_2\left(1-x_s\frac{m_1}{m_2}\right) + Li_2\left(1-x_s\frac{m_2}{m_1}\right) + \frac{1}{2}\ln^2\left(\frac{m_2}{m_1}x_s\right) + \ln(x_s)\left(\ln(1-x_s^2)^2 - \ln(-x_s^2)\right) + Li_2\left(x_s^2\right) + \frac{\pi^2}{12} + \frac{1}{4}\ln^2(-x_s^2)\right\} \\
&= \frac{x_s}{m_1m_2(1-x_s^2)}\left\{-\frac{\Gamma(1+\varepsilon)}{\varepsilon}\left(\frac{4\pi\mu^2}{m_1m_2}\right)^{\varepsilon}\left(\frac{m_2}{m_1}\right)^{\varepsilon}\ln(x_s) + Li_2\left(1-x_s\frac{m_1}{m_2}\right) + Li_2\left(1-x_s\frac{m_2}{m_1}\right) + Li_2(x_s^2) + 2\ln(x_s)\ln(1-x_s^2) - \ln(x_s)\ln(-x_s^2) + \frac{\pi^2}{12} + \frac{1}{4}\ln^2(-x_s^2) + \frac{1}{2}\ln^2\left(\frac{m_2}{m_1}x_s\right)\right\}.
\end{aligned} \tag{61}
$$

Note that:

$$
\ln(-x_s^2) = \ln(x_s^2) \pm i\pi \tag{62}
$$

and because of that, we have

$$
\begin{aligned}
&-\ln(x_s)\ln(-x_s^2) + \frac{1}{4}\ln^2(-x_s^2) + \frac{\pi^2}{12} = -\ln(x_s)(\ln(x_s^2) \pm i\pi) + \frac{1}{4}(\ln x_s^2 \pm i\pi)^2 + \frac{\pi^2}{12} = -2\ln^2(x_s) \mp i\pi\ln(x_s) + \frac{1}{4}\left(\ln^2(x_s^2) - \pi^2 \pm 2i\pi\ln(x_s^2)\right) + \frac{\pi^2}{12} \\
&= -2\ln^2(x_s) \mp i\pi\ln(x_s) + \frac{1}{4}\left(4\ln^2 x_s - \pi^2 \pm 4i\pi\ln x_s\right) + \frac{\pi}{12} = -\frac{\pi^2}{6} - \ln^2(x_s)
\end{aligned} \tag{63}
$$

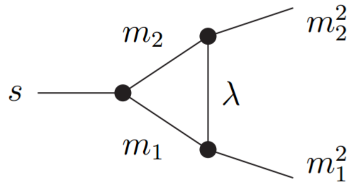
Next,

$$
\begin{aligned}
-\frac{\Gamma(1+\varepsilon)}{\varepsilon}\left(\frac{4\pi\mu^2}{m_1m_2}\right)^{\varepsilon}\left(\frac{m_2}{m_1}\right)^{\varepsilon}\ln(x_s) + \frac{1}{2}\ln^2\left(\frac{m_2}{m_1}x_s\right) &\simeq -\frac{\Gamma(1+\varepsilon)}{\varepsilon}\left(\frac{4\pi\mu^2}{m_1m_2}\right)^{\varepsilon}\left(1+\varepsilon\ln\left(\frac{m_2}{m_1}\right)\right)\ln(x_s) + \frac{1}{2}\left(\ln^2(x_s^2) + \ln^2\left(\frac{m_2}{m_1}\right) + 2\ln(x_s)\ln\left(\frac{m_2}{m_1}\right)\right) \\
&= -\frac{\Gamma(1+\varepsilon)}{\varepsilon}\left(\frac{4\pi\mu^2}{m_1m_2}\right)^{\varepsilon}\ln(x_s) - \ln x_s\ln\left(\frac{m_2}{m_1}\right)) + \frac{1}{2}\left(\ln^2(x_s^2) + \ln^2\left(\frac{m_2}{m_1}\right) + 2\ln(x_s)\ln\left(\frac{m_2}{m_1}\right)\right) \\
&= -\frac{\Gamma(1+\varepsilon)}{\varepsilon}\left(\frac{4\pi\mu^2}{m_1m_2}\right)^{\varepsilon}\ln(x_s) + \frac{1}{2}\left(\ln^2(x_s^2) + \ln^2\left(\frac{m_2}{m_1}\right)\right).
\end{aligned} \tag{64}
$$

To conclude, the final result is

$$
C_0 = \frac{x_s}{m_1m_2(1-x_s^2)}\left\{-\frac{\Gamma(1_+\varepsilon)}{\varepsilon}\left(\frac{4\pi\mu^2}{m_1m_2}\right)^{\varepsilon}\ln(x_s) + Li_2\left(1-x_s\frac{m_1}{m_2}\right) + Li_2\left(1-x_s\frac{m_2}{m_1}\right) + Li_2(x_s^2) + 2\ln(x_s)\ln(1-x_s^2) - \frac{\pi^2}{6} + \frac{1}{2}\ln^2\left(\frac{m_2}{m_1}\right) - \frac{1}{2}\ln^2(x_s^2)\right\}. \tag{65}
$$

5.2.2 B5



These two cases, B.5 and B.6, are the same given in two regularization schemes (B.6 dimensional regularization and B.5 mass regularization). To get rid of the lengthy calculation, the bridge between two regularization schemes is used

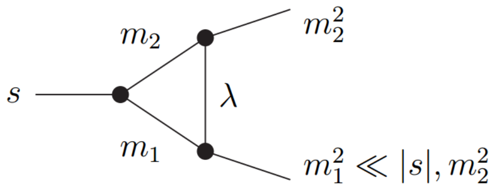
$$\ln \frac{\lambda^2}{m_1 m_2} \leftrightarrow \frac{\Gamma(1+\varepsilon)}{\varepsilon}\left(\frac{4\pi\mu^2}{m_1 m_2}\right)^{\varepsilon}|_{\varepsilon\to 0} = \frac{1}{\varepsilon}|_{\epsilon\to 0}. \tag{66}$$

Our result for this case will be

$$C_0 = \frac{x_s}{m_1 m_2(1-x_s^2)}\left\{-\ln\left(\frac{\lambda^2}{m_1 m_2}\right)\ln(x_s) + Li_2\left(1 - x_s\frac{m_1}{m_2}\right) + Li_2\left(1 - x_s\frac{m_2}{m_1}\right) + Li_2(x_s^2) + 2\ln(x_s)\ln(1-x_s^2) - \frac{\pi^2}{6} + \frac{1}{2}\ln^2\left(\frac{m_2}{m_1}\right) - \frac{1}{2}\ln^2(x_s^2)\right\}. \tag{67}$$

5.3 Overlapping Collinear and Soft singularities

5.3.1 B.8



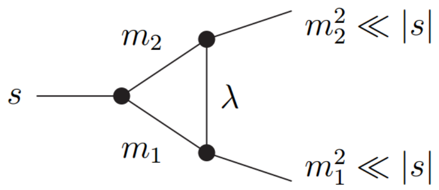
In this case, $m_1^2 << |s|, m_2^2$, x_s becomes

$$x_s = \frac{\sqrt{1-4m_1m_2/[\bar{s}-(m_1-m_2)^2]}-1}{\sqrt{1-4m_1m_2/[\bar{s}-(m_1-m_2)^2]}+1} \simeq \frac{\sqrt{1-4m_1m_2/[\bar{s}-m_2^2]}-1}{\sqrt{1-4m_1m_2/[\bar{s}-m_2^2]}+1} \simeq \frac{1-2m_1m_2/[\bar{s}-m_2^2]-1}{1-2m_1m_2/[\bar{s}-m_2^2]+1} = -\frac{m_1m_2/[\bar{s}-m_2^2]}{1-m_1m_2/[\bar{s}-m_2^2]} \simeq \frac{m_1m_2}{m_2^2-\bar{s}}. \tag{68}$$

Here $|\bar{s}-m_2^2| >> m_1m_2$ was used. From $B.5$ result,

$$\begin{aligned}
C_0 &= \frac{x_s}{m_1m_2(1-x_s^2)}\left\{-\ln\left(\frac{\lambda^2}{m_1m_2}\right)\ln(x_s) + Li_2\left(1-x_s\frac{m_1}{m_2}\right) + Li_2\left(1-x_s\frac{m_2}{m_1}\right) + Li_2(x_s^2) + 2\ln(x_s)\ln(1-x_s^2) - \frac{\pi^2}{6} + \frac{1}{2}\ln^2\left(\frac{m_2}{m_1}\right) - \frac{1}{2}\ln^2(x_s)\right\} \\
&\simeq \frac{m_1m_2}{m_1m_2(m_2^2-\bar{s})}\left\{-\frac{1}{2}\ln^2 x_s + 2\ln(1)\ln x_s - \ln\frac{\lambda^2}{m_1m_2}\ln x_s - \frac{\pi^2}{6} + Li_2(0) + \frac{1}{2}\ln^2\frac{m_1}{m_2} + Li_2(1) + Li_2\left(1-x_s\frac{m_2}{m_1}\right)\right\} \\
&= \frac{1}{m_2^2-\bar{s}}\left\{-\frac{1}{2}\ln^2\frac{m_1m_2}{m_2^2-\bar{s}} - \ln\frac{\lambda^2}{m_1m_2}\ln\frac{m_1m_2}{m_2^2-\bar{s}} + \frac{1}{2}\ln^2\frac{m_1}{m_2} + Li_2\left(1-\frac{m_2^2}{m_2^2-\bar{s}}\right)\right\} \\
&= \frac{1}{m_2^2-\bar{s}}\left\{-\frac{1}{2}\ln^2\frac{m_1m_2}{m_2^2-\bar{s}} - \ln\frac{\lambda^2}{m_1m_2}\ln\frac{m_1m_2}{m_2^2-\bar{s}} + \frac{1}{2}\ln^2\frac{m_1}{m_2} + Li_2\left(\frac{\bar{s}}{\bar{s}-m_2^2}\right)\right\} \\
&= \frac{1}{m_2^2-\bar{s}}\left\{-Li_2\left(\frac{\bar{s}}{m_2^2}\right) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2\left(-\frac{\bar{s}}{m_2^2}\right) - \frac{\pi^2}{6} - \frac{\pi^2}{6} + \ln\frac{m_2^2}{\bar{s}}\ln\left(1-\frac{m_2^2}{\bar{s}}\right) - \frac{1}{2}\ln^2\frac{\bar{s}}{m^2-\bar{s}} - \frac{1}{2}\ln^2\left(\frac{m_1m_2}{m_2^2-\bar{s}}\right) - \ln\frac{\lambda^2}{m_1m_2}\ln\left(\frac{m_1m_2}{m_2^2-\bar{s}}\right) + \frac{1}{2}\ln^2\frac{m_1}{m_2}\right\} \\
&= \frac{1}{m_2^2-\bar{s}}\left\{-Li_2\left(\frac{\bar{s}}{m_2^2}\right) - \frac{1}{2}\ln^2\left(\frac{m_2^2-\bar{s}}{m_2^2}\right) - \frac{1}{2}\ln^2\left(\frac{m_1m_2}{m_2^2-\bar{s}}\right) - \ln\frac{\lambda^2}{m_1m_2}\ln\frac{m_1m_2}{m_2^2-\bar{s}} + \frac{1}{2}\ln^2\frac{m_1}{m_2}\right\} \\
&= \frac{1}{m_2^2-\bar{s}}\left\{-Li_2\left(\frac{\bar{s}}{m_2^2}\right) + \frac{1}{2}\left(\ln\frac{m_1}{m_2} + \ln\left(\frac{m_2^2-\bar{s}}{m_2^2}\right)\right)\left(\ln\frac{m_1}{m_2} - \ln\left(\frac{m_2^2-\bar{s}}{m_2^2}\right)\right) - \frac{1}{2}\ln^2\left(\frac{m_1m_2}{m_2^2-\bar{s}}\right) - \ln\frac{\lambda^2}{m_1m_2}\ln\frac{m_1m_2}{m_2^2-\bar{s}}\right\} \\
&= \frac{1}{m_2^2-\bar{s}}\left\{-Li_2\left(\frac{\bar{s}}{m_2^2}\right) + \frac{1}{2}\ln\frac{m_1m_2}{m_2^2-\bar{s}}\ln\frac{m_1(m_2^2-\bar{s})}{m_2^3} - \frac{1}{2}\ln^2\frac{m_1m_2}{m_2^2-\bar{s}} - \ln\frac{\lambda^2}{m_1m_2}\ln\frac{m_1m_2}{m_2^2-\bar{s}}\right\} \\
&= \frac{1}{m_2^2-\bar{s}}\left\{-Li_2\left(\frac{\bar{s}}{m_2^2}\right) + \ln\frac{m_1m_2}{m_2^2-\bar{s}}\ln\frac{m_2^2-\bar{s}}{m_2^2} - \ln\frac{\lambda^2}{m_1m_2}\ln\frac{m_1m_2}{m_2^2-\bar{s}}\right\} \\
&= \frac{1}{\bar{s}-m_2^2}\left\{Li_2\left(\frac{\bar{s}}{m_2^2}\right) + \ln\left(\frac{m_1(m_2^2-\bar{s})}{\lambda^2 m_2}\right)\ln\left(\frac{m_2^2-\bar{s}}{m_1m_2}\right)\right\}.
\end{aligned} \tag{69}$$

5.3.2 B.9



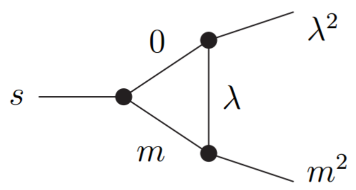
In this case, $m_1^2 << |s|$ and $m_2^2 << |s|$ then

$$x_s \simeq \frac{\sqrt{1-4m_1m_2/\bar{s}}-1}{\sqrt{1-4m_1m_2/\bar{s}}+1} \simeq \frac{1-2m_1m_2/\bar{s}-1}{1-2m_1m_2/\bar{s}+1} = -\frac{m_1m_2}{\bar{s}}\frac{1}{1-m_1m_2/\bar{s}} \simeq -\frac{m_1m_2}{\bar{s}}. \tag{70}$$

From $B.5$ result,

$$\begin{aligned}
C_0 &= \frac{x_s}{m_1m_2(1-x_s^2)}\left\{-\ln\left(\frac{\lambda^2}{m_1m_2}\right)\ln(x_s) + Li_2\left(1-x_s\frac{m_1}{m_2}\right) + Li_2\left(1-x_s\frac{m_2}{m_1}\right) + Li_2(x_s^2) + 2\ln(x_s)\ln(1-x_s^2) - \frac{\pi^2}{6} + \frac{1}{2}\ln^2\left(\frac{m_2}{m_1}\right) - \frac{1}{2}\ln^2(x_s)\right\} \\
&\simeq -\frac{m_1m_2}{m_1m_2\bar{s}}\left\{-\ln\frac{\lambda^2}{m_1m_2}\ln\left(\frac{m_1m_2}{-\bar{s}}\right) + \frac{\pi^2}{6} + \frac{1}{2}\ln^2\frac{m_2}{m_1} - \frac{1}{2}\ln^2\frac{m_1m_2}{-\bar{s}}\right\} \\
&= -\frac{1}{\bar{s}}\left\{-\ln\frac{\lambda^2}{-\bar{s}}\ln\frac{m_1m_2}{-\bar{s}} - \ln\frac{-\bar{s}}{m_1m_2}\ln\frac{m_1m_2}{-\bar{s}} + \frac{\pi^2}{6} + \frac{1}{2}\ln^2\frac{m_2}{m_1} - \frac{1}{2}\ln^2\frac{m_1m_2}{-\bar{s}}\right\} \\
&= -\frac{1}{\bar{s}}\left\{-\ln\frac{\lambda^2}{-\bar{s}}\ln\frac{m_1m_2}{-\bar{s}} + \frac{\pi^2}{6} + \frac{1}{2}\ln^2\frac{m_1m_2}{-\bar{s}} + \frac{1}{2}\ln^2\frac{m_2}{m_1}\right\} \\
&= -\frac{1}{\bar{s}}\left\{-\ln\frac{\lambda^2}{\bar{s}}\ln\frac{m_1m_2}{-\bar{s}} + \frac{\pi^2}{6} + \frac{1}{4}\left(\ln\frac{m_1m_2}{-\bar{s}} + \ln\frac{m_2}{m_1}\right)^2 + \frac{1}{4}\left(\ln\frac{m_1m_2}{-\bar{s}} - \ln\frac{m_2}{m_1}\right)^2\right\} \\
&= \frac{1}{\bar{s}}\left\{\ln\frac{\lambda^2}{\bar{s}}\ln\frac{m_1m_2}{-\bar{s}} + \frac{\pi^2}{6} - \frac{1}{4}\ln^2\frac{m_1^2}{-\bar{s}} - \frac{1}{4}\ln^2\frac{m_2^2}{-\bar{s}}\right\}.
\end{aligned} \tag{71}$$

5.3.3 B.10



where $p_2^2=\lambda^2$, $2p_1p_2=s+\lambda^2-m^2$ and $m_0=0, m_1=m$ and $m_2=\lambda$ and

$$M^2 = m^2y^2 - xy(s-m^2-\lambda^2) + \lambda^2(1-x)^2 - \lambda^2 y. \tag{72}$$

Our integral reads

$$\begin{aligned}
C_0 &= -\int_0^1 dx \int_0^{1-x} dy \left\{m^2y^2 - xy(s-m^2-\lambda^2) + \lambda^2(1-x)^2 - \lambda^2 y - i\delta\right\}^{-1} \\
&= -\int_0^1 dx \int_0^{1-x} dy \left\{m^2y^2 - xy(\bar{s}-m^2-\lambda^2) + \lambda^2(1-x)^2 - \lambda^2 y - i\delta(1+xy)\right\}^{-1} \\
&= -\int_0^1 dx \int_0^{1-x} dy \left\{m^2y^2 - xy(\bar{s}-m^2-\lambda^2) + \lambda^2(1-x)^2 - \lambda^2 y\right\}^{-1}.
\end{aligned} \tag{73}$$

Using the usual transformation,

$$y = \omega\eta, \quad \omega = 1-x, \tag{74}$$

the integral becomes

$$\begin{aligned}
C_0 &= -\int_0^1 \omega d\omega \int_0^1 d\eta \left\{m^2\omega^2\eta^2 + \lambda^2\omega^2 - \omega\eta(\bar{s}-m^2) + \omega^2\eta(\bar{s}-m^2-\lambda^2)\right\}^{-1} \\
&= -\int_0^1 d\omega \int_0^1 d\eta \left\{\omega\left(m^2\eta^2 + \eta(\bar{s}-m^2-\lambda^2) + \lambda^2\right) - \eta(\bar{s}-m^2)\right\}^{-1} \\
&= -\int_0^1 \frac{d\eta}{\eta^2m^2 + \eta(\bar{s}-m^2-\lambda^2) + \lambda^2} \left\{\ln\left(m^2\eta^2 - \lambda^2\eta + \lambda^2\right) - \ln[(m^2-\bar{s})\eta]\right\} \\
&\simeq -\frac{1}{m^2(-\lambda^2/(\bar{s}-m^2) + (\bar{s}-m^2)/m^2)} \int_0^1 \left(\frac{1}{\eta + \lambda^2/(\bar{s}-m^2)} - \frac{1}{\eta + (\bar{s}-m^2)/m^2}\right) \left\{\ln(\eta-\eta_+) + \ln(\eta-\eta_-) - \ln\eta - \ln\left(\frac{m^2-\bar{s}}{m^2}\right)\right\} \\
&= -\frac{1}{m^2(-\lambda^2/(\bar{s}-m^2) + (\bar{s}-m^2)/m^2)} (I-J)
\end{aligned} \tag{75}$$

where

$$\eta_\pm = \pm i\frac{\lambda}{m} \tag{76}$$

are two roots of quadratic equation $m^2\eta^2 - \lambda^2\eta + \lambda^2 = 0$. About I

$$\begin{aligned}
I &\simeq \int_0^1 \frac{d\eta}{\eta + \lambda^2/(\bar{s}-m^2)} \ln\left(\eta - i\frac{\lambda}{m}\right)\left(\eta + i\frac{\lambda}{m}\right) - \int_0^1 \frac{d\eta}{\eta + \lambda^2/(\bar{s}-m^2)} \left\{\ln\eta + \ln\left(\frac{m^2-\bar{s}}{m^2}\right)\right\} \\
&= \int_0^1 \frac{d\eta}{\eta + \lambda^2/(\bar{s}-m^2)} \ln\left(\eta^2 + \frac{\lambda^2}{m^2}\right) - \int_0^1 \frac{d\eta}{\eta + \lambda^2/(\bar{s}-m^2)} \left\{\ln\eta + \ln\left(\frac{m^2-\bar{s}}{m^2}\right)\right\} \\
&= \int_0^1 \frac{d\eta}{\eta} \ln\left(1 + \frac{m^2}{\lambda^2}\eta\right) - \int_0^1 \frac{d\eta}{\eta + \lambda^2/(\bar{s}-m^2)} \left\{\ln\eta + \ln\left(\frac{m^2-\bar{s}}{m^2}\right)\right\} \\
&= -\frac{1}{2}Li_2\left(-\frac{m^2}{\lambda^2}\right) - \left\{\ln\left(\frac{m^2-\bar{s}}{m^2}\right)\ln\left(\eta + \frac{\lambda}{\bar{s}-m^2}\right) + Li_2\left(\frac{\lambda^2/(\bar{s}-m^2)}{\eta + \lambda^2/(\bar{s}-m^2)}\right) + \frac{1}{2}\ln^2\left(\eta + \frac{\lambda^2}{\bar{s}-m^2}\right)\right\}\Big|_0^1 \\
&= \frac{1}{2}Li_2\left(-\frac{\lambda^2}{m^2}\right) + \frac{\pi^2}{12} + \frac{1}{4}\ln^2\left(\frac{\lambda^2}{m^2}\right) - \left[-\ln\left(\frac{m^2-\bar{s}}{m^2}\right)\ln\left(\frac{\lambda}{\bar{s}-m^2}\right) - Li_2(1) - \frac{1}{2}\ln^2\left(\frac{\lambda^2}{\bar{s}-m^2}\right)\right] \\
&= \frac{\pi^2}{12} + \frac{\pi^2}{6} + \ln^2\left(\frac{\lambda}{m}\right) + \ln\left(\frac{m^2-\bar{s}}{m^2}\right)\ln\left(\frac{\lambda^2}{\bar{s}-m^2}\right) + \frac{1}{2}\ln^2\left(\frac{\lambda^2}{\bar{s}-m^2}\right) \\
&= \frac{\pi^2}{4} + \ln^2\left(\frac{\lambda}{m}\right) - \ln\left(\frac{m^2-\bar{s}}{m^2}\right)\ln\left(\frac{\bar{s}-m^2}{m^2}\right) + \frac{1}{2}\ln^2\left(\frac{\bar{s}-m^2}{\lambda^2}\right) \\
&= \frac{\pi^2}{4} + \ln^2\left(\frac{\lambda}{m}\right) - \ln\left(\frac{m^2-\bar{s}}{m^2}\right)\ln\left(\frac{\bar{s}-m^2}{\lambda^2}\right) + \frac{1}{2}\left(\ln^2\frac{m^2-\bar{s}}{\lambda^2} - \pi^2 + 2i\pi\ln\frac{m^2-\bar{s}}{\lambda^2}\right) \\
&= \frac{\pi^2}{4} + \ln^2\left(\frac{\lambda}{m}\right) - \frac{\pi^2}{2} + \frac{1}{2}\ln^2\frac{m^2-\bar{s}}{\lambda^2} - \ln\frac{m^2-\bar{s}}{m^2}\left(\ln\frac{\bar{s}-m^2}{\lambda^2} - i\pi\right) \\
&= -\frac{\pi^2}{4} + \ln^2\frac{\lambda}{m} + \frac{1}{2}\ln\left(\frac{m^2-\bar{s}}{\lambda^2}\right) - \ln^2\frac{m^2-\bar{s}}{\lambda^2}\ln\frac{m^2-\bar{s}}{m^2}.
\end{aligned} \tag{77}$$

About J

$$
\begin{aligned}
J \simeq& \int_0^1 \frac{d\eta}{\eta + (\bar{s}-m^2)/m^2} \left\{ \ln\left(\eta + i\frac{\lambda}{m}\right) + \ln\left(\eta - i\frac{\lambda}{m}\right) - \ln\eta - \ln\left(\frac{m^2-\bar{s}}{m^2}\right) \right\} \\
=& \left\{ Li_2\left(\frac{-i\lambda/m + (\bar{s}-m^2)/m^2}{\eta + (\bar{s}-m^2)/m^2}\right) + Li_2\left(\frac{i\lambda/m + (\bar{s}-m^2)/m^2}{\eta + (\bar{s}-m^2)/m^2}\right) - Li_2\left(\frac{(\bar{s}-m^2)/m^2}{\eta + (\bar{s}-m^2)/m^2}\right) - \ln\left(\frac{m^2-\bar{s}}{m^2}\right) \ln\left(\eta + \frac{\bar{s}-m^2}{m^2}\right) \right\} |_0^1 \\
=& Li_2\left(\frac{\bar{s}-m^2}{\bar{s}}\right) - Li_2\left(1 - \frac{im\lambda}{m^2-\bar{s}}\right) + Li_2\left(\frac{\bar{s}-m^2}{\bar{s}}\right) - Li_2\left(1 + \frac{im\lambda}{m^2-\bar{s}}\right) + Li_2(1) - Li_2\left(\frac{\bar{s}-m^2}{\bar{s}}\right) - \ln\left(\frac{m^2-\bar{s}}{m^2}\right) \ln\left(\frac{\bar{s}-m^2}{\bar{s}}\right) \\
=& Li_2\left(\frac{\bar{s}}{m^2}\right) - \frac{\pi^2}{6} + \ln\left(\frac{\bar{s}}{m^2}\right) \ln\left(\frac{m^2-\bar{s}}{m^2}\right) - \frac{1}{2}\ln^2\frac{\bar{s}}{m^2} + \frac{1}{2}\ln^2\left(\frac{\bar{s}}{m^2}\right) - \frac{1}{2}\ln^2\frac{\bar{s}-m^2}{m^2} - \frac{\pi^2}{6} + \ln\frac{m^2-\bar{s}}{m^2} \ln\left(\frac{\bar{s}-m^2}{\bar{s}}\right) \\
=& Li_2\left(\frac{\bar{s}}{m^2}\right) - \frac{\pi^2}{6} + \ln\frac{\bar{s}}{m^2} \ln\frac{m^2-\bar{s}}{m^2} - \frac{1}{2}\left(\ln^2\frac{m^2-\bar{s}}{m^2} - \pi^2 = 2i\pi\ln\frac{m^2-\bar{s}}{m^2}\right) + \ln\frac{m^2-\bar{s}}{m^2} \ln\frac{\bar{s}-m^2}{\bar{s}} \\
=& Li_2\left(\frac{\bar{s}}{m^2}\right) - \frac{\pi^2}{3} + \frac{\pi^2}{2} + \ln\frac{\bar{s}}{m^2} \ln\frac{m^2-\bar{s}}{m^2} - \frac{1}{2}\ln^2\frac{m^2-\bar{s}}{m^2} + \ln\frac{m^2-\bar{s}}{m^2}\left(\ln\frac{\bar{s}-m^2}{\bar{s}} - i\pi\right) \\
=& Li_2\left(\frac{\bar{s}}{m^2}\right) + \frac{\pi^2}{6} + \ln\frac{\bar{s}}{m^2} \ln\frac{m^2-\bar{s}}{m^2} - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{m^2}\right) + \ln\frac{m^2-\bar{s}}{m^2} \ln\frac{m^2-\bar{s}}{\bar{s}} \\
=& Li_2\left(\frac{\bar{s}}{m^2}\right) + \ln^2\left(\frac{m^2-\bar{s}}{m^2}\right) - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{m^2}\right) + \frac{\pi^2}{6} \\
=& Li_2\left(\frac{\bar{s}}{m^2}\right) + \frac{1}{2}\ln^2\frac{m^2-\bar{s}}{m^2} + \frac{\pi^2}{6}.
\end{aligned} \tag{78}
$$

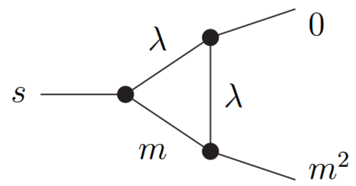
Then

$$
\begin{aligned}
C_0 =& -\frac{1}{m^2(-\lambda^2/(\bar{s}-m^2) + (\bar{s}-m^2)/m^2)} \left\{ -\frac{\pi^2}{4} + \ln^2\frac{\lambda}{m} + \frac{1}{2}\ln\left(\frac{m^2-\bar{s}}{\lambda^2}\right) - \ln^2\frac{m^2-\bar{s}}{\lambda^2} \ln\frac{m^2-\bar{s}}{m^2} - Li_2\left(\frac{\bar{s}}{m^2}\right) - \frac{1}{2}\ln^2\frac{m^2-\bar{s}}{m^2} - \frac{\pi^2}{6} \right\} \\
=& -\frac{1}{m^2(-\lambda^2/(\bar{s}-m^2) + (\bar{s}-m^2)/m^2)} \left\{ -\frac{5\pi^2}{12} - Li_2\left(\frac{\bar{s}}{m^2}\right) + \ln^2\frac{\lambda}{m} + \frac{1}{2}\ln\frac{m^2-\bar{s}}{\lambda^2} \ln\frac{m^2-\bar{s}}{m^2} - \frac{1}{2}\ln^2\frac{m^2-\bar{s}}{m^2} \right\} \\
=& \frac{1}{\bar{s}-m^2} \left\{ \frac{5\pi^2}{12} + Li_2\left(\frac{\bar{s}}{m^2}\right) + \ln^2\left(\frac{m^2-\bar{s}}{m\lambda}\right) \right\}.
\end{aligned} \tag{79}
$$

Note that

$$
\begin{aligned}
& \ln^2\frac{\lambda}{m} - \ln\frac{m^2-\bar{s}}{m^2} \ln\frac{m^2-\bar{s}}{\lambda^2} + \frac{1}{2}\ln\frac{m^2-\bar{s}}{\lambda^2} - \frac{1}{2}\ln^2\frac{m^2-\bar{s}}{m^2} \\
&= \ln^2\frac{\lambda}{m} - \frac{1}{2}\left(\ln\frac{m^2-\bar{s}}{m^2} + \ln\frac{m^2-\bar{s}}{\lambda^2}\right)^2 + \ln^2\frac{m^2-\bar{s}}{\lambda^2} \\
&= \ln^2\frac{\lambda}{m} - 2\ln\frac{m^2-\bar{s}}{m\lambda} + \ln^2\frac{m^2-\bar{s}}{\lambda^2} \\
&= \ln\frac{m^2-\bar{s}}{\lambda^2} \ln\frac{m^2-\bar{s}}{m^2} + \ln\frac{m}{\lambda}\left(\ln\frac{m^2-\bar{s}}{\lambda} - \ln\frac{m^2-\bar{s}}{m\lambda}\right) \\
&= \ln\frac{m^2-\bar{s}}{\lambda^2} \ln\frac{m^2-\bar{s}}{m\lambda} - \ln\frac{m}{\lambda} \ln\frac{m^2-\bar{s}}{m\lambda} \\
&= \ln\frac{m^2-\bar{s}}{m\lambda} \ln\frac{m^2-\bar{s}}{m\lambda} = \ln^2\frac{m^2-\bar{s}}{m\lambda}.
\end{aligned} \tag{80}
$$

5.3.4 B.11



where $m_0 = m_2 = \lambda$, $m_1 = m$, $p_2^2 = \lambda^2$, $2p_1p_2 = s + \lambda^2 - m^2$ and

$$M^2 = m^2y^2 + \lambda^2(1-y) - xy(s-m^2) \tag{81}$$

so

$$\begin{aligned}
C_0 &= -\int_0^1 dx \int_0^{1-x} dy \left\{m^2y^2 + \lambda^2(1-y) - xy(s-m^2) - i\delta\right\}^{-1} = -\int_0^1 dx \int_0^{1-x} dy \left\{m^2y^2 + \lambda^2(1-y) - xy(\bar{s}-m^2) - i\delta(1-xy)\right\}^{-1} \\
&= -\int_0^1 dx \int_0^{1-x} dy \left\{m^2y^2 + \lambda^2(1-y) - xy(\bar{s}-m^2)\right\}^{-1} = -\int_0^1 dy \int_0^{1-y} dx \left\{m^2y^2 + \lambda^2(1-y) - xy(\bar{s}-m^2)\right\}^{-1} \\
&= \frac{1}{\bar{s}-m^2}\int_0^1 \frac{dy}{y}\left\{\ln\left(y^2\bar{s} - y(\bar{s}-m^2+\lambda^2) + \lambda^2\right) - \ln\left(m^2y^2 - \lambda^2 y + \lambda^2\right)\right\} \\
&= \frac{1}{\bar{s}-m^2}\int_0^1 \frac{dy}{y}\left\{\ln\frac{\bar{s}}{m^2} + \ln(y-y_{1+})(y-y_{1-}) - \ln(y-y_{2+})(y-y_{2-})\right\} \\
&= \frac{1}{\bar{s}-m^2}\int_0^1 \frac{dy}{y}\left\{\ln(y_{1+}-y)(y_{1-}-y) - \ln(y_{1+}y_{1-}) - \ln(y_{2-}-y)(y_{2+}-y) + \ln(y_{2-}y_{2+})\right\} \\
&= \frac{1}{\bar{s}-m^2}\int_0^1 \frac{dy}{y}\left\{\ln(y_{1+}-y) - \ln(y_{1+}) + \ln(y_{1-}-y) - \ln(y_{1-}) - (\ln(y_{2+}-y) - \ln(y_{2+})) - (\ln(y_{2-}-y) - \ln(y_{2-}))\right\} \\
&= \frac{1}{\bar{s}-m^2}\int_0^1 \frac{dy}{y}\left\{\ln\left(1-\frac{y}{y_{1+}}\right) + \ln\left(1-\frac{y}{y_{1-}}\right) - \ln\left(1-\frac{y}{y_{2+}}\right) - \ln\left(1-\frac{y}{y_{2-}}\right)\right\} \\
&= -\frac{1}{\bar{s}-m^2}\left\{Li_2\left(\frac{\bar{s}}{\bar{s}-m^2}\right) + Li_2\left(\frac{\bar{s}-m^2}{\lambda^2}\right) + Li_2\left(\frac{1}{i\lambda/m}\right) + Li_2\left(-\frac{1}{i\lambda/m}\right)\right\} \\
&= -\frac{1}{\bar{s}-m^2}\left\{Li_2\left(\frac{\bar{s}}{\bar{s}-m^2}\right) + Li_2\left(\frac{\bar{s}-m^2}{\lambda^2}\right) - \frac{1}{2}Li_2\left(-\frac{m^2}{\lambda^2}\right)\right\} \\
&= -\frac{1}{\bar{s}-m^2}\left\{-Li_2\left(-\frac{\lambda^2}{m^2-\bar{s}}\right) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\lambda^2}\right) - Li_2\left(-\frac{m^2-\bar{s}}{\bar{s}}\right) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\bar{s}}\right) + \frac{1}{2}Li_2\left(-\frac{\lambda^2}{m^2}\right) + \frac{\pi^2}{12} + \frac{1}{4}\ln^2\left(\frac{\lambda^2}{m^2}\right)\right\} \\
&\simeq -\frac{1}{\bar{s}-m^2}\left\{-\frac{\pi^2}{4} - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\lambda^2}\right) + \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\bar{s}}\right) + \ln^2\left(\frac{\lambda}{m}\right) - Li_2\left(\frac{\bar{s}-m^2}{\bar{s}}\right)\right\} \\
&= -\frac{1}{\bar{s}-m^2}\left\{-\frac{\pi^2}{4} - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\lambda^2}\right) + \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\bar{s}}\right) + \ln^2\left(\frac{\lambda}{m}\right) + Li_2\left(1-\frac{\bar{s}}{m^2}\right) + \frac{1}{2}\ln^2\left(\frac{\bar{s}}{m^2}\right)\right\} \\
&= -\frac{1}{\bar{s}-m^2}\left\{-\frac{\pi^2}{4} - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\lambda^2}\right) + \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\bar{s}}\right) + \ln^2\left(\frac{\lambda}{m}\right) - Li_2\left(\frac{\bar{s}}{m^2}\right) + \frac{\pi^2}{6} - \ln\left(\frac{\bar{s}}{m^2}\right)\ln\left(1-\frac{\bar{s}}{m^2}\right) + \frac{1}{2}\ln^2\left(\frac{\bar{s}}{m^2}\right)\right\} \\
&= -\frac{1}{\bar{s}-m^2}\left\{-\frac{\pi^2}{12} - Li_2\left(\frac{\bar{s}}{m^2}\right) - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\lambda^2}\right) - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\bar{s}}\right) + \ln^2\frac{\lambda}{m} + \frac{1}{2}\ln^2\left(\frac{\bar{s}}{m^2}\right) - \ln\left(\frac{\bar{s}}{m^2}\right)\ln\left(\frac{m^2-\bar{s}}{m^2}\right)\right\} \\
&= \frac{1}{\bar{s}-m^2}\left\{\frac{\pi^2}{12} + Li_2\left(\frac{\bar{s}}{m^2}\right) + \ln^2\left(\frac{m^2-\bar{s}}{\lambda m}\right)\right\}
\end{aligned}$$

where we have used

$$\begin{aligned}
-\frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\lambda^2}\right) - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\bar{s}}\right) + \ln^2\frac{\lambda}{m} + \frac{1}{2}\ln^2\left(\frac{\bar{s}}{m^2}\right) - \ln\left(\frac{\bar{s}}{m^2}\right)\ln\left(\frac{m^2-\bar{s}}{m^2}\right) &= -\frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\lambda^2}\right) - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\bar{s}}\right) + \ln^2\frac{\lambda}{m} + \frac{1}{2}\left(\ln^2\left(\frac{m^2-\bar{s}}{m^2}\right) - \ln\left(\frac{\bar{s}}{m^2}\right)\right)^2 - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{m^2}\right. \\
&= -\frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\lambda^2}\right) + \ln^2\left(\frac{\lambda}{m}\right) - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{m^2}\right) \\
&= -\frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{\lambda^2}\right) + \frac{1}{2}\ln^2\left(\frac{\lambda}{m}\right) + \frac{1}{2}\ln^2\left(\frac{\lambda}{m}\right) - \frac{1}{2}\ln^2\left(\frac{m^2-\bar{s}}{m^2}\right) \\
&= -\frac{1}{2}\ln\frac{m^2-\bar{s}}{\lambda m}\ln\left(\frac{(m^2-\bar{s})m}{\lambda^3}\right) - \frac{1}{2}\ln\frac{m^2-\bar{s}}{\lambda m}\ln\left(\frac{(m^2-\bar{s})\lambda}{m^3}\right) \\
&= -\frac{1}{2}\ln\left(\frac{m^2-\bar{s}}{m\lambda}\right)\left[\ln\left(\frac{(m^2-\bar{s})m}{\lambda^3}\right) + \ln\left(\frac{(m^2-\bar{s})\lambda}{m^3}\right)\right] \\
&= -\frac{1}{2}\ln\left(\frac{m^2-\bar{s}}{m\lambda}\right)\ln\left(\frac{(m^2-\bar{s})^2}{m^2\lambda^2}\right) \\
&= -\ln^2\left(\frac{m^2-\bar{s}}{m\lambda}\right).
\end{aligned} \tag{82}$$

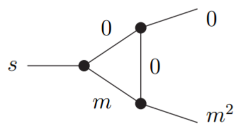
Note that

$$Im\left(\frac{(m^2-\bar{s})^2}{m^2\lambda^2}\right) = -\frac{\delta}{\lambda^2}\left(\frac{1}{m^2}-2\right) > 0. \tag{83}$$

so $\eta\left(\frac{(m^2-\bar{s})m}{\lambda^3}, \frac{(m^2-\bar{s})\lambda}{m^3}\right) = 0$. Moreover, y_{1+}, y_{1-}, y_{2+}, y_{2-} are the roots of equation $y^2\bar{s} - y(\bar{s}-m^2+\lambda^2) + \lambda^2 = 0$ and $m^2y^2 - \lambda^2 y + \lambda^2 = 0$

$$\begin{aligned}
&y_{1+} = \frac{\bar{s}-m^2}{\bar{s}}, \quad y_{1-} = \frac{\lambda^2}{\bar{s}-m^2}, \quad y_{1+}y_{1-} = \frac{\lambda^2}{\bar{s}}, \\
&y_{2+} = i\frac{\lambda}{m}, \quad y_{2-} = -i\frac{\lambda}{m}, \quad y_{2+}y_{2-} = \frac{\lambda^2}{m^2}.
\end{aligned} \tag{84}$$

5.3.5 B.12



We have $p_0 = 0$, $p_1^2 = s$, $p_2^2 = 0$ and $2p_1p_2 = s - m^2$

$$M^2 = y^2m^2 + 2y(1-x)(s-m^2) - y(s-m^2) = y\left(ym^2 - x(s-m^2)\right). \tag{85}$$

$$\begin{aligned} C_0 &= -(1)^3(4\pi\mu)^\varepsilon\Gamma(1+\varepsilon)\int_0^1 dx\int_0^{1-x} dy\left(ym^2 - x(s-m^2) - i\delta\right)^{-1-\varepsilon} \\ &= -(4\pi\mu)^\varepsilon\Gamma(1+\varepsilon)\int_0^1 dx\int_0^x dy\left(ym^2 - (1-x)(s-m^2) - i\delta\right)^{-1-\varepsilon}. \end{aligned} \tag{86}$$

Next, we change the order of integration,

$$\int_0^1 dx\int_0^x dy \to \int_0^1 dy\int_y^1 dx \tag{87}$$

then the integral is evaluated over x

$$\int_y^1 dx\left(ym^2 - (1-x)(s-m^2) - i\delta\right)^{-1-\varepsilon} = \frac{1}{\varepsilon(s-m^2)}\left[(ym^2 - (1-y)(s-m^2) - i\delta)^{-\varepsilon} - (ym^2 - i\delta)^{-\varepsilon}\right]. \tag{88}$$

So

$$\begin{aligned} C_0 &= -(4\pi\mu)^\varepsilon\Gamma(1+\varepsilon)\frac{1}{\varepsilon(s-m^2)}\int_0^1 dyy^{-1-\varepsilon}\left[(ym^2 - (1-y)(s-m^2) - i\delta)^{-\varepsilon} - (ym^2 - i\delta)^{-\varepsilon}\right] \\ &= -(4\pi\mu)^\varepsilon\Gamma(1+\varepsilon)\frac{1}{\varepsilon(s-m^2)}\left\{\int_0^1 dyy^{-1-\varepsilon}\left((ym^2 - (1-y)(s-m^2) - i\delta)^{-\varepsilon} - (-s+m^2-i\delta)^{-\varepsilon}\right)\right. \\ &\quad \left. -\int_0^1 dyy^{-1-\varepsilon}(ym^2 - i\delta)^{-\varepsilon} + \int_0^1 dyy^{-1-\varepsilon}(-s+m^2-i\delta)^{-\varepsilon})\right\} \\ &= -(4\pi\mu)^\varepsilon\Gamma(1+\varepsilon)\frac{1}{\varepsilon(s-m^2)}\left\{\int_0^1 dyy^{-1-\varepsilon}\left((ym^2 - (1-y)(s-m^2+i\delta') - i\delta + i(1-y)\delta)^{-\varepsilon} - (-(s-m^2) - i\delta)^{-\varepsilon}\right)\right. \\ &\quad \left. -\int_0^1 dyy^{-1-\varepsilon}(ym^2 - i\delta)^{-\varepsilon} + \int_0^1 dyy^{-1-\varepsilon}(-(s-m^2)-i\delta)^{-\varepsilon}\right\} \\ &\simeq -(4\pi\mu)^\varepsilon\Gamma(1+\varepsilon)\frac{1}{\varepsilon(s-m^2)}\left\{\int_0^1 dyy^{-1-\varepsilon}\left((ym^2 - (1-y)(\bar{s}-m^2))^{-\varepsilon} - (-\bar{s}+m^2)^{-\varepsilon}\right)\right. \\ &\quad \left. -\int_0^1 dyy^{-1-\varepsilon}(ym^2)^{-\varepsilon} + \int_0^1 dyy^{-1-\varepsilon}(-\bar{s}+m^2)^{-\varepsilon}\right\} \end{aligned} \tag{89}$$

where $\bar{s} = s + i\delta'$. Expanding the integrand in the first integral to ε

$$\begin{aligned} \textbf{First integrand} &= \frac{1}{y}(1 - \varepsilon\ln y + O(\varepsilon^2))\left[\left(1 - \varepsilon\ln(ym^2 - (1-y)(\bar{s}-m^2))\right) - \right. \\ &\quad \left.(1 - \varepsilon\ln(-\bar{s}+m^2)) + O(\varepsilon^2)\right] \\ &= -\frac{\varepsilon}{y}\ln\left(\frac{ym^2 - (1-y)(\bar{s}-m^2)}{-\bar{s}+m^2}\right) + O(\varepsilon^2) \\ &= -\frac{\varepsilon}{y}\ln\left(1 - y\frac{\bar{s}}{\bar{s}-m^2}\right) + O(\varepsilon^2). \end{aligned} \tag{90}$$

Evaluating the second and third integral, we have

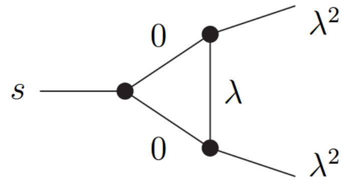
$$\int_0^1 dyy^{-1-2\varepsilon}(m^2)^{-\varepsilon} = -(m^2)^{-\varepsilon}\frac{y^{-2\varepsilon}}{2\varepsilon}|_0^1 = -\frac{m^{-2\varepsilon}}{2\varepsilon}, \tag{91}$$

$$\int_0^1 dyy^{-1-\varepsilon}(-\bar{s}+m^2)^{-\varepsilon}) = -(-\bar{s}+m^2))^{-\varepsilon}\frac{y^{-\varepsilon}}{\varepsilon}|_0^1 = -\frac{(-\bar{s}+m^2)^{-\varepsilon}}{\varepsilon}. \tag{92}$$

To sum up

$$\begin{aligned} C_0 &= \frac{1}{s-m^2}\left\{(4\pi\mu)^\varepsilon\Gamma(1+\varepsilon)\int_0^1\frac{dy}{y}\ln\left(1 - y\frac{\bar{s}}{\bar{s}-m^2}\right) + \frac{\Gamma(1+\varepsilon)}{\varepsilon^2}\left(\frac{4\pi\mu}{m^2-\bar{s}}\right)^\varepsilon - \frac{\Gamma(1+\varepsilon)}{2\varepsilon^2}\left(\frac{4\pi\mu}{m^2}\right)^\varepsilon + O(\varepsilon)\right\} \\ &= \frac{1}{\bar{s}-m^2}\left\{(1 + \varepsilon\ln(4\pi\mu) + O(\varepsilon^2))(1 - \gamma_E\varepsilon + O(\varepsilon^2))\int_0^1\frac{1}{y}\ln\left(1 - y\frac{\bar{s}}{\bar{s}-m^2}\right)dy + \frac{\Gamma(1+\varepsilon)}{\varepsilon^2}\left(\frac{4\pi\mu}{m^2-\bar{s}}\right)^\varepsilon - \right. \\ &\quad \left.\frac{\Gamma(1+\varepsilon)}{2\varepsilon^2}\left(\frac{4\pi\mu}{m}\right)^\varepsilon + O(\varepsilon)\right\} \\ &= \frac{1}{\bar{s}-m^2}\left\{-Li_2\left(\frac{\bar{s}}{\bar{s}-m^2}\right) + \frac{\Gamma(1+\varepsilon)}{\varepsilon^2}\left(\frac{4\pi\mu}{m^2-\bar{s}}\right)^\varepsilon - \frac{\Gamma(1+\varepsilon)}{2\varepsilon^2}\left(\frac{4\pi\mu}{m^2}\right)^\varepsilon + O(\varepsilon)\right\}. \end{aligned} \tag{93}$$

5.3.6 B13



where $m_0 = m_1 = 0$, $m_2 = \lambda$, $p_2^2 = \lambda^2$, $2p_1p_2 = s$ and

$$M^2 = \lambda^2(1-x-y)^2 - sxy \tag{94}$$

so

$$\begin{aligned} C_0 &= -\int_0^1 dx \int_0^{1-x} dy \left\{\lambda^2(1-x-y)^2 - sxy - i\delta\right\}^{-1} = -\int_0^1 dx \int_0^{1-x} dy \left\{\lambda^2(1-x-y)^2 - \bar{s}xy - i\delta(1-xy)\right\}^{-1} \\ &= -\int_0^1 dx \int_0^{1-x} dy \left\{\lambda^2(1-x-y)^2 - \bar{s}xy\right\}^{-1}. \end{aligned} \tag{95}$$

Using the transformation,

$$y = (1-x)\eta, \quad \omega = 1-x, \quad J = \omega. \tag{96}$$

After that, our integral becomes

$$\begin{aligned} C_0 &= -\int_0^1 \omega d\omega \int_0^1 d\eta \left\{-\bar{s}(1-\omega)\omega\eta + \lambda^2(1-\eta)^2\omega^2 - i\delta\right\}^{-1} = -\int_0^1 d\omega \int_0^1 d\eta \left\{-\bar{s}(1-\eta) + [\bar{s}(1-\eta) + \lambda^2\eta^2]\omega\right\}^{-1} \\ &= -\int_0^1 \frac{d\eta}{\lambda^2\eta^2 + \bar{s}(1-\eta)} \left\{\ln\left(\lambda^2\eta^2\right) - \ln\left(-\bar{s}(1-\eta)\right)\right\} = -\int_0^1 \frac{d\eta}{\lambda^2\eta^2 + \bar{s}(1-\eta)} \left\{\ln(\lambda^2\eta^2) - \ln(-\bar{s}) - \ln\left(1-\eta\right)\right\} \\ &= -\int_0^1 \frac{d\eta}{\lambda^2(\eta-\eta_+)(\eta-\eta_-)} \left\{\ln(\lambda^2\eta^2) - \ln(-\bar{s}) - \ln\left(1-\eta\right)\right\} \\ &= -\frac{1}{\lambda^2(\eta_+-\eta_-)} \int_0^1 d\eta \left(\frac{1}{\eta-\eta_+} - \frac{1}{\eta-\eta_-}\right) \left\{\ln\left(-\frac{\lambda^2}{\bar{s}}\eta^2\right) - \ln\left(1-\eta\right)\right\} \\ &= -\frac{1}{\lambda^2(\eta_+-\eta_-)} \int_0^1 d\eta \left(\frac{1}{\eta-\eta_+} - \frac{1}{\eta-\eta_-}\right) \left\{\ln\left(-\frac{\lambda^2}{\bar{s}}\right) + 2\ln\eta - \ln\left(1-\eta\right)\right\} \\ &= -\frac{1}{\lambda^2\left(\eta_+-\eta_-\right)} \left\{\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\left(\ln\frac{1-\eta_+}{-\eta_+} - \ln\frac{1-\eta_-}{-\eta_-}\right) + 2\int_0^1 d\eta \left(\frac{1}{\eta-\eta_+} - \frac{1}{\eta-\eta_-}\right)\ln\eta - \int_0^1 d\eta\left(\frac{1}{\eta-\eta_+} - \frac{1}{\eta-\eta_-}\right)\ln(1-\eta)\right\} \\ &= -\frac{1}{\lambda^2(\eta_+-\eta_-)} \left\{\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\left(\ln\frac{1-\eta_+}{-\eta_+} - \ln\frac{1-\eta_-}{-\eta_-}\right) + 2\left(I_+ - I_-\right) - \left(J_+ - J_-\right)\right\} \end{aligned} \tag{97}$$

where η_+ and η_- are two roots of quadiarac equation $\lambda^2\eta^2 + \bar{s}(1-\eta) = 0$,

$$\eta_\pm = \frac{1}{2\lambda^2}\left(\bar{s} \pm \sqrt{s^2 - 4\lambda^2 s}\right) = \frac{\bar{s}}{2\lambda^2}\left(1 \pm \sqrt{1 - \frac{4\lambda^2}{\bar{s}}}\right) \simeq \frac{\bar{s}}{2\lambda^2}\left(1 \pm 1 \mp \frac{2\lambda^2}{\bar{s}} \mp \frac{2\lambda^4}{\bar{s}^2}\right) = \begin{cases} \eta_+ = \dfrac{\bar{s}}{\lambda^2} - 1 - \dfrac{\lambda^2}{\bar{s}} \\ \eta_- = 1 + \dfrac{\lambda^2}{\bar{s}} \end{cases} \tag{98}$$

and further

$$\begin{aligned} \frac{1-\eta_+}{-\eta_+} &= 1 - \frac{1}{\eta_+} = 1 - \frac{1}{\bar{s}/\lambda^2 - 1 - \lambda^2/\bar{s}} = 1 - \frac{\lambda^2}{\bar{s}} \simeq 1 \\ \frac{1-\eta_-}{\eta_-} &= 1 - \frac{1}{\eta_-} \simeq 1 - \frac{1}{1+\lambda^2/\bar{s}} = 1 - \left(1 - \frac{\lambda^2}{\bar{s}}\right) = \frac{\lambda^2}{\bar{s}} \\ 1 - \eta_+ &= 2 - \frac{\bar{s}}{\lambda^2} + \frac{\lambda^2}{\bar{s}} \simeq -\frac{\bar{s}}{\lambda^2} \\ 1 - \eta_- &= -\frac{\lambda^2}{\bar{s}}. \end{aligned} \tag{99}$$

Note that

$$\int_0^A \frac{dz}{z-b}\ln\left(a-z\right) = \left\{Li_2\left(\frac{b-a}{b-z}\right) + \frac{1}{2}\ln^2\left(b-z\right)\right\}|_0^A + \int_0^A \eta\left(a-z, \frac{1}{b-z}\right)\frac{dz}{z-b}, \tag{100}$$

$$\int_0^A \frac{dz}{z-b}\ln\left(z-a\right) = \left\{Li_2\left(\frac{a-b}{z-b}\right) + \frac{1}{2}\ln^2\left(z-b\right)\right\}|_0^A - \int_0^A \eta\left(z-a, \frac{1}{z-b}\right)\frac{dz}{z-b}. \tag{101}$$

To avoid misunderstand between η (Riemann sheet function) and η (integration variable), we change integration variable to z. Applying (101),

$$I_+ = \left\{Li_2\left(\frac{-\eta_+}{z-\eta_+}\right) + \frac{1}{2}\ln^2\left(z-\eta_+\right) - \int_0^1 \eta\left(z, \frac{1}{z-\eta_+}\right)\frac{1}{z-\eta_+)}\right\}|_0^1. \tag{102}$$

Similarity,

$$I_- = \left\{Li_2\left(\frac{-\eta_-}{z-\eta_-}\right) + \frac{1}{2}\ln^2\left(z-\eta_-\right) - \int_0^1 \eta\left(z, \frac{1}{z-\eta_-}\right)\frac{dz}{z-\eta_-}\right\}|_0^1 \tag{103}$$

then

$$\begin{aligned} I_+ - I_- &= \left\{Li_2\left(\frac{-\eta_+}{z-\eta_+}\right) - Li_2\left(\frac{-\eta_-}{z-\eta_-}\right) + \frac{1}{2}\left(\ln^2(z-\eta_+) - \ln^2\left(z-\eta_-\right)\right)\right\}|_0^1 \\ &= Li_2\left(\frac{-\eta_+}{1-\eta_+}\right) - Li_2\left(\frac{-\eta_-}{1-\eta_-}\right) - \left(Li_2\left(\frac{-\eta_+}{-\eta_+}\right) - Li_2\left(\frac{-\eta_-}{-\eta_-}\right)\right) + \frac{1}{2}\left(\ln^2(1-\eta_+) - \ln^2(1-\eta_-) - (\ln^2(-\eta_+) - \ln^2(-\eta_-))\right) \\ &= -\left(Li_2\left(\frac{1-\eta_+}{-\eta_+}\right) - Li_2\left(\frac{1-\eta_-}{-\eta_-}\right)\right) - \frac{1}{2}\left(\ln^2\left(\frac{1-\eta_+}{\eta_+}\right) - \ln^2\left(\frac{1-\eta_-}{\eta_-}\right)\right) + \frac{1}{2}\left(\ln^2(1-\eta_+) - \ln^2(1-\eta_-) - (\ln^2(-\eta_+) - \ln^2(-\eta_-))\right). \end{aligned} \tag{104}$$

Applying (100),

$$J_+ = \left\{ Li_2\left(\frac{\eta_+ - 1}{\eta_+ - z}\right) + \frac{1}{2}\ln^2(\eta_+ - z) + \eta\left(1-z, \frac{1}{\eta_+ - z}\right)\ln(\eta_+ - z) \right\}|_0^1$$
$$J_- = \left\{ Li_2\left(\frac{\eta_- - 1}{\eta_- - z}\right) + \frac{1}{2}\ln^2(\eta_- - z) + \eta\left(1-z, \frac{1}{\eta_- - z}\right)\ln(\eta_- - z) \right\}|_0^1 \tag{105}$$

then

$$\begin{aligned} J_+ - J_- &= \left\{ Li_2\left(\frac{\eta_+ - 1}{\eta_+ - z}\right) - Li_2\left(\frac{\eta_- - 1}{\eta_- - z}\right) + \frac{1}{2}\left(\ln^2(\eta_+ - z) - \ln^2(\eta_- - z)\right) \right\}|_0^1 \\ &= Li_2\left(\frac{\eta_+ - 1}{\eta_+ - 1}\right) - Li_2\left(\frac{\eta_- - 1}{\eta_- - 1}\right) - \left(Li_2\left(\frac{\eta_+ - 1}{\eta_+}\right) - Li_2\left(\frac{\eta_- - 1}{\eta_-}\right)\right) + \frac{1}{2}\left(\ln^2(\eta_+ - 1) - \ln^2(\eta_- - 1) - (\ln^2(\eta_+) - \ln^2(\eta_-))\right) \\ &= -\left(Li_2\left(\frac{\eta_+ - 1}{\eta_+}\right)\right) - Li_2\left(\frac{\eta_- - 1}{\eta_-}\right) + \frac{1}{2}\left(\ln^2(\eta_+ - 1) - \ln^2(\eta_- - 1) - (\ln^2(\eta_+) - \ln^2(\eta_-))\right) \end{aligned} \tag{106}$$

so

$$\begin{aligned} 2(I_+ - I_-) - (J_+ - J_-) &= -\left(Li_2\left(\frac{1-\eta_+}{-\eta_+}\right) - Li_2\left(\frac{1-\eta_-}{\eta_-}\right)\right) + 2\left(\ln(1-\eta_+)\ln(\eta_+) - \ln(1-\eta_-)\ln(\eta_-)\right) - \left(\ln^2(-\eta_+) - \ln^2(-\eta_-)\right) + \frac{1}{2}\left[\ln^2(\eta_+ - 1) - \ln^2(\eta_- - 1) + (\ln^2(\eta_+) - \ln^2(\eta_-))\right] \\ &= -\left(Li_2\left(1 - \frac{\lambda^2}{\bar{s}}\right) - Li_2\left(\frac{\lambda^2}{\bar{s}}\right)\right) + 2\left(\ln\left(-\frac{\bar{s}}{\lambda^2}\right)\ln\left(\frac{\bar{s}}{\lambda^2}\right) - \ln\left(-\frac{\lambda^2}{\bar{s}}\right)\ln\left(1 + \frac{\lambda^2}{\bar{s}}\right)\right) - \left(\ln^2\left(-\frac{\bar{s}}{\lambda^2}\right) - \ln^2\left(-1 - \frac{\lambda^2}{\bar{s}}\right)\right) \\ &+ \frac{1}{2}\left[\ln^2\left(\frac{\bar{s}}{\lambda^2}\right) - \ln^2\left(\frac{\lambda^2}{\bar{s}}\right) - \left(\ln^2\left(\frac{\bar{s}}{\lambda^2}\right) - \ln^2\left(1 + \frac{\lambda^2}{\bar{s}}\right)\right)\right] \\ &\simeq -Li_2(1) + 2\left(\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\ln\left(\frac{\lambda^2}{\bar{s}}\right) - \ln\left(-\frac{\lambda^2}{\bar{s}}\right)\ln\left(1 + \frac{\lambda^2}{\bar{s}}\right)\right) - \left(\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) - \ln^2\left(-1 - \frac{\lambda^2}{\bar{s}}\right)\right) - \frac{1}{2}\ln^2\left(\frac{\lambda^2}{\bar{s}}\right) \\ &= -\frac{\pi^2}{6} - 2\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\ln\left(1 + \frac{\bar{s}}{\lambda^2}\right) - \left(\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) - \ln^2\left(-1 - \frac{\lambda^2}{\bar{s}}\right)\right) - \frac{1}{2}\ln^2\left(\frac{\lambda^2}{\bar{s}}\right) \\ &\simeq -\frac{\pi^2}{6} - 2\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\ln\left(\frac{\bar{s}}{\lambda^2}\right) - \left(\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) - \ln^2\left(-1 - \frac{\lambda^2}{\bar{s}}\right)\right) - \frac{1}{2}\ln^2\left(\frac{\lambda^2}{\bar{s}}\right) \\ &= -\frac{\pi^2}{6} + 2\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\ln\left(\frac{\lambda^2}{\bar{s}}\right) - \left(\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) - \ln^2\left(-1 - \frac{\lambda^2}{\bar{s}}\right)\right) - \frac{1}{2}\ln^2\left(\frac{\lambda^2}{\bar{s}}\right) \\ &= -\frac{\pi^2}{6} + 2\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\ln\left(\frac{\lambda^2}{\bar{s}}\right) - \left(\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) - \ln^2\left(-1 - \frac{\lambda^2}{\bar{s}}\right)\right) - \frac{1}{2}\ln^2\left(\frac{\lambda^2}{\bar{s}}\right) \\ &= -\frac{\pi^2}{6} + 2\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\left(\ln\left(-\frac{\lambda^2}{\bar{s}}\right) - i\pi\right) - \left(\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) - \left[\ln\left(1 + \frac{\lambda^2}{\bar{s}}\right) - i\pi\right]^2\right] - \frac{1}{2}\left(\ln\left(-\frac{\lambda^2}{\bar{s}}\right) - i\pi\right)^2 \\ &= -\frac{\pi^2}{6} + 2\ln\left(-\frac{\lambda^2}{\bar{s}}\right) - 2i\pi\ln\left(-\frac{\lambda^2}{\bar{s}}\right) - \frac{1}{2}\left[\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) - \pi^2 - 2i\pi\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\right] - \left(\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) + \pi^2\right) \\ &= -\frac{\pi^2}{6} + \frac{\pi^2}{2} - \pi^2 + \frac{1}{2}\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) - i\pi\ln\left(-\frac{\lambda^2}{\bar{s}}\right) \\ &= -\frac{2\pi^2}{3} + \frac{1}{2}\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) - i\pi\ln\left(-\frac{\lambda^2}{\bar{s}}\right) \end{aligned} \tag{107}$$

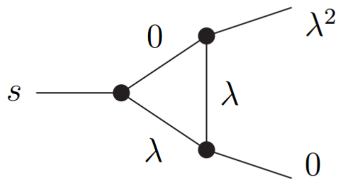
where we have used

$$\ln(-z) = \ln z - i\pi\sigma_{ImZ}$$

with σ_{ImZ} is the sign of imaginary part of Z. Then, our integral now reads

$$\begin{aligned} C_0 &\simeq -\frac{1}{\lambda(\bar{s}/\lambda^2)}\left\{\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\left(\ln(1) - \ln\left(\frac{\lambda^2}{\bar{s}}\right)\right) - \frac{2\pi^2}{3} - \frac{1}{2}\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) - i\pi\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\right\} \\ &= -\frac{1}{\bar{s}}\left\{-\ln\left(-\frac{\lambda^2}{s}\right)\left(\ln\left(-\frac{\lambda^2}{\bar{s}}\right) - i\pi\right) - \frac{2\pi^2}{3} + \frac{1}{2}\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) - i\pi\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\right\} \\ &= \frac{1}{\bar{s}}\left\{\frac{1}{2}\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right) + \frac{2\pi^2}{3}\right\}. \end{aligned} \tag{108}$$

5.3.7 B.14



where $p_1^2 = s$, $m_1 = m_2 = \lambda$, $m_0 = 0$, $p_2^2 = \lambda^2$ and $2p_1p_2 = s + \lambda^2$. M^2 now reads

$$M^2 = \lambda^2(1-x)^2 - xy(s-\lambda^2) \tag{109}$$

so

$$C = -(4\pi\mu^2)^\varepsilon \Gamma(1+\varepsilon) \int_0^1 dx \int_0^{1-x} dy \left\{\lambda^2(1-x)^2 - xy(s-\lambda^2) - i\delta\right\}^{-1-\varepsilon}. \tag{110}$$

The integral finite when $\varepsilon \to 0$, so

$$\begin{aligned} C_0 &= -\int_0^1 dx \int_0^{1-x} \left\{\lambda^2(1-x)^2 - xy(s-\lambda^2) - i\delta\right\}^{-1} \\ &= -\int_0^1 dx \int_0^{1-x} \left\{\lambda^2(1-x)^2 - xy(s-\lambda^2) - i\delta(1+xy)\right\}^{-1} \\ &\simeq -\int_0^1 dx \int_0^{1-x} \left\{\lambda^2(1-x)^2 - xy(s-\lambda^2)\right\}^{-1} \end{aligned} \tag{111}$$

if

$$y = (1-x)\eta, \quad \omega = 1-x \tag{112}$$

the integral now becomes

$$C_0 = -\int_0^1 \omega d\omega \int_0^1 d\eta \left\{\lambda^2\omega^2 - (1-\omega)\omega\eta(\bar{s}-\lambda^2)\right\}^{-1} = -\int_0^1 d\omega \int_0^1 d\eta \left\{\lambda^2\omega - (1-\omega)\eta(\bar{s}-\lambda^2)\right\}^{-1}. \tag{113}$$

There is a pole in the integrand when ω and η tend to be zero. Using decomposition sector method,

$$\begin{aligned} C_0 &= -\int_0^1 d\omega \int_0^1 d\eta \left\{\lambda^2\omega - (1-\omega)\eta(\bar{s}-\lambda^2)\right\}^{-1} (\theta(\omega-\eta) + \theta(\eta-\omega)) \\ &= -(I+J). \end{aligned} \tag{114}$$

About I, put $\eta = \omega t$

$$\begin{aligned} I &= \int_0^1 d\omega \int_0^1 dt \left\{\lambda^2 - (1-\omega)t(\bar{s}-\lambda^2)\right\}^{-1} = -\int_0^1 \frac{d\omega}{(\bar{s}-\lambda^2)(1-\omega)} \left\{\ln\left(\lambda^2 - (1-\omega)(\bar{s}-\lambda^2)\right) - \ln\left(\lambda^2\right)\right\} \\ &= -\int_0^1 \frac{d\omega}{(\bar{s}-\lambda^2)(1-\omega)} \ln\left(1-(1-\omega)\frac{\bar{s}-\lambda^2}{\lambda^2}\right) = -\int_0^1 \frac{d(1-\omega)}{(\bar{s}-\lambda^2)(1-\omega)} \ln\left(1-(1-\omega)\frac{\bar{s}-\lambda^2}{\lambda^2}\right) \\ &= \frac{1}{\bar{s}-\lambda^2} Li_2\left(\frac{\bar{s}-\lambda^2}{\lambda^2}\right) = -\frac{1}{\bar{s}-\lambda^2} Li_2\left(\frac{\lambda^2}{\bar{s}-\lambda^2}\right) - \frac{1}{2(\bar{s}-\lambda^2)} \ln^2\left(-\frac{\lambda^2}{\bar{s}-\lambda^2}\right) - \frac{\pi^2}{6(\bar{s}-\lambda^2)}. \end{aligned} \tag{115}$$

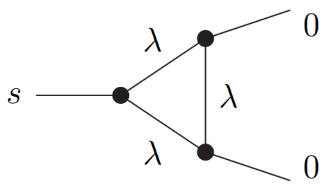
About J

$$\begin{aligned} J &= \int_0^1 dt \int_0^1 d\eta \left\{\lambda^2 t - (1-\eta t)(\bar{s}-\lambda^2)\right\}^{-1} = \frac{1}{\bar{s}-\lambda^2} \int_0^1 \frac{dt}{t} \left\{\ln\left(\lambda^2 t - (1-\eta)(\bar{s}-\lambda^2)\right) - \ln(-(\bar{s}-\lambda^2))\right\} \\ &= \frac{1}{\bar{s}-\lambda^2} \int_0^1 \frac{dt}{t} \ln\left(1 - t\frac{\bar{s}}{\bar{s}-\lambda^2}\right) = -\frac{1}{\bar{s}-\lambda^2} Li_2\left(\frac{\bar{s}}{\bar{s}-\lambda^2}\right) \end{aligned} \tag{116}$$

with $\omega = \eta t$. Combining all results above,

$$\begin{aligned} C_0 &= \frac{1}{\bar{s}-\lambda^2} \left\{ Li_2\left(\frac{\lambda^2}{\bar{s}-\lambda^2}\right) + \frac{1}{2}\ln^2\left(-\frac{\lambda^2}{\bar{s}-\lambda^2}\right) + \frac{\pi^2}{6} + Li_2\left(\frac{\bar{s}}{\bar{s}-\lambda^2}\right) \right\} \\ &\simeq \frac{1}{\bar{s}} \left\{ Li_2(0) + \frac{1}{2}\ln\left(-\frac{\lambda^2}{\bar{s}}\right) + \frac{\pi^2}{6} + Li_2(1) \right\} \\ &= \frac{1}{\bar{s}} \left\{ \frac{1}{2}\ln\left(-\frac{\lambda^2}{\bar{s}}\right) + \frac{\pi^2}{3} \right\}. \end{aligned} \tag{117}$$

5.3.8 B.15



where $p_1^2 = s$, $m_0 = m_1 = m_2 = \lambda$, $p_2^2 = 0$, $2p_1p_2 = s$ and

$$\begin{aligned} M^2 &= ((p_1 - p_2)y + p_2(1-x))^2 - \left[-x\lambda^2 + y(s-\lambda^2) - (1-x-y)\lambda^2\right] = \lambda^2 - sy + 2y(1-x)(p_1-p_2)p_2 \\ &= \lambda^2 - sy + y(1-x)s = \lambda^2 - sxy \end{aligned} \tag{118}$$

then the integral now reads

$$\begin{aligned} C_0 &= -(4\pi\mu^2)^\varepsilon \Gamma(1+\varepsilon) \int_0^1 dx \int_0^{1-x} dy \left\{\lambda^2 - sxy - i\delta\right\}^{-1-\varepsilon} \\ &= -(4\pi\mu^2)^\varepsilon \Gamma(1+\varepsilon)(-s-i\delta')^{-1-\varepsilon} \int_0^1 dx \int_0^{1-x} dy \left\{\frac{\lambda^2}{-s-i\delta'} + xy + \frac{i\delta(1-xy)}{s+i\delta'}\right\}^{-1-\varepsilon} \\ &= -(4\pi\mu^2)^\varepsilon \Gamma(1+\varepsilon)(-\bar{s})^{-1-\varepsilon} \int_0^1 dx \int_0^{1-x} dy \left\{-\frac{\lambda^2}{\bar{s}} + xy + \frac{i\delta(1-xy)}{\bar{s}}\right\}^{-1-\varepsilon} \\ &\simeq -(4\pi\mu^2)^\varepsilon \Gamma(1+\varepsilon)(-\bar{s})^{-1-\varepsilon} \int_0^1 dx \int_0^{1-x} dy \left\{-\frac{\lambda^2}{\bar{s}} + xy\right\}^{-1-\varepsilon} \end{aligned} \tag{119}$$

when $x = y = 0$, the integrand equals to $\frac{\lambda}{\bar{s}}$ so this integral still finite when it is worked in 4 dimensional.

$$\begin{aligned} C_0 &= \frac{1}{\bar{s}} \int_0^1 dx \int_0^{1-x} dy \left\{xy - \frac{\lambda^2}{\bar{s}}\right\}^{-1} = \frac{1}{\bar{s}} \int_0^1 dx \int_0^{1-x} dy \left\{-xy + \frac{\lambda^2}{\bar{s}}\right\}^{-1} \\ &= \frac{1}{\bar{s}} \int_0^1 \frac{dx}{x} \left\{\ln\left(x^2 - x + \frac{\lambda^2}{\bar{s}}\right) - \ln\left(\frac{\lambda^2}{\bar{s}}\right)\right\}. \end{aligned} \tag{120}$$

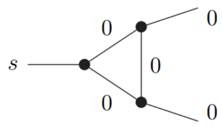
Because the equation $x^2 - x + \frac{\lambda^2}{\bar{s}} = 0$ has solutions

$$x_+ = 1 - \frac{\lambda}{\bar{s}}, \quad x_- = \frac{\lambda}{\bar{s}}, \quad x_+x_- = \frac{\lambda}{\bar{s}} \tag{121}$$

then

$$\begin{aligned} C_0 &= \frac{1}{\bar{s}} \int_0^1 \frac{dx}{x} \left\{\ln(x-x_+)(x-x_-) - \ln(x_+x_-)\right\} = \frac{1}{\bar{s}} \int_0^1 \frac{dx}{x} \left\{\ln(x_+-x)(x_--x) - \ln(x_+x_-)\right\} \\ &= \frac{1}{\bar{s}} \int_0^1 \frac{dx}{x} \left\{\ln(x_+-x) - \ln(x_+) + \ln(x_--x) - \ln(x_-)\right\} \\ &= \frac{1}{\bar{s}} \int_0^1 \left\{\ln\left(1 - \frac{x}{x_+}\right) + \ln\left(1 - \frac{x}{x_-}\right)\right\} \\ &= \frac{1}{\bar{s}} \left(-Li_2\left(\frac{1}{x_+}\right) - Li_2\left(\frac{1}{x_-}\right)\right) \\ &= \frac{1}{\bar{s}} \left\{-Li_2\left(\frac{1}{1-\lambda^2/\bar{s}}\right) - Li_2\left(\frac{\bar{s}}{\lambda^2}\right)\right\} \\ &= \frac{1}{\bar{s}} \left\{-Li_2\left(\frac{1}{1-\lambda^2/\bar{s}}\right) + Li_2\left(\frac{\lambda^2}{\bar{s}}\right) + \frac{\pi^2}{6} + \frac{1}{2}\ln\left(-\frac{\lambda^2}{\bar{s}}\right)\right\} \\ &\simeq \frac{1}{\bar{s}} \left(-Li_2(1) + \frac{\pi^2}{6} + Li_2(0) + \frac{1}{2}\ln^2\left(-\frac{\lambda^2}{\bar{s}}\right)\right) \\ &= \frac{1}{2\bar{s}} \ln\left(-\frac{\lambda^2}{\bar{s}}\right). \end{aligned} \tag{122}$$

5.3.9 B.16



If $p_0 = 0$, then $p_2^2 = 0, p_1^2 = s$ and $(p_2 - p_1)^2 = -2p_1p_2 + p_1^2 = 0$ or $2p_1p_2 = p_1^2 = s$. From that, we get

$$\begin{aligned}
M^2 &= (p_0x + p_1y + p_2z)^2 - \left[(p_0^2 - m_0^2)x + (p_1^2 - m_1^2)y + (p_2^2 - m_2^2)z\right] \\
&= (p_1y + p_2(1-x-y))^2 - p_1^2 y = (y(p_1 - p_2) + p_2(1-x))^2 - p_1^2 y - \\
&= y^2(p_1 - p_2)^2 + p_2^2(1-x)^2 + 2y(1-x)(p_1p_2 - p_2^2) - p_1^2 y \\
&= -y\left[p_1^2 - 2p_1p_2(1-x) + p_2^2 - p_2^2\right] = -y\left[(p_1 - p_2)^2 - p_2^2 + 2xp_1p_2\right] \\
&= -2xyp_1p_2 - = -p_1^2 xy = -sxy.
\end{aligned} \tag{123}$$

Then

$$\begin{aligned}
C_0 &= (-1)^3(4\pi\mu^2)^\varepsilon \Gamma(3 - 2 + \varepsilon) \int_0^1 dx \int_0^{1-x} dy(-sxy - i\delta)^{-1-\varepsilon} \\
&= (-1)^3(4\pi\mu^2)^\varepsilon \Gamma(3 - 2 + \varepsilon) \int_0^1 dx \int_0^{1-x} dy(-\bar{s}xy - i\delta(1 - xy))^{-1-\varepsilon} \\
&= (-1)^3(4\pi\mu^2)^\varepsilon \Gamma(3 - 2 + \varepsilon) \int_0^1 dx \int_0^{1-x} dy(-\bar{s}xy)^{-1-\varepsilon} \\
&= \left(\frac{4\pi\mu^2}{-\bar{s}}\right)^\varepsilon \frac{\Gamma(1+\varepsilon)}{\bar{s}\varepsilon} \int_0^1 \frac{dx}{x} \{x(1-x)\}^{-\varepsilon} \\
&\simeq \left(\frac{4\pi\mu^2}{-\bar{s}}\right)^\varepsilon \frac{\Gamma(1+\varepsilon)}{s\varepsilon} \int_0^1 dx x^{-1-\varepsilon} (1-x)^{-\varepsilon} \\
&= \left(\frac{4\pi\mu^2}{-\bar{s}}\right)^\varepsilon \frac{\Gamma(1+\varepsilon)}{s\varepsilon^2} \frac{\Gamma(-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \\
&= \left(\frac{4\pi\mu^2}{-\bar{s}}\right)^\varepsilon \frac{\Gamma(1+\varepsilon)}{s\varepsilon^2} \frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}.
\end{aligned} \tag{124}$$

Next, using Mathematica, we get

$$\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} = 1 - \frac{\pi^2\varepsilon^2}{6} + O(\varepsilon^3). \tag{125}$$

As a result,

$$C_0 = \frac{1}{s}\left(\frac{\Gamma(1+\varepsilon)}{\varepsilon^2}\left(\frac{4\pi\mu}{-\bar{s}}\right)^\varepsilon - \frac{\pi^2}{6} + O(\varepsilon)\right). \tag{126}$$

6 Appendices

6.1 A1-Schwinger trick

$$\frac{1}{A} = \int_0^\infty dv e^{-Av} \tag{127}$$

for $ReA > 0$, where the integral is well-defined. We can apply this procedure to a product of propagators:

$$\prod_{i=1}^{n} \frac{1}{A_i} = \left(\prod_{i=1}^{n} \int_0^\infty dv_i \right) e^{-\sum_{i=1}^{N} A_i v_i}. \tag{128}$$

Let $v = \sum v_i$ and $\alpha_i = v_i/v$. Then

$$\prod_{i=0}^{N} dv_i = v^{N-1} dv \prod_{i=1}^{N} d\alpha_i \delta \left(1 - \sum_{i=1}^{N} \alpha_i\right) \tag{129}$$

so

$$\prod_{i=1}^{n} \frac{1}{A_i} = \left(\prod_{i=1}^{N} \int_0^\infty d\alpha_i \right) \delta \left(1 - \sum_{i=1}^{N} \alpha_i\right) \int_0^\infty v^{N-1} dv e^{-v \sum \alpha_i A_i} \tag{130}$$

but

$$\int_0^\infty t^{z-1} e^{-bt} dt = \frac{1}{b^z} \Gamma(z) \tag{131}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt = (z-1)! \tag{132}$$

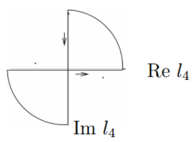
is Euler Gamma function. So all together

$$\prod_{i=1}^{n} \frac{1}{A_i} = \Gamma(N) \left(\prod_{i=1}^{N} \int_0^\infty d\alpha_i \right) \delta \left(1 - \sum_{i=1}^{N} \alpha_i\right) \left(\sum \alpha_i A_i \right)^{-N}. \tag{133}$$

6.2 A1.2 Wick Rotation

The Minkowski metric $\{1, -1, -1, -1\}$ does not offer any simple means of integration over solid angles over all four dimensions. A calculation procedure is much more easier if the integration over solid angles over four dimensions is taken in ordinary Euclidean metric $\{1, 1, 1, 1\}$.

There are two poles whose location are determined by $i\delta$ prescription of propagators. We have been using the most common prescription, the Feynman prescription. The pole



appears when $q^2 = M^2 - i\varepsilon$, where the q_0 component corresponds to $q_0^2 = \mathbf{q}^2 + M^2 - i\varepsilon$. Therefore, for the integration variable q_0 we have poles at

$$q_0 = \pm\sqrt{\mathbf{q^2} + M^2 - i\delta} \simeq \sqrt{\mathbf{q}^2 + M^2} \left(1 - \frac{i\delta}{\mathbf{q}^2 + M^2}\right) \tag{134}$$

which corresponds to poles in the lower, right quadrature and upper, left quadrature.

If we integrate along a close contour that follows the real and imaginary axes and connects these at infinity in upper, right and lower left quadrature we may close the curve without involving any poles. Calling the lower right curve quadrature is C_{UR}, upper left curve quadrature is C_{LL}, the imaginary is represented by subscript I and R for the real axis. Utilizing Cauchy's Integral theorem on the 0 component of integral momentum vector in the complex plane.

$$\oint \frac{d^D l}{(l^2 - M^2 + i\delta)^n} = \int_{-\infty}^{\infty} \frac{d^D l}{(l^2 - M^2 + i\delta)^n} + \int_{C_{UR}} \frac{d^D l}{(l^2 - M^2 + i\delta)^n} - \int_{-i\infty}^{i\infty} \frac{d^D l}{(l^2 - M^2 + i\delta)^n} + \int_{C_{LL}} \frac{d^D l}{(l^2 - M^2 + i\delta)^n} = 0. \tag{135}$$

The integration over the curves C_{UR} and C_{LL} fall off sufficiently rapidly at large $|l_0|$, i.e $\int_{C_{LL}} = \int_{C_{UR}} = 0$. So

$$\int_{-\infty}^{\infty} \frac{d^D l}{(l^2 - M^2 + i\delta)^n} = \int_{-i\infty}^{i\infty} \frac{d^D l}{(l^2 - M^2 + i\delta)^n} \tag{136}$$

We then define a Euclidean 4-momentum variable l_E:

$$l^0 = i l_E^0; \qquad \mathbf{l} = \mathbf{l}_E \tag{137}$$

The Minkowski metric $\{1, -1, -1, -1\}$, $l^2 = l_0^2 - \mathbf{l}^2$ changes to $-l_E^2 = l_0^2 + \mathbf{l}^2$ as usual in ordinary Euclidean metric $\{1, 1, 1, 1\}$. Using these, we have identity

$$\int_{-i\infty}^{i\infty} \frac{d^D l}{(l^2 - M^2 + i\delta)^n} = (-1)^n \int_{-\infty}^{\infty} \frac{i d^D l_E}{(l_E^2 + M^2 - i\delta)^n}. \tag{138}$$

All combining these result,

$$\int_{-\infty}^{\infty} \frac{d^D l}{(l^2 - M^2 + i\delta)^n} = (-1)^n \int_{-\infty}^{\infty} \frac{i d^D l_E}{(l_E^2 + M^2 - i\delta)^n}. \tag{139}$$

The expression now turned into Euclidean space and we can therefore carry out a change of variables to spherical coordinates.

$$\int d^D q_E = \int_0^\infty d|q_E| |q_E|^{D-1} \int d\Omega^{D-1}. \tag{140}$$

6.3 A.1.3-Angular integration in D dimension

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 ... \int_{-\infty}^{\infty} dx_D \exp\left[-(x_1^2 + x_2^2 + .. + x_d^2)\right] \tag{141}$$

since

$$\int_{\infty}^{\infty} dx \exp[-x^2] = \sqrt{\pi} \tag{142}$$

so

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 ... \int_{-\infty}^{\infty} dx_D \exp\left[-(x_1^2 + x_2^2 + .. + x_d^2)\right] = \pi^{D/2} \tag{143}$$

(141) also write in another way

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx_1 ... \int_{-\infty}^{\infty} dx_D \exp\left[-(x_1^2 + x_2^2 + ... + x_D^2)\right] \\
&= \int d\Omega^{D-1} \int_0^{\infty} d|r||r|^{D-1} \exp[-r^2] = \int d\Omega^{D-1} \int_0^{\infty} \frac{dt}{2\sqrt{2}} t^{(D-1)/2} \exp[-t] \\
&= \frac{1}{2} \int d\Omega^{D-1} \int_0^{\infty} t^{-1+D/2} \exp[-t] dt \\
&= \frac{\Gamma(D/2)}{2} \int d\Omega^{D-1}
\end{aligned} \tag{144}$$

so

$$\int d\Omega^{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{145}$$

Radial integration
For the radial component, we are left with an integral of the form:

$$\int_0^{\infty} d|q| \frac{q_E^{D-1}}{(q_E^2 + m^2)^n} \tag{146}$$

we may evaluate this by a series of variable substitutions:
- $|q_E| \to m.y$

$$\int_0^{\infty} \frac{|q_E|^{D-1} dq}{(|q_E|^2 + m^2)^n} = \frac{m^D}{m^{2n}} \int_0^{\infty} dy \frac{y^{D-1} dy}{(1+y^2)^n}. \tag{147}$$

- $y = \sinh u$

where we use the following relations:

$$\begin{aligned}
1 + y^2 &= 1 + \sinh^2 u = \cosh^2 u = (1 - \tanh^2 u)^{-1}, \\
y &= (1 + y^2 - 1)^{1/2} = \left(\frac{1}{1 - \tanh^2 u} - 1\right)^{1/2} = \left(\frac{\tanh^2 u}{1 - \tanh^2 u}\right)^{1/2}, \\
dy &= \cosh u du = (1 - \tanh^2 u)^{-1/2} du,
\end{aligned} \tag{148}$$

- $v = \tanh^2 u$:

$$\begin{aligned}
1 + y^2 &= (1 - v)^{-1}, \\
y &= \left(\frac{v}{1 - v}\right)^{1/2}, \\
dv &= 2 \tanh u \cosh^{-2} u du = 2 \tanh u (1 - \tanh^2 u) du, \\
dy &= (1 - \tanh^2 u)^{-1/2} du = (1 - v)^{-1/2} \frac{1}{2(v)^{1/2}(1 - v)}.
\end{aligned} \tag{149}$$

Giving the integral

$$\begin{aligned}
\frac{m^D}{m^{2n}} \int_0^{\infty} dy \frac{y^{D-1}}{(1+y^2)^n} &= \frac{m^D}{m^{2n}} \int_0^1 dv \frac{(1-v)^{-1/2}}{2(v)^{1/2}(1-v)} \times \left(\frac{v}{1-v}\right)^{(D-1)/2} \times (1-v)^n \\
&= (m^2)^{D/2-n} \int_0^1 dv \frac{1}{2} v^{d/2-1} (1-v)^{n-\frac{D}{2}-1} \\
&= \frac{(m^2)^{\frac{d}{2}-1}}{2} \frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)}
\end{aligned} \tag{150}$$

where we have used the definition of Beta function

$$\int_0^1 dt t^{x-1} (1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{151}$$

Combining all results above, our integral now reads

$$\int \frac{d^D q}{[q^2 - M^2 + i\delta]^n} = (-1)^n i\pi^{D/2} \frac{\Gamma(n - D/2)}{\Gamma(n)} (M^2 - i\delta)^{D/2-n}. \tag{152}$$

6.4 A.1.4-Spence function

The logarithms occurring in this note have a cut along the negative real axis. The rule for the logarithm of a product is

$$\ln ab = \ln a + \ln b + \eta(a, b) \tag{153}$$
$$\eta(x + iy, u + iv) = 2i\pi \left[\theta(-y)\theta(-v)\theta(xv + uy) - \theta(y)\theta(v)\theta(-xv - uy)\right] \tag{154}$$

Two basic Spence function's identities

$$Li_2(z) = -Li_2(1 - x) + \frac{\pi^2}{6} - \ln x \ln(1 - x), \tag{155}$$
$$Li_2(x) = -Li_2\left(\frac{1}{z}\right) - \frac{\pi^2}{6} - \frac{1}{2} \ln^2(-z). \tag{156}$$

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