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α^3-CONTRIBUTION TO THE ANGULAR ASYMMETRY IN $e^+e^- \to \mu^+\mu^-$

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Abstract: The complete α^3-contribution to the angular asymmetry in the differential cross section for $e^+e^- \to \mu^+\mu^-$ is calculated, including hard photon emission. The results are of interest for e^+e^- colliding beam experiments in which the charges of the outgoing muons are detected. Numerical results are given for the case of experiments with small energy resolutions.

1. Introduction

If in future e^+e^- colliding beam experiments the charges of the particles in the final state are detected, more refined tests of quantum electrodynamics will be possible. In the case that there are two charged particles in the final state, the charges of which are not determined, information is obtained on the function

$$S(\theta) = \frac{d\sigma(\theta)}{d\Omega} + \frac{d\sigma(\pi-\theta)}{d\Omega} . \tag{1}$$

In case one measures the differential cross section with charge detection, one also knows the asymmetry function

$$D(\theta) = \frac{d\sigma(\theta)}{d\Omega} - \frac{d\sigma(\pi-\theta)}{d\Omega} . \tag{2}$$

In this paper we shall discuss in particular the function $D(\theta)$ for the reaction

$$e^+(p_+) + e^-(p_-) \to \mu^+(q_+) + \mu^-(q_-) , \tag{3}$$

as an extensive study of $S(\theta)$ has already been made [1].

It has first been noticed by Putzolu [2] that charge conjugation invariance can be invoked to show that only the interference terms between the lowest order graph

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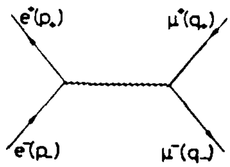

Fig. 1. Lowest order Feynman diagram for μ-pair production.

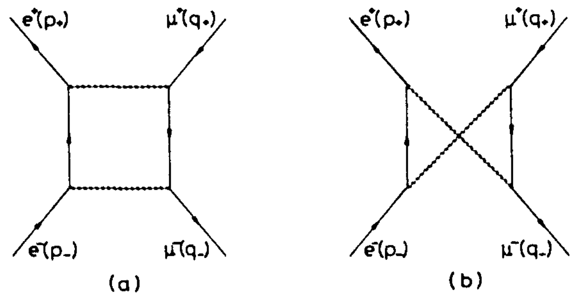

Fig. 2. Feynman diagrams which interfere with the lowest order one to produce an angular asymmetry.

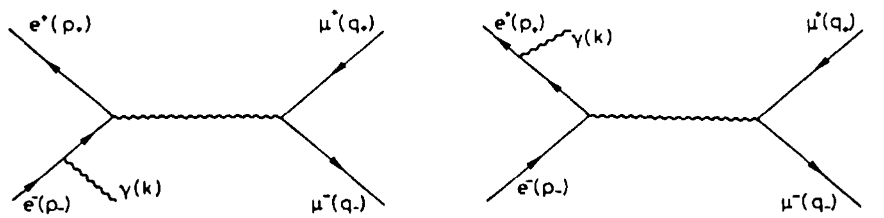

Fig. 3. Bremsstrahlung diagrams producing muons in a C-odd state.

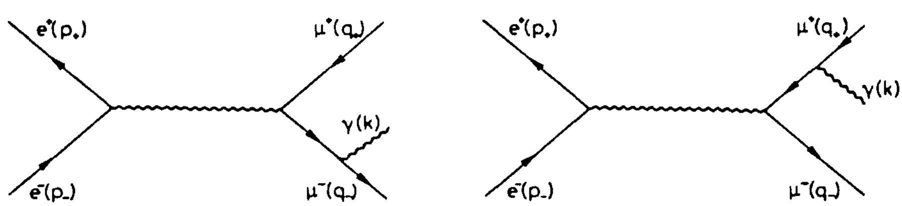

Fig. 4. Bremsstrahlung diagrams producing muons in a C-even state.

(fig. 1) and the two-photon graphs (fig. 2) contribute to D to order α^3, as far as the virtual radiative corrections are concerned. Similarly, for the bremsstrahlung contribution, only the interference between the C-odd muon graphs of fig. 3 and the C-even muon graphs of fig. 4 has to be computed. In sect. 2 we present the complete analytic calculation of the interference of the two box graphs with the lowest order matrix element. This evaluation is valid for all energies and scattering angles, in contrast to some recent approximate calculations [3–5].

Then in sect. 3 the inelastic part of the cross section is given, this contribution being necessary to cancel the infrared divergences in the virtual corrections. This calculation was performed both for hard and soft photons. In the latter case one assumes that the energy loss is so small that recoil effects on the electrons and muons can be neglected. Finally, in sect. 4, numerical results are presented both for the functions S and D under the same experimental conditions and some conclusions are drawn. More technical aspects connected with the calculations of the Feynman integrals are developed in the appendix.

2. Virtual radiative corrections

As explained in the introduction, the only α^3-contribution to $D(\theta)$ due to virtual radiative corrections arises from the two box graphs of fig. 2. By standard methods their matrix elements are given by

$$M^{\mathrm{a}} = \left(\frac{\alpha}{\pi}\right)^2 \int \mathrm{d}^4 k \, \frac{\bar{u}(q_-)\gamma_\alpha(\not{k} - \not{Q} + \mu)\gamma_\beta v(q_+)\bar{v}(p_+)\gamma^\beta(\not{k} - \not{\Delta} + m)\gamma^\alpha u(p_-)}{(\Delta)(Q)(+)(-)} \,,$$

$$M^{\mathrm{b}} = \left(\frac{\alpha}{\pi}\right)^2 \int \mathrm{d}^4 k \, \frac{\bar{u}(q_-)\gamma_\beta(-\not{k} - \not{Q} + \mu)\gamma_\alpha v(q_+)\bar{v}(p_+)\gamma^\beta(\not{k} - \not{\Delta} + m)\gamma^\alpha u(p_-)}{(\Delta)(Q')(+)(-)} \,. \tag{4}$$

We have introduced the following symbols:

$$P = \tfrac{1}{2}(p_+ + p_-)\,, \qquad \Delta = \tfrac{1}{2}(p_+ - p_-)\,, \qquad Q = \tfrac{1}{2}(q_+ - q_-)\,,$$

$$(\Delta) = k^2 - 2k\cdot\Delta - P^2 + i\epsilon\,, \qquad (Q) = k^2 - 2k\cdot Q - P^2 + i\epsilon\,,$$

$$(Q') = k^2 + 2k\cdot Q - P^2 + i\epsilon\,, \qquad (\pm) = k^2 \pm 2k\cdot P + P^2 - \lambda^2 + i\epsilon\,, \tag{5}$$

where λ is a small fictitious photon mass, introduced to regularise the infrared divergence. Also, m and μ are the electron and the muon mass.

The contribution of these diagrams to the cross section is given by

$$\frac{\mathrm{d}\sigma^{\mathrm{a,b}}}{\mathrm{d}\Omega_\mu} = \frac{1}{16\pi^2 s}\frac{|q_+|}{|p_+|} m^2\mu^2 \sum_{\text{spins}} 2\mathrm{Re}\,(M^* M^{\mathrm{a,b}})\,, \tag{6}$$

where $s = (p_+ + p_-)^2$ and M is the lowest order matrix element:

$$M = i(4\pi\alpha/s)\bar{u}(q_-)\gamma_\mu v(q_+)\bar{v}(p_+)\gamma^\mu u(p_-) = i(4\pi\alpha/s)T_0\,. \tag{7}$$

It can easily be seen that the contribution of diagram (b) can be obtained by substituting $(Q,\mu) \to (-Q,-\mu)$ in the expression for $\mathrm{d}\sigma^{\mathrm{a}}/\mathrm{d}\Omega_\mu$ and by adding an overall minus sign. Note that in the final expressions only even powers of the masses appear. If the final result is expressed in terms of the Mandelstam variables $t = (p_+ - q_+)^2$ and $u = (p_+ - q_-)^2$, this is tantamount to

$$\frac{d\sigma^b(s,t)}{d\Omega_\mu} = -\frac{d\sigma^a(s,u)}{d\Omega_\mu} . \tag{8}$$

It thus suffices to calculate M^a.

We proceed by splitting M^a in three parts according to the number of times the vector k appears in the numerator. We define

$$[J;J_\mu;J_{\mu\nu}] = \int d^4k \frac{[1;k_\mu;k_\mu k_\nu]}{(\Delta)(Q)(+)(-)} . \tag{9}$$

and write the matrix element M^a in the form

$$M^a = \left(\frac{\alpha}{\pi}\right)^2 (JT + J^\mu T_\mu + J^{\mu\nu} T_{\mu\nu}) . \tag{10}$$

A tedious, but straightforward trace calculation provides us with the quantities

$$[X;X_\mu;X_{\mu\nu}] = m^2\mu^2 \sum_{\text{spins}} T_0^* [T;T_\mu;T_{\mu\nu}] . \tag{11}$$

and we have

$$\frac{d\sigma^a}{d\Omega_\mu} = \frac{\alpha^3}{\pi} \frac{1}{s^2} \frac{|q_+|}{|p_+|} \frac{1}{2\pi^2} \operatorname{Im} [XJ + X_\mu J^\mu + X_{\mu\nu} J^{\mu\nu}] . \tag{12}$$

The integral J is infrared divergent. It is convenient to write it as a sum of two terms

$$J = (F + G)/2P^2 , \tag{13}$$

where

$$F = \int d^4k (P^2 - k^2)/(\Delta)(Q)(+)(-) ,$$

$$G = \int d^4k/(\Delta)(Q)(+) . \tag{14}$$

Of these two integrals only G is infrared divergent. To write J as in eq. (13) we have made use of the fact that J_μ can be written as

$$J_\mu = J_\Delta \Delta_\mu + J_Q Q_\mu , \tag{15}$$

without a term proportional to P_μ. Multiplying eq. (15) with Δ_μ and Q_μ, and solving for J_Δ and J_Q, we find

$$J_\Delta = [\tau(F_\Delta + F) - Q^2(F_Q + F)]/2\Lambda\,,$$
$$J_Q = [\tau(F_Q + F) - \Delta^2(F_\Delta + F)]/2\Lambda\,, \tag{16}$$

where we have introduced $\tau = \Delta \cdot Q$, $\Lambda = \Delta^2 Q^2 - \tau^2$, and

$$F_{\Delta, Q} = \int \mathrm{d}^4 k/(\Delta, Q)(+)(-)\,. \tag{17}$$

Similarly, for $J_{\mu\nu}$ we write down the decomposition

$$J_{\mu\nu} = K_O g_{\mu\nu} + K_P P_\mu P_\nu + K_\Delta \Delta_\mu \Delta_\nu + K_Q Q_\mu Q_\nu + K_X(Q_\mu \Delta_\nu + \Delta_\mu Q_\nu)\,. \tag{18}$$

Also in this case terms linear in P_μ are absent. We can again solve for the coefficients of the tensors and find

$$K_O = -\tfrac{1}{2}(F - G + H_P + H_\Delta + H_Q) + P^2(J_\Delta + J_Q)\,,$$

$$K_P = (F - G + 2H_P + H_\Delta + H_Q)/2P^2 - (J_\Delta + J_Q)\,,$$

$$K_\Delta = [Q^2(\tfrac{1}{2}H_\Delta - P^2 J_\Delta - K_O) - \tau(\tfrac{1}{2}H_\Delta - P^2 J_\Delta - \tfrac{1}{2}G_\Delta)]/\Lambda\,,$$

$$K_Q = [\Delta^2(\tfrac{1}{2}H_Q - P^2 J_Q - K_O) - \tau(\tfrac{1}{2}H_Q - P^2 J_Q - \tfrac{1}{2}G_Q)]/\Lambda\,,$$

$$K_X = [\Delta^2(\tfrac{1}{2}H_\Delta - P^2 J_\Delta - \tfrac{1}{2}G_\Delta) - \tau(\tfrac{1}{2}H_\Delta - P^2 J_\Delta - K_O)]/\Lambda\,. \tag{19}$$

To obtain $J_{\mu\nu}$ we have therefore to calculate the following simpler integrals:

$$H_\mu = \int \mathrm{d}^4 k\, k_\mu/(\Delta)(Q)(-) = H_P P_\mu + H_\Delta \Delta_\mu + H_Q Q_\mu\,,$$

$$G_{\Delta\mu} = \int \mathrm{d}^4 k\, k_\mu/(\Delta)(+)(-) = G_\Delta \Delta_\mu\,,$$

$$G_{Q\mu} = \int \mathrm{d}^4 k\, k_\mu/(Q)(+)(-) = G_Q Q_\mu\,. \tag{20}$$

The analytic expressions for the integrals F, G, H_P, $H_{\Delta,Q}$, $F_{\Delta,Q}$ and $G_{\Delta,Q}$ are given in the appendix. It follows from these expressions that

$$-G + H_P + H_\Delta + H_Q = 0\,, \tag{21}$$

and consequently that K_O and K_P can be simplified to

$$K_O = -\tfrac{1}{2}F + P^2(J_\Delta + J_Q)\,, \qquad K_P = (\tfrac{1}{2}H_P - K_O)/P^2\,. \tag{22}$$

The last step in the process of calculating $d\sigma^a/d\Omega_\mu$ is now to do the contraction of X_μ and $X_{\mu\nu}$ with the different tensors of the decomposition of J_μ and $J_{\mu\nu}$:

$$
\begin{aligned}
&X = (4\tau + \tfrac{1}{2}s)X_0 - s^2\tau\,,\\
&X_\mu \Delta^\mu = -(2\Delta^2 + 4\tau)X_0 - 2s\tau^2 + (2s^2 + 8m^2\mu^2)\tau + 2s\Delta^2(\mu^2 + \tfrac{1}{4}s)\,,\\
&X_\mu Q^\mu = -(2Q^2 + 4\tau)X_0 - 2s\tau^2 + (2s^2 + 8m^2\mu^2)\tau + 2sQ^2(m^2 + \tfrac{1}{4}s^2)\,,\\
&X_{\mu\nu}g^{\mu\nu} = 10X_0 + 12s\tau\,, \qquad X_{\mu\nu}P^\mu P^\nu = \tfrac{1}{2}s(X_0 + 2s\tau)\,,\\
&X_{\mu\nu}\Delta^\mu\Delta^\nu = 2\Delta^2 X_0 + 8m^2\tau^2 + 4s\Delta^2\tau + 2m^2\Delta^2 s\,,\\
&X_{\mu\nu}Q^\mu Q^\nu = 2Q^2X_0 + 8\mu^2\tau^2 + 4sQ^2\tau + 2\mu^2 Q^2 s\,,\\
&X_{\mu\nu}(Q^\mu\Delta^\nu + \Delta^\mu Q^\nu) = 6\tau X_0 + 6s\tau^2 + 8(m^2\mu^2 - s^2)\tau + 2s\Delta^2 Q^2\,,
\end{aligned} \tag{23}
$$

with

$$X_0 = \tfrac{1}{4}s^2 + 4\tau^2 + s(m^2 + \mu^2)\,. \tag{24}$$

We now isolate the infrared divergent part of $d\sigma^a/d\Omega_\mu$. Notice that G is infrared divergent and that the infrared divergence of H_P can be identified with G. All other integrals being convergent, we have for the divergent piece of $d\sigma^a/d\Omega_\mu$:

$$\left(\frac{d\sigma^a}{d\Omega_\mu}\right)_\lambda = \frac{2\alpha^3}{\pi}\frac{|q_+|}{|p_+|}\frac{4\tau + s}{s^3}X_0\frac{1}{2\pi^2}\,\mathrm{Im}\,G\,. \tag{25}$$

Note finally that with the definition (24) of X_0 the lowest order cross section derived from the Feynman diagram of fig. 1 can be written as

$$\frac{d\sigma^0}{d\Omega_\mu} = \frac{\alpha^2}{s^3}\frac{|q_+|}{|p_+|}X_0\,. \tag{26}$$

It follows that

$$\left(\frac{d\sigma^a}{d\Omega_\mu}\right)_\lambda = \frac{d\sigma^0}{d\Omega_\mu}\frac{2\alpha}{\pi}(4\tau + s)\frac{1}{2\pi^2}\,\mathrm{Im}\,G\,. \tag{27}$$

In the appendix we show that $\mathrm{Im}\,G$ can be rewritten with the help of two new functions $A(s,t)$ and $B(s,t)$, in such a way that eq. (27) becomes

$$\left(\frac{d\sigma^a}{d\Omega_\mu}\right)_\lambda = -\frac{d\sigma^0}{d\Omega_\mu}\frac{\alpha}{2\pi}(4\tau + s)\left[A(s,t)\log\left(\frac{s}{\lambda^2}\right) + B(s,t)\right]. \tag{28}$$

The angular asymmetry due to the virtual radiative corrections thus becomes

$$D^{\mathrm{V}}(\theta) = 2\left[\left(\frac{\mathrm{d}\sigma^{\mathrm{a}}}{\mathrm{d}\Omega_\mu}\right)_{\mathrm{f}} + \left(\frac{\mathrm{d}\sigma^{\mathrm{a}}}{\mathrm{d}\Omega_\mu}\right)_{\lambda} - (t \leftrightarrow u)\right] , \tag{29}$$

where $(\mathrm{d}\sigma^{\mathrm{a}}/\mathrm{d}\Omega_\mu)_{\mathrm{f}}$ is obtained from eq. (12) and subsequent formulae by omitting all integrals G and that part of H_P proportional to G. We have verified that in the limit $E \gg \mu$, m and for $\sin\theta \gg \mu/E$, m/E ($E = \frac{1}{2}\sqrt{s}$), our expressions for the two photon graphs reduce to those existing in the literature [3–5].

3. Soft and hard photon emission

As is well-known, the infrared divergence in virtual radiative corrections cancels against the similar divergence which arises from the cross section for the inelastic reaction

$$\mathrm{e}^+(p_+) + \mathrm{e}^-(p_-) \to \mu^+(q_+) + \mu^-(q_-) + \gamma(k) . \tag{30}$$

When the extra photon escapes detection, the cross section for reaction (30) has to be added to the elastic one. The specific experimental set-up used to measure the angular distribution determines how much of the differential cross section for reaction (30) has to be added to the virtual corrections. In the following we consider differential cross section experiments, where besides the charges also the energies of the muons are measured. One could then define those events where the energies of the muons are larger than some threshold value E_{th} (e.g. $E_{\mathrm{th}} = 0.9\, p_{+0}$) as elastic events, thus adding some events of reaction (30) to the events of reaction (3).

The contribution to the α^3-differential cross section from those inelastic events is given by

$$\int \frac{\partial^5 \sigma^B}{\partial\Omega_\mu \partial\Omega_\gamma \partial k}\, \mathrm{d}\Omega_\gamma \mathrm{d}k , \tag{31}$$

where the integration limits are determined by

$$q_{+0}, q_{-0} \geq E_{\mathrm{th}} . \tag{32}$$

Also, $k = |\boldsymbol{k}|$. The integrand itself can be written as

$$\frac{\partial^5 \sigma^B}{\partial\Omega_\mu \partial\Omega_\gamma \partial k} = \frac{\alpha^3}{2\pi^2} \frac{m^2\mu^2}{[s(s-4m^2)]^{\frac{1}{2}}} \frac{|\boldsymbol{q}_+|k}{2p_{+0} - k + (q_{+0}/|\boldsymbol{q}_+|)k\cos\theta_\gamma} \sum_{i,j=1}^{4} \frac{F_{ij}}{D_i D_j} . \tag{33}$$

In eq. (33), the polar and azimuthal photon angles, θ_γ and φ_γ, are taken with respect to a frame where $\boldsymbol{q}_+$ defines the z-axis, and $\boldsymbol{q}_+ \wedge \boldsymbol{p}_+$ the y-axis. For the expressions F_{ij} and D_i, we refer to ref. [1].

We can divide the contribution from eq. (31) to the elastic α^3-cross section into a part which contributes to D and a part which contributes to S. The latter is obtained from eq. (33) by taking in the sum only the terms with $i, j = 1, 2$ and $i, j = 3,4$. Similarly, the terms contributing to D are given by the combinations $i = 1,2, j = 3,4$ and $i = 3, 4, j = 1, 2$. These two terms in eq. (33) will be denoted by the subscript S and D. They cancel the infrared divergences in S^V and D^V separately. The α^3-expression for S^V originating from the virtual corrections was already given in ref. [1].

The integration limits determined by eq. (32) can also be obtained from the formulae in ref. [1]. The full φ_γ range is allowed, but k and $\cos\theta_\gamma$ values will be restricted. The allowed phase space region is conveniently visualized in the Dalitz plot of fig. 5. The upper triangle, denoted by (I), allows all θ_γ values, whereas in the lower one (II), θ_γ is restricted. We have

$$\begin{aligned}
&(\mathrm{I})\quad 0 \leqslant k \leqslant k_1 , \qquad && -1 \leqslant \cos\theta_\gamma \leqslant 1 , \\
&(\mathrm{II})\; k_1 \leqslant k \leqslant k_2 , && g_1(k, E_{\mathrm{th}}) \leqslant \cos\theta_\gamma \leqslant g_2(k, E_{\mathrm{th}}) ,
\end{aligned} \tag{34}$$

with

$$\begin{aligned}
&k_1 = 2p_{+0} - E_{\mathrm{th}} - B_+(E_{\mathrm{th}}, 1) , \qquad k_2 = 2p_{+0} - 2E_{\mathrm{th}} , \\
&B_+(q_{-0}, 1) = \tfrac{1}{2}\left[2p_{+0} - q_{-0} + |q_-| - \frac{\mu^2}{q_{-0} - 2p_{+0} - |q_-|}\right] ,
\end{aligned} \tag{35}$$

and where the functions $g_1(k, E_{\mathrm{th}})$ and $g_2(k, E_{\mathrm{th}})$ are given by the right-hand side of the equation

$$\cos\theta_\gamma = \frac{2p_{+0}(p_{+0} - k - q_{+0}) + kq_{+0}}{k\,|q_+|} , \tag{36}$$

by inserting the values

$$\begin{aligned}
&q_{+0} = 2p_{+0} - E_{\mathrm{th}} - k , \\
&q_{+0} = E_{\mathrm{th}} ,
\end{aligned} \tag{37}$$

respectively.

As will become clear below, it is useful to know the soft photon limit of expression (33). The part contributing to D takes in this limit the form

$$\begin{aligned}
\frac{\partial^5 \sigma_D^S}{\partial\Omega_\mu \partial\Omega_\gamma \partial k} &= \frac{\alpha^3}{2\pi^2} \frac{m^2\mu^2}{[s(s-4m^2)]^{\frac{1}{2}}} \frac{|q_+|k^2}{2p_{+0}k_0} \frac{Y}{s^2} \\
&\times\left[\frac{2(p_+q_-)}{(p_+k)(q_-k)} + \frac{2(p_-q_+)}{(p_-k)(q_+k)} - \frac{2(p_+q_+)}{(p_+k)(q_+k)} - \frac{2(p_-q_-)}{(p_-k)(q_-k)}\right] ,
\end{aligned} \tag{38}$$

where

$$Y = \frac{-1}{m^2\mu^2}\,[2(p_+q_+)^2 + 2(p_+q_-)^2 + s(m^2 + \mu^2)] . \tag{39}$$

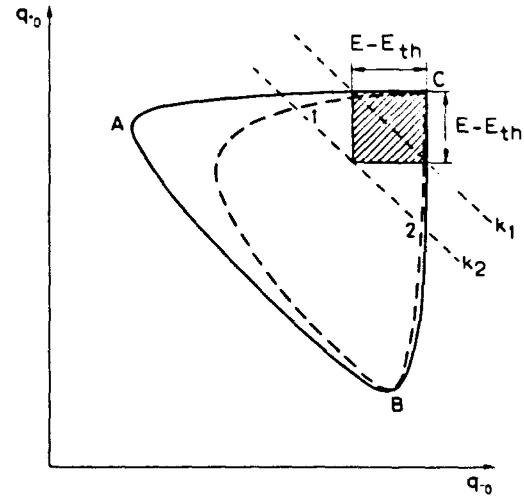

Fig. 5. Dalitz plot for the μ pair. The shaded area is the integration region, part I lying above the line $k = k_1$, part II lying below this line. The lines 1 and 2 give the curves where $\theta_\gamma = 0$ and $\theta_\gamma = \pi - \theta$.

In eqs. (38) and (39) the four-vectors q_+ and q_- have the values from the elastic reaction. Also, a photon mass λ has been introduced into k_0. It is then possible to integrate eq. (38) over the isotropic phase space (I) in eq. (34). The result is

$$\frac{d\sigma_D^S}{d\Omega_\mu} = \frac{d\sigma^0}{d\Omega_\mu}\frac{\alpha}{2\pi}\left\{(4\tau+s)\left[A(s,t)\log\left(\frac{2k_1}{\lambda}\right)^2 + C(s,t)\right] - (t \leftrightarrow u)\right\}. \tag{40}$$

For the details of the calculation we refer to the appendix. From eq. (28) it is explicitly seen that the λ-dependence goes out. The expression for $d\sigma_S^S/d\Omega_\mu$ has been given elsewhere [1].

The total inelastic contribution to D may be written in a form suitable for numerical integration:

$$2\int_{(II)} \frac{\partial^5\sigma_D^B}{\partial\Omega_\mu\partial\Omega_\gamma\partial k}\, d\Omega_\gamma dk + 2\int_{(I)} \frac{\partial^5(\sigma_D^B - \sigma_D^S)}{\partial\Omega_\mu\partial\Omega_\gamma\partial k}\, d\Omega_\gamma dk + 2\frac{d\sigma_D^S}{d\Omega_\mu}, \tag{41}$$

where only the last term is infrared divergent. The other two terms will be evaluated numerically.

We can now write the total expression for D in the form

$$D(\theta) = 2\frac{d\sigma^0}{d\Omega_\mu}[\delta_D^A + \delta_D^I + \delta_D^{II}] = 2\frac{d\sigma^0}{d\Omega_\mu}\delta_D^T. \tag{42}$$

Here δ_D^A is an analytic expression originating from the addition of the virtual correction, eq. (29), and the soft photon bremsstrahlung, eq. (40). Also, δ_D^{II} and δ_D^{I} come from the first two terms in eq. (41). Of these different contributions only δ_D^{A} has so far been given in the literature, and then only in the extreme relativistic limit, which also restricts the angular values [3–5].

Similarly, $S(\theta)$ can be written as

$$S(\theta) = 2\,\frac{\mathrm{d}\sigma^0}{\mathrm{d}\Omega_\mu}\,[1 + \delta_S^{\mathrm{A}} + \delta_S^{\mathrm{I}} + \delta_S^{\mathrm{II}}] = 2\,\frac{\mathrm{d}\sigma^0}{\mathrm{d}\Omega_\mu}\,[1 + \delta_S^{\mathrm{T}}]\ , \tag{43}$$

where δ_S^{A} has been given before [1]. Finally we note that in the extreme relativistic limit expression (40) reduces to the result of ref. [3].

One may wonder whether existing expressions for $e\mu$ scattering [6] could be used to obtain the results of sects. 2 and 3. This can not easily be done since those expressions do not lend themselves to a straightforward analytic continuation and moreover the phase space integrals are different.

4. Results

In this section we give the results of the analytic and total corrections both to the functions S and D i.e. the quantities δ_S^{A}, δ_S^{T}, δ_D^{A}, δ_D^{T} in eqs. (43) and (42). From these and the tabulated values $2(\mathrm{d}\sigma^0/\mathrm{d}\Omega_\mu)$ (table 1) one finds the values of S and D. The numerical evaluation of the integrals I and II in eq. (41) was performed in a way similar to that of ref. [1]. The energies were taken from 0.5 to 5 GeV and the angles varied between 5° and 90°. We have assumed that the threshold energy for accepting the muons as a 2 muon event is given by $E_{\mathrm{th}} = 0.9\, p_{+0}$.

From table 2 it is clear that the analytic correction δ_D^{A} is only slightly different from the total correction δ_D^{T}, whereas δ_S^{A} and δ_S^{T} differ appreciably. Thus it is important to consider hard photon corrections for the function S, but in the case of the function D they may be neglected for the kinematical values considered here. Although we have not tabulated the values for δ_S^{I}, δ_S^{II}, $\delta_{\mathrm{A}}^{\mathrm{I}}$ and $\delta_{\mathrm{A}}^{\mathrm{II}}$ it is interesting to note that the integrals over region II (eq. (41)) are at least twice as important as the ones over region I, but often they are 10 times as large. So it means that the hard and soft matrix element in the top triangular region of phase space (fig. 5) are about equal. In the analytic expression the lower triangular region in phase space is not taken into account, which turns out to be a bad approximation for S, in particular for larger scattering angles. This latter point is clear, since for larger angles θ, the regions $\theta_\gamma \approx \theta$ and $\theta_\gamma \approx \pi - \theta$ lie to their maximal extent inside the experimental phase space, thus increasing δ_S^{II}.

In conclusion, if one wants to perform measurements at the percentage level of the angular distribution of the charged muons with a small energy resolution, it is necessary to take into account hard photon corrections for the symmetric part of the differential cross sections, but the analytic expression δ_D^{A} is sufficiently accurate

Table 1
$2d\sigma^0/d\Omega_\mu$ (in nb) for different values of the beam energy, p_{+0} (in GeV) and the scattering angle θ.

θ \ p_{+0}	0.5	1.0	2.0	3.0	5.0
5°	20.19	5.14	1.29	0.57	0.21
20°	19.14	4.86	1.22	0.54	0.20
40°	16.27	4.10	1.03	0.46	0.16
60°	13.01	3.24	0.81	0.36	0.13
85°	10.66	2.62	0.66	0.29	0.10

for the asymmetric part. It is clear from the results that the effects of the higher order corrections can be most easily observed in the forward direction.

Appendix

A.1. The bremsstrahlung integral

We first consider the integral appearing in the expression for the bremsstrahlung cross section (38). Using the Feynman trick we find

$$R = \int_{|\mathbf{k}|<\Delta E} \frac{d^3k}{k^0} \frac{1}{(p_+k)(q_+k)} = \int_0^1 dx \int_{|\mathbf{k}|<\Delta E} \frac{d^3k}{k^0} \frac{1}{(p_xk)^2}\,, \tag{A.1}$$

with $k^0 = \sqrt{\mathbf{k}^2 + \lambda^2}$ and $p_x = xp_+ + (1-x)q_+$. Performing the k-integration and omitting terms of order λ gives

$$R = 2\pi \int_0^1 dx\, [a(x) \log \left(\frac{2\Delta E}{\lambda}\right)^2 + c(x)]\,, \tag{A.2}$$

with

$$a(x) = (p_{x0}^2 - \mathbf{p}_x^2)^{-1}\,,$$

$$c(x) = \frac{1}{p_{x0}^2 - \mathbf{p}_x^2} \frac{p_{x0}}{|\mathbf{p}_x|} \log \frac{p_{x0} - |\mathbf{p}_x|}{p_{x0} + |\mathbf{p}_x|}\,. \tag{A.3}$$

Noting that $p_{x0} = E$ and that

Table 2
The percentage corrections δ_S^A, δ_S^T, δ_D^A and δ_D^T at threshold energy $E_{th} = 0.9\, p_{+0}$.

p_{+0}	0.5		1.0		2.0		3.0		5.0	
θ	δ_S^A	δ_S^T	δ_S^A	δ_S^T	δ_S^A	δ_S^T	δ_S^A	δ_S^T	δ_S^A	δ_S^T
5°	−8.7	−8.3 ± 0.1	−10.4	−10.0 ± 0.1	−12.0	−11.6 ± 0.1	−13.0	−12.6 ± 0.1	−14.1	−13.7 ± 0.1
20°	−8.7	−8.1 ± 0.1	−10.4	− 9.8 ± 0.1	−12.0	−11.4 ± 0.1	−13.0	−12.4 ± 0.1	−14.1	−13.6 ± 0.1
40°	−8.7	−7.6 ± 0.1	−10.4	− 9.2 ± 0.1	−12.0	−10.8 ± 0.1	−13.0	−11.7 ± 0.1	−14.1	−12.9 ± 0.1
60°	−8.8	−6.7 ± 0.1	−10.4	− 8.2 ± 0.1	−12.0	− 9.7 ± 0.1	−13.0	−10.7 ± 0.1	−14.1	−11.7 ± 0.1
85°	−8.8	−5.3 ± 0.1	−10.4	− 6.6 ± 0.1	−12.0	− 7.8 ± 0.1	−13.0	− 8.5 ± 0.1	−14.1	− 9.7 ± 0.1

p_{+0}	0.5		1.0		2.0		3.0		5.0	
θ	δ_D^A	δ_D^T	δ_D^A	δ_D^T	δ_D^A	δ_D^T	δ_D^A	δ_D^T	δ_D^A	δ_D^T
5°	11.2	10.9 ± 0.1	14.4	14.0 ± 0.1	16.3	15.8 ± 0.1	16.8	16.4 ± 0.1	17.1	16.6 ± 0.1
20°	7.9	7.4 ± 0.1	8.5	8.0 ± 0.1	8.7	8.1 ± 0.1	8.7	8.1 ± 0.1	8.7	8.2 ± 0.1
40°	4.7	4.2 ± 0.1	4.8	4.3 ± 0.1	4.8	4.3 ± 0.1	4.9	4.3 ± 0.1	4.9	4.3 ± 0.1
60°	2.5	2.1 ± 0.1	2.6	2.2 ± 0.1	2.6	2.2 ± 0.1	2.6	2.2 ± 0.1	2.6	2.2 ± 0.1
85°	0.4	0.3 ± 0.1	0.4	0.3 ± 0.1	0.4	0.3 ± 0.1	0.4	0.3 ± 0.1	0.4	0.3 ± 0.1

$$p_{x0}^2 - p_x^2 = t(x^2 - x) + \mu^2(1-x) + m^2 x \,, \tag{A.4}$$

we find

$$A(s,t) = \int_0^1 \mathrm{d}x\, a(x) = [\lambda(t,\mu^2,m^2)]^{-\frac{1}{2}} \log \left| \frac{t - \mu^2 - m^2 - [\lambda(t,\mu^2,m^2)]^{\frac{1}{2}}}{t - \mu^2 - m^2 + [\lambda(t,\mu^2,m^2)]^{\frac{1}{2}}} \right| , \tag{A.5}$$

where

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy \,. \tag{A.6}$$

The integral of $c(x)$ is somewhat more involved. Going over to the new integration variable y, defined by

$$[-tx^2 + (t + \mu^2 - m^2)x + E^2 - \mu^2]^{\frac{1}{2}}/E = y - x(-t)^{\frac{1}{2}}/E \,, \tag{A.7}$$

the integral of $c(x)$ can be evaluated as

$$C(s,t) = \int_0^1 \mathrm{d}x\, c(x) = [\lambda(t,\mu^2,m^2)]^{-\frac{1}{2}} \sum_{i,j=1}^{4} \epsilon_i \delta_j U_{ij}(\eta_0, \eta_1, y_i, y_j). \tag{A.8}$$

We have introduced the symbols

$$\epsilon_i = (+1,-1,-1,+1) \,, \qquad \delta_j = (-1,-1,+1,+1) \,,$$

$$\eta_0 = (1 - \mu^2/E^2)^{\frac{1}{2}} \,, \qquad \eta_1 = (1 - m^2/E^2)^{\frac{1}{2}} + (-t)^{\frac{1}{2}}/E \,,$$

$$U_{ij}(\eta_0, \eta_1, y_i, y_j) = \int_{\eta_0}^{\eta_1} \frac{\log|y - y_j|}{y - y_i}\, \mathrm{d}y \,, \tag{A.9}$$

$$y_1 = -1 - \{t + \mu^2 - m^2 - [\lambda(t,\mu^2,m^2)]^{\frac{1}{2}}\}/2E(-t)^{\frac{1}{2}} \,,$$

$$y_2 = -1 - \{t + \mu^2 - m^2 + [\lambda(t,\mu^2,m^2)]^{\frac{1}{2}}\}/2E(-t)^{\frac{1}{2}} \,,$$

$$y_3 = y_1 + 2 \,, \qquad y_4 = y_2 + 2 \,.$$

The functions U_{ij} can be expressed in terms of dilogarithms as follows:

$$U_{ij} = \mathrm{Re}\left[\mathrm{Li}_2\left(\frac{\eta_0 - y_i}{y_j - y_i}\right) - \mathrm{Li}_2\left(\frac{\eta_1 - y_i}{y_j - y_i}\right)\right] + \log|y_i - y_j| \log \left|\frac{\eta_1 - y_i}{\eta_0 - y_i}\right| \qquad i \neq j,$$

$$U_{ii} = \tfrac{1}{2}(\log|\eta_1 - y_i|)^2 - \tfrac{1}{2}(\log|\eta_0 - y_i|)^2 \,. \tag{A.10}$$

Putting all together this we have

$$R = 2\pi\left[A(s,t) \log\left(\frac{2\Delta E}{\lambda}\right)^2 + C(s,t)\right] . \tag{A.11}$$

A.2. The integral G

Using the Feynman parametrization and performing the k-integration, we find

$$G = -i\pi^2 \int_0^1 dy \int_0^1 dx\, y/[y^2p_x^2 + (1-y)\lambda] \,, \tag{A.12}$$

where p_x is the same four-vector as in eq. (A.1). After the y-integration, we obtain in the limit $\lambda \to 0$

$$G = -\frac{i\pi^2}{2} \int_0^1 dx \frac{1}{p_x^2} \log\left(\frac{p_x}{\lambda}\right)^2 = -\frac{i\pi^2}{2} [A(s,t) \log\frac{s}{\lambda^2} + B(s,t)] \,. \tag{A.13}$$

where $A(s,t)$ is the function defined by eq. (A.5) and

$$B(s,t) = \int_0^1 dx \frac{1}{p_x^2} \log\left(\frac{p_x^2}{s}\right). \tag{A.14}$$

Using the same technique as for the evaluation of eq. (A.8) we find

$$B(s,t) = A(s,t) \log\left(\frac{-t}{s}\right) + [\lambda(t,\mu^2,m^2)]^{-\frac{1}{2}} \sum_{i,j=1}^{2} \rho_i U_{ij}(0,1,x_i,x_j) \,, \tag{A.15}$$

with the functions U_{ij} defined in eq. (A.9) and

$$\begin{aligned} x_1 &= \{t + \mu^2 - m^2 + [\lambda(t,\mu^2,m^2)]^{\frac{1}{2}}\}/2t \,, \\ x_2 &= \{t + \mu^2 - m^2 - [\lambda(t,\mu^2,m^2)]^{\frac{1}{2}}\}/2t \,, \\ \rho_i &= (+1, -1) \,. \end{aligned} \tag{A.16}$$

Thus,

$$\frac{1}{2\pi^2} \operatorname{Im} G = -\tfrac{1}{4}[A(s,t) \log\left(\frac{s}{\lambda^2}\right) + B(s,t)] \,. \tag{A.17}$$

A.3. The integral F

Standard methods allow us to write

$$F = i\pi^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{y^2z(P^2 + Y^2 + 2C)}{(Y^2 + C)^2} \,, \tag{A.18}$$

where

$$Y^2 = P^2(1+2yz-4y+4y^2-4y^2z+y^2z^2) + y^2z^2Q_x^2\,,$$

$$Q_x = (1-x)Q + x\Delta\,, \qquad C = P^2(2yz-1)\,. \tag{A.19}$$

The y-integration is readily done, yielding

$$F = i\pi^2 \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}z\, z\left\{\frac{1}{z^2p_x^2 - s(z-1)} - \frac{1}{2z^2p_x^2[z^2p_x^2-(z-1)s]}\right\}$$

$$+\, i\pi^2 \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}z\, z\left\{\frac{s(2-z)}{2[z^2p_x^2-(z-1)s]^2}\log\frac{z^2p_x^2}{s(z-1)}\right\}. \tag{A.20}$$

Again, p_x has been defined in eq. (A.1).

Let us call the first term in eq. (A.20) A_1 and the other one A_2. For A_1 the z-integration gives

$$A_1 = \frac{i\pi^2}{2}\int_0^1 \mathrm{d}x\,\frac{1}{p_x^2}\log\frac{p_x^2}{s} = \frac{i\pi^2}{2}B(s,t)\,, \tag{A.21}$$

with the help of eq. (A.14). It also turns out that (see eq. (A.5))

$$A_2 = -\frac{\pi^3}{2}A(s,t)\,, \tag{A.22}$$

so that

$$\frac{1}{2\pi^2}\,\mathrm{Im}\,F = \tfrac{1}{4}B(s,t)\,. \tag{A.23}$$

A.4. The integrals H_P, H_Δ and H_Q

The integral H_μ is readily written in the form

$$H_\mu = -i\pi^2 \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\,\frac{yP_\mu + y^2p'_{x\mu}}{y^2p_x^2+(1-y)\lambda^2}\,, \tag{A.24}$$

where

$$p_x = xp_+ + (1-x)q_+\,, \qquad p'_x = x\Delta + (1-x)Q - P\,. \tag{A.25}$$

Separating the numerator in the three vectors and doing the y-integration, we obtain

$$H_\mu = P_\mu\left[G + i\pi^2 \int_0^1 \frac{\mathrm{d}x}{p_x^2}\right] - i\pi^2 \Delta_\mu \int_0^1 \mathrm{d}x\,\frac{x}{p_x^2} - i\pi^2 Q_\mu \int_0^1 \mathrm{d}x\,\frac{1-x}{p_x^2}. \tag{A.26}$$

Repeated use of eq. (A.5) gives

$$\frac{1}{2\pi^2}\,\mathrm{Im}\,H_P = \frac{1}{2\pi^2}\,\mathrm{Im}\,G + \tfrac{1}{2}A(s,t)\,,$$

$$\frac{1}{2\pi^2}\,\mathrm{Im}\,H_\Delta = -\tfrac{1}{2}A_x(s,t)\,,$$

$$\frac{1}{2\pi^2}\,\mathrm{Im}\,H_Q = -\tfrac{1}{2}\left[A(s,t) - A_x(s,t)\right]\,, \tag{A.27}$$

where

$$A_x(s,t) = \int_0^1 \mathrm{d}x\,\frac{x}{p_x^2} = \left[\log\left(\frac{m}{\mu}\right)^2 + (t+\mu^2-m^2)A(s,t)\right]/2t\,. \tag{A.28}$$

Interchanging m and μ in A_x gives $A - A_x$, exhibiting the symmetry in H_Δ and H_Q.

A.5. *The integrals $F_{\Delta,Q}$ and $G_{\Delta,Q}$*

It is clear from the definition of these integrals that F_Q and G_Q can be obtained from F_Δ and G_Δ by replacing $m \to \mu$. It thus suffices to calculate F_Δ and G_Δ. One easily proves that

$$F_\Delta = -\frac{i\pi^2}{2}\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\; 2y/\left[y^2(1-2x)^2P^2 - y^2P^2 + (1-y^2)m^2 - i\epsilon\right]\,. \tag{A.29}$$

The x-integration yields

$$F_\Delta = -\frac{i\pi^2}{2P^2}\int_0^1 \mathrm{d}y\,\frac{1}{y\sqrt{1-a((1-y)/y)^2}}\log\left|\frac{\sqrt{1-a((1-y)/y)^2}-1}{\sqrt{1-a((1-y)/y)^2}+1}\right|, \tag{A.30}$$

where $a = (m^2 - i\epsilon)/P^2$. It is profitable now to assume $a < 0$, so that afterwards an analytic continuation has to be made. A change of variables leads to

$$\frac{1}{2\pi^2}\,\mathrm{Im}\,F_\Delta = \frac{-2}{s(1-\beta^2)}\int_0^1 \mathrm{d}z\,\frac{\log z}{z^2 + 2z\left[(1+\beta^2)/(1-\beta^2)\right] + 1}\,,$$

$$\beta = \left[1 - m^2/P^2\right]^{\frac{1}{2}}\,, \tag{A.31}$$

which can be readily evaluated in terms of dilogarithms to yield

$$\frac{1}{2\pi^2}\,\mathrm{Im}\,F_\Delta = [\mathrm{Li}_2((\beta-1)/(\beta+1)) - \mathrm{Li}_2((\beta+1)/(\beta-1))]/2s\beta\,. \tag{A.32}$$

From the definition of G_Δ follows

$$G_\Delta = F_\Delta/\beta^2 + 2\{\int \mathrm{d}^4k/(+)(-) - \int \mathrm{d}^4k/(\Delta)(+)\}/s\beta^2\,, \tag{A.33}$$

which eventually leads to

$$\frac{1}{2\pi^2}\,\mathrm{Im}\,G_\Delta = \frac{1}{\beta^2}\frac{1}{2\pi^2}\,\mathrm{Im}\,F_\Delta - \frac{1}{s\beta^2}\log\left(\frac{s}{m^2}\right)\,. \tag{A.34}$$

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