Four-point Function

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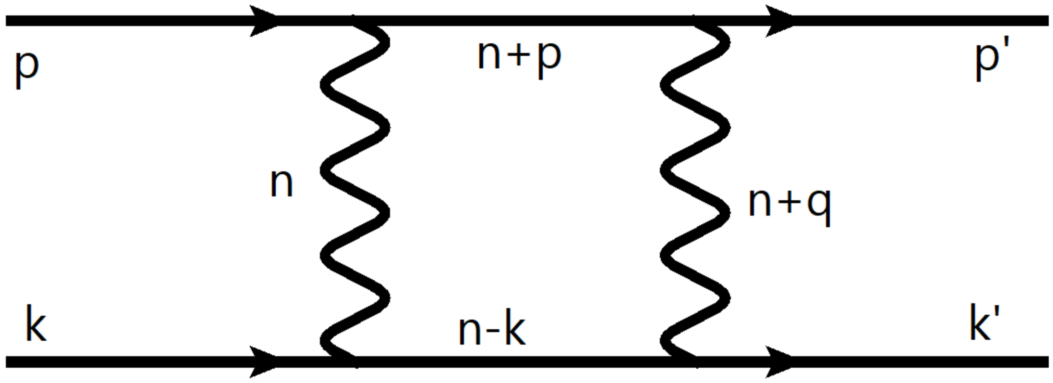

$$D_0(q,-k,p,\lambda,\lambda,m_e,m_\mu) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D n \frac{1}{(n^2-\lambda^2)\left[(n+q)^2-\lambda^2\right]\left[(n-k)^2-m_e^2\right]\left[(n+p)^2-m_\mu^2\right]} \tag{1}$$

$$= \langle | \left\{(n^2-\lambda^2)\left[(n+q)^2-\lambda^2\right]\left[(n-k)^2-m_e^2\right]\left[(n+p)^2-m_\mu^2\right]\right\}^{-1} | \rangle_n \tag{2}$$

$$= 3!\int d^3x \frac{(2\pi\mu)^{4-D}}{i\pi^2} I_4(A), \tag{3}$$

where $\int d^3x = \int_0^1 dy \int_0^{1-y} dz \int_0^{1-y-z} dx$, $A = (qx-ky+pz)^2 - xq^2 + (1-y-z)\lambda^2$, and $I_n(A)$ is the Basic Integral:

$$I_n(A) = i(-1)^n \pi^{D/2}\frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)}(A-i\epsilon)^{D/2-n}, \tag{4}$$

$$\Rightarrow D_0 = \int d^3x \left[(qx-ky+pz)^2 - xq^2 + (1-y-z)\lambda^2 - i\epsilon\right]^{-2} \tag{5}$$

$$= \int d^3x \left\{tx(x+y+z-1) + (m_e y + m_\mu z)^2 + \left[(m_\mu - m_e)^2 - s\right]yz + \lambda^2(1-y-z) - i\epsilon\right\}^{-2}, \tag{6}$$

$$\tag{7}$$

with $t = q^2 = (k' - k)^2$, $s = (k + p)^2$,

$$= \int_0^1 dy \int_0^{1-y} dz \int_0^z dx \left\{ tx(x - z) + [m_e y + m_\mu(1 - y - z)]^2 + \left[(m_\mu - m_e)^2 - s\right] y(1 - y - z) + \lambda^2 z - i\epsilon \right\}^{-2} \tag{8}$$

$$= \int_0^1 dz \int_0^z dy \int_0^{1-z} dx \left\{ tx(x + z - 1) + [m_e y + m_\mu(z - y)]^2 + \left[(m_\mu - m_e)^2 - s\right] y(z - y) + \lambda^2(1 - z) - i\epsilon \right\}^{-2} \tag{9}$$

$$= \int_0^1 dz \int_0^1 dy \int_0^{1-z} \frac{z dx}{\{tx(x + z - 1) + z^2[m_e y + m_\mu(1 - y)]^2 + [(m_\mu - m_e)^2 - s]\, z^2 y(1 - y) + \lambda^2(1 - z) - i\epsilon\}^2} \tag{10}$$

$$= \int_0^1 dz \int_0^1 dy \int_0^{1-z} dx \frac{z}{\left\{z^2 P_y^2 + tx(x + z - 1) + \lambda^2(1 - z) - i\epsilon\right\}^2} \tag{11}$$

$$= \int_0^1 dz \int_0^1 dy \int_0^{1-z} dx \frac{z}{\left\{z^2 \bar{P}_y^2 + \bar{t}x(x + z - 1) + \lambda^2(1 - z)\right\}^2}, \tag{12}$$

where $\bar{P}_y^2 = [py - (1 - y)k]^2 - i\epsilon = \bar{s}y^2 + (m_e^2 - m_\mu^2 - \bar{s})y + m_\mu^2$, $\bar{t} = t + i\epsilon$,

$$= \int_0^1 dz \int_0^1 dy \int_0^z dx \frac{1 - z}{\left\{(1 - z)^2 \bar{P}_y^2 + \bar{t}x(x - z) + \lambda^2 z\right\}^2} \tag{13}$$

$$= \int_0^1 dz \int_0^1 dy \int_0^z dx \frac{1}{z^3}\left(\frac{1}{z} - 1\right) \left\{ \left(\frac{1}{z} - 1\right)^2 \bar{P}_y^2 + \bar{t}\frac{x}{z}(\frac{x}{z} - 1) + \frac{\lambda^2}{z} \right\}^{-2} \tag{14}$$

$$= \int_1^\infty dz \int_0^1 dy \int_0^1 dx (z - 1) \left\{(1 - z)^2 \bar{P}_y^2 + \lambda^2 z + \bar{t}x(x - 1)\right\}^{-2} \tag{15}$$

$$\xrightarrow{\lambda \to 0} \int_1^\infty dz \int_0^1 dy \int_0^1 dx \frac{1}{-2\bar{P}_y^2} \frac{-2(z - 1)\bar{P}_y^2 - \lambda^2}{\left\{(1 - z)^2 \bar{P}_y^2 + \lambda^2 z + \bar{t}x(x - 1)\right\}^2} \tag{16}$$

$$= \int_0^1 dy \int_0^1 dx \frac{1}{2\bar{P}_y^2} \frac{1}{\lambda^2 + \bar{t}x(x - 1)}. \tag{17}$$

Calling $x_1, x_2 = \frac{\bar{t} \pm \sqrt{\bar{t}^2 - 4\lambda^2 \bar{t}}}{2\bar{t}}$ are the solutions of equation: $\bar{t}x(x - 1) + \lambda^2$, when $\lambda \to 0$: $x_1, x_2 = \frac{\lambda^2}{\bar{t}}, 1 - \frac{\lambda^2}{\bar{t}}$. We have:

$$= \int_0^1 dy \int_0^1 dx \frac{1}{2\bar{t}\bar{P}_y^2} \frac{1}{(x - x_1)(x - x_2)} \tag{18}$$

$$= \int_0^1 dy \frac{1}{2\bar{t}\bar{P}_y^2 (x_2 - x_1)} \left[\ln(x_2 - x) - \ln(x - x_1)\right]\Big|_0^1 \tag{19}$$

$$\xrightarrow{\lambda \to 0} \int_0^1 dx \frac{1}{\bar{t}\bar{P}_x^2} \ln\left(\frac{\lambda^2}{-\bar{t}}\right). \tag{20}$$

We calculate $\int_0^1 dx \frac{1}{\bar{P}_x^2}$:

$$\int_0^1 \frac{dx}{\bar{P}_x^2} = \int_0^1 \frac{dx}{\bar{s}x^2 + (m_e^2 - m_\mu^2 - \bar{s})x + m_\mu^2} \tag{21}$$

$$= \int_0^1 \frac{dx}{\bar{s}(x - x_1)(x - x_2)} = \frac{1}{\bar{s}(x_1 - x_2)} \left[\ln\left(\frac{x_1 - 1}{x_1}\right) - \ln\left(\frac{x_2 - 1}{x_2}\right)\right], \tag{22}$$

with x_1,x_2 are the solutions of :

$$\bar{s}x^2 + (m_e^2 - m_\mu^2 - \bar{s})x + m_\mu^2 = 0, \tag{23}$$

$$\begin{aligned}
\triangle &= (\bar{s} - m_e^2 - m_\mu^2)^2 - 4m_\mu^2 m_e^2 && (24)\\
&= \left[\bar{s} - (m_e - m_\mu)^2 - 4m_\mu m_e\right]\left[\bar{s} - (m_e - m_\mu)^2\right] && (25)\\
&= \left[\bar{s} - (m_e - m_\mu)^2\right]^2 \left[1 - \frac{4m_e m_\mu}{\bar{s} - (m_e - m_\mu)^2}\right], && (26)
\end{aligned}$$

$$\Rightarrow x_1, x_2 = \frac{\bar{s} + m_\mu^2 - m_e^2 \pm \sqrt{\triangle}}{2\bar{s}}. \tag{27}$$

And :

$$x_s = \frac{\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}} - 1}{\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}} + 1}.$$

We have :

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$$1 - x_s^2 = \frac{4\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\left(\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}} + 1\right)^2} \tag{28}$$

$$\Rightarrow \frac{(1-x_s^2)}{-x_s} = \frac{\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}} + 1}{1 - \sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}} \frac{4\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\left(\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}} + 1\right)^2} \tag{29}$$

$$= \frac{4\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{1 - \left(\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}\right)^2} = \frac{\left[\bar{s} - (m_e - m_\mu)^2\right]\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{m_e m_\mu}, \tag{30}$$

$$\Rightarrow \frac{(1-x_s^2)m_e m_\mu}{-x_s} = \left[\bar{s} - (m_e - m_\mu)^2\right]\sqrt{1 - \frac{4m_e m_\mu}{\bar{s} - (m_e - m_\mu)^2}} = \sqrt{\triangle} = (x_1 - x_2)\bar{s}. \tag{31}$$

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$$\begin{aligned}
&\frac{x_1 - 1}{x_1} . \frac{x_2}{x_2 - 1} = \frac{x_1 x_2 - x_2}{x_1 x_2 - x_1} = \frac{\bar{s} - m_\mu^2 - m_e^2 - \sqrt{\triangle}}{\bar{s} - m_\mu^2 - m_e^2 + \sqrt{\triangle}} && (32)\\
&\Rightarrow 1 - \frac{\bar{s} - m_\mu^2 - m_e^2 - \sqrt{\triangle}}{\bar{s} - m_\mu^2 - m_e^2 + \sqrt{\triangle}} = \frac{2\sqrt{\triangle}}{\bar{s} - m_\mu^2 - m_e^2 + \sqrt{\triangle}} && (33)\\
&= \frac{2\left[\bar{s} - (m_\mu - m_e)^2\right]\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\bar{s} - m_\mu^2 - m_e^2 + \sqrt{\triangle}} && (34)\\
&= \frac{4\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\frac{2(\bar{s}-m_\mu^2-m_e^2)+2\sqrt{\triangle}}{\bar{s}-(m_\mu-m_e)^2}} = \frac{4\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\frac{2[\bar{s}-(m_\mu-m_e)^2-2m_\mu m_e]}{\bar{s}-(m_\mu-m_e)^2} + 2\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}} && (35)\\
&= \frac{4\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{1 + 1 - \frac{4m_e m_\mu}{\bar{s}-(m_\mu-m_e)^2} + 2\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}} && (36)\\
& && (37)
\end{aligned}$$

$$= \frac{4\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{1+\left(\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}\right)^2+2\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}} = \frac{4\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\left(\sqrt{1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}+1\right)^2} \tag{38}$$

$$= 1 - x_s^2 \tag{39}$$

$$\Rightarrow \frac{\bar{s}-m_\mu^2-m_e^2-\sqrt{\triangle}}{\bar{s}-m_\mu^2-m_e^2+\sqrt{\triangle}} = \frac{x_1-1}{x_1}.\frac{x_2}{x_2-1} = x_s^2 \tag{40}$$

Using Eq. (40) and Eq. (31), we get the final result of Eq .(22) :

$$\frac{1}{\bar{s}(x_1-x_2)}\left[\ln\left(\frac{x_1-1}{x_1}\right)-\ln\left(\frac{x_2-1}{x_2}\right)\right] = \frac{-2x_s}{(1-x_s^2)m_e m_\mu}\ln(x_s). \tag{41}$$

Finally, we obtain the result of four-point function:

$$D_0(q,-k,p,\lambda,\lambda,m_e,m_\mu) = \frac{-2x_s}{(1-x_s^2)\bar{t}m_e m_\mu}\ln(x_s)\ln\left(\frac{\lambda^2}{-\bar{t}}\right). \tag{42}$$

- Prove :

$$\int_1^\infty dz\int_0^1 dy\int_0^1 dx\frac{1}{-2\bar{P}_y^2}\frac{\lambda^2}{\left\{(1-z)^2\bar{P}_y^2+\lambda^2 z+\bar{t}x(x-1)\right\}^2} \xrightarrow{\lambda\to 0} 0. \tag{43}$$

We have:

$$\int_1^\infty dz\int_0^1 dy\int_0^1 dx\frac{1}{-2\bar{P}_y^2}\frac{\lambda^2}{\left\{(1-z)^2\bar{P}_y^2+\lambda^2 z+\bar{t}x(x-1)\right\}^2} \tag{44}$$

$$= \int_1^\infty dz\int_0^1 dy\int_{-1/2}^{1/2} dx\frac{1}{-2\bar{P}_y^2}\frac{\lambda^2}{\left\{(1-z)^2\bar{P}_y^2+\lambda^2 z+\bar{t}(x+1/2)(x-1/2)\right\}^2} \tag{45}$$

$$= \int_1^\infty dz\int_0^1 dy\int_{-1/2}^{1/2} dx\frac{1}{-2\bar{P}_y^2}\frac{\lambda^2}{\left\{(1-z)^2\bar{P}_y^2+\lambda^2 z+\bar{t}x^2-\frac{\bar{t}}{4}\right\}^2} \tag{46}$$

$$= \int_1^\infty dz\int_0^1 dy\int_{-1/2}^{1/2} dx\frac{\lambda^2}{-2\bar{t}^2\bar{P}_y^2}\frac{1}{[(x-A)(x+A)]^2} \tag{47}$$

$$= \int_1^\infty dz\int_0^1 dy\int_{-1/2}^{1/2} dx\frac{\lambda^2}{-8\bar{t}^2\bar{P}_y^2 A^2}\left[\frac{1}{x-A}-\frac{1}{x+A}\right]^2 \tag{48}$$

$$= \int_1^\infty dz\int_0^1 dy\frac{\lambda^2}{-8\bar{t}^2\bar{P}_y^2 A^2}\left[\frac{-2}{A}\ln\left(\frac{1/2-A}{1/2+A}\right)-\frac{2}{1/4-A^2}\right] \tag{49}$$

$$= \int_1^\infty dz\int_0^1 dy\frac{\lambda^2}{-8\bar{t}^2\bar{P}_y^2 A^2}\left\{\left[\frac{-2}{A}\ln\left(\frac{1/2-A}{1/2+A}\right)-8\right]+\left[8-\frac{2}{1/4-A^2}\right]\right\} \tag{50}$$

with:

$$A = \sqrt{\frac{1}{4}-(1-z)^2\frac{\bar{P}_y^2}{\bar{t}}-\frac{\lambda^2}{\bar{t}}z}. \tag{51}$$

@ Consider:

$$f_\lambda(y,z) = \frac{\lambda^2}{-8\bar{t}^2\bar{P}_y^2 A^2(y,z)}\left[\frac{-2}{A(y,z)}\ln\left(\frac{1/2-A(y,z)}{1/2+A(y,z)}\right)-8\right], \tag{52}$$

we can see that $\forall y,z$ in the integral area $[y,z]=\{[0,1];[1,\infty]\}$:

$$\lim_{\lambda\to 0} f_\lambda(y,z) = 0, \tag{53}$$

$$\Rightarrow \lim_{\lambda\to 0}\int_1^\infty dz\int_0^1 dy f_\lambda(y,z) = \int_1^\infty dz\int_0^1 dy \lim_{\lambda\to 0} f_\lambda(y,z) = 0. \tag{54}$$

@ Consider:

$$\int_1^\infty dz\int_0^1 dy\frac{\lambda^2}{-8\bar{t}^2\bar{P}_y^2A^2}\left[8-\frac{2}{1/4-A^2}\right] = \int_1^\infty dz\int_0^1 dy\frac{\lambda^2}{-8\bar{t}^2\bar{P}_y^2A^2}\frac{-A^2}{1/4-A^2} \tag{55}$$

$$= \int_1^\infty dz\int_0^1 dy\frac{\lambda^2}{8\bar{t}^2\bar{P}_y^2}\frac{1}{(1-z)^2\frac{\bar{P}_y^2}{\bar{t}}+\frac{\lambda^2}{\bar{t}}z} = \int_1^\infty dz\int_0^1 dy\frac{\lambda^2}{8\bar{t}\bar{P}_y^4(z_2-z_1)}\left(\frac{1}{z-z_2}-\frac{1}{z-z_1}\right) \tag{56}$$

$$= \int_0^1 dy\frac{-\lambda^2}{8\bar{t}\bar{P}_y^4(z_2-z_1)}\ln\left(\frac{1-z_2}{z-z_1}\right), \tag{57}$$

where $z_1, z_2 = \frac{\left(2-\lambda^2/\bar{P}_y^2\right)\pm\sqrt{\triangle}}{2}$ are the solutions of equation: $z^2 - z\left(2-\frac{\lambda^2}{\bar{P}_y^2}\right)+1$ with $\triangle = -4\frac{\lambda^2}{\bar{P}_y^2}+\frac{\lambda^4}{\bar{P}_y^4}$. We get :

$$\lim_{\lambda\to 0} f_\lambda(y) = \frac{-\lambda^2}{8\bar{t}\bar{P}_y^4(z_2-z_1)}\ln\left(\frac{1-z_2}{1-z_1}\right) \to 0 \qquad \forall y\in[0,1], \tag{58}$$

$$\Rightarrow \lim_{\lambda\to 0}\int_0^1 dy\frac{-\lambda^2}{8\bar{t}\bar{P}_y^4(z_2-z_1)}\ln\left(\frac{1-z_2}{z-z_1}\right) = \int_0^1 dy\lim_{\lambda\to 0}\left[\frac{-\lambda^2}{8\bar{t}\bar{P}_y^4(z_2-z_1)}\ln\left(\frac{1-z_2}{1-z_1}\right)\right] = 0 \tag{59}$$