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BACHELOR'S DEGREE THESIS

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Electron muon elastic scattering in one-loop QED with soft-photon corrections

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Ho Chi Minh City-2020

Acknowledgement

To have the opportunity to complete this thesis and write such words in here, I really want to say thank you all, who side by side have been supporting, helping, believing in me.

First of all, I would like to express my sincere gratitude to my supervisor, Dr. Le Duc Ninh, who is not only a teacher but be also a brother. Besides professional valuable knowledge, skills I have learned from him, it is also enthusiasm for working, candidness, meticulous but simple, and enjoying in your own works, to become a good scientist, a good citizen. Really thank you very much.

I am also grateful to Prof. Jean Tran Thanh Van for his contributions to science and giving me a chance to work in Institute For Interdisciplinary Research in Science and Education (IFIRSE) to accomplish my thesis. I place gratefully on record the financial support and accommodation from IFIRSE.

In the time when I live in ICISE, I would like to thank all ICISE staff members for their thoughtful assistance and IFIRSE members Dr. Dao Thi Nhung, Tran Van Ngoc, Nguyen Tran Quang Thong, Ho Quoc Trung for their advice, support and giving me relaxing moments, this memorable memory I would keep in my mind.

I take this opportunity to express gratitude to all of my teachers at Department of Theoretical Physics: Dr. Vu Quang Tuyen, Dr. Phan Hong Khiem, Dr. Nguyen Huu Nha, Dr. Vo Quoc Phong. Without their inspiring lectures, there might be not my passion today.

Send to all my friends, I really want to say ”thank you” words, thanks for becoming a part of my life, especially, my K16VLLT classmates, who have given me a ”little family” among the bustling city center.

The last, to my parents, my family, who are always by my side, concern, believe, favor me to the way of my dream becoming a theoretical physicist. All my gratitude is not enough to describe their invaluable love for me, which makes me stronger every single day, dare to cope with all difficulties.

Ho Chi Minh City, August 2020

LE Duc Truyen

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Introduction

In particle physics, quantum electrodynamics (QED) is the relativistic quantum field theory of electrodynamics. In essence, it describes how light and matter interact and is the first theory where a combination of quantum mechanics and special relativity is achieved. QED mathematically describes all phenomena involving electrically charged particles interacting by means of exchange of photons.

Recently, the new experiment MUonE has been proposed [1] to measure precisely the running of the fine-structure constant at space-like momenta via the electron-muon elastic scattering. In this experiment, a high-energy muon beam scatters on atomic electrons of low-Z target. The motivation is to resolve the current 3-sigma anomaly on the muon (g-2) measurement. For the MUonE measurement we have to calculate the cross section of electron-muon elastic scattering very precisely, at the level of 10 ppm relative accuracy [2]. At this level of accuracy, the electron mass has to be kept.

To achieve this, we first attempt to calculate the cross section of electron-muon elastic scattering in QED at next-to-leading order with the soft-photon emission. At the next-to-leading order (NLO) level, we have to solve the problems of UV and IR divergences. These topics will be discussed in this thesis.

In this report, we present the main contents into the following sections :

Section Quantum electrodynamics: Introduction of Quantum Electrodynamics and some concepts, formulas, which we will use in this thesis.

Section Leading order results: In this section, we work in the lowest order scattering and evaluate some numerical results. The t-channel divergence will be shown here.

Section Next-to-leading order calculation: In this section, we do calculation in next-to-leading order and present all analytical results and some cross-checked numerical results.

Section UV-divergence cancellation: We will discuss the UV divergences in this section and show how they are cancelled out analytically.

Section IR-divergence cancellation: Discussing the IR divergences and showing how they are cancelled out analytically.

Section Conclusion and Outlook: We present our conclusion and outlook.

Finally, in the appendices we show a second method to obtain the amplitude expressions using the Wick's theorem at NLO level. We also provide all necessary mathematics and one-loop integrals.

1

Quantum electrodynamics

1.1 An overview of QED

Under the electromagnetic gauge $U(1)$ symmetry, we obtain the invariant Lagrangian describing an interaction of a vector field (the photon) with the spinor fields :

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e\bar{\psi}\gamma^{\mu}\psi A_{\mu},\end{aligned} \tag{1.1.1}$$

with

$$\begin{cases} D_{\mu} = \partial_{\mu} + ieA_{\mu} \\ F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} \end{cases}. \tag{1.1.2}$$

The above Lagrangian can be split into three individual parts :

- The free Dirac term :
$$\mathcal{L}_{D} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi. \tag{1.1.3}$$
This term is a dynamical term describing free fermion particles, like electron, muon, tau, quarks etc.

- The free electromagnetic term :
$$\mathcal{L}_{E} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{1.1.4}$$
Where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is electromagnetic tensor in classical electrodynamics with A_{μ} is the photon field obeying Maxwell's equation.

- The interaction term :
$$\mathcal{L}_{int} = -e\bar{\psi}\gamma^{\mu}\psi A_{\mu}, \tag{1.1.5}$$
representing the fundamental interaction between matter particles and the photon.

1.2 Feynman rules

Feynman diagram is a pictorial representation of the mathematical expressions describing the behavior and interaction of subatomic particles. The method is named after American physicist Richard Feynman, who introduced the diagrams in 1948 [3]. Feynman rules allow us to graphically represent the terms in the Wick's expansion of the perturbative S-matrix to directly derive scattering amplitude. In the following sections, we will use the below Feynman rules to obtain the amplitudes expressions. We have also used the Wick's theorem to get the amplitudes directly and checked that both methods give the same results. We refer the reader to Appendix A for the Wick's theorem method.

Feynman Rules for QED [4]

1. For each vertex, include a factor : $-ie\gamma^{\mu}$
2. For each internal photon line, include a factor : $iD^{F}_{\mu\nu} = \frac{-ig_{\mu\nu}}{k^2+i\varepsilon}$
3. For each internal fermion line, write a factor : $iS_F = i\frac{\not{p}+m}{p^2-m^2+i\varepsilon}$
4. For each external line, write one of the following spinor factors, where p and k indicate basis states of correspongding 3-momenta
   - For each initial particle : $u^r(p)$
   - For each final particle : $\bar{u}^r(p)$
   - For each initial anti-particle : $\bar{v}^r(p)$
   - For each final anti-particle : $v^r(p)$
   - For each initial photon : $\varepsilon^r_{\mu}(k)$
   - For each final photon : $\varepsilon^r_{\mu}(k)$
5. The spinor factors (γ matrices, S_F fuctions, spinors) for each fermion line are ordered so that, reading from right to left, they appear in the same sequence as following the fermion line in the direction of its arrows through the vertex.
6. The four-momentum at each vertex are conserved
7. For each closed loop of internal fermions only (without photons inside the loop itself), take the trace (in spinor space) of the resulting matrix and multiply by a factor of (-1) originated from Fermi-Dirac distribution.
8. For each four-momentum q which is not fixed by four-momentum conservation, carry out the integration $\frac{1}{(2\pi)^4}\int d^4q$. One such integration for each closed loop (fermion/fermion or fermion/photon loop).

1.3 LSZ FORMULA

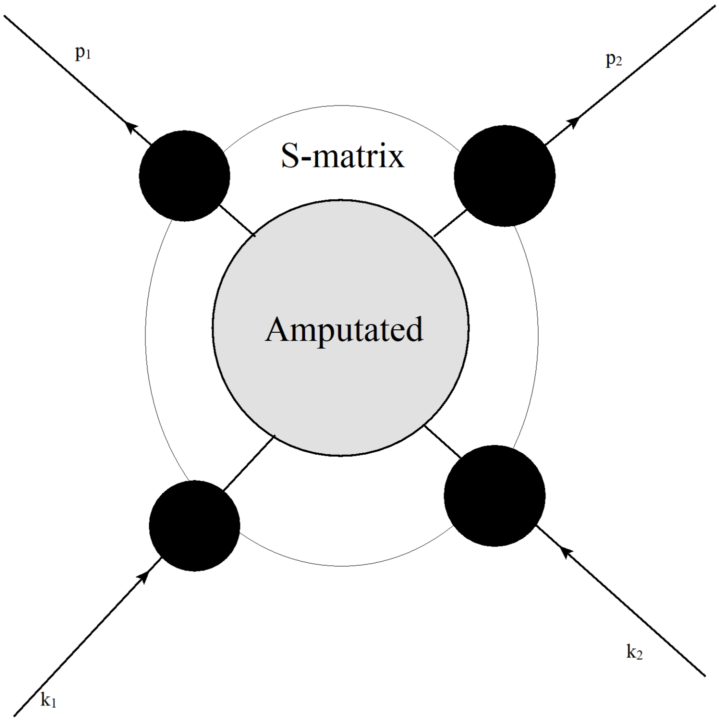


Figure 1: Four-point correlation function

In general, the scattering amplitude at any order in the perturbative expansion can be defined using the Lehmann-Symanzik-Zimmermann (LSZ) formula as [5] :

$$i\mathcal{M}_{total} = \prod_{i=p_1,p_2,k_1,k_2} \sqrt{\tilde{Z}_i} i\mathcal{M}_{amputated}. \tag{1.3.1}$$

where the LSZ factors $\sqrt{\tilde{Z}_i}$ are related to quantum corrections to the external legs. At leading order (LO) we have $\sqrt{\tilde{Z}_i} = 1$. At NLO we have :

$$\tilde{Z}_p = 1 - \frac{d\hat{\Sigma}^{ff}(\not{p})}{d\not{p}}|_{\not{p}=m_\mu}, \tag{1.3.2}$$

where $\hat{\Sigma}^{ff}(\not{p})$ is the renormalized one-loop self-energy correction of a fermion. A detailed calculation of this $\sqrt{\tilde{Z}_i}$ factors is provide in Section 3.2.

2

Leading order results

In this thesis, we are going to calculate the scattering process $e^-\mu^- \to e^-\mu^-$. First of all, let's start at the leading order.

The QED Lagrangian for $e^-\mu^- \to e^-\mu^-$ scattering process :

$$\begin{aligned}\mathcal{L} &= \bar{\psi}_e\left(i\gamma^\nu D_\nu - m_e\right)\psi_e + \bar{\psi}_\mu\left(i\gamma^\nu D_\nu - m_\mu\right)\psi_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= \bar{\psi}_e\left(i\gamma^\nu\partial_\nu - m_e\right)\psi_e + \bar{\psi}_\mu\left(i\gamma^\nu\partial_\nu - m_\mu\right)\psi_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e\bar{\psi}_e\gamma^\nu\psi_e A_\nu - e\bar{\psi}_\mu\gamma^\nu\psi_\mu A_\nu.\end{aligned} \tag{2.0.1}$$

From this Lagrangian Eq.(2.0.1), we use the Feynman rules to obtain the diagram in Fig (2) :

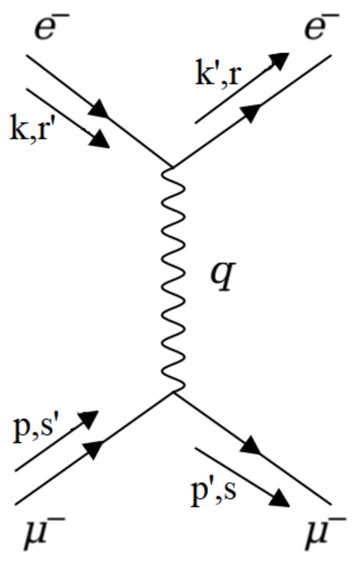


Figure 2: Feynman diagram for $e^-\mu^-$ elastic scattering at leading order

Basing on this diagram, we derive the scattering amplitude at LO :

$$i\mathcal{M}_{LO} = \bar{u}_e^r(k')(-ie\gamma^\alpha)u_e^{r'}(k)\left(\frac{-ig_{\alpha\beta}}{q^2}\right)\bar{u}_\mu^s(p')(-ie\gamma^\beta)u_\mu^{s'}(p) = \frac{ie^2}{q^2}\bar{u}_e^r(k')\gamma^\alpha u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\alpha u_\mu^{s'}(p), \tag{2.0.2}$$

with $q = p - p' \to q^2 = 2(m_\mu^2 - p.p')$. We note that the same result is obtained using Wick's theorem, see Eq. (A.2.15). Finally we get the averaged squared amplitude :

$$
\begin{aligned}
|\bar{\mathcal{M}}|^2_{LO} &= \frac{1}{4}\frac{e^4}{q^4} Tr\left[(\not{k}' + m_e)\gamma^\alpha(\not{k} + m_e)\gamma^\beta\right] Tr\left[(\not{p}' + m_\mu)\gamma_\alpha(\not{p} + m_\mu)\gamma_\beta)\right] \\
&= \frac{e^4}{4q^4}\left[Tr(\not{k}'\gamma^\alpha\not{k}\gamma^\beta) + m_e^2 Tr(\gamma^\alpha\gamma^\beta)\right]\left[Tr(\not{p}'\gamma_\alpha\not{p}\gamma_\beta) + m_\mu^2 Tr(\gamma_\alpha\gamma_\beta)\right] \\
&= \frac{e^4}{4q^4}\left[Tr(\not{k}'\gamma^\alpha\not{k}\gamma^\beta)Tr(\not{p}'\gamma_\alpha\not{p}\gamma_\beta) + m_\mu^2 Tr(\gamma_\alpha\gamma_\beta)Tr(\not{k}'\gamma^\alpha\not{k}\gamma^\beta) + m_e^2 Tr(\gamma^\alpha\gamma^\beta)Tr(\not{p}'\gamma_\alpha\not{p}\gamma_\beta) + 64m_e^2 m_\mu^2\right]
\end{aligned}
\tag{2.0.3}
$$

$$
\Rightarrow |\bar{\mathcal{M}}|^2_{LO} = \frac{8e^4}{q^4}\left[(p'.k'p.k + p'.kp.k') - m_\mu^2 k.k' - m_e^2 p.p' + 2m_e^2 m_\mu^2\right]. \tag{2.0.4}
$$

We use the Lorentz-invariant Mandelstam variables :

$$
s = (p+k)^2 \to s = m_\mu^2 + m_e^2 + 2p.k = m_e^2 + m_\mu^2 + 2p'.k' \tag{2.0.5}
$$

$$
u = (p-k')^2 \to u = m_\mu^2 + m_e^2 - 2p.k' = m_\mu^2 + m_e^2 - 2k.p' \tag{2.0.6}
$$

$$
t = (p-p')^2 \to t = 2(m_\mu^2 - p.p') = 2(m_e^2 - k'.k) = q^2, \tag{2.0.7}
$$

to rewrite the averaged squared matrix :

$$
|\bar{\mathcal{M}}|^2_{LO} = \frac{8e^4}{t^2}\left[\frac{s^2}{4} + \frac{u^2}{4} + (m_\mu^2 + m_e^2)t - \frac{\left(m_\mu^2 + m_e^2\right)^2}{2}\right]. \tag{2.0.8}
$$

We obtain the differential cross section in Center of Mass frame (CMF) (see [5] for the definition of the cross section) :

$$
\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{LO} = \frac{\overline{|\mathcal{M}|}^2_{LO}}{64\pi^2 s} = \frac{2\alpha^2}{t^2 s}\left[\frac{s^2}{4} + \frac{u^2}{4} + (m_\mu^2 + m_e^2)t - \frac{\left(m_\mu^2 + m_e^2\right)^2}{2}\right]. \tag{2.0.9}
$$

The CMF is shown in Fig. (3).

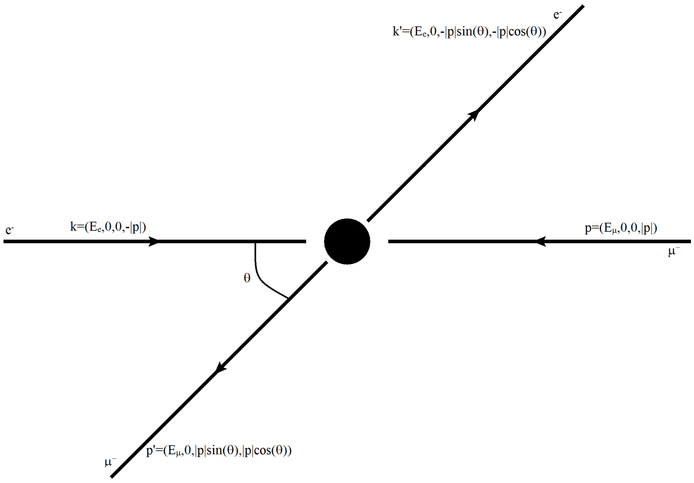


Figure 3: $e^- \mu^- \to e^- \mu^-$ scattering in Center of Mass Frame

$$t = -2|\vec{p}|^2 \left(1 - \cos\theta\right), \tag{2.0.10}$$

$$u = 2\left(m_\mu^2 + m_e^2\right) - s + 2|\vec{p}|^2 \left(1 - \cos\theta\right) \tag{2.0.11}$$

$$|\vec{p}|^2 = \frac{\left[m_e^2 + m_\mu^2 - \frac{1}{2}\left(\frac{\left(m_e^2 - m_\mu^2\right)^2}{s} + s\right)\right]}{-2} \tag{2.0.12}$$

with $|\vec{p}|$ is the momentum in CMF. We also have :

$$\Rightarrow \frac{d\sigma}{d\cos\theta} = 4\pi\frac{\alpha^2}{t^2 s}\left[\frac{s^2}{4} + \frac{u^2}{4} + (m_\mu^2 + m_e^2)t - \frac{\left(m_\mu^2 + m_e^2\right)^2}{2}\right], \tag{2.0.13}$$

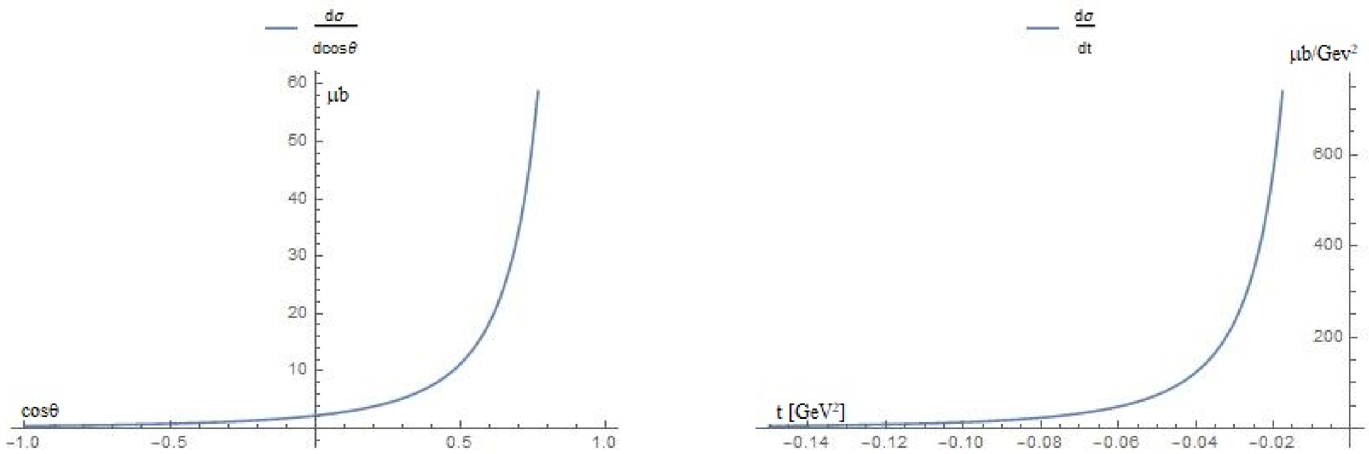
or :

$$\frac{d\sigma}{dt} = 4\pi\frac{1}{2|\vec{p}|^2}\frac{\alpha^2}{t^2 s}\left[\frac{s^2}{4} + \frac{u^2}{4} + (m_\mu^2 + m_e^2)t - \frac{\left(m_\mu^2 + m_e^2\right)^2}{2}\right]. \tag{2.0.14}$$

For the numerical calculation, the input parameters are set to, the same as in [6] :

$$\begin{aligned} \alpha \equiv \alpha(0) = 1/137.03599907430637, \quad & m_e = 0.510998928\text{MeV}, \\ m_\mu = 105.6583715\text{MeV}, \quad & \sqrt{s} = 0.405541158\text{GeV}. \end{aligned} \tag{2.0.15}$$

The $\sqrt{s}$ value above is the colliding energy in the CMF of the MUonE experiment. We get the plots of both differential cross sections in Fig (4) .



(a) $\frac{d\sigma}{d\cos\theta}(\cos\theta)$ (b) $\frac{d\sigma}{dt}(t)$

Figure 4: Differential cross section distributions in CMF

We also provide a table of the differential cross section at different values of $\cos\theta$ and t in Tab. (1-2) :

$\mu^- e^- \to \mu^- e^-$		
$\cos\theta$	$\frac{d\sigma}{d\cos\theta}(\cos\theta)$ $[\mu b]$	
	$m_e \neq 0$	$m_e = 0$
-1	0.3960810768531	0.3960806244802
-0.5	0.8012992639148	0.8012968782068
0	2.2195626924442	2.2195537662498
0.5	11.336969413281	11.336919475097
0.9602	2256.8970604919	2256.8871114676

Table 1: Differential cross section at different values of $\cos\theta$ (in the CMF)

$\mu^- e^- \to \mu^- e^-$		
$t[GeV^2]$	$\frac{d\sigma}{dt}(t)$ $[\mu b/GeV^2]$	
	$m_e \neq 0$	$m_e = 0$
-0.14	5.8129190296129	5.8129416310709
-0.11	10.500804086884	10.500836463598
-8.6×10^{-2}	19.570795327441	19.570838963674
-4.4×10^{-2}	98.689601365371	98.689674694482
-2.12×10^{-2}	671.308540450954	671.307895469414

Table 2: Differential cross section at different values of t

For $m_e = 0$ case, we get rid of all m_e in Eq. (2.0.13-2.0.14) and :

$$t = -2|\vec{p}|^2 (1 - \cos\theta) , \tag{2.0.16}$$

$$u = 2m_\mu^2 - s + 2|\vec{p}|^2 (1 - \cos\theta) \tag{2.0.17}$$

$$|\vec{p}|^2 = \frac{\left[m_\mu^2 - \frac{1}{2}\left(\frac{m_\mu^4}{s} + s\right)\right]}{-2}. \tag{2.0.18}$$

The differential cross section at $t = 0$ or $\cos\theta = 1$, which means our scattering event occurs at zero angles(or equivalent to has no photon exchange at tree level) is divergent. The essence of this singularity is due to the long-range of the electromagnetic interaction.

For the MuonE experiment, we have the scattering of high-energy muons on electrons at rest. The energy of the incoming muon, we require $E_\mu^{\text{Beam}} = E_\mu = 150$ GeV corresponding to $\sqrt{s} \simeq 0.405541$ GeV. In Lab frame (where the inital electron is at rest), we request : $\theta_e, \theta_\mu < 100$ mrad and $E_e > 0.2$ GeV for the final particles [6]. The angular cuts model the typical acceptance conditions of the experiment and the electron energy threshold is imposed to guarantee the presence of two charged tracks in the detector [6].
Using the Lorentz boost, we transform the momenta in CMF to Lab frame. The Lorentz boost matrix :

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \frac{E_\mu + m_e}{\sqrt{s}} & 0 & 0 & \sqrt{\frac{E_\mu^2 - m_\mu^2}{s}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{\frac{E_\mu^2 - m_\mu^2}{s}} & 0 & 0 & \frac{E_\mu + m_e}{\sqrt{s}} \end{pmatrix}, \tag{2.0.19}$$

We obtain the analytical formula for the differential cross section with constraints in the Lab frame :

$$\begin{aligned} \frac{d\sigma}{d\cos\theta} =& \Theta\left[\cos\theta_e - \cos 0.1\right] \Theta\left[\cos\theta_\mu - \cos 0.1\right] \Theta\left[E_e - 0.2\right] \\ & \times 4\pi \frac{\alpha^2}{t^2 s}\left[\frac{s^2}{4} + \frac{u^2}{4} + (m_\mu^2 + m_e^2)t - \frac{\left(m_\mu^2 + m_e^2\right)^2}{2}\right], \end{aligned} \tag{2.0.20}$$

where $\cos\theta_e$, $\cos\theta_\mu$ and E_e are functions of $\cos\theta$. Note that $\cos\theta$ is defined in the CMF.

$\mu^- e^- \to \mu^- e^-$		
Cross section	Analytical result using Mathematica	Monte-Carlo simulation [6]
$\sigma_{LO}^{QED}(m_e \neq 0)$	1265.0603541	1265.060312(7)
$\sigma_{LO}^{QED}(m_e = 0)$	1264.9381128	(-)

Table 3: Leading order cross section for MuonE experiment.

Using the above MuonE experiment setup, we can now compare our analytical result with the numerical simulation of [6]. This is shown in Tab. (3). Although the agreement is better than the 10ppm accuracy level, we notice a difference at the level of 6 standard deviations. In Tab. (3), we see that the result for the case of $m_e = 0$ agrees with the exact result at the level of 4 digits, not enough for the accuracy of the MuonE experiment.

3

Next-to-leading order calculation

To increase our precision, we have to include higher order corrections. In practice, this approach is very important, because the tree-level results are often not good enough to compare to experiment measurements. In this section, we are going to use the following notations for N-point functions of loop integral results :

Notations [7]

$$\frac{16\pi^2}{i}\int\frac{d^4q}{(2\pi)^4}\ldots \rightarrow \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\ldots = \langle|\ldots|\rangle_q \tag{3.0.1}$$

$$A_0(m) = \langle|\left(q^2 - m^2 + i\varepsilon\right)^{-1}|\rangle_q \tag{3.0.2}$$

$$B_0(p, m_0, m_1) = \langle|\left[(q^2 - m_0^2 + i\varepsilon)\left((q+p)^2 - m_1^2 + i\varepsilon\right)\right]^{-1}|\rangle_q \tag{3.0.3}$$

$$B_\mu(p, m_0, m_1) = \langle|\, q_\mu\left[(q^2 - m_0^2 + i\varepsilon)\left((q+p)^2 - m_1^2 + i\varepsilon\right)\right]^{-1}|\rangle_q \tag{3.0.4}$$

$$B_{\mu\nu}(p, m_0, m_1) = \langle|\, q_\mu q_\nu\left[(q^2 - m_0^2 + i\varepsilon)\left((q+p)^2 - m_1^2 + i\varepsilon\right)\right]^{-1}|\rangle_q \tag{3.0.5}$$

$$C_0(p, p', m_0, m_1, m_2) = \langle|\left[(q^2 - m_0^2 + i\varepsilon)\left((q+p)^2 - m_1^2 + i\varepsilon\right)\left((q+p')^2 - m_2^2 + i\varepsilon\right)\right]^{-1}|\rangle_q \tag{3.0.6}$$

$$C_\mu(p, p', m_0, m_1, m_2) = \langle|\, q_\mu\left[(q^2 - m_0^2 + i\varepsilon)\left((q+p)^2 - m_1^2 + i\varepsilon\right)\left((q+p')^2 - m_2^2 + i\varepsilon\right)\right]^{-1}|\rangle_q \tag{3.0.7}$$

$$C_{\mu\nu}(p, p', m_0, m_1, m_2) = \langle|\, q_\mu q_\nu\left[(q^2 - m_0^2 + i\varepsilon)\left((q+p)^2 - m_1^2 + i\varepsilon\right)\left((q+p')^2 - m_2^2 + i\varepsilon\right)\right]^{-1}|\rangle_q \tag{3.0.8}$$

$$\begin{aligned} D_0(p, p_1, p_2, m_0, m_1, m_2, m_3) = \langle|&\left[(q^2 - m_0^2 + i\varepsilon)\left((q+p)^2 - m_1^2 + i\varepsilon\right)\left((q+p_1)^2 - m_2^2 + i\varepsilon\right)\right. \\ &\left.\left((q+p_2)^2 - m_3^2 + i\varepsilon\right)\right]^{-1}|\rangle_q \end{aligned} \tag{3.0.9}$$

$$\begin{aligned} D_\mu(p, p_1, p_2, m_0, m_1, m_2, m_3) = \langle|&q_\mu\left[(q^2 - m_0^2 + i\varepsilon)\left((q+p)^2 - m_1^2 + i\varepsilon\right)\left((q+p_1)^2 - m_2^2 + i\varepsilon\right)\right. \\ &\left.\left((q+p_2)^2 - m_3^2 + i\varepsilon\right)\right]^{-1}|\rangle_q \end{aligned} \tag{3.0.10}$$

$$\begin{aligned} D_{\mu\nu}(p, p_1, p_2, m_0, m_1, m_2, m_3) = \langle|&q_\mu q_\nu\left[(q^2 - m_0^2 + i\varepsilon)\left((q+p)^2 - m_1^2 + i\varepsilon\right)\left((q+p_1)^2 - m_2^2 + i\varepsilon\right)\right. \\ &\left.\left((q+p_2)^2 - m_3^2 + i\varepsilon\right)\right]^{-1}|\rangle_q\,. \end{aligned} \tag{3.0.11}$$

3.1 Renormalization procedure

In this thesis, we would like to do the calculation at next-to-leading order. This requires us to calculate the one-loop amplitudes. At one-loop level, we will however meet some interesting problems of one-loop integrals with UV and IR-divergence. To resolve the UV-singularities, we have to use an extra procedure, the Renormalization method. The renormalization is actually a collection of techniques in QFT, it is going to introduce some new terms which will cancel the above UV-singularities in the loop integrals. To do that, first of all, we must renormalize the QED Lagrangian.

1. The Renormalized Lagrangian
   Replacing the initial quantities or now called the bare quantities $X_0 \to Z_X X$ by renormalized quantities X with renormalization factor Z_X (subscript 0 denotes the bare quantity), this will introduce new terms (called counterterm) in the Lagrangian. The counterterms are determined through renormalization conditions. These can be chosen arbitrarily, however, since the renormalized parameters (m_e, m_μ, e in this calculation) are used as the input parameter and are defined by the chosen renormalization conditions, we have to choose a sets of condions corresponding to the values of m_e, m_μ, and α given in Eq. (2.0.15). This is called the on-shell renormalization sheme.

$$\mathcal{L}_0 = \bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e_0\bar{\psi}_0 A\!\!\!/ \psi_0 \to \mathcal{L}_R = Z_\psi\bar{\psi}(i\partial\!\!\!/ - Z_m.m)\psi - \frac{1}{4}Z_A F^{\mu\nu}F_{\mu\nu} - Z_e Z_\psi\sqrt{Z_A}e\bar{\psi}A\!\!\!/\psi, \tag{3.1.1}$$

with :

$$\begin{cases} \psi_0 = \sqrt{Z_\psi}\psi = \sqrt{1+\delta_\psi}\psi \\ A_0^\mu = \sqrt{Z_A}A^\mu = \sqrt{1+\delta_A}A^\mu \\ m_0 = Z_m.m = m + \delta_m \\ e_0 = Z_e.e = e + \delta_e \end{cases}, \tag{3.1.2}$$

where we have expanded perturbatively at one-loop order $Z_i = 1 + \delta_i(\alpha)$. The renormalized Lagrangian reads :

$$\begin{aligned} \Rightarrow \mathcal{L}_R &= \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e\bar{\psi}A\!\!\!/\psi - \bar{\psi}\delta_m\psi + \delta_\psi\bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}\delta_A F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}A\!\!\!/\psi(\delta_e + \delta_\psi + \frac{1}{2}\delta_A) \\ &= \mathcal{L}_R^0 + \mathcal{L}_{counterterm}. \end{aligned} \tag{3.1.3}$$

We have splitted the renormalized Lagrangian $\mathcal{L}_R$ into the basic Lagrange $\mathcal{L}_R^0$ and the counterterm Lagrangian $\mathcal{L}_{counter}$. The basic $\mathcal{L}_R^0$ has the same form as $\mathcal{L}_0$ but depends on renormalized parameters and fields instead. The counterterm Lagrange will give new diagrams with some new Feynman rules, which will contribute to the total amplitude to cancel the UV-singularities.

2. Counterterm Feynman rules :

From Eq. (3.1.3) we derive the new Feynman rules in Fig. (5-6-7)

- Fermion-fermion vertex :

$$i\delta_\psi(\not p - m) - i\delta_m, \tag{3.1.4}$$

Figure 5: Fermion-fermion counterterm vertex

- Fermion-fermion-photon vertex :

$$-ie\gamma^\mu(\delta_e + \delta_\psi + \frac{1}{2}\delta_A), \tag{3.1.5}$$

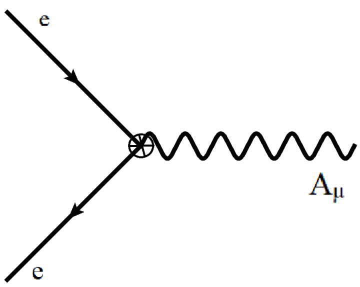


Figure 6: Fermion-fermion-photon counterterm vertex

- Photon-photon vertex :

$$-i\delta_A q^2 g^{\mu\nu}. \tag{3.1.6}$$

Figure 7: Photon-photon counterterm vertex

3. Renormalization conditions [8]
In order to define the renormalization conditions, we first define the following basic fuctions :

$$\begin{aligned} \hat{\Gamma}^{ff}(p) &= \Gamma^{ff}(p) + \delta\Gamma^{ff}(p) = i(\not p - m) + i\Sigma^{ff}(p) + i\delta_\psi(\not p - m) - i\delta_m \\ \hat{\Gamma}^{Aff}_\mu(p,p') &= \Gamma^{Aff}_\mu(p,p') + \delta\Gamma^{Aff}_\mu(p,p') = -ie\gamma_\mu - ie\Lambda_\mu(p,p') - ie\gamma_\mu(\delta_e + \delta_\psi + \frac{1}{2}\delta_A) \\ \hat{\Gamma}^{AA}_{\mu\nu}(k) &= \Gamma^{AA}_{\mu\nu}(k) + \delta^{AA}_{\mu\nu} = -ik^2 g_{\mu\nu} + -i\Sigma^{AA}_{\mu\nu}(k) - ik^2 g_{\mu\nu}\delta_A, \end{aligned} \tag{3.1.7}$$

where the normalized functions are denoted with a hat, and :

$$\Sigma^{ff}(p) = \Sigma^{ff}(\not p) = ie^2\mu^{4-D}\int \frac{d^D q}{(2\pi)^D}\frac{\gamma^\nu(\not p - \not q + m)\gamma_\nu}{[(p-q)^2 - m^2]\, q^2}, \tag{3.1.8}$$

$$\Lambda_\mu(p,p') = -ie^2\mu^{4-D}\int \frac{d^Dq}{(2\pi)^D}\frac{\gamma^\alpha(\not{p}'+\not{q}+m)\gamma_\mu(\not{p}+\not{q}+m)\gamma_\alpha}{q^2\left[(q+p')^2-m^2\right]\left[(q+p)^2-m^2\right]}, \tag{3.1.9}$$

$$\Sigma^{AA}_{\mu\nu}(k) = -ie^2\mu^{4-D}\int \frac{d^Dq}{(2\pi)^D}\frac{Tr\left[\gamma_\mu(\not{q}+\not{k}+m)\gamma_\nu(\not{q}+m)\right]}{(q^2-m)\left[(q+k)^2-m^2\right]}. \tag{3.1.10}$$

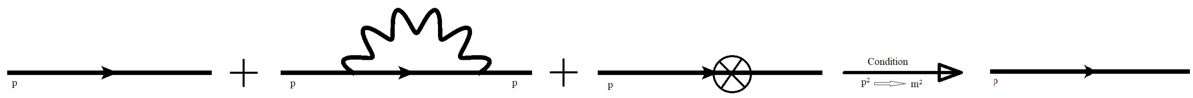
Note that, we have used dimensional regularization [7] to replace $\frac{d^4q}{(2\pi)^4} \to \mu^{4-D}\frac{d^Dq}{(2\pi)^D}$ for the loop integrals. The Lorentz indices of the gamma matrices associated to the integrated momentum q^μ have to be promoted to D-dimension as well. We are now in the position to impose the following conditions on the functions in Eq. (3.1.7). These conditions require that those renormalized fuctions have a tree-level form in the on-shell limit ($p^2 = m^2$). This is the on-shell renormalization sheme :

- Condition 1 - Dirac equation :

$$\tilde{Re}\hat{\Gamma}^{ff}(p)u(p)|_{p^2=m^2} = 0 \tag{3.1.11}$$

$$\Rightarrow \delta_m = \tilde{Re}\Sigma^{ff}(m) = \frac{e^2}{8\pi^2}\left[mB_1(m^2,0,m) - mB_0(m^2,0,m) + \frac{m}{2}\right]. \tag{3.1.12}$$

- Condition 2 :

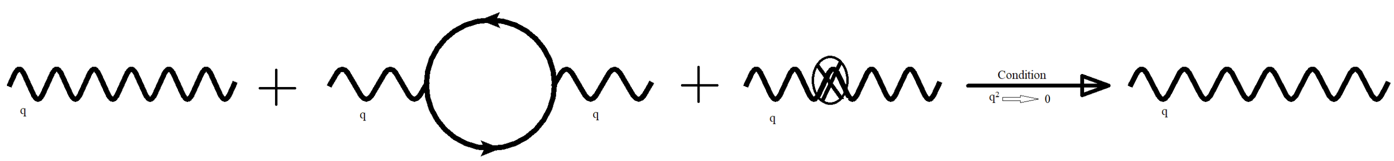


$$\lim_{p^2\to m^2}\frac{\not{p}+m}{p^2-m^2}\tilde{Re}\hat{\Gamma}^{ff}(p)u(p) = iu(p) \tag{3.1.13}$$

$$\begin{aligned}
\Rightarrow \delta_{\psi_j} &= -2m_j\tilde{Re}\frac{\partial\Sigma^{ff}(p)}{\partial p^2}|_{\not{p}=m_j} = -2m_j\frac{\partial\Sigma^{ff}(p)}{\partial p^2}|_{\not{p}=m_j}\\
&= \frac{m_je^2}{4\pi^2}\left[-\frac{B_1(m_j^2,0,m_j)}{2m_j} - \frac{B_0(m_j^2,0,m_j)}{2m_j} - m_j\frac{\partial B_1(p^2,0,m_j)}{\partial p^2}\bigg|_{p^2=m_j^2}\right.\\
&\quad\left. + m_j\frac{\partial B_0(p^2,0,m_j)}{\partial p^2}\bigg|_{p^2=m_j^2} + \frac{1}{4m_j}\right].
\end{aligned} \tag{3.1.14}$$

In the two conditions, the on-shell photon appears, we thus need a photon's counterterm factor δ_A and also the photon's counterterm diagram.

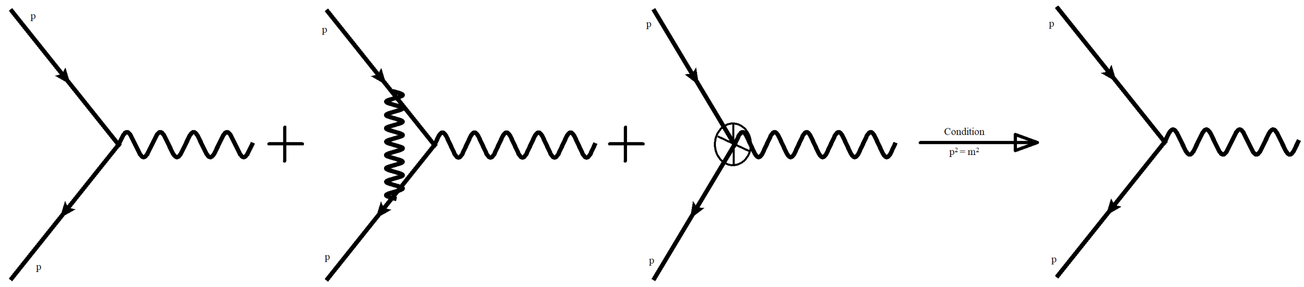
- Condition 3:



$$\lim_{q^2\to 0}\frac{1}{q^2}Re\hat{\Gamma}^{AA}_{\mu\nu}(q)\varepsilon^\nu(q) = -i\varepsilon_\mu(q) \tag{3.1.15}$$

$$\Rightarrow \delta_A = \frac{-e^2}{4\pi^2} \sum_{j=e,\mu} Re\left[-\frac{1}{9} + \frac{2}{3} m_j^2 B_0'(q^2, m_j, m_j)|_{q^2=0} + \frac{B_1(0, m_j, m_j)}{3} + \frac{B_0(0, m_j, m_j)}{2}\right]. \tag{3.1.16}$$

- Condition 4 :



$$\bar{u}(p)\hat{\Gamma}_\mu^{Aff}(p,p)u(p)\Big|_{p^2=m^2} = -ie\bar{u}(p)\gamma_\mu u(p) \tag{3.1.17}$$

$$\Rightarrow \delta_e = -\frac{1}{2}\delta_A. \tag{3.1.18}$$

We can see that, for example, $e_0 = e + \delta_e$ despite the bare e_0 has a non-intuitive value, but essentially, it has to be a constant, and because $\delta_e \equiv \delta_e(q^2)$, the measurable e is thus a function of q^2. Furthermore, base on Eqs. (3.1.16-3.1.18), we can see the reason causing the energy-scale dependence of the physics parameter $e(q^2)$ (or $\alpha(q^2)$) is the Vacuum Polarization process. The renormalization conditions show us an important consequence that the physical parameters are not constant, these depend on energy scales. In the on-shell sheme we have $\delta_e = \delta_e(q^2 = 0)$, therefore we have to use $\alpha = \alpha(0)$ as an input parameter.

3.2 Virtual-correction amplitudes

The higher order correction amplitudes are actually the higher order terms in Wick's expansion of S-matrix. In this section we use the Feynman rules given in Section 1.3 and Section 3.1 to write down the amplitudes for the virtual corrections. We have used the Wick's theorem to cross check the results. This method is presented in Appendix A.3.

The virtual amplitude reads :

$$i\mathcal{M}_{Virt} = i\mathcal{M}_{vp} + i\mathcal{M}_{vc} + i\mathcal{M}_{bd} + i\mathcal{M}_{ct}, \tag{3.2.1}$$

where we have classified the corrections into Vacuum polarization (vp), Vertex correction (vc), Box diagrams (bd), and Counterterm diagrams (ct).

1. **Vacuum polarization**

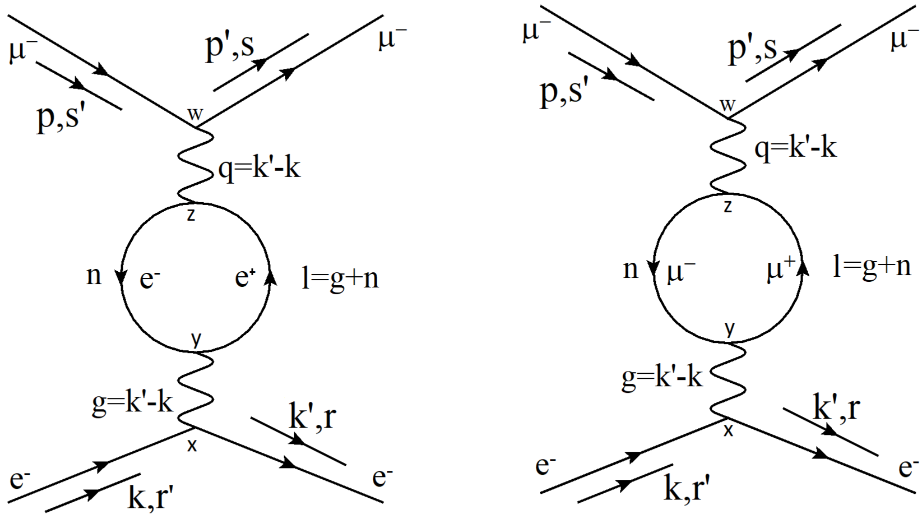


Figure 8: The Vacuum polarization diagrams

First diagram's amplitude of Fig. (8):

$$\begin{aligned} i\mathcal{M}_{1,vp} &= \bar{u}_e^r(k')(-ie\gamma^\rho)u_e^{r'}(k)\frac{-ig_{\rho\nu}}{g^2}(-e^2)\int\frac{d^4n}{(2\pi)^4}\frac{tr\left[\gamma^\nu(\not{l}+m_e)\gamma^\alpha(\not{n}+m_e)\right]}{(n^2-m_e^2)(l^2-m_e^2)}\frac{-ig_{\alpha\beta}}{q^2}\bar{u}_\mu^s(p')(-ie\gamma^\beta)u_\mu^{s'}(p) \\ &= -e^4\int\frac{d^4n}{(2\pi)^4}\bar{u}_e^r(k')\gamma^\rho u_e^{r'}(k)\frac{-ig_{\rho\nu}}{g^2}tr\left[\gamma^\nu\frac{i}{\not{l}-m_e}\gamma^\alpha\frac{i}{\not{n}-m_e}\right]\frac{-ig_{\alpha\beta}}{q^2}\bar{u}_\mu^s(p')\gamma^\beta u_\mu^{s'}(p), \end{aligned} \tag{3.2.2}$$

with :

$$\begin{cases} g = k'-k \\ l = k'-k+n \\ q = k'-k \end{cases} . \tag{3.2.3}$$

Similiarly for the second :

$$i\mathcal{M}_{2,vp} = -e^4\int\frac{d^4n}{(2\pi)^4}\bar{u}_e^r(k')\gamma^\rho u_e^{r'}(k)\frac{-ig_{\rho\nu}}{g^2}tr\left[\gamma^\nu\frac{i}{\not{l}-m_\mu}\gamma^\alpha\frac{i}{\not{n}-m_\mu}\right]\frac{-ig_{\alpha\beta}}{q^2}\bar{u}_\mu^s(p')\gamma^\beta u_\mu^{s'}(p), \tag{3.2.4}$$

with :

$$\begin{cases} g = k'-k \\ l = k'-k+n \\ q = k'-k \end{cases} . \tag{3.2.5}$$

The total Vacuum polarization amplitude reads :

$$i\mathcal{M}_{vp} = i\mathcal{M}_{1,vp} + i\mathcal{M}_{2,vp}. \tag{3.2.6}$$

2. **Vertex correction**

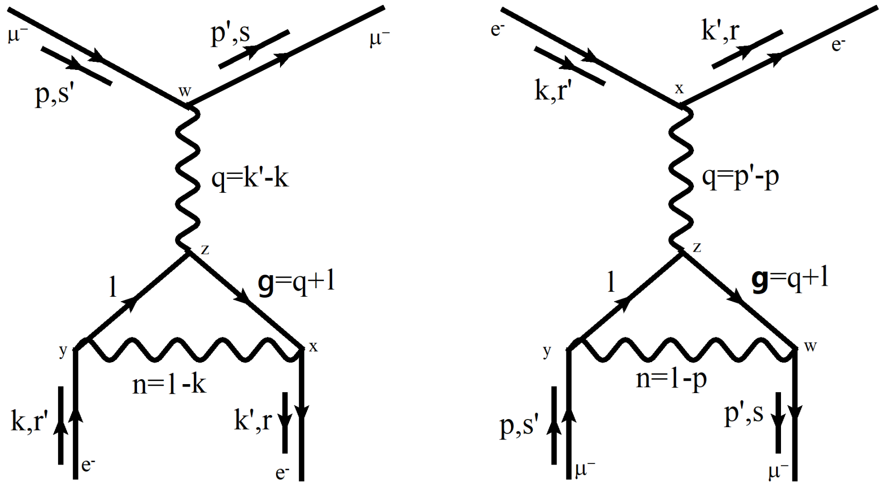


Figure 9: The Vertex correction diagrams

First diagram's amplitude of Fig. (9)

$$\begin{aligned} i\mathcal{M}_{1,vc} &= \bar{u}_e^r(k')(-e^3)\int \frac{d^4 l}{(2\pi)^4} g_{\rho\nu} \frac{\gamma^\nu(\not{g}+m_e)\gamma^\alpha(\not{l}+m_e)\gamma^\rho}{n^2(g^2-m_e)(l^2-m_e)} u_e^{r'}(k)\frac{-ig_{\alpha\beta}}{q^2}.\bar{u}_\mu^s(p')(-ie\gamma^\beta)u_\mu^{s'}(p) \\ &= e^4\int \frac{d^4 l}{(2\pi)^4}\frac{-g_{\rho\nu}g_{\alpha\beta}}{n^2q^2}\bar{u}_e^r(k')\gamma^\nu\frac{i}{\not{g}-m_e}\gamma^\alpha\frac{i}{\not{l}-m_e}\gamma^\rho u_e^{r'}(k).\bar{u}_\mu^s(p')\gamma^\beta u_\mu^{s'}(p), \end{aligned} \tag{3.2.7}$$

with :

$$\begin{cases} n = l-k \\ g = k'-k+l \\ q = k'-k \end{cases}. \tag{3.2.8}$$

Similar for the second :

$$i\mathcal{M}_{2,vc} = e^4\int \frac{d^4 l}{(2\pi)^4}\frac{-g_{\beta\nu}g_{\alpha\rho}}{n^2q^2}\bar{u}_\mu^s(p')\gamma^\nu\frac{i}{\not{g}-m_\mu}\gamma^\alpha\frac{i}{\not{l}-m_\mu}\gamma^\beta u_\mu^{s'}(p).\bar{u}_e^r(k')\gamma^\rho u_e^{r'}(k), \tag{3.2.9}$$

with :

$$\begin{cases} n = l-p \\ g = p'-p+l \\ q = p'-p \end{cases}. \tag{3.2.10}$$

The total Vertex correction amplitude is :

$$i\mathcal{M}_{vc} = i\mathcal{M}_{1,vc} + i\mathcal{M}_{2,vc}. \tag{3.2.11}$$

3. **Box diagrams**

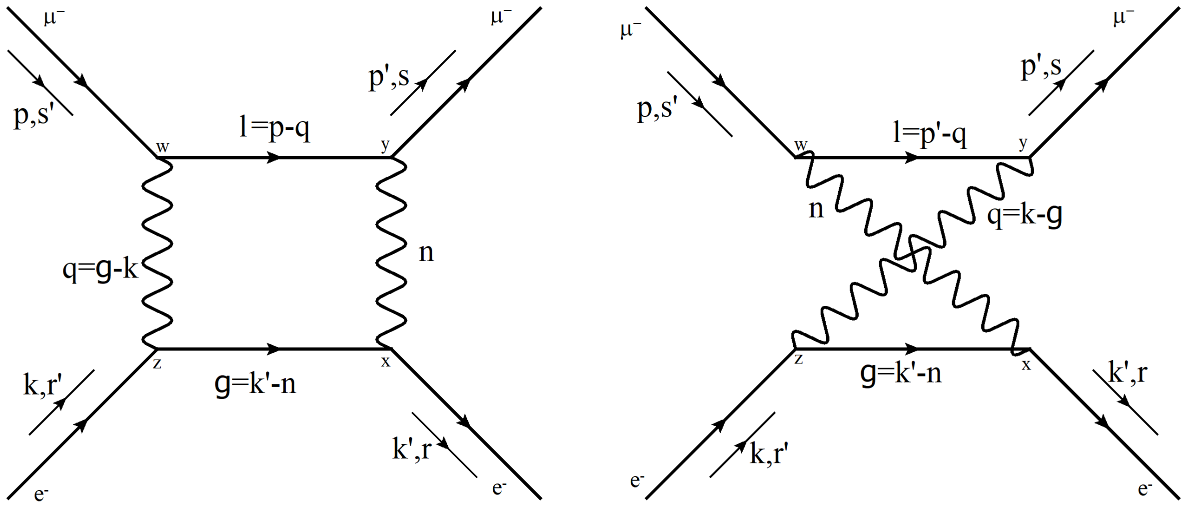


Figure 10: The Box diagrams

First diagram's amplitude of Fig. (10):

$$\begin{aligned} i\mathcal{M}_{1,bd} &= \int \frac{d^4n}{(2\pi)^4}\bar{u}_e^r(k')(-ie\gamma^\rho)\frac{i(\not{g}+m_e)}{g^2-m_e^2}(-ie\gamma^\alpha)u_e^{r'}(k)\frac{-ig_{\rho\nu}}{q^2}\frac{-ig_{\alpha\beta}}{n^2}\bar{u}_\mu^s(p')(-ie\gamma^\nu)\frac{i(\not{l}+m_\mu)}{l^2-m_\mu^2}(-ie\gamma^\beta)u_\mu^{s'}(p) \\ &= e^4\int \frac{d^4n}{(2\pi)^4}\frac{-g_{\alpha\beta}g_{\rho\nu}}{q^2n^2}\bar{u}_e^r(k')\gamma^\rho\frac{i(\not{g}+m_e)}{g^2-m_e^2}\gamma^\alpha u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma^\nu\frac{i(\not{l}+m_\mu)}{l^2-m_\mu^2}\gamma^\beta u_\mu^{s'}(p), \end{aligned} \tag{3.2.12}$$

with :

$$\begin{cases} g = k' - n \\ q = k' - k - n = g - k \\ l = k - k' + n + p = p - q \end{cases} . \tag{3.2.13}$$

Similar for the second :

$$i\mathcal{M}_{2,bd} = e^4\int \frac{d^4n}{(2\pi)^4}\frac{-g_{\alpha\nu}g_{\rho\beta}}{q^2n^2}\bar{u}_e^r(k')\gamma^\rho\frac{i(\not{g}+m_e)}{g^2-m_e^2}\gamma^\alpha u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma^\nu\frac{i(\not{l}+m_\mu)}{l^2-m_\mu^2}\gamma^\beta u_\mu^{s'}(p), \tag{3.2.14}$$

with :

$$\begin{cases} g = k' - n \\ q = k - k' + n = k - g \\ l = p' + k' - k - n = p' - q \end{cases} . \tag{3.2.15}$$

The total Box diagram amplitude reads :

$$i\mathcal{M}_{bd} = i\mathcal{M}_{1,bd} + i\mathcal{M}_{2,bd}. \tag{3.2.16}$$

4. **Counterterm diagrams**

The counterterm diagrams are a consequence of renormalization, which are used to cancel the UV-divergent parts of the other diagrams. From renormalized Lagrangian Eq. (3.1.3), we can obtain the following amplitudes :

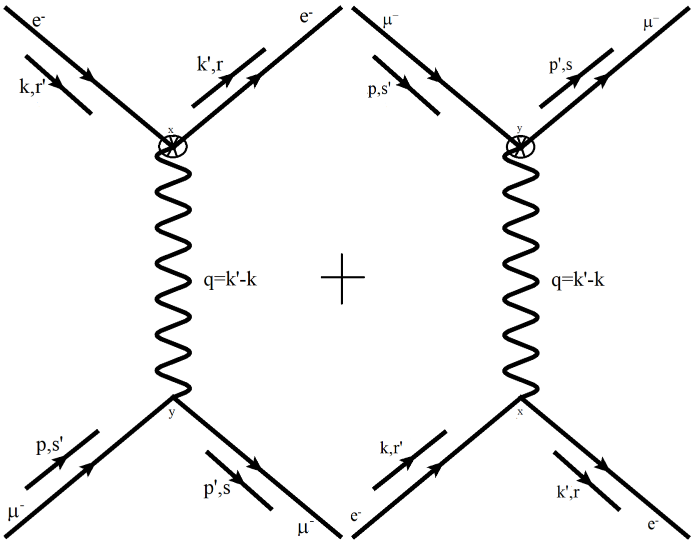


Figure 11: The Vertex counterterm tree diagrams

$$iM_{vct} = \frac{ie^2}{q^2}\bar{u}^r_e(k')\gamma^\nu(\delta_e+\delta_{\psi_e}+\frac{1}{2}\delta_A)u^{r'}_e(k)\bar{u}^s_\mu(p')\gamma_\nu u^{s'}_\mu(p)+\frac{ie^2}{q^2}\bar{u}^r_e(k')\gamma^\nu(\delta_e+\delta_{\psi_\mu}+\frac{1}{2}\delta_A)u^{r'}_e(k)\bar{u}^s_\mu(p')\gamma_\nu u^{s'}_\mu(p), \tag{3.2.17}$$

and :

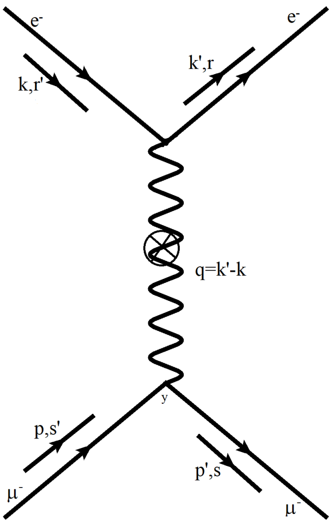


Figure 12: The Photon counterterm tree diagrams

$$iM_{pct} = \frac{-ie^2\delta_A}{q^2}\bar{u}^r_e(k')\gamma^\nu u^{r'}_e(k)\bar{u}^s_\mu(p')\gamma_\nu u^{s'}_\mu(p). \tag{3.2.18}$$

The total Counterterm diagram amplitude reads :

$$i\mathcal{M}_{ct} = i\mathcal{M}_{vct} + i\mathcal{M}_{pct}. \tag{3.2.19}$$

We remark that the δ_A is cancelled out in the sum. This is because there is no external photon in our process.

5. **The renormalized LSZ factors :** We are using LSZ reduction to compute amplitude, so that the external-leg-correction diagrams will be excluded, instead, the LSZ factors must be added. Because the LSZ factors also consist of UV divergent quantities, these factors are renormalized, consequently, these all are equal to **unity** after renormalization, in one-loop approximation, the renormalized LSZ factors [5] read :

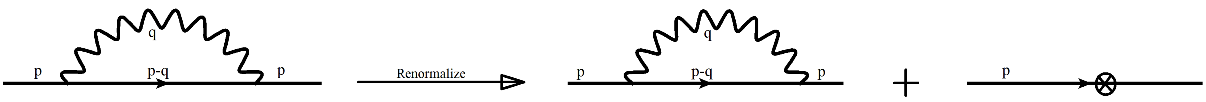


Figure 13: Renormalized wave function correction diagram

$$\hat{\Sigma}^{ff}(p) = \hat{\Sigma}^{ff}(\not{p}) = \Sigma^{ff}(\not{p}) + \delta\Sigma = \Sigma^{ff}(\not{p}) + \delta_\psi(\not{p} - m) - \delta_m \tag{3.2.20}$$

$$\begin{aligned} \Rightarrow \tilde{Z}_p &= 1 - \frac{d\hat{\Sigma}^{ff}(\not{p})}{d\not{p}}\Big|_{\not{p}=m} = 1 - \frac{d\Sigma^{ff}(\not{p})}{d\not{p}}\Big|_{\not{p}=m} - \delta_\psi = 1 - \frac{d\Sigma^{ff}(\not{p})}{d\not{p}}\Big|_{\not{p}=m} + 2m\frac{\partial\Sigma^{ff}(p)}{\partial p^2}\Big|_{\not{p}=m} \\ &= 1 - \frac{d\Sigma^{ff}(d\not{p})}{d\not{p}}\Big|_{\not{p}=m} + \frac{\partial\Sigma^{ff}(p)}{\partial p^2}\frac{2\not{p}\partial\not{p}}{\partial\not{p}}\Big|_{\not{p}=m} = 1 - \frac{d\Sigma^{ff}(\not{p})}{d\not{p}}\Big|_{\not{p}=m} + \frac{\partial\Sigma^{ff}(p)}{\partial p^2}\frac{\partial p^2}{\partial\not{p}}\Big|_{\not{p}=m} = 1 \end{aligned} \tag{3.2.21}$$

3.3 Final NLO analytical results

After deriving the virtual amplitudes, the next step is to re-write these amplitudes in terms of N-point functions. This re-writing makes it easier to calculate the squared amplitude $\mathcal{M}_{Virt}.\mathcal{M}^*_{LO}$ using the program FORM [9].

3.3.1 Full one-loop amplitudes

Vacuum polarization

$$\begin{aligned} i\mathcal{M}_{1,vp} = \frac{-ie^4}{q^4 4\pi^2}\bar{u}^r_e(k')\gamma^\mu u^{r'}_e(k)\bar{u}^s_\mu(p')\gamma^\nu u^{s'}_\mu(p)\Big\{&2B_{\mu\nu}(q, m_e, m_e) + q_\mu B_\nu(q, m_e, m_e) + q_\nu B_\mu(q, m_e, m_e) \\ &+ \left[\frac{q^2}{2}B_0(q^2, m_e, m_e) - A_0(m_e)\right] g_{\mu\nu}\Big\}, \end{aligned} \tag{3.3.1}$$

and :

$$i\mathcal{M}_{2,vp} = \frac{-ie^4}{q^4 4\pi^2}\bar{u}_e^r(k')\gamma^\mu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma^\nu u_\mu^{s'}(p)\left\{2B_{\mu\nu}(q,m_\mu,m_\mu) + q_\mu B_\nu(q,m_\mu,m_\mu) + q_\nu B_\mu(q,m_\mu,m_\mu) + \left[\frac{q^2}{2}B_0(q^2,m_\mu,m_\mu) - A_0(m_\mu)\right]g_{\mu\nu}\right\}. \tag{3.3.2}$$

Vertex correction

$$\begin{aligned} i\mathcal{M}_{1,vc} = \frac{ie^4}{q^2 16\pi^2}\bar{u}_e^r(k')\Big\{&-2\gamma^\alpha - 2\gamma^\mu\gamma^\alpha\gamma^\nu\Big[C_{\mu\nu}(k,k',0,m_e,m_e) + k_\mu C_\nu(k,k',0,m_e,m_e) + k_\nu C_\mu(k,k',0,m_e,m_e) \\ &+k_\mu k_\nu C_0(k,k',0,m_e,m_e) + q_\nu C_\mu(k,k',0,m_e,m_e) + q_\nu k_\mu C_0(k,k',0,m_e,m_e)\Big] + 8m_e\left[C^\alpha(k,k',0,m_e,m_e)\right. \\ &\left.+k^\alpha C_0(k,k',0,m_e,m_e)\right] + 4m_e q^\alpha C_0(k,k',0,m_e,m_e) - 2m_e^2\gamma^\alpha C_0(k,k',0,m_e,m_e)\Big\}\, u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\alpha u_\mu^{s'}(p), \end{aligned} \tag{3.3.3}$$

and :

$$\begin{aligned} i\mathcal{M}_{2,vc} = \frac{ie^4}{q^2 16\pi^2}\bar{u}_e^r(k')\gamma_\alpha u_\mu^{r'}(k)\bar{u}_\mu^s(p')\Big\{&-2\gamma^\alpha - 2\gamma^\mu\gamma^\alpha\gamma^\nu\Big[C_{\mu\nu}(p,p',0,m_\mu,m_\mu) + p_\mu C_\nu(p,p',0,m_\mu,m_\mu) \\ &+p_\nu C_\mu(p,p',0,m_\mu,m_\mu) + p_\mu p_\nu C_0(p,p',0,m_\mu,m_\mu) - q_\nu C_\mu(p,p',0,m_\mu,m_\mu) - q_\nu p_\mu C_0(p,p',0,m_\mu,m_\mu)\Big] \\ &-4m_\mu q^\alpha C_0(p,p',0,m_\mu,m_\mu) + 8m_\mu\left[C^\alpha(p,p',0,m_\mu,m_\mu) + p^\alpha C_0(p,p',0,m_\mu,m_\mu)\right] \\ &-2m_\mu^2\gamma^\alpha C_0(p,p',0,m_\mu,m_\mu)\Big\}\, u_\mu^{s'}(p). \end{aligned} \tag{3.3.4}$$

Box diagrams

$$\begin{aligned} i\mathcal{M}_{1,bd} = \frac{ie^4}{16\pi^2}\Big[&4\bar{u}_e^r(k')\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\nu u_\mu^{s'}(p)k'p'D_0(-q,-k',p',0,0,m_e,m_\mu) \\ &+\bar{u}_e^r(k')\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')2\not{k}'\gamma^\alpha\gamma_\nu u_\mu^{s'}(p)D_\alpha(-q,-k',p',0,0,m_e,m_\mu) \\ &-\bar{u}_e^r(k')2\not{p}'\gamma^\alpha\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\nu u_\mu^{s'}(p)D_\alpha(-q,-k',p',0,0,m_e,m_\mu) \\ &-\bar{u}_e^r(k')\gamma^\mu\gamma^\alpha\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\mu\gamma^\beta\gamma_\nu u_\mu^{s'}(p)D_{\alpha\beta}(-q,-k',p',0,0,m_e,m_\mu)\Big], \end{aligned} \tag{3.3.5}$$

and :

$$\begin{aligned} i\mathcal{M}_{2,bd} = \frac{ie^4}{16\pi^2}\Big[&4\bar{u}_e^r(k')\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\nu u_\mu^{s'}(p)k'pD_0(-q,-k',-p,0,0,m_e,m_\mu) \\ &-\bar{u}_e^r(k')\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma^\nu\gamma_\alpha 2\not{k}' u_\mu^{s'}(p)D_\alpha(-q,-k',-p,0,0,m_e,m_\mu) \\ &-\bar{u}_e^r(k')2\not{p}\gamma^\alpha\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\nu u_\mu^{s'}(p)D_\alpha(-q,-k',-p,0,0,m_e,m_\mu) \\ &+\bar{u}_e^r(k')\gamma^\mu\gamma^\alpha\gamma^\nu u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\nu\gamma^\beta\gamma_\mu u_\mu^{s'}(p)D_{\alpha\beta}(-q,-k',-p,0,0,m_e,m_\mu)\Big]. \end{aligned} \tag{3.3.6}$$

Counterterm diagrams

$$i\mathcal{M}_{ct} = \frac{ie^2}{q^2}\bar{u}_e^r(k')\gamma_\alpha u_\mu^{r'}(k)\bar{u}_\mu^s(p')\gamma^\alpha u_\mu^{s'}(p)\left(-\delta_A+\delta_{\psi_e}+\delta_{\psi_\mu}\right), \tag{3.3.7}$$

with the counterterm factors are determined by renormalization conditions Eq. (3.1.11-3.1.18) :

$$\delta_e = \frac{e^2}{8\pi^2}\sum_{j=e,\mu}\mathrm{Re}\left[\frac{-1}{9}+\frac{2}{3}m_j^2\frac{\partial}{\partial q^2}B_0(q^2,m_j,m_j)\Big|_{q^2=0}+\frac{B_1(0,m_j,m_j)}{3}+\frac{B_0(0,m_j,m_j)}{2}\right], \tag{3.3.8}$$

$$\delta_{\psi_j} = \frac{m_je^2}{4\pi^2}\left[-\frac{B_1(m_j^2,0,m_j)}{2m_j}-\frac{B_0(m_j^2,0,m_j)}{2m_j}-m_j\frac{\partial B_1(p^2,0,m_j)}{\partial p^2}\Big|_{p^2=m_j^2}+m_j\frac{\partial B_0(p^2,0,m_j)}{\partial p^2}\Big|_{p^2=m_j^2}+\frac{1}{4m_j}\right], \tag{3.3.9}$$

$$\delta_A = \frac{-e^2}{4\pi^2}\sum_{j=e,\mu}Re\left[-\frac{1}{9}+\frac{2}{3}m_j^2B_0'(q^2,m_j,m_j)|_{q^2=0}+\frac{B_1(0,m_j,m_j)}{3}+\frac{B_0(0,m_j,m_j)}{2}\right], \tag{3.3.10}$$

$$\frac{\partial B_1(p^2,0,m_j)}{\partial p^2}\Big|_{p^2=m_j^2} = \frac{A_0(m_j)}{2m_j^4}-\frac{1}{2m_j^2}B_0(m_j^2,0,m_j^2). \tag{3.3.11}$$

The tensor N-point functions reads :

$$B_\mu(q,m_1,m_2) = q_\mu B_1(q,m_1,m_2), \tag{3.3.12}$$

$$B_{\mu\nu}(q,m_1,m_2) = g_{\mu\nu}B_{00}(q,m_1,m_2)+q_\mu q_\nu B_{11}(q,m_1,m_2), \tag{3.3.13}$$

$$C_\mu(p_1,p_2,m_0,m_1,m_2) = \sum_{i=1}^{2}p_{i\mu}C_i(p_1,p_2,m_0,m_1,m_2), \tag{3.3.14}$$

$$C_{\mu\nu}(p_1,p_2,m_0,m_1,m_2) = g_{\mu\nu}C_{00}(p_1,p_2,m_0,m_1,m_2)+\sum_{i,j=1}^{2}p_{i\mu}p_{j\nu}C_{ij}(p_1,p_2,m_0,m_1,m_2), \tag{3.3.15}$$

$$D_\mu(p_1,p_2,p_3,m_0,m_1,m_2,m_3) = \sum_{i=1}^{2}p_{i\mu}D_i(p_1,p_2,p_3,m_0,m_1,m_2,m_3), \tag{3.3.16}$$

$$D_{\mu\nu}(p_1,p_2,p_3,m_0,m_1,m_2,m_3) = g_{\mu\nu}D_{00}(p_1,p_2,p_3,m_0,m_1,m_2,m_3)+\sum_{i,j=1}^{3}p_{i\mu}p_{j\nu}D_{ij}(p_1,p_2,p_3,m_0,m_1,m_2,m_3), \tag{3.3.17}$$

because the second rank tensors are symmetric in the Lorentz indices, so $C_{ij}=C_{ji}$, $D_{ij}=D_{ji}$.

We can now define the virtual cross section are follows :

$$\frac{d\sigma_{Virt}}{d\Omega} = \frac{2Re\left[\mathcal{M}_{Virt}.\mathcal{M}_{LO}^*\right]}{64\pi^2 s}. \tag{3.3.18}$$

I have written a FORM code using Eq. (3.3.1-3.3.7) as input to produce the result for $\mathcal{M}_{virt}.\mathcal{M}_{LO}^*$ in terms of scalar integrals A_0, B_0, C_0, D_0 and tensors coefficients $B_i, B_{ij}, C_i, C_{ij}, D_i, D_{ij}$. This FORM code is provided in Appendix F.

3.3.2 Soft-photon corrections

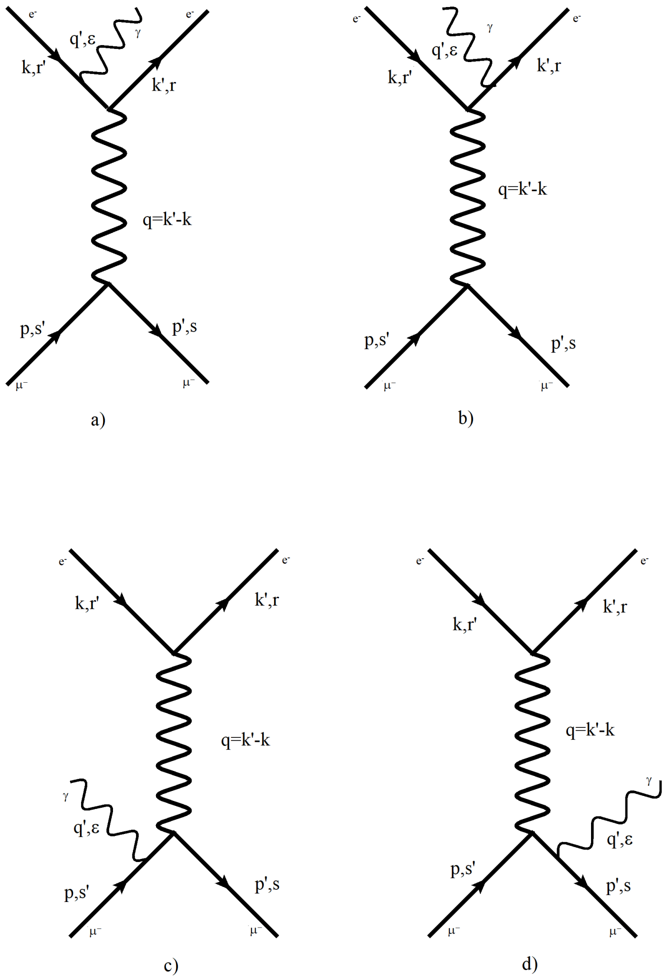


Figure 14: The photon emission diagrams

Because of the electrons and muons are charged particles, they always emit photons (electromagnetic radiation). The photon emission is therefore an essential part of QED scattering processes. The $e^-\mu^- \to e^-\mu^-$ scattering without photon emission is actually unphysical and we can't observe this process separately.

For the calculation at NLO, we have to include the emission of one additional photon. This $2 \to 3$ process is depicted in Fig. (14). We split this real emission process into two parts as follows :

$$d\sigma_{real}(\alpha^3) = d\sigma_{Soft}(\alpha^3) + d\sigma_{hard}(\alpha^3), \tag{3.3.19}$$

where the soft-photon region is defined by $E_\gamma \leq \triangle E$ with $\triangle E$ being a cutoff parameter. The value of $\triangle E$ must be very small compared to the colliding energy. Later, we will choose $\triangle E = 10^{-3}\sqrt{s}/2$ or $\triangle E = 10^{-4}\sqrt{s}/2$ for numerical results.

In this thesis, we include only the soft-photon corrections to our NLO results. The hard photon corrections are left for future work. We will show that $d\sigma_{Soft}$ is IR divergent, but the sum $d\sigma_{Virt} + d\sigma_{Soft}$ is IR finite (see

Section IR-divergence cancellation). From Fig. (14), the soft-photon amplitudes can be calculated as shown in Appendix E. In this calculation, we neglect the momentum q' of the radiative photon everywhere except in the denominator of the fermion propagator. We then get the following result for the differential cross section :

$$\left(\frac{d\sigma}{d\Omega}\right)_{Soft} = -\left(\frac{d\sigma}{d\Omega}\right)_{\rm LO} \cdot \frac{e^2}{(2\pi)^3} \int_{|\mathbf{q}'| \leq \triangle E} \frac{d^3q'}{2\omega_{q'}} \left[\frac{k^2}{(q'k)^2} + \frac{k'^2}{(q'k')^2} - \frac{2kk'}{q'k.q'k'} + \frac{p^2}{(q'p)^2} + \frac{p'^2}{(q'p')^2} - \frac{2pp'}{q'k.q'k'} \right.$$
$$\left. +2\mathrm{Re}\left(\frac{p'k'}{p'q'.k'q'} - \frac{p'k}{p'q'.kq'} - \frac{pk'}{pq'.k'q'} + \frac{pk}{pq'.kq'} \right) \right] (\alpha^3), \tag{3.3.20}$$

with $\omega_{q'} = \sqrt{|\vec{q}'|^2 + \lambda^2}$ where λ is the photon mass. Performing the integration over q' gives :

$$
\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{Soft} = & \frac{-e^2}{16\pi^3}\left(\frac{d\sigma}{d\Omega}\right)_{LO}\Bigg\{2pp'\Bigg[Re\left(4\pi\frac{x_{t\mu}}{m_\mu^2(1-x_{t\mu}^2)}\log\left(x_{t\mu}\right)\log\left(\frac{2\triangle E}{\lambda}\right)^2\right) - \frac{2\pi\alpha_1}{v_1 l_1}\left(\frac{1}{4}\log^2\left(\frac{P_x^0-|P|_x}{P_x^0+|P|_x}\right)\right. \\
& \left.\left.+\mathrm{Li}_2\left(1-\frac{P_x^0-|P|_x}{v_1}\right)+\mathrm{Li}_2\left(1-\frac{P_x^0+|P|_x}{v_1}\right)\right)\Bigg|_{P_x=p'}^{P_x=\alpha_1 p}\right] + 2\pi\log\left(\frac{2\triangle E}{\lambda}\right)^2 - 4\pi\frac{p^0}{|p|}\tanh^{-1}\left(\frac{|p|}{p^0}\right) \\
& +2\pi\log\left(\frac{2\triangle E}{\lambda}\right)^2 - 4\pi\frac{p'^0}{|p'|}\tanh^{-1}\left(\frac{|p'|}{p'^0}\right) + 2kk'\Bigg[Re\left(4\pi\frac{x_{te}}{m_e^2(1-x_{te}^2)}\log(x_{te})\log\left(\frac{2\triangle E}{\lambda}\right)^2\right) \\
& -\frac{2\pi\alpha_2}{v_2 l_2}\left(\frac{1}{4}\log^2\left(\frac{P_x^0-|P|_x}{P_x^0+|P|_x}\right)+\mathrm{Li}_2\left(1-\frac{P_x^0-|P|_x}{v_2}\right)+\mathrm{Li}_2\left(1-\frac{P_x^0+|P|_x}{v_2}\right)\right)\Bigg|_{P_x=k'}^{P_x=\alpha_2 k}\Bigg] \\
& +2\pi\log\left(\frac{2\triangle E}{\lambda}\right)^2 - 4\pi\frac{k^0}{|k|}\tanh^{-1}\left(\frac{|k|}{k^0}\right) + 2\pi\log\left(\frac{2\triangle E}{\lambda}\right)^2 - 4\pi\frac{k'^0}{|k'|}\tanh^{-1}\left(\frac{|k'|}{k'^0}\right) \\
& +2p'k'\Bigg[Re\left(4\pi\frac{x_s}{m_e m_\mu(1-x_s^2)}\log(x_s)\log\left(\frac{2\triangle E}{\lambda}\right)^2\right) - \frac{2\pi\alpha_3}{v_3 l_3}\left(\frac{1}{4}\log^2\left(\frac{P_x^0-|P|_x}{P_x^0+|P|_x}\right)\right. \\
& \left.+\mathrm{Li}_2\left(1-\frac{P_x^0-|P|_x}{v_3}\right)+\mathrm{Li}_2\left(1-\frac{P_x^0+|P|_x}{v_3}\right)\right)\Bigg|_{P_x=p'}^{P_x=-\alpha_3 k'}\Bigg] \\
& +2pk\Bigg[Re\left(4\pi\frac{x_s}{m_e m_\mu(1-x_s^2)}\log(x_s)\log\left(\frac{2\triangle E}{\lambda}\right)^2\right) - \frac{2\pi\alpha_4}{v_4 l_4}\left(\frac{1}{4}\log^2\left(\frac{P_x^0-|P|_x}{P_x^0+|P|_x}\right)\right. \\
& \left.+\mathrm{Li}_2\left(1-\frac{P_x^0-|P|_x}{v_4}\right)+\mathrm{Li}_2\left(1-\frac{P_x^0+|P|_x}{v_4}\right)\right)\Bigg|_{P_x=-k}^{P_x=\alpha_4 p}\Bigg] \\
& +2p'k\Bigg[Re\left(4\pi\frac{x_u}{m_e m_\mu(1-x_u^2)}\log(x_u)\log\left(\frac{2\triangle E}{\lambda}\right)^2\right) - \frac{2\pi\alpha_5}{v_5 l_5}\left(\frac{1}{4}\log^2\left(\frac{P_x^0-|P|_x}{P_x^0+|P|_x}\right)\right. \\
& \left.+\mathrm{Li}_2\left(1-\frac{P_x^0-|P|_x}{v_5}\right)+\mathrm{Li}_2\left(1-\frac{P_x^0+|P|_x}{v_5}\right)\right)\Bigg|_{P_x=p'}^{P_x=\alpha_5 k}\Bigg] \\
& +2pk'\Bigg[Re\left(4\pi\frac{x_u}{m_e m_\mu(1-x_u^2)}\log(x_u)\log\left(\frac{2\triangle E}{\lambda}\right)^2\right) - \frac{2\pi\alpha_6}{v_6 l_6}\left(\frac{1}{4}\log^2\left(\frac{P_x^0-|P|_x}{P_x^0+|P|_x}\right)\right. \\
& \left.+\mathrm{Li}_2\left(1-\frac{P_x^0-|P|_x}{v_6}\right)+\mathrm{Li}_2\left(1-\frac{P_x^0+|P|_x}{v_6}\right)\right)\Bigg|_{P_x=k'}^{P_x=\alpha_6 p}\Bigg]\Bigg\},
\end{aligned}
\tag{3.3.21}
$$

with :

$$v_i = \frac{a_i^2 P_i^2 - P_i'^2}{2l_i} \quad , \quad l_i = a_i P_i^0 - P_i'^0 \quad , \quad a_i \text{ is the solution of : } (a_i P_i - P_i')^2 = 0, \text{ such that : } \frac{a_i P_i^0 - P_i'^0}{P_i'^0} > 0,$$

$$x_{tj} = \frac{\sqrt{1 - \frac{4m_j^2}{t+i\varepsilon}} - 1}{\sqrt{1 - \frac{4m_j^2}{t+i\varepsilon}} + 1}, \qquad x_s = \frac{\sqrt{1 - \frac{4m_e m_\mu}{s+i\varepsilon-(m_e-m_\mu)^2}} - 1}{\sqrt{1 - \frac{4m_e m_\mu}{s+i\varepsilon-(m_e-m_\mu)^2}} + 1}, \qquad x_u = \frac{\sqrt{1 - \frac{4m_e m_\mu}{u+i\varepsilon-(m_e-m_\mu)^2}} - 1}{\sqrt{1 - \frac{4m_e m_\mu}{u+i\varepsilon-(m_e-m_\mu)^2}} + 1} \tag{3.3.22}$$

The values of P_i are given in Tab (4). Eq. (3.3.21) is now ready for numerical calculation.

i	1	2	3	4	5	6
P_i	p	k	-k'	p	k	p
P'_i	p'	k'	p'	-k	p'	k'

Table 4: Momentum notation for soft-photon cross section

Finally, we define here the NLO cross section :

$$d\sigma_{NLO} = d\sigma_{LO}(\alpha^2) + d\sigma_{virt}(\alpha^3) + d\sigma_{soft}(\alpha^3). \tag{3.3.23}$$

We will prove in Section 4 and 5 that this NLO cross section is UV and IR finite. After all divergences are cancelled, we get a finite result. The numerical evaluation of this result is presented next.

3.4 NLO numerical results

For numerical calculation, I have written a FORM program to calculate the squared one-loop amplitude $2Re\left[\mathcal{M}_{Virt} M_{LO}^*\right]$. The out put of this FORM program depends on the kinematical variables s, t, u and the one-loop functions $A_0, B_0, C_0, D_0, B_1, C_1, \ldots$. To calculate this numerically, we have to calculate these loop functions. I was able to calculate almost all these loop integrals analytically (see Appendix D). The exception is the D_0 function, for which my calculation has not been successful. However, the result for the D_0 function is known in [10] and has been implemented in the LoopInts program [11]. To cross check the squared amplitude output of my FORM program, we have used the FormCalc-6.2 [12] and FeynArts-3.4 [13] to generate another expression for the squared amplitude, we denote this expression FormCalc and call the expression of my FORM program FORM. The scalar and tensor loop integrals for both FormCalc and FORM results are calculated by the LoopInts library [1]. The input parameters are the same as in Eq. (2.0.15) with the MUonE experiment energy. The comparisons are are shown in Tab (5-6) :

[1] I thank Le Duc Ninh for providing the FormCalc results and for performing the numerical calculation for results of FORM

$e^-\mu^- \to e^-\mu^-$			
$t[GeV^{-2}]$	$(d\sigma/dt)_{NLO} - (d\sigma/dt)_{LO}$ $[\mu bGeV^{-2}]$		
	FORM ($\triangle E = 10^{-4}\sqrt{s}/2$)	FORM ($\triangle E = 10^{-3}\sqrt{s}/2$)	FormCalc ($\triangle E = 10^{-3}\sqrt{s}/2$)
-0.141465	-1.809186	-1.275878	-1.275878
-0.118602	-3.476991	-2.489794	-2.489794
-8.573608×10^{-2}	-8.538580	-6.139342	-6.139342
-4.429698×10^{-2}	-41.612983	-29.946273	-29.946273
-2.143402×10^{-2}	-195.395369	-140.529858	-140.529858

Table 5: NLO correction at different values of t

$e^-\mu^- \to e^-\mu^-$			
$\cos\theta$	$(d\sigma/d\cos\theta)_{NLO} - (d\sigma/d\cos\theta)_{LO}$ $[\mu b]$		
	FORM ($\triangle E = 10^{-4}\sqrt{s}/2$)	FORM ($\triangle E = 10^{-3}\sqrt{s}/2$)	FormCalc ($\triangle E = 10^{-3}\sqrt{s}/2$)
-0.9	-0.156633	-0.111254	-0.111254
-0.5	-0.334016	-0.239698	-0.239698
0	-0.968162	-0.696526	-0.696526
0.5	-4.773719	-3.434969	-3.434969
0.9	-122.715550	-88.148013	-88.148013

Table 6: NLO correction at differential values at $\cos\theta$ (in the CMF)

We see that the NLO corrections are negative.

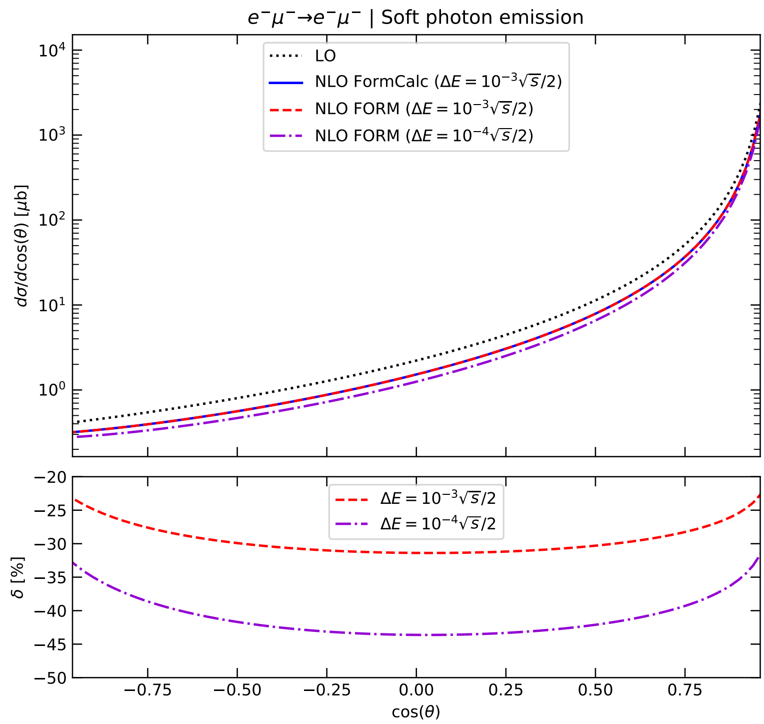


Figure 15: LO and NLO differential cross section in $\cos\theta$

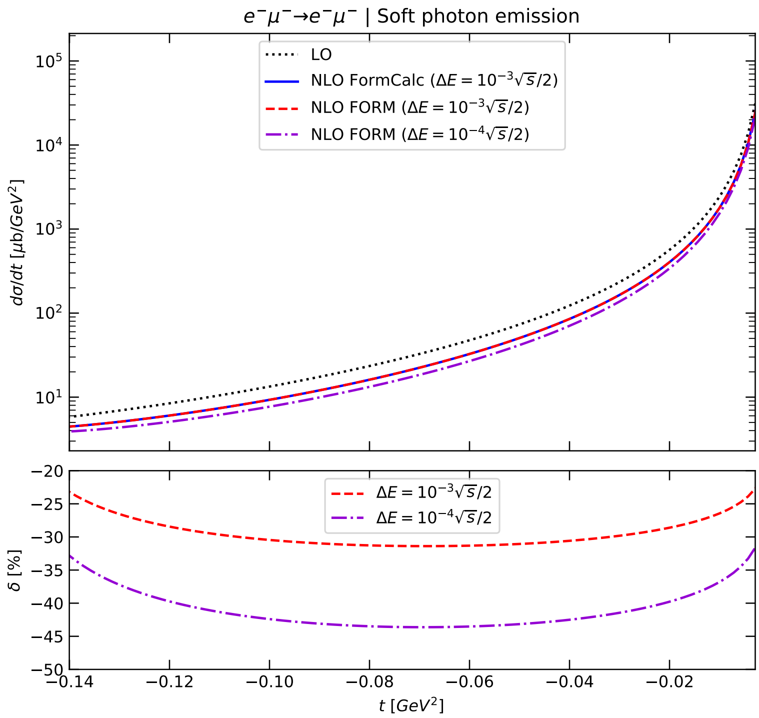


Figure 16: LO and NLO differential cross section in t

In Fig. (15-16), the δ is defined as :

$$\delta = \frac{d\sigma_{NLO} - d\sigma_{LO}}{d\sigma_{LO}}. \tag{3.4.1}$$

Take a look in the two figures, the FORM results are identical to FormCalc, which also shown clearly in Tab. (5-6). The relative correction δ can change dramatically as the cutoff $\triangle E$ be smaller because of $\log(\triangle E)$ dependence in $d\sigma_{Soft}$.

4

UV-DIVERGENCE CANCELLATION

In order to handle UV divergences, which occur in the loop corrections, we have used the multiplicative renormalization method as mentioned above. This method introduces the renormalization factors δ_m, δ_ψ and δ_e. In this section, from the final results in Section 3.3.1, we extract the UV-divergent parts of the virtual amplitude. We will then show that the UV-divergences are completely cancelled out in the final sum. Using dimensional regularization to parameterize divergent values, we obtain :

UV convergent quantities : C_0, C_μ, D_0, D_μ, $D_{\mu\nu}$.

UV divergent parts of N-point functions : (Represent UV divergent term by $\triangle$)

- $A_0(m) = m^2\triangle$,
- $B_0(p^2, m_0, m_1) = \triangle$,
- $B_\mu(p, m_0, m_1) = \frac{-1}{2}p_\mu\triangle$,
- B_1 $(B_\mu = p_\mu.B_1) = \frac{-1}{2}\triangle$,
- $B_{\mu\nu}(p, m_0, m_1) = \frac{-g_{\mu\nu}}{12}\left[p^2 - 3(m_0^2 + m_1^2)\right]\triangle + \frac{p_\mu p_\nu}{3}\triangle$,
- $C_{\mu\nu}(p, p', m_0, m_1, m_2) = g_{\mu\nu}C_{00}(p, p', m_0, m_1, m_2) = \frac{g_{\mu\nu}}{4}\triangle$,

where $\triangle = \frac{2}{4-D} - \gamma_E + 1$, D is the dimensions of the loop integrals and γ_E is the Euler constant. Substituting the UV divergent parts of N-point functions to Eq. (3.3.1-3.3.7), we get the UV divergent value of each correction :

- The Vacuum polariztion
$$i\mathcal{M}_{vp} = \frac{e^2}{4\pi^2}\frac{-2}{3}\triangle.i\mathcal{M}_{LO}. \tag{4.0.1}$$
- The Vertex correction
$$i\mathcal{M}_{vc} = \frac{e^2}{4\pi^2}\frac{1}{2}\triangle.i\mathcal{M}_{LO}. \tag{4.0.2}$$
- The Box diagrams
$$i\mathcal{M}_{bd} = 0.i\mathcal{M}_{LO}. \tag{4.0.3}$$
- LSZ factor
$$\tilde{Z}_i = 1. \tag{4.0.4}$$

- The Counterterm diagrams

$$i\mathcal{M}_{ct} = \frac{e^2}{4\pi^2}\frac{1}{6}\triangle.i\mathcal{M}_{LO}. \tag{4.0.5}$$

The UV divergence of total amplitude:

Combining all results, we obtain :

$$\begin{aligned} i\mathcal{M}_{total} &= \prod_{i=1}^{4}\sqrt{\tilde{Z}_i}(i\mathcal{M}_0 + i\mathcal{M}_{vp} + i\mathcal{M}_{vc} + i\mathcal{M}_{bd} + i\mathcal{M}_{ct}) = 1.\left[1 - \frac{e^2}{4\pi^2}\frac{2}{3}\triangle + \frac{e^2}{4\pi^2}\frac{1}{2}\triangle + \frac{e^2}{4\pi^2}\frac{1}{6}\triangle\right].i\mathcal{M}_{LO} \\ &= i\mathcal{M}_{LO} \rightarrow \textbf{UV convergent}. \end{aligned} \tag{4.0.6}$$

The mathematical trick in renormalization is the absorption of all divergent values into renormalization factors of physics parameters and fields. This still leaves us a lot of freedom in choosing how much of the finite part of the loop integrals can be absorbed by the renormalization factors. There are no unique choice to fix these factors, and any such choice is called a renormalization scheme.

5

IR-DIVERGENCE CANCELLATION

In this section we provide the IR-divergent parts of the virtual amplitude. We will then show that the IR divergences are cancelled out in the sum of the virtual cross section and the soft-photon emission cross section. To regularize the IR divergences, we introduce a photon mass λ in the one-loop integrals and also in Eq. (3.3.21) for the soft-photon correction.

The basic reason of IR divergence is the massless photon in loop integrals, which occurs in any process with charged external particles. We can calculate the IR divergent quantities with a regularized photon mass $\lambda \to 0$ (details are shown in Appendices) :

- $B'_0(m_j^2, \lambda, m_j) = \frac{\partial B_0(p^2,\lambda,m_j)}{\partial p^2}|_{p^2=m_j^2} = -\frac{1}{m_j^2}\log\left(\frac{\lambda}{m_j}\right)$.

- $C_0(p, p', \lambda, m_j, m_j) = C_0(m_j^2, t, m_j^2, \lambda, m_j, m_j) = \frac{-2x_{tj}}{m^2(1-x_{tj}^2)}\log\left(x_{tj}\right)\log\left(\frac{\lambda}{m_j}\right)$,
  with :
$$x_{tj} = \frac{\sqrt{1-\frac{4m_j^2}{t+i\varepsilon}}-1}{\sqrt{1-\frac{4m_j^2}{t+i\varepsilon}}+1}.$$

- $D_0(q, p_1, p_2, \lambda, \lambda, m_1, m_2) = \frac{2}{q^2}C_0(m_1^2, (p_1-p_2)^2, m_2^2, \lambda, m_1, m_2) = \frac{-2x_{12}\log(x_{12})}{m_1 m_2 q^2(1-x_{12}^2)}\log\left(\frac{\lambda^2}{-q^2-i\varepsilon}\right)$ [10],
  with :
$$x_{12} = \frac{\sqrt{1-\frac{4m_1m_2}{(p_1-p_2)^2+i\varepsilon-(m_1-m_2)^2}}-1}{\sqrt{1-\frac{4m_1m_2}{(p_1-p_2)^2+i\varepsilon-(m_1-m_2)^2}}+1}.$$

- $D_\mu(q, p_1, p_2, \lambda, \lambda, m_1, m_2) \sim q_\mu D_1(q, p_1, p_2, \lambda, \lambda, m_1, m_2) = q_\mu \frac{-C_0(m_1^2,(p_1-p_2)^2,m_2^2,\lambda,m_1,m_2)}{q^2}$.

- $D_{\mu\nu}(q, p_1, p_2, \lambda, \lambda, m_1, m_2) \sim q_\mu q_\nu D_{11}(q, p_1, p_2, \lambda, \lambda, m_1, m_2) = q_\mu q_\nu \frac{C_0(m_1^2,(p_1-p_2)^2,m_2^2,\lambda,m_1,m_2)}{q^2}$,

the other functions are IR convergent. Substituting these functions into the Eq. (3.3.1-3.3.7), we acquire :

The IR divergences of virtual differential cross section

- The Vacuum polariztion

$$i\mathcal{M}_{vp}^{IR} = 0.i\mathcal{M}_{LO}. \tag{5.0.1}$$

- The Vertex correction

$$i\mathcal{M}^{IR}_{vc} = \frac{e^2}{4\pi^2}\left[-2k'.k\frac{x_{te}}{m_e^2(1-x_{te}^2)}\log(x_{te})\log\left(\frac{\lambda}{m_e}\right) - 2p'.p\frac{x_{t\mu}}{m_\mu^2(1-x_{t\mu}^2)}\log\left(x_{t\mu}\right)\log\left(\frac{\lambda}{m_\mu}\right)\right].i\mathcal{M}_{LO}. \tag{5.0.2}$$

- The Box diagrams

$$\begin{aligned} i\mathcal{M}^{IR}_{bd} = \frac{e^2}{4\pi^2}\Bigg[&-2k'p'\frac{x_s}{m_e m_\mu(1-x_s^2)}\log(x_s)\log\left(\frac{\lambda^2}{-q^2-i\varepsilon}\right) \\ &-2k'p\frac{x_u}{m_e m_\mu(1-x_u^2)}\log(x_u)\log\left(\frac{\lambda^2}{-q^2-i\varepsilon}\right)\Bigg].i\mathcal{M}_{LO}. \end{aligned} \tag{5.0.3}$$

- The counterterm diagrams

$$i\mathcal{M}^{IR}_{ct} = \frac{e^2}{4\pi^2}\left[-\log\frac{\lambda}{m_e} - \log\frac{\lambda}{m_\mu}\right].i\mathcal{M}_{LO}. \tag{5.0.4}$$

The total virtual differential cross section

$$\begin{aligned} &\left(\frac{d\sigma}{d\Omega}\right)^{IR}_{\text{Virt}} = \frac{e^2}{4\pi^2}\left(\frac{d\sigma}{d\Omega}\right)_{\text{LO}} \\ &\times 2Re\left[-2k'.k\frac{x_{te}}{m_e^2(1-x_{te}^2)}\log(x_{te})\log\left(\frac{\lambda}{m_e}\right) - 2p'.p\frac{x_{t\mu}}{m_\mu^2(1-x_{t\mu}^2)}\log\left(x_{t\mu}\right)\log\left(\frac{\lambda}{m_\mu}\right) - \log\frac{\lambda}{m_e} - \log\frac{\lambda}{m_\mu}\right. \\ &\left.-2k'p'\frac{x_s}{m_e m_\mu(1-x_s^2)}\log(x_s)\log\left(\frac{\lambda^2}{-q^2-i\varepsilon}\right) - 2k'p\frac{x_u}{m_e m_\mu(1-x_u^2)}\log(x_u)\log\left(\frac{\lambda^2}{-q^2-i\varepsilon}\right)\right](\alpha^3). \end{aligned} \tag{5.0.5}$$

$\Rightarrow$ The total virtual differential cross section at α^3 order is IR-divergent. To solve the divergent problem, we have to add the *Soft-photon emission* as above discussed..

The full squared amplitude of soft-photon radiation read:

$$\left|i\mathcal{M}_p + i\mathcal{M}_{p'} + i\mathcal{M}_k + i\mathcal{M}_{k'}\right|^2 = \left|i\mathcal{M}_{p,p'} + i\mathcal{M}_{k,k'}\right|^2 = \left|i\mathcal{M}_{p,p'}\right|^2 + |i\mathcal{M}_{k,k'}|^2 + 2Re\left[\mathcal{M}^*_{p,p'}\mathcal{M}_{k,k'}\right]. \tag{5.0.6}$$

In the soft-photon emission, the first two terms we cancel with the IR divergences of the vertex correction and counterterm, and the last term cancels with the box diagrams. We will show this in the following.

$$\begin{aligned} \left|i\mathcal{M}_{p,p'}\right|^2 + |i\mathcal{M}_{k,k'}|^2 \to \left(\frac{d\sigma}{d\Omega}\right)^{IR}_{1,Soft} = \frac{-e^2}{4\pi^2}\left(\frac{d\sigma}{d\Omega}\right)_{LO}.Re\Bigg\{&4\log\left(\frac{2\triangle E}{\lambda}\right) + 4kk'\frac{x_{te}}{m_e^2(1-x_t^2)}\log(x_{te})\log\left(\frac{2\triangle E}{\lambda}\right) \\ &+4pp'\frac{x_{t\mu}}{m_\mu^2(1-x_{t\mu}^2)}\log\left(x_{t\mu}\right)\log\left(\frac{2\triangle E}{\lambda}\right)\Bigg\}, \end{aligned} \tag{5.0.7}$$

This IR-divergent part will cancels out the IR divergences of the vertex correction Eq. (5.0.2) and the counterterm diagrams Eq. (5.0.4). For the last term, we have :

$$2Re\left[\mathcal{M}^{*}_{p,p'}\mathcal{M}_{k,k'}\right] \to \left(\frac{d\sigma}{d\Omega}\right)^{IR}_{2,Soft} = \frac{-e^2}{4\pi^2}\left(\frac{d\sigma}{d\Omega}\right)_{LO}.2Re\left[2k'p'\frac{x_s}{m_e m_\mu(1-x_s^2)}\log(x_s)\log\left(\frac{2\triangle E}{\lambda}\right)^2 + 2k'p\frac{x_u}{m_e m_\mu(1-x_u^2)}\log(x_u)\log\left(\frac{2\triangle E}{\lambda}\right)^2\right](\alpha^3). \tag{5.0.8}$$

This IR-divergent part cancels the IR-divergences of the Box diagrams.

The total IR-divergent part of the *Soft-photon radiation* differential cross section reads :

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)^{IR}_{Soft} &= \left(\frac{d\sigma}{d\Omega}\right)^{IR}_{1,Soft} + \left(\frac{d\sigma}{d\Omega}\right)^{IR}_{2,Soft}\\ &= \frac{-e^2}{4\pi^2}\left(\frac{d\sigma}{d\Omega}\right)_{LO}.Re\left\{4\log\left(\frac{2\triangle E}{\lambda}\right) + 4kk'\frac{x_{te}}{m_e^2(1-x_t^2)}\log(x_{te})\log\left(\frac{2\triangle E}{\lambda}\right)\right.\\ &+ 4pp'\frac{x_{t\mu}}{m_\mu^2(1-x_{t\mu}^2)}\log(x_{t\mu})\log\left(\frac{2\triangle E}{\lambda}\right) + 2\left[2k'p'\frac{x_s}{m_e m_\mu(1-x_s^2)}\log(x_s)\log\left(\frac{2\triangle E}{\lambda}\right)^2\right.\\ &\left.\left.+2k'p\frac{x_u}{m_e m_\mu(1-x_u^2)}\log(x_u)\log\left(\frac{2\triangle E}{\lambda}\right)^2\right]\right\}(\alpha^3),\end{aligned} \tag{5.0.9}$$

We can see that the cross section of soft-photon radiation process as IR divergent as the virtual corrections, but with a sign difference.

The IR-convergence of the final NLO cross section

The IR-divergent part of the NLO cross section reads :

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)^{IR}_{\rm NLO} &= \left(\frac{d\sigma}{d\Omega}\right)^{IR}_{\rm Virt} + \left(\frac{d\sigma}{d\Omega}\right)^{IR}_{Soft} = \frac{-e^2}{4\pi^2}\left(\frac{d\sigma}{d\Omega}\right)_{\rm LO} Re\left[\log\left(\frac{2\triangle E}{m_e}\right)^2 + \log\left(\frac{2\triangle E}{m_\mu}\right)^2\right.\\ &+4kk'\frac{x_{te}}{m_e^2(1-x_{te}^2)}\log(x_{te})\log\left(\frac{2\triangle E}{m_e}\right) + 4pp'\frac{x_{t\mu}}{m_\mu^2(1-x_{t\mu}^2)}\log(x_{t\mu})\log\left(\frac{2\triangle E}{m_\mu}\right)\\ &\left.+ 4k'p'\frac{x_s}{m_e m_\mu(1-x_s^2)}\log(x_s)\log\left(\frac{4\triangle E^2}{-q^2-i\varepsilon}\right) + 4k'p\frac{x_u}{m_e m_\mu(1-x_u^2)}\log(x_u)\log\left(\frac{4\triangle E^2}{-q^2-i\varepsilon}\right)\right](\alpha^3),\end{aligned} \tag{5.0.10}$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{\rm NLO} \text{ is IR convergent.}$$

Finally, we get the IR-convergent result of the NLO (α^3) differential cross section. At the α^4, α^5 etc., orders, we always obtain the finite cross section result by taking more photons into account. The final result depends on the cutoff $\triangle E$, the smaller $\triangle E$, the less photon contribution, the larger correction (as shown in Section 3.4). If the $\triangle E$ reaches to zero, our cross section will be IR-divergent. In practice, all detectors have a sensitive threshold, it means that the cutoff $\triangle E$ has a finite value at all times.

6

Conclusion and Outlook

Conclusion

From the work of this thesis, we can draw the following conclusions :

- The t-channel divergence occurring in the LO total cross section is due to the infinite range of the electromagnetic interaction.
- We have successfully cancelled out all divergences occurring at next-to-leading order, UV divergence is cancelled by renormalization and IR divergence by adding soft-photon corrections. We note that the photon radiation is an indispensable part of the scattering process of charged particles.
- Our final NLO amplitudes are identical to results of FormCalc program.

Outlook

The next step is to include the missing hard photon corrections. After that, we will include the electroweak corrections in the Standard Model. For the NLO numerical results presented in this thesis, the Fortran program LoopInts has been used. I plan to write a Mathematica code to calculated all the one-loop integrals of the $e^-\mu^- \to e^-\mu^-$ process. This will allow to use high precision to get more accurate results. The program LoopInts has a limit on the precision (quadruple precision at most) and this is likely not enough for calculating the electroweak corrections keeping the electron mass.

APPENDICES

A

Wick's theorem

A.1 Wick's theorem

The S operator containing scattering information can be expanded to Dyson's series [4]:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} T\left\{H_I^I(t_1)H_I^I(t_2)\dots H_I^I(t_n)\right\} d^4x_1 d^4x_2 \dots d^4x_n. \tag{A.1.1}$$

In QED $H_I^I(t_1) = \left(e\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{t_1}$. The time ordering in the integrand is indicated by operator T, because of hard mathematical manipulation, the time ordering operator can be handle via a handy theorem developed by Italian physicist Gian-Carlo Wick. Wick's theorem converts time ordered products of operators into normal ordered products of operators and some things called "contraction". Take example for two scalar fields [5]:

$$\langle 0| T\varphi(x)\varphi(y) |0\rangle \,, \tag{A.1.2}$$

set :

$$\varphi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}} e^{-ipx} \quad \varphi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a^{\dagger}_{\mathbf{p}} e^{ipx}, \tag{A.1.3}$$

first, we consider the case $x^0 > y^0$. The time-odered product of two fields is then

$$\begin{aligned} T\varphi(x)\varphi(y) &= \varphi^+(x)\varphi^+(y) + \varphi^+(x)\varphi^-(y) + \varphi^-(x)\varphi^+(y) + \varphi^-(x)\varphi^-(y) \\ &= \varphi^+(x)\varphi^+(y) + \varphi^+(x)\varphi^-(y) + \varphi^+(y)\varphi^-(x) + \varphi^-(x)\varphi^-(y) + \left[\varphi^-(x)\varphi^+(y)\right]. \end{aligned} \tag{A.1.4}$$

We define *normal order* operator is that placing all destruction operators to the right hand side inside a given term :

$$N\left(a_p a_k^{\dagger} a_q\right) = a_k^{\dagger} a_p a_q, \tag{A.1.5}$$

rewrite Eq.(A.1.4)

$$T\varphi(x)\varphi(y) = N\left(\varphi(x)\varphi(y)\right) + \left[\varphi^{-}(x)\varphi^{+}(y)\right], \tag{A.1.6}$$

similarly for case $x^0 < y^0$:

$$T\varphi(x)\varphi(y) = N\left(\varphi(x)\varphi(y)\right) + \left[\varphi^{-}(y)\varphi^{+}(x)\right], \tag{A.1.7}$$

the contraction of two fields that is actually Feynman propagator, as follows :

$$\overbracket{\varphi(x)\varphi}(y) = \begin{cases} [\varphi^{-}(x)\varphi^{+}(y)] & \text{for} x^0 > y^0 \\ [\varphi^{-}(y)\varphi^{+}(x)] & \text{for} x^0 < y^0 \end{cases} = D_F(x-y), \tag{A.1.8}$$

finally, we get the general Wick's theorem for two scalar fiels :

$$T\varphi(x)\varphi(y) = N\left(\varphi(x)\varphi(y)\right) + \overbracket{\varphi(x)\varphi}(y). \tag{A.1.9}$$

Generalize Wick's theorem for N Dirac fields [5] :

$$T\left[\psi_1\bar{\psi}_2\psi_3..\right] = N\left[\psi_1\bar{\psi}_2\psi_3.. + \text{all possible contractions}\right], \tag{A.1.10}$$

where the contraction of two Dirac fields :

$$\overbracket{\psi(x)\bar{\psi}}(y) \equiv \begin{cases} \left\{\psi^{+}(x)\bar{\psi}^{-}(y)\right\} & \text{for} x^0 > y^0 \\ -\left\{\bar{\psi}^{+}(y)\psi^{-}(x)\right\} & \text{for} x^0 < y^0 \end{cases} = S_F(x-y); \tag{A.1.11}$$

$$\overbracket{\psi(x)\psi}(y) = \overbracket{\bar{\psi}(x)\bar{\psi}}(y) = 0. \tag{A.1.12}$$

A.2 Leading order scattering amplitude

We have formula of amplitude scattering :

$$\langle k'p'|\,S\,|kp\rangle = \lim_{T\to\infty} \langle k'p'|\,T\left[exp\left(-i\int_{-T}^{T} dt\mathcal{H}_I(t)\right)\right]|kp\rangle \stackrel{\text{connected}}{=\!=\!=} i\mathcal{M}(2\pi)^4\delta^4(p_i-p_f). \tag{A.2.1}$$

With $\mathcal{H}_I = e\int d^3x(\bar{\psi}_e\gamma^a\psi_eA_a + \bar{\psi}_\mu\gamma^a\psi_\mu A_a)$. Using the second order of S-matrix only, we get the result :

$$\begin{aligned} &\langle k'p'|\,T\left[\frac{1}{2!}(-ie)\int d^4x(\bar{\psi}^x_e\gamma^a\psi^x_eA_a + \bar{\psi}^x_\mu\gamma^a\psi^x_\mu A_a)(-ie)\int d^4y(\bar{\psi}^y_e\gamma^\beta\psi^y_eA_\beta + \bar{\psi}^y_\mu\gamma^\beta\psi^y_\mu A_\beta)\right]|kp\rangle \\ &= i\mathcal{M}(2\pi)^4\delta^4(p_i-p_f) \end{aligned} \tag{A.2.2}$$

$$\begin{aligned} &\frac{-e^2}{2!}\int d^4x\int d^4y\,\langle 0|\,T\left[a^r_{p'}a^{s'}_{k'}\left(\bar{\psi}^x_e\gamma^a\psi^x_eA_a\bar{\psi}^y_\mu\gamma^\beta\psi^y_\mu A_\beta + \bar{\psi}^x_\mu\gamma^a\psi^x_\mu A_a\bar{\psi}^y_e\gamma^\beta\psi^y_eA_\beta\right)a^{s\dagger}_k a^{r'\dagger}_p\right]|0\rangle \\ &= i\mathcal{M}(2\pi)^4\delta^4(p_i-p_f). \end{aligned} \tag{A.2.3}$$

Because the term $\left(\bar{\psi}^x_e\gamma^\alpha\psi^x_e A_\alpha \bar{\psi}^y_\mu \gamma^\beta \psi^y_\mu A_\beta\right)$ and $\left(\bar{\psi}^x_\mu\gamma^\alpha\psi^x_\mu A_\alpha \bar{\psi}^y_e \gamma^\beta \psi^y_e A_\beta\right)$ are the same results, so we'll combine them to cancel the fractor $\frac{1}{2!}$:

$$\Rightarrow -e^2\int d^4x\int d^4y\, \langle 0|\, T\left(a^r_{p'}a^{s'}_{k'}\bar{\psi}^x_e\gamma^\alpha\psi^x_e A_\alpha \bar{\psi}^y_\mu \gamma^\beta \psi^y_\mu A_\beta a^{s\dagger}_k a^{r'\dagger}_p\right)|0\rangle = i\mathcal{M}(2\pi)^4\delta^4(p_i - p_f) \tag{A.2.4}$$

Using the Wick's theorem [5] :

$$T(a_{p'}a_{k'}\overline{\psi}^x_e\gamma^\alpha\psi^x_e A_\alpha \overline{\psi}^y_\mu \gamma^\beta \psi^y_\mu A_\beta a^\dagger_k a^\dagger_p) = N(a_{p'}a_{k'}\overline{\psi}^x_e\gamma^\alpha\psi^x_e A_\alpha \overline{\psi}^y_\mu \gamma^\beta \psi^y_\mu A_\beta a^\dagger_k a^\dagger_p + \text{all possible contractions}) \tag{A.2.5}$$

only following types of full contractions give non-zero:

$$a_{p'}a_{k'}\overline{\psi}^x_e\gamma^\alpha\psi^x_e A_\alpha \overline{\psi}^y_\mu \gamma^\beta \psi^y_\mu A_\beta a^\dagger_k a^\dagger_p \tag{A.2.6}$$

we just retain the above non-zero terms to obtain the result on the left-hand side Eq.(A.2.4) :

$$LHS = -e^2\int d^4x\int d^4y \langle k', p'|\overline{\psi}^x_e\gamma^\alpha\psi^x_e A_\alpha \overline{\psi}^y_\mu \gamma^\beta \psi^y_\mu A_\beta|k, p\rangle, \tag{A.2.7}$$

To untangle the contractions, we use the formula [5] :

$$a_{p'}a_{k'}\overline{\psi}_y\psi_x = -a_{p'}a_{k'}\psi_x\overline{\psi}_y \quad \text{and} \quad a_{p'}A_\mu\overline{\psi}_y A_\nu = a_{p'}\overline{\psi}_y A_\mu A_\nu. \tag{A.2.8}$$

Untangling the contractions Eq.(A.2.6), we get the result :

$$\begin{aligned} LHS &= -e^2\int d^4x\int d^4y \quad \bar{u}^r_e(k')e^{ik'x}\gamma^\alpha u^{r'}_e(k)e^{-ikx}\int \frac{d^4q}{(2\pi)^4}e^{-iq(x-y)}\left(\frac{-ig_{\alpha\beta}}{q^2}\right)\bar{u}^s_\mu(p')e^{ip'y}\gamma^\beta u^{s'}_\mu(p)e^{-ipy} \\ &= -e^2\int \frac{d^4q}{(2\pi)^4}\left(\frac{-ig_{\alpha\beta}}{q^2}\right) \quad \bar{u}^r_e(k')\gamma^\alpha u^{r'}_e(k)\bar{u}^s_\mu(p')\gamma^\beta u^{s'}_\mu(p)\int d^4x e^{i(k'-k-q)x}\int d^4y e^{i(p'-p+q)y} \\ &= \int \frac{d^4q}{(2\pi)^4}\frac{ie^2}{q^2}\bar{u}^r_e(k')\gamma^\alpha u^{r'}_e(k)g_{\alpha\beta}\bar{u}^s_\mu(p')\gamma^\beta u^{s'}_\mu(p)(2\pi)^8\delta^4(k'-k-q)\delta^4(p'-p+q) \end{aligned} \tag{A.2.9}$$

$$\Rightarrow LHS = \left[\frac{ie^2}{q^2}\bar{u}^r_e(k')\gamma^\alpha u^{r'}_e(k)\bar{u}^s_\mu(p')\gamma_\alpha u^{s'}_\mu(p)\right](2\pi)^4\delta^4(p'+k'-p-k), \tag{A.2.10}$$

with $q = k' - k$ and the formula of contraction :

$$\psi_I(x)|p, s\rangle = e^{-ipx}u^s(p)\,|0\rangle \tag{A.2.11}$$

$$\left(\gamma^0\psi_I(x)|p, s\rangle\right)^\dagger = \langle 0|\, e^{ipx}\bar{u}^s(p) \tag{A.2.12}$$

$$A_\mu A_\nu = G_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\frac{-ig_{\mu\nu}}{p^2+i\varepsilon}. \tag{A.2.13}$$

We have :

$$LHS = i\mathcal{M}(2\pi)^4\delta^4(p_i - p_f), \tag{A.2.14}$$

associating to Eq.(A.2.10), we get result of amplitude scattering :

$$i\mathcal{M} = \frac{ie^2}{q^2}\bar{u}_e^r(k')\gamma^\alpha u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma_\alpha u_\mu^{s'}(p), \tag{A.2.15}$$

it's easy to see above equation is same as eq.(2.0.2).

A.3 One-loop order scattering amplitudes

We have formula of amplitude scattering [5] :

$$\langle k'p'|\, S\, |kp\rangle = \lim_{T\to\infty}\langle k'p'|\, T\left[exp\left(-i\int_{-T}^{T} dt\mathcal{H}_I(t)\right)\right]|kp\rangle = i\mathcal{M}(2\pi)^4\delta^4(p_i - p_f). \tag{A.3.1}$$

With $\mathcal{H}_I = e\int d^3x(\bar{\psi}_e\gamma^\alpha\psi_e A_\alpha + \bar{\psi}_\mu\gamma^\alpha\psi_\mu A_\alpha)$, using the Next-to-leading order (NLO) of S-matrix, we get the result :

$$\begin{aligned}\langle k'p'|\, T[\frac{1}{4!}(-ie)\int d^4x(\bar{\psi}_x^e\gamma^\rho\psi_x^e A_\rho + \bar{\psi}_x^\mu\gamma^\rho\psi_x^\mu A_\rho)(-ie)\int d^4y(\bar{\psi}_y^e\gamma^\nu\psi_y^e A_\nu + \bar{\psi}_y^\mu\gamma^\nu\psi_y^\mu A_\nu).\\ (-ie)\int d^4z(\bar{\psi}_z^e\gamma^\alpha\psi_z^e A_\alpha + \bar{\psi}_z^\mu\gamma^\alpha\psi_z^\mu A_\alpha)(-ie)\int d^4w(\bar{\psi}_z^e\gamma^\alpha\psi_z^e A_\alpha + \bar{\psi}_w^\mu\gamma^\beta\psi_w^\mu A_\beta)]\,|kp\rangle\\ = i\mathcal{M}(2\pi)^4\delta^4(p_i - p_f)\end{aligned} \tag{A.3.2}$$

$$\begin{aligned}\frac{e^4}{4!}\int d^4x\int d^4y\int d^4z\int d^4w\,\langle 0|\, T[a_{p'}^s a_{k'}^{s'}(\bar{\psi}_x^e\gamma^\rho\psi_x^e A_\rho + \bar{\psi}_x^\mu\gamma^\rho\psi_x^\mu A_\rho)(\bar{\psi}_y^e\gamma^\nu\psi_y^e A_\nu + \bar{\psi}_y^\mu\gamma^\nu\psi_y^\mu A_\nu).\\ (\bar{\psi}_z^e\gamma^\alpha\psi_z^e A_\alpha + \bar{\psi}_z^\mu\gamma^\alpha\psi_z^\mu A_\alpha)(\bar{\psi}_w^e\gamma^\beta\psi_w^e A_\beta + \bar{\psi}_w^\mu\gamma^\beta\psi_w^\mu A_\beta)a_k^{r\dagger}a_p^{r'\dagger}]\,|0\rangle\\ = i\mathcal{M}(2\pi)^4\delta^4(p_i - p_f).\end{aligned} \tag{A.3.3}$$

Using the Wick's theorem [5] :

$$\begin{aligned}&T(a_{p'}a_{k'}\overline{\psi}_x\gamma^\rho\psi_x A_\rho\overline{\psi}_y\gamma^\nu\psi_y A_\nu\overline{\psi}_z\gamma^\alpha\psi_z A_\alpha\overline{\psi}_w\gamma^\beta\psi_w A_\beta a_k^\dagger a_p^\dagger)\\ &= N(a_{p'}a_{k'}\overline{\psi}_x\gamma^\rho\psi_x A_\rho\overline{\psi}_y\gamma^\nu\psi_y A_\nu\overline{\psi}_z\gamma^\alpha\psi_z A_\alpha\overline{\psi}_w\gamma^\beta\psi_w A_\beta a_k^\dagger a_p^\dagger + \text{all possible contractions}),\end{aligned} \tag{A.3.4}$$

we'll meet four type of loop diagram :

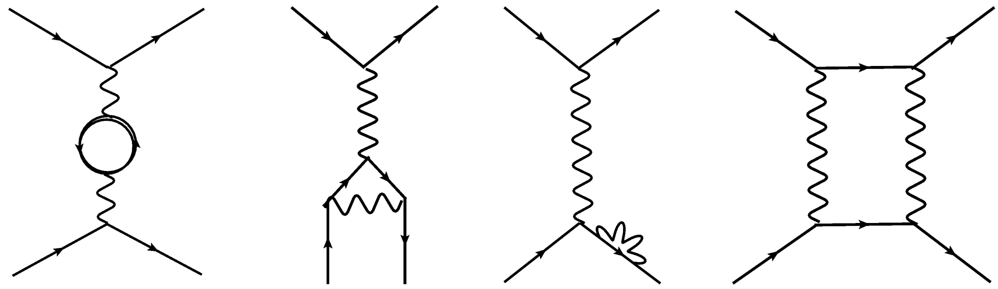


Figure 17: Four types of loop diagrams in NLO e^-e^- scattering process

The third one is *external leg corrections*. We'll neglect it in the amplitude calculation because it's not amputated as required by formula (4.90) in Peskin's book [5] for S-matrix elements.

1. **Vacuum polarization**
   The first diagram is called the *vacuum polarization*. Using Wick's theorem in Eq.(A.3.4) we get two diagrams in this type:

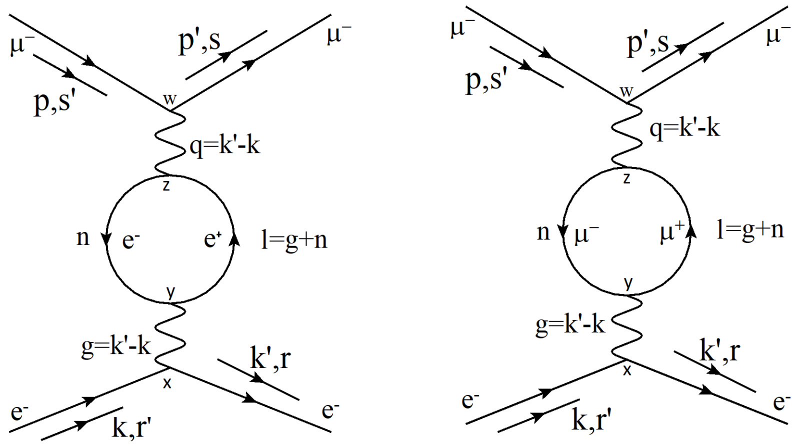


Figure 18: The Vacuum polarization diagrams

corresponding to following two short-term after expanding using Wick's theorem in Eq.(A.3.3) :

$$a_{p'} a_{k'} \overline{\psi}^e_x \gamma^\rho \psi^e_x A_\rho \overline{\psi}^e_y \gamma^\nu \psi^e_y A_\nu \overline{\psi}^e_z \gamma^\alpha \psi^e_z A_\alpha \overline{\psi}^\mu_w \gamma^\beta \psi^\mu_w A_\beta a^\dagger_k a^\dagger_p \tag{A.3.5}$$

$$a_{p'} a_{k'} \overline{\psi}^e_x \gamma^\rho \psi^e_x A_\rho \overline{\psi}^\mu_y \gamma^\nu \psi^\mu_y A_\nu \overline{\psi}^\mu_z \gamma^\alpha \psi^\mu_z A_\alpha \overline{\psi}^\mu_w \gamma^\beta \psi^\mu_w A_\beta a^\dagger_k a^\dagger_p \tag{A.3.6}$$

- We rewrite fully Eq.(A.3.5):

$$e^4 \int d^4x \int d^4y \int d^4z \int d^4w \, \langle 0| \, a_{p'} a_{k'} \overline{\psi}^e_x \gamma^\rho \psi^e_x A_\rho \overline{\psi}^e_y \gamma^\nu \psi^e_y A_\nu \overline{\psi}^e_z \gamma^\alpha \psi^e_z A_\alpha \overline{\psi}^\mu_w \gamma^\beta \psi^\mu_w A_\beta a^\dagger_k a^\dagger_p \, |0\rangle \tag{A.3.7}$$
$$= i\mathcal{M}_{1,vp} (2\pi)^4 \delta^4(p_i - p_f),$$

the $\frac{1}{4!}$ factor is canceled by 4! ways of interchanging vertices to obtain the same contraction. To untangle the contractions, we use the formula :

$$
\begin{gathered}
a_{p'}a_{k'}\overline{\psi}_y\psi_x = -a_{p'}a_{k'}\psi_x\overline{\psi}_y \quad \text{and} \quad a_{p'}A_\rho\overline{\psi}_yA_\nu = a_{p'}\overline{\psi}_yA_\rho A_\nu \\
\psi_I(x)|p,s\rangle = e^{-ipx}u^s(p)\,|0\rangle \quad \text{and} \quad \left(\gamma^0\psi_I(x)|p,s\rangle\right)^\dagger = \langle 0|\,e^{ipx}\overline{u}^s(p) \\
A_\rho A_\nu = g_{\rho\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\frac{-ig_{\rho\nu}}{p^2+i\varepsilon} \\
\psi_x\overline{\psi}_y = S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\frac{i\left(\not{p}+m_e\right)}{p^2-m_e^2+i\varepsilon}.
\end{gathered}
\tag{A.3.8}
$$

Untangling the contractions in Eq(A.3.7), we get the results in the LHS :

$$
\begin{gathered}
e^4\int d^4x\int d^4y\int d^4z\int d^4w.e^{ik'x}\overline{u}_e^r(k')\gamma^\rho e^{-ikx}u_e^{r'}(k)\int\frac{d^4g}{(2\pi)^4}e^{-ig(x-y)}\frac{-ig_{\rho\nu}}{g^2}. \\
(-1)tr\left[\gamma^\nu S_F(y-z)\gamma^\alpha S_F(z-y)\right]\int\frac{d^4q}{(2\pi)^4}e^{-iq(z-w)}\frac{-ig_{\alpha\beta}}{q^2}e^{ip'w}\overline{u}_\mu^s(p')\gamma^\beta e^{-ipw}u_\mu^{s'}(p),
\end{gathered}
\tag{A.3.9}
$$

with :

$$
\begin{aligned}
S_F(y-z) &= \int\frac{d^4l}{(2\pi)^4}\frac{i}{\not{l}-m_e}e^{-il(y-z)} \\
S_F(z-y) &= \int\frac{d^4n}{(2\pi)^4}\frac{i}{\not{n}-m_e}e^{-in(z-y)}.
\end{aligned}
\tag{A.3.10}
$$

Eq.(A.3.9)

$$
\begin{aligned}
&\Rightarrow -e^4\int\frac{d^4g\quad d^4l\quad d^4n\quad d^4q}{(2\pi)^{16}}\overline{u}_e^r(k')\gamma^\rho u_e^{r'}(k)\frac{-ig_{\rho\nu}}{g^2}tr\left[\gamma^\nu\frac{i}{\not{l}-m_e}\gamma^\alpha\frac{i}{\not{n}-m_e}\right]\frac{-ig_{\alpha\beta}}{q^2}\overline{u}_\mu^s(p')\gamma^\beta u_\mu^{s'}(p) \\
&\int d^4xe^{-i(k'-k-g)x}\int d^4ye^{-i(g-l+n)y}\int d^4ze^{-i(l-n-q)z}\int d^4we^{-i(q+p'-p)w} \\
&= -e^4\int\frac{d^4g\quad d^4l\quad d^4n\quad d^4q}{(2\pi)^{16}}\overline{u}_e^r(k')\gamma^\rho u_e^{r'}(k)\frac{-ig_{\rho\nu}}{g^2}tr\left[\gamma^\nu\frac{i}{\not{l}-m_e}\gamma^\alpha\frac{i}{\not{n}-m_e}\right]\frac{-ig_{\alpha\beta}}{q^2}\overline{u}_\mu^s(p')\gamma^\beta u_\mu^{s'}(p) \\
&(2\pi)^{16}\delta^4(k'-k-g)\delta^4(g-l+n)\delta^4(l-n-q)\delta^4(q+p'-p)
\end{aligned}
\tag{A.3.11}
$$

finally we get :

$$
\begin{aligned}
&(2\pi)^4\delta^4(k'-k+p'-p).e^4\int\frac{d^4n}{(2\pi)^4}\overline{u}_e^r(k')\gamma^\rho u_e^{r'}(k)\frac{ig_{\rho\nu}}{g^2}tr\left[\gamma^\nu\frac{i}{\not{l}-m_e}\gamma^\alpha\frac{i}{\not{n}-m_e}\right]\frac{-ig_{\alpha\beta}}{q^2}\overline{u}_\mu^s(p')\gamma^\beta u_\mu^{s'}(p) \\
&= i\mathcal{M}_{1,\nu p}(2\pi)^4\delta^4(p_i-p_f)
\end{aligned}
\tag{A.3.12}
$$

$$\Rightarrow i\mathcal{M}_{1,vp} = -e^4 \int \frac{d^4n}{(2\pi)^4} \bar{u}_e^r(k')\gamma^\rho u_e^{r'}(k) \frac{-ig_{\rho\nu}}{g^2} tr\left[\gamma^\nu \frac{i}{\not{l} - m_e}\gamma^\alpha \frac{i}{\not{n} - m_e}\right] \frac{-ig_{\alpha\beta}}{q^2} \bar{u}_\mu^s(p')\gamma^\beta u_\mu^{s'}(p), \tag{A.3.13}$$

with :

$$\begin{cases} g = k' - k \\ l = k' - k + n \\ q = k' - k \end{cases} . \tag{A.3.14}$$

- Similar for Eq.(A.3.6) corresponding to the second diagram, it will get further (-1) factor :

$$i\mathcal{M}_{2,vp} = -e^4 \int \frac{d^4n}{(2\pi)^4} \bar{u}_e^r(k')\gamma^\rho u_e^{r'}(k) \frac{-ig_{\rho\nu}}{g^2} tr\left[\gamma^\nu \frac{i}{\not{l} - m_\mu}\gamma^\alpha \frac{i}{\not{n} - m_\mu}\right] \frac{-ig_{\alpha\beta}}{q^2} \bar{u}_\mu^s(p')\gamma^\beta u_\mu^{s'}(p), \tag{A.3.15}$$

with :

$$\begin{cases} g = k' - k \\ l = k' - k + n \\ q = k' - k \end{cases} . \tag{A.3.16}$$

The amplitude for *Vaccum polarization* diagrams :

$$i\mathcal{M}_{vp} = i\mathcal{M}_{1,vp} + i\mathcal{M}_{2,vp}. \tag{A.3.17}$$

2. **Vertex correction**

The second diagram in fig[17] is called the *vertex correction*. Using Wick's theorem in Eq.(A.3.4) we get two diagrams in this type :

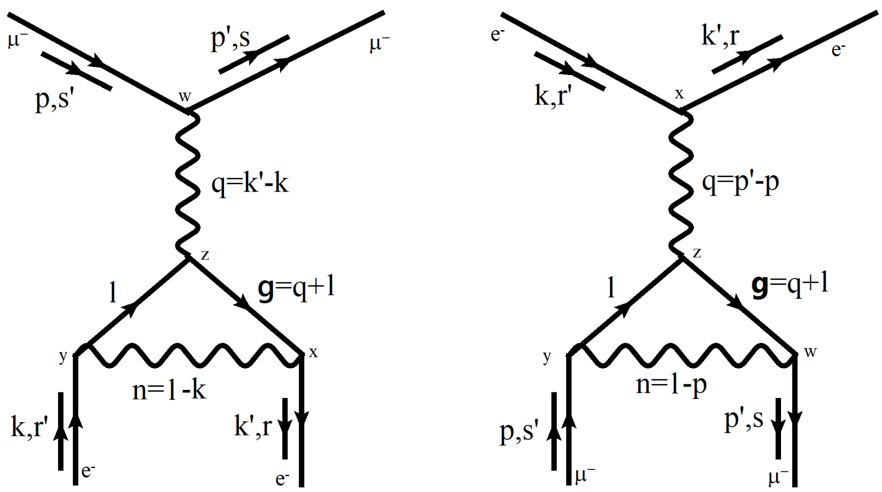


Figure 19: The Vertex correction diagrams

corresponding to two short-term below after expanding using Wick's theorem in Eq.(A.3.3) :

$$a_{p'}a_{k'}\overline{\psi}^e_x\gamma^\rho\psi^e_x A_\rho\overline{\psi}^e_y\gamma^\nu\psi^e_y A_\nu\overline{\psi}^e_z\gamma^\alpha\psi^e_z A_\alpha\overline{\psi}^\mu_w\gamma^\beta\psi^\mu_w A_\beta a^\dagger_k a^\dagger_p \tag{A.3.18}$$

$$a_{k'}a_{p'}\overline{\psi}^\mu_w\gamma^\beta\psi^\mu_w A_\beta\overline{\psi}^\mu_y\gamma^\nu\psi^\mu_y A_\nu\overline{\psi}^\mu_z\gamma^\alpha\psi^\mu_z A_\alpha\overline{\psi}^e_x\gamma^\rho\psi^e_x A_\rho a^\dagger_p a^\dagger_k \tag{A.3.19}$$

- We rewrite fully Eq.(A.3.18) :

$$\begin{aligned}&e^4\int d^4x\int d^4y\int d^4z\int d^4w\,\langle 0|\,a_{p'}a_{k'}\overline{\psi}^e_x\gamma^\rho\psi^e_x A_\rho\overline{\psi}^e_y\gamma^\nu\psi^e_y A_\nu\overline{\psi}^e_z\gamma^\alpha\psi^e_z A_\alpha\overline{\psi}^\mu_w\gamma^\beta\psi^\mu_w A_\beta a^\dagger_k a^\dagger_p\,|0\rangle\\&= iM_{1,vc}(2\pi)^4\delta^4(p_i-p_f),\end{aligned} \tag{A.3.20}$$

the $\frac{1}{4!}$ is canceled by $4!$ ways of interchanging vertices to obtain the same contraction. To untanglethe contractions, we use the formula Eq.(A.3.8) to untangle the contractions in Eq.(A.3.20), we get the results in the LHS :

$$\begin{aligned}&e^4\int d^4x\int d^4y\int d^4z\int d^4w.e^{ik'y}\bar{u}^r_e(k')\gamma^\nu\int\frac{d^4g}{(2\pi)^4}\frac{i}{\not{g}-m_e}e^{-ig(y-z)}\gamma^\alpha\int\frac{d^4l}{(2\pi)^4}\frac{i}{\not{l}-m_e}e^{-il(z-x)}\\&\gamma^\rho e^{-ikx}u^{r'}_e(k).e^{ip'w}\bar{u}^s_\mu(p')\gamma^\beta e^{-ipw}u^{s'}_\mu(p)\int\frac{d^4n}{(2\pi)^4}\frac{-ig_{\rho\nu}e^{-in(x-y)}}{n^2}\int\frac{d^4q}{(2\pi)^4}\frac{-ig_{\alpha\beta}e^{-iq(z-w)}}{q^2}\end{aligned} \tag{A.3.21}$$

$$\begin{aligned}=&e^4\int\frac{d^4g\quad d^4l\quad d^4n\quad d^4q}{(2\pi)^{16}}.\bar{u}^r_e(k')\gamma^\nu\frac{i}{\not{g}-m_e}\gamma^\alpha\frac{i}{\not{l}-m_e}\gamma^\rho u^{r'}_e(k).\bar{u}^s_\mu(p')\gamma^\beta u^{s'}_\mu(p).\frac{-g_{\rho\nu}g_{\alpha\beta}}{n^2q^2}.\\&\int d^4xe^{i(l-k-n)x}\int d^4ye^{i(k'-g+l)y}\int d^4ze^{i(g-l-q)z}\int d^4we^{i(q+p'-p)w}\end{aligned} \tag{A.3.22}$$

$$\begin{aligned}=&e^4\int\frac{d^4g\quad d^4l\quad d^4n\quad d^4q}{(2\pi)^{16}}.\bar{u}^r_e(k')\gamma^\nu\frac{i}{\not{g}-m_e}\gamma^\alpha\frac{i}{\not{l}-m_e}\gamma^\rho u^{r'}_e(k).\bar{u}^s_\mu(p')\gamma^\beta u^{s'}_\mu(p).\frac{-g_{\rho\nu}g_{\alpha\beta}}{n^2q^2}.\\&(2\pi)^{16}\delta^4(l-k-n)\delta^4(k'-g+n)\delta^4(g-l-q)\delta^4(q+p'-p),\end{aligned} \tag{A.3.23}$$

finally we get :

$$\begin{aligned}&(2\pi)^4\delta^4(k'-k+p'-p).e^4\int\frac{d^4l}{(2\pi)^4}\frac{-g_{\rho\nu}g_{\alpha\beta}}{n^2q^2}\bar{u}^r_e(k')\gamma^\nu\frac{i}{\not{g}-m_e}\gamma^\alpha\frac{i}{\not{l}-m_e}\gamma^\rho u^{r'}_e(k).\bar{u}^s_\mu(p')\gamma^\beta u^{s'}_\mu(p)\\&= i\mathcal{M}_{1,vc}(2\pi)^4\delta^4(p_i-p_f)\end{aligned} \tag{A.3.24}$$

$$\Rightarrow i\mathcal{M}_{1,vc} = e^4 \int \frac{d^4l}{(2\pi)^4} \frac{-g_{\rho\nu}g_{\alpha\beta}}{n^2q^2} \bar{u}_e^r(k')\gamma^\nu \frac{i}{\not{g}-m_e}\gamma^\alpha \frac{i}{\not{l}-m_e}\gamma^\rho u_e^{r'}(k).\bar{u}_\mu^s(p')\gamma^\beta u_\mu^{s'}(p), \tag{A.3.25}$$

with :

$$\begin{cases} n = l-k \\ g = k'-k+l \\ q = k'-k \end{cases} . \tag{A.3.26}$$

- Similar for Eq.(A.3.19) corresponding to the second diagram :

$$i\mathcal{M}_{2,vc} = e^4 \int \frac{d^4l}{(2\pi)^4} \frac{-g_{\beta\nu}g_{\alpha\rho}}{n^2q^2} \bar{u}_\mu^s(p')\gamma^\nu \frac{i}{\not{g}-m_\mu}\gamma^\alpha \frac{i}{\not{l}-m_\mu}\gamma^\beta u_\mu^{s'}(p).\bar{u}_e^r(k')\gamma^\rho u_e^{r'}(k) \tag{A.3.27}$$

with :

$$\begin{cases} n = l-p \\ g = p'-p+l \\ q = p'-p \end{cases} . \tag{A.3.28}$$

The full *Vertex correction* amplitude :

$$i\mathcal{M}_{vc} = i\mathcal{M}_{1,vc} + i\mathcal{M}_{2,vc}. \tag{A.3.29}$$

3. **External leg correction**

The third diagram in fig[17] is called *the external leg correction*. Using Wick's theorem in Eq.(A.3.4), we'll get four diagrams in this type :

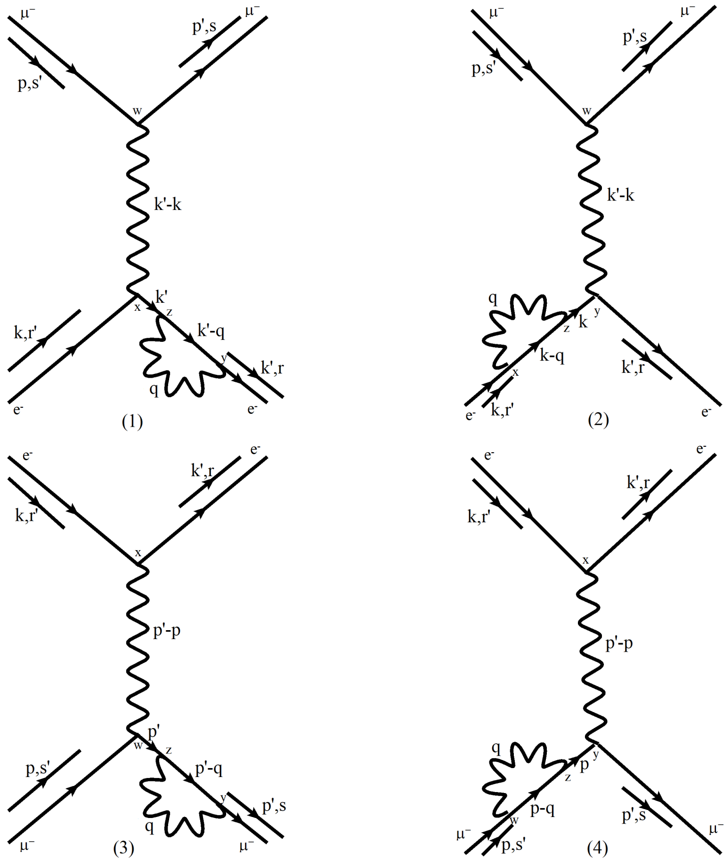


Figure 20: The External leg correction diagrams

corresponding to four contraction terms after expanding using Wick's theorem in Eq.(A.3.3) :

$$a_{p'} a_{k'} \bar{\psi}^e_x \gamma^\rho \psi^e_x A_\rho \bar{\psi}^e_y \gamma^\nu \psi^e_y A_\nu \bar{\psi}^e_z \gamma^\alpha \psi^e_z A_\alpha \bar{\psi}^\mu_w \gamma^\beta \psi^\mu_w A_\beta a^\dagger_k a^\dagger_p \tag{A.3.30}$$

$$a_{p'} a_{k'} \bar{\psi}^e_x \gamma^\rho \psi^e_x A_\rho \bar{\psi}^e_y \gamma^\nu \psi^e_y A_\nu \bar{\psi}^e_z \gamma^\alpha \psi^e_z A_\alpha \bar{\psi}^\mu_w \gamma^\beta \psi^\mu_w A_\beta a^\dagger_k a^\dagger_p \tag{A.3.31}$$

$$a_{k'} a_{p'} \bar{\psi}^\mu_w \gamma^\beta \psi^\mu_w A_\beta \bar{\psi}^\mu_y \gamma^\nu \psi^\mu_y A_\nu \bar{\psi}^\mu_z \gamma^\alpha \psi^\mu_z A_\alpha \bar{\psi}^e_x \gamma^\rho \psi^e_x A_\rho a^\dagger_p a^\dagger_k \tag{A.3.32}$$

$$a_{k'} a_{p'} \bar{\psi}^\mu_w \gamma^\beta \psi^\mu_w A_\beta \bar{\psi}^\mu_y \gamma^\nu \psi^\mu_y A_\nu \bar{\psi}^\mu_z \gamma^\alpha \psi^\mu_z A_\alpha \bar{\psi}^e_x \gamma^\rho \psi^e_x A_\rho a^\dagger_p a^\dagger_k \tag{A.3.33}$$

- We write fully Eq.(A.3.30) :

$$e^4\int d^4x\int d^4y\int d^4z\int d^4w\,\langle 0|\,a_{p'}a_{k'}\overline{\psi}^e_x\gamma^\rho\psi^e_xA_\rho\overline{\psi}^e_y\gamma^\nu\psi^e_yA_\nu\overline{\psi}^e_z\gamma^\alpha\psi^e_zA_\alpha\overline{\psi}^\mu_w\gamma^\beta\psi^\mu_wA_\beta a^\dagger_k a^\dagger_p\,|0\rangle \tag{A.3.34}$$
$$= i\mathcal{M}_{1,elc}(2\pi)^4\delta^4(p_i-p_f),$$

the $\frac{1}{4!}$ is again canceled by 4! ways of interchanging vertices to obtain the same contraction. Untangling the contractions in above equation, we get the result :

$$e^4\int d^4x\int d^4y\int d^4z\int d^4w.e^{ik'y}\bar{u}^r_e(k')\gamma^\nu\int\frac{d^4g}{(2\pi)^4}\frac{i}{\not{g}-m_e}e^{-ig(y-z)}\gamma^\alpha\int\frac{d^4l}{(2\pi)^4}\frac{i}{\not{l}-m_e}e^{-il(z-x)}$$
$$\gamma^\rho e^{-ikx}u^{r'}_e(k).e^{ip'w}\bar{u}^s_\mu(p')\gamma^\beta e^{-ipw}u^{s'}_\mu(p)\int\frac{d^4n}{(2\pi)^4}\frac{-ig_{\rho\beta}e^{-in(x-w)}}{n^2}\int\frac{d^4q}{(2\pi)^4}\frac{-ig_{\nu\alpha}e^{-iq(y-z)}}{q^2} \tag{A.3.35}$$

$$=e^4\int\frac{d^4g\quad d^4l\quad d^4n\quad d^4q}{(2\pi)^{16}}.\bar{u}^r_e(k')\gamma^\nu\frac{i}{\not{g}-m_e}\gamma^\alpha\frac{i}{\not{l}-m_e}\gamma^\rho u^{r'}_e(k).\bar{u}^s_\mu(p')\gamma^\beta u^{s'}_\mu(p).\frac{-g_{\rho\beta}g_{\nu\alpha}}{n^2q^2}.$$
$$\int d^4xe^{i(l-n-k)x}\int d^4ye^{i(k'-g-q)y}\int d^4ze^{i(g-l+q)z}\int d^4we^{i(n+p'-p)w} \tag{A.3.36}$$

$$=e^4\int\frac{d^4g\quad d^4l\quad d^4n\quad d^4q}{(2\pi)^{16}}.\bar{u}^r_e(k')\gamma^\nu\frac{i}{\not{g}-m_e}\gamma^\alpha\frac{i}{\not{l}-m_e}\gamma^\rho u^{r'}_e(k).\bar{u}^s_\mu(p')\gamma^\beta u^{s'}_\mu(p).\frac{-g_{\rho\beta}g_{\nu\alpha}}{n^2q^2}.$$
$$(2\pi)^{16}\delta^4(l-k-n)\delta^4(k'-g-q)\delta^4(g-l+q)\delta^4(n+p'-p), \tag{A.3.37}$$

finally we get :

$$(2\pi)^4\delta^4(k'-k+p'-p).e^4\int\frac{d^4q}{(2\pi)^4}\frac{-g_{\rho\beta}g_{\nu\alpha}}{n^2q^2}\bar{u}^r_e(k')\gamma^\nu\frac{i}{\not{g}-m_e}\gamma^\alpha\frac{i}{\not{l}-m_e}\gamma^\rho u^{r'}_e(k).\bar{u}^s_\mu(p')\gamma^\beta u^{s'}_\mu(p)$$
$$= i\mathcal{M}_{1,elc}(2\pi)^4\delta^4(p_i-p_f) \tag{A.3.38}$$

$$\Rightarrow i\mathcal{M}_{1,elc} = e^4\int\frac{d^4q}{(2\pi)^4}\frac{-g_{\rho\beta}g_{\nu\alpha}}{n^2q^2}\bar{u}^r_e(k')\gamma^\nu\frac{i}{\not{g}-m_e}\gamma^\alpha\frac{i}{\not{l}-m_e}\gamma^\rho u^{r'}_e(k).\bar{u}^s_\mu(p')\gamma^\beta u^{s'}_\mu(p), \tag{A.3.39}$$

with :

$$\begin{cases} l = k' \\ g = k'-q \\ n = k'-k \end{cases}. \tag{A.3.40}$$

- Similar for Eq.(A.3.31) corresponding to the second diagram :

$$i\mathcal{M}_{2,elc} = e^4 \int \frac{d^4q}{(2\pi)^4} \frac{-g_{\rho\alpha}g_{\nu\beta}}{q^2n^2} \bar{u}_e^r(k')\gamma^\nu \frac{i}{\not{g} - m_e}\gamma^\alpha \frac{i}{\not{l} - m_e}\gamma^\rho u_e^{r'}(k).\bar{u}_\mu^s(p')\gamma^\beta u_\mu^{s'}(p), \tag{A.3.41}$$

with :

$$\begin{cases} l = k - q \\ g = k \\ n = k' - k \end{cases} . \tag{A.3.42}$$

- Similar for Eq.(A.3.32) corresponding to the third diagram :

$$i\mathcal{M}_{3,elc} = e^4 \int \frac{d^4q}{(2\pi)^4} \frac{-g_{\rho\beta}g_{\nu\alpha}}{n^2q^2} \bar{u}_\mu^s(p')\gamma^\nu \frac{i}{\not{g} - m_\mu}\gamma^\alpha \frac{i}{\not{l} - m_\mu}\gamma^\beta u_\mu^{s'}(p).\bar{u}_e^r(k')\gamma^\rho u_e^{r'}(k) \tag{A.3.43}$$

with :

$$\begin{cases} l = p' \\ g = p' - q \\ n = p' - p \end{cases} . \tag{A.3.44}$$

- Similar for Eq.(A.3.33) corresponding to the fourth diagram :

$$i\mathcal{M}_{4,elc} = e^4 \int \frac{d^4q}{(2\pi)^4} \frac{-g_{\rho\alpha}g_{\nu\beta}}{q^2n^2} \bar{u}_\mu^s(p')\gamma^\nu \frac{i}{\not{g} - m_\mu}\gamma^\alpha \frac{i}{\not{l} - m_\mu}\gamma^\beta u_\mu^{s'}(p).\bar{u}_e^r(k')\gamma^\rho u_e^{r'}(k) \tag{A.3.45}$$

with :

$$\begin{cases} l = p - q \\ g = p \\ n = p' - p \end{cases} . \tag{A.3.46}$$

The full *External leg correction* amplitude :

$$i\mathcal{M}_{elc} = i\mathcal{M}_{1,elc} + i\mathcal{M}_{2,elc} + i\mathcal{M}_{3,elc} + i\mathcal{M}_{4,elc}. \tag{A.3.47}$$

4. **Box diagrams**

The fourth diagram in fig[17] is called the *Box diagram*. Using Wick's theorem in Eq.(A.3.4), we get two diagrams in this type :

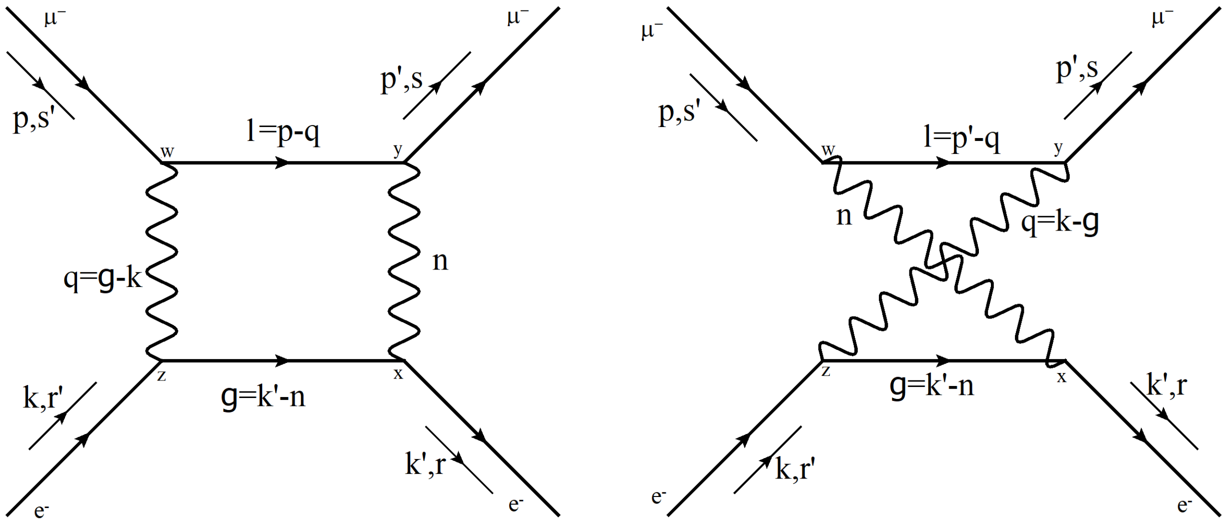


Figure 21: The Box diagrams

corresponding to two short-term below after expanding using Wick's theorem in Eq.(A.3.3) :

$$a_{p'} a_{k'} \bar{\psi}^e_x \gamma^\rho \psi^e_x A_\rho \bar{\psi}^\mu_y \gamma^\nu \psi^\mu_y A_\nu \bar{\psi}^e_z \gamma^\alpha \psi^e_z A_\alpha \bar{\psi}^\mu_w \gamma^\beta \psi^\mu_w A_\beta a^\dagger_k a^\dagger_p \tag{A.3.48}$$

$$a_{p'} a_{k'} \bar{\psi}^e_x \gamma^\rho \psi^e_x A_\rho \bar{\psi}^\mu_y \gamma^\nu \psi^\mu_y A_\nu \bar{\psi}^e_z \gamma^\alpha \psi^e_z A_\alpha \bar{\psi}^\mu_w \gamma^\beta \psi^\mu_w A_\beta a^\dagger_k a^\dagger_p \tag{A.3.49}$$

- We rewrite fully Eq.(A.3.48) :

$$\begin{aligned} &e^4 \int d^4x \int d^4y \int d^4z \int d^4w \, \langle 0| \, a_{p'} a_{k'} \bar{\psi}^e_x \gamma^\rho \psi^e_x A_\rho \bar{\psi}^\mu_y \gamma^\nu \psi^\mu_y A_\nu \bar{\psi}^e_z \gamma^\alpha \psi^e_z A_\alpha \bar{\psi}^\mu_w \gamma^\beta \psi^\mu_w A_\beta a^\dagger_k a^\dagger_p \, |0\rangle \\ &= i\mathcal{M}_{1,bd} (2\pi)^4 \delta^4 (p_i - p_f), \end{aligned} \tag{A.3.50}$$

similarly above, the $\frac{1}{4!}$ is also canceled by 4! ways of interchanging vertices to obtain the same contraction. To untangle the contractions, using also the formula Eq.(A.3.8) to untangle the contractions in Eq.(A.3.50) and we get the results :

$$\begin{aligned} &(2\pi)^4 \delta^4 (k' - k + p' - p).e^4 \int \frac{d^4n}{(2\pi)^4} \frac{-g_{\alpha\beta} g_{\rho\nu}}{q^2 n^2} \bar{u}^r_e(k') \gamma^\rho \frac{i(\not{g} + m_e)}{g^2 - m_e^2} \gamma^\alpha u^{r'}_e(k) \bar{u}^s_\mu(p') \gamma^\nu \frac{i(\not{l} + m_\mu)}{l^2 - m_\mu^2} \gamma^\beta u^{s'}_\mu(p) \\ &= i\mathcal{M}_{1,bd} (2\pi)^4 \delta^4 (p_i - p_f) \end{aligned} \tag{A.3.51}$$

$$\Rightarrow i\mathcal{M}_{1,bd} = e^4 \int \frac{d^4n}{(2\pi)^4} \frac{-g_{\alpha\beta} g_{\rho\nu}}{q^2 n^2} \bar{u}^r_e(k') \gamma^\rho \frac{i(\not{g} + m_e)}{g^2 - m_e^2} \gamma^\alpha u^{r'}_e(k) \bar{u}^s_\mu(p') \gamma^\nu \frac{i(\not{l} + m_\mu)}{l^2 - m_\mu^2} \gamma^\beta u^{s'}_\mu(p), \tag{A.3.52}$$

with :

$$\begin{cases} g = k' - n \\ q = k' - k - n = g - k \\ l = k - k' + n + p = p - q \end{cases} . \tag{A.3.53}$$

- Similar for Eq.(A.3.49) corresponding to the second diagram :

$$i\mathcal{M}_{2,bd} = e^4 \int \frac{d^4n}{(2\pi)^4} \frac{-g_{\alpha\nu}g_{\rho\beta}}{q^2n^2} \bar{u}_e^r(k')\gamma^\rho \frac{i(\not{g} + m_e)}{g^2 - m_e^2} \gamma^\alpha u_e^{r'}(k)\bar{u}_\mu^s(p')\gamma^\nu \frac{i(\not{l} + m_\mu)}{l^2 - m_\mu^2} \gamma^\beta u_\mu^{s'}(p) \tag{A.3.54}$$

with :

$$\begin{cases} g = k' - n \\ q = k - k' + n = k - g \\ l = p' + k' - k - n = p' - q \end{cases} . \tag{A.3.55}$$

The full *Box diagrams* amplitude :

$$i\mathcal{M}_{bd} = i\mathcal{M}_{1,bd} + i\mathcal{M}_{2,bd}. \tag{A.3.56}$$

And the full one-loop amplitude scattering :

$$i\mathcal{M}_{NLO} = i\mathcal{M}_{vp} + i\mathcal{M}_{vc} + i\mathcal{M}_{bd} + i\mathcal{M}_{elc}. \tag{A.3.57}$$

B

Mathematics

The formulas here are taken from [14].

B.1 Logarithms and Powers

The natural logarithm $\log(z)$ is defined as

$$\log(z) = \log(|z|) + i\arg(z), \tag{B.1.1}$$

with $-\pi < \arg(z) \leq \pi$. The logarithm $\log(z)$ has a branch cut along the negative real axis. The general power $w = z^a$ (a is a complex constant) is defined follow the exponential function :

$$z^a = \left(e^{\log(z)}\right)^a = e^{a\log(z)}. \tag{B.1.2}$$

With the above definitions having the following properties :

$$\log(z_1 z_2) = \log(z_1) + \log(z_2) + \eta(z_1, z_2), \tag{B.1.3}$$

$$\eta(z_1, z_2) = 2\pi i\left[\theta(-\mathrm{Im}z_1)\theta(-\mathrm{Im}z_2)\theta(\mathrm{Im}z_1 z_2) - \theta(\mathrm{Im}z_1)\theta(\mathrm{Im}z_2)\theta(-\mathrm{Im}z_1 z_2)\right], \tag{B.1.4}$$

$$(z_1 z_2)^a = e^{a\log(z_1 z_2)} = e^{a[\log(z_1)+\log(z_2)+\eta(z_1,z_2)]} = e^{a\eta(z_1,z_2)} z_1^a z_2^a, \tag{B.1.5}$$

so :

$$\log(z_1 z_2) = \log(z_1) + \log(z_2) \quad \text{If } \mathrm{Im}(z_1) \text{ and } \mathrm{Im}(z_2) \text{ have different sign} \tag{B.1.6}$$

$$\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2) \quad \text{If } \mathrm{Im}(z_1) \text{ and } \mathrm{Im}(z_2) \text{ have the same sign} \tag{B.1.7}$$

$$(z_1 z_2)^a = z_1^a . z_2^a \quad \text{If } \mathrm{Im}(z_1) \text{ and } \mathrm{Im}(z_2) \text{ have different sign.} \tag{B.1.8}$$

$$\tag{B.1.9}$$

For $-z = a - i\rho$ with a is real and $\rho \to 0^+$:

$$\log(-z) = \begin{cases} \log(|a|) & \text{If } a > 0 \\ \log(|a|) - i\pi & \text{If } a < 0 \end{cases} \tag{B.1.10}$$

$$\arg(-z) = \arg z - \pi \tag{B.1.11}$$

$$\log(-z) = \log(|z|) + i\arg(-z) = \log(|z|) + i\arg(z) - i\pi = \log(z) - i\pi \tag{B.1.12}$$

$$(-z)^a = e^{-i\pi a}e^{a\log(z)} = e^{-i\pi a}z^a. \tag{B.1.13}$$

If A and B are real then :

$$\log(AB - i\rho) = \log(A - i\rho') + \log(B - i\rho/A), \tag{B.1.14}$$

where ρ' is infinitesimal and has the same sign as rho. We get :

$$(AB - i\rho)^a = e^{a\log(AB - i\rho)} = e^{a[\log(A - i\rho') + \log(B - i\rho/A)]} = (A - i\rho')^a(B - i\rho/A)^a. \tag{B.1.15}$$

B.2 Dilogarithms

The dilogarithm or Spence function is defined [8]

$$Sp(z) = -\int_0^1 dt \frac{\log(1 - zt)}{t}, \tag{B.2.1}$$

where z is complex number. The dilogarithm function has a cut along the positive real axis from 1 to $+\infty$ due to branch cut of logarithm function. When one is in a problematic situation, the following transformation formulate maybe useful :

$$Sp(z) = -Sp(\frac{1}{z}) - \frac{1}{6}\pi^2 - \frac{1}{2}\log^2(-z), \tag{B.2.2}$$

$$Sp(z) = -Sp(1 - z) + \frac{1}{6}\pi^2 - \log(1 - z)\log(z). \tag{B.2.3}$$

B.3 Gamma and Beta functions

The Gamma function $\Gamma(z)$ is one commonly used extension of the factorial function to complex numbers. The gamma function is defined for all complex numbers except the non-positive integers. For positive integer n:

$$\Gamma(n + 1) = n\Gamma(n)! = n!, \tag{B.3.1}$$

follow the derivation by Daniel Bernoulli, for complex numbers z with a positive real part $\mathrm{Re}(z)$ the gamma function is defined by :

$$\Gamma(z) = \int_0^\infty dt e^{-t}t^{z-1}. \tag{B.3.2}$$

$\Gamma(z)$ is analytical everywhere, except at the points $z = 0, -1, -1, -3, \ldots$ The following properties of the gamma function are very useful :

$$\Gamma(z + 1) = z\Gamma(z) \tag{B.3.3}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{B.3.4}$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + O(\varepsilon), \qquad \varepsilon \to 0, \tag{B.3.5}$$

where $\gamma_E = -\Gamma'(1)$ is Euler constant.

The beta function is defined by

$$B(p,q) = \int_0^1 dt t^{p-1}(1-t)^{q-1}, \tag{B.3.6}$$

where Re$p > 0$ and Re$q > 0$; the principal values of the various powers are to be taken. The analytical continuation of $B(p,q)$ onto the left halves of the p and q planes is achieved by using

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{B.3.7}$$

C

D-dimensional Integral

The below formulas are cited form [7].

C.1 D-dimension properties

Transforming 4-dimension to D-dimension ($D = 4 - 2\varepsilon$, $\varepsilon \to 0$, D can be complex number) with some changed properties :

- Tensor metric $g^{\mu\nu}$ is D-dimension with $\mu, \nu = 0, 1, 2, .., D-1$.
  And $g_{\mu\nu}g^{\mu\nu} = D$.
- Dirac matrix γ^{μ}:
$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}\mathbf{1}$$
$$\mathrm{Tr}\{\mathbf{1}\} = 4.$$
- Remaining Lorentz invariant, Gauge invariant
- Integral :
$$\int \frac{d^4q}{(2\pi)^4} \to \mu^{4-D}\frac{d^Dq}{(2\pi)^D}$$
  with μ is normalized factor (mass scale of dimensional regularization).

C.2 Basic integral

$$I_n(A) = \int d^Dk \frac{1}{(k^2 - A + i\varepsilon)^n}, \qquad A > 0. \tag{C.2.1}$$

1. Wick rotation

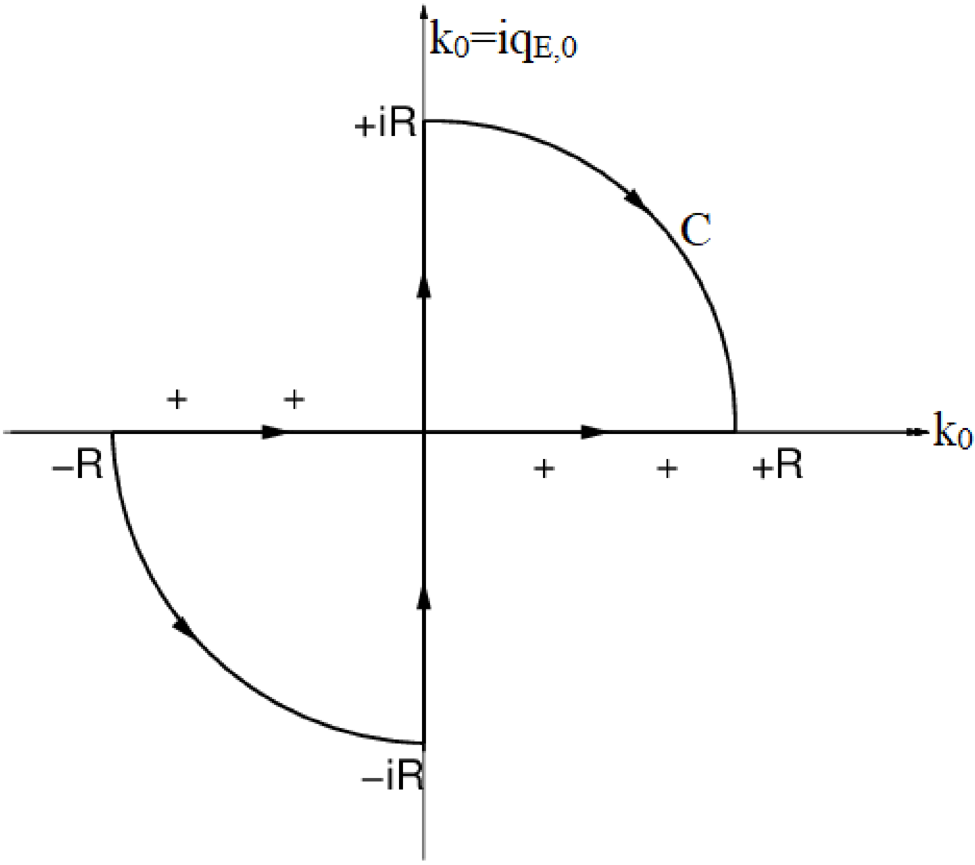


Figure 22: Wick rotation

k_0 has two poles :

$$k_0 = \pm\sqrt{\vec{k}^2 + A} \mp i\varepsilon'. \tag{C.2.2}$$

In Complex Calculus, we get the result :

$$\oint_C dk_0 (k^2 - A + i\varepsilon)^{-n} = 0 \tag{C.2.3}$$

$$\rightarrow \int_{-\infty}^{\infty} dk_0 \ldots = \int_{-i\infty}^{i\infty} dk_0 \ldots \tag{C.2.4}$$

Substitution :

$$k_0 = iq_{E,0}, \quad \vec{k} = \vec{q}_E, \tag{C.2.5}$$

$$k^2 = -q_E^2 \leq 0, \tag{C.2.6}$$

$$\int_{-i\infty}^{i\infty} dk_0 \ldots = i\int_{-\infty}^{\infty} dq_{E,0} \ldots \tag{C.2.7}$$

$$\rightarrow I_n(A) = \int_{-\infty}^{\infty} dk_0 \int d^{D-1}\vec{k}(k^2 - A + i\varepsilon)^{-n} \tag{C.2.8}$$

$$= i\int_{-\infty}^{\infty} dq_{E,0}\int d^{D-1}\vec{q}_E(-q_{E,0}^2-\vec{q}_E^{\,2}-A+i\varepsilon)^{-n} \tag{C.2.9}$$
$$= i\int d^D q_E(-1+i\varepsilon')^n(q_E^2+A-i\varepsilon)^{-n} = i\int d^D q_E(-1)^n(q_E^2+A-i\varepsilon)^{-n}, \tag{C.2.10}$$

with n is a integer .

2. Integrals in D-Dimensional Euclid space

$$\int d^D q_E = \int_{\Omega_D} d\Omega_D\int_0^{\infty} dq_E q_E^{D-1} = \int_{\Omega_D} d\Omega_D\int_0^{\infty} dq_E^2\frac{1}{2}(q_E^2)^{D-1}, \tag{C.2.11}$$

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} = \text{D-dimensional solid angle.} \tag{C.2.12}$$

Proof :

$$\sqrt{\pi}^D = (\int_0^{\infty} dx.e^{-x^2})^D = \int dx_1...dx_D.e^{-\sum_i x_i^2} = \int d^D x.e^{-x^2} = \int d\Omega_D\int_0^{\infty} dx x^{D-1}e^{-x^2} \tag{C.2.13}$$
$$= \int d\Omega_D\frac{1}{2}\int_0^{\infty} dx^2(x^2)^{(D-2)/2}e^{-x^2} = \int d\Omega_D\frac{1}{2}\Gamma(D/2) \to \Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{C.2.14}$$

$$\to I_n(A) = i(-1)^n\Omega_D\int_0^{\infty} dq_E^2\frac{1}{2}(q_E^2)^{D/2-1}(q_E^2+A-i\varepsilon)^{-n} \tag{C.2.15}$$
$$= i(-1)^n\frac{2\pi^{D/2}}{\Gamma(D/2)}\int_0^{\infty} dx\frac{1}{2}x^{D/2-1}(x+A-i\varepsilon)^{-n} \tag{C.2.16}$$
$$= i(-1)^n\frac{\pi^{D/2}}{\Gamma(D/2)}(A-i\varepsilon)^{D/2-n}\int_0^{\infty} dy(y+1)^{-n}y^{D/2-1} \qquad \text{set} : y = \frac{x}{A-i\varepsilon} \tag{C.2.17}$$
$$= i(-1)^n\frac{\pi^{D/2}}{\Gamma(D/2)}(A-i\varepsilon)^{D/2-n}B\left(\frac{D}{2},n-\frac{D}{2}\right) \qquad \text{Beta function} : B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{C.2.18}$$
$$= i(-1)^n\pi^{D/2}\frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)}(A-i\varepsilon)^{D/2-n}. \tag{C.2.19}$$

Similarly, another D-dimensional integrals [5] :

$$\int\frac{d^Dq}{(2\pi)^D}\frac{q^2}{(q^2-A)^n} = \frac{i(-1)^{n-1}}{(4\pi)^{D/2}}\frac{D}{2}\frac{\Gamma(n-\frac{D}{2}-1)}{\Gamma(n)}(A)^{1+\frac{D}{2}-n}, \tag{C.2.20}$$
$$\int\frac{d^Dq}{(2\pi)^D}\frac{q^\mu q^\nu}{(q^2-A)^n} = \frac{i(-1)^{n-1}}{(4\pi)^{D/2}}\frac{g^{\mu\nu}}{2}\frac{\Gamma(n-\frac{D}{2}-1)}{\Gamma(n)}(A)^{1+\frac{D}{2}-n}, \tag{C.2.21}$$
$$\int\frac{d^Dq}{(2\pi)^D}\frac{(q^2)^2}{(q^2-A)^n} = \frac{i(-1)^n}{(4\pi)^{D/2}}\frac{D(D+2)}{4}\frac{\Gamma(n-\frac{D}{2}-2)}{\Gamma(n)}(A)^{2+\frac{D}{2}-n}, \tag{C.2.22}$$
$$\int\frac{d^Dq}{(2\pi)^D}\frac{q^\mu q^\nu q^\rho q^\eta}{(q^2-A)^n} = \frac{i(-1)^n}{(4\pi)^{D/2}}\frac{\Gamma(n-\frac{D}{2}-2)}{\Gamma(n)}(A)^{2+\frac{D}{2}-n}\frac{1}{4}(g^{\mu\nu}g^{\rho\eta}+g^{\mu\rho}g^{\nu\eta}+g^{\mu\eta}g^{\rho\nu}). \tag{C.2.23}$$

In some cases, these integrals will be UV-divergent behaviour when $D \to 4$, we should use the following expanded identities :

$$A^x = 1 + x \log(A) + O(x^2) \tag{C.2.24}$$

$$\Gamma(x) = \frac{1}{x} - \gamma_E + O(x), \tag{C.2.25}$$

with $x \to 0$.

3. Feynman parametrization :
   Using Feynman's trick to transform complicated integrals to basic integral :

$$\frac{1}{A_1 A_2} = \int_0^1 dx \frac{1}{(A_1 x + (1-x)A_2)^2}, \tag{C.2.26}$$

general formula :

$$\frac{1}{A_1 \ldots A_n} = (n-1)! \int_0^1 dx_1 \ldots \int_0^1 dx_n \frac{\delta(1 - \sum_{k=1}^n x_k)}{(\sum_{k=1}^n x_k A_k)^n}. \tag{C.2.27}$$

D

N-POINT INTEGRALS

D.1 One-point function

$$A_0(m) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq(q^2 - m^2 + i\varepsilon)^{-1} = \frac{(2\pi\mu)^{4-D}}{i\pi^2} I_1(m^2) = -m^2\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}}\Gamma(\frac{2-D}{2}), \qquad \text{(D.1.1)}$$

when $D \to 4$:

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}} = 1 + \frac{D-4}{2}\log\left(\frac{m^2}{4\pi\mu^2}\right) + O((D-4)^2), \qquad \text{(D.1.2)}$$

$$\Gamma\left(\frac{2-D}{2}\right) = -\left(\frac{2}{4-D} - \gamma_E + 1\right) + O(D-4), \qquad \text{(D.1.3)}$$

$$\Rightarrow A_0(m) = m^2\left[\underbrace{\frac{2}{4-D} - \gamma_E + \log(4\pi)}_{=\triangle} - \log\left(\frac{m^2}{\mu^2}\right) + 1\right] + O(D-4) \qquad \text{(D.1.4)}$$

$$= m^2\left[\triangle - \log\left(\frac{m^2}{\mu^2}\right) + 1\right] + O(D-4). \qquad \text{(D.1.5)}$$

Note : $(D-4)A_0(m) = -2m^2 + O(D-4)$.

D.2 Two-point functions

D.2.1 Scalar

$$B_0(p, m_0, m) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq \frac{1}{(q^2 - m_0^2 + i\varepsilon)[(q+p)^2 - m^2 + i\varepsilon]}, \qquad \text{(D.2.1)}$$

using Feynman parametrization :

$$\to B_0(p, m_0, m) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\int_0^1 dx \frac{1}{[(q+xp)^2 - x^2p^2 + x(p^2 - m^2 + m_0^2) - m_0^2 + i\varepsilon]^2} \qquad \text{(D.2.2)}$$

$$= \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int_0^1 \underbrace{\int d^Dq' \frac{1}{(q'^2 - A + i\varepsilon)^2}}_{=I_2(A)} \qquad \text{set} : q' = q + xp \tag{D.2.3}$$

$$= (4\pi\mu^2)^{\frac{4-D}{2}}\Gamma\left(\frac{4-D}{2}\right)\int_0^1 dx \left[x^2p^2 - x(p^2 - m^2 + m_0^2) + m_0^2 - i\varepsilon\right]^{\frac{D-4}{2}} \tag{D.2.4}$$

$$= \triangle - \int_0^1 dx \log\left[\frac{x^2p^2 - x(p^2 - m^2 + m_0^2) + m_0^2 - i\varepsilon}{\mu^2}\right] + O(D-4) \tag{D.2.5}$$

$$= \triangle + 2 - \log\left(\frac{m^2}{\mu^2}\right) + X_1 \log\left(\frac{X_1 - 1}{X_1}\right) + X_2 \log\left(\frac{X_2 - 1}{X_2}\right) \tag{D.2.6}$$

$$\text{with: } X_1 = \frac{p^2 + i\varepsilon + m_0^2 - m^2 + \sqrt{(p^2 + i\varepsilon - m_0^2 - m^2)^2 - 4m_0^2m^2}}{2p^2}, \tag{D.2.7}$$

$$X_2 = \frac{p^2 + i\varepsilon + m_0^2 - m^2 - \sqrt{(p^2 + i\varepsilon - m_0^2 - m^2)^2 - 4m_0^2m^2}}{2p^2}. \tag{D.2.8}$$

Alternative notation $B_0(p, m_0, m) \to B_0(p^2, m_0, m)$ and $(D-4)B_0 = -2 + O(D-4)$. Some special cases of Two-point fucntions :

$$B_0(p^2, 0, m) = \triangle - \log\left(\frac{m^2}{\mu^2}\right) + 2 + \frac{m^2 - p^2}{p^2}\log\left(\frac{m^2 - p^2 - i\varepsilon}{m^2}\right) \tag{D.2.9}$$

$$B_0(0, 0, m) = \triangle - \log\left(\frac{m^2}{\mu^2}\right) + 1 \tag{D.2.10}$$

$$B_0(m^2, 0, m) = \triangle - \log\left(\frac{m^2}{\mu^2}\right) + 2 \tag{D.2.11}$$

$$B_0(p^2, 0, 0) = \triangle - \log\left(\frac{-p^2 - i\varepsilon}{\mu^2}\right) + 2 \tag{D.2.12}$$

$$B_0(0, m, m) = \triangle - \log\left(\frac{m^2}{\mu^2}\right) \tag{D.2.13}$$

$$A_0(m) = m^2 B_0(0, 0, m) = m^2(B_0(0, m, m) + 1) \tag{D.2.14}$$

$$B_0(m^2, 0, m) = B_0(0, m, m) + 2. \tag{D.2.15}$$

D.2.2 Tensor

In all Tensor integral cases, we should use the Passarino-Veltman reduction method [7].

1.

$$B_\mu(p_1, m_0, m_1) = \langle| \frac{q_\mu}{(q^2 - m_0^2)\left[(q + p_1)^2 - m_1^2\right]} |\rangle_q = p_{1\mu}B_1(p_1, m_0, m_1) \tag{D.2.16}$$

Multiply by p_1^μ :

$$p_1^2 B_1(p_1^2, m_0, m_1) = \langle| \frac{qp_1}{(q^2 - m_0^2)\left[(q + p_1)^2 - m_1^2\right]} |\rangle_q \tag{D.2.17}$$

$$= \langle| \frac{\frac{1}{2}\left[(q + p_1)^2 - m_1^2\right] - \frac{1}{2}(q^2 - m_0^2) - \frac{1}{2}(p_1^2 - m_1^2 + m_0^2)}{(q^2 - m_0^2)\left[(q + p_1)^2 - m_1^2\right]} |\rangle_q \tag{D.2.18}$$

$$= \frac{1}{2}A(m_0) - \frac{1}{2}A(m_1) - \frac{1}{2}(p_1^2 - m_1^2 + m_0^2)B_0(p_1^2, m_0, m_1) \tag{D.2.19}$$

$$\Rightarrow B_1(p_1^2, m_0, m_1) = \frac{1}{2p^2}\left[A(m_0) - A(m_1) - (p_1^2 - m_1^2 + m_0^2)B_0(p_1^2, m_0, m_1)\right]. \tag{D.2.20}$$

When $D \to 4 : (D-4)B_1(p_1^2, m_0, m_1) = 1 + 0(D-4)$.

2.

$$B_{\mu\nu}(p_1, m_0, m_1) = \langle| \frac{q_\mu q_\nu}{(q^2 - m_0^2)\left[(q+p_1)^2 - m_1^2\right]} |\rangle_q = g_{\mu\nu}B_{00}(p_1^2, m_0, m_1) + p_{1\mu}p_{1\nu}B_{11}(p_1^2, m_0, m_1) \tag{D.2.21}$$

- Multiply by $g^{\mu\nu}$:

$$\begin{aligned} DB_{00}(p_1^2, m_0, m_1) + p_1^2 B_{11}(p_1^2, m_0, m_1) &= \langle| \frac{(q^2 - m_0^2) + m_0^2}{(q^2 - m_0^2)\left[(q+p_1)^2 - m_1^2\right]} |\rangle_q \\ &= A_0(m_1) + m_0^2 B_1(p_1^2, m_0, m_1) \end{aligned} \tag{D.2.22}$$

- Multiply by p_1^μ

$$\begin{aligned} & p_{1\nu}\left[B_{00}(p_1^2, m_0, m_1) + p_1^2 B_{11}(p_1^2, m_0, m_1)\right] \\ &= \langle| \frac{q_\nu\left\{\frac{1}{2}\left[(q+p_1)^2 - m_1^2\right] - \frac{1}{2}(q^2 - m_0^2) - \frac{1}{2}(p_1^2 - m_1^2 + m_0^2)\right\}}{(q^2 - m_0^2)\left[(q+p_1)^2 - m_1^2\right]} |\rangle_1 \\ &= p_{1\nu}\left[\frac{1}{2}A_0(m_1) - \frac{1}{2}(p_1^2 - m_1^2 + m_0^2)B_1(p_1^2, m_0, m_1)\right], \end{aligned} \tag{D.2.23}$$

we get set of equations :

$$\begin{cases} DB_{00}(p_1^2, m_0, m_1) + p_1^2 B_{11}(p_1^2, m_0, m_1) = A_0(m_1) + m_0^2 B_1(p_1^2, m_0, m_1) \\ B_{00}(p_1^2, m_0, m_1) + p_1^2 B_{11}(p_1^2, m_0, m_1) = \frac{1}{2}A_0(m_1) - \frac{1}{2}(p_1^2 - m_1^2 + m_0^2)B_1(p_1^2, m_0, m_1) \end{cases}, \tag{D.2.24}$$

$$\Rightarrow \begin{cases} B_{00} = \frac{1}{2(D-1)}\left[A_0(m_1) + 2m_0^2 B_0(p_1^2, m_0, m_1) + (p_1^2 - m_1^2 + m_0^2)B_1(p_1^2, m_0, m_1)\right] \\ B_{11} = \frac{1}{2(D-1)p_1^2}\left[(D-2)A_0(m_1) - 2m_0^2 B_0(p_1^2, m_0, m_1) - D(p_1^2 - m_1^2 + m_0^2)B_1(p_1^2, m_0, m_1)\right] \end{cases}. \tag{D.2.25}$$

When $D \to 4$:

$$\begin{cases} B_{00} = \frac{1}{6}\left[A_0(m_1) + 2m_0^2 B_0 + (p_1^2 - m_1^2 + m_0^2)B_1 + m_0^2 + m_1^2 - \frac{p_1^2}{3}\right] \\ B_{11} = \frac{1}{6p_1^2}\left[2A_0(m_1) - 2m_0^2 B_0 - 4(p_1^2 - m_1^2 + m_0^2)B_1 - m_0^2 - m_1^2 + \frac{p_1^2}{3}\right] \end{cases}, \tag{D.2.26}$$

and :

$$\begin{cases} (D-4)B_{00}(p_1^2,m_0,m_1) = \frac{-1}{6}\left(p_1^2-3m_0^2-3m_1^2\right)+O(D-4) \\ (D-4)B_{11}(p_1^2,m_0,m_1) = \frac{2}{3} \qquad\qquad\qquad\qquad\quad +O(D-4) \end{cases}. \tag{D.2.27}$$

D.2.3 The derivative of two-point function

1.

$$B_0(p^2,\lambda,m) = \langle|\,(q^2-\lambda^2)\left[(q+p)^2-m^2\right]|\rangle_q \tag{D.2.28}$$

$$B_0'(p^2,\lambda,m)\Big|_{p^2=m^2} = \frac{\partial B_0(p^2,\lambda,m)}{\partial p^2}\Big|_{p^2=m} \tag{D.2.29}$$

$$= \frac{\partial}{\partial p^2}\left\{-\int_0^1 dx \quad \log\left[\frac{x^2p^2-x(p^2-m^2+\lambda^2)+\lambda^2+i\varepsilon}{\mu^2}\right]\right\}\Big|_{p^2=m^2} \tag{D.2.30}$$

$$= \frac{\partial}{\partial p^2}\left\{-\int_0^1 dx \quad \log\left[\frac{x^2p^2}{m^2}-x\frac{p^2-m^2+\lambda^2}{m^2}+\frac{\lambda^2}{m^2}-i\varepsilon\right]\right\}\Big|_{p^2=m^2} \tag{D.2.31}$$

$$= -\int_0^1 dx\frac{1}{m^2}\frac{x^2-x}{x^2-\frac{\lambda^2}{m^2}x+\frac{\lambda^2}{m^2}-i\varepsilon} \approx -\int_0^1 dx\frac{1}{m^2}\frac{x^2-x}{x^2+\frac{\lambda^2}{m^2}-i\varepsilon} \tag{D.2.32}$$

Because $\frac{\partial B_0(p^2,\lambda,m)}{\partial p^2}\Big|_{p^2=m^2}$ is real,

proof:

$$\frac{\partial B_0(p^2,\lambda,m)}{\partial p^2}\Big|_{p^2=m^2} - \frac{\partial B_0^*(p^2,\lambda,m)}{\partial p^2}\Big|_{p^2=m^2} = -\int_0^1 dx\frac{1}{m^2}\left[\frac{x^2-x}{x^2-\frac{\lambda^2}{m^2}x+\frac{\lambda^2}{m^2}-i\varepsilon}-\frac{x^2-x}{x^2-\frac{\lambda^2}{m^2}x+\frac{\lambda^2}{m^2}+i\varepsilon}\right] \tag{D.2.33}$$

$$= -2i\varepsilon\int_0^1 dx\frac{1}{m^2}\frac{x^2-x}{\left|x^2-\frac{\lambda^2}{m^2}x+\frac{\lambda^2}{m^2}-i\varepsilon\right|^2} \quad \xrightarrow{\varepsilon\to 0} \quad 0. \tag{D.2.34}$$

So equation Eq.(D.2.32) become :

$$Re\left[-\int_0^1 dx\frac{1}{m^2}\frac{x^2-x}{\left(x-\frac{i\lambda}{m}\right)\left(x+\frac{i\lambda}{m}\right)}\right] = Re\left[-\frac{1}{2m^2}\int_0^1 dx\left(\frac{x-1}{x-\frac{i\lambda}{m}}+\frac{x-1}{x+\frac{i\lambda}{m}}\right)\right] \tag{D.2.35}$$

$$\begin{aligned} &= -\frac{1}{2m^2}Re\left\{1+\left(\frac{i\lambda}{m}-1\right)\left[\log\left(1-\frac{i\lambda}{m}\right)-\log\left(\frac{-i\lambda}{m}\right)\right]\right. \\ &\qquad\left.+1+\left(\frac{-i\lambda}{m}-1\right)\left[\log\left(1+\frac{i\lambda}{m}\right)-\log\left(\frac{i\lambda}{m}\right)\right]\right\} \end{aligned} \tag{D.2.36}$$

$$= \frac{-1}{2m^2}\left[2+2\log\left(\frac{\lambda}{m}\right)\right] = \frac{-1}{m^2}\left[\log\left(\frac{\lambda}{m}\right)+1\right]. \tag{D.2.37}$$

2.

$$B_0'(0,m,m) = \frac{\partial}{\partial p^2} B_0(p^2,m,m)\Big|_{p^2=0} = \frac{1}{6m^2}. \tag{D.2.38}$$

D.3 Three-point functions

D.3.1 Scalar

1.

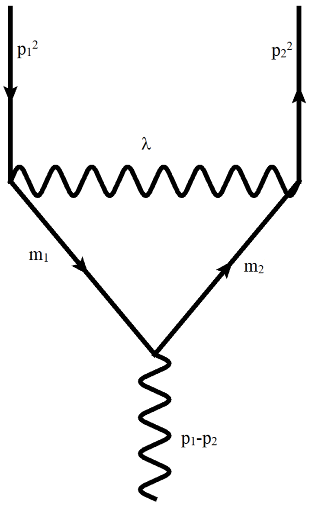


Figure 23: Singularity Three-point function

We will use the mass-regularization to compute it :

$$C_0(p_1,p_2,\lambda,m,m) = \frac{1}{i\pi^2}\int d^4q \left\{(q^2-\lambda^2)[(q+p_1)^2-m^2][(q+p_2)^2-m^2]\right\}^{-1} \tag{D.3.1}$$

$$= \langle| \left\{(q^2-\lambda^2)[(q+p_1)^2-m^2][(q+p_2)^2-m^2]\right\}^{-1} |\rangle_q \tag{D.3.2}$$

Using the Feynman parametrization [15]:

$$\left\{(q^2-\lambda^2)[(q+p_1)^2-m^2][(q+p_2)^2-m^2]\right\}^{-1} \tag{D.3.3}$$

$$= 2\int_0^1 dx \int_0^{1-x} dy \left\{(q^2-\lambda^2)x + [(q+p_1)^2-m^2](1-x-y) + [(q+p_2)^2-m^2]y\right\}^{-3} \tag{D.3.4}$$

$$= 2\int_0^1 dx \int_0^{1-x} dy \left\{[q+p_1(1-x-y)+yp_2]^2 - p_1^2(1-x-y)^2 - y^2p_2^2 - 2p_1p_2y(1-x-y) - \lambda^2 x + i\varepsilon\right\}^{-3} \tag{D.3.5}$$

with $p_1^2 = p_2^2 = m^2$.

$$\Rightarrow C_0(p_1,p_2,\lambda,m,m) \tag{D.3.6}$$

$$= 2\int_0^1 dx \int_0^{1-x} dy \, \langle| \left\{[q+p_1(1-x-y)+yp_2]^2 - p_1^2(1-x-y)^2\right. \tag{D.3.7}$$

$$
-y^2p_2^2 - 2p_1p_2y(1-x-y) - \lambda^2 x + i\varepsilon\Big\}^{-3}|\rangle_q
$$

$$
= -\int_0^1 dx \int_0^{1-x} dy \left\{[p_1(1-x-y)+yp_2]^2 + \lambda^2 x - i\varepsilon\right\}^{-1} \tag{D.3.8}
$$

$$
= -\int_0^1 dx \int_0^{x} dy \left\{[p_1(x-y)+yp_2]^2 + (1-x)\lambda^2 - i\varepsilon\right\}^{-1} \tag{D.3.9}
$$

$$
= -\int_0^1 dx \int_0^{1} dy \quad x\left\{[p_1x(1-y)+xyp_2]^2 + (1-x)\lambda^2 - i\varepsilon\right\}^{-1} \qquad (\text{set } y_n = \frac{y_o}{x}) \tag{D.3.10}
$$

$$
= -\int_0^1 dx \int_0^{1} dy \frac{y}{y^2\,[p_1(1-x)+xp_2]^2 + (1-y)\lambda^2 - i\varepsilon} \qquad (x \leftrightarrow y). \tag{D.3.11}
$$

Set $P_x^2 = [p_1(1-x)+xp_2]^2$ and $\bar{P}_x^2 = [p_1(1-x)+xp_2]^2 - i\varepsilon$:

$$
\Rightarrow C_0(p_1,p_2,\lambda,m,m) = -\int_0^1 dx \int_0^1 dy \frac{y}{y^2\bar{P}_x^2 + (1-y)\lambda^2} \tag{D.3.12}
$$

$$
= -\int_0^1 dx \int_0^1 dy \frac{1}{2\bar{P}_x^2}\left[\frac{2y\bar{P}_x^2 - \lambda^2}{y^2\bar{P}_x^2 + (1-y)\lambda^2} + \frac{\lambda^2}{y^2\bar{P}_x^2 + (1-y)\lambda^2}\right], \tag{D.3.13}
$$

take the limit $\lambda \to 0$, the second term be vanished :

$$
C_0(p_1,p_2,\lambda,m,m) = -\int_0^1 dx \frac{1}{2\bar{P}_x^2}\left(\log \bar{P}_x^2 - \log \lambda^2\right) \tag{D.3.14}
$$

$$
= \log\left(\frac{\lambda^2}{m^2}\right)\int_0^1 dx \frac{1}{2\bar{P}_x^2} - \int_0^1 dx \frac{1}{2\bar{P}_x^2} \log \frac{\bar{P}_x^2}{m^2}. \tag{D.3.15}
$$

Calculating the divergent term $\log\left(\frac{\lambda^2}{m^2}\right)\int_0^1 dx \frac{1}{2\bar{P}_x^2}$:

$$
\int_0^1 \frac{dx}{\bar{P}_x^2} = \int_0^1 \frac{dx}{[p_1(1-x)+xp_2]^2 - i\varepsilon} = \int_0^1 \frac{dx}{\bar{t}x^2 - \bar{t}x + m^2} = \int_0^1 \frac{dx}{\bar{t}\,(x-x_1)\,(x-x_2)}, \tag{D.3.16}
$$

with x_1, x_2 is solution of $\bar{P}_x^2 = m^2 - \bar{t}x + \bar{t}x^2 = 0$ and $\bar{t} = t + i\varepsilon = (p_1-p_2)^2 + i\varepsilon$.

$$
\Rightarrow \int_0^1 \frac{dx}{\bar{P}_x^2} = \int_0^1 \frac{dx}{\bar{t}\,(x_1-x_2)}\left[\frac{1}{x-x_1} - \frac{1}{x-x_2}\right] = \frac{1}{\bar{t}\,(x_1-x_2)}\left[\log\left(\frac{x_1-1}{x_1}\right) - \log\left(\frac{x_2-1}{x_2}\right)\right]. \tag{D.3.17}
$$

With:

$$
x_t = \frac{\sqrt{1-4m^2/\bar{t}}-1}{\sqrt{1-4m^2/\bar{t}}+1} = \frac{x_1-1}{x_1} = \frac{x_2}{x_2-1} \tag{D.3.18}
$$

$$
\text{and:} \qquad \frac{x_t}{m^2(1-x_t^2)} = \frac{1}{\bar{t}\,(x_2-x_1)}. \tag{D.3.19}
$$

$$
\Rightarrow \int_0^1 \frac{dx}{\bar{P}_x^2} = \frac{-x_t}{m^2\,(1-x_t^2)} 2\log(x_t), \tag{D.3.20}
$$

we get final result of divergent term containing the singularity of λ :

$$\log\left(\frac{\lambda^2}{m^2}\right)\int_0^1 dx\frac{1}{2\bar{P}_x^2}=\frac{-x_t}{m^2\left(1-x_t^2\right)}\log(x_t)\log\left(\frac{\lambda^2}{m^2}\right). \tag{D.3.21}$$

Calcualting the finite term $-\int_0^1 dx\frac{1}{2\bar{P}_x^2}\log\frac{\bar{P}_x^2}{m^2}$:

$$-\int_0^1 dx\frac{1}{2\bar{P}_x^2}\log\frac{\bar{P}_x^2}{m^2}=-\int_0^1\frac{dx}{2}\frac{1}{\bar{t}\,(x_1-x_2)}\left[\frac{1}{x-x_1}-\frac{1}{x-x_2}\right]\left[\log(x-x_1)+\log(x-x_2)+\log\frac{\bar{t}}{m^2}\right] \tag{D.3.22}$$

$$=\frac{x_t}{m^2(1-x_t^2)}\int_0^1\frac{dx}{2}\left\{\left[\frac{1}{x-x_1}-\frac{1}{x-x_2}\right][\log(x-x_1)+\log(x-x_2)]+\left[\frac{1}{x-x_1}-\frac{1}{x-x_2}\right]\log\frac{\bar{t}}{m^2}\right\} \tag{D.3.23}$$

$$=\frac{x_t}{2m^2(1-x_t^2)}\left\{\left[\log\left(\frac{x_1-1}{x_1}\right)-\log\left(\frac{x_2-1}{x_2}\right)\right]\log\frac{\bar{t}}{m^2}\right\}+\frac{x_t}{2m^2(1-x_t^2)}\int_0^1 dx\left[\frac{1}{x-x_1}-\frac{1}{x-x_2}\right] \tag{D.3.24}$$

$$\times\left\{[\log(x-x_1)+\log(x-x_2)-\log(x_1-x_2)-\log(x_2-x_1)]+[\log(x_2-x_1)+\log(x_1-x_2)]\right\} \tag{D.3.25}$$

$$=\frac{x_t}{2m^2(1-x_t^2)}\left\{\left[\log\frac{\bar{t}}{m^2}+\log(x_1-x_2)+\log(x_2-x_1)\right]2\log(x_t)\right\} \tag{D.3.26}$$

$$+\frac{x_t}{2m^2(1-x_t^2)}\int_0^1 dx\left[\frac{1}{x-x_1}-\frac{1}{x-x_2}\right][\log(x-x_1)+\log(x-x_2)-\log(x_1-x_2)-\log(x_2-x_1)]. \tag{D.3.27}$$

We have :

$$\frac{\partial}{\partial z}\mathrm{Li}_2\left(\frac{z-a}{z-b}\right)=-\left(\frac{1}{z-a}-\frac{1}{z-b}\right)\log\left(\frac{a-b}{z-b}\right) \tag{D.3.28}$$

$$\Rightarrow -\mathrm{Li}_2\left(\frac{A-a}{A-b}\right)+\mathrm{Li}_2\left(\frac{a}{b}\right)=\int_0^A dz\left(\frac{1}{z-a}-\frac{1}{z-b}\right)[\log(a-b)-\log(z-b)]. \tag{D.3.29}$$

$$\Rightarrow -\int_0^1 dx\frac{1}{2\bar{P}_x^2}\log\frac{\bar{P}_x^2}{m^2}=\frac{x_t}{2m^2(1-x_t^2)}\left\{\left[\log\frac{\bar{t}}{m^2}+\log(x_1-x_2)+\log(x_2-x_1)\right]2\log(x_t)\right. \tag{D.3.30}$$

$$\left.+\left[\mathrm{Li}_2\frac{x_2}{x_1}-\mathrm{Li}_2\frac{1-x_2}{1-x_1}+\mathrm{Li}_2\frac{1-x_1}{1-x_2}-\mathrm{Li}_2\frac{x_1}{x_2}\right]\right\} \tag{D.3.31}$$

$$=\frac{x_t}{2m^2(1-x_t^2)}\left\{\log\left[\frac{-\bar{t}}{m^2}(x_1-x_2)^2\right]2\log(x_t)+\left[\mathrm{Li}_2(-x_t)-\mathrm{Li}_2\left(\frac{-1}{x_t}\right)+\mathrm{Li}_2(-x_t)-\mathrm{Li}_2\left(\frac{-1}{x_t}\right)\right]\right\} \tag{D.3.32}$$

$$=\frac{x_t}{2m^2(1-x_t^2)}\left[2\log(x_t)\log\frac{(1+x_t)^2}{x_t}+4\mathrm{Li}_2(-x_t)+\frac{2\pi^2}{6}+\log^2(x_t)\right] \tag{D.3.33}$$

$$=\frac{x_t}{m^2(1-x_t^2)}\left\{\log(x_t)\left[2\log(1+x_t)-\frac{1}{2}\log(x_t)\right]+2\mathrm{Li}_2(-x_t)+\frac{\pi^2}{6}\right\}. \tag{D.3.34}$$

We used some below correlation :

$$\log\frac{\bar{t}}{m^2}+\log(x_1-x_2)+\log(x_2-x_1)=\log\left[\frac{-\bar{t}}{m^2}(x_1-x_2)^2\right] \tag{D.3.35}$$

$$\mathrm{Li}_2(-x_t)-\mathrm{Li}_2\left(\frac{-1}{x_t}\right)+\mathrm{Li}_2(-x_t)-\mathrm{Li}_2\left(\frac{-1}{x_t}\right)=4\mathrm{Li}_2(-x_t)+\frac{2\pi^2}{6}+\log^2 x_t \tag{D.3.36}$$

$$\log\left[\frac{-\bar{t}}{m^2}(x_1-x_2)^2\right]=\log(1+x_t)^2-\log(x_t) \tag{D.3.37}$$

$$-\mathrm{Li}_2\frac{1}{z}-\frac{\pi^2}{6}-\frac{1}{2}\log^2(-z)=\mathrm{Li}_2(z). \tag{D.3.38}$$

2. Convergent three-point function

$$C_0(q,p,0,0,m)=\langle|\frac{1}{n^2(n+q)^2\left[(n+p)^2-m^2\right]}|\rangle_n \qquad \text{with : } p^2=m^2 \tag{D.3.39}$$

$$=-\int_0^1 dx\int_0^1 dyx\left\{[px(1-y)+yxq]^2-q^2yx\right\}^{-1} \tag{D.3.40}$$

$$=-\int_0^1 dx\int_0^1 dy\left\{x\left[p(1-y)+yq\right]^2-q^2y\right\}^{-1} \tag{D.3.41}$$

$$=-\int_0^1 dy\frac{\log\left[\frac{(p(1-y)+yq)^2-q^2y}{-q^2y}\right]}{(p(1-y)+yq)^2} \qquad \text{apply : } pq=\frac{q^2}{2} \tag{D.3.42}$$

$$=-\int_0^1 dy\frac{\log\left[\frac{p^2(1-y)^2}{-q^2y}\right]}{p^2(1-y)^2+q^2y}=-\int_0^1 dy\frac{\log\left[\frac{p^2y^2}{-q^2(1-y)}\right]}{p^2y^2+q^2(1-y)}. \tag{D.3.43}$$

Consider :

$$\int_0^1\frac{\log A(x)-\log B(x)}{A(x)-B(x)}dx=\int_0^1\frac{\log A(x)-\log B(x)}{a(x-x_1)(x-x_2)}dx, \tag{D.3.44}$$

with x_1,x_2 are the solutions of equation $A(x)-B(x)=0$.

$$\int_0^1\frac{\log A(x)-\log B(x)}{A(x)-B(x)}dx=\int_0^1\left[\log A(x)-\log B(x)\right]\left(\frac{1}{x-x_1}-\frac{1}{x-x_2}\right)\frac{1}{a(x_1-x_2)}, \tag{D.3.45}$$

with the x_1 terms :

$$\int_0^1\frac{\log A(x)-\log B(x)}{x-x_1}dx=\int_0^1\frac{\log A(x)-\log A(x_1)}{x-x_1}dx+\int_0^1\frac{\log B(x_1)-\log B(x)}{x-x_1}dx, \tag{D.3.46}$$

similar for x_2.

From Eq. (D.3.43), we get $0 < y_1 < 1, y_2 < 0$ are the solutions of equation $y^2 + \frac{q^2}{p^2}(1-y) = 0$. We split into two parts corresponding to y_1, y_2 :

- $$\int_0^1 dy \frac{\log\left[\frac{p^2 y^2}{-q^2(1-y)}\right]}{y-y_1} = \int_0^1 dy \frac{\log(p^2y^2) - \log\left[-q^2(1-y)\right]}{y-y_1} \tag{D.3.47}$$

  $+$

  $$\int_0^1 \frac{2\log y - 2\log y_1}{y-y_1} dy = \int_{-y_1}^{1-y_1} \frac{2\log\left(1+\frac{y}{y_1}\right)}{y} dy = \left(\int_0^{1-y_1} - \int_0^{-y_1}\right) \frac{2\log\left(1+\frac{y}{y_1}\right)}{y} dy \tag{D.3.48}$$

  $$= \int_0^1 \frac{dy}{y} 2\log\left(1+y\frac{1-y_1}{y_1}\right) - \int_0^1 \frac{dy}{y} 2\log(1-y) = -2\mathrm{Li}_2\left(\frac{y_1-1}{y_1}\right) + 2\mathrm{Li}_2(1) \tag{D.3.49}$$

  $+$

  $$\int_0^1 \frac{\log[-q^2(1-y_1)] - \log[-q^2(1-y)]}{y-y_1} dy = -\int_0^1 \frac{\log y - \log(1-y_1)}{1-y_1-y} dy \tag{D.3.50}$$

  $$= \int_{y_1-1}^{y_1} \frac{\log\left(1+\frac{y}{1-y_1}\right)}{y} dy = \left(\int_0^{y_1} - \int_0^{y_1-1}\right) \frac{\log\left(1+\frac{y}{1-y_1}\right)}{y} dy \tag{D.3.51}$$

  $$= \int_0^1 \frac{dy}{y} \log\left(1+y\frac{y_1}{1-y_1}\right) - \int_0^1 \frac{dy}{y} \log(1-y) = -\mathrm{Li}_2\left(\frac{y_1}{y_1-1}\right) + \mathrm{Li}_2(1). \tag{D.3.52}$$

- $$\int_0^1 dy \frac{\log\left[\frac{p^2y^2}{-q^2(1-y)}\right]}{y-y_2} = \int_0^1 dy \frac{\log\left(\frac{p^2}{-q^2}\right) + \log\left(\frac{y^2}{1-y}\right)}{y-y_2} \tag{D.3.53}$$

  $+$

  $$\int_0^1 dy \frac{\log\left(\frac{p^2}{-q^2}\right)}{y-y_2} = \log\left(\frac{p^2}{-q^2}\right) \log\left(\frac{y_2-1}{y_2}\right) \tag{D.3.54}$$

  $+$

  $$\int_0^1 dy \frac{\log\left(\frac{y^2}{1-y}\right)}{y-y_2} = \int_0^1 \frac{2\log y}{y-y_2} dy + \int_0^1 \frac{\log y}{y-(1-y_2)} dy \tag{D.3.55}$$

  $$= \int_0^1 dy \int_1^y 2\frac{dt}{t}\frac{1}{y-y_2} + \int_0^1 dy \int_1^y \frac{dt}{t}\frac{1}{y-(1-y_2)} \tag{D.3.56}$$

  $$= -2\int_0^1 \frac{dt}{t}\int_0^t dy \frac{1}{y-y_2} - \int_0^1 \frac{dt}{t}\int_0^t dy \frac{1}{y-(1-y_2)} \tag{D.3.57}$$

  $$= -2\int_0^1 \frac{dt}{t} \log\left(1-\frac{t}{y_2}\right) - \int_0^1 \frac{dt}{t} \log\left(1-\frac{t}{1-y_2}\right) \tag{D.3.58}$$

$$= 2\mathrm{Li}_2\left(\frac{1}{y_2}\right) + \mathrm{Li}_2\left(\frac{1}{1-y_2}\right). \tag{D.3.59}$$

$$\begin{aligned}\Rightarrow C_0(q,p,0,0,m) &= \langle| \frac{1}{n^2(n+q)^2\left[(n+p)^2-m^2\right]} |\rangle_n \\ &= \frac{1}{p^2(y_2-y_1)}\Bigg[-2\mathrm{Li}_2\left(\frac{y_1-1}{y_1}\right) + 3\mathrm{Li}_2(1) - \mathrm{Li}_2\left(\frac{y_1}{y_1-1}\right) - \log\left(\frac{p^2}{-q^2}\right)\log\left(\frac{y_2-1}{y_2}\right) \\ &\qquad -2\mathrm{Li}_2\left(\frac{1}{y_2}\right) - \mathrm{Li}_2\left(\frac{1}{1-y_2}\right)\Bigg].\end{aligned} \tag{D.3.60}$$

D.3.2 Tensor

1.

$$\begin{aligned}C_\mu(p_1,p_2,m_0,m_1,m_2) &= \langle| \frac{q_\mu}{(q^2-m_0^2)\left[(q+p_1)^2-m_1^2\right]\left[(q+p_2)^2-m_2^2\right]} |\rangle_q \\ &= p_{1\mu}C_1 + p_{2\mu}C2,\end{aligned} \tag{D.3.61}$$

with $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$.

- Multiply by p_1^μ :

$$p_1^2C_1 + p_1p_2C_2 = \langle| \frac{\frac{1}{2}\left[(q+p_1)^2-m_1^2\right] - \frac{1}{2}(q^2-m_0^2) - \frac{1}{2}m_0^2}{(q^2-m_0^2)\left[(q+p_1)^2-m_1^2\right]\left[(q+p_2)^2-m_2^2\right]} |\rangle_q \tag{D.3.62}$$

$$= \frac{1}{2}B_0(p_1^2,m_0,m_1) - \frac{1}{2}B_0((p_1-p_2)^2,m_2,m_1) - \frac{1}{2}m_0^2C_0. \tag{D.3.63}$$

- Multiply by p_2^μ :

$$p_1p_2C_1 + p_2^2C_2 = \langle| \frac{\frac{1}{2}\left[(q+p_2)^2-m_2^2\right] - \frac{1}{2}(q^2-m_0^2) - \frac{1}{2}m_0^2}{(q^2-m_0^2)\left[(q+p_1)^2-m_1^2\right]\left[(q+p_2)^2-m_2^2\right]} |\rangle_q \tag{D.3.64}$$

$$= \frac{1}{2}B_0(p_2^2,m_0,m_2) - \frac{1}{2}B_0((p_1-p_2)^2,m_1,m_2) - \frac{1}{2}m_0^2C_0. \tag{D.3.65}$$

The solutions $C_1(p_1,p_2,m_0,m_1,m_2), C_2(p_1,p_2,m_0,m_1,m_2)$:

$$\begin{pmatrix} p_1^2 & p_1p_2 \\ p_1p_2 & p_2^2\end{pmatrix}\begin{pmatrix} C_1 \\ C_2\end{pmatrix} = \begin{pmatrix}\frac{1}{2}B_0(p_1^2,m_0,m_1) - \frac{1}{2}B_0((p_1-p_2)^2,m_2,m_1) - \frac{1}{2}m_0^2C_0 \\ \frac{1}{2}B_0(p_2^2,m_0,m_2) - \frac{1}{2}B_0((p_1-p_2)^2,m_1,m_2) - \frac{1}{2}m_0^2C_0\end{pmatrix}. \tag{D.3.66}$$

When $D \to 4$: $(D-4)C_\mu = O(D-4) \Rightarrow$ UV-convergent $C\mu(p_1,p_2,m_0,m_1,m_2)$. Special case : $m_0 = 0$ $\to$ IR-convergent $C_\mu(p_1,p_2,0,m_1,m_2)$.

2. Convergent 1st-order tensor three-point function

$$\begin{aligned} C_\beta(-q,p',0,0,m) &= \langle| \frac{n_\beta}{n^2(n-q)^2\left[(n+p')^2-m^3\right]} |\rangle_n \qquad \text{with} : p'^2 = m^2 \text{ and} : (q+p')^2 = p^2 = m^2 \\ &= -q_\beta C_1(-q,p',0,0,m) + p_\beta C_2(-q,p',0,0,m). \end{aligned} \tag{D.3.67}$$

- Multiply by q^β :

$$\begin{aligned} -q^2C_1 + qp'C_2 &= \langle| \frac{qn}{n^2(n-q)^2\left[(n+p')^2-m^2\right]} |\rangle_n \\ &= \frac{1}{2}\langle| \frac{n^2+q^2-(n-q)^2}{n^2(n-q)^2\left[(n+p')^2-m^2\right]} |\rangle_n \\ &= \frac{1}{2}\left[B_0((q+p')^2,0,m) - B_0(p'^2,0,m) + q^2C_0(-q,p',0,0,m)\right] \\ &= \frac{1}{2}q^2C_0(-q,p',0,0,m). \end{aligned} \tag{D.3.68}$$

- Multiply by p^β :

$$\begin{aligned} -p'qC_1 + m^2C_2 &= \langle| \frac{p'n}{n^2(n-q)^2\left[(n+p')^2-m^2\right]} |\rangle_n \\ &= \frac{1}{2}\langle| \frac{(n+p')^2-n^2-m^2}{n^2(n-q)^2\left[(n+p')^2-m^2\right]} |\rangle_n \\ &= \frac{1}{2}\left[B_0(q^2,0,0) - B_0((q+p')^2,0,m)\right]. \end{aligned} \tag{D.3.69}$$

The solutions of $C_1(-q,p,0,0,m)$, $C_2(-q,p,0,0,m)$:

$$\begin{pmatrix} -q^2 & qp' \\ -qp' & p'^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}q^2C_0(-q,p',0,0,m) \\ \frac{1}{2}\left[B_0(q^2,0,0) - B_0((q+p')^2,0,m)\right] \end{pmatrix}. \tag{D.3.70}$$

$\Rightarrow C_\beta(-q,p',0,0,m) \rightarrow$ convergent.

3.

$$\begin{aligned} C_{\mu\nu}(p_1,p_2,0,m,m) &= \langle| \frac{q_\mu q_\nu}{q^2\left[(q+p_1)^2-m^2\right]\left[(q+p_2)^2-m^2\right]} |\rangle_q \\ &= g_{\mu\nu}C_{00} + p_{1\mu}p_{1\nu}C_{11} + p_{2\mu}p_{2\nu}C_{22} + (p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu})C_{12}, \end{aligned} \tag{D.3.71}$$

with $p_1^2 = p_2^2 = m^2$.

- Multiply by $g^{\mu\nu}$:

$$DC_{00} + p_1^2C_{11} + p_2^2C_{22} + 2p_1p_2C_{12} = B_0((p_1-p_2)^2,m,m) \tag{D.3.72}$$

- Multiply by p_1^μ :

$$\begin{cases} C_{00} + p_1^2 C_{11} + p_1 p_2 C_{12} = \frac{1}{2}\left[B_1((p_1-p_2)^2, m, m) + B_0((p_1-p_2)^2, m, m)\right] \\ p_1^2 C_{12} + p_1 p_2 C_{22} = \frac{1}{2}\left[B_1(p_2^2, 0, m) - B_1((p_1-p_2)^2, m, m)\right] \end{cases} \tag{D.3.73}$$

- Multiply by p_2^μ :

$$\begin{cases} C_{00} + p_2^2 C_{22} + p_1 p_2 C_{12} = \frac{1}{2}\left[B_1((p_1-p_2)^2, m, m) + B_0((p_1-p_2)^2, m, m)\right] \\ p_2^2 C_{12} + p_1 p_2 C_{11} = \frac{1}{2}\left[B_1(p_1^2, 0, m) - B_1((p_1-p_2)^2, m, m)\right] \end{cases} \tag{D.3.74}$$

$$\Rightarrow C_{00} = \frac{1}{2-D}\left[B_1((p_1-p_2)^2, m, m)\right] \tag{D.3.75}$$

$$= \frac{1}{2}\left[-B_1((p_1-p_2)^2, m, m) + \frac{1}{2}\right] + O(D-4). \tag{D.3.76}$$

$$= \frac{\triangle}{4} + \text{finite term} \tag{D.3.77}$$

When $D \to 4$: $(D-4)C_{00} = -\frac{1}{2}$. The C_{11}, C_{22}, C_{12} solutions:

$$\begin{pmatrix} p_2^2 & p_1 p_2 \\ p_1 p_2 & p_1^2 \end{pmatrix} \begin{pmatrix} C_{22} \\ C_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left[B_0((p_1-p_2)^2, m, m) - B_1((p_1-p_2)^2, m, m)\right] + (1-D)C_{00} \\ \frac{1}{2}\left[B_1(p_2^2, 0, m) - B_1((p_1-p_2)^2, m, m)\right] \end{pmatrix}, \tag{D.3.78}$$

and :

$$\begin{pmatrix} p_1^2 & p_1 p_2 \\ p_1 p_2 & p_2^2 \end{pmatrix} \begin{pmatrix} C_{11} \\ C_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left[B_0((p_1-p_2)^2, m, m) - B_1((p_1-p_2)^2, m, m)\right] + (1-D)C_{00} \\ \frac{1}{2}\left[B_1(p_1^2, 0, m) - B_1((p_1-p_2)^2, m, m)\right] \end{pmatrix}. \tag{D.3.79}$$

$\Rightarrow C_{11}, C_{22}, C_{12}$: convergent.

D.4 Four-point functions

D.4.1 Scalar

Scalar four-point function in the first Box diagram Fig. (10) :

$$\begin{aligned} D_0 &= \langle | \frac{1}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \\ &= \langle | \frac{1}{(n^2-\lambda^2)\left[(n+q)^2-\lambda^2\right]\left[(n-k)^2-m_e^2\right]\left[(n+p)^2-m_\mu^2\right]} |\rangle_n . \end{aligned} \tag{D.4.1}$$

Because of $C_0(-k', p', \lambda, m_e, m_\mu) = C_0(-k, p, \lambda, m_e, m_\mu) = C_0(m_e^2, s, m_\mu^2, \lambda, m_e, m_\mu)$.

$$q^2 D_0 = \langle | \frac{-2nq}{(n^2-\lambda^2)\left[(n+q)^2-\lambda^2\right]\left[(n-k)^2-m_e^2\right]\left[(n+p)^2-m_\mu^2\right]} | \rangle_n \tag{D.4.2}$$

$$\Rightarrow \frac{-q^2}{2} D_0(q,-k,p,\lambda,\lambda,m_e,m_\mu) =$$

$$\langle | \frac{\left(n+\frac{1}{2}q\right)^2-\frac{1}{4}q^2}{(n^2-\lambda^2)\left[(n+q)^2-\lambda^2\right]\left[(n-k)^2-m_e^2\right]\left[(n+p)^2-m_\mu^2\right]} | \rangle_n - C_0(k'^2,s,p'^2,\lambda,m_e,m_\mu) \tag{D.4.3}$$

We split into two parts, the finite term $\langle | \frac{\left(n+\frac{1}{2}q\right)^2-\frac{1}{4}q^2}{(n^2-\lambda^2)\left[(n+q)^2-\lambda^2\right]\left[(n-k)^2-m_e^2\right]\left[(n+p)^2-m_\mu^2\right]} | \rangle_n$ and the IR-divergent term $C_0(k'^2,s,p'^2,\lambda,m_e,m_\mu)$.

The IR-divergent term

We apply the same calculating process $C_0(p_1,p_2,\lambda,m,m)$ Eq. (D.3.2) :

$$C_0(k'^2,s,p'^2,\lambda,m_e,m_\mu) = -\int_0^1 \frac{dx}{2\bar{P}_x^2}\left(\log \bar{P}_x - \log\lambda^2\right) \tag{D.4.4}$$

$$= \log\left(\frac{\lambda^2}{-\bar{q}^2}\right)\int_0^1 \frac{dx}{2\bar{P}_x^2} - \int_0^1 \frac{dx}{2\bar{P}_x^2}\log\left(\frac{\bar{P}_x^2}{-\bar{q}^2}\right) \tag{D.4.5}$$

$$= \log\left(\frac{\lambda^2}{-\bar{t}}\right)\int_0^1 \frac{dx}{2\bar{P}_x^2} - \int_0^1 \frac{dx}{2\bar{P}_x^2}\log\left(\frac{\bar{P}_x^2}{-\bar{t}}\right). \tag{D.4.6}$$

The second finite term : $-\int_0^1 \frac{dx}{2\bar{P}_x^2}\log\left(\frac{\bar{P}_x^2}{-\bar{t}}\right)$ should be cancelled by the initial finite term : $\langle | \frac{\left(n+\frac{1}{2}q\right)^2-\frac{1}{4}q^2}{(n^2-\lambda^2)\left[(n+q)^2-\lambda^2\right]\left[(n-k)^2-m_e^2\right]\left[(n+p)^2-m_\mu^2\right]} | \rangle_n$, but I have not been able to prove this. We will only consider the first IR-divergent term :

$$\log\left(\frac{\lambda^2}{-\bar{t}}\right)\int_0^1 \frac{dx}{2\bar{P}_x^2} = \frac{1}{2}\log\left(\frac{\lambda^2}{-\bar{t}}\right)\int_0^1 \frac{dx}{\bar{s}x^2+(m_e^2-m_\mu^2-\bar{s})x+m_\mu^2} \tag{D.4.7}$$

$$= \frac{1}{2}\log\left(\frac{\lambda^2}{-\bar{t}}\right)\int_0^1 \frac{dx}{\bar{s}(x-x_1)(x-x_2)} \tag{D.4.8}$$

$$= \frac{1}{2}\log\left(\frac{\lambda^2}{-\bar{t}}\right)\frac{1}{\bar{s}(x_1-x_2)}\left[\log\left(\frac{x_1-1}{x_1}\right)-\log\left(\frac{x_2-1}{x_2}\right)\right], \tag{D.4.9}$$

with x_1,x_2 are the solutions of :

$$\bar{s}x^2+(m_e^2-m_\mu^2-\bar{s})x+m_\mu^2 = 0, \tag{D.4.10}$$

$$\triangle = (\bar{s}-m_e^2-m_\mu^2)^2-4m_\mu^2 m_e^2 \tag{D.4.11}$$

$$= \left[\bar{s}-(m_e-m_\mu)^2-4m_\mu m_e\right]\left[\bar{s}-(m_e-m_\mu)^2\right] \tag{D.4.12}$$

$$= \left[\bar{s}-(m_e-m_\mu)^2\right]^2\left[1-\frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}\right], \tag{D.4.13}$$

$$\Rightarrow x_1, x_2 = \frac{\bar{s} + m_\mu^2 - m_e^2 \pm \sqrt{\triangle}}{2\bar{s}}. \tag{D.4.14}$$

And :

$$x_s = \frac{\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}} - 1}{\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}} + 1}.$$

We have :

(+)

$$1 - x_s^2 = \frac{4\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\left(\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}} + 1\right)^2} \tag{D.4.15}$$

$$\Rightarrow \frac{(1 - x_s^2)}{-x_s} = \frac{\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}} + 1}{1 - \sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}} \frac{4\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\left(\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}} + 1\right)^2} \tag{D.4.16}$$

$$= \frac{4\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{1 - \left(\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}\right)^2} = \frac{\left[\bar{s} - (m_e - m_\mu)^2\right]\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{m_e m_\mu}, \tag{D.4.17}$$

$$\Rightarrow \frac{(1 - x_s^2) m_e m_\mu}{-x_s} = \left[\bar{s} - (m_e - m_\mu)^2\right]\sqrt{1 - \frac{4m_e m_\mu}{\bar{s} - (m_e - m_\mu)^2}} = \sqrt{\triangle} = (x_1 - x_2)\bar{s}. \tag{D.4.18}$$

(+)

$$\frac{x_1 - 1}{x_1} . \frac{x_2}{x_2 - 1} = \frac{x_1 x_2 - x_2}{x_1 x_2 - x_1} = \frac{\bar{s} - m_\mu^2 - m_e^2 - \sqrt{\triangle}}{\bar{s} - m_\mu^2 - m_e^2 + \sqrt{\triangle}} \tag{D.4.19}$$

$$\Rightarrow 1 - \frac{\bar{s} - m_\mu^2 - m_e^2 - \sqrt{\triangle}}{\bar{s} - m_\mu^2 - m_e^2 + \sqrt{\triangle}} = \frac{2\sqrt{\triangle}}{\bar{s} - m_\mu^2 - m_e^2 + \sqrt{\triangle}} \tag{D.4.20}$$

$$= \frac{2\left[\bar{s} - (m_\mu - m_e)^2\right]\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\bar{s} - m_\mu^2 - m_e^2 + \sqrt{\triangle}} \tag{D.4.21}$$

$$= \frac{4\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\frac{2(\bar{s}-m_\mu^2-m_e^2)+2\sqrt{\triangle}}{\bar{s}-(m_\mu-m_e)^2}} = \frac{4\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\frac{2\left[\bar{s}-(m_\mu-m_e)^2-2m_\mu m_e\right]}{\bar{s}-(m_\mu-m_e)^2} + 2\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}} \tag{D.4.22}$$

$$= \frac{4\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{1 + 1 - \frac{4m_e m_\mu}{\bar{s}-(m_\mu-m_e)^2} + 2\sqrt{1 - \frac{4m_e m_\mu}{\bar{s}-(m_e-m_\mu)^2}}} \tag{D.4.23}$$

$$= \frac{4\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{1+\left(\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}\right)^2+2\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}} = \frac{4\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}}{\left(\sqrt{1-\frac{4m_em_\mu}{\bar{s}-(m_e-m_\mu)^2}}+1\right)^2} \tag{D.4.24}$$

$$= 1 - x_s^2 \tag{D.4.25}$$

$$\Rightarrow \frac{\bar{s}-m_\mu^2-m_e^2-\sqrt{\triangle}}{\bar{s}-m_\mu^2-m_e^2+\sqrt{\triangle}} = \frac{x_1-1}{x_1}.\frac{x_2}{x_2-1} = x_s^2 \tag{D.4.26}$$

Using Eq. (D.4.26) and Eq. (D.4.18), we get the final result of Eq .(D.4.9) :

$$\log\left(\frac{\lambda^2}{-\bar{t}}\right)\frac{1}{2\bar{s}(x_1-x_2)}\left[\log\left(\frac{x_1-1}{x_1}\right)-\log\left(\frac{x_2-1}{x_2}\right)\right] = \frac{-x_s}{(1-x_s^2)m_em_\mu}\log(x_s)\log\left(\frac{\lambda^2}{-\bar{t}}\right), \tag{D.4.27}$$

or :

$$D_0(q,-k,p,\lambda,\lambda,m_e,m_\mu) = \frac{-2x_s}{(1-x_s^2)q^2m_em_\mu}\log(x_s)\log\left(\frac{\lambda^2}{-\bar{t}}\right). \tag{D.4.28}$$

The same result for $D_0(-q,-k',-p,\lambda,\lambda,m_e,m_\mu)$ in the second Box diagram with substitution of $x_s \to x_u$.

D.4.2 Tensor

1. Fist rank tensor in the Box diagrams Fig. (10) :

$$\begin{aligned} D_\mu(-q,-k',p',\lambda,\lambda,m_e,m_\mu) &= \langle| \frac{n_\mu}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right][(n-k')^2-m_e^2]\left[(n+p')^2-m_\mu^2\right]}|\rangle_n \\ &= -q_\mu D_1 - k'_\mu D_2 + p'_\mu D_3. \end{aligned} \tag{D.4.29}$$

We are using below identities :

$$k'q = \frac{q^2}{2} \qquad kq = \frac{-q^2}{2}, \tag{D.4.30}$$

$$pq = \frac{q^2}{2} \qquad p'q = \frac{-q^2}{2}. \tag{D.4.31}$$

- Multilpy by q^μ :

$$-q^2D_1 - \frac{q^2}{2}D_2 - \frac{q^2}{2}D_3 = \langle| \frac{nq}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right][(n-k')^2-m_e^2]\left[(n+p')^2-m_\mu^2\right]}|\rangle_n \tag{D.4.32}$$

$$= \frac{1}{2}\langle| \frac{-(n-q)^2+n^2+q^2}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right][(n-k')^2-m_e^2]\left[(n+p')^2-m_\mu^2\right]}|\rangle_n \tag{D.4.33}$$

$$= \frac{1}{2}\Bigg[\langle| \frac{-1}{(n^2-\lambda^2)\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \tag{D.4.34}$$

$$+ \langle| \frac{1}{\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n + q^2 D_0(-q,-k',p',\lambda,\lambda,m_e,m_\mu)\Bigg]$$

$$= \frac{1}{2} q^2 D_0(-q,-k',p',\lambda,\lambda,m_e,m_\mu), \tag{D.4.35}$$

- Multilpy by k'^μ :

$$-\frac{q^2}{2}D_1 - m_e^2 D_2 + k'p' D_3 = \langle| \frac{nk'}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \tag{D.4.36}$$

$$= \frac{1}{2}\langle| \frac{n^2-\left[(n-k')^2-m_e^2\right]}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \tag{D.4.37}$$

$$= \frac{1}{2}\left[C_0(-k,p,\lambda,m_e,m_\mu) - \langle| \frac{1}{n^2(n-q)^2\left[(n+p')^2-m_\mu^2\right]} |\rangle_n\right]. \tag{D.4.38}$$

- Multilpy by p'^μ :

$$\frac{q^2}{2}D_1 - p'k' D_2 + m_\mu^2 D_3 = \langle| \frac{np'}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \tag{D.4.39}$$

$$= \frac{1}{2}\langle| \frac{-n^2+\left[(n+p')^2-m_\mu^2\right]}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right]\left[(n-k')^2-m_e^2\right]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \tag{D.4.40}$$

$$= \frac{1}{2}\left[-C_0(-k,p,\lambda,m_e,m_\mu) + \langle| \frac{1}{n^2(n-q)^2\left[(n-k')^2-m_e^2\right]} |\rangle_n\right]. \tag{D.4.41}$$

The solutions $D_1(-q,-k',p',\lambda,\lambda,m_e,m_\mu)$, $D_2(-q,-k',p',\lambda,\lambda,m_e,m_\mu)$, $D_3(-q,-k',p',\lambda,\lambda,m_e,m_\mu)$:

$$\begin{pmatrix} -q^2 & -\frac{q^2}{2} & -\frac{q^2}{2} \\ -\frac{q^2}{2} & -m_e^2 & k'p' \\ \frac{q^2}{2} & -p'k' & m_\mu^2 \end{pmatrix}\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}q^2 D_0(-q,-k',p',\lambda,\lambda,m_e,m_\mu) \\ \frac{1}{2}\left[C_0(-k,p,\lambda,m_e,m_\mu) - \langle| \frac{1}{n^2(n-q)^2\left[(n+p')^2-m_\mu^2\right]} |\rangle_n\right] \\ \frac{1}{2}\left[-C_0(-k,p,\lambda,m_e,m_\mu) + \langle| \frac{1}{n^2(n-q)^2\left[(n-k')^2-m_e^2\right]} |\rangle_n\right] \end{pmatrix}. \tag{D.4.42}$$

$$\begin{aligned} &\Rightarrow D_1(-q,-k',p',\lambda,\lambda,m_e,m_\mu) \sim \frac{-C_0(-k,p,\lambda,m_e,m_\mu)}{q^2} \\ &\rightarrow D_\mu(-q,-k',p',\lambda,\lambda,m_e,m_\mu) \sim q_\mu \frac{C_0(-k,p,\lambda,m_e,m_\mu)}{q^2}. \end{aligned} \tag{D.4.43}$$

The same for $D_\mu(-q,-k',-p,\lambda,\lambda,m_e,m_\mu)$.

2. Second rank tensor in the Box diagrams Fig. (10) :

$$D_{\alpha\beta}(-q,-k',p',\lambda,\lambda,m_e,m_\mu) = \langle| \frac{n_\alpha n_\beta}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right][(n-k')^2-m_e^2]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n$$
$$= g_{\alpha\beta}D_{00} + q_\alpha q_\beta D_{11} + k'_\alpha k'_\beta D_{22} + p'_\alpha p'_\beta D_{33}$$
$$+ (q_\alpha k'_\beta + k'_\alpha q_\beta)D_{12} - (q_\alpha p'_\beta + p'_\alpha q_\beta)D_{13} - (k'_\alpha p'_\beta + p'_\alpha k'_\beta)D_{23}. \tag{D.4.44}$$

- Multiply by $g^{\alpha\beta}$:

$$4D_{00} + q^2D_{11} + m_e^2D_{22} + m_\mu^2D_{33} + q^2D_{12} + q^2D_{13} - 2k'p'D_{23} \tag{D.4.45}$$
$$= \langle| \frac{n^2}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right][(n-k')^2-m_e^2]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n = C_0(-k,p,\lambda,m_e,m_\mu). \tag{D.4.46}$$

- Multiply by q^α :

$$q_\beta D_{00} + q^2q_\beta D_{11} + \frac{q^2}{2}k'_\beta D_{22} - \frac{q^2}{2}p'_\beta D_{33} + (q^2k'_\beta + \frac{q^2}{2}q_\beta)D_{12} - (q^2p'_\beta - \frac{q^2}{2}q_\beta)D_{13} \tag{D.4.47}$$
$$-(\frac{q^2}{2}p'_\beta - \frac{q^2}{2}k'_\beta)D_{23} = \langle| \frac{n_\beta nq}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right][(n-k')^2-m_e^2]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \tag{D.4.48}$$
$$= \frac{1}{2}\langle| \frac{n_\beta\left[n^2+q^2-(n-q)^2\right]}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right][(n-k')^2-m_e^2]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n = \frac{1}{2}\Big[C_\beta(-k,p,\lambda,m_e,m_\mu) \tag{D.4.49}$$
$$+q_\beta C_0(-k,p,\lambda,m_e,m_\mu) - C_\beta(-k',p',\lambda,m_e,m_\mu) + q^2D_\beta(-q,-k',p',\lambda,\lambda,m_e,m_\mu)\Big]. \tag{D.4.50}$$

- Multiply by k'^α :

$$k'_\beta D_{00} + \frac{q^2}{2}q_\beta D_{11} + m_e^2k'_\beta D_{22} + k'p'p'_\beta D_{33} + (\frac{q^2}{2}k'_\beta + m_e^2q_\beta)D_{12} - (\frac{q^2}{2}p'_\beta + k'p'q_\beta)D_{13} \tag{D.4.51}$$
$$-(m_e^2p'_\beta + k'p'k'_\beta)D_{23} = \langle| \frac{n_\beta nk'}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right][(n-k')^2-m_e^2]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \tag{D.4.52}$$
$$= \frac{1}{2}\langle| \frac{n_\beta\left(n^2-[(n-k')^2-m_e^2]\right)}{(n^2-\lambda^2)\left[(n-q)^2-\lambda^2\right][(n-k')^2-m_e^2]\left[(n+p')^2-m_\mu^2\right]} |\rangle_n \tag{D.4.53}$$
$$= \frac{1}{2}\left[C_\beta(-k,p,\lambda,m_e,m_\mu) + q_\beta C_0(-k,p,\lambda,m_e,m_\mu) - C_\beta(-q,p',\lambda,\lambda,m_\mu)\right]. \tag{D.4.54}$$

- Multiply by p'^{α} :

$$p'_\beta D_{00} - \frac{q^2}{2} q_\beta D_{11} + p'k'k'_\beta D_{22} + m_\mu^2 p'_\beta D_{33} + (-\frac{q^2}{2} k'_\beta + k'p'q_\beta) D_{12} - (-\frac{q^2}{2} p'_\beta + m_\mu^2 q_\beta) D_{13} \tag{D.4.55}$$

$$- (k'p'p'_\beta + m_\mu^2 k'_\beta) D_{23} = \langle | \frac{n_\beta n p'}{(n^2 - \lambda^2)\left[(n-q)^2 - \lambda^2\right][(n-k')^2 - m_e^2]\left[(n+p')^2 - m_\mu^2\right]} | \rangle_n \tag{D.4.56}$$

$$= \frac{1}{2} \langle | \frac{n_\beta \left(\left[(n+p')^2 - m_\mu^2\right] - n^2\right)}{(n^2 - \lambda^2)\left[(n-q)^2 - \lambda^2\right][(n-k')^2 - m_e^2]\left[(n+p')^2 - m_\mu^2\right]} | \rangle_n \tag{D.4.57}$$

$$= \frac{1}{2}\left[C_\beta(-q, -k', \lambda, \lambda, m_e) - q_\beta C_0(-k, p, \lambda, m_e, m_\mu) - C_\beta(-k, p, \lambda, m_e, m_\mu)\right]. \tag{D.4.58}$$

So we get set of equations :

- $$4D_{00} + q^2 D_{11} + m_e^2 D_{22} + m_\mu^2 D_{33} + q^2 D_{12} + q^2 D_{13} - 2k'p'D_{23} = C_0(-k, p, \lambda, m_e, m_\mu) \tag{D.4.59}$$

- $$\begin{aligned} D_{00} + q^2 D_{11} + \frac{q^2}{2} D_{12} + \frac{q^2}{2} D_{13} = \frac{1}{2} \Big[& C_0(-k, p, \lambda, m_e, m_\mu) + C_1(-k, p, \lambda, m_e, m_\mu) \\ & + C_2(-k, p, \lambda, m_e, m_\mu) - q^2 D_1(-q, -k', p', \lambda, \lambda, m_e, m_\mu)\Big] \end{aligned} \tag{D.4.60}$$

- $$\begin{aligned} \frac{q^2}{2} D_{22} + q^2 D_{12} + \frac{q^2}{2} D_{23} = & \frac{1}{2}\Big[C_1(-k', p', \lambda, m_e, m_\mu) - C_1(-k, p, \lambda, m_e, m_\mu) \\ & - q^2 D_2(-q, -k', p', \lambda, \lambda, m_e, m_\mu)\Big] \end{aligned} \tag{D.4.61}$$

- $$\begin{aligned} \frac{-q^2}{2} D_{33} - q^2 D_{13} - \frac{q^2}{2} D_{23} = & \frac{1}{2}\Big[-C_2(-k', p', \lambda, m_e, m_\mu) + C_2(-k, p, \lambda, m_e, m_\mu) \\ & + q^2 D_3(-q, -k', p', \lambda, \lambda, m_e, m_\mu)\Big] \end{aligned} \tag{D.4.62}$$

- $$\begin{aligned} \frac{q^2}{2} D_{11} + m_e^2 D_{12} - k'p'D_{13} = \frac{1}{2} \Big[& C_0(-k, p, \lambda, m_e, m_\mu) + C_1(-k, p, \lambda, m_e, m_\mu) \\ & + C_2(-k, p, \lambda, m_e, m_\mu) + C_1(-q, p', \lambda, \lambda, m_\mu)\Big] \end{aligned} \tag{D.4.63}$$

- $$D_{00}+m_e^2D_{22}+\frac{q^2}{2}D_{12}-k'p'D_{23}=-\frac{1}{2}C_1(-k,p,\lambda,m_e,m_\mu) \tag{D.4.64}$$

- $$k'p'D_{33}-\frac{q^2}{2}D_{13}-m_e^2D_{23}=\frac{1}{2}\left[C_2(-k,p,\lambda,m_e,m_\mu)-C_2(-q,p',\lambda,\lambda,m_\mu)\right] \tag{D.4.65}$$

- $$\begin{aligned}\frac{-q^2}{2}D_{11}+k'p'D_{12}-m_\mu^2D_{13}=-\frac{1}{2}\Big[&C_1(-q,-k',\lambda,\lambda,m_e)+C_1(-k,p,\lambda,m_e,m_\mu)\\&+C_0(-k,p,\lambda,m_e,m_\mu)\Big]\end{aligned} \tag{D.4.66}$$

- $$p'k'D_{22}-\frac{q^2}{2}D_{12}-m_\mu^2D_{23}=\frac{1}{2}\left[C_1(-k,p,\lambda,m_e,m_\mu)-C_2(-q,-k',\lambda,\lambda,m_e)\right] \tag{D.4.67}$$

- $$D_{00}+m_\mu^2D_{33}+\frac{q^2}{2}D_{13}-k'p'D_{23}=-\frac{1}{2}C_2(-k,p,\lambda,m_e,m_\mu). \tag{D.4.68}$$

From the above set of equations, we can easy deduce out which parts being proportional to IR-divergent value :

$$\begin{aligned}\Rightarrow D_{11}(-q,-k',p',\lambda,\lambda,m_e,m_\mu)&\sim\frac{C_0(-k,p,\lambda,m_e,m_\mu)}{q^2}\\ \to D_{\mu\nu}(-q,-k',p',\lambda,\lambda,m_e,m_\mu)&\sim q_\mu q_\nu\frac{C_0(-k,p,\lambda,m_e,m_\mu)}{q^2}.\end{aligned} \tag{D.4.69}$$

Similar for $D_{\mu\nu}(-q,-k',-p,\lambda,\lambda,m_e,m_\mu)$.

D.5 Proving the relations

- Eq.(D.3.28) :

$$\frac{\partial}{\partial z}\mathrm{Li}_2\frac{z-a}{z-b}=-\frac{\partial}{\partial z}\left\{\int_0^1\frac{dt}{t}\log\left(1-\frac{z-a}{z-b}t\right)\right\}=\int_0^1\frac{dt}{t}\frac{\left(\frac{z-a}{z-b}\right)'t}{1-\frac{z-a}{z-b}t}=-\int_0^1dt\left(\frac{b-a}{z-b}\right)\frac{1}{z-b-(z-a)t} \tag{D.5.1}$$

$$\left(\frac{b-a}{z-b}\right)\frac{\log\frac{z-b-(z-a)}{z-b}}{z-a}=\frac{b-a}{(z-b)(z-a)}\log\frac{a-b}{z-b}=\left[\frac{1}{z-b}-\frac{1}{z-a}\right]\log\frac{a-b}{z-b}. \tag{D.5.2}$$

- Eq.(D.3.35), we have :

$$x_1 - x_2 = \sqrt{1 - 4m^2/\bar{t}} \tag{D.5.3}$$

$$\Rightarrow \log\frac{\bar{t}}{m^2} + \log(x_1 - x_2) + \log(x_2 - x_1) = \log\frac{\bar{t}}{m^2} + \log\left[-(x_1 - x_2)^2\right] \tag{D.5.4}$$

$$= \log\frac{\bar{t}}{m^2} + \log\left(4m^2/\bar{t} - 1\right) = \log\left[\frac{\bar{t}}{m^2}\left(4m^2/\bar{t} - 1\right)\right] = \log\left[-\frac{\bar{t}}{m^2}(x_2 - x_1)^2\right]. \tag{D.5.5}$$

- Eq.(D.3.36) :

$$\mathrm{Li}_2(-x_t) - \mathrm{Li}_2\frac{-1}{x_t} + \mathrm{Li}_2(-x_t) - \mathrm{Li}_2\frac{-1}{x_t} = 2\mathrm{Li}_2(-x_t) - 2\mathrm{Li}_2\left(\frac{-1}{x_t}\right) \tag{D.5.6}$$

$$= 2\mathrm{Li}_2(-x_t) - 2\left[\log(-x_t) + \frac{\pi^2}{6} + \frac{1}{2}\log^2\frac{1}{x_t}\right] = 4\mathrm{Li}_2(-x_t) + \frac{\pi^2}{3} + \log^2 x_t. \tag{D.5.7}$$

- Eq.(D.3.37), considering :

$$1 + x_t = \frac{x_1 - x_2}{1 - x_2} \Rightarrow \frac{(x_1 - x_2)^2}{x_t} = \frac{-x_1}{x_2}\frac{(x_1 - x_2)^2}{(1 - x2)^2} = \frac{(x_1 - x_2)^2}{-x_1 x_2}, \tag{D.5.8}$$

because :

$$x_t = \frac{x_1 - 1}{1 - x_2} = \frac{-x_2}{x_1} \qquad x_1 + x_2 = 1; \tag{D.5.9}$$

and

$$x_2 . x_1 = \frac{m^2}{\bar{t}}. \tag{D.5.10}$$

$$\Rightarrow \frac{(1 + x_t)^2}{x_t} = \frac{(x_1 - x_2)^2}{\frac{-m^2}{\bar{t}}} = \frac{-\bar{t}}{m^2}(x_1 - x_2)^2, \tag{D.5.11}$$

$$\Rightarrow \log\left[\frac{-\bar{t}}{m^2}(x_1 - x_2)^2\right] = \log\frac{(1 + x_t)^2}{x_t}. \tag{D.5.12}$$

We have :

$$\frac{(1 + x_t)^2}{x_t} = \frac{-\bar{t}}{m^2}(x_1 - x_2)^2 = 4 - \frac{\bar{t}}{m^2} \tag{D.5.13}$$

$$\Rightarrow \mathrm{Im}\left[\frac{(1 + x_t)^2}{x_t}\right] < 0; \tag{D.5.14}$$

$$x_t = \frac{\sqrt{1-4m^2/\bar{t}}-1}{\sqrt{1-4m^2/\bar{t}}+1} \Rightarrow x_t - 1 = \frac{-1}{x_1} \tag{D.5.15}$$

$$\operatorname{Im}(x_1) = \operatorname{Im}\left(\frac{\sqrt{1-4m^2/\bar{t}}}{2}\right) > 0 \tag{D.5.16}$$

$$\Rightarrow \operatorname{Im}(x_t) = \operatorname{Im}(x_t - 1) = \operatorname{Im}\left(\frac{-1}{x_1}\right) > 0 \tag{D.5.17}$$

$$\Rightarrow \log\left[\frac{-\bar{t}}{m^2}(x_1 - x_2)^2\right] = \log\frac{(1+x_t)^2}{x_t} = \log(1+x_t)^2 - \log x_t, \tag{D.5.18}$$

because $\operatorname{Im}(\frac{(1+x_t)^2}{x_t})$ and $\operatorname{Im}(\frac{1}{x_t})$ are always same sign.

E

Soft-photons emission

The amplitude matrix element without soft photons be [8]:

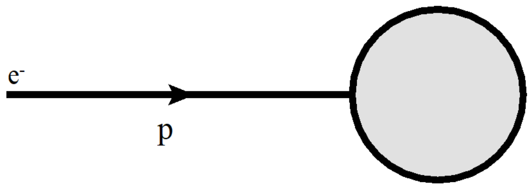


Figure 24: No photon emission

$$i\mathcal{M}_0 = A(p)u^s(p), \tag{E.0.1}$$

the $u^s(p)$ is the incoming fermion line with $p^2 = m^2$,$A(p)$ is the remainder of the amplitude matrix. Consider one photon emission process :

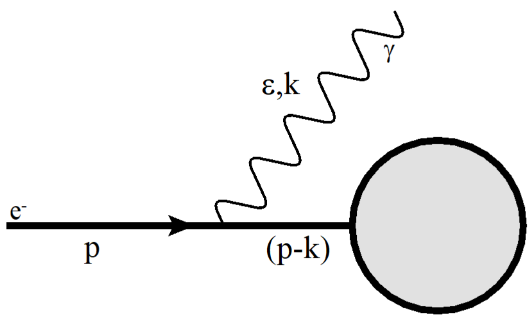


Figure 25: Bremsstrahlung process

$$i\mathcal{M} = A(p-k)\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2}\left(-ie\gamma^\mu\varepsilon_\mu\right)u^s(p). \tag{E.0.2}$$

The numerator:

$$(\not{p}-\not{k}+m)\gamma^\mu\varepsilon_\mu = (-\gamma^\mu\not{p}+2p^\mu+\frac{1}{2}\gamma^\mu\not{k}-\frac{1}{2}\not{k}\gamma^\mu+m\gamma^\mu)\varepsilon_\mu \qquad (\text{Gauge condtion } \varepsilon_\mu k^\mu = 0) \tag{E.0.3}$$

$$= \left(-m\gamma^\mu+2p\varepsilon-i\varepsilon_\mu k_\nu\sigma^{\mu\nu}\right) \qquad (\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu\gamma^\nu]) \tag{E.0.4}$$

$$= 2p\varepsilon - i\varepsilon_\mu k_\nu\sigma^{\mu\nu}. \tag{E.0.5}$$

$$\Rightarrow i\mathcal{M} = A(p-k)\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2}\left(-ie\gamma^{\mu}\varepsilon_{\mu}\right)u^s(p) = e\frac{A(p-k)}{-2pk}\left(2p\varepsilon - i\varepsilon_{\mu}k_{\nu}\sigma^{\mu\nu}\right)u^s(p). \tag{E.0.6}$$

In the soft-photon approximation, we neglect all terms proportional to photon momentum k in everywhere, except in the denominator because of IR-singularity. We can see that the second term is proportional to magnetic moment interaction [8]:

$$\mu\tilde{B}(k) \sim \sigma_i\tilde{B}^i(k) \sim \sigma_i(-i\varepsilon^{ijl}k_j\tilde{A}_l(k)) \sim -i\varepsilon^{ijl}\sigma_ik_j\varepsilon_l(k) \tag{E.0.7}$$

$$\left(\mu = g\frac{e}{2m}\vec{S}, B(x) = \nabla\times\vec{A}, \tilde{B}(k) = i\varepsilon^{ijl}k_l\tilde{A}_l(k)\right),$$

physically, this approximation does not take magnetic moment interaction into account. The soft-photon approximate amplitude :

$$i\mathcal{M} = e\frac{A(p)}{-2pk}2p\varepsilon u^s(p) = \frac{-ep\varepsilon}{pk}i\mathcal{M}_0. \tag{E.0.8}$$

Similarly for the outgoing fermion line, the soft-photon approximate amplitude :

$$i\mathcal{M} = \frac{ep'\varepsilon}{p'k}i\mathcal{M}_0. \tag{E.0.9}$$

We obtain the amplitude in two cases :

$$i\mathcal{M} = ei\mathcal{M}_0\left(\frac{p'\varepsilon}{p'k} - \frac{p\varepsilon}{pk}\right) \tag{E.0.10}$$

$$\Rightarrow |\bar{\mathcal{M}}|^2 = |\bar{\mathcal{M}}_0|^2\sum_{\varepsilon}e^2\left|\frac{p'\varepsilon}{p'k} - \frac{p\varepsilon}{pk}\right|^2. \tag{E.0.11}$$

The soft-photon cross section :

$$\left(\frac{d\sigma}{d\Omega}\right)_{soft} = \left(\frac{d\sigma}{d\Omega}\right)_{LO}\int_{|\vec{k}|\leq\triangle E}\frac{d^3k}{(2\pi)^3}\frac{e^2}{2\omega_k}\sum_{\varepsilon}\left|\frac{p'\varepsilon}{p'k} - \frac{p\varepsilon}{pk}\right|^2, \tag{E.0.12}$$

with $\omega_k = \sqrt{\vec{k}^2+\lambda^2}$ and $\triangle E$ is a cut-off parameter which must be small enough.

Consider this below integral term :

$$\frac{e^2}{(2\pi)^3}\int_{|\vec{k}|\leq\triangle E}\frac{d^3k}{2\omega_k}\sum_{\varepsilon}\left[\frac{\varepsilon^*p.\varepsilon p}{(kp)^2} + \frac{\varepsilon^*p'.\varepsilon p'}{(kp')^2} - \frac{\varepsilon^*p'.\varepsilon p}{kp.kp'} - \frac{\varepsilon^*p.\varepsilon p'}{kp.kp'}\right] \tag{E.0.13}$$

$$= \frac{-e^2}{(2\pi)^3}\int_{|\vec{k}|\leq\triangle E}\frac{d^3k}{2\omega_k}\left[\frac{p^2}{(kp)^2} + \frac{p'^2}{(kp')^2} - \frac{2pp'}{kp.kp'}\right] \tag{E.0.14}$$

Calculating the general term :

$$\int_{|\vec{k}|\leq\triangle E}\frac{d^3k}{\omega_k}\frac{1}{kp.kp'}=\int_0^1 dx\int_{|\vec{k}|\leq\triangle E}\frac{d^3k}{\omega_k}\frac{1}{[kpx+(1-x)kp']^2}\qquad\text{(Feynman parametrization)}\tag{E.0.15}$$

$$=\int_0^1 dx\int_{|\vec{k}|\leq\triangle E}\frac{d^3k}{\omega_k}\frac{1}{(kP_x)^2}\qquad(\text{set}:P_x=px+(1-x)p')\tag{E.0.16}$$

$$=\int_0^1 dx\int_{|\vec{k}|\leq\triangle E}\frac{d^3k}{\omega_k}\frac{1}{\left(k^0P_x^0-\vec{k}\vec{P}_x\right)^2}=\int_0^1 dx\int_{k\leq\triangle E}\frac{d\Omega k^2dk}{\sqrt{k^2+\lambda^2}}\frac{1}{(k^0P_x^0-k\mathbf{P}_xcos\theta)^2}\tag{E.0.17}$$

$$=2\pi\int_0^1 dx\int_0^{\triangle E}\frac{k^2dk}{\sqrt{k^2+\lambda^2}}\int_\pi^0\frac{dcos\theta}{(k^0P_x^0-k\mathbf{P}_xcos\theta)^2}=4\pi\int_0^1 dx\int_0^{\triangle E}\frac{k^2dk}{\sqrt{k^2+\lambda^2}}\frac{1}{k^{02}P_x^{02}-k^2\mathbf{P}_x^2}\tag{E.0.18}$$

$$=4\pi\int_0^1 dx\int_0^{\triangle E}\frac{k^2dk}{\sqrt{k^2+\lambda^2}}\frac{1}{k^2(P_x^{02}-\mathbf{P}_x^2)+\lambda^2P_x^{02}}=\int_0^1 dx\frac{4\pi}{P_x^2}\int_0^{\triangle E}\frac{k^2dk}{\sqrt{k^2+\lambda^2}}\frac{1}{k^2+\lambda^2\frac{P_x^{02}}{P_x^2}},\tag{E.0.19}$$

with $P_x^2=P_x^{02}-\mathbf{P}_x^2$ and $B=\frac{P_x^{02}}{P_x^2}$.

$$\Rightarrow\int_{|\vec{k}|\leq\triangle E}\frac{d^3k}{\omega_k}\frac{1}{kp.kp'}=\int_0^1 dx\frac{4\pi}{P_x^2}\int_0^{\triangle E}\frac{dk}{\sqrt{k^2+\lambda^2}}\left[\frac{k^2+\lambda^2B}{k^2+\lambda^2B}-\frac{\lambda^2B}{k^2+\lambda^2B}\right]\tag{E.0.20}$$

§ The IR-divergent term

$$\int_0^1 dx\frac{4\pi}{P_x^2}\int_0^{\triangle E}\frac{dk}{\sqrt{k^2+\lambda^2}}\frac{k^2+\lambda^2B}{k^2+\lambda^2B}=\int_0^1 dx\frac{4\pi}{P_x^2}\int_0^{\triangle E}\frac{dk}{\sqrt{k^2+\lambda^2}}=\int_0^1 dx\frac{4\pi}{P_x^2}\log\left(k+\sqrt{k^2+\lambda^2}\right)\Big|_0^{\triangle E}\tag{E.0.21}$$

$$=\int_0^1 dx\frac{4\pi}{P_x^2}\left[\log\left(\triangle E+\sqrt{\triangle E^2+\lambda^2}\right)-\log\lambda\right]=\int_0^1 dx\frac{2\pi}{P_x^2}\log\left(\frac{2\triangle E}{\lambda}\right)^2\tag{E.0.22}$$

$$=Re\left[\int_0^1 dx\frac{2\pi}{P_x^2-i\varepsilon}\log\left(\frac{2\triangle E}{\lambda}\right)^2\right]=Re\left[-4\pi\frac{x_t}{m^2(1-x_t^2)}\log(x_t)\log\left(\frac{2\triangle E}{\lambda}\right)^2\right],\tag{E.0.23}$$

(the integral $\int_0^1\frac{dx}{P_x^2}$ we have done it before Eq. (D.3.20)).

§ The finite term

$$\int_0^1 dx\frac{4\pi}{P_x^2}\int_0^{\triangle E}\frac{dk}{\sqrt{k^2+\lambda^2}}\frac{-\lambda^2B}{k^2+\lambda^2B}=\int_0^1 dx\frac{4\pi}{P_x^2}\int_0^{\frac{\triangle E}{\lambda}}\frac{dy(-B)}{\sqrt{y^2+1}(y^2+B)},\qquad y=\frac{k}{\lambda}\tag{E.0.24}$$

$$\xrightarrow[A=-B]{\lambda\to 0}\int_0^1 dx\frac{4\pi}{P_x^2}\int_0^\infty\frac{dyA}{\sqrt{y^2+1}(y^2-A)}=-\int_0^1 dx\frac{4\pi}{P_x^2}\frac{\tanh^{-1}\left(\sqrt{\frac{A+1}{A}}\right)}{\sqrt{1+\frac{1}{A}}}=-\int_0^1 dx\frac{4\pi}{P_x^2}\frac{P_x^0}{\mathbf{P}_x}\tanh^{-1}\left(\frac{\mathbf{P}_x}{P_x^0}\right).\tag{E.0.25}$$

Using the relation :

$$\tanh^{-1}z=\frac{1}{2}\log\left(\frac{z+1}{1-z}\right)\tag{E.0.26}$$

$$\Rightarrow \int_0^1 dx \frac{4\pi}{P_x^2} \int_0^{\triangle E} \frac{dk}{\sqrt{k^2+\lambda^2}} \frac{-\lambda^2 B}{k^2+\lambda^2 B} = \int_0^1 \frac{2\pi dx}{P_x^2} \frac{P_x^0}{\mathbf{P}_x} \log\left(\frac{P_x^0 - \mathbf{P}_x}{P_x^0 + \mathbf{P}_x}\right). \tag{E.0.27}$$

We'll use and transform some variables for convenience [16]. Get $q = ap$ such that $(q-p')^2 = 0$ and $\frac{q_0 - p'_0}{p'_0} > 0$:

$$\text{Eq.(E.0.27)} = a \int_0^1 \frac{2\pi dx}{P_x^2} \frac{P_x^0}{\mathbf{P}_x} \log\left(\frac{P_x^0 - \mathbf{P}_x}{P_x^0 + \mathbf{P}_x}\right). \tag{E.0.28}$$

Using :

$$\begin{aligned} v &= \frac{q^2 - p'^2}{2l} \qquad l = q_0 - p'_0 = \pm|\vec{q} - \vec{p}'| \\ M &= P_x^0 - \mathbf{P}_x \qquad \Rightarrow P_x^0 + \mathbf{P}_x = \frac{v[M - (2vp'_0 - p'^2)/v]}{M - v} \\ &\Rightarrow P_x^2 = 2vP_x^0 - 2vp'_0 + p'^2 \qquad \Rightarrow x = \frac{P_x^2 - p'^2}{2vl}. \end{aligned}$$

$$\begin{aligned} a \int_0^1 \frac{2\pi dx}{P_x^2} \frac{P_x^0}{\mathbf{P}_x} \log\left(\frac{P_x^0 - \mathbf{P}_x}{P_x^0 + \mathbf{P}_x}\right) = 2\pi a \int_{p'_0 - \mathbf{p}'}^{q_0 - \mathbf{q}} \frac{dM}{4vl} \left\{ 2\left[\frac{1}{M} + \frac{1}{M - v} - \frac{1}{M - (2p'_0 v - p'^2)/v}\right] - \frac{4}{M - v}\right\} \\ \times \log\left[\frac{M - v}{v} . \frac{M}{M - (2vp'^0 - p'^2)/v}\right] \end{aligned} \tag{E.0.29}$$

$$\begin{aligned} = 2\pi a \Bigg\{ \frac{1}{4vl} \log^2 \left(\frac{M - v}{v} . \frac{M}{M - (2p'_0 v - p'^2)/v}\right)\Bigg|_{p'_0 - \mathbf{p}'}^{q_0 - \mathbf{q}} \\ - \int_{p'_0 - \mathbf{p}'}^{q_0 - \mathbf{q}} \frac{dM}{vl(M - v)} \log\left(\frac{(M - v)M}{v[M - (2vp'^0 - p'^2)/v]}\right)\Bigg\}, \end{aligned} \tag{E.0.30}$$

applying two identities (a is a constant):

$$\frac{d}{dM}\mathrm{Li}_2[a(M+v)] = \frac{-1}{M+v} \log[1 - a(M+v)] \tag{E.0.31}$$

$$\log(M - a) = \log\left(1 - \frac{M - v}{a - v}\right) + \log(v - a), \tag{E.0.32}$$

extend :

$$\log\left(\frac{(M - v)M}{v[M - (2vp'^0 - p'^2)/v]}\right) = \log\left(\frac{M - v}{v}\right) + \log(M) - \log\left(M - \frac{2p'_0 v - p'^2}{v}\right) \tag{E.0.33}$$

counter each part:

–

$$- \int_{p'_0 - \mathbf{p}'}^{q_0 - \mathbf{q}} \frac{dM}{vl(M - v)} \log\left(\frac{M - v}{v}\right) = \frac{-1}{2vl} \log^2\left(\frac{M - v}{v}\right)\Bigg|_{p'_0 - \mathbf{p}'}^{q_0 - \mathbf{q}}. \tag{E.0.34}$$

–

$$-\int_{p_0'-\mathbf{p}'}^{q_0-\mathbf{q}} \frac{dM}{\nu l(M-\nu)}\log(M) = \frac{1}{\nu l}\left[\mathrm{Li}_2\left(\frac{\nu-M}{\nu}\right) - \log(\nu)\log(M-\nu)\right]\Bigg|_{p_0'-\mathbf{p}'}^{q_0-\mathbf{q}}. \tag{E.0.35}$$

–

$$\int_{p_0'-\mathbf{p}'}^{q_0-\mathbf{q}} \frac{dM}{\nu l(M-\nu)}\log\left(M-\frac{2p_0'\nu-p'^2}{\nu}\right) = \frac{-1}{\nu l}\left[\mathrm{Li}_2\left(\frac{M-\nu}{(2p_0'\nu-p'^2)/\nu-\nu}\right)\right. \\ \left. -\log\left(\nu-\frac{2p_0'\nu-p'^2}{\nu}\right)\log(A-\nu)\right]\Bigg|_{p_0'-\mathbf{p}'}^{q_0-\mathbf{q}}. \tag{E.0.36}$$

The result of finite term:

$$2\pi a\left\{\frac{1}{4\nu l}\log^2\left(\frac{M-\nu}{\nu}.\frac{M}{M-(2p_0'\nu-p'^2)/\nu}\right) - \frac{1}{2\nu l}\log^2\left(\frac{M-\nu}{\nu}\right) + \frac{1}{\nu l}\left[\mathrm{Li}_2\left(\frac{\nu-M}{\nu}\right)\right.\right. \\ \left.\left. -\log(\nu)\log(M-\nu) - \mathrm{Li}_2\left(\frac{M-\nu}{(2p_0'\nu-p'^2)/\nu-\nu}\right) + \log\left(\nu-\frac{2p_0'\nu-p'^2}{\nu}\right)\log(A-\nu)\right]\right\}\Bigg|_{p_0'-\mathbf{p}'}^{q_0-\mathbf{q}}. \tag{E.0.37}$$

After some simple algebra, we change M variable to P_x. The final result of finite term :

$$\frac{2\pi a}{\nu l}\left\{\frac{1}{4}\log^2\left(\frac{P_x^0-\mathbf{P}_x}{P_x^0+\mathbf{P}_x}\right) + \mathrm{Li}_2\left(1-\frac{P_x^0-\mathbf{P}_x}{\nu}\right) + \mathrm{Li}_2\left(1-\frac{P_x^0+\mathbf{P}_x}{\nu}\right)\right\}\Bigg|_{P_x=p'}^{P_x=ap}. \tag{E.0.38}$$

F

FORM CODE

My FORM program calculating virtual amplitudes is shown below with the example of calculating Vertex correction amplitude :

```
*
***************    L.D Truyen 12/7/2020    ************************************
*
*******************************************************************************
***********************************    DECLARE    ******************************
*******************************************************************************
*  Parameters and Mandelstam variables
Symbols a,t,s,u,e,mme,mmu,pi;
*  Particles momentum
vector k,k1,p,p1,q;
vector km,k1m,pm,p1m,qm;
*  Dirac spinor
function U,V;
Cfunction Ubar,Vbar,
*  Particles mass
m;
*********************
*  N-point function *
*********************
*  4-point function
Cfunction Dget, Dval, D0, D1, D2, D3 , D00, D11, D22, D33, D12, D23, D13;
symbol dd0,dd1,dd2,dd3,dd00,dd11,dd22,dd33,dd12,dd23,dd13,id1,id2;
* Box s
set Q1: qm,k1m,p1,qm,k1m,qm,p1,k1m,p1;
set P1: qm,k1m,p1,k1m,qm,p1,qm,p1,k1m;
* Box u
set Q2: qm,k1m,pm,qm,k1m,qm,pm,k1m,pm;
set P2: qm,k1m,pm,k1m,qm,pm,qm,pm,k1m;

set Di: D1,D2,D3;
set Dii: D11,D22,D33,D12,D12,D13,D13,D23,D23;
set D: D0,D1,D2,D3,D00,D11,D22,D33,D12,D23,D13;
set dd: dd0,dd1,dd2,dd3,dd00,dd11,dd22,dd33,dd12,dd23,dd13;
*  3-point function
```

```
Cfunction  Cget,Cval,C0,C1,C2,C00,C11,C22,C12;
symbol  cc0,cc1,cc2,cc00,cc11,cc22,cc12,ic1,ic2;
*  Ver  e
set  Q3:  k,k1,k,k1;
set  P3:  k,k1,k1,k;
*  Ver  mu
set  Q4:  p,p1,p,p1;
set  P4:  p,p1,p1,p;

set  Ci:  C1,C2;
set  Cii:  C11,C22,C12,C12;
set  C:  C0,C1,C2,C00,C11,C22,C12;
set  cc:  cc0,cc1,cc2,cc00,cc11,cc22,cc12;
*   2-point  function
Cfunction  Bget,  Bval,  B0,  B1,  B00,  B11,  dB0,  dB1;
symbol  bb0,bb1,bb00,bb11,dbb0,dbb1,ib1,ib2,ib3,ib4,ib5,ib6;
set  B:  B0,B1,B00,B11,dB0,dB1;
set  bb:  bb0,bb1,bb00,bb11,dbb0,dbb1;
*   1-point  function
Cfunction  Aget,  Aval,  A0;
symbol  aa0,ia1,ia2;
********************
*   Tensor  index
indices  i,i1,i2,j,j1,j2,mu,nu,mu1,mu2,nu1,nu2,rho,sigma;
*
***********************************************************
***************   Feynman  Amplitude   ************
***********************************************************
*    NLO_Amplitude
*******************
local  [M_NLO]=
************************
*    Vertex1  Amplitude
************************
i_*e^4/(t*16*pi^2)*Ubar(1,k1)*(-2*g_(1,mu)-
2*g_(1,i,mu,j)*(sum_(a,1,4,Q3[a](i)*P3[a](j)*Cii[a](k.k,t,k1.k1,0,mme^2,mme^2))+
d_(i,j)*C00(k.k,t,k1.k1,0,mme^2,mme^2)+
k(i)*sum_(a,1,2,Q3[a](j)*Ci[a](k.k,t,k1.k1,0,mme^2,mme^2))+
k(j)*sum_(a,1,2,Q3[a](i)*Ci[a](k.k,t,k1.k1,0,mme^2,mme^2))+
k(i)*k(j)*C0(k.k,t,k1.k1,0,mme^2,mme^2)+
q(j)*(sum_(a,1,2,Q3[a](i)*Ci[a](k.k,t,k1.k1,0,mme^2,mme^2))+
k(i)*C0(k.k,t,k1.k1,0,mme^2,mme^2)))+
4*mme*q(mu)*C0(k.k,t,k1.k1,0,mme^2,mme^2)-
2*mme^2*g_(1,mu)*C0(k.k,t,k1.k1,0,mme^2,mme^2)+
8*mme*(sum_(a,1,2,Q3[a](mu)*Ci[a](k.k,t,k1.k1,0,mme^2,mme^2))+
k(mu)*C0(k.k,t,k1.k1,0,mme^2,mme^2)))*U(1,k)*Ubar(2,p1)*g_(2,mu)*U(2,p)+
```

```
************************
*    Vertex2  Amplitud
i_*e^4/(t*16*pi^2)*Ubar(2,p1)*(-2*g_(2,mu)-
2*g_(2,i,mu,j)*(sum_(a,1,4,Q4[a](i)*P4[a](j)*Cii[a](p.p,t,p1.p1,0,mmu^2,mmu^2))+
d_(i,j)*C00(p.p,t,p1.p1,0,mmu^2,mmu^2)+
p(i)*sum_(a,1,2,Q4[a](j)*Ci[a](p.p,t,p1.p1,0,mmu^2,mmu^2))+
p(j)*sum_(a,1,2,Q4[a](i)*Ci[a](p.p,t,p1.p1,0,mmu^2,mmu^2))+
p(i)*p(j)*C0(p.p,t,p1.p1,0,mmu^2,mmu^2)-
q(j)*(sum_(a,1,2,Q4[a](i)*Ci[a](p.p,t,p1.p1,0,mmu^2,mmu^2))+
p(i)*C0(p.p,t,p1.p1,0,mmu^2,mmu^2)))-
4*mmu*q(mu)*C0(p.p,t,p1.p1,0,mmu^2,mmu^2)-
2*mmu^2*g_(2,mu)*C0(p.p,t,p1.p1,0,mmu^2,mmu^2)+
8*mmu*(sum_(a,1,2,Q4[a](mu)*Ci[a](p.p,t,p1.p1,0,mmu^2,mmu^2))+
p(mu)*C0(p.p,t,p1.p1,0,mmu^2,mmu^2)))*U(2,p)*Ubar(1,k1)*g_(1,mu)*U(1,k);
************************
*  LO Amplitude conjugate
************************
local [M0*] = -i_*e^2*
*      electron line
Ubar(1,k)*g_(1,rho)*U(1,k1)*
*      photon propagator
d_(rho,sigma)/t*
*      muon line
Ubar(2,p)*g_(2,sigma)*U(2,p1);
************************
*  Square Amplitude
************************
local [M_NLO^2]=2*1/4*[M0*]*[M_NLO];
id k1m = -k1;
id pm = -p;
id qm = -q;
bracket e,t;
print;
.sort
************************
*      identities        *
************************
*  Dirac equation
id U(i?,k?)*Ubar(i?,k?)=(g_(i,k)+m(k));
id V(i?,k?)*Vbar(i?,k?)=(g_(i,k)-m(k));
***********************************
*****   Trace calculation   **********
Trace4,1;
Trace4,2;
*
*******************************************
```

```
*        Mandelstam variable transform        *
id k?{k1,p}.q = t/2;
id k?{k,p1}.q = -t/2;
id q.q = t;
id k1.k = mme^2-t/2;
id p1.p = mmu^2-t/2;
id k.p = (s-mmu^2-mme^2)/2;
id k1.p1 = (s-mmu^2-mme^2)/2;
id k1.p = (mmu^2+mme^2-u)/2;
id k.p1 = (mmu^2+mme^2-u)/2;
id only s=2*mmu^2+2*mme^2-u-t;
*  mass identification
id k?.k?=m(k)*m(k);
id m(k?{k,k1})=mme;
id m(k?{p,p1})=mmu;
*********************************************
****  Numerical calculation conventions  ****
id D0?D[t](?a) = Dval(dd[t],Dget(?a));
id D0?C[t](?a) = Cval(cc[t],Cget(?a));
id D0?B[t](?a) = Bval(bb[t],Bget(?a));
id A0(?a) = Aval(aa0,Aget(?a));
*     Shortcut variables
argument;
id Dget(k1.k1,k.k,p.p,p1.p1,t,s,0,mme^2,0,mmu^2) = id1;
id Dget(k1.k1,k.k,p1.p1,p.p,t,u,0,mme^2,0,mmu^2) = id2;
id Cget(k.k,t,k1.k1,0,mme^2,mme^2) = ic1;
id Cget(p.p,t,p1.p1,0,mmu^2,mmu^2) = ic2;
id Bget(q.q,mme^2,mme^2) = ib1;
id Bget(q.q,mmu^2,mmu^2) = ib2;
id Bget(mme^2,0,mme^2) = ib3;
id Bget(mmu^2,0,mmu^2) = ib4;
id Bget(0,mme^2,mme^2) = ib5;
id Bget(0,mmu^2,mmu^2) = ib6;
id Aget(mme^2) = ia1;
id Aget(mmu^2) = ia2;
endargument;
*
*****   export result   *******
*
bracket e,t,u,pi;
format mathematica;
print +s [M_NLO^2];
```

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