# 1D Ising Model (classical & quantum)

The Ising chain provides an exactly solvable playground to explore the role of thermal & quantum fluctuations.

Consider first the classical 1d Ising model,

$$Z = \sum_{\{\sigma_i^z = \pm 1\}} e^{-\beta H}$$

$$\beta H = -K \sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=1}^{N} \sigma_i^z \qquad (K = \beta J, \quad h = \beta B)$$

This model can be solved exactly. Suppose the chain has periodic boundaries, $\sigma_{N+1}^z = \sigma_1^z$. Then it is convenient to write $Z$ as

$$Z = \sum_{\{\sigma_i = \pm 1\}} \prod_{i=1}^{N} \left[ e^{\frac{h}{2}\sigma_i^z} e^{K \sigma_i^z \sigma_{i+1}^z} e^{\frac{h}{2}\sigma_{i+1}^z} \right]$$

$$= \sum_{\{\sigma_i = \pm 1\}} \left( \prod_{i=1}^{N} M(\sigma_i, \sigma_{i+1}) \right)$$

$$= \sum_{\{\sigma_i = \pm 1\}} \langle \sigma_1 | \hat{M} | \sigma_2 \rangle \langle \sigma_2 | \hat{M} | \sigma_3 \rangle \dots \langle \sigma_N | \hat{M} | \sigma_1 \rangle$$

$$= \mathrm{Tr} \left( \hat{M}^N \right)$$

where $\hat{M}$ is the transfer matrix

$$\hat{M} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}$$

Thus

$$Z = \lambda_+^N + \lambda_-^N$$

where $\lambda_\pm$ are the eigenvalues of $\hat{M}$.

Eigenvalues of $\hat{M}$:

$$\left(e^{K+h} - \lambda\right)\left(e^{K-h} - \lambda\right) - e^{-2K} = 0$$

$$\Rightarrow \lambda^2 - 2\lambda e^{+K} \cosh h + e^{2K} - e^{-2K} = 0$$

$$\Rightarrow \qquad \lambda_{\pm} = e^{K}\left(\cosh h \pm \sqrt{\sinh^2 h + e^{-4K}}\right)$$

Unless $K = \infty$ and $h = 0$, $\lambda_+ > \lambda_-$. Thus, in the thermodynamic limit $N \to \infty$, $Z = \lambda_+^N + \lambda_-^N \to \lambda_+^N$.

$$\Rightarrow \qquad \beta F = -\ln Z = -N \ln \lambda_+$$

The free energy is analytic. This indicates that there are no phase transitions in the range $0 < K < \infty$.

Lets compute the magnetization and the zero field susceptibility:

$$m = \frac{1}{N}\sum_i \langle \sigma_i^z \rangle = \frac{1}{N}\frac{\partial}{\partial h}\ln Z = \frac{1}{\lambda_+}\frac{\partial \lambda_+}{\partial h}$$

$$= \frac{1}{\cosh h + \sqrt{\sinh^2 h + e^{-4K}}}\left(\sinh h + \frac{\sinh h \cosh h}{\sqrt{\sinh^2 h + e^{-4K}}}\right)$$

$$\chi = \left.\frac{\partial m}{\partial h}\right|_{h=0} = \frac{1}{1 + e^{-2K}}\left(1 + \frac{1}{e^{-2K}}\right) = e^{2K}$$

Thus, the chain is a paramagnet even for an arbitrarily low (but non-zero) temperature (i.e. $K < \infty$).

We can also compute the spin correlation function

$$C(r) = \langle \sigma_j^z \sigma_{j+r}^z \rangle$$

$$= \frac{1}{Z} \sum_{\{\sigma_i = \pm 1\}} \sigma_j^z \sigma_{j+r}^z \, e^{-\beta H}$$

$$= \frac{1}{Z} \sum_{\{\sigma_j = \pm 1\}} \left( \prod_{i=1}^{j-1} M(\sigma_i, \sigma_{i+1}) \right) \sigma_j \left( \prod_{i=j}^{j+r-1} M(\sigma_i, \sigma_{i+1}) \right) \sigma_{j+r} \left( \prod_{i=j+r}^{N} M(\sigma_i, \sigma_{i+1}) \right)$$

$$= \frac{1}{Z} \, \mathrm{Tr} \left( \hat{M}^j \, \hat{\sigma}^z \, \hat{M}^r \, \hat{\sigma}^z \, \hat{M}^{N-r-j} \right)$$

$$= \frac{1}{Z} \, \mathrm{Tr} \left( \hat{\sigma}^z \, \hat{M}^r \, \hat{\sigma}^z \, \hat{M}^{N-r} \right)$$

where $\hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

It is convenient to evaluate the trace in the eigenbasis of $\hat{M}$. Focus on $h=0$. Then

$$\hat{M} = \begin{pmatrix} e^{K} & e^{-K} \\ e^{-K} & e^{K} \end{pmatrix}, \qquad \lambda_\pm = e^{K} \pm e^{-K} = \begin{cases} 2 \cosh K \\ 2 \sinh K \end{cases}$$

$$\hat{M} \, |\pm\rangle = \lambda_\pm \, |\pm\rangle \quad \Rightarrow \quad |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Note that $\hat{\sigma}^z$ flips $|+\rangle$ and $|-\rangle$:

$$\hat{\sigma}^z \, |+\rangle = |-\rangle, \qquad \hat{\sigma}^z \, |-\rangle = |+\rangle$$

Hence:

$$C(r) = \frac{1}{\lambda_+^N + \lambda_-^N} \left( \lambda_-^r \lambda_+^{N-r} + \lambda_+^r \lambda_-^{N-r} \right)$$

For $K \neq 0$, and in the thermodynamic limit,

$$C(r) = \frac{1}{\lambda_+^N} \lambda_-^r \lambda_+^{N-r} = \left(\frac{\lambda_-}{\lambda_+}\right)^r = (\tanh K)^r$$

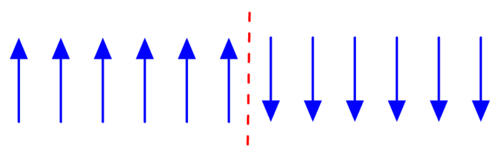
$$= \exp(-r/\xi)$$

correlation length $\quad \xi = -\dfrac{1}{\log (\tanh K)}$

Short range correlations. As $T \to 0$, $K \to \infty$ and $\xi \to \frac{1}{2} e^{2K}$ so $\xi$ can be quite large at small $T$.

## Why?

We found that an infinitesimal temperature is sufficient to disorder the Ising chain. We can understand this by considering a domain wall (take open BC's):



It costs energy $2J$ to introduce such a domain wall. On the other hand, we gain entropy $k_B \log N$ arising from counting the number of locations where we can place the domain wall. Hence, the change in free energy for adding one domain wall is

$$\Delta F = \Delta E - T \Delta S = 2J - k_B T \log N$$

and in the thermodynamic limit it always pays off to introduce domain walls. This is true for any 1d Ising model, provided the interactions are short-ranged.

What is the density of domain walls? At small $T$, we expect DWs to be dilute. Suppose there are $n$ DWs, $n \ll N$. Then the energy cost will be

$$\Delta E = 2nJ$$

whereas the configurational entropy is

$$\Delta S = k_B \log \frac{N!}{n!\,(N-n)!} \simeq k_B\left(N \log N - n \log n - (N-n)\log(N-n)\right)$$

Minimizing $\Delta F = \Delta E - T\Delta S$ w.r.t. $n$ yields

$$0 = \frac{dF}{dn} = 2J - k_B T\,(-\log n - 1 + \log(N-n) + 1)$$

$$\Rightarrow \quad \log\left[\frac{n}{N-n}\right] = -2\frac{J}{k_B T} = -2K$$

$$\Rightarrow \quad \frac{\rho}{1-\rho} = e^{-2K},$$ where $\rho = \frac{n}{N}$ = density of DWs

$\Rightarrow$ At low $T$, $\rho \approx e^{-2K}$

$\Rightarrow$ average distance between DWs $= \frac{1}{\rho} = e^{2K}$

$\Rightarrow$ distance between a given point and the nearest DW $= \frac{1}{2}e^{2K}$ which is the value of $\xi$ we found earlier.

# Ising chain in a transverse field

Reading: S. Sachdev "Quantum Phase Transitions" Ch. 4

We will study the following 1d Hamiltonian at $T=0$:

$$H_Q = -J \sum_i \left( \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + g\, \hat{\sigma}_i^x \right)$$

In contrast to previous case, now the field is along x and the two terms in H do not commute.

The unitary operator

$$\hat{P} = \prod_j \hat{\sigma}_j^x$$

flips all spins from $|\uparrow\rangle$ to $|\downarrow\rangle$ and vice versa. It acts on the operators of the model as,

$$\hat{\sigma}_i^x \to \hat{P}^+ \hat{\sigma}_i^x \hat{P} = \sigma_i^x \quad ; \quad \hat{\sigma}_i^z \to \hat{P}^+ \hat{\sigma}_i^z \hat{P} = -\hat{\sigma}_i^z$$

Hence it commutes with $H_Q$. It generates a global $\mathbb{Z}_2$ symmetry of the Hamiltonian.

Clearly, at zero temperature there are no thermal fluctuations. However, quantum fluctuations are present in the form of the term $-g\sum_i \sigma_i^x$, which flips spins and can destroy the Ising order.

We will provide an exact solution to the model, but first lets consider two limiting cases.

# Strong field, $g \gg 1$

In this limit all the spins want to align with the field. The ground state for $g = \infty$ is unique: (see ⊛ below)

$$|0\rangle = \prod_j |\rightarrow\rangle_j = |\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rangle$$

For $g = \infty$, we can also write the excited states exactly. The lowest-lying excitations involve one spin flip:

$$|i\rangle = |\leftarrow\rangle_i \prod_{j \neq i} |\rightarrow\rangle_j = |\rightarrow\rightarrow\rightarrow\rightarrow\underset{i}{\leftarrow}\rightarrow\rightarrow\rightarrow\rightarrow\rangle$$

We call these "single particle states", and they are degenerate for $g = \infty$. If $g$ is large but finite, these states are no longer eigenstates of $H$, since the term $\sigma_i^z \sigma_{i+1}^z$ hops the particle:

$$\langle i | H_Q | i+1 \rangle = -J$$

[This term also creates pairs of particles when it acts elsewhere in the chain, but provided $g \gg 1$ we can ignore mixing of states with different particle number].

We can write approximate wavefunctions for the excitations,

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_i e^{ikx_i} |i\rangle$$

with energy

$$\varepsilon_k = Jg\left[2 - \frac{2}{g}\cos ka + \mathcal{O}\left(\frac{1}{g^2}\right)\right]$$

above the ground state energy.

⊛ $\quad |\rightarrow\rangle_j = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_j + |\downarrow\rangle_j\right) \quad ; \quad |\leftarrow\rangle_j = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_j - |\downarrow\rangle_j\right)$

## Weak field limit, $g \ll 1$

Consider first $g=0$. The ground state has all the spins aligned, but we have two possibilities:

$$|\Uparrow\rangle = \prod_j |\uparrow\rangle_j = |\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$$

$$|\Downarrow\rangle = \prod_j |\downarrow\rangle_j = |\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\rangle$$

One of these states is chosen spontaneously.

Note that the $\mathbb{Z}_2$ generator, $\hat{P} = \prod_j \hat{\sigma}_j^x$, which commutes with $H_Q$, transforms the states $|\Uparrow\rangle$ and $|\Downarrow\rangle$ into one another.

When $g$ is small but non-zero, $g$ will mix a small fraction of spins of the opposite direction. You may worry that the true ground state is some symmetrized combination of majority up and majority down states. However, to reach $|\Downarrow\rangle$ starting from $|\Uparrow\rangle$ one must act with $g\sum_i \hat{\sigma}_i^x$ $N$ times. In the thermodynamic limit the two states are strictly degenerate, to all orders in perturbation theory in $g$.

What are the basic excitations? In this case, it is best to think in terms of domain walls,

$$|i\rangle = |\uparrow\uparrow\uparrow\underset{i}{\uparrow}\underset{i+1}{\downarrow}\downarrow\downarrow\downarrow\rangle$$

As before, if we think of this as a "particle", then the Hamiltonian hops this particle at order $g$:

$$\langle i|H|i+1\rangle = -Jg$$

Then, if we construct plane wave states out of these "particles" as before, we obtain excitations with energy

$$\mathcal{E}_k = J(2 - 2g\cos ka + \mathcal{O}(g^2))$$

We've found that the weak & strong field limits of $H$ behave qualitatively different. There must be at least one quantum phase transition separating the two. One can show using <u>duality</u> that the transition occurs at $g_c = 1$.

We will then give an exact solution of the model.

# Duality (in pictures)

| | $g \gg 1$ | $g \ll 1$ |
|---|---|---|
| Ground state | $\lvert \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \rangle$ | $\lvert \downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow \rangle$ (or $\lvert \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \rangle$ ) |
| Basic excitation | Spin flip: $\lvert \rightarrow\rightarrow\rightarrow\rightarrow\underset{i}{\leftarrow}\rightarrow\rightarrow \rangle$ | Domain wall $\lvert \uparrow\uparrow\uparrow\underset{i}{\uparrow}\vdots\underset{i+1}{\downarrow}\downarrow\downarrow\downarrow \rangle$ |
| ... created by | $\hat\sigma_i^z$ | $\prod_{j\le i} \hat\sigma_j^x \triangleq \hat\tau_i^z$ |
| ... counted by | $\hat\sigma_i^x$ | $\hat\sigma_i^z \hat\sigma_{i+1}^z \triangleq \hat\tau_i^x$ |

The operators $\hat\tau_i^z$, $\hat\tau_i^x$ satisfy

$$\{\hat\tau_i^\alpha, \hat\tau_j^\beta\} = 2\delta^{\alpha\beta},$$

that is, the same algebra as $\hat\sigma_i^z$, $\hat\sigma_i^x$. Note that

$$\hat\tau_{i-1}^z \hat\tau_i^z = \Big(\prod_{j<i} \hat\sigma_j^x\Big)\Big(\prod_{j<i+1} \hat\sigma_j^x\Big) = \hat\sigma_i^x$$

$$\Rightarrow \boxed{H_Q = -J\sum_i \left[\hat\sigma_i^z \hat\sigma_{i+1}^z + g\,\hat\sigma_i^x\right] = -Jg\sum_i \left[\frac{1}{g}\hat\tau_i^x + \hat\tau_{i-1}^z\hat\tau_i^z\right]}$$

The model, written in terms of domain wall variables, takes on the same form as the original, but with $\boxed{g \leftrightarrow \frac{1}{g}}$.

$\Rightarrow$ For every phase transition at $g_c$, there must be one at $\frac{1}{g_c}$.

$\Rightarrow$ If only one transition separates the ferromagnet from the paramagnet, it must be at $g_c = 1$.

# Jordan - Wigner transformation

The main idea behind the exact solution of the transverse field Ising model is the realization that spin-$\frac{1}{2}$ degrees of freedom are very similar to spinless fermions, through the following association:

| spins | | | fermions |
|---|---|---|---|
| spin down | $\downarrow$ | $\longleftrightarrow$ | $\bullet$ occupied site |
| spin up | $\uparrow$ | $\longleftrightarrow$ | $\sqcup$ empty site |

Or, in terms of operators ($c_i, c_i^+$ create/annihilate a fermion)

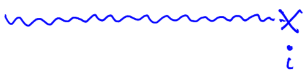
$$\hat{\sigma}_i^z = 1 - 2c_i^+ c_i$$

We are tempted to complete the mapping by associating $\hat{\sigma}_i^+ = (\hat{\sigma}_i^x + i\,\hat{\sigma}_i^y)/2$ with $c_i$ and $\sigma_i^- = (\hat{\sigma}_i^x - i\hat{\sigma}_i^y)/2$ with $c_i^+$. This gives the correct spin algebra on a single site, but when multiple sites are present, it leads to a problem: $\hat{\sigma}_i^\pm$ commute on different sites, whereas $c_i$, $c_i^+$ anticommute.

Jordan and Wigner provided the solution — the naive single-site correspondence must be modified by a "string" operator,

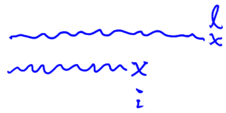
$$\sigma_i^+ = \left[\prod_{j<i} (1 - 2\, c_j^+ c_j)\right] c_i$$

$$\sigma_i^- = \left[\prod_{j<i} (1 - 2\, c_j^+ c_j)\right] c_i^+$$




The string $\prod_{j<i} (1 - 2c_j^+ c_j)$ takes on values $\pm 1$, depending on whether an even/odd number of fermions is present to the left of site $i$.

The string solves the statistics problem. For example, consider (without loss of generality) $l > i$. Then




$$\sigma_i^+ \sigma_l^- = \left(\prod_{j<i} (1-2c_j^\dagger c_j)\right) c_i \left(\prod_{m<l} (1-2c_m^\dagger c_m)\right) c_l^\dagger$$

If we move $c_i$ all the way to the right, we'll pick up two minus signs, one from switching the parity of the $l$-string and the second from anticommuting with $c_l^\dagger$. Note that the $i$-string commutes with everything else and we can move it at will. Hence

$$\sigma_i^+ \sigma_l^- = \left(\prod_{m<l} (1-2c_m^\dagger c_m)\right) c_l^\dagger \left(\prod_{j<i} (1-2c_j^\dagger c_j)\right) c_i = \sigma_l^- \sigma_i^+$$

Thus we have succeeded in mapping spins into fermions. The price we paid is using a highly non-local representation in terms of fermionic operators.

We can also write the inverse relation,

$$c_i = \left(\prod_{j<i} \hat{\sigma}_j^z\right) \hat{\sigma}_i^+$$

$$c_i^\dagger = \left(\prod_{j<i} \hat{\sigma}_j^z\right) \hat{\sigma}_i^-$$

You can check that the relations

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \qquad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$$

imply

$$[\hat{\sigma}_i^+, \hat{\sigma}_j^-] = \delta_{ij} \hat{\sigma}_i^z, \qquad [\hat{\sigma}_i^z, \hat{\sigma}_j^\pm] = \pm 2\delta_{ij} \hat{\sigma}_i^\pm$$

and vice versa.

# Solution of the transverse field Ising model

We presented earlier the standard form of the Jordan-Wigner transformation. For our purposes, it is best to rotate the spin axis,

$$\hat{\sigma}^z \to \hat{\sigma}^x, \qquad \hat{\sigma}^x \to -\hat{\sigma}^z$$

and to write the mapping:

$$\hat{\sigma}_i^x = 1 - 2c_i^\dagger c_i$$

$$\hat{\sigma}_i^z = -\prod_{j<i} (1 - 2c_j^\dagger c_j)\,(c_i + c_i^\dagger)$$

We want to insert this into $H_Q$. The $\hat{\sigma}_i^x$ term is straightforward. For the $\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$ term

$$\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z = \left[\prod_{j<i} (1 - 2c_j^\dagger c_j)\right](c_i + c_i^\dagger)\left[\prod_{j<i+1} (1 - 2c_j^\dagger c_j)\right](c_{i+1} + c_{i+1}^\dagger)$$

$$= (c_i + c_i^\dagger)(1 - 2c_i^\dagger c_i)(c_{i+1} + c_{i+1}^\dagger)$$

$$= (-c_i + c_i^\dagger)(c_{i+1} + c_{i+1}^\dagger)$$

$$= c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i$$

Therefore,

$$H_Q = -J \sum_i \left( c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i - 2g\, c_i^\dagger c_i + g \right)$$

and we see that $H_Q$ is a single-particle operator, i.e. it describes non-interacting fermions!

The anomalous terms $c^\dagger c^\dagger$ and $cc$ imply that particle number is not conserved (although parity is). These arise because $\sigma_i^z \sigma_{i+1}^z$ can flip $|\rightarrow\rangle$ spins in pairs.

In order to diagonalize, we first go to a plane wave basis,

$$c_k = \frac{1}{\sqrt{N}} \sum_j c_j \, e^{-ikr_j}$$

$$\Rightarrow \quad H_Q = J \sum_k \left(2\,(g - \cos ka)\, c_k^\dagger c_k - i \sin ka \left[c_{-k}^\dagger c_k^\dagger + c_{-k} c_k\right] - g\right)$$

$$= 2J \sum_{k>0} \left\{ (c_k^\dagger \;\; c_{-k}) \begin{pmatrix} (g - \cos ka) & i \sin ka \\ -i \sin ka & -(g - \cos ka) \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} - g + (g - \cos ka) \right\} + 2J(g-1)\, c_0^\dagger c_0$$

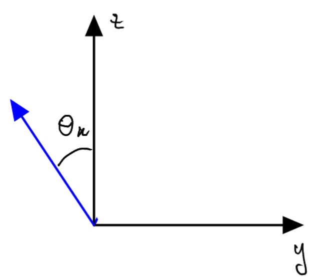
note sum only over half of $k$ points

$k=0$ term

The $k \neq 0$ part involves a matrix

$$\begin{pmatrix} (g - \cos ka) & i \sin ka \\ -i \sin ka & -(g - \cos ka) \end{pmatrix} = (g - \cos ka) \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\mu_z} - \sin ka \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\mu_y}$$

This can be diagonalized by a unitary transformation, amounting to a rotation in the $yz$ plane by angle $\theta_k$




$$\tan \theta_k = \frac{\sin ka}{g - \cos ka}$$

$$U_k = e^{-i\theta_k \frac{\mu_x}{2}} = \begin{pmatrix} \cos \frac{\theta_k}{2} & -i \sin \frac{\theta_k}{2} \\ -i \sin \frac{\theta_k}{2} & \cos \frac{\theta_k}{2} \end{pmatrix}$$

$$U_k^+ \begin{pmatrix} (g-\cos ka) & i\sin ka \\ -i\sin ka & -(g-\cos ka) \end{pmatrix} U_k = \sqrt{(g-\cos ka)^2+\sin^2 ka}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We use $U_k$ to perform a Bogoliubov transformation:

$$\begin{pmatrix} c_k \\ c_{-k}^+ \end{pmatrix} = U_k \begin{pmatrix} \gamma_k \\ \gamma_{-k}^+ \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta_k}{2} & -i\sin\frac{\theta_k}{2} \\ -i\sin\frac{\theta_k}{2} & \cos\frac{\theta_k}{2} \end{pmatrix}\begin{pmatrix} \gamma_k \\ \gamma_{-k}^+ \end{pmatrix}$$

Note that this is different from bosonic case. The $\gamma$s satisfy fermionic anticommutation relations, e.g.

$$\{c_k, c_{k'}^+\} = \left\{\cos\frac{\theta_k}{2}\gamma_k - i\sin\frac{\theta_k}{2}\gamma_{-k}^+ ,\ \cos\frac{\theta_{k'}}{2}\gamma_{k'}^+ + i\sin\frac{\theta_{k'}}{2}\gamma_{-k'}\right\}$$

$$= \cos^2\frac{\theta_k}{2}\underbrace{\{\gamma_k, \gamma_{k'}^+\}}_{\delta_{k,k'}} + \sin^2\frac{\theta_k}{2}\underbrace{\{\gamma_{-k}^+, \gamma_{-k'}\}}_{\delta_{k,k'}}$$

$$= \delta_{k,k'}$$

$$H_Q = \sum_{k>0} \varepsilon_k\left(\gamma_k^+\gamma_k - \gamma_{-k}\gamma_{-k}^+\right) + \text{const.}$$

$$\Rightarrow \quad \boxed{H_Q = \sum_k \varepsilon_k\left(\gamma_k^+\gamma_k - \tfrac{1}{2}\right) + \text{const}}$$

where

$$\boxed{\varepsilon_k = 2J\sqrt{(g-\cos ka)^2+\sin^2 ka} = 2J\sqrt{1+g^2-2g\cos ka}}$$

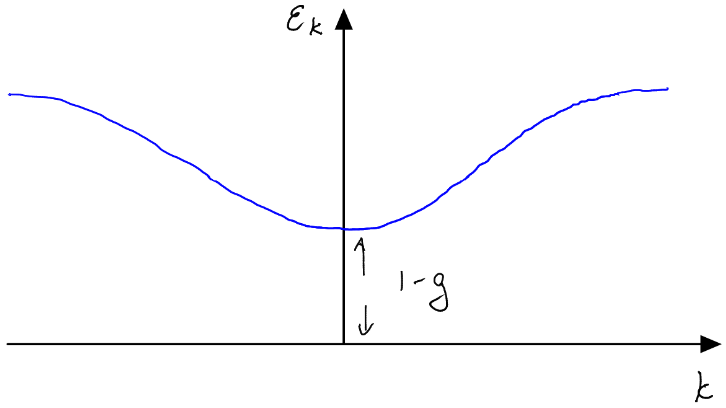
Limits:

* $g \ll 1 \quad \Rightarrow \quad \varepsilon_k = 2J(1-g\cos ka) + O(g^2)$
* $g \gg 1 \quad \Rightarrow \quad \varepsilon_k = 2Jg\left(1-\frac{1}{g}\cos ka + O\left(\frac{1}{g^2}\right)\right)$

matching what we obtained earlier.

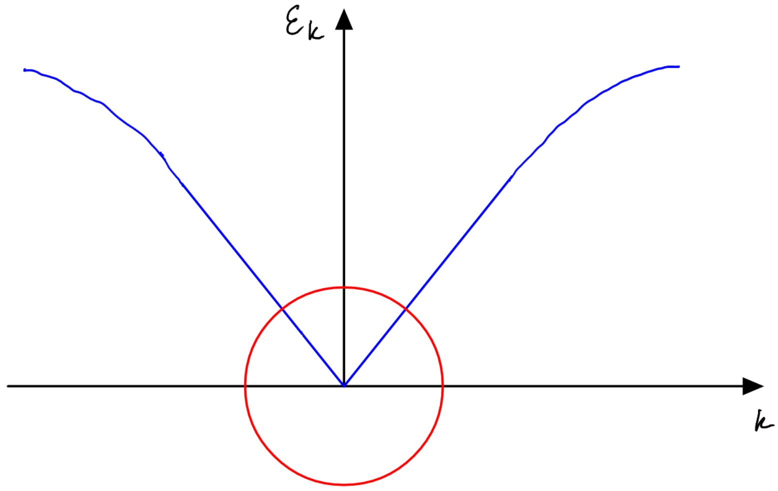
# Single quasiparticle spectrum

**Ferromagnetic phase:** $g < 1$




$\Delta = \text{gap} = 1-g$

**Critical point:** $g = 1$



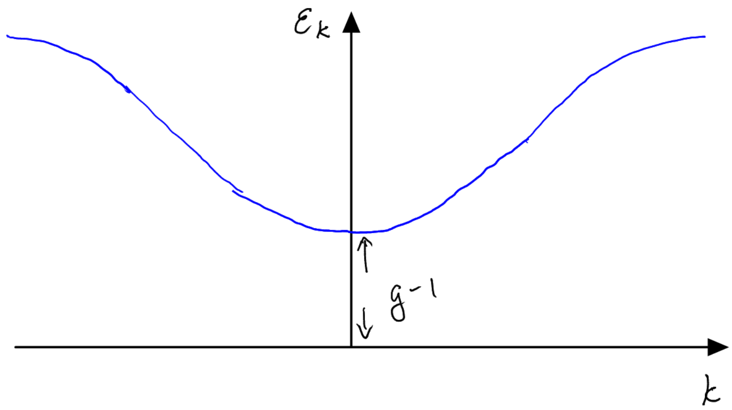

gapless

linearly dispersing mode

$$\varepsilon_k = 2\sqrt{2}\, J \sqrt{1-\cos ka}$$

$$\simeq 2 J a |k| \qquad (ka \ll 1)$$

**Paramagnetic phase:** $g > 1$




$\Delta = g-1$

The full spectrum is obtained by adding multiple quasiparticles (which don't interact with each other).

There is a <u>quantum phase transition</u> at $g_c = 1$. The ground state undergoes a qualitative change at $g_c$, between a phase that breaks the global $\mathbb{Z}_2$ symmetry (ferromagnet) to a phase that doesn't (paramagnet). In either of these phases, there is a gap to excitations, $\Delta = |g - g_c|$.

As the transition is approached, the gap vanishes, corresponding to a divergent correlation time

$$\tau = \frac{\hbar}{\Delta},$$

and also to a divergent correlation length

$$\xi = \frac{c\hbar}{\Delta}$$

where $c = \frac{2Ja}{\hbar}$ is the speed of the gapless excitations at criticality. These divergent scales can be used to go to the continuum limit near $g_c$, and to write the model as a field theory in $D = 1+1$ dimensions. See S. Sachdev's book.

There is one outstanding question in our treatment. The ferromagnetic phase $g < 1$ has a two-fold degenerate spectrum (corresponding to excitations starting from either of the two broken symmetry states). But these are nowhere to be seen in our solution. The reason is that we were a bit careless when treating the string with periodic BCs. We'll instead consider open BCs to understand the degeneracy.

# Majorana operators

Our fermionized Hamiltonian, with open BCs:

$$H_Q = -J\left[\sum_{i=1}^{N-1}(-c_i+c_i^{\dagger})(c_{i+1}+c_{i+1}^{\dagger}) + g\sum_{i=1}^{N}(1-2c_i^{\dagger}c_i)\right]$$

Let $\begin{cases} a_j = c_j + c_j^{\dagger} \\ b_j = i\,(c_j^{\dagger} - c_j) \end{cases}$

Then $a_j^{\dagger} = a_j$ and $b_j^{\dagger} = b_j$,

$$\{a_j, a_{j'}\} = \{c_j + c_j^{\dagger}, c_{j'} + c_{j'}^{\dagger}\} = \{c_j^{\dagger}, c_{j'}\} + \{c_j, c_{j'}^{\dagger}\} = \delta_{j,j'}$$

$$\{b_j, b_{j'}\} = \delta_{j,j'}$$

$$\{b_j, a_{j'}\} = i\,\{c_j^{\dagger} - c_j, c_{j'} + c_{j'}^{\dagger}\} = i\left(\{c_j^{\dagger}, c_{j'}\} - \{c_j, c_{j'}^{\dagger}\}\right) = 0$$

So the $a_j$, $b_j$ are a set of $2N$ independent anticommuting Hermitean operators, each of which squares to 1. We call such operators "Majorana operators".

We can combine pairs of Majoranas to obtain a conventional fermion, e.g.

$$\begin{cases} c_j = \frac{1}{2}(a_j + ib_j) \\ c_j^{\dagger} = \frac{1}{2}(a_j - ib_j) = (c_j)^{\dagger} \end{cases}$$

Then $c_j^{\dagger}c_j = \frac{1}{4}(a_j - ib_j)(a_j + ib_j) = \frac{1}{4}\left(a_j^2 + b_j^2 - 2ia_jb_j\right) = \frac{1}{2}(1 - ia_jb_j)$

$$\Rightarrow \quad -ia_jb_j = 2c_j^{\dagger}c_j - 1$$

Write $H_Q$ using Majoranas:

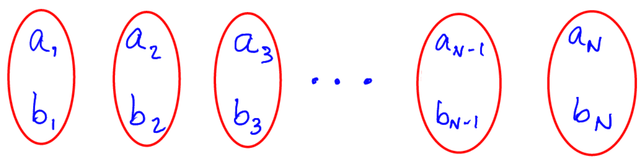
$$H_Q = -J\left[\sum_{i=1}^{N-1}(-c_i + c_i^\dagger)(c_{i+1} + c_{i+1}^\dagger) + g\sum_{i=1}^{N}(1 - 2c_i^\dagger c_i)\right]$$

$$= -J\left[\sum_{i=1}^{N-1}(-i b_i a_{i+1}) + g\sum_{i=1}^{N} i a_i b_i\right]$$

Limits:

* $\underline{g \gg 1}$ $\qquad H_Q \simeq Jg\sum_{i=1}^{N}(-i a_i b_i) = 2Jg\sum_{i=1}^{N} c_i^\dagger c_i + \text{const}$

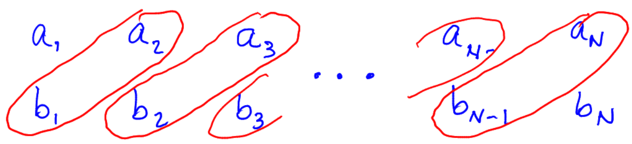
Here, it is most natural to combine Majoranas to reconstitute the original $c$, $c^\dagger$ fermions:




* $\underline{g \ll 1}$ $\qquad H_Q = J\sum_{i=1}^{N-1}(i b_i a_{i+1}) = 2J\sum_{i=1}^{N-1} d_i^\dagger d_i + \text{const}$

where we've defined a new type of fermion, built out of $b_i$ and $a_{i+1}$:

$$\begin{cases} d_i = \frac{1}{2}(b_i + i a_{i+1}) \\ d_i^\dagger = \frac{1}{2}(b_i - i a_{i+1}) \end{cases}$$




The end result looks similar to the $g \gg 1$ case, but note that there are only $N-1$ $d_i$ fermions. We can define

$$d_N = \frac{1}{2}(a_1 + i b_N), \qquad d_N^\dagger = \frac{1}{2}(a_1 - i b_N)$$

Although $d_N$, $d_N^+$ do not appear in $H_Q$, they are necessary to account for all $2^N$ states in the Hilbert space. The energy does not depend on whether $d_N$ is occupied or not. This makes the entire spectrum two-fold degenerate.

What if $g \neq 0$ (but $g < 1$)? Now the Majoranas at the edges, $a_1$ and $b_N$, will hybridize, leading to a term

$$\eta \, i \, a_1 b_N \propto \eta \, d_N^+ d_N$$

in the Hamiltonian. Thus, the degeneracy is lifted for finite systems. However, the hybridization strength $\eta$ is exponentially small in the system size.

$$\eta \sim e^{-N/\xi}$$

where $\xi$ is the correlation length, and for long chains it takes a correspondingly long time for a system initially prepared in the majority up state to reach the minority down state.

The fermionized form of the Ising chain has a striking resemblence to a 1d p-wave superconductor. Majorana modes were first predicted in this context by A. Kitaev, arXiv:cond-mat/0010440.