Cross-Currency Discount Curve Construction Algorithms

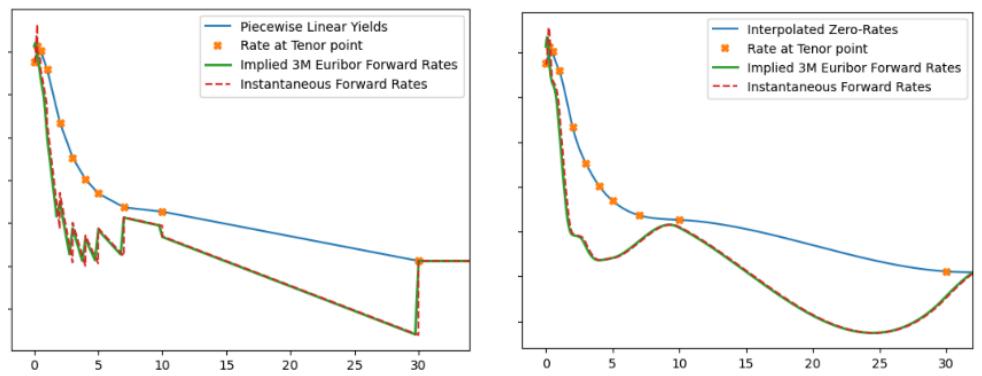
Lee S.

Term Structures and Curve Construction

Bootstrapping/Curve-Calibration is the process of extracting market observable information from quoted instruments to extrapolate a full and continuous Term-Structure of Yields, Forward Rates, and Discount Factors inferred from a mixture of various contracts, such as FRAs, Swaps, and Bonds. This is unlikely to create continuous curves that are needed for valuations as cash-flow dates don't necessarily align with extrapolated rates. There is a need to estimate appropriate curves that can correctly price back the observed market quoted instruments, but also be used in the valuation for new derivatives. The most common estimation method is to interpolate the Zero-Rates as the state-variable and get corresponding Forward and Discounting curves from the definitions:

$$D(t) = e^{-z_t \times t} \qquad f(t_{i-1}, t_i) = \frac{1}{\Delta t}\left(\frac{D(t_{i-1})}{D(t_i)} - 1\right)$$

Linearly and NC-Splines interpolated EURIBOR rates on 03-06-2024.



Cross-Currency Basis Swaps

XCCY Basis Swaps are floating-floating IRS with each leg exchanging IR of different currencies. Prior to the 2007 Financial Crisis, credit/liquidity risk was regarded a less significant concern but due to changing market sentiments, failure of major banks, and scandals involving exploitative fixings of expected lending rates, adjustments had to be made to the structure of contracts. The decrease in confidence within the interbank derivatives market led to adoptions of the Credit Support Annex (rules and conditions to mitigate risk exposure for unsecured OTC exchanges) as common practice. Modifications to XCCY Basis Swaps are:

- Conventional risk-free rate benchmarks no longer reference surveyed, forward-looking IBORs but historical, backward-looking OIS-rates. LIBORs are no longer published, but EURIBOR still continues to be a benchmark. If a XCCY Swap references EURIBOR for the EUR leg but an OIS-rate (SOFR, SONIA) for the other, coupon payments would need to align by having EURIBOR-in-Arrears and OIS-Rates-with-Delays. The spread between current EURIBOR and ESTR is typically considered negligible within these contracts.
- The addition of a basis spread term to a leg, expressing the underlying risk premium between currencies. This was thought to be negligible prior to 2007, but very quickly reached large magnitudes (up to +/- 400 bps) for certain currency pairs, and never again stabilised to low levels pre-crisis.
- The use of a modified XCCY Discount Factor / CSA-curve to the leg(s) of non-collateral currencies. XCCY Basis Swaps are collateralised contracts, secured by liquid capital that offsets credit exposure of a counterparty. Hence there is no longer a straightforward relationship between the Interest Rates and the XCCY Discount Factors used to determine the future value of cash-flows.

$$D_A^{OIS}(t_n) + \sum_{i=1}^{n}\left(D_A^{OIS}(t_i) \times f_A(t_{i-1}, t_i) \times \Delta t_i\right)$$
$$= N\left(D_B^{XCCY}(t_n) + \sum_{i=1}^{n}\left(D_B^{XCCY}(t_i) \times (f_B(t_{i-1}, t_i) + s_n) \times \Delta t_i\right)\right)$$

is a standard structure of a Constant Notional Float-Float XCCY Basis Swap that aligns coupon payment dates. D_A^{OIS} and D_B^{XCCY} are the risk-free and XCCY Discount Factors applied to collateral currency A and non-collateral currency B respectively. f_A, f_B are the Forward Rates over the accrual periods Δt_i and s_n is the XCCY basis spread associated with the maturity t_n. The FX-Rate between currencies is entirely expressed by an appropriate notional amount N.

Bootstrapping Algorithm

The lack of market observable inter-currency derivatives creates a difficulty in constructing the CSA-Curves required to price the non-collateral currency floating payments. Assumptions are imposed on the swap equation to extrapolate the full XCCY Discount Curve. For a set of spreads $\{s_k\}$ associated with consecutive tenors $\{\tau_k\}$ and coupon payments at 3-month intervals, the XCCY Swap priced at par is

$$1 = D(\tau_K) + \frac{1}{4}\sum_{k=1}^{K}\sum_{j=1}^{J(k)}\left(D\left(\tau_{k-1} + \tfrac{j}{4}\right)\left(f_B\left(\tau_{k-1} + \tfrac{j-1}{4},\ \tau_{k-1} + \tfrac{j}{4}\right) + s_k\right)\right)$$

where D is the XCCY Discount Factor applied to currency B flows collateralised in currency A, $J(k) = 4(\tau_k - \tau_{k-1})$ are the number of quarters between two consecutive tenor points. To build the full CSA-curve, it is necessary to impose a relationship between Discount Factors at consecutive tenors, and we assume piecewise-constant quarterly discount rates:

$$\frac{D\left(\tau_{k-1} + \frac{1}{4}\right)}{D(\tau_{k-1})} = \frac{D\left(\tau_{k-1} + \frac{2}{4}\right)}{D\left(\tau_{k-1} + \frac{1}{4}\right)} = \cdots = \frac{D(\tau_k)}{D\left(\tau_k - \frac{1}{4}\right)}$$

and from this there is a sequence of polynomials functions with constant terms:

$$P_k(x) = x^{J(K)} + \frac{1}{4}\sum_{j=1}^{J(K)} D(\tau_{k-1})^{1-\frac{j}{J(K)}}\left(f_B\left(\tau_{k-1} + \tfrac{j-1}{4}, \tau_{k-1} + \tfrac{j}{4}\right) + s_k\right)x^j$$
$$+ \frac{1}{4}\sum_{i=1}^{t_n = \tau_{k-1} - \frac{1}{4}}\left(D\left(\tfrac{i}{4}\right)\left(f_B(t_{i-1}, t_i) + s_k\right)\right) - 1$$

associated with basis spread $\{s_k\}$ at tenors $\{\tau_k\}$ whose solutions $x = D(\tau_k)^{\frac{1}{J(K)}}$ are roots of the XCCY Discount Factors subsequently calculated to derive the intermediate quarterly discounts from the piecewise-constant assumption. The entire CSA-Curve is constructed by iterating through available spreads at tenors given the priorly computed preceding XCCY Discount Factors. The validity of this algorithm is vindicated by the constructed curves closely matching the quoted CSA-Curves on Bloomberg **within +/- 10 bps and less than 0.25% difference** throughout all maturities. Assuming piecewise-constant quarterly discounts is equivalent to a constant rate of change of XCCY Zero-Rates imposing continuity

$$z_k\left(\tau_{k-1} + \tfrac{j}{4}\right) = \frac{1}{\tau_{k-1} + \frac{j}{4}}\left[\tau_{k-1} z(\tau_{k-1})\frac{\tau_k - \left(\tau_{k-1} + \frac{j}{4}\right)}{\tau_k - \tau_{k-1}} + \tau_k z(\tau_k)\frac{\frac{j}{4}}{\tau_k - \tau_{k-1}}\right]$$

and exhibiting smoothness of the XCCY Zero-Rate curve between tenors of quoted basis spreads. The algorithm is functionally an estimation method (analogous to interpolation) that establishes continuity between nodal points.

Simulation for Risk-Measurement

The Bootstrapping Algorithm above only considers the IR in the non-collateral currency since the floating leg is priced at par, suggesting that the IR differential of two economies can be characterised by the spreads. The rates in the non-collateral currency as well as the basis spreads must be simulated to measure the risk inherent within instruments involving XCCY CSA-Curves. This introduces additional complexity to models that rely on generating potential movements of IR since the spreads are non-negligible. On a practical aspect, the runtime of curve construction is not immaterial when applied to thousands of simulations. Investigation has determined that the main computational bottleneck of the Bootstrapping Algorithm is the framework of the IDE implemented to solve the sequence of polynomials $\{P_k\}$.

$$x_{n+1} = x_n - \frac{P_k(x_n)}{P_k'(x_n)}$$

The above is an iterative step of Newton-Raphson, an efficient root-finding algorithm. From initial value x_0, each successive iteration outputs a better approximate solution of P_k until an acceptable threshold of error is achieved. For the polynomial, the number of iterations for convergence can be greatly reduced by an informed choice of initial parameter. When solving for the XCCY Discount Factor at $x_k = [D(\tau_k)]^{\frac{1}{J(K)}}$, the optimal initialisation is $x_0 = x_{k-1} = [D(\tau_{k-1})]^{\frac{1}{J(k-1)}}$, the variable of the prior polynomial in the sequence. This reliably converges within an upper limit of 10 iterations and an error threshold of 10^{-15}.

(Adjusted) Covered Interest Rate Parity

Covered Interest Rate Parity was a fundamental no-arbitrage condition in international markets but since the emergence of XCCY Basis Spreads during the Global Financial Crisis, it has been systematically violated and no longer holds.

$$X_T = X_t \times \frac{1 + f_A(t, T) \times \Delta t}{1 + f_B(t, T) \times \Delta t}$$

It was historically formulated under the presumptions of high liquidity and market efficiency particularly for the short-term dynamics of IR and FX-Rates and understood to be less robust farther into the future due to structural frictions and IR unpredictability from regulatory, geopolitical, or macroeconomic changes. Sentiments regarding counterparty risk causes a prevalent inconsistency between the empirical IR / FX-Rates and those implied by CIP. The basis is the standard metric that reflects the disparity in the cost of funding between two currencies. Bloomberg/NY-Fed/CME defines the XCCY/FX Basis Spread as the difference between market quoted IR and the theoretical IR implied by CIP:

$$s \equiv \left[\frac{1}{\Delta t} \times \left(\frac{X_t}{X_0} \times (1 + f_A(0, t) \times \Delta t) - 1\right)\right] - f_B(0, t)$$

This allows a reformulation of the CIP no-arbitrage condition that acknowledges liquidity/credit risk and the existence of the required collateral in a CSA contract.

$$X_t = X_0 \times \frac{1 + f_A(0, t) \times \Delta t}{1 + (f_B(0, t) + s) \times \Delta t}$$

This Adjusted Covered Interest Rate Parity accurately conforms to market conditions and is an equivalently annualised calculation analogous to the Forward Rates from which is it defined. The Adjusted-CIP describes a relationship on sequential FX-rates and can be decomposed into a product sum that converges as

$$X_\tau = X_0 \lim_{\substack{\tau = n\,\Delta t \\ n \to \infty \\ \Delta t \to 0}} \prod_{i=1}^{n}\left(\frac{1 + f_A(t_{i-1}, t_i)\Delta t}{1 + (f_B(t_{i-1}, t_i) + s)\Delta t}\right) = X_0\,\frac{D_B^{OIS}(\tau)\exp(-s\,\tau)}{D_A^{OIS}(\tau)}$$

providing an expression between the forward/spot FX-rates, the risk-free OIS Discount Factors in each currency, and the XCCY basis spread for the period.

Curve Construction from Explicit Calculation

The CSA-curve is foundationally distinct from the single-currency Discount-curve, due to the auxiliary burden of describing the time-value of the liquid collateral that secures the exchange and offsets credit exposure. The counterparty that is In-the-Money at payment dates receives a collateral amount, determined by the structure of the exchange. If fair amounts of currencies are exchanged on an agreed upon future date, the non-collateral currency (or both currencies for exchanges collateralised in a third currency) no longer discounts with the risk-free rate but shifts to describe collateralisation.

$$N_A^{PV} = X_0 N_B^{PV} \implies N_A^{FV} D_A^{OIS} = X_0 N_B^{FV} D_B^{XCCY} \implies D_B^{XCCY} = \frac{X_t}{X_0} D_A^{OIS}$$

With the Adjusted-CIP formula, it can be demonstrated that the XCCY Discount Factors has an explicit calculation

$$D_B^{XCCY}(\tau_k) = D_B^{OIS}(\tau_k) \times exp(-s_k \times \tau_k)$$

at tenors for which there are quoted basis spreads. Bootstrapping the CSA-curve can be bypassed on regions of the Term-Structure described unambiguously by the derived equation. Estimation via Interpolation Schemes on the XCCY Zero-Rates are used to construct the full CSA-curve.

Risk Drivers of the Cross-Currency Discount Curve

Notice that both the Bootstrapping Algorithm and the Explicit Calculation of the XCCY Discount Factors aren't invariant to the IR in the collateral currency but it is the basis spread that integrates this variable in the CSA-curve. The derived formula also characterises the deviation of the XCCY Discount Factors from the risk-free Discounts; it is proportional to the exponential of the XCCY basis spreads and coincides with the OIS Discount Factors if the basis is close to 0. This approximation holds greater weight on the short end of the curve because of the nearer-sighted forecasting nature of CIP, but also because equivalent spreads would inherently cause greater deviation from the OIS discounts for the far future.

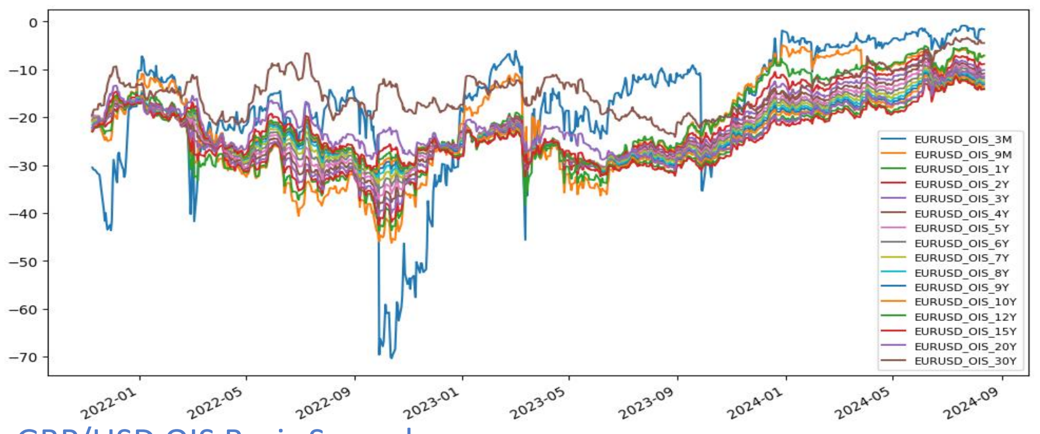
$$\frac{\partial}{\partial s_k} D_B^{XCCY}(\tau_k) = -\tau_k \times D_B^{OIS}(\tau_k) \times \exp(-s_k \times \tau_k) = -\tau_k \times D_B^{XCCY}(\tau_k)$$

For the purposes of risk-measurement, the simulations of IR without considering the basis creates less error on the nearer term when constructing the XCCY Discount Factors, but it is important to recognise the relationship and the compounding dependency on the XCCY basis spreads farther into the future.

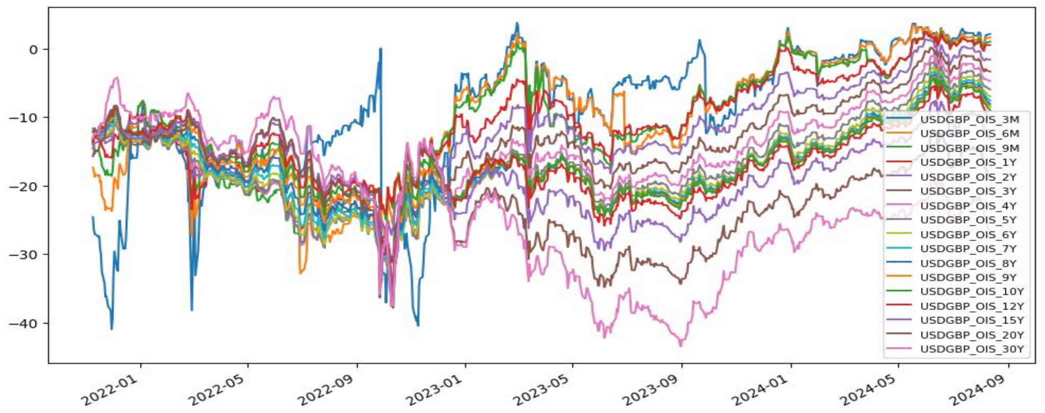
Historical Basis Spreads

Few market data repositories currently publish long-dated historical XCCY basis spreads, even for discontinued IBOR fixings, although it is not a complex task to find figures that illustrate fluctuations of spreads since the blowout in 2007. The graphs below show recent XCCY basis spreads for EUR/USD and GBP/USD pairs for the major OIS-rates, SONIA for GBP, SOFR for USD, and ESTR for EUR.

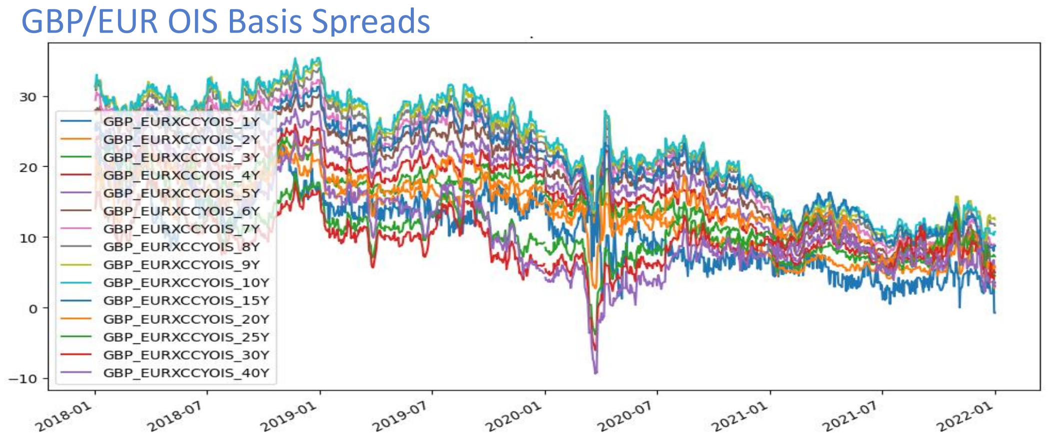
EUR/USD OIS Basis Spreads



GBP/USD OIS Basis Spreads



GBP/EUR OIS Basis Spreads



EUR/GBP spreads do not continue past 2022, unlike the EUR/USD and GBP/USD pairs as it is no longer commonly quoted on the market. There is a unique need for EUR/GBP Basis Swaps to have CSA contracts that are specifically tailored to needs of counterparties. This may involve triangulation of the XCCY Discount Factors applied on a floating leg, collateralisation of both currency cash flows in USD, or stripping the XCCY basis spreads directly as defined. The additional operational complexity can be worthwhile since the traded volume of EUR/GBP Basis Swaps is not insignificant despite being considered a "fringe" commodity.

Combining Curve Construction Methods

There isn't a straightforward explicit relationship between spreads of shorter and longer tenors. Various XCCY Discount Curve Construction Algorithms in practice strip both basis spreads and the CSA-curve simultaneously, especially relevant to the longer end of the curve, for which the Adjusted-CIP relationship becomes tenuous. When optimising for accuracy, it is reasonable to implement the Bootstrapping Algorithm on longer maturities while explicitly calculating the XCCY Discount Factors in the short/medium term.

Term	3M	6M	9M	1Y	2Y	3Y	4Y	5Y	6Y
CSA-Cst	0.9911	0.9835	0.9768	0.9709	0.9510	0.9325	0.9142	0.8956	0.8767
CSA-BB	0.9910	0.9832	0.9767	0.9708	0.9509	0.9325	0.9141	0.8955	0.8767
Diff (%)	0.010%	0.034%	0.006%	0.006%	0.017%	0.010%	0.009%	0.005%	0.001%
Term	7Y	8Y	9Y	10Y	12Y	15Y	20Y	30Y	50Y
CSA-Cst	0.8576	0.8383	0.8186	0.7989	0.7593	0.7024	0.6249	0.5103	0.3551
CSA-BB	0.8577	0.8382	0.8187	0.7990	0.7594	0.7025	0.6244	0.5090	0.3541
Diff (%)	-0.003%	0.006%	-0.005%	-0.011%	-0.013%	-0.017%	0.083%	0.270%	0.292%

The table compares the constructed CSA-curve with the quoted Bloomberg EUR/USD curve with spot date of 03-08-2024. The **green** represents explicitly calculated values, and the **orange** are XCCY Discount Factors from the Bootstrapping Algorithm. Combining the two methods act complementarily to one another to create a XCCY Term-Structure more accurate than either method implemented individually. On latter maturities, the more accurately calculated prior values allow the Bootstrapping Algorithm to optimise for reduced error.