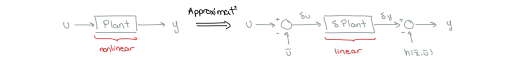


**Modelling CS:** control input,  $y$ : plant output  
**State variable** CS is in state variable form if  
 $\dot{x}_1 = f_1(t, x_1, \dots, x_n, u), \dots, \dot{x}_n = f_n(t, x_1, \dots, x_n, u)$   
 $y = h(t, x_1, \dots, x_n, u)$  is a collection of  $n$  1st order ODEs.  
**Time-Invariant (TI)** CS is TI if  $f_i(\cdot)$  does not depend on  $t$ .  
**State space (SS)** TI CS is in SS form if  $\dot{x} = f(x, u), y = h(x, u)$  where  $x(t) \in \mathbb{R}^n$  is called the state.  
**Single-input-single-output (SISO)** CS is SISO if  $u(t), y(t) \in \mathbb{R}$ .  
**LTI** CS in SS form is LTI if  $\dot{x} = Ax + Bu, y = Cx + Du$   
 $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}$   
**\*SISO:**  $p = 1, m = 1$   
 $*x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}, y(t) \in \mathbb{R}^p$ .  
**Input-Output (IO)** LTI CS is in IO form if  
 $\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + \dots + b_1 \frac{du}{dt} + b_0 u$   
 $*m \leq n$  (causality)  
**IO to SS Model** 1. Define  $x$  s.t. highest order derivative in  $\dot{x}$   
 2.1 If LTI, then  
 \*Write  $\dot{x} = Ax + Bu = f(x, u)$  by isolating for components of  $\dot{x}$   
 \*Write  $y = Cx + Du = h(x, u)$  by setting measurement output  $y$  to component of  $x$   
 2.2 If not LTI, then  
 \*Write  $\dot{x} = f(x, u)$  by isolating for components of  $\dot{x}$   
 \*Write  $y = h(x, u)$  by setting measurement output  $y$  to component of  $x$   
**Equilibrium**  $y_d$  (steady state) b/c if  $y(0) = y_d$  at  $t = 0$ , then  $y(t) = y_d \forall t \geq 0$ .

**Equilibrium pair** Consider the system  $\dot{x} = f(x, u)$ . The pair  $(\bar{x}, \bar{u})$  is an equilibrium pair if  $f(\bar{x}, \bar{u}) = 0$ .  
**Equilibrium point**  $\bar{x}$  is an equilibrium point w/ control  $u = \bar{u}$ .  
**\*If**  $u = \bar{u}$  and  $x(0) = \bar{x}$  then  $x(t) = \bar{x} \forall t \geq 0$  (i.e. a system that starts at equilibrium remains at equilibrium).  
**Find Equilibrium Pair/Point** 1. Set  $f(x, u) = 0$   
 2. Solve  $f(x, u) = 0$  to find  $(x, u) = (\bar{x}, \bar{u})$ .  
 If specific  $u = \bar{u}$ , then find  $x = \bar{x}$  by solving  $f(x, \bar{u}) = 0$ .

**Linearization of Nonlinear System** Consider system  $\dot{x} = f(x, u)$  w/ equ. pair  $(\bar{x}, \bar{u})$ , then error coordinates around equ. pair  $\delta x = x - \bar{x}, \delta u = u - \bar{u}, \delta y = y - h(\bar{x}, \bar{u})$   $\delta \dot{x} = \dot{x} - f(\bar{x}, \bar{u})$  w/  
 $\delta \dot{x} = A\delta x + B\delta u, A = \frac{\partial f(\bar{x}, \bar{u})}{\partial x} \in \mathbb{R}^{n_1 \times n_1}, B = \frac{\partial f(\bar{x}, \bar{u})}{\partial u} \in \mathbb{R}^{n_1},$   
 $\delta y = C\delta x + D\delta u, C = \frac{\partial h(\bar{x}, \bar{u})}{\partial x} \in \mathbb{R}^{1 \times n_1}, D = \frac{\partial h(\bar{x}, \bar{u})}{\partial u} \in \mathbb{R}$   
**\*Only valid at equ. pairs.**

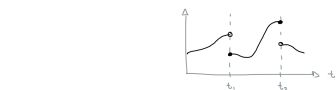


**Linear Approx.** Given a diff. fcn.  $f: \mathbb{R} \rightarrow \mathbb{R}$ , its linear approx. at  $\bar{x}$  is  $f_{lin} = f(\bar{x}) + f'(\bar{x})(x - \bar{x})$ .  
**\*Remainder** Thm:  $f(x) = f_{lin} + r(x)$  where  $\lim_{x \rightarrow \bar{x}} \frac{r(x)}{x - \bar{x}} = 0$ .



**\*Note:** Can provide a good approx. near  $\bar{x}$  but not globally.  
**\*Gen. f:**  $\mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}, f(x) = f(\bar{x}) + \frac{\partial f}{\partial x}(\bar{x})(x - \bar{x}) + R(x)$   
**\*Jacobian:**  $\frac{\partial f}{\partial x}(x) = \left[ \frac{\partial f}{\partial x_1}(x) \quad \dots \quad \frac{\partial f}{\partial x_{n_1}}(x) \right] \in \mathbb{R}^{n_2 \times n_1}$   
**Linearization Steps** 1. Find equ. pair  $(\bar{x}, \bar{u})$   
 2. Derive  $A, B, C, D$  and then evaluate at  $(\bar{x}, \bar{u})$   
 3. Write  $\delta \dot{x} = A\delta x + B\delta u$  and  $\delta y = C\delta x + D\delta u$

**Laplace Transform** Given a fcn  $f: \mathbb{R}_+ \rightarrow [0, \infty) \rightarrow \mathbb{R}^n$ , its Laplace transform is  $F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st} dt, s \in \mathbb{C}$ .  
**\* $\mathcal{L}: f(t) \mapsto F(s), t \in \mathbb{R}_+$  (time dom.) &  $s \in \mathbb{C}$  (Laplace dom.).**  
**P.W. CTS:** A fcn  $f: \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is p.w. cts if on every finite interval of  $\mathbb{R}$ ,  $f(t)$  has at most a finite # of discontinuity points  $(t_i)$  and the limits  $\lim_{t \rightarrow t_i^+} f(t), \lim_{t \rightarrow t_i^-} f(t)$  are finite.



**Exp. Order** A function  $f: \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is of **exp. order** if  $\exists$  constants  $K, \rho, T > 0$  s.t.  $\|f(t)\| \leq Ke^{\rho t}, \forall t \geq T$ .  
**Existence of LT Thm** If  $f(t)$  is p.w. cts and of exp. order w/ constants  $K, \rho, T > 0$ , then  $F(\cdot)$  exists and is defined  $\forall s \in D := \{s \in \mathbb{C} : \text{Re}(s) > \rho\}$  and  $F(\cdot)$  is analytic on  $D$ .  
**\*Analytic** fcn iff differentiable fcn.  
**\*D:** Region of convergence (ROC), open half plane.



**Unit Step**  $1(t) := \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$   
**Table of Common Laplace Transforms:**  $f(t) \mid F(s)$   
 $1(t) \mapsto \frac{1}{s} \quad t1(t) \mapsto \frac{1}{s^2} \quad t^k 1(t) \mapsto \frac{k!}{s^{k+1}} \quad e^{at} 1(t) \mapsto \frac{1}{s-a}$   
 $t^n e^{at} 1(t) \mapsto \frac{n!}{(s-a)^{n+1}} \quad \sin(at) 1(t) \mapsto \frac{a}{s^2+a^2}$   
 $\cos(at) 1(t) \mapsto \frac{s}{s^2+a^2} \quad \frac{1}{2\omega} [\sin(\omega t) - \omega t \cos(\omega t)] 1(t) \mapsto \frac{1}{(s^2+\omega^2)^2}$

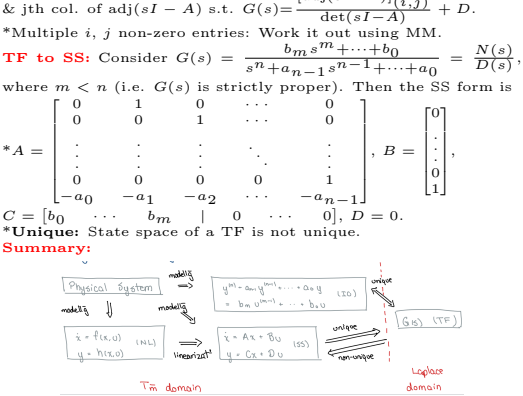
**Prop. of Laplace Transform Linearity:**  $\mathcal{L}\{cf(t) + g(t)\} = c\mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}, c \sim \text{constant}$ .  
**Differentiation:** If the Laplace transform of  $f'(t)$  exists, then  $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^-)$ .  
 If the Laplace transform of  $f^{(n)}(t) := \frac{d^n f}{dt^n}(t)$  exists, then  $\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - \sum_{i=1}^n s^{n-i} f(i-1)(0^-)$ .  
**Integration:**  $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} \mathcal{L}\{f(t)\}$ .  
**Convolution:** Let  $(f * g)(t) := \int_0^t f(\tau)g(t-\tau) d\tau = \int_0^t f(t-\tau)g(\tau) d\tau$ , then  $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ .  
**Time Delay:**  $\mathcal{L}\{f(t-T)1(t-T)\} = e^{-Ts} \mathcal{L}\{f(t)\}, T \geq 0$ .  
**Multiplication by t:**  $\mathcal{L}\{tf(t)\} = -\frac{d}{ds} \mathcal{L}\{f(t)\}$ .  
**Shift in s:**  $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}|_{s \rightarrow s-a} = F(s-a)$ , where  $F(s) = \mathcal{L}\{f(t)\}$  &  $a$  const.  
**Trig. Id.**  $\frac{1}{2} \sin(2t) = \sin(t) \cos(t), \sin(a-b) = \sin(a) \cos(b) - \cos(a) \sin(b), \cos(a-b) = \cos(a) \cos(b) + \sin(a) \sin(b)$   
**Complete the Square:**  $ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c$

**Inverse Laplace Transform** Given  $F(s)$ , its inverse LT is  $f(t) = \mathcal{L}^{-1}\{F(s)\} := \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds$   
 $= \lim_{w \rightarrow \infty} \frac{1}{2\pi j} \int_{c-jw}^{c+jw} F(s)e^{st} ds, c \in \mathbb{C}$  is selected s.t. the line  $L := \{s \in \mathbb{C} : s = c + j\omega, \omega \in \mathbb{R}\}$  is inside the ROC of  $F(s)$ .  
**Zero:**  $z \in \mathbb{C}$  is a zero of  $F(s)$  if  $F(z) = 0$ .  
**Pole:**  $p \in \mathbb{C}$  is a pole of  $F(s)$  if  $\frac{1}{F(p)} = 0$ .  
**Cauchy's Residue THM** If  $F(s)$  is analytic (complex diff.) everywhere except at isolated poles  $\{p_1, \dots, p_N\}$ , then  $\mathcal{L}^{-1}\{F(s)\} = \sum_{i=1}^N \text{Res}\left[F(s)e^{st}, s = p_i\right] 1(t)$ ,  
**\*Res** $[F(s)e^{st}, s = p_i]$ : Residue of  $F(s)e^{st}$  at  $s = p_i$ .  
**Residue Computation** Let  $G(s)$  be a complex analytic fcn w/ a pole at  $s = p, r$  be the multiplicity of the pole  $p$ . Then  $\text{Res}[G(s), s = p] = \frac{1}{(r-1)!} \lim_{s \rightarrow p} \frac{d^{r-1}}{ds^{r-1}} [G(s)(s-p)^r]$ .  
**Inv. LT Partial Frac.:** 1. Factorize  $F(s)$  into partial fractions.  
 2. Find coefficients and use LT table to find inverse LT.  
**\*Complete the square.**  
**Inv. LT Residue:** 1. Find poles of  $F(s)$  and their residues.  
 2. Use Cauchy's Residue THM to find inverse LT.  
**\*Note:** Complex Conjugate (CC) poles  $\rightarrow$  CC residues (use Euler).  
 $*\cos(x) = \frac{e^{jx} + e^{-jx}}{2}, \sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$   
**Transfer Function:** Consider a CS in IO form. Assume zero initial conds.  $y(0) = \dots = \frac{d^{(m-1)} y}{dt^{(m-1)}}(0) = 0$  and

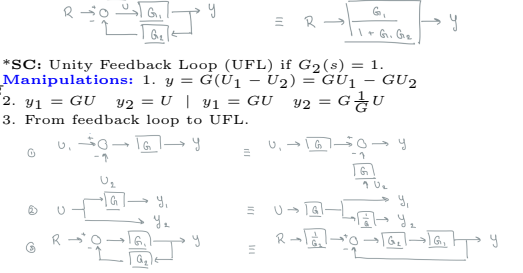
$u(0) = \dots = \frac{d^{(m-1)} u}{dt^{(m-1)}}(0) = 0$ . Then the TF from  $u$  to  $y$  is  
 $G(s) := \frac{Y(s)}{U(s)} = \frac{b_m s^m + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0}$ .  
**\*0 Ini. Conds.:**  $y_0(s) = G(s)u(s)$   
**\* $\emptyset$  Ini. Conds.:**  $y_\emptyset(s) = y_0(s) + \frac{\text{poly. based on initial conds.}}{s^n + a_{n-1} s^{n-1} + \dots + a_0}$

**TF Steps (IO to TF):** 1. Given IO form of CS, assume zero initial conds.  
 2. Find  $G(s)$  by taking LT of IO form and forming  $Y(s)/U(s)$ .  
**\*Careful:**  $Y(s)/U(s) = G(s)$  not  $U(s)/Y(s) = G(s)$ .  
**Impulse Response:** Given CS modeled by TF  $G(s)$ , its IR is  $g(t) := \mathcal{L}^{-1}\{G(s)\}$ .  
**\* $\mathcal{L}\{\delta(t)\} = 1$** , then if  $u(t) = \delta(t)$ , then  $Y(s) = U(s)G(s) = G(s)$ .  
**SS to TF:**  $G(s) = C(sI - A)^{-1}B + D$  s.t.  $y(s) = G(s)U(s)$ .  
**\*Assume**  $x(0) = 0 \in \mathbb{R}^n$  (zero initial conds.).  
**\*LTI:**  $G(s)$  of an LTI system is always a rational fcn.  
**\*Not Invertible:** Values of  $s$  s.t.  $sI - A$  non invertible can correspond to poles of  $G(s)$ .  
**Inverse:** 1. For  $A \in \mathbb{R}^{n \times n}$ , find  $[\text{cof}(A)]_{(i,j)} = (-1)^{i+j} \det(A_{(i,j)})$ .  
**\* $A_{(i,j)}$ :**  $A$  w/ row  $i$  and col.  $j$  removed.  
 2. Assemble  $\text{cof}(A)$  and find  $\det(A) = \sum_{j=1}^n a_{ij} [\text{cof}(A)]_{(i,j)}$  w/ fixed  $i$  or  $\det(A) = \sum_{i=1}^n a_{ij} [\text{cof}(A)]_{(i,j)}$  w/ fixed  $j$ .  
 3. Find  $A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{\det(A)} [\text{cof}(A)]^T$ .  
**\*2 x 2:**  $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

**TF (SS to TF):** 1. Given SS form, assume zero initial conds.  
 2. Solve  $G(s) = C(sI - A)^{-1}B + D$ .  
**\*If**  $C = [0 \quad I_k \quad 0]$  &  $B = [0 \quad I_j \quad 0]$ , then only need  $i$ th row &  $j$ th col. of  $\text{adj}(sI - A)$  s.t.  $G(s) = \frac{[\text{adj}(sI - A)]_{(i,j)}}{\det(sI - A)} + D$ .  
**\*Multiple**  $i, j$  non-zero entries: Work it out using MM.  
**TF to SS:** Consider  $G(s) = \frac{b_m s^m + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0} = \frac{N(s)}{D(s)}$ , where  $m < n$  (i.e.  $G(s)$  is strictly proper). Then the SS form is  
 $A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$   
 $C = [b_0 \quad \dots \quad b_m \quad 0 \quad \dots \quad 0], D = 0$ .  
**\*Unique:** State space of a TF is not unique.  
**Summary:**



**Block Diagram Types of Blocks:**  
**Cascade:**  $y_2 = (G_1(s)G_2(s))U \stackrel{\text{SISO}}{=} y_2 = (G_2(s)G_1(s))U$   
 $U \mapsto \boxed{G_1} \xrightarrow{y_1} \boxed{G_2} \xrightarrow{y_2} \dots \equiv U \mapsto \boxed{G_1 G_2} \xrightarrow{y_2}$   
**Parallel**  $y = (G_1(s) + G_2(s))U$   
 $U \mapsto \boxed{G_1} \xrightarrow{y_1} \boxed{G_2} \xrightarrow{y_2} \dots \equiv U \mapsto \boxed{G_1 + G_2} \xrightarrow{y}$   
**Feedback**  $y = \left( \frac{G_1(s)}{1 + G_1(s)G_2(s)} \right) R$   
 $R \mapsto \boxed{G_1} \xrightarrow{y_1} \boxed{G_2} \xrightarrow{y_2} \dots \equiv R \mapsto \boxed{G_1 + G_1 G_2} \xrightarrow{y}$



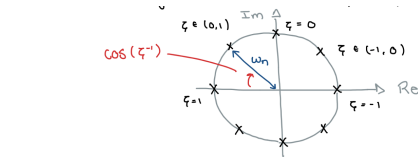
**\*SC:** Unity Feedback Loop (UFL) if  $G_2(s) = 1$ .  
**Manipulations:** 1.  $y = G(U_1 - U_2) = GU_1 - GU_2$   
 2.  $y_1 = GU \quad y_2 = U \mid y_1 = GU \quad y_2 = G \frac{1}{G} U$   
 3. From feedback loop to UFL.  
 $0 \quad U \xrightarrow{\frac{1}{G}} \boxed{G_1} \xrightarrow{y} \dots \equiv U \mapsto \boxed{G_1} \xrightarrow{\frac{1}{G}} U$   
 $1 \quad U \mapsto \boxed{G_1} \xrightarrow{y_1} \dots \equiv U \mapsto \boxed{G_1} \xrightarrow{\frac{1}{G}} U$   
 $2 \quad U \mapsto \boxed{G_1} \xrightarrow{y_1} \dots \equiv U \mapsto \boxed{G_1} \xrightarrow{\frac{1}{G}} U$   
 $3 \quad R \mapsto \boxed{G_1} \xrightarrow{y_1} \dots \equiv R \mapsto \boxed{G_1} \xrightarrow{\frac{1}{G}} U$

**Find TF from Block Diagram:** 1. Start from in  $\rightarrow$  out, making simplifications using block diagram rules.  
 2. Simplify until you get the form  $U(s) \rightarrow \boxed{G(s)} \rightarrow Y(s)$ .  
**Time Response of Elementary Terms:**  $1(t) \leftarrow$  pole @ 0  
 $t^n 1(t) \leftarrow$  pole @ 0 w/ mult.  $n \mid e^{at} 1(t) \leftarrow$  pole @  $a$   
 $\sin(\omega t + \phi) 1(t) \leftarrow$  pole @  $\pm j\omega \mid \cos(\omega t + \phi) 1(t) \leftarrow$  pole @  $\pm j\omega$

**Real Pole:**  $y(s) = \frac{1}{s+a}$ , real pole at  $s = -a$ , then  $y(t) = e^{-at} 1(t)$   
 1.  $a > 0 \implies \lim_{t \rightarrow \infty} y(t) = 0 \mid 2. a < 0 \implies \lim_{t \rightarrow \infty} y(t) = \infty$ .  
 3.  $a = 0 \implies y(t) = 1(t)$  is constant.



**Time Constant:**  $\tau = \frac{1}{a}$  of the pole  $s = -a$  for  $a > 0$   
**Pair of Comp. Conj. Poles:**  
 $y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2}, |\zeta| < 1$ , then  
 $y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t) 1(t)$   
**\*Poles:**  $s_{1,2} = -\zeta\omega_n \pm j\omega_n \sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$   
**\* $\zeta = \frac{\sigma}{\omega_n}$ :** Damping ratio (or damping coefficient)  
**\* $\sigma = \zeta\omega_n$ :** Decay/growth rate  $\mid \omega_d$ : Freq. of oscillation  
**\* $\omega_n = \sqrt{\sigma^2 + \omega_d^2} \left[ \frac{\text{radians}}{\text{seconds}} \right]$ :** Undamped natural freq.  
**\* $\omega_d = \omega_n \sqrt{1-\zeta^2} \left[ \frac{\text{radians}}{\text{seconds}} \right]$ :** Damped natural freq.  
**\* $|s_{1,2}|^2 = \omega_n^2$ :** Mag. of poles is  $\omega_n$ .  
**\* $\cos^{-1}(\zeta)$ :** Angle of  $s_1$  on complex plane CW from -ve Re axis.



**Damping Ratio Effect:**  $0 < \zeta_1 < \zeta_2 < 1$ , then  

 -  $1 < \zeta_4 < \zeta_3 < 0$ , then  $\sigma = \zeta\omega_n < 0$ , (exp. envelop  $\uparrow$ )

**Class. of 2nd Order Sys.:**  $y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ , w/ any  $|\zeta|$   

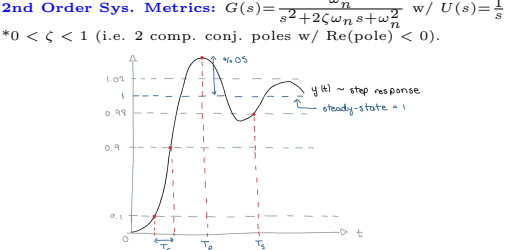
 -  $0 < \zeta < 1$ : underdamped (damped oscillat'n)  
 -  $\zeta = 1$ : critically damped (no oscillat'n)  
 -  $\zeta > 1$ : overdamped (no oscillat'n)  
 -  $\zeta < 0$ : underdamped (undamped oscillat'n)

**Loc. of Poles and Behavior:**  

 - faster oscillat'n  
 - faster exp. decay  
 - constant  
 - faster exp. growth  
 - faster oscillat'n

**Control Spec. of 2nd Order Sys.: Step Response:** Given a TF  $G(s)$ , its SR is  $y(t)$  resulting from applying the input  $u(t) = 1(t)$ , i.e.  $\mathcal{L}^{-1}\left\{G(s)\frac{1}{s}\right\}$ .  
**Control Spec.** A control spec. is a criterion specifying how we would like a CS to behave.  
**2nd Order Sys. Metrics:**  $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$  w/  $U(s) = \frac{1}{s}$   
**\* $0 < \zeta < 1$**  (i.e. 2 comp. conj. poles w/  $\text{Re}(\text{pole}) < 0$ ).  

 -  $y(t) \sim$  step response  
 - steady-state = 1



**Rise Time (RT):**  $T_r$  is the time it takes  $y(t)$  to go from 10% to 90% of its steady-state value.  
**RT:** 1. Find  $t_1 > 0$  s.t.  $y(t_1) = 0.1, t_2 > 0$  s.t.  $y(t_2) = 0.9$ .  
 3. Compute  $T_r = t_2 - t_1$ .  $T_r \approx \frac{1.8}{\omega_n}$ .  
**Settling Time (ST):**  $T_s$  is the time required to reach and stay w/in 2% of the steady-state value.  
**ST:** 1. Find when it's first that  $|y(t) - 1| \leq 0.02$  indefinitely.  
 $T_s \approx \frac{4}{\zeta\omega_n}$   
**Peak Time:**  $T_p$  is time req'd to reach the max (peak) value.  
**Peak Time:** 1. Find the first time when  $\dot{y}(t) = 0$ .  
 $* T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$

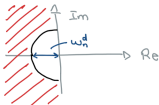
**% Overshoot:**  $\%OS = \frac{[\text{peak value}] - [\text{steady-state value}]}{[\text{steady-state value}]} \times 100\%$   
**\*% OS = OS  $\times 100\%$ .**  
 $* OS = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \iff \zeta = \frac{-\ln(OS)}{\sqrt{\pi^2 + (\ln(OS))^2}}$

**LT Steps:** 1. Write  $f(t)$  as a sum and use linearity  
\*Trig. id. may be useful.  
2. Use prop. of LT and common LT to find  $F(s)$

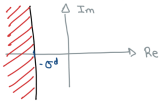
**Transient Performance Sat.:** Given performance spec.  $T_r \leq T_r^d$ ,  $T_s \leq T_s^d$ ,  $OS \leq OS^d$ , find loc. of poles of  $G(s)$ .

\*Admissible region for the poles of  $G(s)$  s.t. the step response meets all three spec. is the intersection of the above three regions.

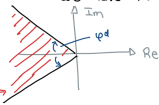
**Rise Time:**  $T_r \approx \frac{1.8}{\omega_n} \leq T_r^d \stackrel{\text{APB}}{\Leftrightarrow} \omega_n \geq \frac{1.8}{T_r^d} \equiv \omega_n^d$



**Settling Time:**  $T_s \approx \frac{4}{\zeta \omega_n} = \frac{4}{\sigma} \leq T_s^d \stackrel{\text{APB}}{\Leftrightarrow} \sigma \geq \frac{4}{T_s^d} \equiv \sigma^d$



**OS:**  $\exp\left(\frac{-\pi \zeta}{\sqrt{1-\zeta^2}}\right) \leq OS^d \stackrel{\text{APB}}{\Leftrightarrow} \zeta \geq \frac{-\ln(OS^d)}{\sqrt{\pi^2 + (\ln(OS^d))^2}} \equiv \zeta^d$



$\phi^d = \cos^{-1}(\zeta^d)$

**Add. Poles & Zeros:** The analysis remains approx. correct under the following assumptions:

- Any add. poles of  $G(s)$  have much more -ve real part (5-10 times) than the real part of the dom. complex conjugate poles.



\*dominant poles, additional poles.

- Real part of zeros are -ve & very diff. from the real part of the two dom. poles.

**Internal Stability:**  $\dot{x} = Ax$  is

- Stable** if  $\forall x(0) \in \mathbb{R}^n$ , the soln.  $x(t)$  is bdd; that is,  $\exists M > 0$  s.t.  $\|x(t)\| \leq M \forall t \geq 0$ .
- Asymp. Stable** if it's stable &  $\forall x(0) \in \mathbb{R}^n$ , the soln.  $x(t)$  converges to the origin; that is,  $\lim_{t \rightarrow \infty} x(t) = 0$ .
- Unstable** if it's not stable; that is,  $\exists x(0) \in \mathbb{R}^n$  s.t.  $x(t)$  is not bdd.

**Asymptotic Stability Thm.**  $\dot{x} = Ax$  is A.S. iff  $\text{eig}(A) \subseteq \mathbb{C}^- \equiv \{s \in \mathbb{C} \mid \text{Re}(s) < 0\}$ , i.e. open left half plane (OLHP).

**Instability Thm.** If  $\exists$  an eigenvalue  $\lambda$  of  $A$  w/  $\text{Re}(\lambda) > 0$ , then  $\dot{x} = Ax$  is unstable.

**Fact:** Zeros of  $s^2 + a_1s + a_0$  are in  $\mathbb{C}^-$  iff  $a_1, a_0 > 0$ .

**Internal Stability** 1. Linearize around  $(\bar{x}, \bar{u})$  w/  $\bar{u} = 0$ .

- Find  $A$  and determine  $\text{eig}(A) = \lambda$  s.t.  $\det(sI - A) = 0$ .

- Check if  $\text{eig}(A) \subseteq \mathbb{C}^-$  for asymptotic stability

- Check if  $\text{Re}(\text{eig}(A)) > 0$  for instability.

**BIBO Stability:** An LTI system w/ 0 i.c. is Bounded Input Bounded Output (BIBO) stable if for any bdd input  $u(t)$ , the output  $y(t)$  is also bdd.

**BIBO Unstable:** An LTI system w/ 0 i.c. is BIBO unstable if it's not BIBO stable; that is,  $\exists$  a bdd  $u(t)$  s.t.  $y(t)$  is not bdd.

**BIBO Stable Thm.** A system  $y(s) = G(s)U(s)$  is BIBO stable iff  $\text{poles}(G(s)) \subseteq \mathbb{C}^-$ .

**Lemma:** If  $p$  is a pole of  $G(s)$ , then  $p$  is an eig(A). I.e.  $\text{poles}(G(s)) := \{p \in \mathbb{C} \mid p \text{ is a pole of } G(s)\} \subseteq \text{eig}(A)$ .

\***Pole-0 Cancellation:**  $\text{eig}(A)$  need not be a pole of  $G(s)$ .

**Thm.** If  $\text{eig}(A) \subseteq \mathbb{C}^-$ , then  $\forall B, C, D$  the TF  $G(s)$  is BIBO stable. That is, internal asymptotic stability  $\Rightarrow$  BIBO stability.

**BIBO Stability** 1. Find  $G(s)$  from SS form and determine poles.

- Check if  $\text{poles}(G(s)) \subseteq \mathbb{C}^-$ . 1. Check if  $\text{eig}(A) \subseteq \mathbb{C}^-$  since internal asymptotic stability  $\Rightarrow$  BIBO stability.

**Routh-Hurwitz:** Consider  $a(s) = s^n + a_{n-1}s^{n-1} + \dots + a_0$ .

$*s^n$		1	$a_{n-2}$	$a_{n-4}$	$a_{n-6}$	$\dots$	0
$*s^{n-1}$		$a_{n-1}$	$a_{n-3}$	$a_{n-5}$	$a_{n-7}$	$\dots$	0
$*s^{n-2}$		$b_1$	$b_2$	$b_3$	$\dots$		
$*s^{n-3}$		$c_1$	$c_2$	$\dots$			

.		.	.
.		.	.
$*s$		*	0
$*1$		*	0

$b_1 = -\frac{1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{bmatrix}$   $b_2 = -\frac{1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{bmatrix}$

$b_3 = -\frac{1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-6} \\ a_{n-1} & a_{n-7} \end{bmatrix}$   $c_1 = -\frac{1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{bmatrix}$

$c_2 = -\frac{1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{bmatrix}$

**Routh-Hurwitz Stability Criterion:** The roots of  $a(s)$  are in  $\mathbb{C}^-$  iff the 1st col of Routh array has no sign changes. The # of sign changes is equal to the # of roots of  $a(s) \in \mathbb{C}^+ := \{s \in \mathbb{C} : \text{Re}(s) > 0\}$ .

\*If 1st element of a row is 0, Routh array cannot be completed.

**FVT v1:** Suppose  $Y(s) = \mathcal{L}\{y(t)\}$  is a proper rational fcn. If  $y(\infty) := \lim_{t \rightarrow \infty} y(t)$  exists and is finite, then  $y(\infty) = \lim_{s \rightarrow 0} sY(s)$

**FVT v2:** Suppose  $Y(s) = \mathcal{L}\{y(t)\}$  is a proper rational fcn. Moreover, suppose either:

- $\text{poles}(Y(s)) \subseteq \mathbb{C}^-$

- $Y(s)$  has only one pole at  $s = 0$  and all other poles are in  $\mathbb{C}^-$ . Then  $y(\infty) := \lim_{t \rightarrow \infty} y(t)$  exists and is finite and satisfies  $y(\infty) = \lim_{s \rightarrow 0} sY(s)$ .

**FVT** 1. Does  $y(\infty)$  exist? Check if pole at  $s = 0$ , then compute

Routh Array to see if poles are in  $\mathbb{C}^-$ .

- Compute  $\lim_{s \rightarrow 0} sY(s)$  if it exists.