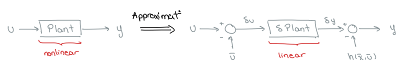
Modelling CS u: control input, y: plant output
State variable CS is in state variable form if
$\dot{x}_1 = f_1(t, x_1, \ldots, x_n, u), \ldots, \dot{x}_n = f_n(t, x_1, \ldots, x_n, u)$
$y = h(t, x_1, \ldots, x_n, u)$ is a collection of n 1st order ODEs.
Time-Invariant (TI) CS is TI if $f_i(\cdot)$ does not depend on t.
State space (SS) TI CS is in SS form if $\dot{x} = f(x, u)$, $y = h(x, u)$ where $x(t) \in \mathbb{R}^n$ is called the state.
Single-input-single-output (SISO) CS is SISO if $u(t), y(t) \in \mathbb{R}$.
LTI CS in SS form is LTI if $\dot{x} = Ax + Bu,\ y = Cx + Du$
$A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}$
where $x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m,\ y(t) \in \mathbb{R}^p$.
Input-Output (IO) LTI CS is in IO form if
$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdot + a_1\frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + \cdot + b_1 \frac{du}{dt} + b_0 u$
where $m \le n$ (causality)

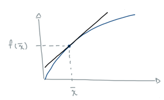
IO to SS Model 1. Define x s.t. highest order derivative in $\dot{x}$
2. Write $\dot{x} = Ax + Bu = f(x, u)$ by isolating for components of $\dot{x}$
3. Write $y = Cx + Du = h(x, u)$ by setting measurement output y to component of x
Equilibria y_d (steady state) b/c if $y(0) = y_d$ at $t = 0$, then $y(t) = y_d\ \forall t \ge 0$.

Equilibrium pair Consider the system $\dot{x} = f(x, u)$. The pair $(\bar{x}, \bar{u})$ is an equilibrium pair if $f(\bar{x}, \bar{u}) = 0$.
Equilibrium point $\bar{x}$ is an equilibrium point w/ control $u = \bar{u}$.
*If $u = \bar{u}$ and $x(0) = \bar{x}$ then $x(t) = \bar{x}\ \forall t \ge 0$ (i.e. a system that starts at equilibrium remains at equilibrium).
Find Equilibrium Pair/Point 1. Set $f(x, u) = 0$
2. Solve $f(x, u) = 0$ to find $(x, u) = (\bar{x}, \bar{u})$.
3. If specific $u = \bar{u}$, then find $x = \bar{x}$ by solving $f(x, \bar{u}) = 0$.

Linearization of Nonlinear System Consider system $\dot{x} = f(x, u)$ w/ equ. pair $(\bar{x}, \bar{u})$, then error coordinates around equ. pair
$\delta x = x - \bar{x}$, $\delta u = u - \bar{u}$, $\delta y = y - h(\bar{x}, \bar{u})$ $\delta\dot{x} = \dot{x} - f(\bar{x}, \bar{u})$ w/
$\delta\dot{x} = A\delta x + B\delta u$, $A = \frac{\partial f(\bar{x}, \bar{u})}{\partial x} \in \mathbb{R}^{n_1 \times n_1}$, $B = \frac{\partial f(\bar{x}, \bar{u})}{\partial u} \in \mathbb{R}^{n_1}$,
$\delta y = C\delta x + D\delta u$, $C = \frac{\partial h}{\partial x}(\bar{x}, \bar{u}) \in \mathbb{R}^{1\times n_1}$, $D = \frac{\partial h(\bar{x}, \bar{u})}{\partial u} \in \mathbb{R}$
*Only valid at equ. pairs.

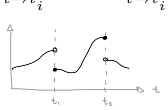


Linear Approx. Given a diff. fcn. $f : \mathbb{R} \to \mathbb{R}$, its linear approx. at $\bar{x}$ is $f_{\text{lin}} = f(\bar{x}) + f'(\bar{x})(x - \bar{x})$.
*Remainder Thm: $f(x) = f_{\text{lin}} + r(x)$ where $\lim_{x\to\bar{x}} \frac{r(x)}{x-\bar{x}} = 0$.

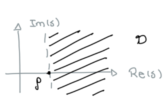


*Note: Can provide a good approx. near $\bar{x}$ but not globally.
*Gen. $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$, $f(x) = f(\bar{x}) + \frac{\partial f}{\partial x}(\bar{x})(x - \bar{x}) + R(x)$
*Jacobian: $\frac{\partial f}{\partial x}(\bar{x}) = \left[\frac{\partial f}{\partial x_1}(\bar{x}) \quad \cdots \quad \frac{\partial f}{\partial x_{n_1}}(\bar{x})\right] \in \mathbb{R}^{n_2 \times n_1}$
Linearization Steps 1. Find equ. pair $(\bar{x}, \bar{u})$
2. Derive A, B, C, D and then evaluate at $(\bar{x}, \bar{u})$
3. Write $\delta\dot{x} = A\delta x + B\delta u$ and $\delta y = C\delta x + D\delta u$

Laplace Transform Given a fcn $f : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}^n$, its Laplace transform is $F(s) = \mathcal{L}\{f(t)\} := \int_{0^-}^{\infty} f(t)e^{-st}\, dt$, $s \in \mathbb{C}$.
*$\mathcal{L} : f(t) \mapsto F(s)$, $t \in \mathbb{R}_+$ (time dom.) & $s \in \mathbb{C}$ (Laplace dom.).
P.W. CTS: A fcn $f : \mathbb{R}_+ \to \mathbb{R}^n$ is **p.w. cts** if on every finite interval of $\mathbb{R}$, $f(t)$ has at most a finite # of discontinuity points (t_i) and the limits $\lim_{t\to t_i^+} f(t)$, $\lim_{t\to t_i^-} f(t)$ are finite.



Exp. Order A function $f : \mathbb{R}_+ \to \mathbb{R}^n$ is of **exp. order** if $\exists$ constants $K, \rho, T > 0$ s.t. $\|f(t)\| \le Ke^{\rho t}$, $\forall t \ge T$.
Existence of LT Thm If $f(t)$ is p.w. cts and of exp. order w/ constants $K, \rho, T > 0$, then $F(\cdot)$ exists and is defined $\forall\, s \in D := \{s \in \mathbb{C} : \text{Re}(s) > \rho\}$ and $F(\cdot)$ is analytic on D.
*Analytic fcn iff differentiable fcn.
*D: Region of convergence (ROC), open half plane.

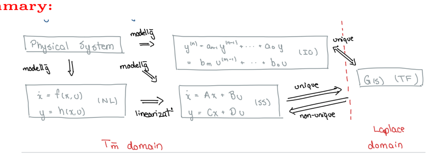


Unit Step $\mathbf{1}(t) := \begin{cases} 1, & \text{if } t \ge 0 \\ 0, & \text{otherwise} \end{cases}$
Table of Common Laplace Transforms: $f(t)$ | $F(s)$
$\mathbf{1}(t) \mapsto \frac{1}{s}$ $\quad t\mathbf{1}(t) \mapsto \frac{1}{s^2}$ $\quad t^k\,\mathbf{1}(t) \mapsto \frac{k!}{s^{k+1}}$ $\quad e^{at}\,\mathbf{1}(t) \mapsto \frac{1}{s-a}$
$t^n e^{at}\,\mathbf{1}(t) \mapsto \frac{n!}{(s-a)^{n+1}}$ $\quad \sin(at)\,\mathbf{1}(t) \mapsto \frac{a}{s^2+a^2}$
$\cos(at)\,\mathbf{1}(t) \mapsto \frac{s}{s^2+a^2}$
Prop. of Laplace Transform Linearity:
$\mathcal{L}\{cf(t) + g(t)\} = c\mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$, $c \sim$ constant.
Differentiation: If the Laplace transform of $f'(t)$ exists, then $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^-)$.
If the Laplace transform of $f^{(n)}(t) := \frac{d^n f}{dt^n}(t)$ exists, then
$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0^-)$.
Integration: $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$.
Convolution: Let $(f * g)(t) := \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau$, then $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$.
Time Delay: $\mathcal{L}\{f(t-T)\mathbf{1}(t-T)\} = e^{-Ts}\mathcal{L}\{f(t)\}, T \ge 0$.
Multiplication by t: $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}[\mathcal{L}\{f(t)\}]$.
Shift in s: $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}\big|_{s\to s-a} = F(s-a)$, where $F(s) = \mathcal{L}\{f(t)\}$ & a const.

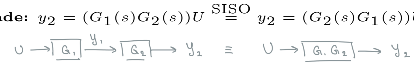
Trig. Id. $2\sin(2t) = 2\sin(t)\cos(t)$, $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$, $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
Complete the Square: $ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c$
LT Steps: 1. Write $f(t)$ as a sum and use linearity
*Trig. id. may be useful.
2. Use prop. of LT and common LT to find $F(s)$
Inverse Laplace Transform Given $F(s)$, its inverse LT is $f(t) = \mathcal{L}^{-1}\{F(s)\} := \frac{1}{2\pi}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\, ds$

$= \lim_{w\to\infty} \frac{1}{2\pi} \int_{c-jw}^{c+jw} F(s)e^{st}ds$, $c \in \mathbb{C}$ is selected s.t. the line $L := \{s \in \mathbb{C} : s = c + j\omega, \omega \in \mathbb{R}\}$ is inside the ROC of $F(s)$.
Zero: $z \in \mathbb{C}$ is a zero of $F(s)$ if $F(z) = 0$.
Pole: $p \in \mathbb{C}$ is a pole of $F(s)$ if $\frac{1}{F(p)} = 0$.
Cauchy's Residue THM If $F(s)$ is analytic (complex diff.) everywhere except at isolated poles $\{p_1, \ldots, p_N\}$, then $\mathcal{L}^{-1}\{F(s)\} = \sum_{i=1}^{N} \text{Res}\left[F(s)e^{st}, s = p_i\right] \mathbf{1}(t)$,
*$\text{Res}[F(s)e^{st}, s = p_i]$: Residue of $F(s)e^{st}$ at $s = p_i$.
Residue Computation Let $G(s)$ be a complex analytic fcn w/ a pole at $s = p$, r be the multiplicity of the pole p. Then $\text{Res}[G(s), s = p] = \frac{1}{(r-1)!} \lim_{s\to p} \frac{d^{r-1}}{ds^{r-1}} [G(s)(s-p)^r]$.
Inv. LT Partial Frac.: 1. Factorize $F(s)$ into partial fractions.
2. Find coefficients and use LT table to find inverse LT.
*Complete the square.
Inv. LT Residue: 1. Find poles of $F(s)$ and their residues.
2. Use Cauchy's Residue THM to find inverse LT.
*Note: Complex Conjugate (CC) poles → CC residues (use Euler).
Transfer Function: Consider a CS in IO form. Assume zero initial conds. $y(0) = \cdots = \frac{d^{(n-1)}y}{dt^{(n-1)}}(0) = 0$ and $u(0) = \cdots = \frac{d^{(m-1)}u}{dt^{(m-1)}}(0) = 0$. Then the TF from u to y is $G(s) := \frac{y(s)}{U(s)} = \frac{b_m s^m + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$.
***0 Ini. Conds.:** $y_0(s) = G(s)u(s)$
***∅ Ini. Conds.:** $y_\emptyset(s) = y_0(s) + \frac{\text{poly. based on initial conds.}}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$

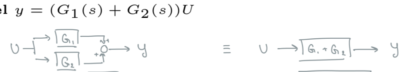
TF Steps (IO to TF): 1. Given IO form of CS, assume zero initial conds.
2. Find $G(s)$ by taking LT of IO form and forming $Y(s)/U(s)$.
*Careful: $Y(s)/U(s) = G(s)$ not $U(s)/Y(s) = G(s)$.
Impulse Response: Given CS modeled by TF $G(s)$, its IR is $g(t) := \mathcal{L}^{-1}\{G(s)\}$.
*$\mathcal{L}\{\delta(t)\} = 1$, then if $u(t)=\delta(t)$, then $Y(s) = U(s)G(s) = G(s)$.
SS to TF: $G(s) = C(sI - A)^{-1}B + D$ s.t. $y(s) = G(s)U(s)$.
*Assume $x(0) = 0 \in \mathbb{R}^n$ (zero initial conds.).
***LTI:** $G(s)$ of an LTI system is always a rational fcn.
***Not Invertible:** Values of s s.t. $sI - A$ not invertible can correspond to poles of $G(s)$.
Inverse: 1. For $A \in \mathbb{R}^{n\times n}$, find $[\text{cof}(A)]_{(i,j)} = (-1)^{i+j}\det(A_{(i,j)})$.
*$A_{(i,j)}$: A w/ row i and col. j removed.
2. Assemble cof(A) and find $\det(A) = \sum_{j=1}^{n} a_{ij}[\text{cof}(A)]_{(i,j)}$ w/ fixed i or $\det(A) = \sum_{i=1}^{n} a_{ij}[\text{cof}(A)]_{(i,j)}$ w/ fixed j.
3. Find $A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{\det(A)}[\text{cof}(A)]^T$.
*$2 \times 2 : A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
TF (SS to TF): 1. Given SS form, assume zero initial conds.
2. Solve $G(s) = C(sI - A)^{-1}B + D$.
*If $C = \begin{bmatrix} 0 & \cdot & 1_i & \cdot & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & \cdot & 1_j & \cdot & 0 \end{bmatrix}$, then only need ith row & jth col. of adj$(sI - A)$ s.t. $G(s) = \frac{[\text{adj}(sI-A)]_{(i,j)}}{\det(sI-A)} + D$.
*Multiple i, j non-zero entries: Work it out using MM.
TF to SS: Consider $G(s) = \frac{b_m s^m + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} = \frac{N(s)}{D(s)}$, where $m < n$ (i.e. $G(s)$ is strictly proper). Then the SS form is
$$*A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$
$C = \begin{bmatrix} b_0 & \cdots & b_m & | & 0 & \cdots & 0 \end{bmatrix}$, $D = 0$.
***Unique:** State space of a TF is not unique.
Summary:



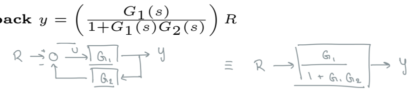
Block Diagram Types of Blocks:
Cascade: $y_2 = (G_1(s)G_2(s))U \overset{\text{SISO}}{=} y_2 = (G_2(s)G_1(s))U$



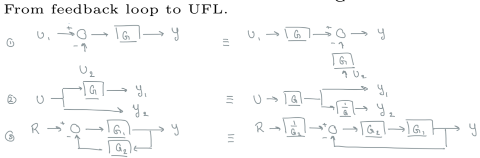
Parallel $y = (G_1(s) + G_2(s))U$



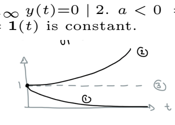
Feedback $y = \left(\frac{G_1(s)}{1+G_1(s)G_2(s)}\right)R$



***SC:** Unity Feedback Loop (UFL) if $G_2(s) = 1$.
Manipulations: 1. $y = G(U_1 - U_2) = GU_1 + GU_2$
2. $y_1 = GU \quad y_2 = U$ | $y_1 = GU \quad y_2 = G\frac{1}{G}U$
3. From feedback loop to UFL.



Find TF from Block Diagram: 1. Start from in → out, making simplifications using block diagram rules.
2. Simplify until you get the form $U(s) \to \boxed{G(s)} \to Y(s)$.
Time Response of Elementary Terms: $\mathbf{1}(t) \leftarrow$ pole @ 0
$t^n\mathbf{1}(t) \leftarrow$ pole @ 0 w/ mult. n | $e^{at}\mathbf{1}(t) \leftarrow$ pole @ a
$\sin(\omega t + \phi)\mathbf{1}(t) \leftarrow$ pole @ $\pm j\omega$ | $\cos(\omega t + \phi)\mathbf{1}(t) \leftarrow$ pole @ $\pm j\omega$
Real Pole: $y(s) = \frac{1}{s+a}$, real pole at $s = -a$, then $y(t) = e^{-at}\mathbf{1}(t)$
1. $a > 0 \implies \lim_{t\to\infty} y(t) = 0$ | 2. $a < 0 \implies \lim_{t\to\infty} y(t) = \infty$.
3. $a = 0 \implies y(t) = \mathbf{1}(t)$ is constant.



Time Constant: When $a > 0$, $\tau = \frac{1}{a}$ is the time constant of the pole $s = -a$.

Pair of Comp. Conj. Poles:

$y(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2+\omega_d^2}$, $|\zeta| < 1$, then

$y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t)\mathbf{1}(t)$

*Poles: $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$

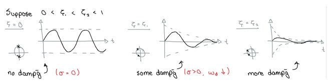
*ζ: Damping ratio (or damping coefficient)

*σ: Decay/growth rate | ω_d: Freq. of oscillation

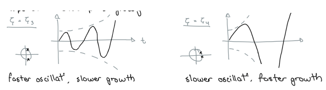
*$\omega_n \left[\frac{\text{radians}}{\text{seconds}}\right]$: Undamped natural freq.

*$\omega_d \left[\frac{\text{radians}}{\text{seconds}}\right]$: Damped natural freq.

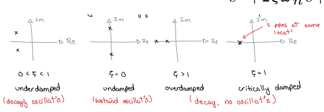
Damping Ratio Effect: $0 < \zeta_1 < \zeta_2 < 1$, then



$-1 < \zeta_4 < \zeta_3 < 0$, then $\sigma = \zeta\omega_n < 0$, (exp. envelop $\uparrow$)

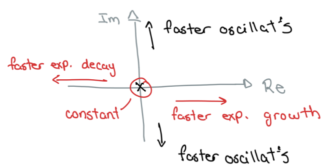


Class. of 2nd Order Systems: $y(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, then



Loc. of Poles and Behavior: $\sigma = \zeta\omega_n$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

*$\zeta = \frac{\sigma}{\sqrt{\sigma^2+\omega_d^2}}$ | $\omega_n = \sqrt{\sigma^2+\omega_d^2}$



Control Spec. of 2nd Order Systems: **Step Response:** Given a TF $G(s)$, its SR is $y(t)$ resulting from applying the input $u(t) = \mathbf{1}(t)$, i.e. $\mathcal{L}^{-1}\left\{G(s)\frac{1}{s}\right\}$.

Control Spec. A control spec. is a criterion specifiying how we would like a CS to behave.

Metrics: Used to quantify the transient performance of 2nd order systems w/ $0 < \zeta < 1$.

Rise Time: T_r is the time it takes $y(t)$ to go from 10% to 90% of its steady-state value.

Rise Time: 1. Find $t_1 > 0$ s.t. $y(t_1) = 0.1$.
2. Find $t_2 > 0$ s.t. $y(t_2) = 0.9$.
3. Compute $T_r = t_2 - t_1$. Approx. $T_r \approx \frac{1.8}{\omega_n}$.

Settling Time: T_s is the time required to reach and stay within 2% of the steady-state value.

Settling Time: 1. Look at $|y(t) - 1|$ and find when it is first that $|y(t) - 1| \leq 0.02$. Approx.: $T_s \approx \frac{4}{\zeta\omega_n}$.

Peak Time: T_p is the time required to reach the maximum (peak) value.

Peak Time: 1. Find the first time when $\dot{y}(t) = 0$.

*$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$.

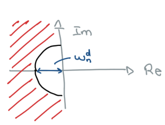
% Overshoot: $\%\text{OS} = \frac{[\text{peak value}]-[\text{steady-state value}]}{[\text{steady-state value}]} \times 100\%$

*% OS = OS $\times$ 100%.

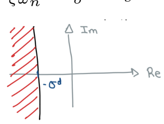
*$\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \iff \zeta = \frac{-\ln(\text{OS})}{\sqrt{\pi^2+(\ln(\text{OS}))^2}}$.

Transient Performance Satisfaction: Admissible region for the poles of $G(s)$ s.t. the step response meets all three spec.

Rise Time: $T_r \approx \frac{1.8}{\omega_n} \leq T_r^d \overset{\text{approx.}}{\iff} \omega_n \geq \frac{1.8}{T_r^d} \equiv \omega_n^d$



Settling Time: $T_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} \leq T_s^d \overset{\text{approx.}}{\iff} \sigma \geq \frac{4}{T_s^d} \equiv \sigma^d$



OS: $\text{OS} = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \leq \text{OS}^d \overset{\text{approx.}}{\iff}$

$\zeta \geq \frac{-\ln(\text{OS}^d)}{\sqrt{\pi^2+(\ln(\text{OS}^d))^2}} \equiv \zeta^d$