# ROB311 Quiz 2

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# Probabilistic Inference Problems

## 1 Bayesian Networks

**Definition**: Vertices represent random variables and edges represent dependencies between variables.

### 1.1 Junction

**Definition**: A **junction** $\mathcal{J}$ consists of three vertices, $X_1$, $X_2$, and $X_3$, connected by two edges, $e_1$ and $e_2$:

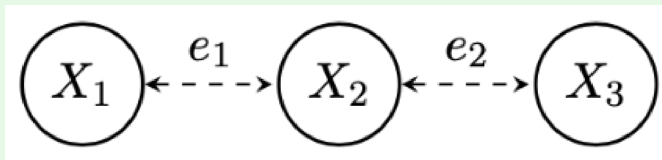


Figure 1

- $X_1$ and $X_2$ are not independent, $X_2$ and $X_3$ are not independent, but when is $X_1$ and $X_3$ independent?

#### 1.1.1 Causal Chain

**Definition**: A causal chain is a junction $\mathcal{J}$ s.t.

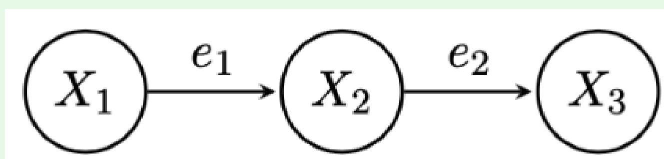


Figure 2

- $X_1$ and $X_3$ are not independent (unconditionally), but are independent given $X_2$.

**Notes**:
- **Analogy:** Given $X_2$, $X_1$ and $X_3$ are independent. Why? $X_2$'s door closes when you know $X_2$, so $X_1$ and $X_3$ are independent.
- **Distinction b/w Causal and Dependence:** $X_1$ and $X_2$ are dependent. However, from a causal perspective, $X_1$ is influencing $X_2$ (i.e. $X_1 \rightarrow X_2$).

#### 1.1.2 Common Cause

**Definition**: A common cause is a junction $\mathcal{J}$ s.t.

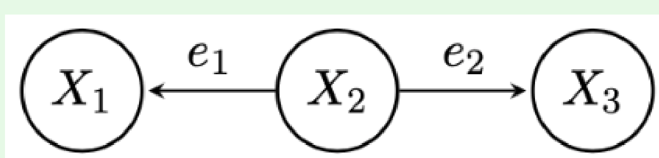


Figure 3

- $X_1$ and $X_3$ are not independent (unconditionally), but are independent given $X_2$.

**Notes**:
- **Analogy:** Given $X_2$, $X_1$ and $X_3$ are independent. Why? Consider the following example:
  - Let $X_2$ represent whether a person smokes or not, $X_1$ represent whether they have yellow teeth, $X_3$ represent whether they have lung cancer.
- Without knowing $X_2$, observing $X_1$ provides information about $X_3$ because yellow teeth are associated with smoking, which in turn increases the likelihood of lung cancer.
- If $X_2$ is known, then knowing whether a person has yellow teeth provides no additional information about whether they have lung cancer beyond what is already known from smoking status.

### 1.1.3 Common Effect

**Definition**: A common effect is a junction $\mathcal{J}$ s.t.

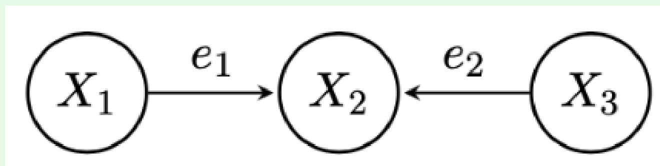


Figure 4

- $X_1$ and $X_3$ are independent (unconditionally), but are not independent given $X_2$ or any of $X_2$'s descendents.

**Notes**:
- **Analogy:** Consider the following example:
  - Let $X_2$ represent whether the grass is wet, $X_1$ represent whether it rained, $X_3$ represent whether the sprinkler was on.
- Without knowing whether the grass is wet ($X_2$), the occurrence of rain ($X_1$) and the sprinkler being on ($X_3$) are independent events. The rain may occur regardless of the sprinkler, and vice versa.
- However, once we observe that the grass is wet ($X_2$), the two events become dependent:
  - If we learn that the sprinkler was not on, then the wet grass must have been caused by rain.
  - If we learn that it did not rain, then the wet grass must have been caused by the sprinkler.

# 2 Dependence Separation

## 2.1 Independence

**Theorem:** Any two variables, $X_1$ and $X_2$, in a Bayesian network, $\mathcal{B} = (\mathcal{V}, \mathcal{E})$, are independent given $\mathcal{K} \subseteq \mathcal{V}$ if every undirected path is blocked.

### 2.1.1 Blocked Undirected Path

**Definition:** An undirected path,

$$p = \langle (X_1, e_1, X_2), \ldots, (X_{|p|-1}, e_{|p|-1,|p|}, X_{|p|}) \rangle,$$

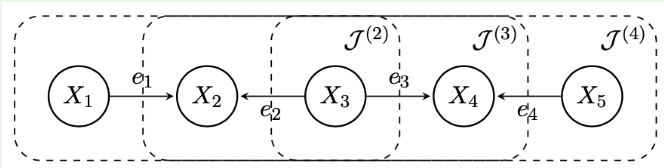


Figure 5

is **blocked** given $\mathcal{K} \subseteq \mathcal{V}$ if any of its junctions,

$$\mathcal{J}^{(n)} = \{(X_{n-1}, X_n, X_{n+1}), (e_{n-1}, e_n)\},$$

is blocked given $\mathcal{K}$.

### 2.1.2 Blocked Junction

**Definition:** $\mathcal{J} = (\{X_1, X_2, X_3\}, \{e_1, e_2\})$ is **blocked** given $\mathcal{K} \subseteq \mathcal{V}$ if $X_1$ and $X_3$ are independent given $\mathcal{K}$.

## 2.2 Consequence of Dependence Separation

**Theorem:** For any variable, $X \in \mathcal{V}$, it can be shown that $X$ is independent of $X$'s non-descendants, $\mathcal{V} \setminus \text{des}(X)$, given $X$'s parents, $\text{pts}(X)$.

**Notes:**

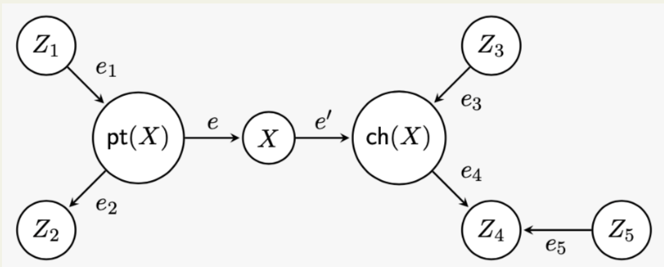


Figure 6

- Given $X$'s parent, apply junction rules to determine that $X$ is independent of its non-descendants.
- $\mathcal{J} = \{(Z_1, \text{pt}(X), X), (e_1, e)\}$ shows that $Z_1$ and $X$ are independent given $\text{pt}(X)$ (causal chain).
- $\mathcal{J} = \{(Z_2, \text{pt}(X), X), (e_2, e)\}$ shows that $Z_2$ and $X$ are independent given $\text{pt}(X)$ (common cause).
- Given $\text{ch}(X)$'s parent, apply junction rules to determine that $\text{ch}(X)$ is independent of its non-descendants.
- $\mathcal{J} = \{\text{pt}(X), X, \text{ch}(X)), (e, e')\}$ shows that $\text{pt}(X)$ and $\text{ch}(X)$ are independent given $X$ (causal chain).

- Given $Z_4$'s parent, apply junction rules to determine that $Z_4$ is independent of its non-descendants.
- $\mathcal{J} = \{X, \text{ch}(X), Z_4, (e', e_4)\}$ shows that $X$ and $Z_4$ are independent given $\text{ch}(X)$ (causal chain).
- CHECK THIS OVER AGAIN WITH THE PROFESSOR.

## 2.3 Canonical Problems

### 2.3.1 Undirected Path Blocked?

**Process:**
1. **Given:** Undirected path $p$ and $\mathcal{K}$
2. Check if any of the junctions on the undirected path are blocked given $\mathcal{K}$.
   - i.e. Check if $X_1$ and $X_3$ of the junction are independent given $\mathcal{K}$.

### 2.3.2 Independence

**Process:**
1. Given a Bayesian network w/ 2 variables to find independence.
2. Find all undirected paths between the 2 variables in the Bayesian network.
3. Identify a set of variables, $\mathcal{K}$, that block at least one junction in all undirected paths.
   - Test a junction by seeing junction given relationships.
4. If all undirected paths are blocked, then the 2 variables are independent given $\mathcal{K}$.

**Warning:**
- Be careful of common effect, in which it is blocked by default.
- Be careful of decesdents of a common effect (i.e. outward arrows from a common effect) as given it may not be blocked.
- Cyclic paths are not blocked by default.
- If $\mathcal{K}$ includes variables that cause all undirected paths to be blocked, then variables are independent. BUT be careful of common effect, in which it is blocked by default so YOU DONT INCLUDE IT IN $\mathcal{K}$.

**Example:**
1. **Given:** Bayesian network.

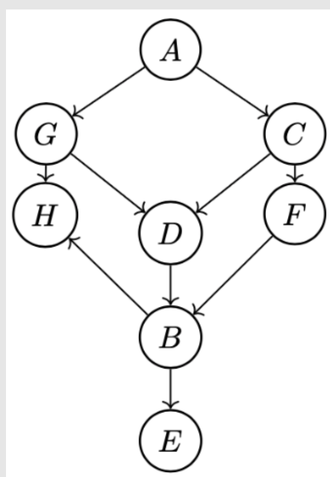


Figure 7

2. **Problem:** $A$ and $E$ are
   - independent if $\mathcal{K} =$
   - not necessarily independent for $\mathcal{K} =$
3. **Soln:**
   (a) **Undirected Paths:**
   - $A \to G \to H \to B \to E$
   - $A \to G \to D \to B \to E$
   - $A \to C \to F \to B \to E$
   - $A \to C \to D \to B \to E$

**Example**: **Independent:**

| $\mathcal{K}$ |
|---|
| $\{G, C\}$ <br>• $A \iff G \iff H \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(A, G, H), (e_1, e_2)\}$ is blocked given $G$ since $A, H$ independent given $G$ (causal chain)<br>• $A \iff G \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(A, G, D), (e_1, e_2)\}$ is blocked given $G$ since $A, D$ independent given $G$ (causal chain)<br>• $A \iff C \iff F \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(A, C, F), (e_1, e_2)\}$ is blocked given $C$ since $A, F$ independent given $C$ (causal chain)<br>• $A \iff C \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(A, C, D), (e_1, e_2)\}$ is blocked given $C$ since $A, D$ independent given $C$ (causal chain) |
| $\{D, F\}$ <br>• $A \iff G \iff H \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(G, H, B), (e_1, e_2)\}$ is blocked NOT given $H$ since $G, B$ independent NOT given $H$ (common effect)<br>• $A \iff G \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(G, D, B), (e_1, e_2)\}$ is blocked given $D$ since $G, B$ independent given $D$ (causal chain)<br>• $A \iff C \iff F \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(C, F, B), (e_1, e_2)\}$ is blocked given $F$ since $C, B$ independent given $F$ (causal chain)<br>• $A \iff C \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(C, D, B), (e_1, e_2)\}$ is blocked given $D$ since $C, B$ independent given $D$ (causal chain) |

**Not Necessarily Independent:**

| $\mathcal{K}$ |
|---|
| $\{H, D, F\}$ <br>• $A \iff G \iff H \iff B \iff E$ is unblocked given $\mathcal{K}$ since $\mathcal{J} = \{(G, H, B), (e_1, e_2)\}$ is unblocked given $H$ since $G, B$ not independent given $H$ (common effect)<br>• $A \iff G \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(G, D, B), (e_1, e_2)\}$ is blocked given $D$ (causal chain) since $G, B$ independent given $D$ (causal chain)<br>• $A \iff C \iff F \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(C, F, B), (e_1, e_2)\}$ is blocked given $F$ since $C, B$ independent given $F$ (causal chain)<br>• $A \iff C \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(C, D, B), (e_1, e_2)\}$ is blocked given $D$ since $C, B$ independent given $D$ (causal chain) |

**Example**: Determine all subsets of $\{B, C, D, F, G, H\}$ for which $A$ and $E$ are independent.

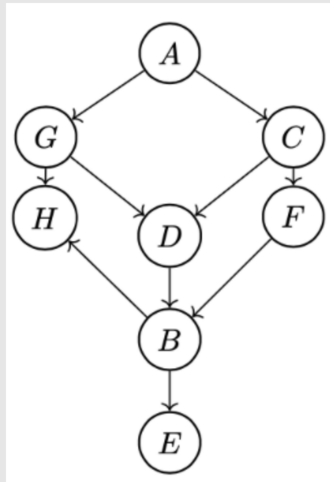


Figure 8

1. **Undirected Paths:**
   - $A \to G \to H \to B \to E$
   - $A \to G \to D \to B \to E$
   - $A \to C \to F \to B \to E$
   - $A \to C \to D \to B \to E$

| $\mathcal{K}$ |
|---|
| $\{B\}$ (Any subset that contains $B$ will be independent)<br>• $AGHBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(H, B, E), (e_1, e_2)\}$ is b since $H, E$ indep. given $B$ (causal chain)<br>• $AGDBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(D, B, E), (e_1, e_2)\}$ is b since $D, E$ indep. given $B$ (causal chain)<br>• $ACFBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(F, B, E), (e_1, e_2)\}$ is b since $F, E$ indep. given $B$ (causal chain)<br>• $ACDBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(D, B, E), (e_1, e_2)\}$ is b since $D, E$ indep. given $B$ (causal chain) |
| $\{C\}$ (Not independent)<br>• $AGDBE$ is ub given $\mathcal{K}$ since $\mathcal{J} = \{(D, B, E), (e_1, e_2)\}$ is ub since $D, E$ NOT indep. given $\mathcal{K}$ (causal chain) |
| $\{D\}$ (Not indepedent)<br>• $ACFBE$ is ub given $\mathcal{K}$ since $\mathcal{J} = \{(F, B, E), (e_1, e_2)\}$ is ub since $F, E$ NOT indep. given $\mathcal{K}$ (causal chain) |
| $\{F\}$ (Not independent)<br>• $AGDBE$ is ub given $\mathcal{K}$ since $\mathcal{J} = \{(D, B, E), (e_1, e_2)\}$ is ub since $D, E$ NOT indep. given $\mathcal{K}$ (causal chain) |
| $\{G\}$ (Not independent)<br>• $ACFBE$ is ub given $\mathcal{K}$ since $\mathcal{J} = \{(F, B, E), (e_1, e_2)\}$ is ub since $F, E$ NOT indep. given $\mathcal{K}$ (causal chain) |
| $\{H\}$ (Not independent)<br>• $ACFBE$ is ub given $\mathcal{K}$ since $\mathcal{J} = \{(F, B, E), (e_1, e_2)\}$ is ub since $F, E$ NOT indep. given $\mathcal{K}$ (common effect) |

**Example**:

| $\mathcal{K}$ |
|---|
| $\{C, D\}$ (Any subset that contains $C$ and $D$ except $H$ will be independent)<br>• $AGHBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(G, H, B), (e_1, e_2)\}$ is b since $G, B$ indep. not given $H$ (common effect)<br>• $AGDBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(G, D, B), (e_1, e_2)\}$ is b since $G, B$ indep. given $D$ (causal chain)<br>• $ACFBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(A, C, F), (e_1, e_2)\}$ is b since $A, F$ indep. given $C$ (causal chain)<br>• $ACDBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(A, C, D), (e_1, e_2)\}$ is b since $A, D$ indep. given $C$ (causal chain) |
| ... |

**Example**:
1. **Given:**

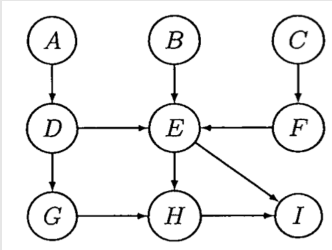


Figure 9

2. **Problem 1:** Is it gauranteed that $A \perp C$?
3. **Solution 1:** True b/c all undirected paths are blocked.
   (a) ADEFC is b since $\mathcal{J} = \{(D, E, F), (e_1, e_2)\}$ is b since $D$, $F$ indep. NOT given $E$ (common effect)
   (b) ADGHEFC is b since $\mathcal{J} = \{(G, H, E), (e_1, e_2)\}$ is b since $G, E$ indep. NOT given $H$ (common effect)
   (c) ADGHIEFC is b since $\mathcal{J} = \{(H, I, E), (e_1, e_2)\}$ is b since $H, E$ indep. NOT given $I$ (common effect)

4. **Problem 2:** Is it gauranteed that $B \perp C \mid I$?
5. **Solution 2:** False b/c BEFC is ub.
   (a) BEFC is ub since $\mathcal{J} = \{(B, E, F), (e_1, e_2)\}$ is ub since $B, F$ NOT indep. given $E$'s descendent, $I$ (common effect)

6. **Problem 3:** Is it gauranteed that $D \perp I \mid \{E, G\}$?
7. **Solution 3:** True b/c all undirected paths are blocked.
   (a) DEI is b since $\mathcal{J} = \{(D, E, I), (e_1, e_2)\}$ is b since $D, I$ indep. given $E$ (causal chain)
   (b) DEHI is b since $\mathcal{J} = \{(D, E, H), (e_1, e_2)\}$ is b since $D, H$ indep. given $E$ (causal chain)
   (c) DGHI is b since $\mathcal{J} = \{(D, G, H), (e_1, e_2)\}$ is b since $D, H$ indep. given $G$ (causal chain)
   (d) DGHEI is b since $\mathcal{J} = \{(D, G, H), (e_1, e_2)\}$ is b since $D, H$ indep. given $G$ (causal chain)

8. **Problem 4:** Is it gauranteed that $C \perp H \mid G$?
9. **Solution 4:** False b/c CFEH is ub.
   (a) CFEH is ub since $\mathcal{J} = \{(C, F, E), (e_1, e_2)\}$ is ub since $C, E$ NOT indep. given $G$ (causal chain)

**Example**:
1. **Problem 5:** Suppose each variable is binary. What is the size of the domain of the joint distribution b/w the variables?
2. **Solution 5:**
   (a) Since 9 variables, each with 2 values, the size of the domain of the joint distribution is $2^9 = 512$.

3. **Problem 6:** Suppose each variable is binary. What is the min # of values that actually need to be stored to represent the joint distribution entirely based on the Bayesian network? Use the fact that probability distributions are normalized.
4. **Solution 6:** 1+1+1+2+8+2+2+4+4 = 25 values need to be stored.
   (a) $P(A)$, $P(B)$, $P(C)$ has 1 value each
      - Since $P(\#)$ can represent 2 values, i.e. $P(0) = 1 - P(1)$, so only need to store 1 value.
   (b) $P(D \mid A)$, $P(F \mid C)$, $P(G \mid D)$ has 2 values each
      - Same idea, can take the complement of the other value for 4 values.
   (c) $P(H \mid G, E)$, $P(I \mid E, H)$ has 4 values each
      - Same idea, can take the complement of the other value for 8 values.
   (d) $P(E \mid D, B, F)$ has 8 values
      - Same idea, can take the complement of the other value for 16 values.