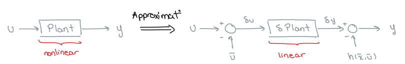
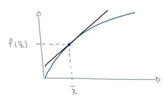
Modelling CS u: control input, y: plant output
State variable CS is in state variable form if
$\dot{x}_1 = f_1(t, x_1, \ldots, x_n, u), \ldots, \dot{x}_n = f_n(t, x_1, \ldots, x_n, u)$
$y = h(t, x_1, \ldots, x_n, u)$ is a collection of n 1st order ODEs.
Time-Invariant (TI) CS is TI if $f_i(\cdot)$ does not depend on t.
State space (SS) TI CS is in SS form if $\dot{x} = f(x, u)$, $y = h(x, u)$ where $x(t) \in \mathbb{R}^n$ is called the state.
Single-input-single-output (SISO) CS is SISO if $u(t), y(t) \in \mathbb{R}$.
LTI CS in SS form is LTI if $\dot{x} = Ax + Bu,\ y = Cx + Du$
*$A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}$
*SISO: $p = 1,\ m = 1$
*$x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m,\ y(t) \in \mathbb{R}^p$.
Input-Output (IO) LTI CS is in IO form if
$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdot + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^m} + \cdot + b_1\frac{du}{dt} + b_0 u$
*$m \le n$ (causality)

IO to SS Model 1. Define x s.t. highest order derivative in $\dot{x}$
2.1 If LTI, then
*Write $\dot{x} = Ax + Bu = f(x, u)$ by isolating for components of $\dot{x}$
*Write $y = Cx + Du = h(x, u)$ by setting measurement output y to component of x
2.2 If not LTI, then
*Write $\dot{x} = f(x, u)$ by isolating for components of $\dot{x}$
*Write $y = h(x, u)$ by setting measurement output y to component of x
Equilibria y_d (steady state) b/c if $y(0) = y_d$ at $t = 0$, then $y(t) = y_d$ $\forall t \ge 0$.

Equilibrium pair Consider the system $\dot{x} = f(x, u)$. The pair $(\bar{x}, \bar{u})$ is an equilibrium pair if $f(\bar{x}, \bar{u}) = 0$.
Equilibrium point $\bar{x}$ is an equilibrium point w/ control $u = \bar{u}$.
*If $u = \bar{u}$ and $x(0) = \bar{x}$ then $x(t) = \bar{x}$ $\forall t \ge 0$ (i.e. a system that starts at equilibrium remains at equilibrium).
Find Equilibrium Pair/Point 1. Set $f(x, u) = 0$
2. Solve $f(x, u) = 0$ to find $(x, u) = (\bar{x}, \bar{u})$.
3. If specific $u = \bar{u}$, then find $x = \bar{x}$ by solving $f(x, \bar{u}) = 0$.
Linearization of Nonlinear System Consider system $\dot{x} = f(x, u)$ w/ equ. pair $(\bar{x}, \bar{u})$, then error coordinates around equ. pair
$\delta x = x - \bar{x}$, $\delta u = u - \bar{u}$, $\delta y = y - h(\bar{x}, \bar{u})$ $\delta\dot{x} = \dot{x} - f(\bar{x}, \bar{u})$ w/
$\delta\dot{x} = A\delta x + B\delta u$, $A = \frac{\partial f(\bar{x},\bar{u})}{\partial x} \in \mathbb{R}^{n_1\times n_1}$, $B = \frac{\partial f(\bar{x},\bar{u})}{\partial u} \in \mathbb{R}^{n_1}$,
$\delta y = C\delta x + D\delta u$, $C = \frac{\partial h}{\partial x}(\bar{x}, \bar{u}) \in \mathbb{R}^{1\times n_1}$, $D = \frac{\partial h(\bar{x},\bar{u})}{\partial u} \in \mathbb{R}$
*Only valid at equ. pairs.

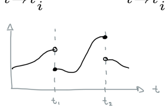


Linear Approx. Given a diff. fcn. $f : \mathbb{R} \to \mathbb{R}$, its linear approx. at $\bar{x}$ is $f_{\text{lin}} = f(\bar{x}) + f'(\bar{x})(x - \bar{x})$.
*Remainder Thm: $f(x) = f_{\text{lin}} + r(x)$ where $\lim_{x\to\bar{x}} \frac{r(x)}{x-\bar{x}} = 0$.

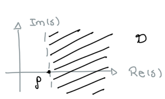


*Note: Can provide a good approx. near $\bar{x}$ but not globally.
*Gen. $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$, $f(x) = f(\bar{x}) + \frac{\partial f}{\partial x}(\bar{x})(x - \bar{x}) + R(x)$
*Jacobian: $\frac{\partial f}{\partial x}(\bar{x}) = \left[\frac{\partial f}{\partial x_1}(\bar{x}) \quad \cdots \quad \frac{\partial f}{\partial x_{n_1}}(\bar{x})\right] \in \mathbb{R}^{n_2\times n_1}$
Linearization Steps 1. Find equ. pair $(\bar{x}, \bar{u})$
2. Derive A, B, C, D and then evaluate at $(\bar{x}, \bar{u})$
3. Write $\delta\dot{x} = A\delta x + B\delta u$ and $\delta y = C\delta x + D\delta u$

Laplace Transform Given a fcn $f : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}^n$, its Laplace transform is $F(s) = \mathcal{L}\{f(t)\} := \int_{0^-}^{\infty} f(t)e^{-st}\,dt$, $s \in \mathbb{C}$.
*$\mathcal{L} : f(t) \mapsto F(s)$, $t \in \mathbb{R}_+$ (time dom.) & $s \in \mathbb{C}$ (Laplace dom.).
P.W. CTS: A fcn $f : \mathbb{R}_+ \to \mathbb{R}^n$ is **p.w. cts** if on every finite interval of $\mathbb{R}$, $f(t)$ has at most a finite # of discontinuity points (t_i) and the limits $\lim_{t\to t_i^+} f(t)$, $\lim_{t\to t_i^-} f(t)$ are finite.



Exp. Order A function $f : \mathbb{R}_+ \to \mathbb{R}^n$ is of **exp. order** if $\exists$ constants $K, \rho, T > 0$ s.t. $\|f(t)\| \le Ke^{\rho t}$, $\forall t \ge T$.
Existence of LT Thm If $f(t)$ is p.w. cts and of exp. order w/ constants $K, \rho, T > 0$, then $F(\cdot)$ exists and is defined $\forall\, s \in D := \{s \in \mathbb{C} : \text{Re}(s) > \rho\}$ and $F(\cdot)$ is analytic on D.
*Analytic fcn iff differentiable fcn.
*D: Region of convergence (ROC), open half plane.



Unit Step $\mathbf{1}(t) := \begin{cases} 1, & \text{if } t \ge 0 \\ 0, & \text{otherwise} \end{cases}$
Table of Common Laplace Transforms: $f(t) \mid F(s)$
$\mathbf{1}(t) \mapsto \frac{1}{s}$ $\quad t\mathbf{1}(t) \mapsto \frac{1}{s^2}$ $\quad t^k\,\mathbf{1}(t) \mapsto \frac{k!}{s^{k+1}}$ $\quad e^{at}\,\mathbf{1}(t) \mapsto \frac{1}{s-a}$
$t^n e^{at}\,\mathbf{1}(t) \mapsto \frac{n!}{(s-a)^{n+1}}$ $\quad \sin(at)\,\mathbf{1}(t) \mapsto \frac{a}{s^2+a^2}$
$\cos(at)\,\mathbf{1}(t) \mapsto \frac{s}{s^2+a^2}$ $\quad \frac{1}{2\omega^3}[\sin(\omega t) - \omega t\cos(\omega t)]\,\mathbf{1}(t) \mapsto \frac{1}{(s^2+\omega^2)^2}$

Prop. of Laplace Transform Linearity:
$\mathcal{L}\{cf(t) + g(t)\} = c\mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$, $c \sim$ constant.
Differentiation: If the Laplace transform of $f'(t)$ exists, then $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^-)$.
If the Laplace transform of $f^{(n)}(t) := \frac{d^n f}{dt^n}(t)$ exists, then $\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0^-)$.
Integration: $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$.
Convolution: Let $(f * g)(t) := \int_0^t f(\tau)g(t - \tau)d\tau = \int_0^t f(t - \tau)g(\tau)d\tau$, then $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$.
Time Delay: $\mathcal{L}\{f(t - T)\mathbf{1}(t - T)\} = e^{-Ts}\mathcal{L}\{f(t)\}$, $T \ge 0$.
Multiplication by t: $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}[\mathcal{L}\{f(t)\}]$.
Shift in s: $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}|_{s\to s-a} = F(s - a)$, where $F(s) = \mathcal{L}\{f(t)\}$ & a const.

Trig. Id. $\frac{1}{2}\sin(2t) = \sin(t)\cos(t)$, $\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b)$, $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
Complete the Square: $ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c$

LT Steps: 1. Write $f(t)$ as a sum and use linearity
*Trig. id. may be useful.
2. Use prop. of LT and common LT to find $F(s)$

Inverse Laplace Transform Given $F(s)$, its inverse LT is $f(t) = \mathcal{L}^{-1}\{F(s)\} := \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds$
$= \lim_{w\to\infty} \frac{1}{2\pi j}\int_{c-jw}^{c+jw} F(s)e^{st}ds$, $c \in \mathbb{C}$ is selected s.t. the line $L := \{s \in \mathbb{C} : s = c + j\omega, \omega \in \mathbb{R}\}$ is inside the ROC of $F(s)$.

Zero: $z \in \mathbb{C}$ is a zero of $F(s)$ if $F(z) = 0$.
Pole: $p \in \mathbb{C}$ is a pole of $F(s)$ if $\frac{1}{F(p)} = 0$.

Cauchy's Residue THM If $F(s)$ is analytic (complex diff.) everywhere except at isolated poles $\{p_1, \ldots, p_N\}$, then $\mathcal{L}^{-1}\{F(s)\} = \sum_{i=1}^{N} \text{Res}\left[F(s)e^{st}, s = p_i\right] \mathbf{1}(t)$,
*Res$[F(s)e^{st}, s = p_i]$: Residue of $F(s)e^{st}$ at $s = p_i$.

Residue Computation Let $G(s)$ be a complex analytic fcn w/ a pole at $s = p$, r be the multiplicity of the pole p. Then Res$[G(s), s = p] = \frac{1}{(r-1)!}\lim_{s\to p}\frac{d^{r-1}}{ds^{r-1}}\left[G(s)(s-p)^r\right]$.

Inv. LT Partial Frac.: 1. Factorize $F(s)$ into partial fractions.
2. Find coefficients and use LT table to find inverse LT.
*Complete the square.

Inv. LT Residue: 1. Find poles of $F(s)$ and their residues.
2. Use Cauchy's Residue THM to find inverse LT.
*Note: Complex Conjugate (CC) poles $\to$ CC residues (use Euler).
*$\cos(x) = \frac{e^{jx}+e^{-jx}}{2}$, $\sin(x) = \frac{e^{jx}-e^{-jx}}{2j}$

Transfer Function: Consider a CS in IO form. Assume zero initial conds. $y(0) = \cdots = \frac{d^{(n-1)}y}{dt^{(n-1)}}(0) = 0$ and $u(0) = \cdots = \frac{d^{(m-1)}u}{dt^{(m-1)}}(0) = 0$. Then the TF from u to y is $G(s) := \frac{y(s)}{U(s)} = \frac{b_m s^m + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$.
***0 Ini. Conds.:** $y_0(s) = G(s)u(s)$
***$\emptyset$ Ini. Conds.:** $y_\emptyset(s) = y_0(s) + \frac{\text{poly. based on initial conds.}}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$

TF Steps (IO to TF): 1. Given IO form of CS, assume zero initial conds.
2. Find $G(s)$ by taking LT of IO form and forming $Y(s)/U(s)$.
*Careful: $Y(s)/U(s) = G(s)$ not $U(s)/Y(s) = G(s)$.

Impulse Response: Given CS modeled by TF $G(s)$, its IR is $g(t) := \mathcal{L}^{-1}\{G(s)\}$.
*$\mathcal{L}\{\delta(t)\} = 1$, then if $u(t)=\delta(t)$, then $Y(s) = U(s)G(s) = G(s)$.

SS to TF: $G(s) = C(sI - A)^{-1}B + D$ s.t. $y(s) = G(s)U(s)$.
*Assume $x(0) = 0 \in \mathbb{R}^n$ (zero initial conds.).
***LTI:** $G(s)$ of an LTI system is always a rational fcn.
***Not Invertible:** Values of s s.t. $sI - A$ not invertible can correspond to poles of $G(s)$.

Inverse: 1. For $A \in \mathbb{R}^{n\times n}$, find $[\text{cof}(A)]_{(i,j)} = (-1)^{i+j}\det(A_{(i,j)})$.
*$A_{(i,j)}$: A w/ row i and col. j removed.
2. Assemble cof(A) and find $\det(A) = \sum_{j=1}^{n} a_{ij}[\text{cof}(A)]_{(i,j)}$ w/ fixed i or $\det(A) = \sum_{i=1}^{n} a_{ij}[\text{cof}(A)]_{(i,j)}$ w/ fixed j.
3. Find $A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{\det(A)}[\text{cof}(A)]^T$.
*$2 \times 2 : A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

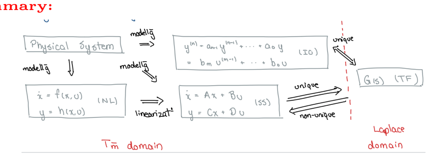
TF (SS to TF): 1. Given SS form, assume zero initial conds.
2. Solve $G(s) = C(sI - A)^{-1}B + D$.
*If $C = \begin{bmatrix} 0 & \cdot & 1_i & \cdot & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & \cdot & 1_j & \cdot & 0 \end{bmatrix}$, then only need ith row & jth col. of adj$(sI - A)$ s.t. $G(s) = \frac{[\text{adj}(sI-A)]_{(i,j)}}{\det(sI-A)} + D$.
*Multiple i, j non-zero entries: Work it out using MM.

TF to SS: Consider $G(s) = \frac{b_m s^m + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} = \frac{N(s)}{D(s)}$, where $m < n$ (i.e. $G(s)$ is strictly proper). Then the SS form is

$$*A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

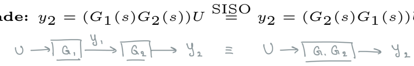
$C = \begin{bmatrix} b_0 & \cdots & b_m & | & 0 & \cdots & 0 \end{bmatrix}$, $D = 0$.
***Unique:** State space of a TF is not unique.

Summary:

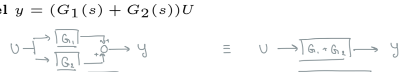


Block Diagram Types of Blocks:

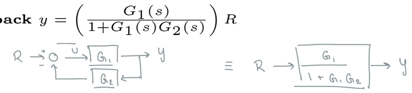
Cascade: $y_2 = (G_1(s)G_2(s))U \overset{\text{SISO}}{=} y_2 = (G_2(s)G_1(s))U$



Parallel $y = (G_1(s) + G_2(s))U$

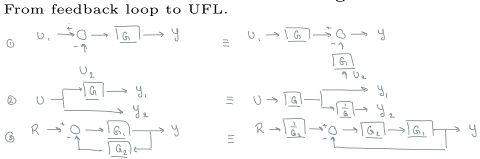


Feedback $y = \left(\frac{G_1(s)}{1+G_1(s)G_2(s)}\right)R$



***SC:** Unity Feedback Loop (UFL) if $G_2(s) = 1$.

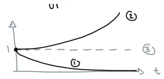
Manipulations: 1. $y = G(U_1 - U_2) = GU_1 - GU_2$
2. $y_1 = GU \quad y_2 = U$ | $y_1 = GU \quad y_2 = G\frac{1}{G}U$
3. From feedback loop to UFL.



Find TF from Block Diagram: 1. Start from in $\to$ out, making simplifications using block diagram rules.
2. Simplify until you get the form $U(s) \to \boxed{G(s)} \to Y(s)$.

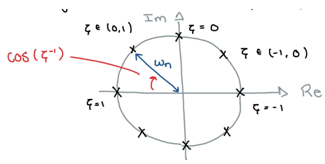
Time Response of Elementary Terms: $\mathbf{1}(t) \leftarrow$ pole @ 0
$t^n\mathbf{1}(t) \leftarrow$ pole @ 0 w/ mult. n | $e^{at}\mathbf{1}(t) \leftarrow$ pole @ a
$\sin(\omega t + \phi)\mathbf{1}(t) \leftarrow$ pole @ $\pm j\omega$ | $\cos(\omega t + \phi)\mathbf{1}(t) \leftarrow$ pole @ $\pm j\omega$

Real Pole: $y(s)=\frac{1}{s+a}$, real pole at $s=-a$, then $y(t)=e^{-at}\mathbf{1}(t)$
1. $a>0 \implies \lim_{t\to\infty} y(t)=0$ | 2. $a<0 \implies \lim_{t\to\infty} y(t)=\infty$.
3. $a=0 \implies y(t)=\mathbf{1}(t)$ is constant.

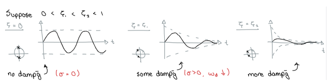


Time Constant: $\tau=\frac{1}{a}$ of the pole $s=-a$ for $a>0$
Pair of Comp. Conj. Poles:
$y(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{(s+\sigma)^2+\omega_d^2}$, $|\zeta|<1$, then
$y(t)=\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\sigma t}\sin(\omega_d t)\mathbf{1}(t)$

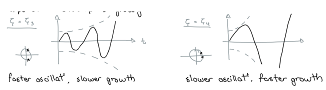
*Poles: $s_{1,2}=-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}=-\sigma \pm j\omega_d$
*$\zeta=\frac{\sigma}{\omega_n}$: Damping ratio (or damping coefficient)
*$\sigma=\zeta\omega_n$: Decay/growth rate | ω_d: Freq. of oscillation
*$\omega_n=\sqrt{\sigma^2+\omega_d^2}\left[\frac{\text{radians}}{\text{seconds}}\right]$: Undamped natural freq.
*$\omega_d=\omega_n\sqrt{1-\zeta^2}\left[\frac{\text{radians}}{\text{seconds}}\right]$: Damped natural freq.
*$|s_{1,2}|^2=\omega_n^2$: Mag. of poles is ω_n.
*$\cos^{-1}(\zeta)$: Angle of s_1 on complex plane CW from -ve Re axis.



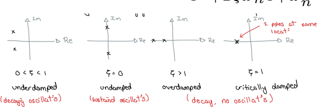
Damping Ratio Effect: $0<\zeta_1<\zeta_2<1$, then



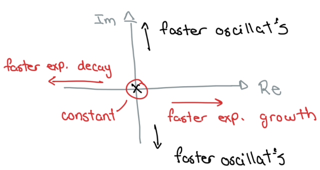
$-1<\zeta_4<\zeta_3<0$, then $\sigma=\zeta\omega_n<0$, (exp. envelop ↑)



Class. of 2nd Order Sys.: $y(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, w/ any $|\zeta|$

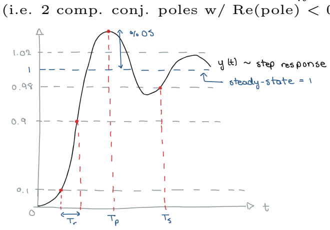


Loc. of Poles and Behavior:



Control Spec. of 2nd Order Sys.: **Step Response:** Given a TF $G(s)$, its SR is $y(t)$ resulting from applying the input $u(t)=\mathbf{1}(t)$, i.e. $\mathcal{L}^{-1}\left\{G(s)\frac{1}{s}\right\}$.
Control Spec. A control spec. is a criterion specifiying how we would like a CS to behave.

2nd Order Sys. Metrics: $G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ w/ $U(s)=\frac{1}{s}$
*$0<\zeta<1$ (i.e. 2 comp. conj. poles w/ Re(pole) <0).



Rise Time (RT): T_r is the time it takes $y(t)$ to go from 10% to 90% of its steady-state value.
RT: 1. Find $t_1>0$ s.t. $y(t_1)=0.1$, $t_2>0$ s.t. $y(t_2)=0.9$.
3. Compute $T_r=t_2-t_1$. $\boxed{T_r\approx\frac{1.8}{\omega_n}}$.

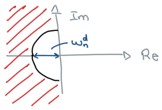
Settling Time (ST): T_s is the time required to reach and stay w/in 2% of the steady-state value.
ST: 1. Find when it's first that $|y(t)-1|\leq 0.02$ indefinitely.
$\boxed{T_s\approx\frac{4}{\zeta\omega_n}}$

Peak Time: T_p is time req'd to reach the max (peak) value.
Peak Time: 1. Find the first time when $\dot{y}(t)=0$.
* $\boxed{T_p=\frac{\pi}{\omega_d}=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$.

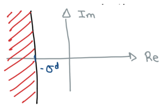
% Overshoot: $\%\text{OS}=\frac{[\text{peak value}]-[\text{steady-state value}]}{[\text{steady-state value}]}\times 100\%$
*% OS = OS × 100%.
* $\boxed{\text{OS}=\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \iff \zeta=\frac{-\ln(\text{OS})}{\sqrt{\pi^2+(\ln(\text{OS}))^2}}}$.

Transient Performance Sat.: Given performance spec. $T_r \leq T_r^d$, $T_s \leq T_s^d$, $\text{OS} \leq \text{OS}^d$, find loc. of poles of $G(s)$.
*Admissible region for the poles of $G(s)$ s.t. the step response meets all three spec. is the intersection of the above three regions.

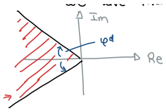
Rise Time: $T_r \approx \frac{1.8}{\omega_n} \leq T_r^d \overset{\text{app.}}{\iff} \omega_n \geq \frac{1.8}{T_r^d} \equiv \omega_n^d$



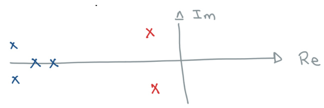
Settling Time: $T_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} \leq T_s^d \overset{\text{app.}}{\iff} \sigma \geq \frac{4}{T_s^d} \equiv \sigma^d$



OS: $\exp\left(\frac{-\pi\zeta}{\sqrt{1\text{-}\zeta^2}}\right) \leq \text{OS}^d \overset{\text{app.}}{\iff} \zeta \geq \frac{-\ln(\text{OS}^d)}{\sqrt{\pi^2 + (\ln(\text{OS}^d))^2}} \equiv \zeta^d$



$\phi^d = \cos^{-1}(\zeta^d)$
Add. Poles & Zeros: The analysis remains approx. correct under the following assumptions:
1. Any add. poles of $G(s)$ have much more -ve real part (5-10 times) than the real part of the dom. complex conjugate poles.



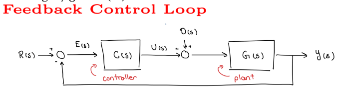
*dominant poles, additional poles.
2. Real part of zeros are -ve & very diff. from the real part of the two dom. poles.

Internal Stablity: $\dot{x} = Ax$ is
1. **Stable** if $\forall x(0) \in \mathbb{R}^n$, the soln. $x(t)$ is bdd; that is, $\exists M > 0$ s.t. $\|x(t)\| \leq M \ \forall t \geq 0$.
2. **Asymp. Stable** if it's stable & $\forall x(0) \in \mathbb{R}^n$, the soln. $x(t)$ converges to the origin; that is, $\lim_{t\to\infty} x(t) = 0$.
3. **Unstable** if it's not stable; that is, $\exists x(0) \in \mathbb{R}^n$ s.t. $x(t)$ is not bdd.

Asymptotic Stablity Thm. $\dot{x} = Ax$ is A.S. iff $\text{eig}(A) \subseteq \mathbb{C}^- \equiv \{s \in \mathbb{C} \mid \text{Re}(s) < 0\}$, i.e. open left half plane (OLHP).
Instability Thm. If $\exists$ an eigenvalue λ of A w/ $\text{Re}(\lambda) > 0$, then $\dot{x} = Ax$ is unstable.
Fact: Zeros of $s^2 + a_1 s + a_0$ are in $\mathbb{C}^-$ iff $a_1, a_0 > 0$.
Internal Stability 1. Linearize around $(\bar{x}, \bar{u})$ w/ $\bar{u} = 0$.
2. Find A and determine $\text{eig}(A) = \lambda$ s.t. $\det(sI - A) = 0$.
3. Check if $\text{eig}(A) \subseteq \mathbb{C}^-$ for asymptotic stability
4. Check if $\text{Re}(\text{eig}(A)) > 0$ for instability.
BIBO Stability: An LTI system w/ 0 i.c. is Bounded Input Bounded Output (BIBO) stable if for any bdd input $u(t)$, the output $y(t)$ is also bdd.
BIBO Unstable: An LTI system w/ 0 i.c. is BIBO unstable if it's not BIBO stable; that is, $\exists$ a bdd $u(t)$ s.t. $y(t)$ is not bdd.
BIBO Stable Thm. A system $y(s) = G(s)U(s)$ is BIBO stable iff $\text{poles}(G(s)) \subseteq \mathbb{C}^-$.
Lemma: If p is a pole of $G(s)$, then p is an $\text{eig}(A)$. I.e. $\text{poles}(G(s)) := \{p \in \mathbb{C} \mid p \text{ is a pole of } G(s)\} \subseteq \text{eig}(A)$.
***Pole-0 Cancellation:** $\text{eig}(A)$ need not be a pole of $G(s)$.
Thm. If $\text{eig}(A) \subseteq \mathbb{C}^-$, then $\forall B, C, D$ the TF $G(s)$ is BIBO stable. That is, internal asymptotic stability $\Rightarrow$ BIBO stability.
BIBO Stability 1. Find $G(s)$ from SS form and determine poles.
2. Check if $\text{poles}(G(s)) \subseteq \mathbb{C}^-$. 1. Check if $\text{eig}(A) \subseteq \mathbb{C}^-$ since internal asymptotic stability $\Rightarrow$ BIBO stability.
Routh-Hurwitz: Consider $a(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$.

$*s^n$	$\mid$ 1	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$	0	
$*s^{n-1}$	$\mid$ a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$	0	
$*s^{n-2}$	$\mid$ b_1	b_2	b_3	$\cdots$			
$*s^{n-3}$	$\mid$ c_1	c_2	$\cdots$				
$\vdots$							
$*s$	$\mid$ $*$	0					
$*1$	$\mid$ $*$	0					

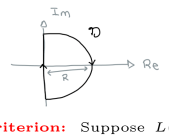
$b_1 = -\frac{1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{bmatrix}$ $b_2 = -\frac{1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{bmatrix}$
$b_3 = -\frac{1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-6} \\ a_{n-1} & a_{n-7} \end{bmatrix}$ $c_1 = -\frac{1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{bmatrix}$
$c_2 = -\frac{1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{bmatrix}$
Routh-Hurwitz Stability Criterion: The roots of $a(s)$ are in $\mathbb{C}^-$ iff the 1st col of Routh array has no sign changes. The # of sign changes is equal to the # of roots of $a(s) \in \mathbb{C}^+ := \{s \in \mathbb{C} : \text{Re}(s) > 0\}$.
*If 1st element of a row is 0, Rooth array cannot be completed.
FVT v1: Suppose $Y(s) = \mathcal{L}\{y(t)\}$ is a proper rational fcn. If $y(\infty) := \lim_{t\to\infty} y(t)$ exists and is finite, then $y(\infty) = \lim_{s\to 0} sY(s)$
FVT v2: Suppose $Y(s) = \mathcal{L}\{y(t)\}$ is a proper rational fcn. Moreover, suppose either:
1. $\text{poles}(Y(s)) \subseteq \mathbb{C}^-$
2. $Y(s)$ has only one pole at $s = 0$ and all other poles are in $\mathbb{C}^-$.
Then $y(\infty) := \lim_{t\to\infty} y(t)$ exists and is finite and satisfies $y(\infty) = \lim_{s\to 0} sY(s)$.
FVT 1. Does $y(\infty)$ exist? Check if pole at $s = 0$, then compute Rooth Array to see if poles are in $\mathbb{C}^-$.
2. Compute $\lim_{s\to 0} sY(s)$ if it exists.
Standard Feedback Control Loop



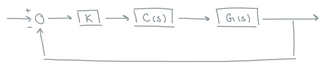
$R(s)$: Ref., $E(s) = R(s) - y(s)$: Err., $C(s)$: Controller, $U(s)$: Control input, $D(s)$: Dist., $G(s)$: Plant, $y(s)$: Plant output.
***Assume:** $R(s)$ and $D(s)$ are strictly proper rational fcns w/ a fixed set of poles but arbitrary zeros & gain.
*$\mathcal{R}, \mathcal{D}$: Classes of ref. and dist. satisfying the above assumption.

Basic Control Prob.: Design $C(s)$ s.t. 3 spec. are met:
1. **Stability:** $\forall$ bdd $r(t), d(t)$, we have $u(t), e(t)$ bdd.
2. **Asymptotic Tracking:** When $d(t) = 0\ \forall t \geq 0$, then $\forall r(t) \in \mathcal{R}$, $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} r(t) - y(t) = 0$.
3. **Disturbance Rejection:** When $r(t) = 0\ \forall t \geq 0$, then $\forall d(t) \in \mathcal{D}$, $\lim_{t\to\infty} y(t) = 0$.
Open-Loop Control: 1. Design $u(t)$ s.t. $y(t)$ tracks ref. $y_r \in \mathbb{R}$, i.e. $\lim_{t\to\infty} y(t) = y_r$.
2. Set $u(t) = \gamma y_r \mathbf{1}(t)$ w/ $\gamma \in \mathbb{R}$ (const. scaling factor)
3. Apply FVT to find γ s.t. $\lim_{t\to\infty} y(t) = y_r$. 4. Determine $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} y_r - y(t)$
Limitations: 1. Req. perfect knowledge of plant paramters.
2. Not robust against parameter var./(unknown) dist.
3. Does not allow us to speed up convergence.
Feedback Control: 1. Design $u(t)$ s.t. $y(t)$ tracks ref. $y_r \in \mathbb{R}$, i.e. $\lim_{t\to\infty} y(t) = y_r$.
2. Set $u(t) = Ke(t) = K(y_r - y(t))$ w/ $K > 0$ (const. gain).
3. Use block mani. to find $y(s)$ in terms of input and $G(s)$.
4. Apply FVT to find K s.t. $\lim_{t\to\infty} y(t) = y_r$.
5. Determine $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} y_r - y(t)$
Advantages: 1. Doesn't req. perfect knowledge of plant param.
2. Robust against param. var./dist. by $\uparrow K$.
3. Allows us to speed up the rate of convergence by $\uparrow K$.
Disadvantages: 1. Feedback can introduce instability.
2. High-gain amplifies noise.
3. Asymptotic tracking doesn't occur.
Integral Control: 1. Design $u(t)$ s.t. $y(t)$ tracks ref. $y_r \in \mathbb{R}$, i.e. $\lim_{t\to\infty} y(t) = y_r$.
2. Set $u(t) = \mathcal{L}^{-1}\{C(s)E(s)\} = Ke(t) + KT_I \int_0^t e(\tau)d\tau$ (prop. int. (PI) controller) w/ $K, T_I > 0$ (const. gains).
*$C(s) = K\left(1 + \frac{T_I}{s}\right)$
3. Use block mani. to find $y(s)$ in terms of input and $G(s)$.
4. Apply FVT to find $\lim_{t\to\infty} y(t) = y_r$ as desired.
BIBO Stability of Closed-Loop System: **Gang of 4 TF:**
$$\begin{bmatrix} E(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{1+C(s)G(s)} & \frac{-G(s)}{1+C(s)G(s)} \\ \frac{C(s)}{1+C(s)G(s)} & \frac{-C(s)G(s)}{1+C(s)G(s)} \end{bmatrix} \begin{bmatrix} R(s) \\ D(s) \end{bmatrix}$$
BIBO Stable of CLS: The std. feedback control loop (CLS) is BIBO stable if all the Gang of 4 TFs are BIBO stable.
CLS is BIBO Stable THM: The CLS is BIBO stable iff
1. Poles of $\frac{1}{1+C(s)G(s)} \subseteq \mathbb{C}^-$
2. $C(s)G(s)$ has no pole-zero cancel. in $\bar{\mathbb{C}}^+ = \{s \in \mathbb{C} : \text{Re}(s) \geq 0\}$.
Practical Considerations:
1. Don't cancel an unstable 0 of $G(s)$ w/ an unstable pole in $C(s)$.
2. Don't cancel an unstable pole of $G(s)$ w/ an unstable 0 in $C(s)$.
Asymp. Tracking of Poly. Suppose $d(t) = 0$ & want to track a <u>poly. ref. signal</u> of the form: $r(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$, that is: $R(s) = \frac{N_R(s)}{s^k}$, w/ $N_R(0) \neq 0$ and $\deg(N_R(s)) \leq k-1$.
***GOAL:** Design $C(s)$ to achieve $\lim_{t\to\infty} e(t) = 0$.
Prop: Suppose $C(s)$ is designed so that:
1. $\frac{1}{1+C(s)G(s)}$ is BIBO stable
2. $C(s)G(s) = \frac{C'(s)G'(s)}{s^k}$ with $C'(0)G'(0) \neq 0$.
Then $\frac{1}{s^k + C'(s)G'(s)}$ is BIBO stable.
Asymp. Tracking of Poly. Thm Suppose $C(s)$ satisfies CLS is BIBO stable THM and $d(t) = 0\ \forall t \geq 0$. For any poly. ref. signal $r(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$, the following hold:
a. If $C(s)G(s)$ has k or more poles at $s = 0$, then $\lim_{t\to\infty} e(t) = 0$.
b. If $C(s)G(s)$ has $k-1$ poles at $s = 0$, then:
$$\lim_{t\to\infty} e(t) = \begin{cases} \frac{N_R(0)}{1+C'(0)G'(0)}, & \text{if } k = 1 \\ \frac{N_R(0)}{C'(0)G'(0)}, & \text{if } k \geq 2 \end{cases}$$
c. If $C(s)G(s)$ has k-2 or fewer poles at $s = 0$, then $\lim_{t\to\infty} |e(t)| = \infty$.
Type k: The TF $C(s)G(s)$ is of type k if it has k poles at $s = 0$.
Dist. Rejection: Suppose $r(t) = 0\ \forall t \geq 0$ and $d(t)$ is a poly. dist. signal of the form: $d(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$, that is: $D(s) = \frac{N_D(s)}{s^k}$, with $N_D(0) \neq 0$ and $\deg(N_D(s)) \leq k-1$.
***GOAL:** Design $C(s)$ to achieve $\lim_{t\to\infty} e(t) = 0$.
Dist. Rejection Thm: Suppose $C(s)$ satisfies CLS is BIBO stable THM and $r(t) = 0\ \forall t \geq 0$. For any poly. dist. signal $d(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$, the following hold:
a. If $C(s)$ has k or more poles at $s = 0$, then $\lim_{t\to\infty} e(t) = 0$.
b. If $C(s)$ has $k-1$ poles at $s = 0$, then $\lim_{t\to\infty} e(t) \neq 0$ exists.
c. If $C(s)$ has k-2 or fewer poles at $s = 0$, then $\lim_{t\to\infty} |e(t)| = \infty$.
General Thm (Internal Model Principle): Suppose $R(s)$ and $D(s)$ are strictly proper rational fns w/ poles in $\overline{\mathbb{C}^+}$. $C(s)$ solves the Basic Control Problem iff:
1) $C(s)$ makes the CLS BIBO stable;
2) $C(s)G(s)$ has the poles($R(s)$) w/ at least same multiplicities;
3) $C(s)$ has the poles($D(s)$) w/ at least same multiplicities.
Corollary: If $G(s)$ has zeros that are also poles of $R(s)$ or $D(s)$, then the Basic Control Problem is unsolvable.
Internal Model: The IMP states if $G(s)$ does not contain the poles of $R(s)$ and $D(s)$, then $C(s)$ must contain these poles. Since these poles enable $C(s)$ to reproduce $r(t)$ and $d(t)$, we say $C(s)$ must contain an <u>internal model</u> of $r(t)$ and $d(t)$.
Proposition: Suppose $G(s)$ is BIBO stable. Let $Y(s) = G(s)U(s)$, where $Y(s) = \mathcal{L}\{y(t)\}$ and $U(s) = \mathcal{L}\{u(t)\}$. If $\lim_{t\to\infty} u(t) = 0$, then $\lim_{t\to\infty} y(t) = 0$.
*Decaying input $\implies$ decaying output so don't worry in IMP.
General Controller Design Procedure: Given $R(s) = \mathcal{L}\{r(t)\}$ and $D(s) = \mathcal{L}\{d(t)\}$:
1. **Feasibility:** Verify no zero of $G(s)$ is an unstable pole of $R(s)$ or $D(s)$.
2. **Internal Model:** Let $p_1, \ldots, p_k$ denote the unstable poles of $R(s)$ or $D(s)$ not in $G(s)$, accounting for multiplicities. Construct:
$$C(s) = C'(s) \cdot \frac{1}{(s-p_1)\ldots(s-p_k)}$$
3. **Stability:** Design $C'(s)$ so that the CLS is BIBO stable.
4. **Performance:** Tune controller parameters to achieve the desired performance specifications.
Argument Principle Let $\mathcal{D}$ be a simple (no self-intersections) closed (no endpoints) path in $\mathcal{C}$ oriented CCW. Suppose $F(s)$ has no poles or zeros on $\mathcal{D}$ & isolated poles inside $\mathcal{D}$. Let $\gamma(\theta)$ be a parametrization of $\mathcal{D}$, i.e. $\mathcal{D} = \{\gamma(\theta) : \theta \in \mathbb{R}\}$ and $\mathcal{F} = \{F(\gamma(\theta)) : \theta \in \mathbb{R}\}$. Then $\mathcal{F}$ encircles the origin $n_e = n_z - n_p$ times CCW.
*n_z: # of zeros of $F(s)$ inside $\mathcal{D}$
*n_p: # of poles of $F(s)$ inside $\mathcal{D}$
Notes:
1. A -ve CCW encirlement is the same as a +ve CW encirlement.
2. If $\mathcal{D}$ is oriented CW, the Argument Principle still holds by replacing $CCW \to CW$ everywhere.

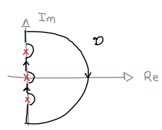
Application to Feedback Loops: To stabilize the CLS, it suffices to consider the FB loop where we require:
-**Zeros of** $1 + C(s)G(s) \subseteq \mathbb{C}^-$ (focus on this)
-$C(s)G(s)$ has no unstable pole-zero cancellations.
See if $\exists$ zeros in $\mathbb{C}^+$. So consider the contour:
$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 = \{j\omega : \omega \in [-R, R]\} \cup \{Re^{j\theta} : \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$
*If $R \to \infty$, then more of $\mathbb{C}^+$ is contained inside $\mathcal{D}$.
*By the **Argument Principle**, if we:
-Count the number of encirclements of $1 + C(s)G(s)$ (n_e).
-Know the number of unstable poles of $1 + C(s)G(s)$ (n_p).
*Then, we can compute the number of zeros in $\mathbb{C}^+$.
Nyquist Contour: The path $\mathcal{D}$ above w/ $R \to \infty$.



Nyquist Stablity Criterion: Suppose $L(s) = C(s)G(s)$ is a strictly proper rational fn and has no poles on the Im axis. Also let $K \in \mathbb{R}$. Then the TF $\frac{1}{1+KL(s)}$ is BIBO stable iff
(1) $\mathcal{L} = \{L(j\omega) : \omega \in \mathbb{R}\}$ does not pass through the pt. $\frac{-1}{k}$
(2) $\mathcal{L}$ encircles $\frac{-1}{k}$ a total of n_p times CCW $= P_{OLS} - P_{CLS}$
*P_{OLS}: # of open-loop poles of $L(s)$ in $\mathbb{C}^+$
*P_{CLS}: # of closed-loop poles of $L(s)$ in $\mathbb{C}^+$
*n_p is the # of poles of $L(s)$ in $\mathbb{C}^+$.



Nyquist Plot of $L(s)$ $\mathcal{L} = \{L(j\omega) : \omega \in \mathbb{R}\}$
Obs. of Nyquist Plot: $L(s) = \frac{b_m s^m + \cdots + b_0}{s^n + \cdots + a_1 s + a_0}$ w/ $m < n$.
(1) When $\omega = 0$, $L(j\omega) = \dfrac{b_0}{a_0} \in \mathbb{R}$ is always on the Re-axis.
(2) As $\omega \to \infty$, $L(j\omega) \to 0$ and $\angle L(j\omega) = \begin{cases} -\frac{\pi}{2}(n-m) & \text{if } b_m > 0 \\ \pi - \frac{\pi}{2}(n-m) & \text{if } b_m < 0 \end{cases}$
(3) Since $\overline{L(j\omega)} = L(\overline{j\omega}) = L(-j\omega)$, $\mathcal{L}$ is symmetric w.r.t the Re-axis.
(4) $\mathcal{L}$ intersects:
*Re-axis when $\angle L(j\omega) \in \{0, \pi\} \mod 2\pi$
*Im-axis when $\angle L(j\omega) \in \{\pm\frac{\pi}{2}\} \mod 2\pi$
Nyquist Stability Criterion w/ Imaginary Poles: Suppose $L(s)$ satisfies all the requirements of the Nyquist Stability Criterion except it has poles on the Im axis. Then the conclusion of the Nyquist Stability Criterion hold true provided we use the **indented Nyquist Contour**.



*X: poles of $L(s)$ on the Im-axis.
*Radius of each semi-circle around each pole on the Im-axis is $\epsilon > 0$ and consider the case when $\epsilon \to 0^+$.
Gain and Phase Margin:
Frequency Response: Given the TF $G(s)$, its frequency response is $G(j\omega)$ where $\omega \in \mathbb{R}$.
Proposition: Suppose the TF $G(s)$ is BIBO stable. If $u(t) = A\sin(\omega t + \phi)\mathbf{1}(t)$, then the steady-state output is $y_{ss}(t) \equiv |G(j\omega)|A\sin(\omega t + \phi + \angle G(j\omega))\mathbf{1}(t)$ that is, $\lim_{t\to\infty} y(t) - y_{ss}(t) = 0$.
Robustness Margins:

$L(s)$: Strictly proper rational fn, has no poles in $\mathbb{C}^+$
Gain Margin (GM): $GM \equiv A_{\max} = \frac{1}{|L_0(j\omega_{pc})|}$
Phase Margin (PM): $PM \equiv \phi_{\max} = \angle L_0(j\omega_{gc}) - (-\pi) = \angle L_0(j\omega_{gc}) + \pi$
***Crossover Frequency:** $\omega_{gc} \in [0, \infty)$: Soln to $|L(j\omega_{gc})| = 1$.