

ROB311 Quiz 3

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Contents

1	Reinforcement Learning	2
1.1	Estimating Q-Star Empirically	2
1.1.1	Running Average Update Rule	2
1.2	Q-Learning Algorithm	3
1.3	Modified Q-Learning Algorithm	3
1.4	Training vs. Testing	4
1.4.1	K Sims, 1 Test	4
1.4.2	K Tests	4
2	Partially Observable MDPs (POMDPs)	5
2.1	Bayesian Network	5
3	Estimating the Optimal Quality Function	8
3.1	Estimating the Optimal Quality Function	8
3.2	Exploration versus Exploitation	8
3.2.1	Simplified Case:	8
3.3	Alternate Policies	10
4	Multi-Agent Problems	11
4.1	Action Equilibria	11
4.1.1	Finding Action Equilibria	11
4.2	Strategy Equilibria	11
4.2.1	Finding Strategy Equilibria	11
4.2.2	Existence of Strategy Equilibria	11
4.2.3	Convergence of Strategy Equilibria	11
4.3	Examples	11
4.3.1	Optimal Action Profiles	11
5	Turn-Based Games	12
5.1	Zero-Sum Turn-Based Games	12
5.2	α/β Pruning	12
5.2.1	α Cuts	12
5.2.2	β Cuts	12
5.3	Examples	13
5.3.1	Zero-Sum Turn-Based Games	14

Partially Observable Probabilistic Decision Problems

1 Reinforcement Learning

Summary: In a RL problem, $p(\cdot | \cdot, \cdot)$ and/or $r(\cdot, \cdot)$ unknown.

1.1 Estimating Q-Star Empirically

Summary:

γ	Equation
0	$q^*(s, a) = \lim_{K \rightarrow \infty} \bar{R}_K$ <ul style="list-style-type: none"> $\bar{R}_K = \frac{1}{K} \sum_{k=1}^K r_k$: empirical average reward. r_k: reward obtained in the k^{th} simulation. K: # of times action a taken in state s (# of simulations)
0	$q^*(s, a) \leftarrow q^*(s, a) + \frac{1}{N(s, a)} (r(s, a, s') - q^*(s, a))$ <ul style="list-style-type: none"> $N(s, a)$: # of times action a taken in state s.
$\neq 0$	$q^*(s, a) \leftarrow q^*(s, a) + \frac{1}{N(s, a)} \left(\left[r(s, a, s') + \gamma \max_{a'} q^*(s', a') \right] - q^*(s, a) \right)$ <ul style="list-style-type: none"> Using old q^* values to estimate.

1.1.1 Running Average Update Rule

Definition:

$$\bar{x} \leftarrow \bar{x} + \alpha(x_{\text{new}} - \bar{x}).$$

- α : learning rate

1.2 Q-Learning Algorithm

Algorithm:

```

1 procedure Q_LEARNING():
2   for each episode do
3     set initial state  $s \leftarrow s_0$ 
4     while  $s \notin \mathcal{T}$  do #  $\mathcal{T}$ : terminal states
5       randomly choose an action in  $\mathcal{A}(s)$ 
6       get next state,  $s'$ , and reward  $r$ 
7       update  $N(s, a)$  and  $q^*(s, a)$  as follows:
8
9        $q^*(s, a) \leftarrow q^*(s, a) + \frac{1}{N(s, a)} \left( r(s, a, s') + \gamma \max_{a'} q^*(s', a') - q^*(s, a) \right)$ 
10
11        $N(s, a) \leftarrow N(s, a) + 1$ 
12
13        $s \leftarrow s'$ 
14     end while
15   end for

```

- **Note:** Possible infinite while loop if \mathcal{T} is not reached.

1.3 Modified Q-Learning Algorithm

Algorithm:

```

1 procedure Q_LEARNING():
2   for each episode do
3      $l \leftarrow 0$ 
4     set initial state  $s \leftarrow s_0$ 
5     while  $s \notin \mathcal{T}$  and  $l < l_{\max}$  do
6       randomly choose an action in  $\mathcal{A}(s)$ 
7       get next state,  $s'$ , and reward  $r$ 
8       update  $N(s, a)$  and  $q^*(s, a)$  as follows:
9
10       $q^*(s, a) \leftarrow q^*(s, a) + \frac{1}{N(s, a)} \left( r(s, a, s') + \gamma \max_{a'} q^*(s', a') - q^*(s, a) \right)$ 
11
12       $N(s, a) \leftarrow N(s, a) + 1$ 
13
14       $l \leftarrow l + 1$ 
15       $s \leftarrow s'$ 
16    end while
17  end for

```

Notes: Choice of γ and l_{\max} are coupled:

- $\gamma \approx 1$ requires large l_{\max}
- $\gamma \approx 0$ requires small l_{\max}

1.4 Training vs. Testing

Notes: Episodes are classified as either:

- training (sim): reward accumulated during episode does not count
- testing (test): reward accumulated during episode counts

1.4.1 K Sims, 1 Test

Notes:

1. select actions randomly during K simulations
2. extract optimal policy, π^*
3. use π^* during test

1.4.2 K Tests

Notes:

- maximize average reward over K tests
- must balance between exploration and exploitation
- Common ways to balance exploration and exploitation: ε -greedy strategy, UCB algorithm

Strategy	Description
ε -greedy	choose optimal action with probability $\varepsilon(k)$ <ul style="list-style-type: none"> • In episode k, choose the optimal action with probability $\varepsilon(k)$, where: <ul style="list-style-type: none"> – $\varepsilon(0) \approx 0$ – $\varepsilon(k)$ is increasing – $\varepsilon(k) \rightarrow 1$ as $k \rightarrow \infty$ • Common choice for $\varepsilon(k)$ is $1 - \frac{1}{k}$.
UCB algorithm	choose action that maximizes $\text{UCB}(\cdot)$ $\text{UCB}(s, a) = \begin{cases} q^*(s, a) + C \sqrt{\frac{\log k}{N(s, a)}}, & \text{if } N(s, a) > 0 \\ \infty, & \text{otherwise} \end{cases}$ <ul style="list-style-type: none"> • In episode k, choose the action that maximizes $\text{UCB}(\cdot)$. • C: exploration parameter • $N(s, a)$: # of times a taken from s.

2 Partially Observable MDPs (POMDPs)

Summary: In a POMDPs, we assume that:

- environment modelled using state space, \mathcal{S}
- single agent
- S_t = state after transition t
- A_t = action inducing transition t
- stochastic state transitions with memoryless property:

$$S_T \perp S_0, A_1, \dots, A_{T-1}, S_{T-2} \mid S_{T-1}, A_T$$

- R_t = reward for transition t , i.e., (S_{T-1}, A_T, S_T)
- O_t = observation of S_t

Name	Function:
Initial state distribution	$p_0(s) := \mathbb{P}[S_0 = s]$
Transition distribution	$p(s' s, a) := \mathbb{P}[S_t = s' A_t = a, S_{t-1} = s]$
Reward function	$r(s, a, s') := \text{reward for transition } (s, a, s')$
<ul style="list-style-type: none"> • Since actual state is unknown, so are legal actions. • Can fix by assuming $\mathcal{A}(s) = \mathcal{A}(s') := \mathcal{A} \forall s, s'$: <ul style="list-style-type: none"> – if $a \notin \mathcal{A}(s)$, then $p(s' s, a) = 0$ for all $s' \neq s$ – if $a \notin \mathcal{A}(s)$, then $r(s, a, s') = 0$ for all s' 	
Policy for choosing actions	$\pi_t(a o_0, \dots, o_t) := \mathbb{P}[A_t = a O_0 = o_0, \dots, O_t = o_t]$
Measurement model	$m(o s) := \mathbb{P}[O_t = o S_t = s]$
<ul style="list-style-type: none"> • Observe that policy is now time-dependent. • Special Case: If we assume the agent cannot use past observations, $A_t \perp O_0, \dots, O_{t-1} \mid O_t$, policy becomes time-independent, $\pi_t(a o_0, \dots, o_t) = \pi_0(a o_t).$ <ul style="list-style-type: none"> – Only need to specify π_0. 	
Belief after t observations	$b_t(s_t a_{1:t}, o_{0:t}) = \mathbb{P}[S_t = s_t A_t = a_t, O_{0:t} = o_{0:t}]$ $b_t(s_t a_{1:t}, o_{0:t}) = m(o_t s_t) \sum_{s_{t-1}} p(s_t s_{t-1}, a_t) b_{t-1}(s_{t-1} a_{1:t-1}, o_{1:t-1})$
<ul style="list-style-type: none"> • b_t: Probability distribution • $b_0(s_0) = \mathbb{P}[S_0 = s_0]$: Initial belief distribution • Only holds for $t \geq 1$. • For $t = 0$ (assuming uniform prior): $b_0(s_0 o_0) = \frac{m(o_0 s_0)}{\sum_s m(o_0 s)}$. 	

2.1 Bayesian Network

Notes: $S_0, O_0, A_1, R_1, S_1, O_1, A_2, R_2, S_2, O_2, \dots$ form a Bayesian network:

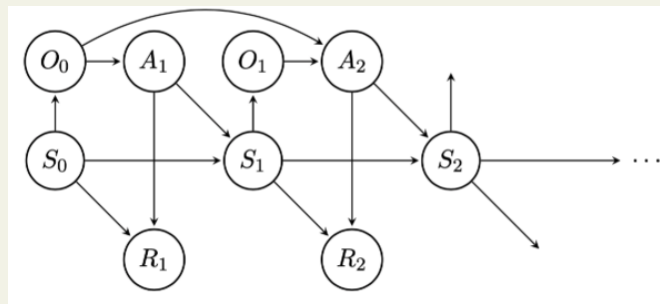


Figure 1

- Assuming $A_t \perp O_0, \dots, O_{t-1} \mid O_t$. WHERE DOES THIS COME INTO PLAY.

Example:

3 Estimating the Optimal Quality Function

3.1 Estimating the Optimal Quality Function

Motivation: The agent need not know the model of the environment. However, it must actually make moves, even when learning.

If the agent doesn't have a model, it must estimate q^* , \mathcal{A}^* , and π^* .

Definition: When the environment is in state s , the agent can take an action a and:

- **Update \hat{q} :** $\hat{q}(s, a; t) \leftarrow (1 - \alpha)\hat{q}(s, a; t) + \alpha \left(r' + \gamma \max_{a'} \hat{q}(s', a'; t + 1) \right)$
– $0 \leq \alpha \leq 1$: learning rate
- **Compute $\hat{\mathcal{A}}$:** $\hat{\mathcal{A}}(s; t) = \arg \max_{a' \in \mathcal{A}(s)} \hat{q}(s, a'; t)$
- **Compute $\hat{\pi}$:** $\hat{\pi}(a' | s; t) = 0 \ \forall a' \notin \hat{\mathcal{A}}(s; t)$

3.2 Exploration versus Exploitation

Motivation: To ensure \hat{q} converges to q^* and the agent's expected return is maximized, the agent must balance exploration and exploitation.

Definition:

- **Exploitation:** Choose the most promising actions based on current knowledge.
– Use optimal policy: $\hat{\pi}(\cdot, \cdot; t)$
- **Exploration:** Choose the least tried actions to improve current knowledge.
– Choose actions randomly

3.2.1 Simplified Case:

Example:

- **Given:** Assume the environment is stateless, but rewards are random.



Figure 2

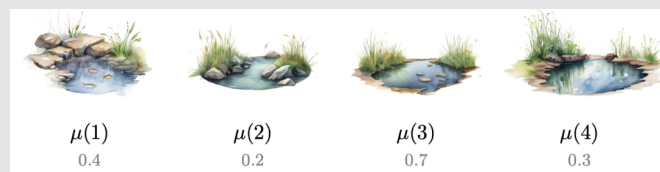


Figure 3

- $\mu(a)$: expected reward for action a (unknown to the agent):
– $0 \leq \mu(a) \leq 1$ for all a .

- **Best-case expected return:** (with $\gamma = 1$ under π^*) from transition t is:

$$u^*(t) := (T - t) \max_{a'} \mu(a')$$

where in this case:

$$\pi^*(a; t) = 0 \quad \text{if } a \notin \arg \max_{a'} \mu(a').$$

- **Estimation of $\mu(\cdot)$.** Since the agent does not have a model, it must estimate $\mu(\cdot)$.

The agent can take an action a and:

1. **Update** $n(\cdot)$ and $\hat{\mu}(\cdot)$:

$$n(a) \leftarrow n(a) + 1$$

$$\hat{\mu}(a) \leftarrow \left(1 - \frac{1}{n(a)}\right) \hat{\mu}(a) + \frac{1}{n(a)} r'$$

2. **Compute $\hat{\pi}$:**

$$\hat{\pi}(a; t) = 0 \quad \text{for all } a \notin \arg \max_{a'} \hat{\mu}(a').$$

- **Alternate Policies** We want to compare the expected return under various policies. The expected return from transition t under a policy ρ is:

$$u^\rho(t) := \mathbb{E}^\pi[G_t] = \sum_{a'} \rho(a'; t) (\mu(a') + u^\rho(t+1)).$$

3.3 Alternate Policies

Summary: To ensure the agent's expected return is maximized, the agent must strike a balance exploration and exploitation.

In the following cases, the expected return from transition t is

$$u^{\text{avg}}(t) \equiv \frac{T-t}{|\mathcal{A}|} \sum_a \mu(a)$$

We want to choose ρ so that $u^\rho > u^{\text{avg}}$.

Policy	Function:
Exploitation only	Choose a random action, same for all transitions
Exploration only	Choose a random action, different for each transition
Softmax	Apply a soft-max over \hat{u} $\rho(a; t) = \left[\sum_{a'} \exp \left(\frac{\hat{\mu}(a')}{\tau} \right) \right]^{-1} \exp \left(\frac{\hat{\mu}(a)}{\tau} \right)$ <ul style="list-style-type: none"> Choose a temperature value decrease with t. $\tau(t) \in [0, \infty), \tau \rightarrow 0$
ϵ -greedy	Use $\hat{\pi}$ w/ prob. $1 - \epsilon$, otherwise take a random action $\rho(a; t) = \epsilon \frac{1}{ \mathcal{A} } + (1 - \epsilon) \hat{\pi}(a; t)$ <ul style="list-style-type: none"> Choose an exploration rate decrease w/ t. $\epsilon(t) \in [0, 1], \epsilon \rightarrow 0$
Upper confidence bound	Choose the action with the highest $\text{ucb}(\cdot)$ $\rho(a; t) = 0 \text{ if } a \notin \arg \max_{a'} \text{ucb}(a'; t)$ <ul style="list-style-type: none"> Compute $\text{ucb}(\cdot)$ for each action. $\text{ucb}(a; t) = \hat{\mu}(a) + \sqrt{\frac{\ln t}{n(a)}}$

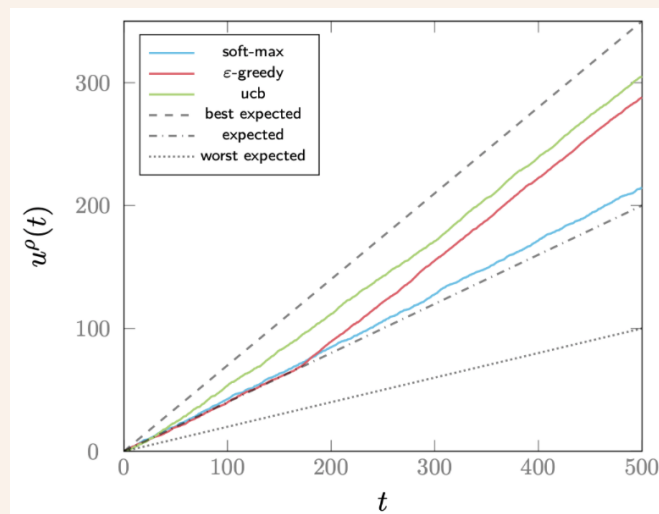


Figure 4

One-Shot Multi-Agent Decision Problems

4 Multi-Agent Problems

Summary: In a **Multi-Agent problem**, we assume that:

- Set of states for environment is \mathcal{S}
- P agents within environment.
- For each state $s \in \mathcal{S}$:
 - possible actions for agent i is $\mathcal{A}_i(s)$
 - set of action profiles is $\mathcal{A}(s) = \prod_{i=1}^P \mathcal{A}_i(s)$
- possible state-action pairs are $\mathcal{T} = \{(s, a) \text{ s.t. } s \in \mathcal{S}, a \in \mathcal{A}(s)\}$
- environment in some origin state, s_0
- environment destroyed after N transitions
- agent j wants to find policy $\pi_j(a_j | s)$ so that $\mathbb{E}[r_j(p)]$ is maximized
- agents act independently given the environmen

Name	Function:
State transition given state-action pair defined by $\text{tr} : \mathcal{T} \rightarrow \mathcal{S}$	$\text{tr}(s, a) = \text{state transition from } s \text{ under } a$
Reward to each agent, i defined by $r_i : \mathcal{Q} \times \mathcal{S} \rightarrow \mathbb{R}_+$	$r_i(s, a, \text{tr}(s, a)) = \text{rwd to agent } i \text{ for } (s, a, \text{tr}(s, a))$
State evolution of environment after N transitions	$p = \langle (s_0, a^{(1)}, s_1), \dots, (s_{N-1}, a^{(N)}, s_N) \rangle$ <ul style="list-style-type: none"> • Given sequence of actions: $p.a = \langle a^{(1)}, \dots, a^{(n)} \rangle$ • $s_N = \tau(s_{n-1}, a^{(n)})$
reward to agent i	$r_i(p) = \sum_{n=1}^N r_i(s_{n-1}, a^{(n)}, s_n)$
expected-reward (value) of playing a from s for agent j	$p(s' s) := \mathbb{P}[S_{t+1} = s' S_t = s]$
Prob. that state of the env. after T transitions is s	$p_T(s) := \mathbb{P}[S_T = s]$ $= \sum_{s'} p_{T-1}(s') p(s s')$ <ul style="list-style-type: none"> • $p_{T-1}(s')$: Prob. s' at $T-1$ (given) <ul style="list-style-type: none"> – $p_0(s)$: Base case • $p(s s')$: Prob. s given s' (from graph)

4.1 Action Equilibria

4.1.1 Finding Action Equilibria

4.2 Strategy Equilibria

4.2.1 Finding Strategy Equilibria

4.2.2 Existence of Strategy Equilibria

4.2.3 Convergence of Strategy Equilibria

4.3 Examples

4.3.1 Optimal Action Profiles

5 Turn-Based Games

5.1 Zero-Sum Turn-Based Games

Summary: In a zero-sum turn-based games, we assume that

- **Agents and Environment:**
 - there are two agents, called the **maximizer** and **minimizer**
 - the environment is always in one of a discrete set of states, \mathcal{S}
 - a subset of the states, $\mathcal{T} \subseteq \mathcal{S}$, are terminal states
 - there is only one decision maker for each non-terminal state, $s \in \mathcal{S} \setminus \mathcal{T}$
 - For each non-terminal state, $s \in \mathcal{S} \setminus \mathcal{T}$, the decision-maker has a discrete set of actions, $\mathcal{A}(s)$
- **Decision Process:** At time-step t , the decision-maker will:
 - **Observe:** Observe the state s_t
 - **Select:** Select an action $a_t \in \mathcal{A}(s_t)$
 - **Move:** Make the move (s_t, a_t)
- **State Transitions:**
 - Environment transitions to a deterministic state, s_{t+1} , based on a stationary fn,

$$s_{t+1} = \text{tr}(s_t, a_t)$$

- Once a terminal state is reached (if $s_{t+1} \in \mathcal{T}$), the maximizer obtains a reward for the final transition based on a reward fn, $r(\cdot, \cdot, \cdot)$:

$$r(s_t, a_t, s_{t+1}) = \text{maximizer's reward for reaching state } s_{t+1}$$

$$-r(s_t, a_t, s_{t+1}) = \text{minimizer's reward for reaching state } s_{t+1}$$

5.2 α/β Pruning

Definition:

- α : Maximum value at s thus far (initially $-\infty$)
- β : Minimum value at s thus far (initially $+\infty$)

5.2.1 α Cuts

Notes:

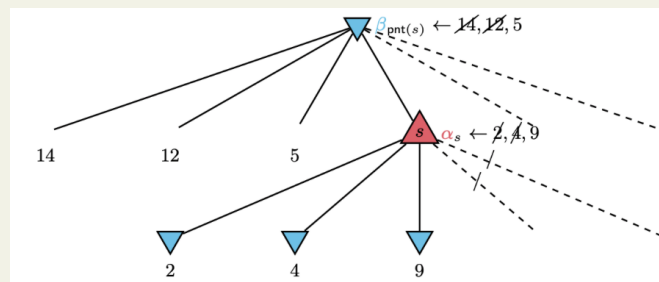


Figure 5

5.2.2 β Cuts

Notes:

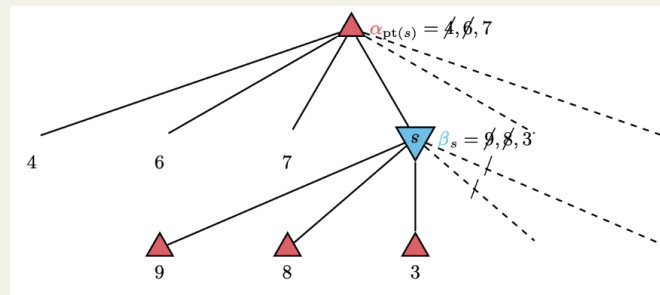


Figure 6

5.3 Examples

Example:

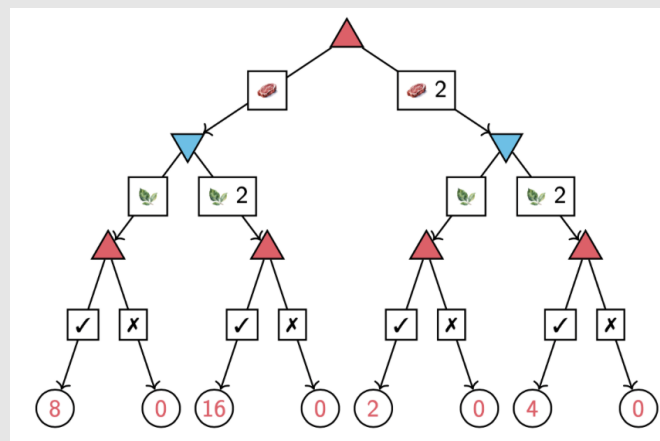


Figure 7

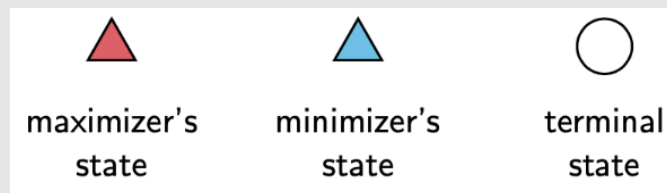


Figure 8



Figure 9

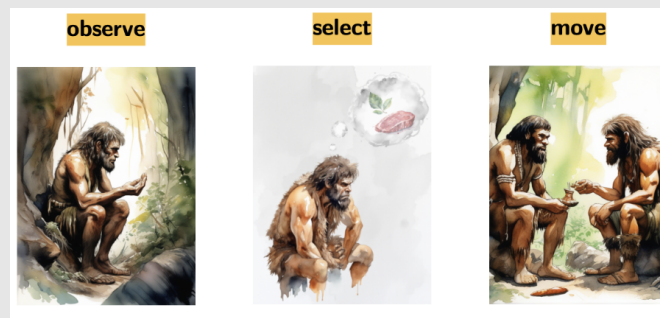


Figure 10

5.3.1 Zero-Sum Turn-Based Games

Example:

- **Given:** Cavemen is injured from his hunt. He has extra food, but needs medicine.
 - He meets another caveman who is willing to trade.
- **Game Tree:**
 - States:
 - * Red triangle:
 - * Blue triangle:
 - * White circles with #s
 - Actions: Square boxes are actions

Warning: In every given non-terminal state, only one agent can make the decision.