ROB311 Quiz 2

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Contents

Probabilistic Inference Problems

1 Bayesian Networks

Definition: Vertices represent random variables and edges represent dependencies between variables.

1.1 Junction

Definition: A **junction** $\mathcal{J}$ consists of three vertices, X_1, X_2, and X_3, connected by two edges, e_1 and e_2:

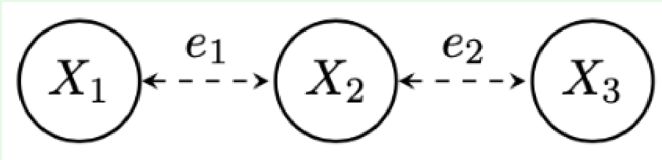

Figure 1

- X_1 and X_2 are not independent, X_2 and X_3 are not independent, but when is X_1 and X_3 independent?

1.1.1 Causal Chain

Definition: A causal chain is a junction $\mathcal{J}$ s.t.

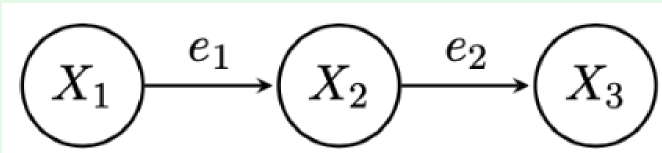

Figure 2

- X_1 and X_3 are not independent (unconditionally), but are independent given X_2.

Notes:
- **Analogy:** Given X_2, X_1 and X_3 are independent. Why? X_2's door closes when you know X_2, so X_1 and X_3 are independent.
- **Distinction b/w Causal and Dependence:** X_1 and X_2 are dependent. However, from a causal perspective, X_1 is influencing X_2 (i.e. $X_1 \rightarrow X_2$).

1.1.2 Common Cause

Definition: A common cause is a junction $\mathcal{J}$ s.t.

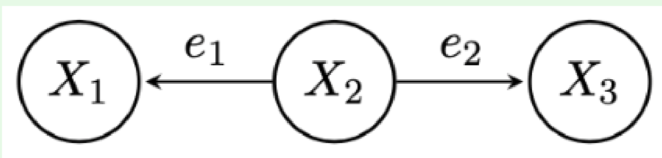

Figure 3

- X_1 and X_3 are not independent (unconditionally), but are independent given X_2.

Notes:
- **Analogy:** Given X_2, X_1 and X_3 are independent. Why? Consider the following example:
  - Let X_2 represent whether a person smokes or not, X_1 represent whether they have yellow teeth, X_3 represent whether they have lung cancer.
- Without knowing X_2, observing X_1 provides information about X_3 because yellow teeth are associated with smoking, which in turn increases the likelihood of lung cancer.
- If X_2 is known, then knowing whether a person has yellow teeth provides no additional information about whether they have lung cancer beyond what is already known from smoking status.

1.1.3 Common Effect

Definition: A common effect is a junction $\mathcal{J}$ s.t.

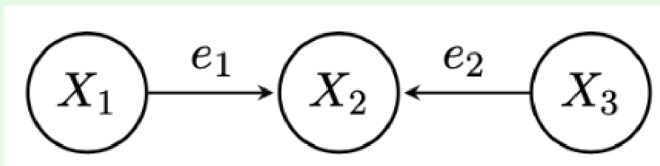

Figure 4

- X_1 and X_3 are independent (unconditionally), but are not independent given X_2 or any of X_2's descendents.

Notes:
- **Analogy:** Consider the following example:
  - Let X_2 represent whether the grass is wet, X_1 represent whether it rained, X_3 represent whether the sprinkler was on.
- Without knowing whether the grass is wet (X_2), the occurrence of rain (X_1) and the sprinkler being on (X_3) are independent events. The rain may occur regardless of the sprinkler, and vice versa.
- However, once we observe that the grass is wet (X_2), the two events become dependent:
  - If we learn that the sprinkler was not on, then the wet grass must have been caused by rain.
  - If we learn that it did not rain, then the wet grass must have been caused by the sprinkler.

2 Dependence Separation

2.1 Independence

Theorem: Any two variables, X_1 and X_2, in a Bayesian network, $\mathcal{B} = (\mathcal{V}, \mathcal{E})$, are independent given $\mathcal{K} \subseteq \mathcal{V}$ if every undirected path is blocked.

2.1.1 Blocked Undirected Path

Definition: An undirected path,

$$p = \langle (X_1, e_1, X_2), \ldots, (X_{|p|-1}, e_{|p|-1,|p|}, X_{|p|}) \rangle,$$

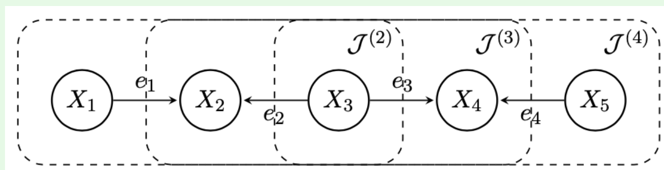

Figure 5

is **blocked** given $\mathcal{K} \subseteq \mathcal{V}$ if any of its junctions,

$$\mathcal{J}^{(n)} = \{(X_{n-1}, X_n, X_{n+1}), (e_{n-1}, e_n)\},$$

is blocked given $\mathcal{K}$.

2.1.2 Blocked Junction

Definition: $\mathcal{J} = (\{X_1, X_2, X_3\}, \{e_1, e_2\})$ is **blocked** given $\mathcal{K} \subseteq \mathcal{V}$ if X_1 and X_3 are independent given $\mathcal{K}$.

2.2 Consequence of Dependence Separation

Theorem: For any variable, $X \in \mathcal{V}$, it can be shown that X is independent of X's non-descendants, $\mathcal{V} \setminus \text{des}(X)$, given X's parents, $\text{pts}(X)$.

Notes:

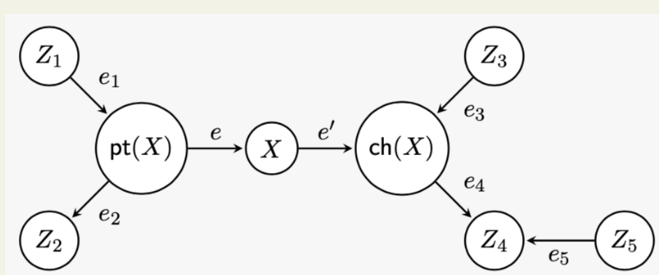

Figure 6

- Given X's parent, apply junction rules to determine that X is independent of its non-descendants.
- $\mathcal{J} = \{(Z_1, \text{pt}(X), X), (e_1, e)\}$ shows that Z_1 and X are independent given $\text{pt}(X)$ (causal chain).
- $\mathcal{J} = \{(Z_2, \text{pt}(X), X), (e_2, e)\}$ shows that Z_2 and X are independent given $\text{pt}(X)$ (common cause).
- Given $\text{ch}(X)$'s parent, apply junction rules to determine that $\text{ch}(X)$ is independent of its non-descendants.
- $\mathcal{J} = \{\text{pt}(X), X, \text{ch}(X)), (e, e')\}$ shows that $\text{pt}(X)$ and $\text{ch}(X)$ are independent given X (causal chain).

- Given Z_4's parent, apply junction rules to determine that Z_4 is independent of its non-descendants.
- $\mathcal{J} = \{X, \mathrm{ch}(X), Z_4, (e', e_4)\}$ shows that X and Z_4 are independent given $\mathrm{ch}(X)$ (causal chain).
- CHECK THIS OVER AGAIN WITH THE PROFESSOR.

2.3 Canonical Problems

2.3.1 Undirected Path Blocked?

Process:
1. **Given:** Undirected path p and $\mathcal{K}$
2. Check if any of the junctions on the undirected path are blocked given $\mathcal{K}$.
   - i.e. Check if X_1 and X_3 of the junction are independent given $\mathcal{K}$.

2.3.2 Independence

Process:
1. Given a Bayesian network w/ 2 variables to find independence.
2. Find all undirected paths between the 2 variables in the Bayesian network.
3. Identify a set of variables, $\mathcal{K}$, that block at least one junction in all undirected paths.
   - Test a junction by seeing junction given relationships.
4. If all undirected paths are blocked, then the 2 variables are independent given $\mathcal{K}$.

Warning:
- Be careful of common effect, in which it is blocked by default.
- Be careful of decesdents of a common effect (i.e. outward arrows from a common effect) as given it may not be blocked.
- Cyclic paths are not blocked by default.

Example:
1. **Given:** Bayesian network.

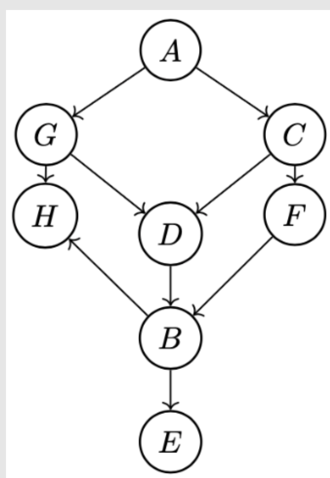

Figure 7

2. **Problem:** A and E are
   - independent if $\mathcal{K} =$
   - not necessarily independent for $\mathcal{K} =$
3. **Soln:**
   (a) **Undirected Paths:**
   - $A \to G \to H \to B \to E$
   - $A \to G \to D \to B \to E$
   - $A \to C \to F \to B \to E$
   - $A \to C \to D \to B \to E$

Example: **Independent:**

$\mathcal{K}$
$\{G, C\}$

- $A \iff G \iff H \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(A, G, H), (e_1, e_2)\}$ is blocked given G since A, H independent given G (causal chain)
- $A \iff G \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(A, G, D), (e_1, e_2)\}$ is blocked given G since A, D independent given G (causal chain)
- $A \iff C \iff F \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(A, C, F), (e_1, e_2)\}$ is blocked given C since A, F independent given C (causal chain)
- $A \iff C \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(A, C, D), (e_1, e_2)\}$ is blocked given C since A, D independent given C (causal chain)

$\{D, F\}$

- $A \iff G \iff H \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(G, H, B), (e_1, e_2)\}$ is blocked NOT given H since G, B independent NOT given H (common effect)
- $A \iff G \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(G, D, B), (e_1, e_2)\}$ is blocked given D since G, B independent given D (causal chain)
- $A \iff C \iff F \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(C, F, B), (e_1, e_2)\}$ is blocked given F since C, B independent given F (causal chain)
- $A \iff C \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(C, D, B), (e_1, e_2)\}$ is blocked given D since C, B independent given D (causal chain)

Not Necessarily Independent:

$\mathcal{K}$
$\{H, D, F\}$

- $A \iff G \iff H \iff B \iff E$ is unblocked given $\mathcal{K}$ since $\mathcal{J} = \{(G, H, B), (e_1, e_2)\}$ is unblocked given H since G, B not independent given H (common effect)
- $A \iff G \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(G, D, B), (e_1, e_2)\}$ is blocked given D (causal chain) since G, B independent given D (causal chain)
- $A \iff C \iff F \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(C, F, B), (e_1, e_2)\}$ is blocked given F since C, B independent given F (causal chain)
- $A \iff C \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(C, D, B), (e_1, e_2)\}$ is blocked given D since C, B independent given D (causal chain)

Example: Determine all subsets of $\{B, C, D, F, G, H\}$ for which A and E are independent.

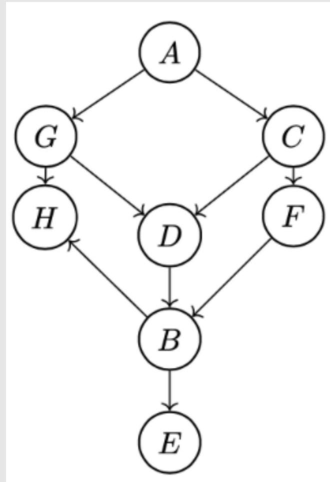

Figure 8

1. **Undirected Paths:**
   - $A \to G \to H \to B \to E$
   - $A \to G \to D \to B \to E$
   - $A \to C \to F \to B \to E$
   - $A \to C \to D \to B \to E$

$\mathcal{K}$
$\{B\}$ (Any subset that contains B will be independent)
• $AGHBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(H, B, E), (e_1, e_2)\}$ is b since H, E indep. given B (causal chain) • $AGDBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(D, B, E), (e_1, e_2)\}$ is b since D, E indep. given B (causal chain) • $ACFBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(F, B, E), (e_1, e_2)\}$ is b since F, E indep. given B (causal chain) • $ACDBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(D, B, E), (e_1, e_2)\}$ is b since D, E indep. given B (causal chain)
$\{C\}$ (Not independent)
• $AGDBE$ is ub given $\mathcal{K}$ since $\forall \mathcal{J}$ on p, all are ub.
$\{D\}$ (Not indepedent)
• $ACFBE$ is ub given $\mathcal{K}$ since $\forall \mathcal{J}$ on p, all are ub.
$\{F\}$ (Not independent)
• $AGDBE$ is ub given $\mathcal{K}$ since $\forall \mathcal{J}$ on p, all are ub.
$\{G\}$ (Not independent)
• $ACFBE$ is ub given $\mathcal{K}$ since $\forall \mathcal{J}$ on p, all are ub.
$\{H\}$ (Not independent)
• $ACFBE$ is ub given $\mathcal{K}$ since $\forall \mathcal{J}$ on p, all are ub.

Example:

$\mathcal{K}$
$\{C, D\}$ (Any subset that contains C and D except H will be independent)   • $AGHBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(G, H, B), (e_1, e_2)\}$ is b since G, B indep. not given H (causal effect)   • $AGDBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(G, D, B), (e_1, e_2)\}$ is b since G, B indep. given D (causal chain)   • $ACFBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(A, C, F), (e_1, e_2)\}$ is b since A, F indep. given C (causal chain)   • $ACDBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(A, C, D), (e_1, e_2)\}$ is b since A, D indep. given C (causal chain)
...

Example: Any set with

$$B \vee [(G \vee (D \wedge (\neg H))) \wedge (C \vee F)]$$

Example:

1. **Given:**

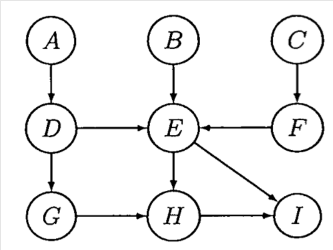

Figure 9

2. **Problem 1:** Is it gauranteed that $A \perp C$?
3. **Solution 1:** True b/c all undirected paths are blocked.
   (a) ADEFC is b since $\mathcal{J} = \{(D, E, F), (e_1, e_2)\}$ is b since D, F indep. NOT given E (common effect)
   (b) ADGHEFC is b since $\mathcal{J} = \{(G, H, E), (e_1, e_2)\}$ is b since G, E indep. NOT given H (common effect)
   (c) ADGHIEFC is b since $\mathcal{J} = \{(H, I, E), (e_1, e_2)\}$ is b since H, E indep. NOT given I (common effect)
4. **Problem 2:** Is it gauranteed that $B \perp C \mid I$?
5. **Solution 2:** False b/c BEFC is ub.
   (a) BEFC is ub since $\mathcal{J} = \{(B, E, F), (e_1, e_2)\}$ is ub since B, F NOT indep. given E's descendent, I (common effect)
6. **Problem 3:** Is it gauranteed that $D \perp I \mid \{E, G\}$?
7. **Solution 3:** True b/c all undirected paths are blocked.
   (a) DEI is b since $\mathcal{J} = \{(D, E, I), (e_1, e_2)\}$ is b since D, I indep. given E (causal chain)
   (b) DEHI is b since $\mathcal{J} = \{(D, E, H), (e_1, e_2)\}$ is b since D, H indep. given E (causal chain)
   (c) DGHI is b since $\mathcal{J} = \{(D, G, H), (e_1, e_2)\}$ is b since D, H indep. given G (causal chain)
   (d) DGHEI is b since $\mathcal{J} = \{(D, G, H), (e_1, e_2)\}$ is b since D, H indep. given G (causal chain)
8. **Problem 4:** Is it gauranteed that $C \perp H \mid G$?
9. **Solution 4:** False b/c CFEH is ub.
   (a) CFEH is ub since $\mathcal{J} = \{(C, F, E), (e_1, e_2)\}$ is ub since C, E NOT indep. given G (causal chain)
10. **Problem 5:** Suppose each variable is binary. What is the size of the domain of the joint distribution b/w the variables?
11. **Solution 5:**
    (a) Since 9 variables, each with 2 values, the size of the domain of the joint distribution is $2^9 = 512$.
12. **Problem 6:** Suppose each variable is binary. What is the min # of values that actually need to be stored to represent the joint distribution entirely based on the Bayesian network? Use the fact that probability distributions are normalized.
13. **Solution 6:** 1+1+1+2+8+2+2+4+4 = 25 values need to be stored.
    (a) $P(A)$, $P(B)$, $P(C)$ has 1 value each
        - Since $P(\#)$ can represent 2 values, i.e. $P(0) = 1 - P(1)$, so only need to store 1 value.
    (b) $P(D \mid A)$, $P(F \mid C)$, $P(G \mid D)$ has 2 values each
        - Same idea, can take the complement of the other value for 4 values.
    (c) $P(H \mid G, E)$, $P(I \mid E, H)$ has 4 values each
        - Same idea, can take the complement of the other value for 8 values.
    (d) $P(E \mid D, B, F)$ has 8 values
        - Same idea, can take the complement of the other value for 16 values.