Modelling CS u: control input, y: plant output

State variable CS is in state variable form if
$\dot{x}_1 = f_1(t, x_1, \ldots, x_n, u), \ldots, \dot{x}_n = f_n(t, x_1, \ldots, x_n, u)$
$y = h(t, x_1, \ldots, x_n, u)$ is a collection of n 1st order ODEs.

Time-Invariant (TI) CS is TI if $f_i(\cdot)$ does not depend on t.

State space (SS) TI CS is in SS form if $\dot{x} = f(x, u)$, $y = h(x, u)$ where $x(t) \in \mathbb{R}^n$ is called the state.

Single-input-single-output (SISO) CS is SISO if $u(t), y(t) \in \mathbb{R}$.

LTI CS in SS form is LTI if $\dot{x} = Ax + Bu$, $y = Cx + Du$
$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$
where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$.

Input-Output (IO) LTI CS is in IO form if
$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt} + \cdot + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^m} + \cdot + b_1\frac{du}{dt} + b_0 u$
where $m \le n$ (causality)

IO to SS Model 1. Define x s.t. highest order derivative in $\dot{x}$
2. Write $\dot{x} = Ax + Bu = f(x, u)$ by isolating for components of $\dot{x}$
3. Write $y = Cx + Du = h(x, u)$ by setting measurement output y to component of x

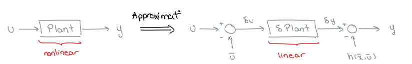
Equilibria y_d (steady state) b/c if $y(0) = y_d$ at $t = 0$, then $y(t) = y_d\ \forall t \ge 0$.

Equilibrium pair Consider the system $\dot{x} = f(x, u)$. The pair $(\bar{x}, \bar{u})$ is an equilibrium pair if $f(\bar{x}, \bar{u}) = 0$.

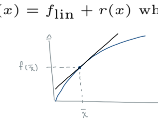
Equilibrium point $\bar{x}$ is an equilibrium point w/ control $u = \bar{u}$.
*If $u = \bar{u}$ and $x(0) = \bar{x}$ then $x(t) = \bar{x}\ \forall t \ge 0$ (i.e. a system that starts at equilibrium remains at equilibrium).

Find Equilibrium Pair/Point 1. Set $f(x, u) = 0$
2. Solve $f(x, u) = 0$ to find $(x, u) = (\bar{x}, \bar{u})$.
3. If specific $u = \bar{u}$, then find $x = \bar{x}$ by solving $f(x, \bar{u}) = 0$.

Linearization of Nonlinear System Consider system $\dot{x} = f(x, u)$ w/ equ. pair $(\bar{x}, \bar{u})$, then error coordinates around equ. pair
$\delta x = x - \bar{x}$, $\delta u = u - \bar{u}$, $\delta y = y - h(\bar{x}, \bar{u})$ $\delta\dot{x} = \dot{x} - f(\bar{x}, \bar{u})$ w/
$\delta\dot{x} = A\delta x + B\delta u$, $A = \frac{\partial f(\bar{x}, \bar{u})}{\partial x} \in \mathbb{R}^{n_1\times n_1}$, $B = \frac{\partial f(\bar{x}, \bar{u})}{\partial u} \in \mathbb{R}^{n_1}$,
$\delta y = C\delta x + D\delta u$, $C = \frac{\partial h}{\partial x}(\bar{x}, \bar{u}) \in \mathbb{R}^{1\times n_1}$, $D = \frac{\partial h(\bar{x}, \bar{u})}{\partial u} \in \mathbb{R}$
*Only valid at equ. pairs.



Linear Approx. Given a diff. fcn. $f : \mathbb{R} \to \mathbb{R}$, its linear approx. at $\bar{x}$ is $f_{\text{lin}} = f(\bar{x}) + f'(\bar{x})(x - \bar{x})$.
*Remainder Thm: $f(x) = f_{\text{lin}} + r(x)$ where $\lim_{x\to\bar{x}} \frac{r(x)}{x - \bar{x}} = 0$.

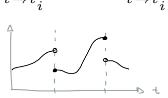


*Note: Can provide a good approx. near $\bar{x}$ but not globally.
*Gen. $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$, $f(x) = f(\bar{x}) + \frac{\partial f}{\partial x}(\bar{x})(x - \bar{x}) + R(x)$
*Jacobian: $\frac{\partial f}{\partial x}(\bar{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial f}{\partial x_{n_1}}(\bar{x}) \end{bmatrix} \in \mathbb{R}^{n_2\times n_1}$

Linearization Steps 1. Find equ. pair $(\bar{x}, \bar{u})$
2. Derive A, B, C, D and then evaluate at $(\bar{x}, \bar{u})$
3. Write $\delta\dot{x} = A\delta x + B\delta u$ and $\delta y = C\delta x + D\delta u$

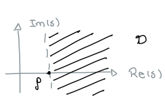
Laplace Transform Given a fcn $f : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}^n$, its Laplace transform is $F(s) = \mathcal{L}\{f(t)\} := \int_{0^-}^{\infty} f(t)e^{-st}\,dt$, $s \in \mathbb{C}$.
*$\mathcal{L} : f(t) \mapsto F(s)$, $t \in \mathbb{R}_+$ (time dom.) & $s \in \mathbb{C}$ (Laplace dom.).

P.W. CTS: A fcn $f : \mathbb{R}_+ \to \mathbb{R}^n$ is **p.w. cts** if on every finite interval of $\mathbb{R}$, $f(t)$ has at most a finite # of discontinuity points (t_i) and the limits $\lim_{t\to t_i^+} f(t)$, $\lim_{t\to t_i^-} f(t)$ are finite.



Exp. Order A function $f : \mathbb{R}_+ \to \mathbb{R}^n$ is of **exp. order** if $\exists$ constants $K, \rho, T > 0$ s.t. $\|f(t)\| \le Ke^{\rho t}$, $\forall t \ge T$.

Existence of LT Thm If $f(t)$ is p.w. cts and of exp. order w/ constants $K, \rho, T > 0$, then $F(\cdot)$ exists and is defined $\forall s \in D := \{s \in \mathbb{C} : \text{Re}(s) > \rho\}$ and $F(\cdot)$ is analytic on D.
*Analytic fcn iff differentiable fcn.
*D: Region of convergence (ROC), open half plane.



Unit Step $\mathbf{1}(t) := \begin{cases} 1, & \text{if } t \ge 0 \\ 0, & \text{otherwise} \end{cases}$

Table of Common Laplace Transforms: $f(t)\ |\ F(s)$
$\mathbf{1}(t) \mapsto \frac{1}{s}$ $\quad t\mathbf{1}(t) \mapsto \frac{1}{s^2}$ $\quad t^k\,\mathbf{1}(t) \mapsto \frac{k!}{s^{k+1}}$ $\quad e^{at}\,\mathbf{1}(t) \mapsto \frac{1}{s-a}$
$t^n e^{at}\,\mathbf{1}(t) \mapsto \frac{n!}{(s-a)^{n+1}}$ $\quad \sin(at)\,\mathbf{1}(t) \mapsto \frac{a}{s^2+a^2}$
$\cos(at)\,\mathbf{1}(t) \mapsto \frac{s}{s^2+a^2}$ $\quad \frac{1}{2\omega^3}[\sin(\omega t) - \omega t\cos(\omega t)]\,\mathbf{1}(t) \mapsto \frac{1}{(s^2+\omega^2)^2}$

Prop. of Laplace Transform Linearity:
$\mathcal{L}\{cf(t) + g(t)\} = c\mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$, $c \sim$ constant.
Differentiation: If the Laplace transform of $f'(t)$ exists, then $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^-)$.
If the Laplace transform of $f^{(n)}(t) := \frac{d^n f}{dt^n}(t)$ exists, then $\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0^-)$.
Integration: $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$.
Convolution: Let $(f * g)(t) := \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau$, then $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$.
Time Delay: $\mathcal{L}\{f(t-T)\mathbf{1}(t-T)\} = e^{-Ts}\mathcal{L}\{f(t)\}$, $T \ge 0$.
Multiplication by t: $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}[\mathcal{L}\{f(t)\}]$.
Shift in s: $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}|_{s\to s-a} = F(s-a)$, where $F(s) = \mathcal{L}\{f(t)\}$ & a const.

Trig. Id. $2\sin(2t) = 2\sin(t)\cos(t)$, $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$, $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

Complete the Square: $ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c$

LT Steps: 1. Write $f(t)$ as a sum and use linearity
*Trig. id. may be useful.
2. Use prop. of LT and common LT to find $F(s)$

Inverse Laplace Transform Given $F(s)$, its inverse LT is $f(t) = \mathcal{L}^{-1}\{F(s)\} := \frac{1}{2\pi}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds$
$= \lim_{w\to\infty} \frac{1}{2\pi}\int_{c-jw}^{c+jw} F(s)e^{st}ds$, $c \in \mathbb{C}$ is selected s.t. the line $L := \{s \in \mathbb{C} : s = c + j\omega, \omega \in \mathbb{R}\}$ is inside the ROC of $F(s)$.
Zero: $z \in \mathbb{C}$ is a zero of $F(s)$ if $F(z) = 0$.
Pole: $p \in \mathbb{C}$ is a pole of $F(s)$ if $\frac{1}{F(p)} = 0$.

Cauchy's Residue THM If $F(s)$ is analytic (complex diff.) everywhere except at isolated poles $\{p_1, \ldots, p_N\}$, then
$\mathcal{L}^{-1}\{F(s)\} = \sum_{i=1}^{N} \text{Res}\left[F(s)e^{st}, s = p_i\right]\mathbf{1}(t)$,
*Res$[F(s)e^{st}, s = p_i]$: Residue of $F(s)e^{st}$ at $s = p_i$.

Residue Computation Let $G(s)$ be a complex analytic fcn w/ a pole at $s = p$, r be the multiplicity of the pole p. Then
Res$[G(s), s = p] = \frac{1}{(r-1)!}\lim_{s\to p}\frac{d^{r-1}}{ds^{r-1}}\left[G(s)(s-p)^r\right]$.

Inv. LT Partial Frac.: 1. Factorize $F(s)$ into partial fractions.
2. Find coefficients and use LT table to find inverse LT.
*Complete the square.

Inv. LT Residue: 1. Find poles of $F(s)$ and their residues.
2. Use Cauchy's Residue THM to find inverse LT.
*Note: Complex Conjugate (CC) poles → CC residues (use Euler).

Transfer Function: Consider a CS in IO form. Assume zero initial conds. $y(0) = \cdots = \frac{d^{(n-1)}y}{dt^{(n-1)}}(0) = 0$ and
$u(0) = \cdots = \frac{d^{(m-1)}u}{dt^{(m-1)}}(0) = 0$. Then the TF from u to y is
$G(s) := \frac{y(s)}{U(s)} = \frac{b_m s^m + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$.
***0 Ini. Conds.:** $y_0(s) = G(s)u(s)$
***$\emptyset$ Ini. Conds.:** $y_\emptyset(s) = y_0(s) + \frac{\text{poly. based on initial conds.}}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$

TF Steps (IO to TF): 1. Given IO form of CS, assume zero initial conds.
2. Find $G(s)$ by taking LT of IO form and forming $Y(s)/U(s)$.
*Careful: $Y(s)/U(s) = G(s)$ not $U(s)/Y(s) = G(s)$.

Impulse Response: Given CS modeled by TF $G(s)$, its IR is $g(t) := \mathcal{L}^{-1}\{G(s)\}$.
*$\mathcal{L}\{\delta(t)\} = 1$, then if $u(t) = \delta(t)$, then $Y(s) = U(s)G(s) = G(s)$.

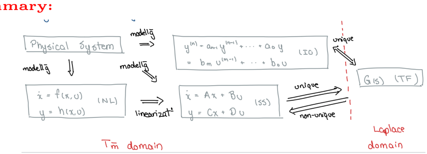
SS to TF: $G(s) = C(sI - A)^{-1}B + D$ s.t. $y(s) = G(s)U(s)$.
*Assume $x(0) = 0 \in \mathbb{R}^n$ (zero initial conds.).
***LTI:** $G(s)$ of an LTI system is always a rational fcn.
***Not Invertible:** Values of s s.t. $sI - A$ not invertible can correspond to poles of $G(s)$.

Inverse: 1. For $A \in \mathbb{R}^{n\times n}$, find $[\text{cof}(A)]_{(i,j)} = (-1)^{i+j}\det(A_{(i,j)})$.
*$A_{(i,j)}$: A w/ row i and col. j removed.
2. Assemble cof(A) and find $\det(A) = \sum_{j=1}^{n} a_{ij}[\text{cof}(A)]_{(i,j)}$ w/ fixed i or $\det(A) = \sum_{i=1}^{n} a_{ij}[\text{cof}(A)]_{(i,j)}$ w/ fixed j.
3. Find $A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{\det(A)}[\text{cof}(A)]^T$.
*2×2: $A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

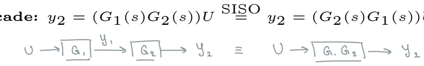
TF (SS to TF): 1. Given SS form, assume zero initial conds.
2. Solve $G(s) = C(sI - A)^{-1}B + D$.
*If $C = \begin{bmatrix} 0 & \cdot & 1_i & \cdot & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & \cdot & 1_j & \cdot & 0 \end{bmatrix}$, then only need ith row & jth col. of adj$(sI - A)$ s.t. $G(s) = \frac{[\text{adj}(sI-A)]_{(i,j)}}{\det(sI-A)} + D$.
*Multiple i, j non-zero entries: Work it out using MM.

TF to SS: Consider $G(s) = \frac{b_m s^m + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} = \frac{N(s)}{D(s)}$, where $m < n$ (i.e. $G(s)$ is strictly proper). Then the SS form is
$$*A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$
$C = \begin{bmatrix} b_0 & \cdots & b_m & | & 0 & \cdots & 0 \end{bmatrix}$, $D = 0$.
***Unique:** State space of a TF is not unique.

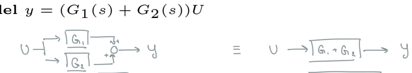
Summary:



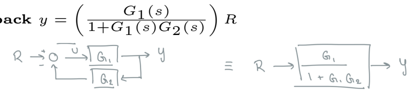
Block Diagram Types of Blocks:
Cascade: $y_2 = (G_1(s)G_2(s))U \overset{\text{SISO}}{=} y_2 = (G_2(s)G_1(s))U$



Parallel $y = (G_1(s) + G_2(s))U$

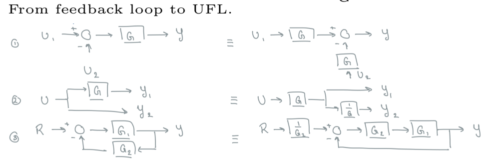


Feedback $y = \left(\frac{G_1(s)}{1 + G_1(s)G_2(s)}\right)R$



*SC: Unity Feedback Loop (UFL) if $G_2(s) = 1$.

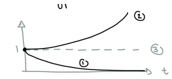
Manipulations: 1. $y = G(U_1 - U_2) = GU_1 + GU_2$
2. $y_1 = GU \quad y_2 = U \ |\ y_1 = GU \quad y_2 = G\frac{1}{G}U$
3. From feedback loop to UFL.



Find TF from Block Diagram: 1. Start from in → out, making simplifications using block diagram rules.
2. Simplify until you get the form $U(s) \to \boxed{G(s)} \to Y(s)$.

Time Response of Elementary Terms: $\mathbf{1}(t) \leftarrow$ pole @ 0
$t^n\mathbf{1}(t) \leftarrow$ pole @ 0 w/ mult. n | $e^{at}\mathbf{1}(t) \leftarrow$ pole @ a
$\sin(\omega t + \phi)\mathbf{1}(t) \leftarrow$ pole @ $\pm j\omega$ | $\cos(\omega t + \phi)\mathbf{1}(t) \leftarrow$ pole @ $\pm j\omega$

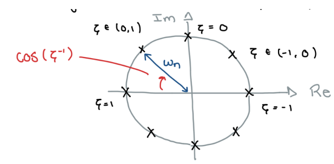
Real Pole: $y(s) = \frac{1}{s+a}$, real pole at $s = -a$, then $y(t) = e^{-at}\mathbf{1}(t)$
1. $a > 0 \implies \lim_{t\to\infty} y(t) = 0$ | 2. $a < 0 \implies \lim_{t\to\infty} y(t) = \infty$.
3. $a = 0 \implies y(t) = \mathbf{1}(t)$ is constant.



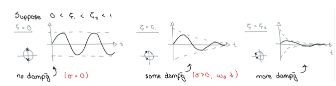
Time Constant: $\tau = \frac{1}{a}$ of the pole $s = -a$ for $a > 0$

Pair of Comp. Conj. Poles:
$y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$, $|\zeta| < 1$, then
$y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\sigma t}\sin(\omega_d t)\mathbf{1}(t)$

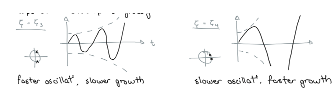
*Poles: $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$
*$\zeta = \frac{\sigma}{\omega_n}$: Damping ratio (or damping coefficient)
*$\sigma = \zeta\omega_n$: Decay/growth rate | ω_d: Freq. of oscillation
*$\omega_n = \sqrt{\sigma^2 + \omega_d^2}\left[\frac{\text{radians}}{\text{seconds}}\right]$: Undamped natural freq.
*$\omega_d = \omega_n\sqrt{1-\zeta^2}\left[\frac{\text{radians}}{\text{seconds}}\right]$: Damped natural freq.
*$|s_{1,2}|^2 = \omega_n^2$: Mag. of poles is ω_n.
*$\cos^{-1}(\zeta)$: Angle of s_1 on complex plane CW from -ve Re axis.



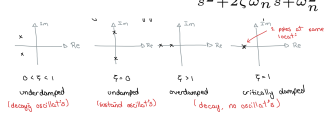
Damping Ratio Effect: $0 < \zeta_1 < \zeta_2 < 1$, then



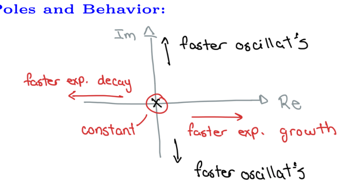
$-1 < \zeta_4 < \zeta_3 < 0$, then $\sigma = \zeta\omega_n < 0$, (exp. envelop ↑)



Class. of 2nd Order Sys.: $y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, w/ $|\zeta| < 1$



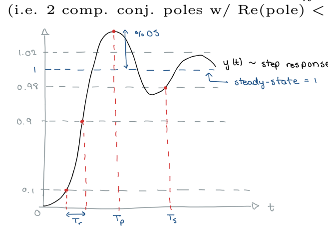
Loc. of Poles and Behavior:



Control Spec. of 2nd Order Sys.: Step Response: Given a TF $G(s)$, its SR is $y(t)$ resulting from applying the input $u(t) = \mathbf{1}(t)$, i.e. $\mathcal{L}^{-1}\left\{G(s)\frac{1}{s}\right\}$.

Control Spec. A control spec. is a criterion specifiying how we would like a CS to behave.

2nd Order Sys. Metrics: $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ w/ $U(s) = \frac{1}{s}$
*$0 < \zeta < 1$ (i.e. 2 comp. conj. poles w/ Re(pole) < 0).



Rise Time (RT): T_r is the time it takes $y(t)$ to go from 10% to 90% of its steady-state value.
RT: 1. Find $t_1 > 0$ s.t. $y(t_1) = 0.1$, $t_2 > 0$ s.t. $y(t_2) = 0.9$.
3. Compute $T_r = t_2 - t_1$. $\boxed{T_r \approx \frac{1.8}{\omega_n}}$.

Settling Time (ST): T_s is the time required to reach and stay w/in 2% of the steady-state value.

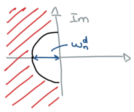
ST: 1. Find when it's first that $|y(t) - 1| \le 0.02$. $\boxed{T_s \approx \frac{4}{\zeta\omega_n}}$.

Peak Time: T_p is time req'd to reach the max (peak) value.
Peak Time: 1. Find the first time when $\dot{y}(t) = 0$.
* $\boxed{T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$

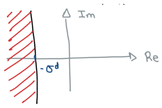
% Overshoot: %OS $= \frac{[\text{peak value}] - [\text{steady-state value}]}{[\text{steady-state value}]} \times 100\%$
*% OS = OS × 100%.
* $\boxed{\%\text{OS} = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \iff \zeta = \frac{-\ln(\text{OS})}{\sqrt{\pi^2 + (\ln(\text{OS}))^2}}}$.

Transient Performance Sat.: Given performance spec. $T_r \leq T_r^d$, $T_s \leq T_s^d$, $\text{OS} \leq \text{OS}^d$, find loc. of poles of $G(s)$.
*Admissible region for the poles of $G(s)$ s.t. the step response meets all three spec. is the intersection of the above three regions.

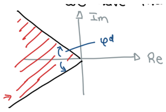
Rise Time: $T_r \approx \frac{1.8}{\omega_n} \leq T_r^d \overset{\text{app.}}{\iff} \omega_n \geq \frac{1.8}{T_r^d} \equiv \omega_n^d$



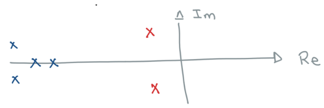
Settling Time: $T_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} \leq T_s^d \overset{\text{app.}}{\iff} \sigma \geq \frac{4}{T_s^d} \equiv \sigma^d$



OS: $\exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \leq \text{OS}^d \overset{\text{app.}}{\iff} \zeta \geq \frac{-\ln(\text{OS}^d)}{\sqrt{\pi^2+(\ln(\text{OS}^d))^2}} \equiv \zeta^d$



Add. Poles & Zeros: The analysis remains approx. correct under the following assumptions:
1. Any add. poles of $G(s)$ have much more -ve real part (5-10 times) than the real part of the dom. complex conjugate poles.



*dominant poles, additional poles.
2. Real part of zeros are -ve & very diff. from the real part of the two dom. poles.

Internal Stablity: $\dot{x} = Ax$ is
1. **Stable** if $\forall x(0) \in \mathbb{R}^n$, the soln. $x(t)$ is bdd; that is, $\exists M > 0$ s.t. $\|x(t)\| \leq M \ \forall t \geq 0$.
2. **Asymp. Stable** if it's stable & $\forall x(0) \in \mathbb{R}^n$, the soln. $x(t)$ converges to the origin; that is, $\lim_{t\to\infty} x(t) = 0$.
3. **Unstable** if it's not stable; that is, $\exists x(0) \in \mathbb{R}^n$ s.t. $x(t)$ is not bdd.

Asymptotic Stablity Thm. $\dot{x} = Ax$ is A.S. iff $\text{eig}(A) \subseteq \mathbb{C}^- \equiv \{s \in \mathbb{C} \mid \text{Re}(s) < 0\}$, i.e. open left half plane (OLHP).
Instability Thm. If $\exists$ an eigenvalue λ of A w/ $\text{Re}(\lambda) > 0$, then $\dot{x} = Ax$ is unstable.
Fact: Zeros of $s^2 + a_1 s + a_0$ are in $\mathbb{C}^-$ iff $a_1, a_0 > 0$.
Internal Stability 1. Linearize around $(\bar{x}, \bar{u})$ w/ $\bar{u} = 0$.
2. Find A and determine $\text{eig}(A) = \lambda$ s.t. $\det(sI - A) = 0$.
3. Check if $\text{eig}(A) \subseteq \mathbb{C}^-$ | $\text{Re}(\text{eig}(A)) > 0$.
BIBO Stability: An LTI system w/ 0 i.c. is Bounded Input Bounded Output (BIBO) stable if for any bdd input $u(t)$, the output $y(t)$ is also bdd.
BIBO Unstable: An LTI system w/ 0 i.c. is BIBO unstable if it's not BIBO stable; that is, $\exists$ a bdd $u(t)$ s.t. $y(t)$ is not bdd.
BIBO Stable Thm. A system $y(s) = G(s)U(s)$ is BIBO stable iff $\text{poles}(G(s)) \subseteq \mathbb{C}^-$.
Lemma: If p is a pole of $G(s)$, then p is an eig(A). I.e. $\text{poles}(G(s)) := \{p \in \mathbb{C} \mid p \text{ is a pole of } G(s)\} \subseteq \text{eig}(A)$.
***Pole-0 Cancellation:** eige(A) need not be a pole of $G(s)$.
Thm. If $\text{eig}(A) \subseteq \mathbb{C}^-$, then $\forall B, C, D$ the TF $G(s)$ is BIBO stable. That is, internal asymptotic stability $\Rightarrow$ BIBO stability.
BIBO Stability 1. Find $G(s)$ from SS form and determine poles.
2. Check if $\text{poles}(G(s)) \subseteq \mathbb{C}^-$.

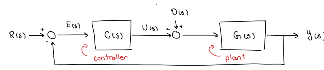
Routh-Hurwitz: Consider $a(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$.

$*s^n$	1	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$	0
$*s^{n-1}$	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$	0
$*s^{n-2}$	b_1	b_2	b_3	$\cdots$		
$*s^{n-3}$	c_1	c_2	$\cdots$			
$\vdots$						
$*s$	$*$	0				
$*1$	$*$	0				

$b_1 = -\frac{1}{a_{n-1}} \det\begin{bmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{bmatrix}$ $b_2 = -\frac{1}{a_{n-1}} \det\begin{bmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{bmatrix}$
$b_3 = -\frac{1}{a_{n-1}} \det\begin{bmatrix} 1 & a_{n-6} \\ a_{n-1} & a_{n-7} \end{bmatrix}$ $c_1 = -\frac{1}{b_1} \det\begin{bmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{bmatrix}$
$c_2 = -\frac{1}{b_1} \det\begin{bmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{bmatrix}$
Routh-Hurwitz Stability Criterion: The roots of $a(s)$ are in $\mathbb{C}^-$ iff the 1st col of Routh array has no sign changes. The # of sign changes is equal to the # of roots of $a(s) \in \mathbb{C}^+ := \{s \in \mathbb{C} : \text{Re}(s) > 0\}$.
*If 1st element of a row is 0, Rooth array cannot be completed.
FVT v1: Suppose $Y(s) = \mathcal{L}\{y(t)\}$ is a proper rational fcn. If $y(\infty) := \lim_{t\to\infty} y(t)$ exists and is finite, then $y(\infty) = \lim_{s\to 0} sY(s)$.
FVT v2: Suppose $Y(s) = \mathcal{L}\{y(t)\}$ is a proper rational fcn. Moreover, suppose either:
1. $\text{poles}(Y(s)) \subseteq \mathbb{C}^-$
2. $Y(s)$ has only one pole at $s = 0$ and all other poles are in $\mathbb{C}^-$.
Then $y(\infty) := \lim_{t\to\infty} y(t)$ exists and is finite and satisfies $y(\infty) = \lim_{s\to 0} sY(s)$.
FVT 1. Does $y(\infty)$ exist? Check if pole at $s = 0$, then compute Rooth Array to see if poles are in $\mathbb{C}^-$.
2. Compute $\lim_{s\to 0} sY(s)$ if it exists.

MIDTERM CUTOFF

Standard Feedback Control Loop



$R(s)$: Ref., $E(s) = R(s) - y(s)$: Err., $C(s)$: Controller, $U(s)$: Control input, $D(s)$: Dist., $G(s)$: Plant, $y(s)$: Plant output.
***Assume:** $R(s)$ and $D(s)$ are strictly proper rational fcns w/ a fixed set of poles but arbitrary zeros & gain.
*$\mathcal{R}, \mathcal{D}$: Classes of ref. and dist. satisfying the above assumption.
Basic Control Prob.: Design $C(s)$ s.t. 3 spec. are met:
1. **Stability:** $\forall$ bdd $r(t), d(t)$, we have $u(t), e(t)$ bdd.
2. **Asymptotic Tracking:** When $d(t) = 0 \ \forall t \geq 0$, then $\forall r(t) \in \mathcal{R}$, $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} r(t) - y(t) = 0$.
3. **Disturbance Rejection:** When $r(t) = 0 \ \forall t \geq 0$, then $\forall d(t) \in \mathcal{D}$, $\lim_{t\to\infty} y(t) = 0$.
Open-Loop Control: 1. Design $u(t)$ s.t. $y(t)$ tracks ref. $y_r \in \mathbb{R}$, i.e. $\lim_{t\to\infty} y(t) = y_r$.
2. Set $u(t) = \gamma y_r \mathbf{1}(t)$ w/ $\gamma \in \mathbb{R}$ (const. scaling factor)
3. Apply FVT to find γ s.t. $\lim_{t\to\infty} y(t) = y_r$. 4. Determine $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} y_r - y(t)$
Limitations: 1. Req. perfect knowledge of plant paramters.
2. Not robust against parameter var./(unknown) dist.
3. Does not allow us to speed up convergence.
Feedback Control: 1. Design $u(t)$ s.t. $y(t)$ tracks ref. $y_r \in \mathbb{R}$, i.e. $\lim_{t\to\infty} y(t) = y_r$.
2. Set $u(t) = Ke(t) = K(y_r - y(t))$ w/ $K > 0$ (const. gain).
3. Use block mani. to find $y(s)$ in terms of input and $G(s)$.
4. Apply FVT to find K s.t. $\lim_{t\to\infty} y(t) = y_r$.
5. Determine $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} y_r - y(t)$
Advantages: 1. Doesn't req. perfect knowledge of plant param.
2. Robust against param. var./dist. by $\uparrow K$.
3. Allows us to speed up the rate of convergence by $\uparrow K$.
Disadvantages: 1. Feedback can introduce instability.
2. High-gain amplifies noise.
3. Asymptotic tracking doesn't occur.
Integral Control: 1. Design $u(t)$ s.t. $y(t)$ tracks ref. $y_r \in \mathbb{R}$, i.e. $\lim_{t\to\infty} y(t) = y_r$.
2. Set $u(t) = \mathcal{L}^{-1}\{C(s)E(s)\} = Ke(t) + KT_I \int_0^t e(\tau)d\tau$ (prop. int. (PI) controller) w/ $K, T_I > 0$ (const. gains).
*$C(s) = K\left(1 + \frac{T_I}{s}\right)$
3. Use block mani. to find $y(s)$ in terms of input and $G(s)$.
4. Apply FVT to find $\lim_{t\to\infty} y(t) = y_r$ as desired.
BIBO Stability of Closed-Loop System: **Gang of 4 TF:**

$$\begin{bmatrix} E(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{1+C(s)G(s)} & \frac{-G(s)}{1+C(s)G(s)} \\ \frac{C(s)}{1+C(s)G(s)} & \frac{-C(s)G(s)}{1+C(s)G(s)} \end{bmatrix} \begin{bmatrix} R(s) \\ D(s) \end{bmatrix}$$

BIBO Stable of CLS: The std. feedback control loop (CLS) is BIBO stable if all the Gang of 4 TFs are BIBO stable.
Thm: The CLS is BIBO stable iff 1. Poles of $\frac{1}{1+C(s)G(s)} \in \mathbb{C}^-$
2. $C(s)G(s)$ has no pole-zero cancel. in $\bar{\mathbb{C}}^+ = \{s \in \mathbb{C} : \text{Re}(s) \geq 0\}$.
Practical Considerations:
1. Don't cancel an unstable 0 of $G(s)$ w/ an unstable pole in $C(s)$.
2. Don't cancel an unstable pole of $G(s)$ w/ an unstable 0 in $C(s)$.
Asymp. Tracking of Poly. Suppose $d(t) = 0$ & want to track a <u>poly. ref. signal</u> of the form: $r(t) = (c_0 + c_1 t + \cdots + c_{k-1}t^{k-1})1(t)$, that is: $R(s) = \frac{N_R(s)}{s^k}$, with $N_R(0) \neq 0$ and $\deg(N_R(s)) \leq k-1$.
***GOAL:** Design $C(s)$ to achieve zero steady-state tracking error, i.e., $\lim_{t\to\infty} e(t) = 0$.
Proposition
Suppose $C(s)$ is designed so that:
1. $\frac{1}{1+C(s)G(s)}$ is BIBO stable; 2. $C(s)G(s) = \frac{C'(s)G'(s)}{s^k}$ with $C'(0)G'(0) \neq 0$.
Then $\frac{1}{s^k + C'(s)G'(s)}$ is BIBO stable.
Altogether, we see that if $C(s)$ is designed such that:
1. $C(s)$ satisfies our stability specifications;
2. $C(s)G(s)$ has k poles at $s = 0$,
then we achieve our asymptotic tracking specification of a polynomial with degree $k-1$.
Theorem
Suppose $C(s)$ satisfies:
1. **(CLS is BIBO stable)** poles $\left(\frac{1}{1+C(s)G(s)}\right) \subset \mathbb{C}^-$;
2. $C(s)G(s)$ has no pole-zero cancellations in $\mathbb{C}^+ := \{s \in \mathbb{C} : \text{Re}(s) \geq 0\}$ and $d(t) = 0$ for all $t \geq 0$.
For any polynomial reference signal:

$$r(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$$

the following hold:
a. If $C(s)G(s)$ has k or more poles at $s = 0$, then $\lim_{t\to\infty} e(t) = 0$.
b. If $C(s)G(s)$ has exactly $k-1$ poles at $s = 0$, then:

$$\lim_{t\to\infty} e(t) = \begin{cases} \frac{N_R(0)}{1+C'(0)G'(0)}, & \text{if } k = 1 \\ \frac{N_R(0)}{C'(0)G'(0)}, & \text{if } k \geq 2 \end{cases}$$

c. Otherwise, $C(s)G(s)$ has $k-2$ or fewer poles at $s = 0$ and $\lim_{t\to\infty} |e(t)| = \infty$.
Definition
The transfer function $C(s)G(s)$ is of type k if it has k poles at $s = 0$.
Theorem (Internal Model Principle)
Suppose $R(s)$ and $D(s)$ are strictly proper rational functions with poles in $\mathbb{C}^+ = \{s \in \mathbb{C} : \text{Re}(s) \geq 0\}$. A controller solves the Basic Control Problem if and only if:
1) $C(s)$ makes the CLS BIBO stable;
2) $C(s)G(s)$ has the poles of $R(s)$ with at least the same multiplicities;
3) $C(s)$ has the poles of $D(s)$ with at least the same multiplicities.
Corollary
If $G(s)$ has zeros that are also poles of $R(s)$ or $D(s)$, then the Basic Control Problem is unsolvable.
Important: The IMP states that if $G(s)$ does not contain the poles of $R(s)$ and $D(s)$, then $C(s)$ must contain these poles. Since these poles enable $C(s)$ to reproduce $r(t)$ and $d(t)$, we say $C(s)$ must contain an <u>internal model</u> of $r(t)$ and $d(t)$.
Proposition
Suppose $G(s)$ is BIBO stable. Let $Y(s) = G(s)U(s)$, where $Y(s) = \mathcal{L}\{y(t)\}$ and $U(s) = \mathcal{L}\{u(t)\}$.
If $\lim_{t\to\infty} u(t) = 0$, then $\lim_{t\to\infty} y(t) = 0$.
General Controller Design Procedure

Given $R(s) = \mathcal{L}\{r(t)\}$ and $D(s) = \mathcal{L}\{d(t)\}$:
1. **Feasibility:** Verify no zero of $G(s)$ is an unstable pole of $R(s)$ or $D(s)$.
2. **Internal Model:** Let $p_1, \ldots, p_k$ denote the unstable poles of $R(s)$ or $D(s)$ not in $G(s)$, accounting for multiplicities. Construct:

$$C(s) = C'(s) \cdot \frac{1}{(s - p_1) \ldots (s - p_k)}$$

3. **Stability:** Design $C'(s)$ so that the CLS is BIBO stable.
4. **Performance:** Tune controller parameters to achieve the desired performance specifications.