

Modelling CS u : control input, y : plant output
State variable CS is in state variable form if
 $\dot{x}_1 = f_1(t, x_1, \dots, x_n, u), \dots, \dot{x}_n = f_n(t, x_1, \dots, x_n, u)$
 $y = h(t, x_1, \dots, x_n, u)$ is a collection of n 1st order ODEs.
Time-Invariant (TI) CS is TI if $f_i(\cdot)$ does not depend on t .
State space (SS) TI CS is in SS form if $\dot{x} = f(x, u), y = h(x, u)$ where $x(t) \in \mathbb{R}^n$ is called the state.
Single-input-single-output (SISO) CS is SISO if $u(t), y(t) \in \mathbb{R}$.
LTI CS in SS form is LTI if $\dot{x} = Ax + Bu, y = Cx + Du$
 $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}$
 where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$.
Input-Output (IO) LTI CS is in IO form if
 $\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + \dots + b_1 \frac{du}{dt} + b_0 u$
 where $m \leq n$ (causality)

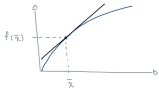
IO to SS Model 1. Define x s.t. highest order derivative in \dot{x}
 2. Write $\dot{x} = Ax + Bu = f(x, u)$ by isolating for components of \dot{x}
 3. Write $y = Cx + Du = h(x, u)$ by setting measurement output y to component of x
Equilibria y_d (steady state) b/c if $y(0) = y_d$ at $t = 0$, then $y(t) = y_d \forall t \geq 0$.

Equilibrium pair Consider the system $\dot{x} = f(x, u)$. The pair (\bar{x}, \bar{u}) is an equilibrium pair if $f(\bar{x}, \bar{u}) = 0$.
Equilibrium point \bar{x} is an equilibrium point w/ control $u = \bar{u}$.
 *If $u = \bar{u}$ and $x(0) = \bar{x}$ then $x(t) = \bar{x} \forall t \geq 0$ (i.e. a system that starts at equilibrium remains at equilibrium).
Find Equilibrium Pair/Point 1. Set $f(x, u) = 0$
 2. Solve $f(x, u) = 0$ to find $(x, u) = (\bar{x}, \bar{u})$.
 3. If specific $u = \bar{u}$, then find $x = \bar{x}$ by solving $f(x, \bar{u}) = 0$.

Linearization of Nonlinear System Consider system $\dot{x} = f(x, u)$ w/ equ. pair (\bar{x}, \bar{u}) , then error coordinates around equ. pair $\delta x = x - \bar{x}, \delta u = u - \bar{u}, \delta y = y - h(\bar{x}, \bar{u})$ $\delta \dot{x} = \dot{x} - f(\bar{x}, \bar{u})$ w/
 $\delta \dot{x} = A \delta x + B \delta u, A = \frac{\partial f(\bar{x}, \bar{u})}{\partial x} \in \mathbb{R}^{n_1 \times n_1}, B = \frac{\partial f(\bar{x}, \bar{u})}{\partial u} \in \mathbb{R}^{n_1},$
 $\delta y = C \delta x + D \delta u, C = \frac{\partial h}{\partial x}(\bar{x}, \bar{u}) \in \mathbb{R}^{1 \times n_1}, D = \frac{\partial h(\bar{x}, \bar{u})}{\partial u} \in \mathbb{R}$
 *Only valid at equ. pairs.



Linear Approx. Given a diff. fcn. $f : \mathbb{R} \rightarrow \mathbb{R}$, its linear approx. at \bar{x} is $f_{lin} = f(\bar{x}) + f'(\bar{x})(x - \bar{x})$.
 *Remainder Thm: $f(x) = f_{lin} + r(x)$ where $\lim_{x \rightarrow \bar{x}} \frac{r(x)}{x - \bar{x}} = 0$.

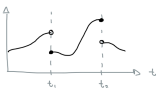


*Note: Can provide a good approx. near \bar{x} but not globally.

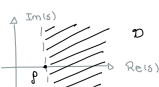
*Gen. $f : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}, f(x) = f(\bar{x}) + \frac{\partial f}{\partial x}(\bar{x})(x - \bar{x}) + R(x)$
 *Jacobian: $\frac{\partial f}{\partial x}(\bar{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\bar{x}) & \dots & \frac{\partial f}{\partial x_{n_1}}(\bar{x}) \end{bmatrix} \in \mathbb{R}^{n_2 \times n_1}$

Linearization Steps 1. Find equ. pair (\bar{x}, \bar{u})
 2. Derive A, B, C, D and then evaluate at (\bar{x}, \bar{u})
 3. Write $\delta \dot{x} = A \delta x + B \delta u$ and $\delta y = C \delta x + D \delta u$

Laplace Transform Given a fcn $f : \mathbb{R}_+ = [0, \infty) \rightarrow \mathbb{R}^n$, its Laplace transform is $F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st} dt, s \in \mathbb{C}$.
 * $\mathcal{L} : f(t) \mapsto F(s), t \in \mathbb{R}_+$ (time dom.) & $s \in \mathbb{C}$ (Laplace dom.).
P.W. CTS: A fcn $f : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is **p.w. cts** if on every finite interval of \mathbb{R} , $f(t)$ has at most a finite # of discontinuity points (t_i) and the limits $\lim_{t \rightarrow t_i^+} f(t), \lim_{t \rightarrow t_i^-} f(t)$ are finite.



Exp. Order A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is of **exp. order** if \exists constants $K, \rho, T > 0$ s.t. $\|f(t)\| \leq K e^{\rho t}, \forall t \geq T$.
Existence of LT Thm If $f(t)$ is p.w. cts and of exp. order w/ constants $K, \rho, T > 0$, then $F(\cdot)$ exists and is defined $\forall s \in D := \{s \in \mathbb{C} : \text{Re}(s) > \rho\}$ and $F(\cdot)$ is analytic on D .
 *Analytic fcn iff differentiable fcn.
 * D : Region of convergence (ROC), open half plane.



Unit Step $1(t) := \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Table of Common Laplace Transforms: $f(t) \mapsto F(s)$
 $1(t) \mapsto \frac{1}{s} \quad t1(t) \mapsto \frac{1}{s^2} \quad t^k 1(t) \mapsto \frac{k!}{s^{k+1}} \quad e^{at} 1(t) \mapsto \frac{1}{s-a}$
 $t^n e^{at} 1(t) \mapsto \frac{n!}{(s-a)^{n+1}} \quad \sin(at) 1(t) \mapsto \frac{a}{s^2+a^2}$
 $\cos(at) 1(t) \mapsto \frac{s}{s^2+a^2} \quad \frac{1}{2\omega^3} [\sin(\omega t) - \omega t \cos(\omega t)] 1(t) \mapsto \frac{1}{(s^2+\omega^2)^2}$

Prop. of Laplace Transform Linearity:
 $\mathcal{L}\{cf(t) + g(t)\} = c\mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}, c \sim \text{constant}$.
Differentiation: If the Laplace transform of $f'(t)$ exists, then $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^-)$.
 If the Laplace transform of $f^{(n)}(t) := \frac{d^n f}{dt^n}(t)$ exists, then $\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - \sum_{i=1}^n s^{n-i} f^{(i-1)}(0^-)$.
Integration: $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} \mathcal{L}\{f(t)\}$.
Convolution: Let $(f * g)(t) := \int_0^t f(\tau)g(t - \tau) d\tau = \int_0^t f(t - \tau)g(\tau) d\tau$, then $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$.
Time Delay: $\mathcal{L}\{f(t - T)1(t - T)\} = e^{-Ts} \mathcal{L}\{f(t)\}, T \geq 0$.
Multiplication by t: $\mathcal{L}\{tf(t)\} = -\frac{d}{ds} [\mathcal{L}\{f(t)\}]$.
Shift in s: $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}|_{s \rightarrow s-a} = F(s - a)$, where $F(s) = \mathcal{L}\{f(t)\}$ & a const.

Trig. Id. $2 \sin(2t) = 2 \sin(t) \cos(t), \sin(a - b) = \sin(a) \cos(b) - \cos(a) \sin(b), \cos(a - b) = \cos(a) \cos(b) + \sin(a) \sin(b)$
Complete the Square: $ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c$
LT Steps: 1. Write $f(t)$ as a sum and use linearity
 *Trig. id. may be useful.
 2. Use prop. of LT and common LT to find $F(s)$
Inverse Laplace Transform Given $F(s)$, its inverse LT is $f(t) = \mathcal{L}^{-1}\{F(s)\} := \frac{1}{2\pi} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds$

$= \lim_{w \rightarrow \infty} \frac{1}{2\pi} \int_{c-jw}^{c+jw} F(s) e^{st} ds, c \in \mathbb{C}$ is selected s.t. the line $L := \{s \in \mathbb{C} : s = c + j\omega, \omega \in \mathbb{R}\}$ is inside the ROC of $F(s)$.
Zero: $z \in \mathbb{C}$ is a zero of $F(s)$ if $F(z) = 0$.
Pole: $p \in \mathbb{C}$ is a pole of $F(s)$ if $\frac{1}{F(p)} = 0$.
Cauchy's Residue THM If $F(s)$ is analytic (complex diff.) everywhere except at isolated poles $\{p_1, \dots, p_N\}$, then $\mathcal{L}^{-1}\{F(s)\} = \sum_{i=1}^N \text{Res}\left[F(s)e^{st}, s = p_i\right] \mathbf{1}(t)$,
 $\text{Res}[F(s)e^{st}, s = p_i]$: Residue of $F(s)e^{st}$ at $s = p_i$.
Residue Computation Let $G(s)$ be a complex analytic fcn w/ a pole at $s = p$, r be the multiplicity of the pole p . Then $\text{Res}[G(s), s = p] = \frac{1}{(r-1)!} \lim_{s \rightarrow p} \frac{d^{r-1}}{ds^{r-1}} [G(s)(s-p)^r]$.
Inv. LT Partial Frac.: 1. Factorize $F(s)$ into partial fractions. 2. Find coefficients and use LT table to find inverse LT.
 *Complete the square.
Inv. LT Residues: 1. Find poles of $F(s)$ and their residues. 2. Use Cauchy's Residue THM to find inverse LT.
 *Note: Complex Conjugate (CC) poles \rightarrow CC residues (use Euler).
Transfer Function: Consider a CS in IO form. Assume zero initial conds. $y(0) = \dots = \frac{d^{(n-1)}}{dt^{(n-1)}} y(0) = 0$ and

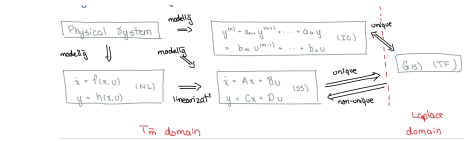
$u(0) = \dots = \frac{d^{(m-1)}}{dt^{(m-1)}} u(0) = 0$. Then the TF from u to y is $G(s) := \frac{y(s)}{U(s)} = \frac{b_m s^m + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0}$.
***0 Ini. Conds.:** $y_0(s) = G(s)u(s)$
*** \emptyset Ini. Conds.:** $y_\emptyset(s) = y_0(s) + \frac{\text{poly. based on initial conds.}}{s^n + a_{n-1} s^{n-1} + \dots + a_0}$

TF Steps (IO to TF): 1. Given IO form of CS, assume zero initial conds. 2. Find $G(s)$ by taking LT of IO form and forming $Y(s)/U(s)$.
 *Careful: $Y(s)/U(s) = G(s)$ not $U(s)/Y(s) = G(s)$.
Impulse Response: Given CS modeled by TF $G(s)$, its IR is $g(t) := \mathcal{L}^{-1}\{G(s)\}$.
 $\mathcal{L}\{\delta(t)\} = 1$, then if $u(t) = \delta(t)$, then $Y(s) = U(s)G(s) = G(s)$.
SS to TF: $G(s) = C(sI - A)^{-1}B + D$ s.t. $y(s) = G(s)U(s)$.
 *Assume $x(0) = 0 \in \mathbb{R}^n$ (zero initial conds).
***LTI:** $G(s)$ of an LTI system is always a rational fcn.
***Not Invertible:** Values of s s.t. $sI - A$ not invertible can correspond to poles of $G(s)$.
Inverse: 1. For $A \in \mathbb{R}^{n \times n}$, find $[\text{cof}(A)]_{(i,j)} = (-1)^{i+j} \det(A_{(i,j)})$.
 $A_{(i,j)}$: A w/ row i and col. j removed.
 2. Assemble $\text{cof}(A)$ and find $\det(A) = \sum_{j=1}^n a_{ij} [\text{cof}(A)]_{(i,j)}$ w/ fixed i or $\det(A) = \sum_{i=1}^n a_{ij} [\text{cof}(A)]_{(i,j)}$ w/ fixed j .
 3. Find $A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{\det(A)} [\text{cof}(A)]^T$.
 2×2 : $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
TF (SS to TF): 1. Given SS form, assume zero initial conds. 2. Solve $G(s) = C(sI - A)^{-1}B + D$.
 *If $C = [0 \cdot 1_i \cdot 0]$ & $B = [0 \cdot 1_j \cdot 0]$, then only need i th row & j th col. of $\text{adj}(sI - A)$ s.t. $G(s) = \frac{[\text{adj}(sI - A)]_{(i,j)}}{\det(sI - A)} + D$.
 *Multiple i, j non-zero entries: Work it out using MM.
TF to SS: Consider $G(s) = \frac{b_m s^m + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0} = \frac{N(s)}{D(s)}$, where $m < n$ (i.e. $G(s)$ is strictly proper). Then the SS form is

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C = [b_0 \quad \dots \quad b_m \quad | \quad 0 \quad \dots \quad 0], D = 0.$$

***Unique:** State space of a TF is not unique.



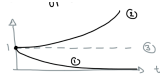
Block Diagram Types of Blocks:
Cascade: $y_2 = (G_1(s)G_2(s))U \stackrel{\text{SISO}}{\cong} y_2 = (G_2(s)G_1(s))U$
 $U \rightarrow \boxed{G_1} \xrightarrow{y_1} \boxed{G_2} \rightarrow y_2 \equiv U \rightarrow \boxed{G_1, G_2} \rightarrow y_2$

Parallel $y = (G_1(s) + G_2(s))U$
 $U \xrightarrow{\begin{matrix} \rightarrow \boxed{G_1} \\ \rightarrow \boxed{G_2} \end{matrix}} y \equiv U \rightarrow \boxed{G_1 + G_2} \rightarrow y$

Feedback $y = \left(\frac{G_1(s)}{1 + G_1(s)G_2(s)} \right) R$
 $R \xrightarrow{\begin{matrix} \rightarrow \boxed{G_1} \\ \rightarrow \boxed{G_2} \end{matrix}} y \equiv R \rightarrow \boxed{\frac{G_1}{1 + G_1 G_2}} \rightarrow y$

***SC:** Unity Feedback Loop (UFL) if $G_2(s) = 1$.
Manipulations: 1. $y = G(U_1 - U_2) = GU_1 + GU_2$
 2. $y_1 = GU \quad y_2 = U \mid y_1 = GU \quad y_2 = G \frac{1}{G} U$
 3. From feedback loop to UFL.
 $\textcircled{a} \quad U \xrightarrow{\begin{matrix} \rightarrow \boxed{G} \\ \rightarrow \boxed{G} \end{matrix}} y \equiv U \xrightarrow{\boxed{G}} \xrightarrow{\begin{matrix} \rightarrow \boxed{G} \\ \rightarrow \boxed{G} \end{matrix}} y$
 $\textcircled{b} \quad U \xrightarrow{\boxed{G}} y_1 \xrightarrow{\boxed{G}} y_2 \equiv U \xrightarrow{\boxed{G}} \xrightarrow{\boxed{G}} y_1 \xrightarrow{\boxed{G}} y_2$
 $\textcircled{c} \quad R \xrightarrow{\begin{matrix} \rightarrow \boxed{G_1} \\ \rightarrow \boxed{G_2} \end{matrix}} y \equiv R \xrightarrow{\boxed{\frac{1}{G_2}}} \xrightarrow{\begin{matrix} \rightarrow \boxed{G_1} \\ \rightarrow \boxed{G_2} \end{matrix}} y$

Find TF from Block Diagram: 1. Start from in \rightarrow out, making simplifications using block diagram rules. 2. Simplify until you get the form $U(s) \rightarrow \boxed{G(s)} \rightarrow Y(s)$.
Time Response of Elementary Terms: $\mathbf{1}(t) \leftarrow$ pole @ 0
 $t^n \mathbf{1}(t) \leftarrow$ pole @ 0 w/ mult. $n \mid e^{at} \mathbf{1}(t) \leftarrow$ pole @ a
 $\sin(\omega t + \phi) \mathbf{1}(t) \leftarrow$ pole @ $\pm j\omega \mid \cos(\omega t + \phi) \mathbf{1}(t) \leftarrow$ pole @ $\pm j\omega$
Real Pole: $y(s) = \frac{1}{s+a}$, real pole at $s = -a$, then $y(t) = e^{-at} \mathbf{1}(t)$
 1. $a > 0 \implies \lim_{t \rightarrow \infty} y(t) = 0 \mid 2. a < 0 \implies \lim_{t \rightarrow \infty} y(t) = \infty$.
 3. $a = 0 \implies y(t) = \mathbf{1}(t)$ is constant.



Time Constant: $\tau = \frac{1}{a}$ of the pole $s = -a$ for $a > 0$

Pair of Comp. Conj. Poles:

$$y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2}, |\zeta| < 1, \text{ then}$$

$$y(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\sigma t} \sin(\omega_d t) \mathbf{1}(t)$$

*Poles: $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} = -\sigma \pm j\omega_d$

* $\zeta = \frac{\sigma}{\omega_n}$: Damping ratio (or damping coefficient)

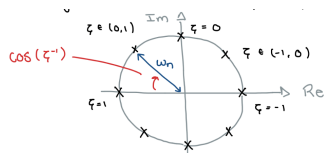
* $\sigma = \zeta\omega_n$: Decay/growth rate | ω_d : Freq. of oscillation

* $\omega_n = \sqrt{\sigma^2 + \omega_d^2} \left[\frac{\text{radians}}{\text{seconds}} \right]$: Undamped natural freq.

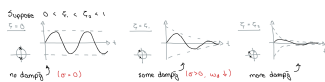
* $\omega_d = \omega_n\sqrt{1 - \zeta^2} \left[\frac{\text{radians}}{\text{seconds}} \right]$: Damped natural freq.

* $|s_{1,2}|^2 = \omega_n^2$: Mag. of poles is ω_n .

* $\cos^{-1}(\zeta)$: Angle of s_1 on complex plane CW from -ve Re axis.



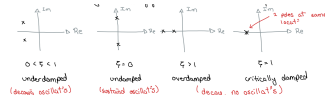
Damping Ratio Effect: $0 < \zeta_1 < \zeta_2 < 1$, then



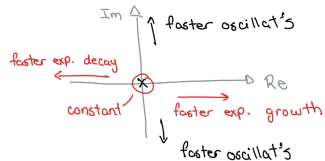
$-1 < \zeta_4 < \zeta_3 < 0$, then $\sigma = \zeta\omega_n < 0$, (exp. envelop \uparrow)



Class. of 2nd Order Sys.: $y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, w/ $|\zeta| < 1$



Loc. of Poles and Behavior:

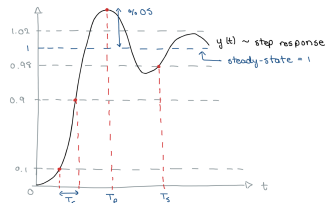


Control Spec. of 2nd Order Sys.: Step Response: Given a TF $G(s)$, its SR is $y(t)$ resulting from applying the input $u(t) = \mathbf{1}(t)$, i.e. $\mathcal{L}^{-1} \left\{ G(s) \frac{1}{s} \right\}$.

Control Spec. A control spec. is a criterion specifying how we would like a CS to behave.

2nd Order Sys. Metrics: $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ w/ $U(s) = \frac{1}{s}$

* $0 < \zeta < 1$ (i.e. 2 comp. conj. poles w/ $\text{Re}(\text{pole}) < 0$).



Rise Time (RT): T_r is the time it takes $y(t)$ to go from 10% to 90% of its steady-state value.

RT: 1. Find $t_1 > 0$ s.t. $y(t_1) = 0.1$, $t_2 > 0$ s.t. $y(t_2) = 0.9$.

3. Compute $T_r = t_2 - t_1$.

$$T_r \approx \frac{1.8}{\omega_n}$$

Settling Time (ST): T_s is the time required to reach and stay w/in 2% of the steady-state value.

ST: 1. Find when it's first that $|y(t) - 1| \leq 0.02$.

$$T_s \approx \frac{4}{\zeta\omega_n}$$

Peak Time: T_p is time req'd to reach the max (peak) value.

Peak Time: 1. Find the first time when $\dot{y}(t) = 0$.

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

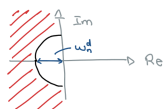
% Overshoot: $\%OS = \frac{[\text{peak value}] - [\text{steady-state value}]}{[\text{steady-state value}]} \times 100\%$

* $\%OS = OS \times 100\%$.

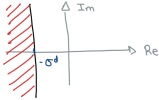
$$\exp\left(-\frac{\pi\zeta}{\sqrt{1 - \zeta^2}}\right) \iff \zeta = \frac{-\ln(OS)}{\sqrt{\pi^2 + (\ln(OS))^2}}$$

Transient Performance Satisfaction: Admissible region for the poles of $G(s)$ s.t. the step response meets all three spec.

Rise Time: $T_r \approx \frac{1.8}{\omega_n} \leq T_r^d \text{ approx.}$ $\omega_n \geq \frac{1.8}{T_r^d} \equiv \omega_n^d$



Settling Time: $T_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} \leq T_s^d \text{ approx.}$ $\sigma \geq \frac{4}{T_s^d} \equiv \sigma^d$



OS: $OS = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \leq OS^d \overset{\text{approx.}}{\iff}$

$$\zeta \geq \frac{-\ln(OS^d)}{\sqrt{\pi^2 + (\ln(OS^d))^2}} \equiv \zeta^d$$

