

ECE368 Cheatsheet

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Contents

1	L1: Probability Review	2
1.1	Sample Space	2
1.2	Probability Definitions	2
1.3	Axioms of Probability	3
1.4	Conditional Probability	3
1.4.1	Consequences of Conditional Probability	3
1.4.2	Independence	3
1.4.3	Importance of Labelling	4
2	L2: Probability Review	5
2.1	Total Probability	5
2.2	Bayes' Rule	5
2.2.1	Posteriori Probability, Priori Probability (Prior), Likelihood	5
2.2.2	Interpretation of Bayes' Rule	5
2.3	Random Variables	6
2.4	Distribution of RV	6
2.4.1	Cumulative Distribution Function (CDF) of RV	6
2.4.2	Discrete RV Probability Mass Function (PMF)	6
2.4.3	Continuous RV Probability Density Function (PDF)	6
2.4.4	Conditional PMF/PDF	7
2.5	Expected Values	7
3	L3: Probability Review	9
3.1	2 RVs	9
3.2	Joint PMF/PDF	9
3.3	Expectations	9
3.3.1	Correlation	9
3.3.2	Covariance	10
3.3.3	Correlation Coefficient	10
3.4	Marginal PMF/PDF	10
3.5	Conditional PMF/PDF	10
3.6	Bayes' Rule	11
3.7	Independent vs. Uncorrelated vs. Orthogonal	11
3.8	Conditional Expectation	11
4	L4: Estimation of Sample Mean	15
4.1	Parameter Estimation:	15
4.2	Estimator:	15
4.2.1	Estimation Error:	15
4.2.2	Unbiased	15
4.2.3	Consistent	15
4.3	Sample Mean & Law of Large Numbers	16
4.3.1	Digression for Sum of RVs (not necessarily independent or identically distributed)	16
4.3.2	Unbiased (i.i.d.)	16
4.3.3	Consistent (i.i.d.)	17

4.3.4	Weak Law of Large Numbers	17
4.3.5	Confidence Interval: Finding n	18
5	L5: Sample Mean and Maximum Likelihood Estimation	19
5.1	Maximum Likelihood Estimation	19
5.1.1	Log-Likelihood	19
6	L6: Maximum Likelihood and Laplace	22
6.1	MLE for Categorical Random Variables	22
6.2	MLE for Gaussian Random Variables	23
6.3	Will the Sun Rise Tomorrow? (Laplace's Problem)	23
6.3.1	Frequentist Approach	24
6.3.2	Bayesian Approach	24
6.4	Sample Mean is Not Always an ML Estimator	25
7	L7: Maximum A Posteriori (MAP) Estimation	26
8	L8: MAP Conjugate Prior	27
9	L9: Least Mean Squares (LMS) Estimation	28

Summary: On second thoughts, the lecture notes he posts are good, so I think I'll just do the cheatsheet.

W1 (LG-IPPR 1.1, 1.2; Murphy 2.1 – 2.3)

1 L1: Probability Review

Summary:

FAQ:

- How to study? Practice, practice.
- What textbooks? Use 2024 version of Murphy, Leon Garcia as main reference, Bishop, 4th textbook is intro.
- How is HW graded? Effort, and tutorials are used to explain soln.

1.1 Sample Space

Motivation: If you have 4 sheep and a flea, the probability that starting from sheep 1, the flea will jump to sheep 4 in 10 steps is 0.2.

- Ambiguous as there are 2 different interpretations for the sample space (i.e. space of probability is not clear):
 - Set of sheep
 - Set of number of steps

1.2 Probability Definitions

Definition:

- **Random Experiment:** An outcome (realization) for each run.
- **Sample Space Ω :** Set of all possible outcomes.
- **Events:** (measurable) subsets of Ω .
- **Probability of Event A :** $P[A] \equiv P[\text{'outcome is in } A\text{'}]$.

Example: Roll Fair Die

- $\Omega = \{1, 2, 3, 4, 5, 6\}$.

- $P[\text{'even number'}] = \frac{1}{2}$.

1.3 Axioms of Probability

Definition:

1. $P[A] \geq 0$ for all $A \in \Omega$.
2. $P[\Omega] = 1$.
3. If $A \cap B = \emptyset$, then $P[A \cup B] = P[A] + P[B]$ for all $A, B \in \Omega$.



Figure 1: 3rd Axiom

1.4 Conditional Probability

Definition:

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \quad (1)$$

- $|$: Given event (data/obs.).

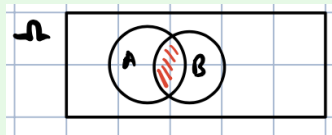


Figure 2: Conditional Probability

Notes:

- Changing sample space to B .
- Conditional probability satisfy the 3 axioms (i.e. are probabilities), can be viewed as probability measure on new sample space B .

1.4.1 Consequences of Conditional Probability

Definition:

$$P[A \cap B] = P[A|B]P[B] = P[B|A]P[A] \quad (2)$$

1.4.2 Independence

Definition: A and B are independent iff

$$P[A \cap B] = P[A]P[B] \iff P[A|B] = P[A] \iff P[B|A] = P[B] \quad (3)$$

1.4.3 Importance of Labelling

Example: Toss 2 Fair Coins

1. **Given:** Given that one of the coins is heads, what is the probability that the other coin is tails?
2. **Wrong Solution:** $\frac{1}{2}$ since $\{HH, HT, TH, TT\}$, so $P[T|H] = \frac{1}{2}$, which assumes that the coins are distinguishable (i.e. coin #1 is heads)
3. **Correct Solution:** $\frac{2}{3}$ since $\{HH, HT, TH\}$ as we didn't specify which coin was heads, so $P[T|H] = \frac{2}{3}$, which assumes that the coins are indistinguishable.

2 L2: Probability Review

2.1 Total Probability

Definition: If H_1, \dots, H_n form a partition of Ω , then

$$P[A] = \sum_{i=1}^n P[A|H_i]P[H_i] \quad (4)$$

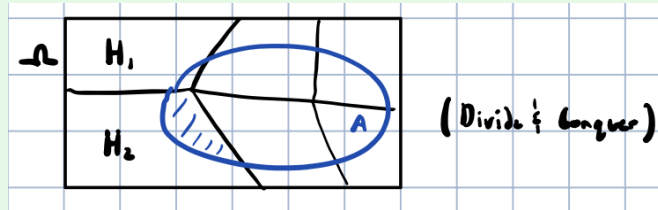


Figure 3: Total Probability

2.2 Bayes' Rule

Definition:

$$P[H_k|A] = \frac{P[H_k \cap A]}{P[A]} = \frac{P[A|H_k]P[H_k]}{\sum_{i=1}^n P[A|H_i]P[H_i]} \quad (5)$$

2.2.1 Posteriori Probability, Priori Probability (Prior), Likelihood

Definition:

- **Posteriori:** $P[H_k|A]$.
- **Priori:** $P[H_k]$.
- **Likelihood:** $P[A|H_k]$.

Example: Suppose a lie detector is 95% accurate, i.e. $P[\text{'out=truth'}|\text{'in=truth'}] = 0.95$ and $P[\text{'out=lie'}|\text{'in=lie'}] = 0.95$. It says that Mr. Ernst is lying. What is the probability Mr. Ernst is actually lying.

- **Observation:** $A = \text{'out=lie'}$.
- **Hypothesis:** $H_0 = \text{'in=lie'}$ and $H_1 = \text{'in=truth'}$.
- **Solution:**
$$P[H_0|A] = \frac{P[A|H_0]P[H_0]}{P[A|H_0]P[H_0] + P[A|H_1]P[H_1]} = \frac{0.95 \times P[H_0]}{0.95 \times P[H_0] + 0.05 \times (1 - P[H_0])}.$$
- $H_0 = 0.01$: i.e. 1% of the population are liars, then
$$P[H_0|A] = \frac{0.95 \times 0.01}{0.95 \times 0.01 + 0.05 \times 0.99} = 0.16.$$

Warning: Need to know priori probability.

2.2.2 Interpretation of Bayes' Rule

Notes: Taking one component of the total probability and normalizing it by the sum of all components.

2.3 Random Variables

Motivation: Coin Toss Mapping of each outcome to a real number

- $w \in \Omega$ is the outcome of a coin toss, and X is the RV, so $H \rightarrow 0$ and $T \rightarrow 1$.



Figure 4: Random Variables

- Mapping is deterministic function. RV is not random or variable.

Definition: Mapping from Ω to \mathbb{R} .

2.4 Distribution of RV

2.4.1 Cumulative Distribution Function (CDF) of RV

Definition:

$$F_X(x) \equiv P[X \leq x] \quad (6)$$

2.4.2 Discrete RV Probability Mass Function (PMF)

Definition:

$$P_X(x_j) \equiv P[X = x_j] \quad j = 1, 2, 3, \dots \quad (7)$$

Example: Binomial RV w/ (n, p)

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (8)$$

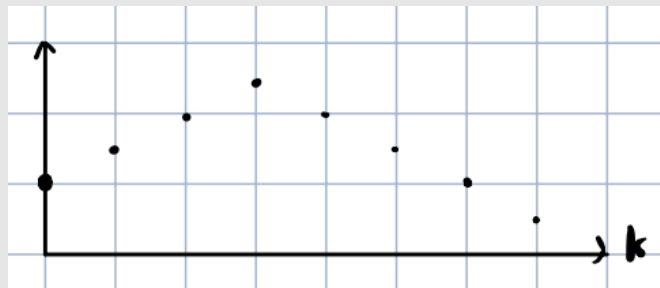


Figure 5: Binomial RV

2.4.3 Continuous RV Probability Density Function (PDF)

Definition:

$$f_X(x) \equiv \frac{d}{dx} F_X(x) \quad (9)$$

$$P[x < X < x + dx] = f_X(x)dx \quad (10)$$

Example: Gaussian RV w/ (μ, σ^2)

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (11)$$

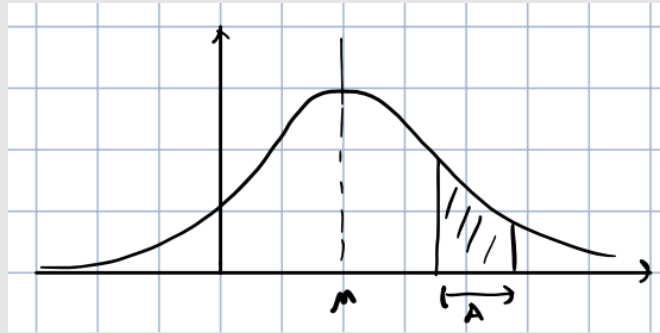


Figure 6: Gaussian RV

- $P[X \in A] = \int_A f_X(x)dx.$

Notes: Discrete RV has pdf w/ δ functions.

2.4.4 Conditional PMF/PDF

Definition:

$$P_X(x|A) \quad (12)$$

$$f_X(x|A) \quad (13)$$

Example: Continuous

$$f(x|X > a) = \begin{cases} \frac{f_X(x)}{P[X > a]} & \text{if } x > a \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

Example: Geometric RV Geometric RV X w/ success probability p

$$P_X(k) = (1-p)^{k-1}p \quad (15)$$

- **Memoryless Property:** $P_X[k|X > m] = \frac{p(1-p)^{k-1}}{(1-p)^m} = p(1-p)^{k-m-1}.$
 - So it only cares about the additional trials (i.e. same as resetting after m trials).

2.5 Expected Values

Definition:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx \stackrel{\text{If int. values}}{=} \sum_{k=-\infty}^{\infty} k f_X(k) \quad (16)$$

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x) dx \stackrel{\text{If int. values}}{=} \sum_{k=-\infty}^{\infty} h(k) f_X(k) \quad (17)$$

$$\text{Var}[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2 \quad (18)$$

$$E[X|A] = \int_{-\infty}^{\infty} x f_X(x|A) dx \quad (19)$$

Example: Lottery Ticket (Geometric RV)

1. **Given:** Buying one lottery ticket per week
 - Each ticket has $10^{-7} = p$ chance of winning the jackpot.
 - X = '# of weeks to win jackpot'.
2. **Problem:** What is the expected number of weeks to win the jackpot?
3. **Solution:** $E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = \dots = \frac{1}{p} = 10^7$ weeks.
4. **Extension (Memoryless Property):** If I have already played for 999999 weeks, what is the expected number of weeks to win the jackpot? $E[X - 999999 | X > 999999] = E[X] = 10^7$ weeks.

3 L3: Probability Review

3.1 2 RVs

Notes: RVs are neither random nor a variable.

$$\underline{Z} = (X, Y)$$

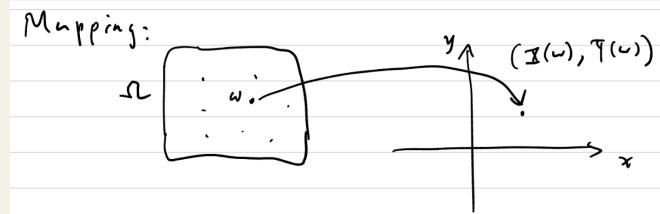


Figure 7: Mapping of RVs

3.2 Joint PMF/PDF

Definition:

$$P_{X,Y}(x, y) = P[X = x, Y = y] \quad (20)$$

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) \quad (21)$$

$$P[(X, Y) \in A] = \int \int_{(x,y) \in A} f_{X,Y}(x, y) dx dy \quad (22)$$

Example: Jointly Gaussian RVs X and Y with $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_1}{\sigma_1} \right)^2 - 2\rho \left(\frac{x-\mu_1}{\sigma_1} \right) \left(\frac{y-\mu_2}{\sigma_2} \right) + \left(\frac{y-\mu_2}{\sigma_2} \right)^2 \right] \right\}$$

3.3 Expectations

Definition:

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy$$

Notes:

- $g(X, Y)$ is also an RV, but inside the integral or sum, you use x and y as dummy variables to vary through the values of the RVs.

3.3.1 Correlation

Definition:

$$E[XY] \quad (23)$$

3.3.2 Covariance

Definition:

$$\text{Cov}[X, Y] = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - \mu_X \mu_Y = E[XY] - E[X]E[Y] \quad (24)$$

Notes:

- Mean shifted to 0.

3.3.3 Correlation Coefficient

Definition:

$$\rho_{X,Y} = E \left[\left(\frac{X - \mu_X}{\sigma_X} \right) \left(\frac{Y - \mu_Y}{\sigma_Y} \right) \right] = \frac{\text{Cov}[X, Y]}{\sigma_X \sigma_Y} \quad (25)$$

- $|\rho_{X,Y}| \leq 1$

Notes:

- Mean shifted to 0 and normalized by the standard deviation.

3.4 Marginal PMF/PDF

Definition:

$$P_X(x) = \sum_{j=1}^{\infty} P_{X,Y}(x, y_j), \quad P_Y(y) = \sum_{j=1}^{\infty} P_{X,Y}(x_j, y) \quad (26)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx \quad (27)$$

Notes:

- Total probability theorem is being used here.

Example: Jointly Gaussian X and Y :

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy \\ &= \dots \quad (\text{completing the square}) \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}, \quad \text{marginally Gaussian} \end{aligned}$$

- Gaussian RVs has a property that the PDF of a single variable is equal to the marginal Gaussian of two variables.

3.5 Conditional PMF/PDF

Definition:

$$P_{X|Y}(x|y) \triangleq P[X = x|Y = y] = \frac{P_{X,Y}(x, y)}{P_Y(y)} \quad (28)$$

$$f_{X|Y}(x|y) \triangleq \frac{f_{X,Y}(x, y)}{f_Y(y)} \quad (29)$$

3.6 Bayes' Rule

Definition:

$$P_{Y|X}(x|y) = \frac{P_{X,Y}(x,y)}{P_X(x)} = \frac{P_{X|Y}(x|y)P_Y(y)}{\sum_{j=1}^{\infty} P_{X,Y}(x,y_j)P_Y(y_j)} \quad (30)$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_{X|Y}(x|y)f_Y(y)}{\int_{-\infty}^{\infty} f_{X|Y}(x|y')f_Y(y') dy'} \quad (31)$$

3.7 Independent vs. Uncorrelated vs. Orthogonal

Definition:

1. Independent:

$$f_{X|Y}(x|y) = f_X(x) \quad \forall y \Leftrightarrow f_{X,Y}(x,y) = f_X(x)f_Y(y) \quad (32)$$

2. Uncorrelated:

$$\text{Cov}[X,Y] = 0 \quad \Leftrightarrow \quad \rho_{X,Y} = 0 \quad (33)$$

3. Orthogonal:

$$E[XY] = 0 \quad (34)$$

Theorem: If independent, then uncorrelated.

Derivation:

$$\begin{aligned} \text{Independent} &\implies E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x,y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_X(x) f_Y(y) dx dy \\ &= \left(\int_{-\infty}^{\infty} x f_X(x) dx \right) \left(\int_{-\infty}^{\infty} y f_Y(y) dy \right) \\ &\implies E[XY] = E[X]E[Y] \\ &\implies \text{Cov}[X,Y] = 0, \quad \text{uncorrelated} \\ &\neq \text{in general.} \end{aligned}$$

Example: Jointly Gaussian RVs X and Y : If uncorrelated, i.e. $\rho_{X,Y} = 0$, then X and Y are independent.

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2} \exp \left\{ -\frac{1}{2} \left[\left(\frac{x-\mu_1}{\sigma_1} \right)^2 + \left(\frac{y-\mu_2}{\sigma_2} \right)^2 \right] \right\} \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}} \\ &= f_X(x)f_Y(y) \quad \text{independent} \end{aligned}$$

3.8 Conditional Expectation

Definition:

$$E[Y] = E[E[Y|X]] \quad (35)$$

$$E[h(Y)] = E[E[h(Y)|X]] \quad (36)$$

Notes:

- $E[E[Y|X]]$ is w.r.t. X .
- $E[Y|X]$ is w.r.t. Y .

Derivation:

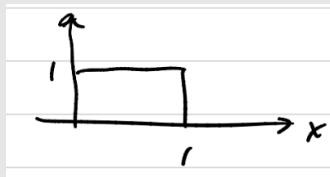
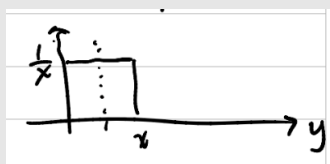
$$\begin{aligned}
 E[Y] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{X,Y}(x,y) dx dy \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{Y|X}(y|x) f_X(x) dx dy \\
 &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy \right) f_X(x) dx \\
 &= \int_{-\infty}^{\infty} E[Y|X=x] f_X(x) dx \quad (\text{using the total probability theorem}) \\
 &= \int_{-\infty}^{\infty} g(x) f_X(x) dx \\
 &= E[g(X)] \\
 &= E[E[Y|X]].
 \end{aligned}$$

Example:

1. **Given:** An unknown voltage. $X \sim \text{Uniform}(0, 1)$. Measurement from a (bad) voltmeter: $Y \sim \text{Uniform}(0, X)$.

$$\begin{aligned}
 f_X(x) &= \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \\
 f_{Y|X}(y|x) &= \begin{cases} \frac{1}{x}, & 0 < y < x \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

- **Note:** Area under PDF is 1.

Figure 8: Uniform Distribution of X Figure 9: Uniform Distribution of Y

2. Expected Value (Average Reading of Bad Voltmeter):

$$\begin{aligned}
 E[Y] &= E[E[Y|X]] \\
 &= E\left[\frac{X}{2}\right] \quad \text{Since in the middle of 0 and x} \\
 &= \frac{1}{2} \cdot E[X] \\
 &= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \quad \text{Since } E[X] \text{ (i.e. mean) is 0.5}
 \end{aligned}$$

3. The Long Way:

$$\begin{aligned}
 f_Y(y) &= \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) dx \\
 &= \int_y^1 f_{Y|X}(y|x) f_X(x) dx \\
 &= \int_y^1 \frac{1}{x} \cdot 1 dx \\
 &= -\ln y. \\
 E[Y] &= \int_0^1 y \cdot (-\ln y) dy = \dots = \frac{1}{4}
 \end{aligned}$$

4. **Question:** Suppose $Y = \frac{1}{8}$. What is "best" given X ? This will be the question for the rest of the course.

4 L4: Estimation of Sample Mean

4.1 Parameter Estimation:

Motivation: The readout of a sensor is $X = \theta + N$ volts

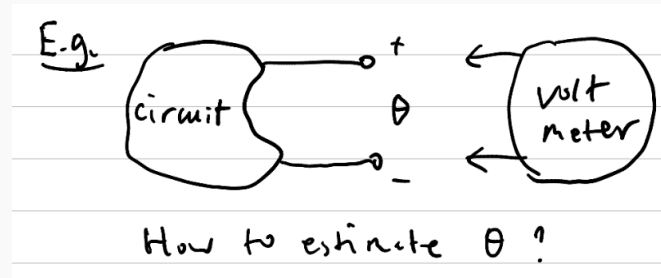


Figure 10:

- There is some noise N in the sensor, so we want to estimate the true value of θ (unknown parameter to be estimated)
 - e.g. Mean and/or variance of X .

4.2 Estimator:

Definition: Perform n independent and identically distributed (i.i.d.) measurements/observations of X : X_1, X_2, \dots, X_n .

$$\hat{\Theta} = \hat{\Theta}(\underline{X}) = g(X_1, X_2, \dots, X_n) \quad (37)$$

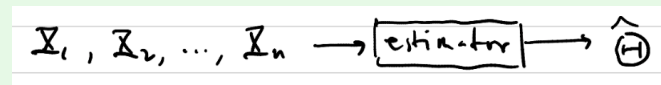


Figure 11:

4.2.1 Estimation Error:

Definition:

$$\hat{\Theta}(\underline{X}) - \theta \quad (38)$$

4.2.2 Unbiased

Definition: The estimator $\hat{\Theta}$ is unbiased if

$$\mathbb{E}[\hat{\Theta}(\underline{X})] = \theta \quad (39)$$

- **Asymptotically Unbiased:** $\lim_{n \rightarrow \infty} \mathbb{E}[\hat{\Theta}(\underline{X})] = \theta$ (big data)

4.2.3 Consistent

Definition: The estimator $\hat{\Theta}$ is consistent if $\hat{\Theta}(\underline{X}) \rightarrow \theta$ as $n \rightarrow \infty$, in probability, i.e., $\forall \epsilon > 0$,

$$\lim_{n \rightarrow \infty} P(|\hat{\Theta}(\underline{X}) - \theta| < \epsilon) \rightarrow 1 \quad (40)$$

as $n \rightarrow \infty$.

4.3 Sample Mean & Law of Large Numbers

Definition: Given a sequence of i.i.d. random variables (RVs), X_1, X_2, \dots, X_n , w/ unknown mean μ , estimate μ . Let $S_n = X_1 + X_2 + \dots + X_n$. The *sample mean* is

$$M_n = \frac{1}{n} S_n$$

- How good is M_n as an estimator of μ ?
 - Use unbiased and consistent to evaluate M_n .

Example: Previous voltage measurement, e.g.,

$$X_i = \mu + N_i$$

where μ is the true value and N_i is the noise.

If we assume N_i are i.i.d. with zero mean,

$$E[X_i] = E[\mu + N_i] = E[\mu] + E[N_i] = \mu + 0 = \mu, \quad \forall i$$

4.3.1 Digression for Sum of RVs (not necessarily independent or identically distributed)

Derivation:

$$\begin{aligned} E[S_n] &= E[X_1 + \dots + X_n] \\ &= E[X_1] + \dots + E[X_n] \end{aligned}$$

Derivation:

$$\begin{aligned} \text{Var}[S_n] &= E[(S_n - E[S_n])^2] \\ &= E\left[\left(\sum_{i=1}^n X_i - E[X_i]\right)^2\right] \\ &= E\left[\sum_{i=1}^n \sum_{j=1}^n (X_i - E[X_i])(X_j - E[X_j])\right] \\ &= \sum_{i=1}^n \sum_{j=1}^n E[(X_i - E[X_i])(X_j - E[X_j])] \\ &= \sum_{i=1}^n \sum_{j=1}^n \text{Cov}[X_i, X_j] \\ &= \sum_{i=1}^n \text{Var}[X_i] + \sum_{i \neq j} \sum_{j=1}^n \text{Cov}[X_i, X_j] \end{aligned}$$

4.3.2 Unbiased (i.i.d.)

Derivation:

$$\begin{aligned}
E[M_n] &= E\left[\frac{1}{n}S_n\right] \\
&= \frac{1}{n} (E[X_1] + \dots + E[X_n]) \\
&= \frac{1}{n}(n\mu) \quad \text{since } X_i \text{ are i.i.d. so same expectation} \\
&= \mu \Rightarrow \text{Unbiased!}
\end{aligned}$$

4.3.3 Consistent (i.i.d.)**Derivation:**

$$\begin{aligned}
\text{Var}[M_n] &= \text{Var}\left[\frac{1}{n}S_n\right] \\
&= \frac{1}{n^2} \text{Var}[S_n] \quad \text{taking out constant requires squaring} \\
&= \frac{1}{n^2} \left(\sum_{i=1}^n \text{Var}[X_i] + \sum_{i \neq j} \text{Cov}[X_i, X_j] \right) \\
&= \frac{1}{n^2} (n\sigma^2) \quad \sigma^2 \triangleq \text{Var}[X_i] \text{ and } X_i \text{ are i.i.d. so covariance is 0} \\
&= \frac{\sigma^2}{n} \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

- This means that there is no variance in the sample mean as n approaches infinity, so it converges to the true mean.

Recall the Chebyshev Inequality:

$$P[|X - E[X]| \geq \epsilon] \leq \frac{\text{Var}[X]}{\epsilon^2}, \quad \forall \epsilon > 0.$$

Substitute in M_n :

$$\begin{aligned}
P[|M_n - E[M_n]| \geq \epsilon] &\leq \frac{\text{Var}[M_n]}{\epsilon^2} \\
P[|M_n - \mu| \geq \epsilon] &\leq \frac{\sigma^2}{n\epsilon^2} \\
\Rightarrow P[|M_n - \mu| < \epsilon] &\geq 1 - \frac{\sigma^2}{n\epsilon^2} \rightarrow 1 \text{ as } n \rightarrow \infty \text{ then it is consistent}
\end{aligned}$$

Warning: Cov = 0 because independence implies uncorrelated.

4.3.4 Weak Law of Large Numbers

Definition: Even if σ is infinite, then $\forall \epsilon > 0$,

$$\lim_{n \rightarrow \infty} P[|M_n - \mu| < \epsilon] = 1$$

4.3.5 Confidence Interval: Finding n

Example: Measure an unknown voltage θ for n times and obtain independent measurements:

$$X_i = \theta + N_i,$$

where N_i are i.i.d. random variables with mean 0 and variance 1.

- We want to determine how many measurements n are sufficient so that

$$P(|M_n - \theta| < 0.1) \geq 0.95,$$

where 0.1 is the desired precision and 0.95 is the confidence level.

- The sample mean is given by:

$$M_n = \frac{1}{n} \sum_{i=1}^n X_i = \theta + \frac{1}{n} \sum_{i=1}^n N_i.$$

- The variance of X_i is:

$$\sigma^2 = \text{Var}[X_i] = \text{Var}[\theta + N_i] = \text{Var}[N_i] = 1.$$

– $\text{Var}[aX + b] = a^2 \text{Var}[X]$, where $a = 1$ and $b = \theta$.

- Using Chebyshev's inequality:

$$1 - \frac{\sigma^2}{n\epsilon^2} \geq 0.95,$$

where $\epsilon = 0.1$ (precision).

- Solving for n :

$$\begin{aligned} 1 - \frac{1}{n(0.1)^2} &\geq 0.95, \\ \frac{1}{n(0.1)^2} &\leq 0.05, \\ n &\geq 2000. \end{aligned}$$

Thus, at least 2000 measurements are needed to achieve the desired precision and confidence level.

5 L5: Sample Mean and Maximum Likelihood Estimation

FAQ:

- Why can we say that it is consistent for the last example?

5.1 Maximum Likelihood Estimation

Motivation: Choose parameter θ that is most likely to generate the observation x_1, x_2, \dots, x_n .

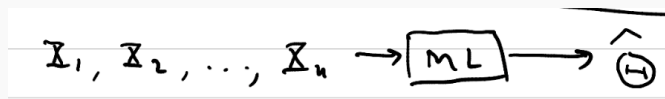


Figure 12:

Definition:

$$\hat{\Theta} = \arg \max_{\theta} P_{\underline{X}}(\underline{x}|\theta), \text{ discrete } X. \quad (41)$$

$$\hat{\Theta} = \arg \max_{\theta} f_{\underline{X}}(\underline{x}|\theta), \text{ continuous } X. \quad (42)$$

5.1.1 Log-Likelihood

Definition:

$$\hat{\theta} = \arg \max_{\theta} \sum_{i=1}^n \log P_X(x_i|\theta) \quad (43)$$

$$\hat{\theta} = \arg \max_{\theta} \sum_{i=1}^n \log f_X(x_i|\theta). \quad (44)$$

Warning: Can only go from argmax of fcn to argmax of log fcn, if it is i.i.d.

Derivation:

$$\begin{aligned}
 \text{i.i.d. } X_1, X_2, \dots, X_n &\implies \\
 p_{\underline{X}}(\underline{x}|\theta) &= \prod_{i=1}^n p_{X_i}(x_i|\theta) \\
 &= \prod_{i=1}^n p_X(x_i|\theta) \quad \text{drop the i due to i.i.d. assumption} \\
 \log p_{\underline{X}}(\underline{x}|\theta) &= \sum_{i=1}^n \log p_X(x_i|\theta).
 \end{aligned}$$

Example:

1. Model and Observations:

- Assume a biased coin with probability θ of showing heads. Find ML estimator for θ .
- Toss the coin n times and obtain Bernoulli random variables X_1, \dots, X_n such that:

$$\text{"heads"} \rightarrow 1, \quad \text{"tails"} \rightarrow 0.$$

- Total number of heads is:

$$k = \sum_{i=1}^n X_i.$$

For example:

$$\underline{x} = (1, 0, 0, 0, 1, 1, 0, 1, 1, 1), \quad k = 6.$$

- Probability of observations x_1, \dots, x_n corresponding to parameter θ is:

$$p_{\underline{X}}(\underline{x}|\theta) = \theta^k (1 - \theta)^{n-k}.$$

- It is sufficient to know only k .
- Note: Don't need the $\binom{n}{k}$ term because we are given the specific sequence of heads and tails.

2. Log-Likelihood and Maximization:

- The log-likelihood function is:

$$\log p_{\underline{X}}(\underline{x}|\theta) = k \log(\theta) + (n - k) \log(1 - \theta).$$

- To maximize the log-likelihood over θ , set:

$$\begin{aligned} 0 &= \frac{\partial}{\partial \theta} \log p_{\underline{X}}(\underline{x}|\theta), \\ 0 &= \frac{k}{\theta} - \frac{n - k}{1 - \theta}, \\ \theta &= \frac{k}{n}. \end{aligned}$$

- Thus, the Maximum Likelihood Estimator (MLE) is:

$$\hat{\Theta} = \frac{k}{n}, \quad \text{where } k = \sum_{i=1}^n X_i.$$

This corresponds to the observed frequency of heads, which is intuitive b/c the more heads we see, the more likely the coin is biased towards heads.

3. Examples:

- For $\underline{x} = (1, 0, 0, 0, 1, 1, 0, 1, 1, 1)$:

$$\begin{aligned} p_X(\underline{x}|\theta) &= \theta^6 (1 - \theta)^4, \\ \hat{\theta} &= \frac{6}{10} = 0.6. \end{aligned}$$

- For $\underline{x} = (0, 1, 1, 1, 0, 0, 1, 0, 1, 0)$:

$$\begin{aligned} p_X(\underline{x}|\theta) &= \theta^5 (1 - \theta)^5, \\ \hat{\theta} &= \frac{5}{10} = 0.5. \end{aligned}$$

Notes:

1. k is a sufficient statistic for this Maximum Likelihood (ML) estimator.

2. The expectation of the estimator $\hat{\theta}$ is:

$$\begin{aligned} E[\hat{\Theta}] &= E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] \\ &= \frac{1}{n} \sum_{i=1}^n E[X_i] \\ &= \frac{1}{n}(n\theta) \\ &= \theta \quad (\text{Unbiased}). \end{aligned}$$

$$- E[X_i] = (1)\theta + (0)(1 - \theta) = \theta$$

3. In fact, $\hat{\Theta} = \frac{1}{n} \sum_{i=1}^n X_i$ is the sample mean, and θ is the true mean. Therefore, $\hat{\theta} \rightarrow \theta$ in probability, which implies that $\hat{\theta}$ is *consistent*.

6 L6: Maximum Likelihood and Laplace

6.1 MLE for Categorical Random Variables

Example:

1. We say that $X \sim \text{Cat}(\underline{\theta})$ if

$$P[X = m] = \theta_m, \quad m = 1, 2, \dots, M.$$

- Going from 2 to M categories is a generalization of the Bernoulli distribution.

The parameter $\underline{\theta}$ is a vector:

$$\underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_M \end{bmatrix},$$

such that $\theta_m \geq 0$ and $\sum_{m=1}^M \theta_m = 1$.

2. Given n i.i.d. observations X_1, \dots, X_n , we aim to find the maximum likelihood estimator (MLE) of $\underline{\theta}$.
3. Define n_m as the number of observations that equal m :

$$n_m = \sum_{i=1}^n 1(x_i = m),$$

where $1(x_i = m)$ is the indicator function. Note that $\sum_{m=1}^M n_m = n$.

4. The likelihood function is:

$$p_{\underline{X}}(\underline{x} | \underline{\theta}) = \prod_{m=1}^M \theta_m^{n_m}.$$

- Similar to the Bernoulli distribution, but with M categories.

Taking the log, we get:

$$\log p_{\underline{X}}(\underline{x} | \underline{\theta}) = \sum_{m=1}^M n_m \log \theta_m.$$

5. To find the optimal $\underline{\theta}$, we minimize the negative log-likelihood:

$$\min_{\underline{\theta}} - \sum_{m=1}^M n_m \log \theta_m,$$

subject to the constraints $\theta_m \geq 0$ for $1 \leq m \leq M$ and $\sum_{m=1}^M \theta_m = 1$.

6. Solving this optimization problem, the MLE is:

$$\hat{\theta}_m = \frac{N_m}{n} = \frac{\sum_{i=1}^n 1(X_i = m)}{n}, \quad \hat{\underline{\theta}} = \begin{bmatrix} \frac{N_1}{n} \\ \vdots \\ \frac{N_M}{n} \end{bmatrix}.$$

6.2 MLE for Gaussian Random Variables

Example:

1. Given n i.i.d. observations X_1, \dots, X_n of a Gaussian random variable with parameters (μ, σ^2) , we aim to find the maximum likelihood estimators (MLEs) of μ and σ^2 .

$$f_{\underline{X}}(\underline{x}|\mu, \sigma^2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right),$$

$$\log f_{\underline{X}}(\underline{x}|\mu, \sigma^2) = \sum_{i=1}^n \left(-\frac{(x_i - \mu)^2}{2\sigma^2} - \log \sigma - \log \sqrt{2\pi}\right).$$

2. To find μ , take the derivative of the log-likelihood with respect to μ and set it to zero:

$$0 = \frac{\partial}{\partial \mu} \sum_{i=1}^n \left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right),$$

$$0 = \frac{1}{n} \sum_{i=1}^n x_i - \mu,$$

$$\mu = \frac{1}{n} \sum_{i=1}^n x_i.$$

3. To find σ^2 , take the derivative of the log-likelihood with respect to σ^2 and set it to zero:

$$0 = \frac{\partial}{\partial \sigma^2} \sum_{i=1}^n \left(-\frac{(x_i - \mu)^2}{2\sigma^2} - \frac{1}{2} \log \sigma^2\right),$$

$$0 = -\frac{1}{2} \sum_{i=1}^n \left(\frac{(x_i - \mu)^2}{\sigma^4}\right) + \frac{1}{2\sigma^2},$$

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2.$$

4. Thus, the MLEs are:

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n X_i, \quad (\text{sample mean})$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \hat{\mu})^2. \quad (\text{sample variance})$$

- Note: The sample variance is biased, so we often use $\frac{1}{n-1}$ instead of $\frac{1}{n}$ to make it unbiased.
- Note: The sample mean is unbiased.

6.3 Will the Sun Rise Tomorrow? (Laplace's Problem)

Example:

- Observation: The Sun has risen for n consecutive days. Estimate the probability that it will rise tomorrow.
- Model: Assume n i.i.d. Bernoulli random variables X_1, \dots, X_n with $P[X_i = 1] = \theta$.

6.3.1 Frequentist Approach

Example:

1. The Maximum Likelihood Estimator (MLE) is:

$$\hat{\theta} = \frac{K}{n} = \frac{\sum_{i=1}^n X_i}{n}.$$

2. If $K = n$ (i.e., the Sun has risen every day so far), then:

$$\hat{\theta} = \frac{n}{n} = 1.$$

3. Conclusion: The Sun will rise tomorrow with probability 1, regardless of what n is, based on the Frequentist approach.
 - This doesn't make sense b/c if $n = 1$ then we are assuming 100% it will rise based on one observation.

6.3.2 Bayesian Approach

Example:

1. Assume that θ is not fixed but drawn from a uniform distribution in $[0, 1]$. This means that the probability of the Sun rising is based on a uniform distribution.
2. We want to find the probability that the sun will rise tomorrow given that it has risen for n consecutive days:

$$P[X_{n+1} = 1 | X_1 = 1, \dots, X_n = 1].$$

Using Bayes' Theorem:

$$P[X_{n+1} = 1 | X_1 = 1, \dots, X_n = 1] = \frac{P[X_1 = 1, \dots, X_{n+1} = 1]}{P[X_1 = 1, \dots, X_n = 1]}.$$

3. Compute $P[X_1 = 1, \dots, X_n = 1]$:

$$P[X_1 = 1, \dots, X_n = 1] = \int_0^1 P[X_1 = 1, \dots, X_n = 1 | \Theta = \theta] f_{\Theta}(\theta) d\theta.$$

- The joint probability is calculated by integrating the product of the likelihood and the prior (i.e. marginalizing over θ).
- The likelihood becomes θ^n because the observations are i.i.d.
- The prior is uniform, so $f_{\Theta}(\theta) = 1$.

Since $f_{\Theta}(\theta) = 1$ (uniform prior) and $P[X_1 = 1, \dots, X_n = 1 | \Theta = \theta] = \theta^n$, we have:

$$P[X_1 = 1, \dots, X_n = 1] = \int_0^1 \theta^n d\theta = \frac{1}{n+1}.$$

4. Compute $P[X_1 = 1, \dots, X_{n+1} = 1]$ similarly:

$$P[X_1 = 1, \dots, X_{n+1} = 1] = \int_0^1 \theta^{n+1} d\theta = \frac{1}{n+2}.$$

5. Combine results:

$$P[X_{n+1} = 1 | X_1 = 1, \dots, X_n = 1] = \frac{\frac{1}{n+2}}{\frac{1}{n+1}} = \frac{n+1}{n+2}.$$

6. Conclusion: As n increases, the probability approaches 1, providing more certainty with more data.

6.4 Sample Mean is Not Always an ML Estimator

Example: Given an unknown voltage x , we measure it using a voltmeter that outputs a random reading Y that is uniform in $[0, x]$. Suppose we make n i.i.d. measurements Y_1, \dots, Y_n and wish to estimate $\mu = \frac{x}{2} = \mathbb{E}[Y]$.

1. **Sample Mean:**

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n Y_i \quad \dots (1)$$

2. **To Find the ML Estimator:**

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y} \mid \mu) &= \prod_{i=1}^n f_Y(y_i \mid \mu) \\ &= \prod_{i=1}^n \frac{1}{2\mu} \cdot 1(0 \leq y_i \leq 2\mu) \\ &= \frac{1}{(2\mu)^n} \prod_{i=1}^n 1(0 \leq y_i \leq 2\mu). \end{aligned}$$

The likelihood is maximized for:

$$\arg \max_{\mu} f_{\mathbf{Y}}(\mathbf{y} \mid \mu) = \max_{1 \leq i \leq n} \frac{1}{2} Y_i.$$

Therefore:

$$\hat{\mu} = \max_{1 \leq i \leq n} \frac{1}{2} Y_i \quad \dots (2)$$

3. Clearly, (1) \neq (2).

7 L7: Maximum A Posteriori (MAP) Estimation

8 L8: MAP Conjugate Prior

9 L9: Least Mean Squares (LMS) Estimation

1. Assume prior:

$$p_{\Theta}(\theta) \text{ or } f_{\Theta}(\theta) \quad (\text{Bayesian})$$

Observations: $\bar{X} = x$.

2. LMS Estimator:

$$\begin{aligned} \hat{\theta} &= g(x) = \mathbb{E}[\Theta \mid \bar{X} = x] \\ \text{or } \hat{\Theta} &= g(\bar{X}) = \mathbb{E}[\Theta \mid \bar{X}]. \end{aligned}$$

Note:

- **MAP:** Use the most probable θ given x .
- **LMS:** Use the expected value (conditional on $\bar{X} = x$) of Θ , i.e., the "Conditional Expectation Estimator."

3. Unbiasedness of LMS Estimator:

$$\begin{aligned} \mathbb{E}[\hat{\Theta}] &= \mathbb{E}[\mathbb{E}[\Theta \mid \bar{X}]] = \mathbb{E}[\Theta], \\ \implies \mathbb{E}[\hat{\Theta} - \Theta] &= 0. \end{aligned}$$

4. LMS Estimator Minimizes Conditional MSE:

$$\mathbb{E}[(\Theta - \hat{\Theta})^2 \mid \bar{X} = x].$$

Proof:

(a) First, suppose no observations: $\hat{\Theta}$ is a constant.

$$\begin{aligned} \hat{\Theta} &= \arg \min_c \mathbb{E}[(\Theta - c)^2], \\ 0 &= \frac{d}{dc} [-2\mathbb{E}[\Theta] + 2c], \\ c &= \mathbb{E}[\Theta]. \end{aligned}$$

(b) Alternate view:

$$\mathbb{E}[(\Theta - c)^2] = \text{Var}[\Theta] + (\mathbb{E}[\Theta] - c)^2.$$

To minimize: Set bias $\mathbb{E}[\Theta] - c$ to zero.

(c) Now, with observations $\bar{X} = x$:

$$\mathbb{E}[(\Theta - g(x))^2 \mid \bar{X} = x] = \text{Var}[\Theta \mid \bar{X} = x] + (\mathbb{E}[\Theta \mid \bar{X} = x] - g(x))^2.$$

To minimize: Set $g(x) = \mathbb{E}[\Theta \mid \bar{X} = x]$.

5. Conclusion:

$$\hat{\Theta} = g(x) = \mathbb{E}[\Theta \mid \bar{X} = x].$$

1. Example: Prior Coin Toss Problem

$$\begin{aligned} \hat{\theta}_{\text{LMS}} &= \mathbb{E}[\Theta \mid X = k] \\ &= \frac{k + \alpha}{n + \alpha + \beta}. \end{aligned}$$

2. Example: Prior Voltage Problem

(a) **Setup:**

- Unknown voltage Θ .
- Prior: $\Theta \sim \text{Uniform}[0, 1]$.

- Volt meter reading Y given Θ : $Y \sim \text{Uniform}[0, \Theta]$.
- Independent measurements: Y_1, \dots, Y_n given Θ .

(b) **Likelihood:**

$$\begin{aligned} f_{Y|\Theta}(\mathbf{y} | \theta) &= \prod_{i=1}^n f_Y(y_i | \theta) \\ &= \frac{1}{\theta^n} \cdot 1(\theta \geq \max_i y_i). \end{aligned}$$

(c) **Posterior:**

$$f_{\Theta|Y}(\theta | \mathbf{y}) = \frac{\frac{1}{\theta^n} \cdot 1(\theta \geq \max_i y_i)}{f_Y(\mathbf{y})}.$$

(d) **Estimators:**

- Maximum Likelihood (ML):

$$\hat{\theta} = \max_{1 \leq i \leq n} y_i.$$

- LMS:

$$\hat{\theta} = \mathbb{E}[\Theta | Y = y] = \int_0^\infty \theta f_{\Theta|Y}(\theta | y) d\theta.$$

(e) **Derivation for LMS:**

- Compute $f_Y(y)$ for $n = 1$:

$$\begin{aligned} f_Y(y) &= \int_y^1 \frac{1}{\theta} d\theta \\ &= \ln(\theta) \Big|_y^1 \\ &= -\ln(y). \end{aligned}$$

- Compute $\hat{\theta}$ for $n = 1$:

$$\begin{aligned} \hat{\theta} &= \int_y^1 \frac{\theta \cdot 1(\theta \geq y)}{-\ln(y)} d\theta \\ &= \frac{1}{-\ln(y)} \int_y^1 \theta d\theta \\ &= \frac{1}{-\ln(y)} \cdot \frac{y^2 - 1}{2}. \end{aligned}$$

(f) **Graphical Interpretation:**

- $f_{\Theta|Y}(\theta | y) = \frac{1}{-\ln(y)} \cdot 1(\theta \geq y) \cdot 1(0 \leq \theta \leq 1)$.
- The MAP estimator corresponds to the most probable θ .
- The LMS estimator minimizes the mean squared error, representing the "safest" choice.