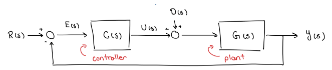
Standard Feedback Control Loop



$R(s)$: Ref., $E(s) = R(s) - y(s)$: Err., $C(s)$: Controller, $U(s)$: Control input, $D(s)$: Dist., $G(s)$: Plant, $y(s)$: Plant output.
***Assume:** $R(s)$ and $D(s)$ are strictly proper rational fcns w/ a fixed set of poles but arbitrary zeros & gain.
*$\mathcal{R}, \mathcal{D}$: Classes of ref. and dist. satisfying the above assumption.
Basic Control Prob.: Design $C(s)$ s.t. 3 spec. are met:
1. **Stability:** $\forall$ bdd $r(t), d(t)$, we have $u(t), e(t)$ bdd.
2. **Asymptotic Tracking:** When $d(t) = 0\ \forall t \geq 0$, then $\forall r(t) \in \mathcal{R}$, $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} r(t) - y(t) = 0$.
3. **Disturbance Rejection:** When $r(t) = 0\ \forall t \geq 0$, then $\forall d(t) \in \mathcal{D}$, $\lim_{t\to\infty} y(t) = 0$.
Open-Loop Control: 1. Design $u(t)$ s.t. $y(t)$ tracks ref. $y_r \in \mathbb{R}$, i.e. $\lim_{t\to\infty} y(t) = y_r$.
2. Set $u(t) = \gamma y_r \mathbf{1}(t)$ w/ $\gamma \in \mathbb{R}$ (const. scaling factor)
3. Apply FVT to find γ s.t. $\lim_{t\to\infty} y(t) = y_r$. 4. Determine $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} y_r - y(t)$
Limitations: 1. Req. perfect knowledge of plant paramters.
2. Not robust against parameter var./(unknown) dist.
3. Does not allow us to speed up convergence.
Feedback Control: 1. Design $u(t)$ s.t. $y(t)$ tracks ref. $y_r \in \mathbb{R}$, i.e. $\lim_{t\to\infty} y(t) = y_r$.
2. Set $u(t) = Ke(t) = K(y_r - y(t))$ w/ $K > 0$ (const. gain).
3. Use block mani. to find $y(s)$ in terms of input and $G(s)$.
4. Apply FVT to find K s.t. $\lim_{t\to\infty} y(t) = y_r$.
5. Determine $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} y_r - y(t)$
Advantages: 1. Doesn't req. perfect knowledge of plant param.
2. Robust against param. var./dist. by $\uparrow K$.
3. Allows us to speed up the rate of convergence by $\uparrow K$.
Disadvantages: 1. Feedback can introduce instability.
2. High-gain amplifies noise.
3. Asymptotic tracking doesn't occur.
Integral Control: 1. Design $u(t)$ s.t. $y(t)$ tracks ref. $y_r \in \mathbb{R}$, i.e. $\lim_{t\to\infty} y(t) = y_r$.
2. Set $u(t) = \mathcal{L}^{-1}\{C(s)E(s)\} = Ke(t) + KT_I \int_0^t e(\tau)d\tau$ (prop. int. (PI) controller) w/ $K, T_I > 0$ (const. gains).
*$C(s) = K\left(1 + \frac{T_I}{s}\right)$
3. Use block mani. to find $y(s)$ in terms of input and $G(s)$.
4. Apply FVT to find $\lim_{t\to\infty} y(t) = y_r$ as desired.
BIBO Stability of Closed-Loop System: Gang of 4 TF:

$$\begin{bmatrix} E(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{1+C(s)G(s)} & \frac{-G(s)}{1+C(s)G(s)} \\ \frac{C(s)}{1+C(s)G(s)} & \frac{-C(s)G(s)}{1+C(s)G(s)} \end{bmatrix} \begin{bmatrix} R(s) \\ D(s) \end{bmatrix}$$

BIBO Stable of CLS: The std. feedback control loop (CLS) is BIBO stable if all the Gang of 4 TFs are BIBO stable.
CLS is BIBO Stable THM: The CLS is BIBO stable iff
1. Poles of $\frac{1}{1+C(s)G(s)} \subseteq \mathbb{C}^-$
2. $C(s)G(s)$ has no pole-zero cancel. in $\bar{\mathbb{C}}^+ = \{s \in \mathbb{C} : \mathrm{Re}(s) \geq 0\}$.
Practical Considerations:
1. Don't cancel an unstable 0 of $G(s)$ w/ an unstable pole in $C(s)$.
2. Don't cancel an unstable pole of $G(s)$ w/ an unstable 0 in $C(s)$.
Asymp. Tracking of Poly. Suppose $d(t) = 0$ & want to track a poly. ref. signal of the form: $r(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$, that is: $R(s) = \frac{N_R(s)}{s^k}$, w/ $N_R(0) \neq 0$ and $\deg(N_R(s)) \leq k - 1$.
***GOAL:** Design $C(s)$ to achieve $\lim_{t\to\infty} e(t) = 0$.
Prop: Suppose $C(s)$ is designed so that:
1. $\frac{1}{1+C(s)G(s)}$ is BIBO stable
2. $C(s)G(s) = \frac{C'(s)G'(s)}{s^k}$ with $C'(0)G'(0) \neq 0$.
Then $\frac{1}{s^k + C'(s)G'(s)}$ is BIBO stable.
Asymp. Tracking of Poly. Thm Suppose $C(s)$ satisfies CLS is BIBO stable THM and $d(t) = 0\ \forall t \geq 0$. For any poly. ref. signal $r(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$, the following hold:
a. If $C(s)G(s)$ has k or more poles at $s = 0$, then $\lim\limits_{t\to\infty} e(t) = 0$.
b. If $C(s)G(s)$ has $k - 1$ poles at $s = 0$, then:

$$\lim_{t\to\infty} e(t) = \begin{cases} \frac{N_R(0)}{1+C'(0)G'(0)}, & \text{if } k = 1 \\ \frac{N_R(0)}{C'(0)G'(0)}, & \text{if } k \geq 2 \end{cases}$$

c. If $C(s)G(s)$ has k-2 or fewer poles at $s = 0$, then $\lim\limits_{t\to\infty} |e(t)| = \infty$.
Type k: The TF $C(s)G(s)$ is of type k if it has k poles at $s = 0$.
Dist. Rejection: Suppose $r(t) = 0\ \forall t \geq 0$ and $d(t)$ is a poly. dist. signal of the form: $d(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$, that is: $D(s) = \frac{N_D(s)}{s^k}$, with $N_D(0) \neq 0$ and $\deg(N_D(s)) \leq k - 1$.
***GOAL:** Design $C(s)$ to achieve $\lim_{t\to\infty} e(t) = 0$.
Dist. Rejection Thm: Suppose $C(s)$ satisfies CLS is BIBO stable THM and $r(t) = 0\ \forall t \geq 0$. For any poly. dist. signal $d(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$, the following hold:
a. If $C(s)$ has k or more poles at $s = 0$, then $\lim_{t\to\infty} e(t) = 0$.
b. If $C(s)$ has $k - 1$ poles at $s = 0$, then $\lim_{t\to\infty} e(t) \neq 0$ exists.
c. If $C(s)$ has k-2 or fewer poles at $s = 0$, then $\lim_{t\to\infty} |e(t)| = \infty$.
General Thm (Internal Model Principle): Suppose $R(s)$ and $D(s)$ are strictly proper rational fns w/ poles in $\overline{\mathbb{C}^+}$. $C(s)$ solves the Basic Control Problem iff:
1) $C(s)$ makes the CLS BIBO stable;
2) $C(s)G(s)$ has the poles($R(s)$) w/ at least same multiplicities;
3) $C(s)$ has the poles($D(s)$) w/ at least same multiplicities.
Corollary: If $G(s)$ has zeros that are also poles of $R(s)$ or $D(s)$, then the Basic Control Problem is unsolvable.
Internal Model: The IMP states if $G(s)$ does not contain the poles of $R(s)$ and $D(s)$, then $C(s)$ must contain these poles. Since these poles enable $C(s)$ to reproduce $r(t)$ and $d(t)$, we say $C(s)$ must contain an internal model of $r(t)$ and $d(t)$.
Proposition: Suppose $G(s)$ is BIBO stable. Let $Y(s) = G(s)U(s)$, where $Y(s) = \mathcal{L}\{y(t)\}$ and $U(s) = \mathcal{L}\{u(t)\}$. If $\lim\limits_{t\to\infty} u(t) = 0$, then $\lim\limits_{t\to\infty} y(t) = 0$.
*Decaying input $\implies$ decaying output so don't worry in IMP.
General Controller Design Procedure: Given $R(s) = \mathcal{L}\{r(t)\}$ and $D(s) = \mathcal{L}\{d(t)\}$:
1. **Feasibility:** Verify no zero of $G(s)$ is an unstable pole of $R(s)$ or $D(s)$.
2. **Internal Model:** Let $p_1, \ldots, p_k$ denote the unstable poles of $R(s)$ or $D(s)$ not in $G(s)$, accounting for multiplicities. Construct:

$$C(s) = C'(s) \cdot \frac{1}{(s - p_1)\ldots(s - p_k)}$$

3. **Stability:** Design $C'(s)$ so that the CLS is BIBO stable.
4. **Performance:** Tune controller parameters to achieve the desired performance specifications.

Argument Principle Let $\mathcal{D}$ be a simple (no self-intersections) closed (no endpoints) path in $\mathcal{C}$ oriented CCW. Suppose $F(s)$ has no poles or zeros on $\mathcal{D}$ & isolated poles inside $\mathcal{D}$. Let $\gamma(\theta)$ be a parametrization of $\mathcal{D}$, i.e. $\mathcal{D} = \{\gamma(\theta) : \theta \in \mathbb{R}\}$ and $\mathcal{F} = \{F(\gamma(\theta)) : \theta \in \mathbb{R}\}$. Then $\mathcal{F}$ encircles the origin $n_e = n_z - n_p$ times CCW.
*n_z: # of zeros of $F(s)$ inside $\mathcal{D}$
*n_p: # of poles of $F(s)$ inside $\mathcal{D}$

Notes:
1. A -ve CCW encirlement is the same as a +ve CW encirlement.
2. If $\mathcal{D}$ is oriented CW, the Argument Principle still holds by replacing $CCW \rightarrow CW$ everywhere.

Nyquist Contour: The path $\mathcal{D}$ above w/ $R \rightarrow \infty$.
Application to Feedback Loops: To stabilize the CLS, it suffices to consider the FB loop where we require:
-**Zeros of** $1 + C(s)G(s) \subseteq \mathbb{C}^-$ (focus on this)
-$C(s)G(s)$ has no unstable pole-zero cancellations.

See if $\exists$ zeros in $\mathbb{C}^+$. So consider the contour:
$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 = \{j\omega : \omega \in [-R, R]\} \cup \{Re^{j\theta} : \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$
*If $R \rightarrow \infty$, then more and more of $\mathbb{C}^+$ is contained inside $\mathcal{D}$.
*By the **Argument Principle**, if we:
-Count the number of encirclements of $1 + C(s)G(s)$ (n_e).
-Know the number of unstable poles of $1 + C(s)G(s)$ (n_p).
*Then, we can compute the number of zeros in $\mathbb{C}^+$.