

**Modelling CS**  $u$ : control input,  $y$ : plant output  
**State variable** CS is in state variable form if  
 $\dot{x}_1 = f_1(t, x_1, \dots, x_n, u), \dots, \dot{x}_n = f_n(t, x_1, \dots, x_n, u)$   
 $y = h(t, x_1, \dots, x_n, u)$  is a collection of  $n$  1st order ODEs.  
**Time-Invariant (TI)** CS is TI if  $f_i(\cdot)$  does not depend on  $t$ .  
**State space (SS)** TI CS is in SS form if  $\dot{x} = f(x, u), y = h(x, u)$  where  $x(t) \in \mathbb{R}^n$  is called the state.  
**Single-input-single-output (SISO)** CS is SISO if  $u(t), y(t) \in \mathbb{R}$ .  
**LTI** CS in SS form is LTI if  $\dot{x} = Ax + Bu, y = Cx + Du$   
 $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}$   
 where  $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$ .  
**Input-Output (IO)** LTI CS is in IO form if  
 $\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + \dots + b_1 \frac{du}{dt} + b_0 u$   
 where  $m \leq n$  (causality)

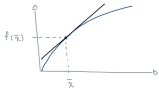
**IO to SS Model** 1. Define  $x$  s.t. highest order derivative in  $\dot{x}$   
 2. Write  $\dot{x} = Ax + Bu = f(x, u)$  by isolating for components of  $\dot{x}$   
 3. Write  $y = Cx + Du = h(x, u)$  by setting measurement output  $y$  to component of  $x$   
**Equilibria**  $y_d$  (steady state) b/c if  $y(0) = y_d$  at  $t = 0$ , then  $y(t) = y_d \forall t \geq 0$ .

**Equilibrium pair** Consider the system  $\dot{x} = f(x, u)$ . The pair  $(\bar{x}, \bar{u})$  is an equilibrium pair if  $f(\bar{x}, \bar{u}) = 0$ .  
**Equilibrium point**  $\bar{x}$  is an equilibrium point w/ control  $u = \bar{u}$ .  
 \*If  $u = \bar{u}$  and  $x(0) = \bar{x}$  then  $x(t) = \bar{x} \forall t \geq 0$  (i.e. a system that starts at equilibrium remains at equilibrium).  
**Find Equilibrium Pair/Point** 1. Set  $f(x, u) = 0$   
 2. Solve  $f(x, u) = 0$  to find  $(x, u) = (\bar{x}, \bar{u})$ .  
 3. If specific  $u = \bar{u}$ , then find  $x = \bar{x}$  by solving  $f(x, \bar{u}) = 0$ .

**Linearization of Nonlinear System** Consider system  $\dot{x} = f(x, u)$  w/ equ. pair  $(\bar{x}, \bar{u})$ , then error coordinates around equ. pair  $\delta x = x - \bar{x}, \delta u = u - \bar{u}, \delta y = y - h(\bar{x}, \bar{u})$   $\delta \dot{x} = \dot{x} - f(\bar{x}, \bar{u})$  w/  
 $\delta \dot{x} = A \delta x + B \delta u, A = \frac{\partial f(\bar{x}, \bar{u})}{\partial x} \in \mathbb{R}^{n_1 \times n_1}, B = \frac{\partial f(\bar{x}, \bar{u})}{\partial u} \in \mathbb{R}^{n_1},$   
 $\delta y = C \delta x + D \delta u, C = \frac{\partial h}{\partial x}(\bar{x}, \bar{u}) \in \mathbb{R}^{1 \times n_1}, D = \frac{\partial h(\bar{x}, \bar{u})}{\partial u} \in \mathbb{R}$   
 \*Only valid at equ. pairs.



**Linear Approx.** Given a diff. fcn.  $f : \mathbb{R} \rightarrow \mathbb{R}$ , its linear approx. at  $\bar{x}$  is  $f_{lin} = f(\bar{x}) + f'(\bar{x})(x - \bar{x})$ .  
 \*Remainder Thm:  $f(x) = f_{lin} + r(x)$  where  $\lim_{x \rightarrow \bar{x}} \frac{r(x)}{x - \bar{x}} = 0$ .

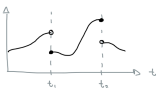


\*Note: Can provide a good approx. near  $\bar{x}$  but not globally.

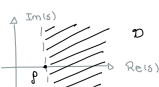
\*Gen.  $f : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}, f(x) = f(\bar{x}) + \frac{\partial f}{\partial x}(\bar{x})(x - \bar{x}) + R(x)$   
 \*Jacobian:  $\frac{\partial f}{\partial x}(\bar{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\bar{x}) & \dots & \frac{\partial f}{\partial x_{n_1}}(\bar{x}) \end{bmatrix} \in \mathbb{R}^{n_2 \times n_1}$

**Linearization Steps** 1. Find equ. pair  $(\bar{x}, \bar{u})$   
 2. Derive  $A, B, C, D$  and then evaluate at  $(\bar{x}, \bar{u})$   
 3. Write  $\delta \dot{x} = A \delta x + B \delta u$  and  $\delta y = C \delta x + D \delta u$

**Laplace Transform** Given a fcn  $f : \mathbb{R}_+ = [0, \infty) \rightarrow \mathbb{R}^n$ , its Laplace transform is  $F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st} dt, s \in \mathbb{C}$ .  
 \* $\mathcal{L} : f(t) \mapsto F(s), t \in \mathbb{R}_+$  (time dom.) &  $s \in \mathbb{C}$  (Laplace dom.).  
**P.W. CTS:** A fcn  $f : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is **p.w. cts** if on every finite interval of  $\mathbb{R}, f(t)$  has at most a finite # of discontinuity points  $(t_i)$  and the limits  $\lim_{t \rightarrow t_i^+} f(t), \lim_{t \rightarrow t_i^-} f(t)$  are finite.



**Exp. Order** A function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is of **exp. order** if  $\exists$  constants  $K, \rho, T > 0$  s.t.  $\|f(t)\| \leq Ke^{\rho t}, \forall t \geq T$ .  
**Existence of LT Thm** If  $f(t)$  is p.w. cts and of exp. order w/ constants  $K, \rho, T > 0$ , then  $F(\cdot)$  exists and is defined  $\forall s \in D := \{s \in \mathbb{C} : \text{Re}(s) > \rho\}$  and  $F(\cdot)$  is analytic on  $D$ .  
 \*Analytic fcn iff differentiable fcn.  
 \* $D$ : Region of convergence (ROC), open half plane.



**Unit Step**  $1(t) := \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$

**Table of Common Laplace Transforms:**  $f(t) \mapsto F(s)$   
 $1(t) \mapsto \frac{1}{s} \quad t1(t) \mapsto \frac{1}{s^2} \quad t^k 1(t) \mapsto \frac{k!}{s^{k+1}} \quad e^{at} 1(t) \mapsto \frac{1}{s-a}$   
 $t^n e^{at} 1(t) \mapsto \frac{n!}{(s-a)^{n+1}} \quad \sin(at) 1(t) \mapsto \frac{a}{s^2+a^2}$   
 $\cos(at) 1(t) \mapsto \frac{s}{s^2+a^2} \quad \frac{1}{2\omega^3} [\sin(\omega t) - \omega t \cos(\omega t)] 1(t) \mapsto \frac{1}{(s^2+\omega^2)^2}$

**Prop. of Laplace Transform Linearity:**  
 $\mathcal{L}\{cf(t) + g(t)\} = c\mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}, c \sim \text{constant}$ .  
**Differentiation:** If the Laplace transform of  $f'(t)$  exists, then  $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^-)$ .  
 If the Laplace transform of  $f^{(n)}(t) := \frac{d^n f}{dt^n}(t)$  exists, then  $\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - \sum_{i=1}^n s^{n-i} f^{(i-1)}(0^-)$ .  
**Integration:**  $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} \mathcal{L}\{f(t)\}$ .  
**Convolution:** Let  $(f * g)(t) := \int_0^t f(\tau)g(t - \tau) d\tau = \int_0^t f(t - \tau)g(\tau) d\tau$ , then  $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ .  
**Time Delay:**  $\mathcal{L}\{f(t - T)1(t - T)\} = e^{-Ts} \mathcal{L}\{f(t)\}, T \geq 0$ .  
**Multiplication by t:**  $\mathcal{L}\{tf(t)\} = -\frac{d}{ds} [\mathcal{L}\{f(t)\}]$ .  
**Shift in s:**  $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}|_{s \rightarrow s-a} = F(s - a)$ , where  $F(s) = \mathcal{L}\{f(t)\}$  &  $a$  const.

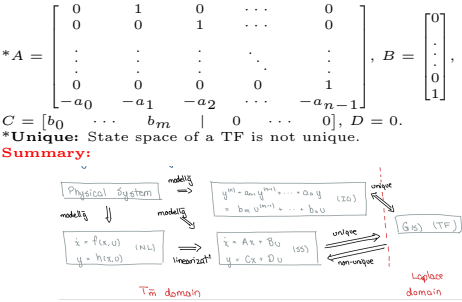
**Trig. Id.**  $2 \sin(2t) = 2 \sin(t) \cos(t), \sin(a - b) = \sin(a) \cos(b) - \cos(a) \sin(b), \cos(a - b) = \cos(a) \cos(b) + \sin(a) \sin(b)$   
**Complete the Square:**  $ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c$   
**LT Steps:** 1. Write  $f(t)$  as a sum and use linearity  
 \*Trig. id. may be useful.  
 2. Use prop. of LT and common LT to find  $F(s)$   
**Inverse Laplace Transform** Given  $F(s)$ , its inverse LT is  $f(t) = \mathcal{L}^{-1}\{F(s)\} := \frac{1}{2\pi} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds$

$= \lim_{w \rightarrow \infty} \frac{1}{2\pi} \int_{c-jw}^{c+jw} F(s) e^{st} ds$ ,  $c \in \mathbb{C}$  is selected s.t. the line  $L := \{s \in \mathbb{C} : s = c + j\omega, \omega \in \mathbb{R}\}$  is inside the ROC of  $F(s)$ .  
**Zero:**  $z \in \mathbb{C}$  is a zero of  $F(s)$  if  $F(z) = 0$ .  
**Pole:**  $p \in \mathbb{C}$  is a pole of  $F(s)$  if  $\frac{1}{F(p)} = 0$ .  
**Cauchy's Residue THM** If  $F(s)$  is analytic (complex diff.) everywhere except at isolated poles  $\{p_1, \dots, p_N\}$ , then  $\mathcal{L}^{-1}\{F(s)\} = \sum_{i=1}^N \text{Res}\left[F(s)e^{st}, s = p_i\right] \mathbf{1}(t)$ ,  
 $\text{Res}[F(s)e^{st}, s = p_i]$ : Residue of  $F(s)e^{st}$  at  $s = p_i$ .  
**Residue Computation** Let  $G(s)$  be a complex analytic fcn w/ a pole at  $s = p$ ,  $r$  be the multiplicity of the pole  $p$ . Then  $\text{Res}[G(s), s = p] = \frac{1}{(r-1)!} \lim_{s \rightarrow p} \frac{d^{r-1}}{ds^{r-1}} [G(s)(s-p)^r]$ .  
**Inv. LT Partial Frac.:** 1. Factorize  $F(s)$  into partial fractions. 2. Find coefficients and use LT table to find inverse LT. \*Complete the square.  
**Inv. LT Residues:** 1. Find poles of  $F(s)$  and their residues. 2. Use Cauchy's Residue THM to find inverse LT.  
**\*Note:** Complex Conjugate (CC) poles  $\rightarrow$  CC residues (use Euler).  
**Transfer Function:** Consider a CS in IO form. Assume zero initial conds.  $y(0) = \dots = \frac{d^{(n-1)}}{dt^{(n-1)}} y(0) = 0$  and

$u(0) = \dots = \frac{d^{(m-1)}}{dt^{(m-1)}} u(0) = 0$ . Then the TF from  $u$  to  $y$  is  $G(s) := \frac{y(s)}{U(s)} = \frac{b_m s^m + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0}$ .  
**\*0 Ini. Conds.:**  $y_0(s) = G(s)u(s)$   
**\* $\emptyset$  Ini. Conds.:**  $y_\emptyset(s) = y_0(s) + \frac{\text{poly. based on initial conds.}}{s^n + a_{n-1} s^{n-1} + \dots + a_0}$

**TF Steps (IO to TF):** 1. Given IO form of CS, assume zero initial conds. 2. Find  $G(s)$  by taking LT of IO form and forming  $Y(s)/U(s)$ . \*Careful:  $Y(s)/U(s) = G(s)$  not  $U(s)/Y(s) = G(s)$ .  
**Impulse Response:** Given CS modeled by TF  $G(s)$ , its IR is  $g(t) := \mathcal{L}^{-1}\{G(s)\}$ .  
 $\mathcal{L}\{\delta(t)\} = 1$ , then if  $u(t) = \delta(t)$ , then  $Y(s) = U(s)G(s) = G(s)$ .  
**SS to TF:**  $G(s) = C(sI - A)^{-1}B + D$  s.t.  $y(s) = G(s)U(s)$ .  
**\*Assume**  $x(0) = 0 \in \mathbb{R}^n$  (zero initial conds).  
**\*LTI:**  $G(s)$  of an LTI system is always a rational fcn.  
**\*Not Invertible:** Values of  $s$  s.t.  $sI - A$  not invertible can correspond to poles of  $G(s)$ .  
**Inverse:** 1. For  $A \in \mathbb{R}^{n \times n}$ , find  $[\text{cof}(A)]_{(i,j)} = (-1)^{i+j} \det(A_{(i,j)})$ .  
**\* $A_{(i,j)}$ :**  $A$  w/ row  $i$  and col.  $j$  removed.  
 2. Assemble  $\text{cof}(A)$  and find  $\det(A) = \sum_{j=1}^n a_{ij} [\text{cof}(A)]_{(i,j)}$  w/ fixed  $i$  or  $\det(A) = \sum_{i=1}^n a_{ij} [\text{cof}(A)]_{(i,j)}$  w/ fixed  $j$ .  
 3. Find  $A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{\det(A)} [\text{cof}(A)]^T$ .  
 $2 \times 2 : A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$   
**TF (SS to TF):** 1. Given SS form, assume zero initial conds. 2. Solve  $G(s) = C(sI - A)^{-1}B + D$ .

\*If  $C = [0 \cdot 1_i \cdot 0]$  &  $B = [0 \cdot 1_j \cdot 0]$ , then only need  $i$ th row &  $j$ th col. of  $\text{adj}(sI - A)$  s.t.  $G(s) = \frac{[\text{adj}(sI - A)]_{(i,j)}}{\det(sI - A)} + D$ .  
**\*Multiple  $i, j$  non-zero entries:** Work it out using MM.  
**TF to SS:** Consider  $G(s) = \frac{b_m s^m + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0} = \frac{N(s)}{D(s)}$ , where  $m < n$  (i.e.  $G(s)$  is strictly proper). Then the SS form is



**Block Diagram Types of Blocks:**  
**Cascade:**  $y_2 = (G_1(s)G_2(s))U \stackrel{\text{SISO}}{\cong} y_2 = (G_2(s)G_1(s))U$

$$U \rightarrow \boxed{G_1} \xrightarrow{y_1} \boxed{G_2} \rightarrow y_2 \quad \equiv \quad U \rightarrow \boxed{G_1, G_2} \rightarrow y_2$$

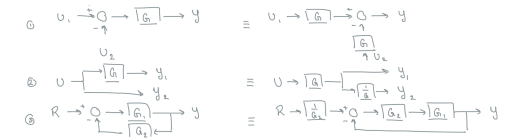
**Parallel**  $y = (G_1(s) + G_2(s))U$

$$U \xrightarrow{\begin{matrix} \rightarrow \boxed{G_1} \\ \rightarrow \boxed{G_2} \end{matrix}} y \quad \equiv \quad U \rightarrow \boxed{G_1 + G_2} \rightarrow y$$

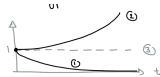
**Feedback**  $y = \left( \frac{G_1(s)}{1 + G_1(s)G_2(s)} \right) R$

$$R \xrightarrow{\begin{matrix} \rightarrow \boxed{G_1} \\ \rightarrow \boxed{G_2} \end{matrix}} y \quad \equiv \quad R \rightarrow \boxed{\frac{G_1}{1 + G_1 G_2}} \rightarrow y$$

**\*SC:** Unity Feedback Loop (UFL) if  $G_2(s) = 1$ .  
**Manipulations:** 1.  $y = G(U_1 - U_2) = GU_1 + GU_2$   
 2.  $y_1 = GU \quad y_2 = U \mid y_1 = GU \quad y_2 = G \frac{1}{G} U$   
 3. From feedback loop to UFL.



**Find TF from Block Diagram:** 1. Start from in  $\rightarrow$  out, making simplifications using block diagram rules. 2. Simplify until you get the form  $U(s) \rightarrow \boxed{G(s)} \rightarrow Y(s)$ .  
**Time Response of Elementary Terms:**  $\mathbf{1}(t) \leftarrow$  pole @ 0  
 $t^n \mathbf{1}(t) \leftarrow$  pole @ 0 w/ mult.  $n \mid e^{at} \mathbf{1}(t) \leftarrow$  pole @  $a$   
 $\sin(\omega t + \phi) \mathbf{1}(t) \leftarrow$  pole @  $\pm j\omega \mid \cos(\omega t + \phi) \mathbf{1}(t) \leftarrow$  pole @  $\pm j\omega$   
**Real Pole:**  $y(s) = \frac{1}{s+a}$ , real pole at  $s = -a$ , then  $y(t) = e^{-at} \mathbf{1}(t)$   
 1.  $a > 0 \implies \lim_{t \rightarrow \infty} y(t) = 0 \mid 2. a < 0 \implies \lim_{t \rightarrow \infty} y(t) = \infty$ .  
 3.  $a = 0 \implies y(t) = \mathbf{1}(t)$  is constant.

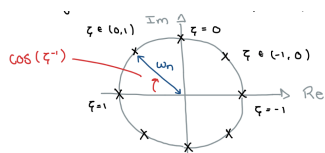


**Time Constant:**  $\tau = \frac{1}{a}$  of the pole  $s = -a$  for  $a > 0$   
**Pair of Comp. Conj. Poles:**

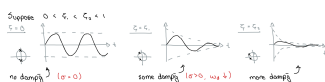
$$y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}, |\zeta| < 1, \text{ then}$$

$$y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t) 1(t)$$

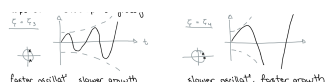
- \*Poles:  $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$
- \* $\zeta = \frac{\sigma}{\omega_n}$ : Damping ratio (or damping coefficient)
- \* $\sigma = \zeta\omega_n$ : Decay/growth rate |  $\omega_d$ : Freq. of oscillation
- \* $\omega_n = \sqrt{\sigma^2 + \omega_d^2} \left[ \frac{\text{radians}}{\text{seconds}} \right]$ : Undamped natural freq.
- \* $\omega_d = \omega_n\sqrt{1-\zeta^2} \left[ \frac{\text{radians}}{\text{seconds}} \right]$ : Damped natural freq.
- \* $|s_{1,2}|^2 = \omega_n^2$ : Mag. of poles is  $\omega_n$ .
- \* $\cos^{-1}(\zeta)$ : Angle of  $s_1$  on complex plane CW from -ve Re axis.



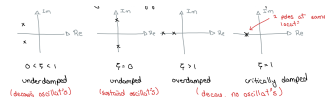
**Damping Ratio Effect:**  $0 < \zeta_1 < \zeta_2 < 1$ , then



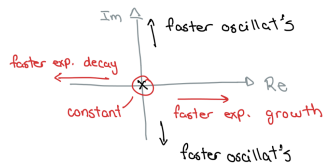
$-1 < \zeta_4 < \zeta_3 < 0$ , then  $\sigma = \zeta\omega_n < 0$ , (exp. envelop  $\uparrow$ )



**Class. of 2nd Order Sys.:**  $y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ , w/  $|\zeta| < 1$



**Loc. of Poles and Behavior:**

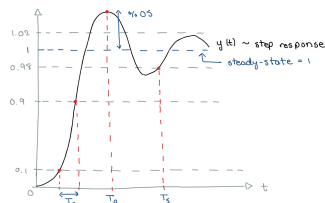


**Control Spec. of 2nd Order Sys.: Step Response:** Given a TF  $G(s)$ , its SR is  $y(t)$  resulting from applying the input  $u(t) = 1(t)$ , i.e.  $\mathcal{L}^{-1}\left\{G(s)\frac{1}{s}\right\}$ .

**Control Spec.** A control spec. is a criterion specifying how we would like a CS to behave.

**2nd Order Sys. Metrics:**  $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$  w/  $U(s) = \frac{1}{s}$

\* $0 < \zeta < 1$  (i.e. 2 comp. conj. poles w/  $\text{Re}(\text{pole}) < 0$ ).



**Rise Time (RT):**  $T_r$  is the time it takes  $y(t)$  to go from 10% to 90% of its steady-state value.

**RT:** 1. Find  $t_1 > 0$  s.t.  $y(t_1) = 0.1$ ,  $t_2 > 0$  s.t.  $y(t_2) = 0.9$ .

3. Compute  $T_r = t_2 - t_1$ .

$$T_r \approx \frac{1.8}{\omega_n}$$

**Settling Time (ST):**  $T_s$  is the time required to reach and stay w/in 2% of the steady-state value.

**ST:** 1. Find when it's first that  $|y(t) - 1| \leq 0.02$ .

$$T_s \approx \frac{4}{\zeta\omega_n}$$

**Peak Time:**  $T_p$  is time req'd to reach the max (peak) value.

**Peak Time:** 1. Find the first time when  $\dot{y}(t) = 0$ .

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

**% Overshoot:**  $\%OS = \frac{[\text{peak value}] - [\text{steady-state value}]}{[\text{steady-state value}]} \times 100\%$

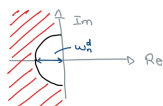
\*% OS = OS  $\times$  100%.

$$\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \iff \zeta = \frac{-\ln(OS)}{\sqrt{\pi^2 + (\ln(OS))^2}}$$

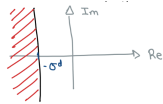
**Transient Performance Sat.:** Given performance spec.  $T_r \leq T_r^d$ ,  $T_s \leq T_s^d$ ,  $OS \leq OS^d$ , find loc. of poles of  $G(s)$ .

\*Admissible region for the poles of  $G(s)$  s.t. the step response meets all three spec. is the intersection of the above three regions.

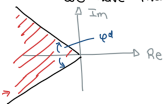
**Rise Time:**  $T_r \approx \frac{1.8}{\omega_n} \leq T_r^d \xrightarrow{\text{app.}} \omega_n \geq \frac{1.8}{T_r^d} \equiv \omega_n^d$



**Settling Time:**  $T_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} \leq T_s^d \xrightarrow{\text{app.}} \sigma \geq \frac{4}{T_s^d} \equiv \sigma^d$

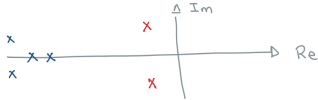


$$\text{OS: } \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \leq \text{OS}^d \xrightarrow{\text{app.}} \zeta \geq \frac{-\ln(\text{OS}^d)}{\sqrt{\pi^2 + (\ln(\text{OS}^d))^2}} \equiv \zeta^d$$



**Add. Poles & Zeros:** The analysis remains approx. correct under the following assumptions:

- Any add. poles of  $G(s)$  have much more -ve real part (5-10 times) than the real part of the dom. comp. conjugate poles.



\*dominant poles, additional poles.

- Real part of zeros are -ve & very diff. from the real part of the two dom. poles.

**Internal Stability:**  $\dot{x} = Ax$  is

- Stable** if  $\forall x(0) \in \mathbb{R}^n$ , the soln.  $x(t)$  is bdd; that is,  $\exists M > 0$  s.t.  $\|x(t)\| \leq M \forall t \geq 0$ .
- Asymp. Stable** if it's stable &  $\forall x(0) \in \mathbb{R}^n$ , the soln.  $x(t)$  converges to the origin; that is,  $\lim_{t \rightarrow \infty} x(t) = 0$ .
- Unstable** if it's not stable; that is,  $\exists x(0) \in \mathbb{R}^n$  s.t.  $x(t)$  is not bdd.

**Asymptotic Stability Thm.**  $\dot{x} = Ax$  is A.S. iff  $\text{eig}(A) \subseteq \mathbb{C}^- \equiv \{s \in \mathbb{C} \mid \text{Re}(s) < 0\}$ , i.e. open left half plane (OLHP).

**Instability Thm.** If  $\exists$  an eigenvalue  $\lambda$  of  $A$  w/  $\text{Re}(\lambda) > 0$ , then  $\dot{x} = Ax$  is unstable.

**Fact:** Zeros of  $s^2 + a_1s + a_0$  are in  $\mathbb{C}^-$  iff  $a_1, a_0 > 0$ .

- Internal Stability** 1. Linearize around  $(\bar{x}, \bar{u})$  w/  $\bar{u} = 0$ .
- Find  $A$  and determine  $\text{eig}(A) = \lambda$  s.t.  $\det(sI - A) = 0$ .

- Check if  $\text{eig}(A) \subseteq \mathbb{C}^- \mid \text{Re}(\text{eig}(A)) > 0$ .

**BIBO Stability:** An LTI system w/ 0 i.e. is Bounded Input Bounded Output (BIBO) stable if for any bdd input  $u(t)$ , the output  $y(t)$  is also bdd.

**BIBO Unstable:** An LTI system w/ 0 i.e. is BIBO unstable if it's not BIBO stable; that is,  $\exists$  a bdd  $u(t)$  s.t.  $y(t)$  is not bdd.

**BIBO Stable Thm.** A system  $y(s) = G(s)U(s)$  is BIBO stable iff  $\text{poles}(G(s)) \subseteq \mathbb{C}^-$ .

**Lemma:** If  $p$  is a pole of  $G(s)$ , then  $p$  is an  $\text{eig}(A)$ . I.e.  $\text{poles}(G(s)) := \{p \in \mathbb{C} \mid p \text{ is a pole of } G(s)\} \subseteq \text{eig}(A)$ .

**\*Pole-0 Cancellation:**  $\text{eig}(A)$  need not be a pole of  $G(s)$ .

**Thm.** If  $\text{eig}(A) \subseteq \mathbb{C}^-$ , then  $\forall B, C, D$  the TF  $G(s)$  is BIBO stable. That is, internal asymptotic stability  $\Rightarrow$  BIBO stability.

- BIBO Stability** 1. Find  $G(s)$  from SS form and determine poles.
- Check if  $\text{poles}(G(s)) \subseteq \mathbb{C}^-$ .

**Routh-Hurwitz:** Consider  $a(s) = s^n + a_{n-1}s^{n-1} + \dots + a_0$ .

$$\begin{array}{c|cccccc} *s^n & 1 & a_{n-2} & a_{n-4} & a_{n-6} & \dots & 0 \\ *s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \dots & 0 \\ *s^{n-2} & b_1 & b_2 & b_3 & \dots & & \\ *s^{n-3} & c_1 & c_2 & \dots & & & \end{array}$$

$$\begin{array}{c|cc} *s & * & 0 \\ *1 & * & 0 \end{array}$$

$$b_1 = -\frac{1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{bmatrix} \quad b_2 = -\frac{1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{bmatrix}$$

$$b_3 = -\frac{1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-6} \\ a_{n-1} & a_{n-7} \end{bmatrix} \quad c_1 = -\frac{1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{bmatrix}$$

$$c_2 = -\frac{1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{bmatrix}$$

**Routh-Hurwitz Stability Criterion:** The roots of  $a(s)$  are in  $\mathbb{C}^-$  iff the 1st col of Routh array has no sign changes. The # of sign changes is equal to the # of roots of  $a(s) \in \mathbb{C}^+ := \{s \in \mathbb{C} : \text{Re}(s) > 0\}$ .

\*If 1st element of a row is 0, Routh array cannot be completed.

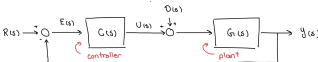
**FVT v1:** Suppose  $Y(s) = \mathcal{L}\{y(t)\}$  is a proper rational fcn. If  $y(\infty) := \lim_{t \rightarrow \infty} y(t)$  exists and is finite, then  $y(\infty) = \lim_{s \rightarrow 0} sY(s)$

**FVT v2:** Suppose  $Y(s) = \mathcal{L}\{y(t)\}$  is a proper rational fcn. Moreover, suppose either:

- $\text{poles}(Y(s)) \subseteq \mathbb{C}^-$
  - $Y(s)$  has only one pole at  $s = 0$  and all other poles are in  $\mathbb{C}^-$ . Then  $y(\infty) := \lim_{t \rightarrow \infty} y(t)$  exists and is finite and satisfies  $y(\infty) = \lim_{s \rightarrow 0} sY(s)$
- FVT** 1. Does  $y(\infty)$  exist? Check if pole at  $s = 0$ , then compute Routh Array to see if poles are in  $\mathbb{C}^-$ .
- Compute  $\lim_{s \rightarrow 0} sY(s)$  if it exists.

## MIDTERM CUTOFF

### Standard Feedback Control Loop



$R(s)$ : Ref.,  $E(s) = R(s) - y(s)$ : Err.,  $C(s)$ : Controller,  $U(s)$ : Control input,  $D(s)$ : Dist.,  $G(s)$ : Plant,  $y(s)$ : Plant output.

**\*Assume:**  $R(s)$  and  $D(s)$  are strictly proper rational fcns w/ a fixed set of poles but arbitrary zeros & gain.

$\mathcal{R}, D$ : Classes of ref. and dist. satisfying the above assumption.

**Basic Control Probl.:** Design  $C(s)$  s.t. 3 spec. are met:

- Stability:**  $\forall$  bdd  $r(t), d(t)$ , we have  $u(t), e(t)$  bdd.
- Asymptotic Tracking:** When  $d(t) = 0 \forall t \geq 0$ , then  $\forall r(t) \in \mathcal{R}$ ,  $\lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} r(t) - y(t) = 0$ .
- Disturbance Rejection:** When  $r(t) = 0 \forall t \geq 0$ , then  $\forall d(t) \in \mathcal{D}$ ,  $\lim_{t \rightarrow \infty} y(t) = 0$ .

**Open-Loop Control:** 1. Design  $u(t)$  s.t.  $y(t)$  tracks ref.  $y_r \in \mathbb{R}$ , i.e.  $\lim_{t \rightarrow \infty} y(t) = y_r$ .

2. Set  $u(t) = \gamma y_r \mathbf{1}(t)$  w/  $\gamma \in \mathbb{R}$  (const. scaling factor)

3. Apply FVT to find  $\gamma$  s.t.  $\lim_{t \rightarrow \infty} y(t) = y_r$ . 4. Determine  $\lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} y_r - y(t)$

**Limitations:** 1. Req. perfect knowledge of plant parameters.

2. Not robust against parameter var./ (unknown) dist.

3. Does not allow us to speed up convergence.

**Feedback Control:** 1. Design  $u(t)$  s.t.  $y(t)$  tracks ref.  $y_r \in \mathbb{R}$ , i.e.  $\lim_{t \rightarrow \infty} y(t) = y_r$ .

2. Set  $u(t) = Ke(t) = K(y_r - y(t))$  w/  $K > 0$  (const. gain).

3. Use block mani. to find  $y(s)$  in terms of input and  $G(s)$ .
4. Apply FVT to find  $K$  s.t.  $\lim_{t \rightarrow \infty} y(t) = y_r$ .
5. Determine  $\lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} y_r - y(t)$

**Advantages:** 1. Doesn't req. perfect knowledge of plant param.

2. Robust against param. var./dist. by  $\uparrow K$ .
3. Allows us to speed up the rate of convergence by  $\uparrow K$ .

**Disadvantages:** 1. Feedback can introduce instability.

2. High-gain amplifies noise.
3. Asymptotic tracking doesn't occur.

**Integral Control:** 1. Design  $u(t)$  s.t.  $y(t)$  tracks ref.  $y_r \in \mathbb{R}$ , i.e.  $\lim_{t \rightarrow \infty} y(t) = y_r$ .

2. Set  $u(t) = \mathcal{L}^{-1}\{C(s)E(s)\} = Ke(t) + KT_I \int_0^t e(\tau) d\tau$  (prop. int. (PI) controller) w/  $K, T_I > 0$  (const. gains).

$*C(s) = K \left(1 + \frac{T_I s}{s}\right)$

3. Use block mani. to find  $y(s)$  in terms of input and  $G(s)$ .
4. Apply FVT to find  $\lim_{t \rightarrow \infty} y(t) = y_r$  as desired.

**BIBO Stability of Closed-Loop System**