Intro: **Random Experiment:** An outcome for each run.
Sample Space Ω: Set of all possible outcomes.
Event: Subsets of Ω.
Prob. of Event A: $P(A) = \frac{\text{Number of outcomes in A}}{\text{Number of outcomes in } \Omega}$
Axioms: $P(A) \geq 0 \ \forall A \in \Omega$, $P(\Omega) = 1$,
If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B) \ \forall A, B \in \Omega$
Cond. Prob. $P(A|B) = \frac{P(A \cap B)}{P(B)}$
$* \ P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$
Independence: $P(A|B) = P(A) \Leftrightarrow P(A \cap B) = P(A)P(B)$
Total Prob. Thm: If $H_1, H_2, \ldots, H_n$ form a partition of Ω, then $P(A) = \sum_{i=1}^{n} P(A|H_i)P(H_i)$.
Bayes' Rule: $P(H_k|A) = \frac{P(H_k \cap A)}{P(A)} = \frac{P(A|H_k)P(H_k)}{\sum_{i=1}^{n} P(A|H_i)P(H_i)}$
*Posteriori: $P(H_k|A)$, Likelihood: $P(A|H_k)$, Prior: $P(H_k)$
1 RV: **CDF:** $F_X(x) = P[X \leq x]$
PMF: $P_X(x_j) = P[X = x_j] \ j = 1, 2, \ldots$
PDF: $f_X(x) = \frac{d}{dx} F_X(x)$
*$P[a \leq X \leq b] = \int_a^b f_X(x)\, dx$ IS THIS CORRECT?
Cond. PMF: $P_X(x|A) = P[X = x|A] = \frac{P[X=x, A]}{P[A]}$ IS THIS CORRECT?
Cond. PDF: $f_X(x|A) = \frac{f_{X,A}(x,a)}{f_A(a)}$ IS THIS CORRECT?
Exp.: $E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x) dx \mid \sum_{k=-\infty}^{\infty} h(k) P_X(x_i = k)$
Variance: $\sigma_X^2 = \text{Var}[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$
Cond. Exp.: $E[X|A] = \int_{-\infty}^{\infty} x f_X(x|A)\, dx$
2 RVs: **Joint PMF:** $P_{X,Y}(x, y) = P[X = x, Y = y]$
Joint PDF: $f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$
*$P[(X, Y) \in A] = \int\int_{(x,y)\in A} f_{X,Y}(x, y)\, dx\, dy$
Expectation: $E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dx\, dy$
Correlation: $E[XY]$
Covariance: $\text{Cov}[X, Y] = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - E[X]E[Y]$
Correlation Coeff.: $\rho_{X,Y} = E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)\left(\frac{Y - \mu_Y}{\sigma_Y}\right)\right] = \frac{\text{Cov}[X,Y]}{\sigma_X \sigma_Y}$
Marginal PMF: $P_X(x) = \sum_{j=1}^{\infty} P_{X,Y}(x, y_j)$
Marginal PDF: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$
Conditional PMF: $P_{X|Y}(x|Y) = P[X = x|Y = y] = \frac{P_{X,Y}(x,y)}{P_Y(y)}$
Conditional PDF: $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Bayes' Rule $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_{X|Y}(x|y) f_Y(y)}{\int_{-\infty}^{\infty} f_{X|Y}(x|y') f_Y(y')\, dy'}$
*$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)} = \frac{P_{X|Y}(x|y) P_Y(y)}{\sum_{j=1}^{\infty} P_{X|Y}(x|y_j) P_Y(y_j)}$
Independent: $f_{X|Y}(x|y) = f_X(x) \ \forall y \Leftrightarrow f_{X,Y}(x, y) = f_X(x) f_Y(y)$
*If independent, then uncorrelated.
Uncorrelated: $\text{Cov}[X, Y] = 0 \Leftrightarrow \rho_{X,Y} = 0$
Orthogonal: $E[XY] = 0$
Conditional Expectation: $E[Y] = E[E[Y|X]]$ or $E[E[h(Y)|X]]$
*$E[E[Y|X]]$ w.r.t. X | $E[Y|X]$ w.r.t. Y.
Estimation: Estimate unknown parameter θ from n i.i.d. measurements $X_1, X_2, \ldots, X_n$, $\hat{\Theta}(\underline{X}) = g(X_1, X_2, \ldots, X_n)$
Estimation Error: $\hat{\Theta}(\underline{X}) - \theta$.
Unbiased: $\hat{\Theta}(\underline{X})$ is unbiased if $E[\hat{\Theta}(\underline{X})] = \theta$.
***Asymptotically unbiased:** $\lim_{n\to\infty} E[\hat{\Theta}(\underline{X})] = \theta$.
Consistent: $\hat{\Theta}(\underline{X})$ is consistent if $\hat{\Theta}(\underline{X}) \to \theta$ as $n \to \infty$ or $\forall \epsilon > 0, \lim_{n\to\infty} P[|\hat{\Theta}(\underline{X}) - \theta| < \epsilon] \to 1$.
Sample Mean: $M_n = \frac{1}{n} S_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.
*Given a sequence of i.i.d. RVs, $X_1, X_2, \ldots, X_n$, M_n is unbiased and consistent.
Chebychev's Inequality: $P[|X - E[X]| \geq \epsilon] \leq \frac{\text{Var}[X]}{\epsilon^2}$
Weak Law of Large #s: $\lim_{n\to\infty} P[|M_n - \mu| < \epsilon] = 1 \ \forall \epsilon > 0$.
Maximum Likelihood Estimation: Choose parameter θ that is most likely to generate the obs. $x_1, x_2, \ldots, x_n$.
*Disc: $\hat{\Theta} = \arg\max_\theta P_{\underline{X}}(\underline{x}|\theta) \xrightarrow{\log} \hat{\theta} = \arg\max_\theta \sum_{i=1}^{n} \log P_X(x_i|\theta)$
*Cont: $\hat{\Theta} = \arg\max_\theta f_{\underline{X}}(\underline{x}|\theta) \xrightarrow{\log} \hat{\theta} = \arg\max_\theta \sum_{i=1}^{n} \log f_X(x_i|\theta)$
Maximum A Posteriori (MAP) Estimation:
*Disc: $\hat{\theta} = \arg\max_\theta P_{\Theta|\underline{X}}(\theta|\underline{x}) = \arg\max_\theta P_{\underline{X}|\Theta}(\underline{x}|\theta) P_\Theta(\theta)$
*Cont: $\hat{\theta} = \arg\max_\theta f_{\Theta|\underline{X}}(\theta|\underline{x}) = \arg\max_\theta f_{\underline{X}|\Theta}(\underline{x}|\theta) f_\Theta(\theta)$
Bayes' Rule: $P_{\Theta|\underline{X}}(\theta|\underline{x}) = \begin{cases} \frac{P_{\underline{X}|\Theta}(\underline{x}|\theta) P_\Theta(\theta)}{P_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ disc.} \\ \frac{f_{\underline{X}|\Theta}(\underline{x}|\theta) P_\Theta(\theta)}{f_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ cont.} \end{cases}$

$$f_{\Theta|\underline{X}}(\theta|\underline{x}) = \begin{cases} \frac{P_{\underline{X}|\Theta}(\underline{x}|\theta) f_\Theta(\theta)}{P_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ disc.} \\ \frac{f_{\underline{X}|\Theta}(\underline{x}|\theta) f_\Theta(\theta)}{f_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ cont.} \end{cases}$$

*Independent of θ: $f_{\underline{X}}(\underline{x}) = \int_{-\infty}^{\infty} f_{\underline{X}|\Theta}(\underline{x}|\theta) f_\Theta(\theta)\, d\theta$

Beta Prior Θ is a Beta R.V. w/ $\alpha, \beta > 0$

$$f_\Theta(\theta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1} & \text{if } 0 < \theta < 1 \\ 0 & \text{otherwise} \end{cases}$$

*$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$
Properties: 1. $\Gamma(x+1) = x\Gamma(x)$. For $m \in \mathbb{Z}^+$, $\Gamma(m+1) = m!$.
2. $\beta(\alpha, \beta) = \frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!} = \beta\binom{\alpha+\beta-1}{\alpha-1}$
3. Expected Value: $E[\Theta] = \frac{\alpha}{\alpha+\beta}$
4. Mode (max of PDF): $\frac{\alpha-1}{\alpha+\beta-2}$
Least Mean Squares (LMS) Estimation: Assume prior $P_\Theta(\theta)$ or $f_\Theta(\theta)$ w/ obs. $\underline{X} = \underline{x}$.
*$\hat{\theta} = g(\underline{x}) = \mathbb{E}[\Theta|\underline{X} = \underline{x}]$ | $\hat{\Theta} = g(\underline{X}) = \mathbb{E}[\Theta|\underline{X}]$