

# ROB311 Quiz 1

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# 1 Prologue

## Summary:

- Variables:
  - State:  $\mathbf{x}(t)$
  - Action(s):  $\mathbf{u}(t)$
  - Measurement:  $\mathbf{y}_k^{(i)}$
  - Context:  $\mathbf{z}_k^{(i)}$
  - Old Context:  $\mathbf{z}_{k-1}^{(i)}$
  - Plan:  $\mathbf{p}_k^{(i)}$
  - (i): Ith agent
- Conversion to DT is necessary because robots are digitalized system and then converted back to CT for execution.

## 1.1 Setup of Planning Problems

**Definition:** In a planning problem, it is assumed that:

- the environment is representable using a discrete set of states,  $\mathcal{S}$
- for each state,  $s \in \mathcal{S}$ , each agent,  $i$ , has a discrete set of actions,  $\mathcal{A}_i(s)$ , with  $\mathcal{A}(s) := \times_i \mathcal{A}_i(s)$  (joint action set)
- **Move:** Any tuple,  $(s, a)$ , where  $s \in \mathcal{S}$  and  $a \in \mathcal{A}(s)$
- **Transition:** Any 3-tuple,  $(s, a, s')$ , where  $s, s' \in \mathcal{S}$  and  $a \in \mathcal{A}(s)$ 
  - the transition resulting from a move may be deterministic/stochastic
- **Reward:**  $\text{rwd}_i(s, a, s')$  is agent  $i$ 's reward for the transition,  $(s, a, s')$
- **Path:** Any sequence of transitions of the form.

$$p = \langle (s^{(0)}, a^{(1)}, s^{(1)}), (s^{(1)}, a^{(2)}, s^{(2)}), \dots \rangle$$

- **Objective:** Each agent wants to realize a path that maximizes its own reward

**Warning:**  $\mathcal{A}(s)$  is the joint action set of all agents at state  $s$ .

## 1.2 Components of a Robotic System

### Summary:

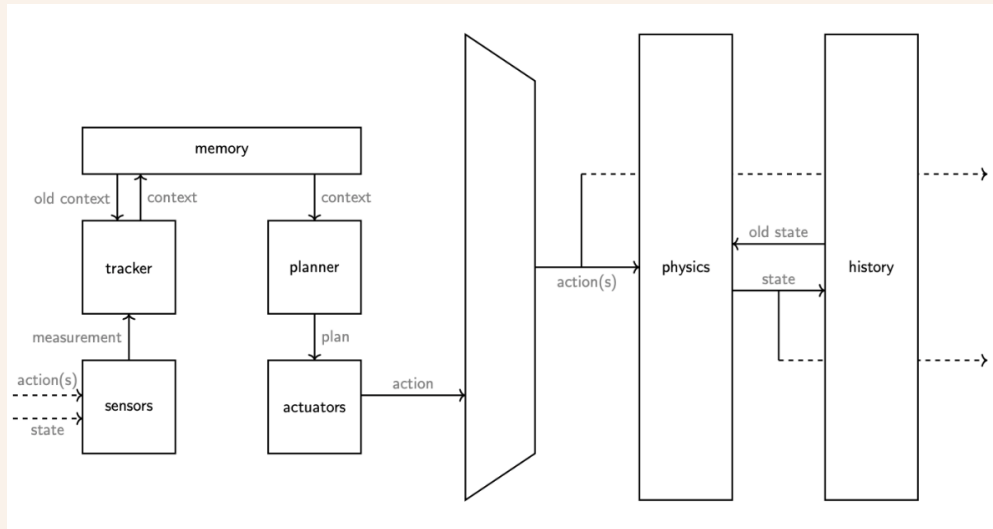


Figure 1: Components of a Robotic System (Words)

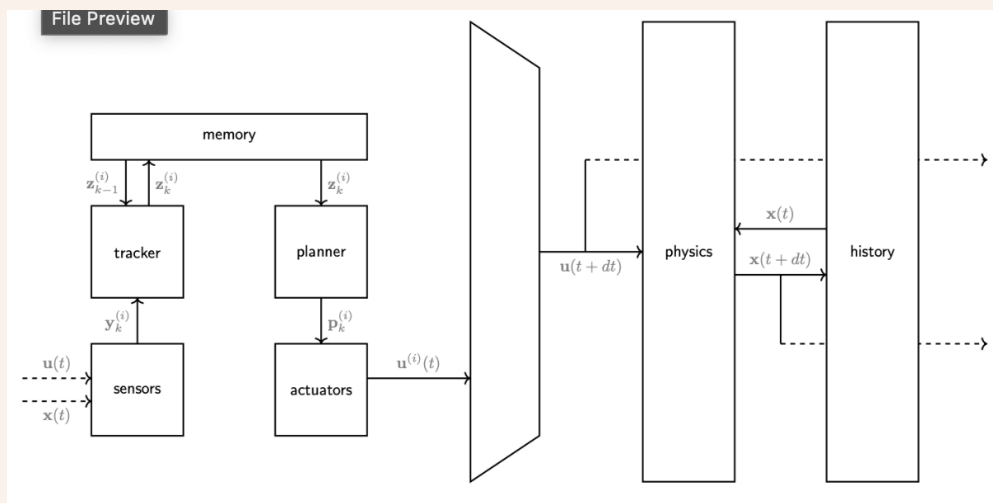


Figure 2: Components of a Robotic System (Math)

### 1.2.1 Overview (Robots, the Environment)

**Definition:**

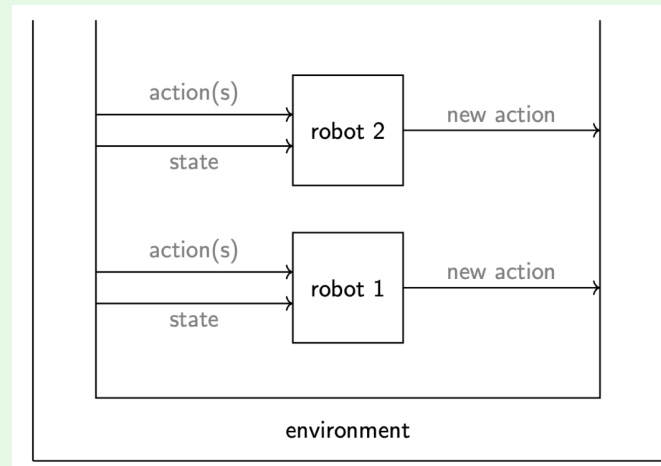


Figure 3: Overview (Robots, the Environment)

**Notes:**

- Environment  $\rightarrow$  previous actions + current state  $\rightarrow$  robot  $\rightarrow$  new action  $\rightarrow$  environment

### 1.2.2 Robot (Sensors, Actuators, the Brain)

**Definition:**

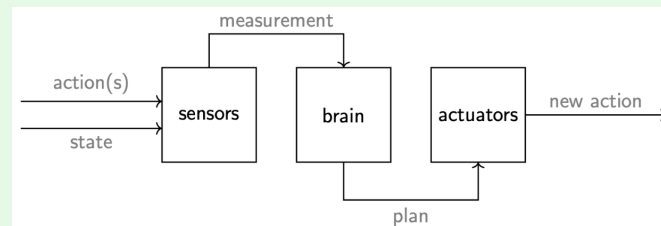


Figure 4: Robot (Sensors, Actuators, the Brain)

**Notes:**

- Measurements can be noisy and inaccurate if not a perfect sensor.
- Measurements go into the brain which can create a plan.

### 1.2.3 Brain (Tracker, Planner, Memory)

#### Definition:

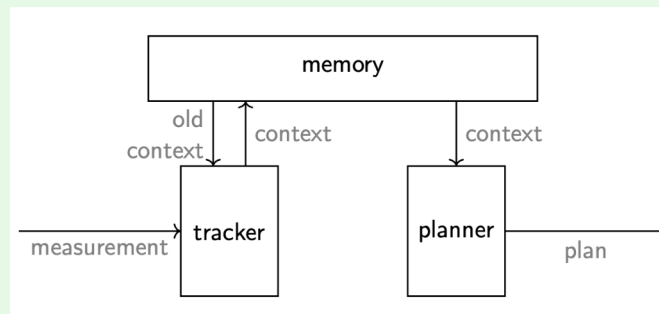


Figure 5: Brain (Tracker, Planner, Memory)

#### Notes:

- The tracker takes in the measurements and old context and updates the context.
- The planner takes in the context and creates a plan.
- The memory stores the context.

### 1.2.4 Environment (Physics, State)

#### Definition:

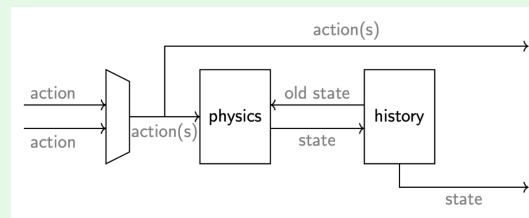


Figure 6: Environment (Physics, State)

## 1.3 Equations of a Robotic System

### 1.3.1 Sensing

**Definition:** Take a measurement:

$$\mathbf{y}^{(i)}(t) = \text{sns}^{(i)}(\mathbf{x}(t), \mathbf{u}(t), t)$$

Convert the measurement into a discrete-time signal using a sampling period of  $T^{(i)}$ :

$$\mathbf{y}_k^{(i)} = \text{dt}(\mathbf{y}^{(i)}(t), t, T^{(i)})$$



Figure 7: Sensing

### 1.3.2 Tracking

**Definition:** Track (update) the context:

$$\mathbf{z}_k^{(i)} = \text{trk}^{(i)}(\mathbf{z}_{k-1}^{(i)}, \mathbf{y}_k^{(i)}, k)$$



Figure 8: Tracking

### 1.3.3 Planning

**Definition:** Make a plan:

$$\mathbf{p}_k^{(i)} = \text{pln}^{(i)}(\mathbf{z}_k^{(i)}, k)$$

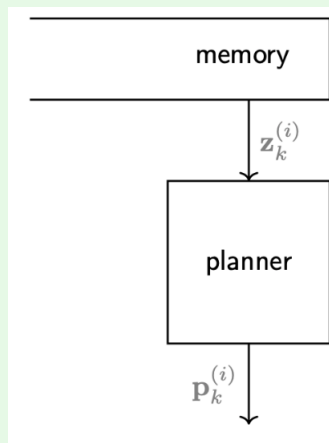


Figure 9: Planning

### 1.3.4 Acting

**Definition:** Convert the plan into a continuous-time signal using a sampling period of  $T^{(i)}$ :

$$\mathbf{p}(t) = \text{ct}(\mathbf{p}_k^{(i)}, t, T^{(i)})$$

Execute the plan:

$$\mathbf{u}^{(i)}(t) = \text{act}^{(i)}(\mathbf{p}^{(i)}(t), t)$$

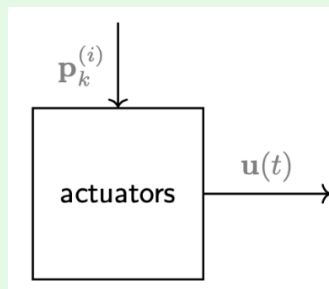


Figure 10: Acting



### 1.3.5 Simulating

**Definition:** Simulate the environment's response:

$$\dot{\mathbf{x}}(t) = \text{phy}(\mathbf{x}(t), \mathbf{u}(t), t)$$



Figure 11: Simulating

## 2 Search Algorithms

### Summary:

Alg.	Halting	Sound	Complete	Optimal	Time	Space
Choose REMOVE( $\cdot$ ) so algo. exhibits the characteristics:						
<ul style="list-style-type: none"> <li>• <b>Halting:</b> Terminates after finitely many nodes explored   <b>Sound:</b> Returned (possibly NULL) soln. is correct</li> <li>• <b>Complete:</b> Halting &amp; sound when a non-NULL soln. exists   <b>Opt.:</b> Returns an opt. soln. when mult. exist</li> <li>• <b>Time:</b> Minimizes nodes <b>explored</b>/expanded/exported   <b>Space:</b> Minimizes nodes simultaneously open</li> </ul>						

Choose REMOVE( $\cdot$ ) so algo. exhibits the characteristics for as many path trees as possible.

- $b$  ( $b < \infty$ ): Branching factor (the maximum number of children a node can have)
- $d$ : Depth (the length of the longest path),  $l^*$ : Length of the shortest solution
- $c^*$ : Cost of the cheapest solution,  $\epsilon$ : Cost of the cheapest edge

### Uninformed Search Algorithms

<b>BFS</b>	$d < \infty$ , non-NULL	always	always	constant cst	$b^{l^*}$	$b^{l^*+1}$
<ul style="list-style-type: none"> <li>• Explores the least-recently expanded open node first.</li> </ul>						
<b>DFS</b>	$d < \infty$	always	$d < \infty$	never	$b^d$	$bd$
<ul style="list-style-type: none"> <li>• Explores the most-recently expanded open node first.</li> </ul>						
<b>IDDFS</b>	always	always	always	constant cst	$b^{l^*}$	$bl^*$
<ul style="list-style-type: none"> <li>• Same as DFS but with iterative deepening.</li> </ul>						
<b>CFS</b>	$d < \infty$ , non-NULL	yes	$\epsilon > 0$	$\epsilon > 0$	$b^{c^*/\epsilon}$	$b^{c^*/\epsilon+1}$
<ul style="list-style-type: none"> <li>• Explores the cheapest open node first.</li> </ul>						

### Informed Search Algorithms

<b>HFS</b>	$d < \infty$	never	never	never	-	-
<ul style="list-style-type: none"> <li>• Explores the node with the smallest hur-value first, <math>ecst(p) = hur(p)</math></li> </ul>						
<b>A*</b>	hur admissible, $\epsilon > 0$	always	hur admissible, $\epsilon > 0$	hur admissible, $\epsilon > 0$	$O(b^{c^*/\epsilon})$	$O(b^{c^*/\epsilon+1})$
<ul style="list-style-type: none"> <li>• Explores the node with the smallest ecst-value first, <math>ecst(p) = cst(p) + hur(p)</math></li> </ul>						
<b>IIA*</b>	always	always	always	always	$b^{l^*}$	$bl^*$
<ul style="list-style-type: none"> <li>• Same as A* but with iterative inflating on ecst.</li> </ul>						
<b>WA*</b>	-	-	-	-	-	-
<ul style="list-style-type: none"> <li>• Same as A* but <math>ecst(s) = wcst(s) + (1 - w)hur(s)</math> w/ <math>w \in [0, 1]</math></li> <li>• <math>w = 0</math>: HFS, <math>w = 0.5</math>: A*, <math>w = 1</math>: CFS, iteratively increasing <math>w</math> from 0 to 1: anytime version of WA*</li> </ul>						

## 2.1 Modifications to Search Algorithms:

### Summary:

---

#### Modifications

---

##### Depth-Limiting

- Enforce a depth limit,  $d_{\max}$ , to any search algorithm.

---

##### Iterative-Deepening

- Iteratively increase the depth-limit to any search algorithm w/ depth-limiting.

---

##### Cost-Limiting

- Enforce a cost limit of  $c_{\max}$  to any search algorithm.

---

##### Iterative Inflating

- Iteratively increase the cost limit,  $c_{\max}$ , to any search algorithm w/ cost-limiting.

---

##### Intra-Path Cycle Checking

- Do not expand a path if it is cyclic.

---

##### Inter-Path Cycle Checking

- Do not expand a path if its destination is that of an explored path.
-

## 2.2 Setup

**Definition:** In a search problem, it is assumed that:

- There is only one agent (us).
- For each state,  $s \in S$ , we have a discrete set of actions,  $\mathcal{A}(s)$ .
- The transition resulting from a move,  $(s, a)$ , is deterministic; the resulting state is  $tr(s, a)$ .
- $cst(s, a, tr(s, a))$  is our cost for the transition,  $(s, a, tr(s, a))$ .
- We want to realize a path that minimizes our cost.

A search problem may have no solutions, in which case, we define the solution as **NULL**.

**Warning:** A **NULL** solution is not the same as  $p = \langle \rangle$  (an empty solution w/  $s^{(0)} \in \mathcal{G}$ ).

## 2.3 Search Graphs

**Definition:** In a search graph (a graph representing a search problem):

- $S$  is defined by the vertices.
- $\mathcal{G}$  is a subset of the vertices.
- $s^{(0)}$  is some vertex.
- $tr(\cdot, \cdot)$  and  $\mathcal{T}$  are defined by the edges.
- $cst(\cdot, \cdot, \cdot)$  is defined by the edge weights.

## 2.4 Path Trees

**Definition:** A search algorithm explores a tree of possible paths.

- In such a tree, each node represents the path from the root to itself.
  - The node may also include other info (such as the path's origin, cost, etc).

## 2.5 Search Algorithms

**Algorithm:** All search algorithms follow the template below:

```

1  $\mathcal{O} \leftarrow \{(\langle \rangle, 0)\}$  ▷ initialize a set of open nodes
2 SEARCH( $\mathcal{O}$ )

•  $\langle \rangle$ : Empty path, 0: Cost of empty path.

1 procedure SEARCH( $\mathcal{O}$ )
2   if  $\mathcal{O} = \emptyset$  then
3     return NULL ▷ the search algorithm failed to find a path to a goal
4    $n \leftarrow \text{REMOVE}(\mathcal{O})$  ▷ "explore" a node  $n$ 
5   if  $\text{DST}(n) \in \mathcal{G}$  then
6     return  $n$  ▷ the search algorithm found a path to a goal
7   for  $n' \in \text{CHL}(n)$  do ▷ "expand"  $n$  and "export" its children
8      $\mathcal{O} \leftarrow \mathcal{O} \cup \{n'\}$ 
9   SEARCH( $\mathcal{O}$ )
```

- Explore: Remove a node from the open set.
- Expand: Generate the children of the node.
- Export: Add the children to the open set.

**Warning:** The key difference is in the order that  $\text{REMOVE}(\cdot)$  removes nodes.

## 2.6 Modifications to Search Algorithms

### 2.6.1 Depth-Limiting

#### Algorithm:

```

1 procedure SEARCHDL( $\mathcal{O}$ ,  $d_{\max}$ ):
2   if  $\mathcal{O} = \emptyset$  then
3     return NULL
4    $n \leftarrow \text{REMOVE}(\mathcal{O})$ 
5   if  $\text{dst}(n) \in \mathcal{G}$  then
6     return  $n$ 
7   for  $n' \in \text{chl}(n)$  do
8     if  $\text{len}(n') \leq d_{\max}$  then
9        $\mathcal{O} \leftarrow \mathcal{O} \cup \{n'\}$ 
10  SEARCHDL( $\mathcal{O}$ ,  $d_{\max}$ )

```

$\triangleright$  the search algorithm failed to find a path to a goal  
 $\triangleright$  "explore" a node,  $n$   
 $\triangleright$  the search algorithm found a path to a goal  
 $\triangleright$  "expand"  $n$  and "export" its children  
 $\triangleright$  unless the child is too long

### 2.6.2 Iterative Deepening

#### Algorithm:

```

1 procedure SEARCHID():
2    $n \leftarrow \text{NULL}$ 
3    $d_{\max} = 0$ 
4    $\triangleright$  while a solution has not been found, reset the open set, run the search algorithm, then increase the
   depth-limit
5   while  $n = \text{NULL}$  do
6      $\mathcal{O} \leftarrow \{(\langle \rangle, 0)\}$ 
7      $n \leftarrow \text{SEARCHDL}(\mathcal{O}, d_{\max})$ 
8      $d_{\max} \leftarrow d_{\max} + 1$ 
9   return  $n$ 

```

**Warning:** Increasing  $d_{\max}$  can be done in different ways.

### 2.6.3 Cost-Limiting

#### Algorithm:

```

1 procedure SEARCHCL( $\mathcal{O}$ ,  $c_{\max}$ ):
2   if  $\mathcal{O} = \emptyset$  then
3     return NULL
4    $n \leftarrow \text{REMOVE}(\mathcal{O})$ 
5   if  $\text{dst}(n) \in \mathcal{G}$  then
6     return  $n$ 
7   for  $n' \in \text{chl}(n)$  do
8     if  $\text{cst}(n') \leq c_{\max}$  then
9        $\mathcal{O} \leftarrow \mathcal{O} \cup \{n'\}$ 
10  SEARCHCL( $\mathcal{O}$ ,  $c_{\max}$ )

```

$\triangleright$  the search algorithm failed to find a path to a goal  
 $\triangleright$  "explore" a node,  $n$   
 $\triangleright$  the search algorithm found a path to a goal  
 $\triangleright$  "expand"  $n$  and "export" its children  
 $\triangleright$  unless the child is too expensive

### 2.6.4 Iterative-Inflating

#### Algorithm:

```

1 procedure SEARCHII():
2    $n \leftarrow \text{NULL}$ 
3    $c_{\max} = 0$ 
4   ▷ while a solution has not been found, reset the open set, run the search algorithm, then increase the
   cost-limit
5   while  $n = \text{NULL}$  do
6      $\mathcal{O} \leftarrow \{(\langle \rangle, 0)\}$ 
7      $n \leftarrow \text{SEARCHCL}(\mathcal{O}, c_{\max})$ 
8      $c_{\max} \leftarrow c_{\max} + \epsilon$ 
9   return  $n$ 

```

**Warning:** Increasing  $c_{\max}$  can be done in different ways.

### 2.6.5 Intra-Path Cycle Checking

#### Algorithm:

```

1 procedure SEARCH( $\mathcal{O}$ ):
2   if  $\mathcal{O} = \emptyset$  then
3     return NULL
4    $n \leftarrow \text{REMOVE}(\mathcal{O})$ 
5   if  $\text{dst}(n) \in \mathcal{G}$  then
6     return  $n$ 
7   for  $n' \in \text{chl}(n)$  do
8     if not CYCLIC( $n'$ ) then
9        $\mathcal{O} \leftarrow \mathcal{O} \cup \{n'\}$ 
10  SEARCH( $\mathcal{O}$ )

```

▷ "expand"  $n$  and "export" its children  
▷ unless the child is cyclic

- Optimality of an algorithm is preserved provided  $\epsilon > 0$ .

### 2.6.6 Inter-Path Cycle Checking

#### Algorithm:

```

1 procedure SEARCH( $\mathcal{O}, \mathcal{C}$ ):
2   if  $\mathcal{O} = \emptyset$  then
3     return NULL
4    $n \leftarrow \text{REMOVE}(\mathcal{O})$ 
5    $\mathcal{C} \leftarrow \mathcal{C} \cup \{n\}$ 
6   if  $\text{dst}(n) \in \mathcal{G}$  then
7     return  $n$ 
8   for  $n' \in \text{chl}(n)$  do
9     if  $n' \notin \mathcal{C}$  then
10       $\mathcal{O} \leftarrow \mathcal{O} \cup \{n'\}$ 
11  SEARCH( $\mathcal{O}, \mathcal{C}$ )

```

▷ add  $n$  to the closed set  
▷ "expand"  $n$  and "export" its children  
▷ unless the child's destination is closed

and then call the algorithm as follows:

```

1  $\mathcal{O} \leftarrow \{(\langle \rangle, 0)\}$ 
2  $\mathcal{C} \leftarrow \{\}$ 
3 SEARCH( $\mathcal{O}, \mathcal{C}$ )

```

▷ initialize a set of closed vertices

## 2.7 Informed Search Algorithms

### 2.7.1 Estimated Cost

**Definition:**  $\text{ecst}(\cdot)$ : estimate the total cost to a goal given a path,  $p$ , based on:

- $\text{cst}(p)$ : Cost of path  $p$
- $\text{hur} : \mathcal{S} \rightarrow \mathbb{R}_+$ : Estimate of the extra cost needed to get to a goal from  $\text{dst}(p)$ 
  - $\text{hur}(s)$  estimates the cost to get to  $\mathcal{G}$  from  $s$  and  $\text{hur}(p)$  means  $\text{hur}(\text{dst}(p))$ .

### 2.7.2 Admissible

**Definition:** A heuristic,  $\text{hur}(\cdot)$ , is said to be **admissible** if

$$\text{hur}(s) \leq \text{hur}^*(s)$$

for all  $s \in \mathcal{S}$  and

$$\text{hur}(s) = 0$$

for all  $s \in \mathcal{G}$ .

### 2.7.3 Consistent

**Definition:** A heuristic,  $\text{hur}(\cdot)$ , is said to be **consistent** if

$$\underbrace{\text{hur}(s) - \text{hur}(\text{tr}(s, a))}_{\text{estimated cost of the transition } (s, a, \text{tr}(s, a))} \leq \underbrace{\text{cst}(s, a, \text{tr}(s, a))}_{\text{true cost of the transition, } (s, a, \text{tr}(s, a))}$$

for all  $s \in \mathcal{S}$ , and  $a \in \mathcal{A}(s)$ , and

$$\text{hur}(s) = 0$$

for all  $s \in \mathcal{G}$ .

**Theorem:** If a heuristic,  $\text{hur}(\cdot)$ , is consistent, then it is also admissible.

### 2.7.4 Domination

**Definition:** If  $\text{hur}_1$  and  $\text{hur}_2$  are admissible, then:

- $\text{hur}_1$  **strongly dominates**  $\text{hur}_2$  if for all  $s \in \mathcal{S} \setminus \mathcal{G}$ :

$$\text{hur}_1(s) > \text{hur}_2(s)$$

- $\text{hur}_1$  **weakly dominates**  $\text{hur}_2$  if for all  $s \in \mathcal{S}$ :

$$\text{hur}_1(s) \geq \text{hur}_2(s)$$

and for some  $s \in \mathcal{S}$ :

$$\text{hur}_1(s) > \text{hur}_2(s)$$

### 2.7.5 Designing Heuristics via Problem Relaxation

**Definition:** Let  $\text{hur}_{\text{ori}}^*$  be the perfect heuristic for a search problem, and  $\text{cst}_{\text{rel}}^*$  be the optimal cost for a relaxed version of the problem. Then

$$\text{cst}_{\text{rel}}^*(s) \leq \text{hur}_{\text{ori}}^*(s) \text{ for all } s \in \mathcal{S}.$$

### 2.7.6 Combining Heuristics

**Definition:** If  $\{\text{hur}_k(\cdot)\}_k$  are admissible (resp. consistent), then  $\max_k \{\text{hur}_k\}(\cdot)$  is also admissible (resp. consistent).

**Definition:** If  $\text{hur}_{\max} \equiv \max\{\text{hur}_1, \text{hur}_2\}$ , then if  $\text{hur}_k$  is consistent:

$$\begin{aligned}\text{hur}_k(s) - \text{hur}_k(\text{tr}(s, a)) &\leq \text{cst}(s, a, \text{tr}(s, a)) \\ \text{hur}_{\max}(s) &= \text{hur}_{\max}(\text{tr}(s, a)) - \text{cst}^*(s, a, \text{tr}(s, a))\end{aligned}$$

### 2.7.7 Anytime Search Algorithms

**Definition:** An **anytime algorithm** finds a solution quickly (even if it is sub-optimal), and then iteratively improves it (if time permits).



## 2.8 Canonical Examples

### 2.8.1 How to setup a search problem?

#### Process:

1. Given a search graph, we need to define the following:
  - $\mathcal{S}$ : set of vertices
  - $\mathcal{G}$ : goal states (subset of  $\mathcal{S}$ )
  - $s^{(0)}$ : initial state
  - $\mathcal{T}$ : set of edges (defined by  $\text{tr}(\cdot, \cdot)$ )
    - $\text{tr}(\cdot, \cdot)$ : transition function
  - $\text{cst}(\cdot, \cdot, \cdot)$ : cost function (defined by edge weights)

#### Example:

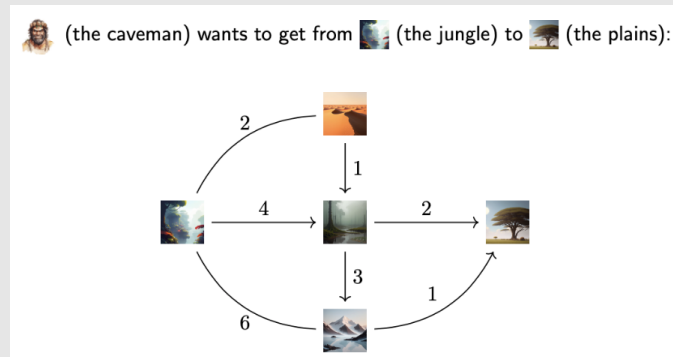


Figure 12

In our example,  $\mathcal{S} = \left\{ \text{🌿}, \text{🏠}, \text{🌅}, \text{🌊}, \text{🌳} \right\}$ ,  $\mathcal{G} = \left\{ \text{🌳} \right\}$ ,  
 $s^{(0)} = \text{🌿}$ , and one possible transition is  $\langle \text{🌿}, \emptyset, \text{🏠} \rangle$ , at a cost of 4.

Figure 13

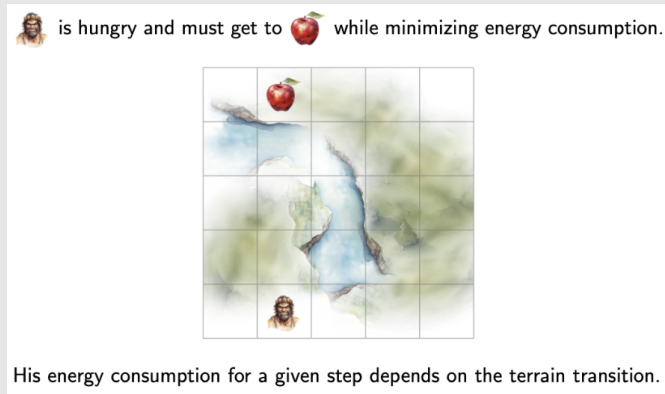
**Example:**

Figure 14

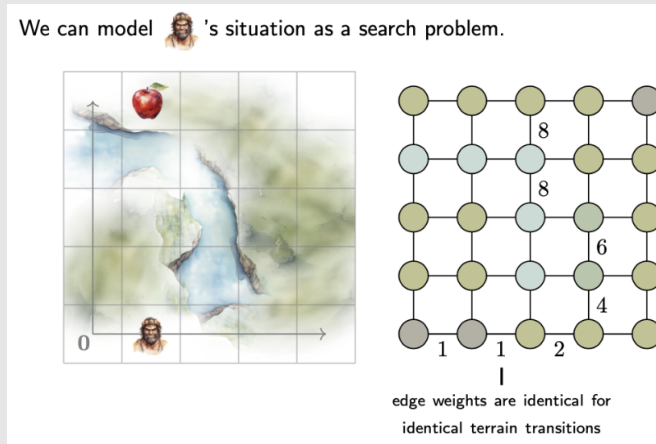


Figure 15

- $\mathcal{S} = \{0, \dots, 4\}^2$
- $\mathcal{G} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$
- $s^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

### 2.8.2 How to setup a path tree?

### Process:

1. Start at  $s^{(0)}$
2. Choose a path until you reach a goal state.
3. Repeat until you have found all paths (probably infinite).

**Example:**

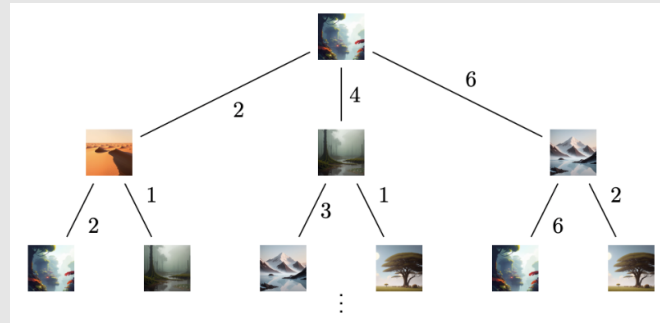


Figure 16

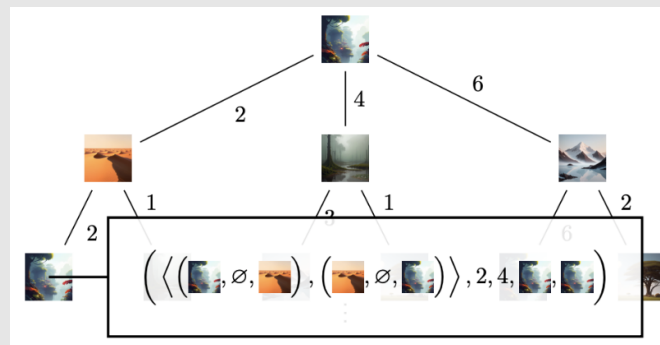


Figure 17

### 2.8.3 When to use each algorithm?

#### Process:

1. Do we have a heuristic?
  - **Yes:** Use informed search algorithms.
  - **No:** Use uninformed search algorithms.
2. Are path costs non-uniform?
  - **Yes:** Eliminate BFS.
  - **No:** Eliminate CFS,  $A^*$
- 3.
4. Is the search space finite or infinite?
  - **Finite:** Use any algorithm.
  - **Infinite:** Use BFS, IDDFS, CFS, or  $A^*$ .
5. Do we need to guarantee finding a solution (completeness)?
  - **Yes:** Use BFS, IDDFS,  $IIA^*$ , CFS (if  $\epsilon > 0$ ).
  - **No:** Use DFS, HFS,  $WA^*$
6. Find properties needed for the problem and match them to the characteristics of the algorithm.
7. Choose the algorithm that best matches the properties.
  - **BFS:** Need shortest path in an unweighted graph.
  - **DFS:** Explore a deep path quickly, and completeness is not needed.
  - **IDDFS:** Want completeness of BFS but with the complexity of DFS.
  - **CFS:** Need the least-cost path in a weighted graph.
  - **HFS:**
  - **$A^*$ :**
  - **$IIA^*$ :**
  - **$WA^*$ :**

#### Example:

## 2.8.4 Tracing Search Algorithms

Example:

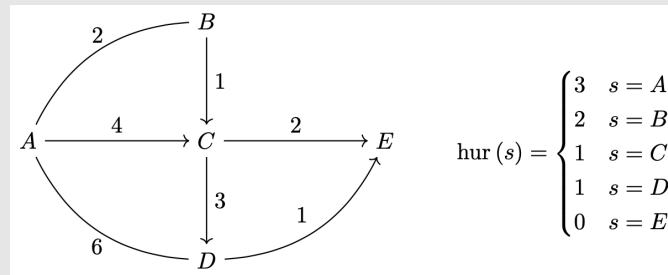


Figure 18

**Process: BFS**

1. Start at  $s_0$  as **current node**
2. Expand all neighboring nodes of the **current node** and add them to the open set (queue).
3. Remove the **current node** from the open set and add it to the path.
4. Choose the least-recently expanded node from the open set as the **current node**.
5. Repeat steps 2 and 4 until the goal state is reached or the open set is empty.

**Example: BFS**

Path	Open Set
	{A}
A	{AB, AC, AD}
AB	{AC, AD, ABA, ABC}
AC	{AD, ABA, ABC, ACD, ACE}
AD	{ABA, ABC, ACD, ACE, ADA, ADE}
ABA	{ABC, ACD, ACE, ADA, ADE, ABAB, ABAC, ABAD}
ABC	{ACD, ACE, ADA, ADE, ABAB, ABAC, ABAD, ABCD, ABCE}
ACD	{ACE, ADA, ADE, ABAB, ABAC, ABAD, ABCD, ABCE, ACDA, ACDE}
ACE	{ADA, ADE, ABAB, ABAC, ABAD, ABCD, ABCE, ACDA, ACDE}

**Intra:**

Path	Open Set
	{A}
A	{AB, AC, AD}
AB	{AC, AD, ABC}
AC	{AD, ABC, ACD, ACE}
AD	{ABC, ACD, ACE, ADE}
ABC	{ACD, ACE, ADE, ABCD, ABCE}
ACD	{ACE, ADE, ABCD, ABCE, ACDE}
ACE	{ADE, ABCD, ABCE, ACDE}

**Process: DFS**

1. Start at  $s_0$  as **current node**
2. Expand all neighboring nodes of the **current node** and add them to the open set (stack).
3. Remove the **current node** from the open set and add it to the path.
4. Choose the most-recently expanded node from the open set as the **current node**.
5. Repeat steps 2 and 4 until the goal state is reached or the open set is empty.

**Example: DFS**

Path	Open Set
	{A}
A	{AB, AC, AD}
AD	{AB, AC, ADA, ADE}
ADE	{AB, AC, ADA, ADE}

**Intra:**

Path	Open Set
	{A}
A	{AB, AC, AD}
AD	{AB, AC, ADE}
ADE	{AB, AC, ADE}

**Inter:**

**Process: IDDFS**

1. Start with a depth limit of 0.
2. Perform DFS up to the current depth limit.
3. If the goal state is not reached, increment the depth limit based on given fcn and repeat step 2.
4. Continue until the goal state is found or all nodes are explored.

**Example: IDDFS**

Depth	Path	Open Set
0		{A}
0	A	{}
1	A	{AB, AC, AD}
1	AD	{AB, AC}
1	AC	{AB}
1	AB	{}
2	AB	{ABA, ABC}
2	ABC	{ABA}
2	ABA	{}
3	ABA	{ABAB, ABAC, ABAD}
3	ABAB	{ABAC, ABAD, ABABC}

**Process: CFS**

1. Start at  $s_0$  as **current node**
2. Expand all neighboring nodes of the **current node** and add them to the open set (priority queue).
3. Remove the **current node** from the open set and add it to the path.
4. Choose the cheapest expanded node from the open set as the **current node**.
5. Repeat steps 2 and 4 until the goal state is reached or the open set is empty.

**Example: CFS****No Inter-Path Cycle Checking, No Intra-Path Cycle Checking**

Path	Open Set
	{A   0}
A	{AB   2, AC   4, AD   6}
AB	{AC   4, AD   6, ABC   3, ABA   4}
ABC	{AC   4, AD   6, ABA   4, ABCE   5, ABCD   6}
AC	{AD   6, ABA   4, ABCE   5, ABCD   6, ACD   7, ACE   6}
ABA	{AD   6, ABCE   5, ABCD   6, ACD   7, ACE   6, ABAB   6, ABAC   8, ABAD   10}
ABCE	{AC   4, ABA   4, AD   6, ABCD   6}

**Intra-Path Cycle Checking:**

Path	Open Set
	{A   0}
A	{AB   2, AC   4, AD   6}
AB	{AC   4, AD   6, ABC   3, <del>ABA</del> intra}
ABC	{AC   4, AD   6, ABCE   5, ABCD   6}
AC	{AD   6, ABCE   5, ABCD   6, ACD   7, ACE   6}
ABCE	{AC   4, AD   6, ABCD   6, ACD   7, ACE   6}

- **Inter-Path Cycle Checking:** Doesn't affect the algorithm in this instance.

**Warning:**

- How to perform shortcut?

**Process: HFS**

1. Start at  $s_0$  as **current node**
2. Expand all neighboring nodes of the **current node** and add them to the open set (priority queue).
3. Remove the **current node** from the open set and add it to the path.
4. Choose the lowest heuristic value expanded node from the open set as the **current node**.
5. Repeat steps 2 and 4 until the goal state is reached or the open set is empty.

**Example: HFS**

Path	Open Set
	$\{A \mid 3\}$
$A$	$\{AB \mid 2, AC \mid 1, AD \mid 1\}$
$AC$	$\{AB \mid 2, AD \mid 1, ACE \mid 0\}$
$ACE$	$\{AB \mid 2, AD \mid 1\}$

- **Inter/Intra-Path Cycle Checking:** Doesn't affect the algorithm in this instance.

**Process: A\***

1. Start at  $s_0$  as **current node**
2. Expand all neighboring nodes of the **current node** and add them to the open set (priority queue).
3. Remove the **current node** from the open set and add it to the path.
4. Choose the lowest  $esc(p) = cst(p) + hur(p)$  expanded node from the open set as the **current node**.
5. Repeat steps 2 and 4 until the goal state is reached or the open set is empty.

**Example: A\*****No Inter-Path Cycle Checking, No Intra-Path Cycle Checking**

Path	Open Set
	$\{A \mid 3\}$
$A$	$\{AB \mid 2+2, AC \mid 4+1, AD \mid 6+1\}$
$AB$	$\{AC \mid 5, AD \mid 7, ABC \mid 3+1, ABA \mid 4+3\}$
$ABC$	$\{AC \mid 5, AD \mid 7, ABCD \mid 6+1, ABCE \mid 5+0, ABA \mid 7\}$
$AC$	$\{AD \mid 7, ABCD \mid 7, ABCE \mid 5, ABA \mid 7, ACD \mid 7+1, ACE \mid 6+0\}$
$ABCE$	$\{AD \mid 7, ABCD \mid 7, ABA \mid 7, ACD \mid 8, ACE \mid 6\}$

**Intra-Path Cycle Checking:**

Path	Open Set
	$\{A \mid 3\}$
$A$	$\{AB \mid 2+2, AC \mid 4+1, AD \mid 6+1\}$
$AB$	$\{AC \mid 5, AD \mid 7, ABC \mid 3+1, \cancel{ABA} \text{ intra}\}$
$ABC$	$\{AC \mid 5, AD \mid 7, ABCD \mid 6+1, ABCE \mid 5+0\}$
$AC$	$\{AD \mid 7, ABCD \mid 7, ABCE \mid 5, ACD \mid 7+1, ACE \mid 6+0\}$
$ABCE$	$\{AD \mid 7, ABCD \mid 7, ACD \mid 8, ACE \mid 6\}$

- **Inter-Path Cycle Checking:** Doesn't affect the algorithm in this instance.



**Process: IIA\***

1. Start with a cost limit of 0.
2. Perform A\* up to the current cost limit.
3. If the goal state is not reached, increment the cost limit based on given fcn and repeat step 2.
4. Continue until the goal state is found or all nodes are explored.

**Example: IIA\*****Inter-Path Cycle Checking, Intra-Path Cycle Checking**

Cost	Path	Open Set
0	$\langle \rangle$	$\{A \mid 0 + 3\}$
1	$\langle \rangle$	$\{A \mid 3\}$
2	$\langle \rangle$	$\{A \mid 3\}$
3	$\langle \rangle$	$\{A \mid 3\}$
3	$A$	$\{AB \mid 2 + 2, AC \mid 4 + 1, AD \mid 6 + 1\}$
4	$AB$	$\{AC \mid 5, AD \mid 7, ABC \mid 3 + 1\}$
4	$ABC$	$\{AC \mid 5, AD \mid 7, ABCD \mid 6 + 1, ABCE \mid 5 + 0\}$
5	$AC$	$\{AD \mid 7, ABCD \mid 7, ABCE \mid 5, ACD \mid 7 + 1, ACE \mid 6 + 0\}$
5	$ABCE$	$\{AD \mid 7, ABCD \mid 7, ACD \mid 8, ACE \mid 6\}$

**Process: WA\***

1. Start at  $s_0$  as **current node**
2. Expand all neighboring nodes of the **current node** and add them to the open set (priority queue).
3. Remove the **current node** from the open set and add it to the path.
4. Choose the lowest  $esct(p) = w \cdot cst(p) + (1 - w) \cdot hur(p)$  expanded node from the open set as the **current node**.
5. Repeat steps 2 and 4 until the goal state is reached or the open set is empty.

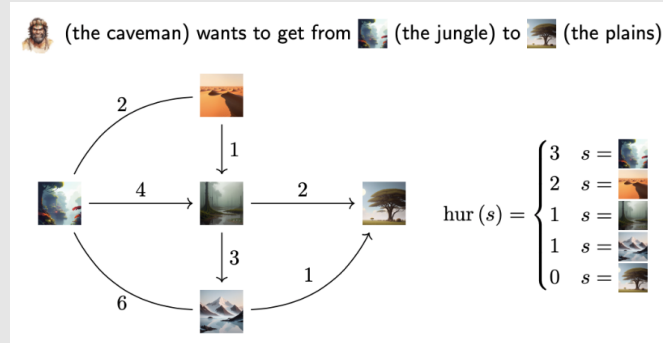
**Process: How to Prove Consistent/Admissible Given a Search Graph?****Admissible:**

1. Given  $\text{hur}(s)$  and search graph with  $\text{cst}(s, a, \text{tr}(s, a))$ . If consistent, then it is admissible.
2. Check  $\forall s \in \mathcal{G}$ ,  $\text{hur}(s) = 0$ . If not, then it is not admissible.
3. For each  $s \in \mathcal{S}$ , calculate  $\text{hur}^*(s)$  (i.e. actual cost of optimal soln.) using the search graph.
  - (a) Start at  $s$  and choose path that gives the lowest cost to  $s \in \mathcal{G}$ .
4. Check if  $\text{hur}(s) \leq \text{hur}^*(s) \forall s \in \mathcal{S}$ . If not, then it is not admissible.
5. Repeat  $\forall s \in \mathcal{S}$ .
6. If all are true, then it is admissible.

**Consistent:**

1. Given  $\text{hur}(s)$  and search graph with  $\text{cst}(s, a, \text{tr}(s, a))$ .
2. Check  $\forall s \in \mathcal{G}$ ,  $\text{hur}(s) = 0$ . If not, then it is not consistent.
3. For each  $s \in \mathcal{S}$ , calculate  $\text{hur}(s) - \text{hur}(\text{tr}(s, a))$ .
  - (a) check if it is  $\leq \text{cst}(s, a, \text{tr}(s, a))$ . If not, then it is not consistent.
  - (b) Repeat  $\forall a \in \mathcal{A}(s)$
4. Repeat  $\forall s \in \mathcal{S}$ .
5. If all are true, then it is consistent.

**Warning:** Be careful of bidirectional edges bc for consistency you need compute the cost of the heuristic edge in both directions.

**Example:**Figure 19: Jungle ( $s^{(0)}$ ), Desert, Swamp, Mountain, Plains (Goal)**Admissible:**

1.  $s = \text{Plains}$ :  $\text{hur}(\text{Plains}) = 0$
2.  $s = \text{Jungle}$ :  $\text{hur}(\text{Jungle}) = 3 \leq \text{hur}^*(\text{Jungle}) = 2 + 1 + 2 = 5$
3.  $s = \text{Desert}$ :  $\text{hur}(\text{Desert}) = 2 \leq \text{hur}^*(\text{Desert}) = 1 + 2$
4.  $s = \text{Swamp}$ :  $\text{hur}(\text{Swamp}) = 1 \leq \text{hur}^*(\text{Swamp}) = 2$
5.  $s = \text{Mountain}$ :  $\text{hur}(\text{Mountain}) = 1 \leq \text{hur}^*(\text{Mountain}) = 1$
6. Therefore, it is admissible.

**Consistent:**

1.  $s = \text{Plains}$ :  $\text{hur}(\text{Plains}) = 0$
2.  $s = \text{Jungle}$ :
  - (a)  $\text{hur}(\text{Jungle}) - \text{hur}(\text{Desert}) = 3 - 2 = 1 \leq \text{cst}(\text{Jungle}, \cdot, \text{Desert}) = 2$
  - (b)  $\text{hur}(\text{Jungle}) - \text{hur}(\text{Swamp}) = 3 - 1 = 2 \leq \text{cst}(\text{Jungle}, \cdot, \text{Swamp}) = 4$
  - (c)  $\text{hur}(\text{Jungle}) - \text{hur}(\text{Mountain}) = 3 - 1 = 2 \leq \text{cst}(\text{Jungle}, \cdot, \text{Mountain}) = 6$
3.  $s = \text{Desert}$ :
  - (a)  $\text{hur}(\text{Desert}) - \text{hur}(\text{Jungle}) = 2 - 3 = -1 \leq \text{cst}(\text{Desert}, \cdot, \text{Jungle}) = 2$
  - (b)  $\text{hur}(\text{Desert}) - \text{hur}(\text{Swamp}) = 2 - 1 = 1 \leq \text{cst}(\text{Desert}, \cdot, \text{Swamp}) = 1$
4.  $s = \text{Swamp}$ :
  - (a)  $\text{hur}(\text{Swamp}) - \text{hur}(\text{Mountain}) = 1 - 1 = 0 \leq \text{cst}(\text{Swamp}, \cdot, \text{Mountain}) = 3$
  - (b)  $\text{hur}(\text{Swamp}) - \text{hur}(\text{Plains}) = 1 - 0 = 1 \leq \text{cst}(\text{Swamp}, \cdot, \text{Plains}) = 2$
5.  $s = \text{Mountain}$ :
  - (a)  $\text{hur}(\text{Mountain}) - \text{hur}(\text{Jungle}) = 1 - 3 = -2 \leq \text{cst}(\text{Mountain}, \cdot, \text{Desert}) = 6$
  - (b)  $\text{hur}(\text{Mountain}) - \text{hur}(\text{Plains}) = 1 - 0 = 1 \leq \text{cst}(\text{Mountain}, \cdot, \text{Plains}) = 1$
6. Therefore, it is consistent.

**Process: How to Design Heuristic via Problem Relaxation?**

1. Make an assumption to simplify the problem as a relaxed problem.
2. Find the cost of the optimal solution of the relaxed problem,  $\text{cst}_{\text{rel}}(s)$ .
3. HOW TO FIND THE COST OF THE OPTIMAL SOLUTION?

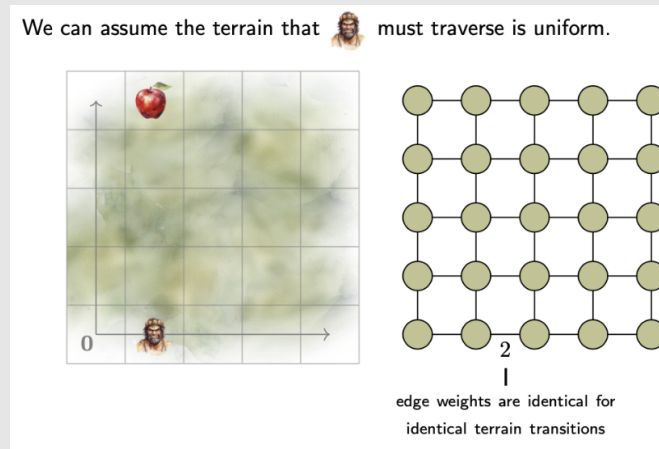
**Example:**

Figure 20

### 3 Constraint Satisfaction Problems

#### 3.1 Setup of CSP

**Definition:** A **constraint satisfaction problem (CSP)** consists of:

- a set of **variables**,  $\mathcal{V}$ , where the domain of  $V \in \mathcal{V}$  is  $\text{dom}(V)$
- a set of **constraints**,  $\mathcal{C}$ , where the scope of  $C \in \mathcal{C}$  is  $\text{scp}(C) \subseteq \mathcal{V}$

#### 3.2 Assignment

**Definition:** An **assignment** is a set of pairs,  $\{(V, v)\}_{V \in \tilde{\mathcal{V}}}$ , where  $v \in \text{dom}(V)$ , and  $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ . It is **complete** if  $\tilde{\mathcal{V}} = \mathcal{V}$ , and **partial** otherwise.

#### 3.3 Consistent

##### 3.3.1 Complete Assignment

**Definition:** A complete assignment,  $A$ , is **consistent** if it satisfies every constraint  $C$  with  $\text{scp}(C) \subseteq \tilde{\mathcal{V}}$ .

**Warning:** A solution to a CSP is any complete and consistent assignment.

##### 3.3.2 Partial Assignment

**Definition:** A (possibly partial) assignment,  $\{(V, v)\}_{V \in \tilde{\mathcal{V}}}$ , is **consistent** if it satisfies every constraint,  $C \in \mathcal{C}$  such that  $\text{scp}(C) \subseteq \tilde{\mathcal{V}}$ .

##### 3.3.3 k-Consistent

**Definition:** A CSP is **k-consistent** if for any consistent assignment of  $k - 1$  variables,  $\{(V, v)\}_{V \in \tilde{\mathcal{V}}}$ , and any  $k^{\text{th}}$  variable,  $V'$ , there is a value,  $v' \in \text{dom}(V')$ , so the assignment,  $\{(V, v)\}_{V \in \tilde{\mathcal{V}}} \cup \{(V', v')\}$  is consistent.

##### 3.3.4 Edge/Arc Consistent

**Definition:** 2-consistent.

### 3.4 Constraint Satisfaction Algorithms

#### 3.4.1 Main

Algorithm:

#### 3.4.2 Satisfy

Algorithm:

#### 3.4.3 Enforce: Enforcing k-Consistency

Algorithm:

#### 3.4.4 EnforceVar: Enforcing k-Consistency


Algorithm:

### 3.5 Setup of CSP

#### Process:

1.

#### Example:

 now wants to find food to meet his nutritional requirements:










		Nutrients (25 g)			
	Supply	 Carbs	 Fat	 Protein	 Vitamins
Minimum	----	8	3	2	1
 Nuts	4	2	1	1	0
 Fruits	5	2	0	0	0
 Legumes	4	2	0	1	1
 Grains	6	3	0	0	0
 Meat	3	0	1	2	0
Maximum	----	10	4	5	1

Figure 21: Information

For our example, the variables could be:

$$\begin{array}{ll}
 \begin{array}{c} \text{Nuts} \\ \text{Fruits} \\ \text{Legumes} \\ \text{Grains} \\ \text{Meat} \end{array} & \begin{array}{l} \in \{0, 1, 2, 3, 4\} \\ \in \{0, 1, 2, 3, 4\} \\ \in \{0, 1, 2, 3, 4\} \\ \in \{0, 1, 2, 3, 4\} \\ \in \{0, 1, 2, 3\} \end{array} \\
 & \begin{array}{l} \text{dom}(\text{Nuts}) \\ \text{dom}(\text{Fruits}) \\ \text{dom}(\text{Legumes}) \\ \text{dom}(\text{Grains}) \\ \text{dom}(\text{Meat}) \end{array}
 \end{array}$$

Figure 22: Variables

For our example, the constraints could be:

$$\begin{array}{ll}
 \text{Carbs} & : \quad 8 \leq 2 \text{ Nuts} + 2 \text{ Fruits} + 2 \text{ Legumes} + 3 \text{ Grains} \leq 10 \\
 & \quad \text{scp}(\text{Carbs}) = \{\text{Nuts}, \text{Fruits}, \text{Legumes}, \text{Grains}\} \\
 \text{Fat} & : \quad 3 \leq \text{Nuts} + \text{Meat} \leq 4 \\
 & \quad \text{scp}(\text{Fat}) = \{\text{Nuts}, \text{Meat}\} \\
 \text{Protein} & : \quad 2 \leq \text{Nuts} + \text{Legumes} + 2 \text{ Meat} \leq 5 \\
 & \quad \text{scp}(\text{Protein}) = \{\text{Nuts}, \text{Legumes}, \text{Meat}\} \\
 \text{Vitamins} & : \quad 1 \leq \text{Legumes} \leq 2 \\
 & \quad \text{scp}(\text{Vitamins}) = \{\text{Legumes}\}
 \end{array}$$

Figure 23: Constraints

**Process:** How to build a hyper-graph?

1. Circle the variables that appear in constraint  $C_i \forall i$ .

**Example:**

We can visualize the constraints using a hyper-graph.

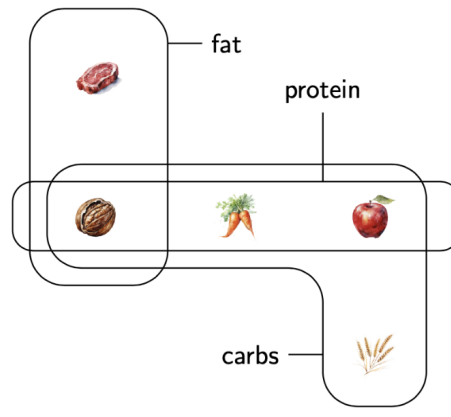


Figure 24

**Process:** How to build a path tree?

- 1.

**Example:**

The path tree traversed lists all possible ways to build a complete assignment:

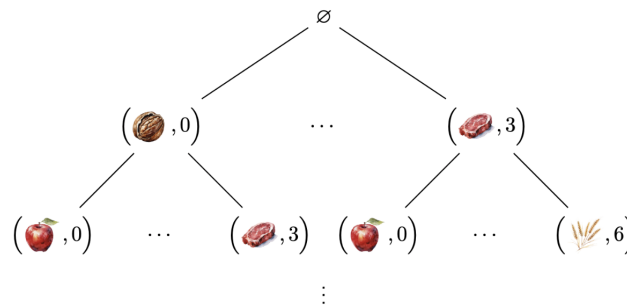


Figure 25

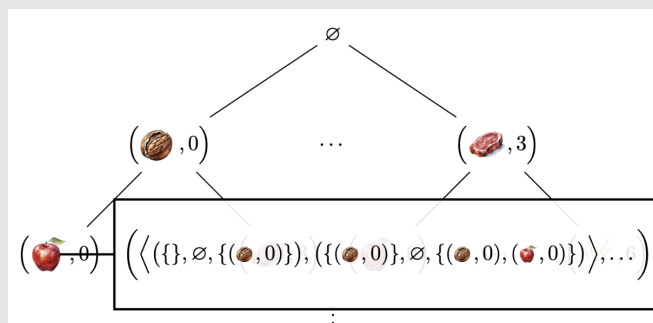


Figure 26



**Process: How to check  $k$ -Consistency?**

1. Given  $\mathcal{V}$  w/  $\text{dom}(V) = \{v_1, \dots, v_{|\text{dom}(V)|}\} \forall V \in \mathcal{V}$  and  $\mathcal{C}$  w/  $\text{scp}(C) = \{V_1, \dots, V_{|\text{scp}(C)|}\} \forall C \in \mathcal{C}$ .
2. Remove all constraints that have  $k + 1$  or more variables.
3. For each  $C \in \mathcal{C}$ , do the following:
  - (a) For each  $V \in \text{scp}(C)$ , do the following:
    - i. For each  $v \in \text{dom}(V)$ , do the following:
      - Fix  $V$  to  $v$  for  $k - 1$  variables.
      - For the last variable  $V$ , find a value in  $\text{dom}(V)$  that satisfies all constraints.
      - **Key:** If there is one combination that doesn't satisfy the constraint, then the CSP is not  $k$ -consistent.
4. If all constraints are satisfied, then the CSP is  $k$ -consistent.

**Process: How to Enforce  $k$ -Consistency?**

1. Given  $\mathcal{V}$  w/  $\text{dom}(V) = \{v_1, \dots, v_{|\text{dom}(V)|}\} \forall V \in \mathcal{V}$  and  $\mathcal{C}$  w/  $\text{scp}(C) = \{V_1, \dots, V_{|\text{scp}(C)|}\} \forall C \in \mathcal{C}$ .
2. Remove all constraints that have  $k + 1$  or more variables.
3. **Pre-pruning:** For each remaining  $C \in \mathcal{C}$ , do the following:
  - (a) For each  $V \in \text{scp}(C)$ , do the following:
    - i. For each  $v \in \text{dom}(V)$ , do the following:
      - Fix  $V$  to  $v$ .
      - For the other  $V \in \text{scp}(C)$ , check if the constraint is satisfied by trying all combinations (need only one).
      - **Key:** If the constraint is not satisfied, then remove the value from  $\text{dom}(V)$ .
4. If you had to remove any values from  $\text{dom}(V)$ , then check with the other constraints.
5. **Pruning:** Every constraint is satisfied.

**Warning:** Can think of checking as picking  $k - 1$  variables, then choosing any value for the  $k^{\text{th}}$  variable that satisfies all constraints. While enforcing is fixing a variable to a value, then checking if there is a combination for the other variables that satisfies all constraints.

**Warning:** Enforcing  $k$ -consistency is enforcing  $k - 1, \dots, 1$ -consistency.

**Process: How to determine a solution to a CSP?**

1. After pre-pruning the domains.
2. Assign variables in alphabetical order and values in numerical order. Prune the pre-pruned domains.
3. If you can assign all variables, then you have a solution.
4. Repeat the process until you find all solutions.

### Example: Pre-Pruning Domains

- $\mathcal{V} = \left\{ \text{wheat}, \text{meat}, \text{carrots} \right\}$
- $\text{dom} \left( \text{wheat} \right) = \{1, 2, 3\}$
- $\text{dom} \left( \text{carrots} \right) = \{2, 3, 4\}$
- $\text{dom} \left( \text{meat} \right) = \{1, 2, 4\}$
- $\mathcal{C} = \left\{ \underbrace{\text{wheat} + \text{carrots}}_C = \text{meat} \right\}$

Figure 27

- $\text{dom} \left( \text{wheat} \right) = \{1, 2, \cancel{3}\}$
- $\text{dom} \left( \text{carrots} \right) = \{2, 3, \cancel{4}\}$
- $\text{dom} \left( \text{meat} \right) = \{\cancel{1}, \cancel{2}, 4\}$

Figure 28: Pre-pruning. Since only one constraint, it is also pruning.

**Example:**

1. **Given:** Consider a CSP in which  $\mathcal{V} = \{A, B, C, D, E\}$ , where:

$$\text{dom}(A) = \{0, 1, 2, 3, 4\}$$

$$\text{dom}(B) = \{0, 1, 2, 3, 4\}$$

$$\text{dom}(C) = \{0, 1, 2, 3\}$$

$$\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}$$

$$\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}$$

and  $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ , where:

$$C_1 : 8 \leq 2a + 2b + 2d + 3e \leq 10$$

$$C_2 : 3 \leq a + c \leq 4$$

$$C_3 : 2 \leq a + b + 2c \leq 5$$

$$C_4 : 1 \leq b \leq 2$$

2. **Problem:** Solve the following CSP using  $k = 4$  consistency. Pre-prune the domains using  $k = 4$  consistency. Assign variables in alphabetical order and values in numerical order.

**Example: 4-Consistency Pre-Pruning**

Fixed Value	Satisfactory Combination?
$C_4 : 1 \leq b \leq 2$	
$b = 0, b = 1, b = 2, b = 3, b = 4$	No, Yes, Yes, No, No
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, 2, 3\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
$C_2 : 3 \leq a + c \leq 4$	
$a = 0, a = 1, a = 2, a = 3, a = 4$	Yes, Yes, Yes, Yes, Yes
$c = 0, c = 1, c = 2, c = 3$	Yes, Yes, Yes, Yes
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, 2, 3\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
$C_3 : 2 \leq a + b + 2c \leq 5$	
$a = 0, a = 1, a = 2, a = 3, a = 4$	Yes, Yes, Yes, Yes, Yes
$b = 1, b = 2$	Yes, Yes
$c = 0, c = 1, c = 2, c = 3$	Yes, Yes, Yes, No
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, 2, \cancel{3}\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	

**Example: 4-Consistency Pre-Pruning Continued:**

Fixed Value	Satisfactory Combination?
$C_2 : 3 \leq a + c \leq 4$	
$a = 0, a = 1, a = 2, a = 3, a = 4$ $c = 0, c = 1, c = 2$	No, Yes, Yes, Yes, Yes Yes, Yes, Yes
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{\emptyset, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, 2, \cancel{3}\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
$C_3 : 2 \leq a + b + 2c \leq 5$	
$a = 1, a = 2, a = 3, a = 4$ $b = 1, b = 2$ $c = 0, c = 1, c = 2$	Yes, Yes, Yes, Yes Yes, Yes Yes, Yes, No
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{\emptyset, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, \cancel{2}, \cancel{3}\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
$C_2 : 3 \leq a + c \leq 4$	
$a = 1, a = 2, a = 3, a = 4$ $c = 0, c = 1$	No, Yes, Yes, Yes Yes, Yes
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{\emptyset, \cancel{1}, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, \cancel{2}, \cancel{3}\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
$C_3 : 2 \leq a + b + 2c \leq 5$	
$a = 2, a = 3, a = 4$ $b = 1, b = 2$ $c = 0, c = 1$	Yes, Yes, Yes Yes, Yes Yes, Yes
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{\emptyset, \cancel{1}, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, \cancel{2}, \cancel{3}\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	

**Example: 4-Consistency Pre-Pruning Continued:**

Fixed Value	Satisfactory Combination?
$C_1 : 8 \leq 2a + 2b + 2d + 3e \leq 10$	
$a = 2, a = 3, a = 4$	Yes, Yes, Yes
$b = 1, b = 2$	Yes, Yes
$d = 0, d = 1, d = 2, d = 3, d = 4, d = 5$	Yes, Yes, Yes, No, No, No
$e = 0, e = 1, e = 2, e = 3, e = 4, e = 5, e = 6$	Yes, Yes, No, No, No, No
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{\emptyset, \cancel{1}, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, \cancel{2}, \cancel{3}\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, \cancel{3}, \cancel{4}, \cancel{5}\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}\}</math></li> </ul>	

4. **Conclusion:** This is the final pruned domain, which means if I fix a variable to a value, then I can find a satisfactory combination for the rest of the variables in this pruned domain.
- $\text{dom}(A) = \{2, 3, 4\}$
  - $\text{dom}(B) = \{1, 2\}$
  - $\text{dom}(C) = \{0, 1\}$
  - $\text{dom}(D) = \{0, 1, 2\}$
  - $\text{dom}(E) = \{0, 1\}$

**Example: 4-Consistency Pre-Pruning**

Queue	Fixed Value	Satisfactory Combination?
$C_4 : 1 \leq b \leq 2$		
$\{C_2, C_3, C_1\}$	$b = 0, b = 1, b = 2, b = 3, b = 4$	No, Yes, Yes, No, No
<ul style="list-style-type: none"> <li><math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li><math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li><math>\text{dom}(C) = \{0, 1, 2, 3\}</math></li> <li><math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li><math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> <li><math>\{C_2, C_3, C_1\}</math></li> </ul>		
$C_2 : 3 \leq a + c \leq 4$		
$\{C_3, C_1\}$	$a = 0, a = 1, a = 2, a = 3, a = 4$	Yes, Yes, Yes, Yes, Yes
-	$c = 0, c = 1, c = 2, c = 3$	Yes, Yes, Yes, Yes
<ul style="list-style-type: none"> <li><math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li><math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li><math>\text{dom}(C) = \{0, 1, 2, 3\}</math></li> <li><math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li><math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> <li><math>\{C_3, C_1\}</math></li> </ul>		
$C_3 : 2 \leq a + b + 2c \leq 5$		
$\{C_1\}$	$a = 0, a = 1, a = 2, a = 3, a = 4$	Yes, Yes, Yes, Yes, Yes
-	$b = 1, b = 2$	Yes, Yes
-	$c = 0, c = 1, c = 2, c = 3$	Yes, Yes, Yes, No
<ul style="list-style-type: none"> <li><math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li><math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li><math>\text{dom}(C) = \{0, 1, 2, \cancel{3}\}</math></li> <li><math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li><math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> <li><math>\{C_1, C_2\}</math></li> </ul>		

**Example: 4-Consistency Continued:**

Queue	Fixed Value	Satisfactory Combination?
$C_1 : 8 \leq 2a + 2b + 2d + 3e \leq 10$		
$\{C_2\}$	$a = 0, a = 1, a = 2, a = 3, a = 4$	Yes, Yes, Yes, Yes, Yes
-	$b = 1, b = 2$	Yes, Yes
-	$d = 0, d = 1, d = 2, d = 3, d = 4, d = 5$	Yes, Yes, Yes, Yes, Yes, No
-	$e = 0, e = 1, e = 2, e = 3, e = 4, e = 5, e = 6$	Yes, Yes, Yes, No, No, No, No
<ul style="list-style-type: none"> <li><math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li><math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li><math>\text{dom}(C) = \{0, 1, 2, \cancel{3}\}</math></li> <li><math>\text{dom}(D) = \{0, 1, 2, 3, 4, \cancel{5}\}</math></li> <li><math>\text{dom}(E) = \{0, 1, 2, \cancel{3}, \cancel{4}, \cancel{5}, \emptyset\}</math></li> <li><math>\{C_2\}</math></li> </ul>		
$C_2 : 3 \leq a + c \leq 4$		
$\{\}$	$a = 0, a = 1, a = 2, a = 3, a = 4$	No, Yes, Yes, Yes, Yes
-	$c = 0, c = 1, c = 2$	Yes, Yes, Yes
<ul style="list-style-type: none"> <li><math>\text{dom}(A) = \{\emptyset, 1, 2, 3, 4\}</math></li> <li><math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li><math>\text{dom}(C) = \{0, 1, 2, \cancel{3}\}</math></li> <li><math>\text{dom}(D) = \{0, 1, 2, 3, 4, \cancel{5}\}</math></li> <li><math>\text{dom}(E) = \{0, 1, 2, \cancel{3}, \cancel{4}, \cancel{5}, \emptyset\}</math></li> <li><math>\{C_3, C_1\}</math></li> </ul>		
$C_3 : 2 \leq a + b + 2c \leq 5$		
$\{C_1\}$	$a = 1, a = 2, a = 3, a = 4$	Yes, Yes, Yes, Yes
-	$b = 1, b = 2$	Yes, Yes
-	$c = 0, c = 1, c = 2$	Yes, Yes, No
<ul style="list-style-type: none"> <li><math>\text{dom}(A) = \{\emptyset, 1, 2, 3, 4\}</math></li> <li><math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li><math>\text{dom}(C) = \{0, 1, \cancel{2}, \cancel{3}\}</math></li> <li><math>\text{dom}(D) = \{0, 1, 2, 3, 4, \cancel{5}\}</math></li> <li><math>\text{dom}(E) = \{0, 1, 2, \cancel{3}, \cancel{4}, \cancel{5}, \emptyset\}</math></li> <li><math>\{C_1, C_2\}</math></li> </ul>		



**Example: 4-Consistency Continued:**

Queue	Fixed Value	Satisfactory Combination?
$C_1 : 8 \leq 2a + 2b + 2d + 3e \leq 10$		
$\{C_2\}$	$a = 1, a = 2, a = 3, a = 4$	Yes, Yes, Yes, Yes
-	$b = 1, b = 2$	Yes, Yes
-	$d = 0, d = 1, d = 2, d = 3, d = 4$	Yes, Yes, Yes, Yes, No
-	$e = 0, e = 1, e = 2$	Yes, Yes, Yes
<ul style="list-style-type: none"> <li><math>\text{dom}(A) = \{\emptyset, 1, 2, 3, 4\}</math></li> <li><math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li><math>\text{dom}(C) = \{0, 1, \cancel{2}, \cancel{3}\}</math></li> <li><math>\text{dom}(D) = \{0, 1, 2, 3, \cancel{4}, \cancel{5}\}</math></li> <li><math>\text{dom}(E) = \{0, 1, 2, \cancel{3}, \cancel{4}, \cancel{5}, \emptyset\}</math></li> <li><math>\{C_2\}</math></li> </ul>		
$C_2 : 3 \leq a + c \leq 4$		
$\{\}$	$a = 1, a = 2, a = 3, a = 4$	No, Yes, Yes, Yes
-	$c = 0, c = 1$	Yes, Yes
<ul style="list-style-type: none"> <li><math>\text{dom}(A) = \{\emptyset, \cancel{1}, 2, 3, 4\}</math></li> <li><math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li><math>\text{dom}(C) = \{0, 1, \cancel{2}, \cancel{3}\}</math></li> <li><math>\text{dom}(D) = \{0, 1, 2, 3, \cancel{4}, \cancel{5}\}</math></li> <li><math>\text{dom}(E) = \{0, 1, 2, \cancel{3}, \cancel{4}, \cancel{5}, \emptyset\}</math></li> <li><math>\{C_3, C_1\}</math></li> </ul>		
$C_3 : 2 \leq a + b + 2c \leq 5$		
$\{C_1\}$	$a = 2, a = 3, a = 4$	Yes, Yes, Yes
-	$b = 1, b = 2$	Yes, Yes
-	$c = 0, c = 1$	Yes, Yes
<ul style="list-style-type: none"> <li><math>\text{dom}(A) = \{\emptyset, \cancel{1}, 2, 3, 4\}, \text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li><math>\text{dom}(C) = \{0, 1, \cancel{2}, \cancel{3}\}, \text{dom}(D) = \{0, 1, 2, 3, \cancel{4}, \cancel{5}\}</math></li> <li><math>\text{dom}(E) = \{0, 1, 2, \cancel{3}, \cancel{4}, \cancel{5}, \emptyset\}</math></li> <li><math>\{C_1\}</math></li> </ul>		
$C_1 : 8 \leq 2a + 2b + 2d + 3e \leq 10$		
$\{\}$	$a = 2, a = 3, a = 4$	Yes, Yes, Yes
-	$b = 1, b = 2$	Yes, Yes
-	$d = 0, d = 1, d = 2, d = 3$	Yes, Yes, Yes, No
-	$e = 0, e = 1, e = 2$	Yes, Yes, No
<ul style="list-style-type: none"> <li><math>\text{dom}(A) = \{\emptyset, \cancel{1}, 2, 3, 4\}, \text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li><math>\text{dom}(C) = \{0, 1, \cancel{2}, \cancel{3}\}, \text{dom}(D) = \{0, 1, 2, \cancel{3}, \cancel{4}, \cancel{5}\}</math></li> <li><math>\text{dom}(E) = \{0, 1, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{5}, \emptyset\}</math></li> <li><math>\{\}</math></li> </ul>		

**Example: 4-Consistency Post-Pre-Pruning:**

$$C_1 : 8 \leq 2a + 2b + 2d + 3e \leq 10$$

$$C_2 : 3 \leq a + c \leq 4$$

$$C_3 : 2 \leq a + b + 2c \leq 5$$

$$C_4 : 1 \leq b \leq 2$$

Solution	Updated Necessary Domains After Assignment
$A = 2$	$\text{dom}(B) = \{1, 2\}, \text{dom}(C) = \{\emptyset, 1\}, \text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 2, B = 1$	$\text{dom}(C) = \{\emptyset, 1\}, \text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 2, B = 1, C = 1$	$\text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 2, B = 1, C = 1, D = 0$	$\text{dom}(E) = \{\emptyset, 1\}$
$A = 2, B = 1, C = 1, D = 0, E = 1$	<b>Solution Found</b>
$A = 2$	$\text{dom}(B) = \{1, 2\}, \text{dom}(C) = \{\emptyset, 1\}, \text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 2, B = 1$	$\text{dom}(C) = \{\emptyset, 1\}, \text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 2, B = 1, C = 1$	$\text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 2, B = 1, C = 1, D = 1$	$\text{dom}(E) = \{0, \cancel{1}\}$
$A = 2, B = 1, C = 1, D = 1, E = 0$	<b>Solution Found</b>
$A = 2$	$\text{dom}(B) = \{1, 2\}, \text{dom}(C) = \{\emptyset, 1\}, \text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 2, B = 1$	$\text{dom}(C) = \{\emptyset, 1\}, \text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 2, B = 1, C = 1$	$\text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 2, B = 1, C = 1, D = 2$	$\text{dom}(E) = \{0, \cancel{1}\}$
$A = 2, B = 1, C = 1, D = 2, E = 0$	<b>Solution Found</b>
$A = 2$	$\text{dom}(B) = \{1, 2\}, \text{dom}(C) = \{\emptyset, 1\}, \text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 2, B = 2$	$\text{dom}(C) = \{\emptyset, \cancel{1}\}, \text{dom}(D) = \{0, 1, \cancel{2}\}, \text{dom}(E) = \{0, \cancel{1}\}$
-	<b>No Solution Found</b>
$A = 3$	$\text{dom}(B) = \{1, 2\}, \text{dom}(C) = \{0, 1\}, \text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 3, B = 1$	$\text{dom}(C) = \{0, \cancel{1}\}, \text{dom}(D) = \{0, 1, \cancel{2}\}, \text{dom}(E) = \{0, \cancel{1}\}$
$A = 3, B = 1, C = 0$	$\text{dom}(D) = \{0, 1, \cancel{2}\}, \text{dom}(E) = \{0, \cancel{1}\}$
$A = 3, B = 1, C = 0, D = 0$	$\text{dom}(E) = \{0, \cancel{1}\}$
$A = 3, B = 1, C = 0, D = 0, E = 0$	<b>Solution Found</b>
$A = 3$	$\text{dom}(B) = \{1, 2\}, \text{dom}(C) = \{0, 1\}, \text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 3, B = 1$	$\text{dom}(C) = \{0, \cancel{1}\}, \text{dom}(D) = \{0, 1, \cancel{2}\}, \text{dom}(E) = \{0, \cancel{1}\}$
$A = 3, B = 1, C = 0$	$\text{dom}(D) = \{0, 1, \cancel{2}\}, \text{dom}(E) = \{0, \cancel{1}\}$
$A = 3, B = 1, C = 0, D = 1$	$\text{dom}(E) = \{0, \cancel{1}\}$
$A = 3, B = 1, C = 0, D = 1, E = 0$	<b>Solution Found</b>
$A = 3$	$\text{dom}(B) = \{1, 2\}, \text{dom}(C) = \{0, 1\}, \text{dom}(D) = \{0, 1, 2\}, \text{dom}(E) = \{0, 1\}$
$A = 3, B = 2$	$\text{dom}(C) = \{0, \cancel{1}\}, \text{dom}(D) = \{0, \cancel{1}, \cancel{2}\}, \text{dom}(E) = \{0, \cancel{1}\}$
$A = 3, B = 2, C = 0$	$\text{dom}(D) = \{0, \cancel{1}, \cancel{2}\}, \text{dom}(E) = \{0, \cancel{1}\}$
$A = 3, B = 2, C = 0, D = 0$	$\text{dom}(E) = \{0, \cancel{1}\}$
$A = 3, B = 2, C = 0, D = 0, E = 0$	<b>Solution Found</b>
$A = 4$	$\text{dom}(B) = \{1, \cancel{2}\}, \text{dom}(C) = \{0, \cancel{1}\}, \text{dom}(D) = \{0, 1, \cancel{2}\}, \text{dom}(E) = \{0, \cancel{1}\}$
$A = 4, B = 1$	$\text{dom}(C) = \{0, \cancel{1}\}, \text{dom}(D) = \{0, \cancel{1}, \cancel{2}\}, \text{dom}(E) = \{0, \cancel{1}\}$
$A = 4, B = 1, C = 0$	$\text{dom}(D) = \{0, \cancel{1}, \cancel{2}\}, \text{dom}(E) = \{0, \cancel{1}\}$
$A = 4, B = 1, C = 0, D = 0$	$\text{dom}(E) = \{0, \cancel{1}\}$
$A = 4, B = 1, C = 0, D = 0, E = 0$	<b>Solution Found</b>

**Example: 3-Consistency**

Fixed Value	Satisfactory Combination?
$C_4 : 1 \leq b \leq 2$	
$b = 0, b = 1, b = 2, b = 3, b = 4$	No, Yes, Yes, No, No
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, 2, 3\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
$C_2 : 3 \leq a + c \leq 4$	
$a = 0, a = 1, a = 2, a = 3, a = 4$	Yes, Yes, Yes, Yes, Yes
$c = 0, c = 1, c = 2, c = 3$	Yes, Yes, Yes, Yes
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, 2, 3\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
$C_3 : 2 \leq a + b + 2c \leq 5$	
$a = 0, a = 1, a = 2, a = 3, a = 4$	Yes, Yes, Yes, Yes, Yes
$b = 1, b = 2$	Yes, Yes
$c = 0, c = 1, c = 2, c = 3$	Yes, Yes, Yes, No
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, 2, \cancel{3}\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	

**Example:**

Fixed Value	Satisfactory Combination?
$C_2 : 3 \leq a + c \leq 4$	
$a = 0, a = 1, a = 2, a = 3, a = 4$ $c = 0, c = 1, c = 2$	No, Yes, Yes, Yes, Yes Yes, Yes, Yes
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{\emptyset, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, 2, \cancel{3}\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
$C_3 : 2 \leq a + b + 2c \leq 5$	
$a = 1, a = 2, a = 3, a = 4$ $b = 1, b = 2$ $c = 0, c = 1, c = 2$	Yes, Yes, Yes, Yes Yes, Yes Yes, Yes, No
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{\emptyset, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, \cancel{2}, \cancel{3}\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
$C_2 : 3 \leq a + c \leq 4$	
$a = 1, a = 2, a = 3, a = 4$ $c = 0, c = 1$	No, Yes, Yes, Yes Yes, Yes
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{\emptyset, \cancel{1}, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, \cancel{2}, \cancel{3}\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
$C_3 : 2 \leq a + b + 2c \leq 5$	
$a = 2, a = 3, a = 4$ $b = 1, b = 2$ $c = 0, c = 1$	Yes, Yes, Yes Yes, Yes Yes, Yes
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{\emptyset, \cancel{1}, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, \cancel{2}, \cancel{3}\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
4. <b>Conclusion:</b> $\text{dom}(A) = \{2, 3, 4\}$ , $\text{dom}(B) = \{1, 2\}$ , $\text{dom}(C) = \{0, 1\}$ , $\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}$ , $\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}$	

**Example: 2-Consistency**

Fixed Value	Satisfactory Combination?
$C_4 : 1 \leq b \leq 2$	
$b = 0, b = 1, b = 2, b = 3, b = 4$	No, Yes, Yes, No, No
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, 2, 3\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
$C_2 : 3 \leq a + c \leq 4$	
$a = 0, a = 1, a = 2, a = 3, a = 4$	Yes, Yes, Yes, Yes, Yes
$c = 0, c = 1, c = 2, c = 3$	Yes, Yes, Yes, Yes
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, 2, 3\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
<p>4. <b>Conclusion:</b> <math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math>, <math>\text{dom}(B) = \{1, 2\}</math>, <math>\text{dom}(C) = \{0, 1, 2, 3\}</math>, <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math>, <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></p>	

**Example: 1-Consistency**

Fixed Value	Satisfactory Combination?
$C_4 : 1 \leq b \leq 2$	
$b = 0, b = 1, b = 2, b = 3, b = 4$	No, Yes, Yes, No, No
<ul style="list-style-type: none"> <li>• <math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math></li> <li>• <math>\text{dom}(B) = \{\emptyset, 1, 2, \cancel{3}, \cancel{4}\}</math></li> <li>• <math>\text{dom}(C) = \{0, 1, 2, 3\}</math></li> <li>• <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math></li> <li>• <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></li> </ul>	
<p>4. <b>Conclusion:</b> <math>\text{dom}(A) = \{0, 1, 2, 3, 4\}</math>, <math>\text{dom}(B) = \{1, 2\}</math>, <math>\text{dom}(C) = \{0, 1, 2, 3\}</math>, <math>\text{dom}(D) = \{0, 1, 2, 3, 4, 5\}</math>, <math>\text{dom}(E) = \{0, 1, 2, 3, 4, 5, 6\}</math></p>	

## Learning Problems

**Definition:** Assume that there is some (unknown) relationship,

$$f : \mathcal{X} \rightarrow \mathcal{Y} \text{ s.t. } x \mapsto_f y$$

- $\mathcal{X}$ : Input Space
- $\mathcal{Y}$ : Output Space (i.e. information we desire about input)

Find  $h : \mathcal{X} \rightarrow \mathcal{Y}$  (hypothesis) s.t.  $h \approx f$ , given some data about  $f$ :

$$\mathcal{D} = \left\{ \left( x^{(i)}, y_i \right), x^{(i)} \in \mathcal{X}, y_i = f \left( x^{(i)} \right) \in \mathcal{Y}, i = 1 \dots N \right\}$$

- $\text{in}(\mathcal{D}) = \{x \text{ s.t. } (x, y) \in \mathcal{D}\}$
- $\text{out}(\mathcal{D}) = \{y \text{ s.t. } (x, y) \in \mathcal{D}\}$

### 3.6 Classification vs. Regression Problems

**Definition:**

- **Classification Problems:**  $\mathcal{X} \subseteq \mathbb{R}^M$  and  $\mathcal{Y} \subseteq \mathbb{N}$
- **Regression Problems:**  $\mathcal{X} \subseteq \mathbb{R}^M$  and  $\mathcal{Y} \subseteq \mathbb{R}$

### 3.7 Feature Spaces

**Definition:** It is often easier to learn relationships from high-level features (instead of the raw input). Need mapping b/w input space and feature space:

$$\phi : \mathcal{X} \rightarrow \mathcal{F}$$

## 4 PAC Learning

### 4.1 Probably Approximately Correct (PAC) Estimations

#### 4.1.1 Hoeffding's Inequality

**Definition:** For any  $\epsilon > 0$ ,

$$\mathbb{P}(|\nu - \mu| \geq \epsilon) \leq 2e^{-2\epsilon^2 N} \quad (1)$$

- $\mu$ : Probability of an event.
- $\nu$ : Relative frequency in a sample size  $N$ .
- $\mu \stackrel{?}{\approx} \nu$ :  $\mu$  is probably approximately equal to  $\nu$ . As  $N \rightarrow \infty$ :  $\nu \rightarrow \mu$
- $\epsilon$ : Tolerance (i.e. how close we want  $\nu$  to be to  $\mu$ ).
  - $\epsilon \rightarrow 0$ :  $\nu = \mu$

**Warning:** Approximate the true distribution with high probability by taking a large enough sample size (i.e. empirical distribution converges to true distribution).

### 4.2 PAC Learning

#### 4.2.1 Error

**Definition:**

- **Out-Sample Error:**

$$E_{\text{out}} = \mathbb{P}[f \neq h]$$

- **In-Sample Error:**

$$E_{\text{in}} = \frac{1}{N} \sum_{i=1}^N \mathbb{I}[f(x^{(i)}) \neq h(x^{(i)})]$$

#### 4.2.2 Union Bound Theorem

**Theorem:**

$$\mathbb{P}[E_1 \vee \dots \vee E_M] \leq \sum_{i=1}^M \mathbb{P}[E_i]$$

**Warning:** If the events are highly correlated, then the union bound is not tight.

#### 4.2.3 Generalization of Hoeffding's Inequality

**Definition:** Assuming that  $h$  is chosen from a set of hypotheses  $\mathcal{H}$ , derive a (loose) upper-bound on  $|E_{\text{out}} - E_{\text{in}}|$ :

$$\begin{aligned} \mathbb{P} \left[ \bigvee_{h \in \mathcal{H}} (|E_{\text{out}} - E_{\text{in}}(h)| > \epsilon) \right] &\leq \sum_{h \in \mathcal{H}} \mathbb{P}[|E_{\text{out}} - E_{\text{in}}(h)| > \epsilon] \\ &\leq \sum_{h \in \mathcal{H}} 2e^{-2\epsilon^2 N} \\ &= 2|\mathcal{H}|e^{-2\epsilon^2 N} \end{aligned}$$

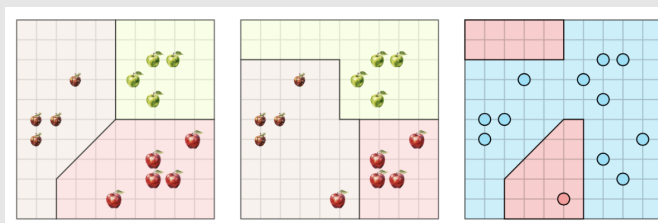
- Endow  $\mathcal{F}$  w/ prob. distribution,  $P : \mathcal{X} \rightarrow [0, 1]$ , then
  - $E_{\text{out}}$  is analogous to  $\mu$
  - $E_{\text{in}}(h)$  is analogous to  $\nu$ .

**Notes:**

- $E_{\text{in}}(h) \stackrel{?}{\approx} E_{\text{out}}$  requires small  $|\mathcal{H}|$  (generalization)
- $E_{\text{in}}(h) \approx 0$  requires large  $|\mathcal{H}|$  (discrimination)

**Example:**

1. **Given:** An opaque box containing red and blue balls. Take  $N$  IID samples.
  - $\mu$ : Probability of drawing a blue ball (unknown).
  - $\nu$ : Relative frequency of blue balls in the sample (known).
2. **Problem 1:** What is  $\nu$  in this case? 8 balls total, 5 are blue.
3. **Solution 1:**  $\nu = \frac{5}{8}$
4. **Problem 2:** How to partition  $\mathcal{F}$  into regions where  $f = h$  and  $f \neq h$ ?
5. **Solution 2:**

Figure 29: LS  $h$ , MS  $f$ 

6. **Problem 3:** What is the out-sample error?
7. **Solution 3:** In words, the probability of the hypothesis being wrong.

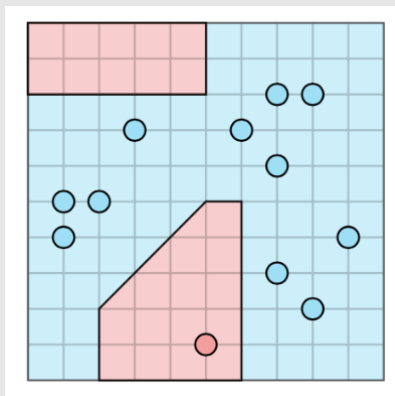


Figure 30

8. **Problem 4:** What is the in-sample error given this sample of 11 balls s.t.  $f = h$ , 1 ball s.t.  $f \neq h$ ?
9. **Solution 4:**  $E_{\text{in}} = \frac{1}{12}$



## 5 Decision Trees

### 5.1 Structure

**Definition:** Each vertex in a decision tree is either:

1. A **condition vertex**: a vertex that sorts points based on a question.
2. A **decision vertex**: a vertex that assigns all points a specific class.

**Notes:** We want to find the minimum # of condition vertices (or questions) needed to "sufficiently discriminate" (identify the class of every point in  $\mathcal{D}$ ).

- More condition vertices improve discrimination.
- Less condition vertices improve generalization.

### 5.2 Building a Decision Tree

**Definition:** Consider determining the class of a randomly chosen target point.

- If we ask a  $K$ -ary question abt. the pts. in  $\mathcal{D}$ , we can form  $K$  subsets,  $\mathcal{D}^{(1)}, \dots, \mathcal{D}^{(K)}$ , using the answers s.t.
  - $|\mathcal{D}^{(k)}| \in \{0, \dots, |\mathcal{D}|\}$
  - $|\mathcal{D}| = \sum_{k=1}^K |\mathcal{D}^{(k)}|$

#### 5.2.1 Special Case

**Notes:** Suppose each pt. belongs to a unique class (i.e. the # of classes is  $|\mathcal{D}|$ ).

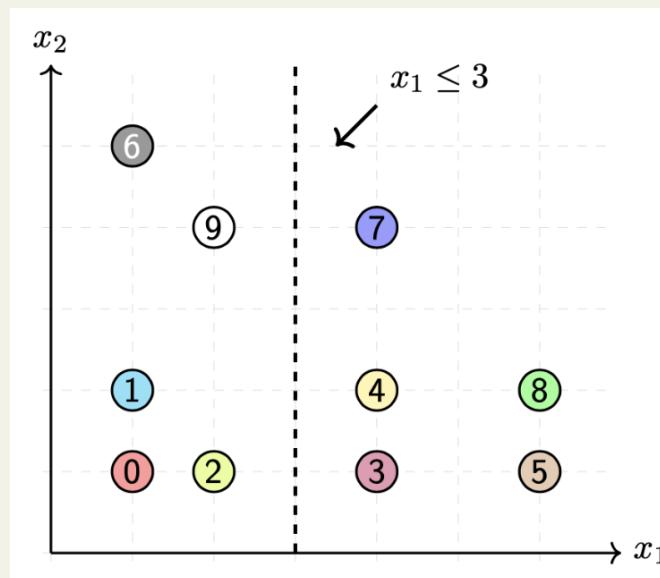


Figure 31

1. Before asking the question:  $|\mathcal{D}|$  possible guesses for the target point's class.
2. After asking the question: Either
  - $|\mathcal{D}^{(1)}|, \dots, |\mathcal{D}^{(K-1)}|$  or
  - $|\mathcal{D}^{(K)}|$
 guesses, depending on the answer for the target point.
3. Goal: Minimize the # of guesses needed in the worst-case, which would be

$$\max\{|\mathcal{D}^{(1)}|, \dots, |\mathcal{D}^{(K)}|\}.$$

4. Given the constraints on  $|\mathcal{D}^{(1)}|, \dots, |\mathcal{D}^{(K)}|$ , we can show that  $\max\{|\mathcal{D}^{(1)}|, \dots, |\mathcal{D}^{(K)}|\}$  is minimized when

$$|\mathcal{D}^{(K)}| \in \left\{ \left\lfloor \frac{|\mathcal{D}|}{K} \right\rfloor, \left\lceil \frac{|\mathcal{D}|}{K} \right\rceil \right\}.$$

Basically, the best question splits  $\mathcal{D}$  into  $K$  sets of (roughly) the same size.

**Warning:** Roughly due to floor/ceil.

**Theorem:** Given a classification data-set,  $\mathcal{D}$ , in which the class of each point is unique (i.e.,  $|\text{out}(\mathcal{D})| = |\mathcal{D}|$ ), the class of a randomly chosen target point can be determined within

$$\lceil \log_K(|\mathcal{D}|) \rceil$$

$K$ -ary questions.

### 5.2.2 General Case

**Motivation:** Suppose points do not necessarily belong to a unique class.

In the context of decision trees:

- $X$  is the class of a randomly chosen target point.
- $Y$  is the answer to a  $K$ -ary question for  $X$ .

Maximize  $IG(X|Y)$  (i.e. choose the question to maximize the information gained).

### 5.2.3 Entropy, Conditional Entropy, and Information Gain

**Definition:** The **entropy** of a random variable  $X$  (in  $K$ -its) is defined as

$$H(X) = - \sum_{\forall x \in X} p_X(x) \log_K(p_X(x)).$$

The **conditional entropy** of a random variable,  $X$ , given a random variable  $Y$ , is

$$H(X|Y) = - \sum_{\forall y \in Y} \sum_{\forall x \in X} p_{X|Y}(x|y) \log_K(p_{X|Y}(x|y)).$$

The **information gain** from  $Y$  is:

$$IG(X|Y) = H(X) - H(X|Y).$$

**Warning:**

- There are  $\infty$  many potential questions, but there are only finite many ways to split the dataset.

**Process:**

1. Calculate  $H(X)$  (i.e. entropy before the split).
2. Calculate  $H(X|Y)$  (i.e. entropy after the split).
  - (a) Calculate entropy for each subset of  $X$  based on the question,  $Y$ .
  - (b) Calculate the weighted average of the entropies.
3. Calculate  $IG(X|Y) = H(X) - H(X|Y)$ .

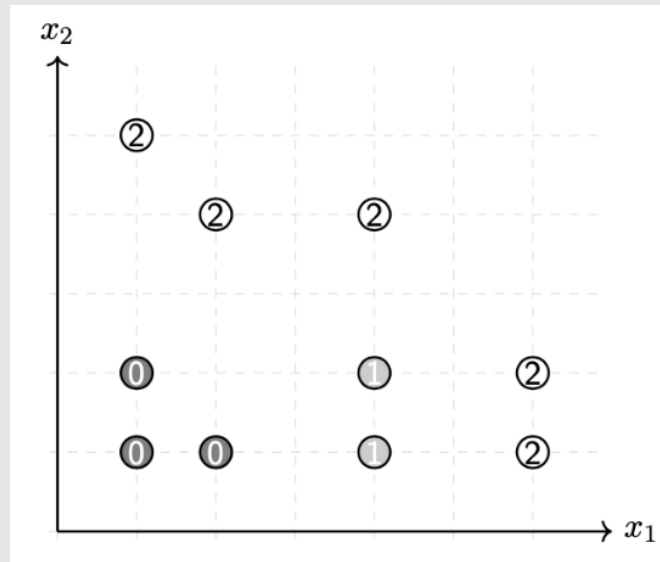
**Example:**

Figure 32

**Example: 2-Ary Question**

1. **Given:**  $X = \{0, 1, 2\}$ ,  $Y = \begin{cases} 1, & \text{if } x_1 \leq 3 \quad (\text{Yes}) \\ 0, & \text{if } x_1 > 3 \quad (\text{No}) \end{cases}$ ,

2. **Problem:**  $IG(X|Y) = ?$

3. **Solution:**

(a) Entropy before the split:  $H(X) = \frac{3}{10} \log_2 \left( \frac{10}{3} \right) + \frac{2}{10} \log_2 \left( \frac{10}{2} \right) + \frac{5}{10} \log_2 \left( \frac{10}{5} \right)$

(b) Entropy after the split:

i.  $H(X | x_1 \leq 3) = \frac{3}{5} \log_2 \left( \frac{5}{3} \right) + \frac{2}{5} \log_2 \left( \frac{5}{2} \right)$

ii.  $H(X | x_1 > 3) = \frac{2}{5} \log_2 \left( \frac{5}{2} \right) + \frac{3}{5} \log_2 \left( \frac{5}{3} \right)$ .

iii. Weighted Avg. Entropy:  $H(X|Y) = \frac{5}{10} H(X | x_1 \leq 3) + \frac{5}{10} H(X | x_1 > 3)$

(c)  $IG(X|Y) = H(X) - H(X|Y)$

**Example: 2-Ary Question**

1. **Given:**  $X = \{0, 1, 2\}$ ,  $Y = \begin{cases} 1, & \text{if } x_2 \leq 3 \quad (\text{Yes}) \\ 0, & \text{if } x_2 > 3 \quad (\text{No}) \end{cases}$ ,

2. **Problem:**  $IG(X|Y) = ?$

3. **Solution:**

(a) Entropy before the split:  $H(X) = \frac{3}{10} \log_2 \left( \frac{10}{3} \right) + \frac{2}{10} \log_2 \left( \frac{10}{2} \right) + \frac{5}{10} \log_2 \left( \frac{10}{5} \right)$

(b) Entropy after the split:

i.  $H(X | x_2 > 3) = \frac{3}{3} \log_2 \left( \frac{3}{3} \right)$

ii.  $H(X | x_2 \leq 3) = \frac{3}{5} \log_2 \left( \frac{5}{3} \right) + \frac{2}{5} \log_2 \left( \frac{5}{2} \right) + \frac{2}{5} \log_2 \left( \frac{5}{2} \right)$ .

iii. Weighted Avg. Entropy:  $H(X|Y) = \frac{3}{10} H(X | x_2 > 3) + \frac{7}{10} H(X | x_2 \leq 3)$

(c)  $IG(X|Y) = H(X) - H(X|Y)$

**Example: 3-Ary Question**

1. **Given:**  $X = \{0, 1, 2\}$ ,  $Y = \begin{cases} 1, & \text{if } x_1 \leq 3 \text{ and } x_2 \leq 3 \\ 2, & \text{if } x_1 \leq 3 \text{ and } x_2 > 3 \\ 3, & \text{if } x_1 > 3 \end{cases}$

2. **Problem:**  $IG(X|Y) = ?$

3. **Solution:**

(a) Entropy before the split:  $H(X) = \frac{3}{10} \log_2 \left( \frac{10}{3} \right) + \frac{2}{10} \log_2 \left( \frac{10}{2} \right) + \frac{5}{10} \log_2 \left( \frac{10}{5} \right)$

(b) Entropy after the split:

i.  $H(X | x_1 \leq 3 \text{ and } x_2 \leq 3) = \frac{3}{3} \log_2 \left( \frac{3}{3} \right)$

ii.  $H(X | x_1 \leq 3 \text{ and } x_2 > 3) = \frac{2}{2} \log_2 \left( \frac{2}{2} \right)$

iii.  $H(X | x_1 > 3) = \frac{2}{5} \log_2 \left( \frac{5}{2} \right) + \frac{3}{5} \log_2 \left( \frac{5}{3} \right)$

iv.  $H(X|Y) = \frac{3}{10} H(X | x_1 \leq 3 \text{ and } x_2 \leq 3) + \frac{2}{10} H(X | x_1 \leq 3 \text{ and } x_2 > 3) + \frac{5}{10} H(X | x_1 > 3)$

(c)  $IG(X|Y) = H(X) - H(X|Y)$

**Example: Decision Tree**

1. **Given:**  $X = \{0, 1, 2\}$
2. **Problem:** Draw a decision tree using binary conditions of the form,  $x_i \leq k$ , where  $i \in \{1, 2\}$  and  $k \in \mathbb{Z}$ , that maximizes the information gained at each level.
3. **Solution (Level 1):**

(a) Entropy before the split:  $H(X) = \frac{3}{10} \log_2 \left( \frac{10}{3} \right) + \frac{2}{10} \log_2 \left( \frac{10}{2} \right) + \frac{5}{10} \log_2 \left( \frac{10}{5} \right) = 1.485[\text{bits}]$

(b) Entropy after the split and information gain (everything in base 2 since 2-ary).

Split	Entropy
$x_1 \leq 1$	$H(X Y) = \frac{3}{10} \left[ \frac{2}{3} \log \left( \frac{3}{2} \right) + \frac{1}{3} \log \left( \frac{3}{1} \right) \right] + \frac{7}{10} \left[ \frac{1}{7} \log \left( \frac{7}{1} \right) + \frac{2}{7} \log \left( \frac{7}{2} \right) + \frac{4}{7} \log \left( \frac{7}{4} \right) \right] = 1.241[\text{bits}]$ <ul style="list-style-type: none"> <li><math>IG(X Y) = 1.485 - 1.241 = 0.244[\text{bits}]</math></li> </ul>
$x_1 \leq 2, 3$	$H(X Y) = \frac{5}{10} \left[ \frac{3}{5} \log \left( \frac{5}{3} \right) + \frac{2}{5} \log \left( \frac{5}{2} \right) \right] + \frac{5}{10} \left[ \frac{2}{5} \log \left( \frac{5}{2} \right) + \frac{3}{5} \log \left( \frac{5}{3} \right) \right] = 0.971[\text{bits}]$ <ul style="list-style-type: none"> <li><math>IG(X Y) = 1.485 - 0.971 = 0.514[\text{bits}]</math></li> </ul>
$x_1 \leq 4, 5$	$H(X Y) = \frac{8}{10} \left[ \frac{3}{8} \log \left( \frac{8}{3} \right) + \frac{2}{8} \log \left( \frac{8}{2} \right) + \frac{3}{8} \log \left( \frac{8}{3} \right) \right] + \frac{2}{10} \left[ \frac{2}{2} \log \left( \frac{2}{2} \right) \right] = 1.249[\text{bits}]$ <ul style="list-style-type: none"> <li><math>IG(X Y) = 1.485 - 1.249 = 0.236[\text{bits}]</math></li> </ul>
$x_1 \leq 6$	$H(X Y) = \frac{10}{10} \left[ \frac{3}{10} \log \left( \frac{10}{3} \right) + \frac{2}{10} \log \left( \frac{10}{2} \right) + \frac{5}{10} \log \left( \frac{10}{5} \right) \right] = 1.485[\text{bits}]$ <ul style="list-style-type: none"> <li><math>IG(X Y) = 1.485 - 1.485 = 0[\text{bits}]</math></li> </ul>
$x_2 \leq 1$	$H(X Y) = \frac{4}{10} \left[ \frac{2}{4} \log \left( \frac{4}{2} \right) + \frac{1}{4} \log \left( \frac{4}{1} \right) + \frac{1}{4} \log \left( \frac{4}{1} \right) \right] + \frac{6}{10} \left[ 2 \cdot \frac{1}{6} \log \left( \frac{6}{1} \right) + \frac{4}{6} \log \left( \frac{6}{4} \right) \right] = 1.351[\text{bits}]$ <ul style="list-style-type: none"> <li><math>IG(X Y) = 1.485 - 1.351 = 0.134[\text{bits}]</math></li> </ul>
$x_2 \leq 2, 3$	$H(X Y) = \frac{7}{10} \left[ \frac{3}{7} \log \left( \frac{7}{3} \right) + \frac{2}{7} \log \left( \frac{7}{2} \right) + \frac{2}{7} \log \left( \frac{7}{2} \right) \right] + \frac{3}{10} \left[ \frac{3}{3} \log \left( \frac{3}{3} \right) \right] = 1.090[\text{bits}]$ <ul style="list-style-type: none"> <li><math>IG(X Y) = 1.485 - 1.090 = 0.395[\text{bits}]</math></li> </ul>
$x_2 \leq 4$	$H(X Y) = \frac{9}{10} \left[ \frac{3}{9} \log \left( \frac{9}{3} \right) + \frac{2}{9} \log \left( \frac{9}{2} \right) + \frac{4}{9} \log \left( \frac{9}{4} \right) \right] + \frac{1}{10} \left[ \frac{1}{1} \log \left( \frac{1}{1} \right) \right] = 1.377[\text{bits}]$ <ul style="list-style-type: none"> <li><math>IG(X Y) = 1.485 - 1.377 = 0.108[\text{bits}]</math></li> </ul>
$x_2 \leq 5$	$H(X Y) = \frac{10}{10} \left[ \frac{3}{10} \log \left( \frac{10}{3} \right) + \frac{2}{10} \log \left( \frac{10}{2} \right) + \frac{5}{10} \log \left( \frac{10}{5} \right) \right] = 1.485[\text{bits}]$ <ul style="list-style-type: none"> <li><math>IG(X Y) = 1.485 - 1.485 = 0[\text{bits}]</math></li> </ul>

**Example: Decision Tree Continued:**

4. **Solution (Level 2):**  $x_1 \leq 2, 3$  has the highest information gain. For clarity, choose  $x_1 \leq 3$  as the question.

(a) Entropy before the split (treat as 2 indep. problems)

$$\text{i. } H(X_L) = \frac{3}{5} \log\left(\frac{5}{3}\right) + \frac{2}{5} \log\left(\frac{5}{2}\right) = 0.971$$

$$\text{ii. } H(X_R) = \frac{2}{5} \log\left(\frac{5}{2}\right) + \frac{3}{5} \log\left(\frac{5}{3}\right) = 0.971$$

(b) Entropy after the split and information gain (everything in base 2 since 2-ary).

**Split      Entropy**

**Left Split**

$$x_1 \leq 1 \quad H(X_L|Y) = \frac{3}{5} \left[ \frac{2}{3} \log\left(\frac{3}{2}\right) + \frac{1}{3} \log\left(\frac{3}{1}\right) \right] + \frac{2}{5} \left[ \frac{1}{2} \log\left(\frac{1}{2}\right) + \frac{1}{2} \log\left(\frac{1}{2}\right) \right] = 0.151[\text{bits}]$$

$$\bullet \quad IG(X|Y) = 0.971 - 0.151 = 0.820[\text{bits}]$$

$$x_2 \leq 1 \quad H(X_L|Y) = \frac{2}{5} \left[ \frac{2}{2} \log\left(\frac{2}{2}\right) \right] + \frac{3}{5} \left[ \frac{1}{3} \log\left(\frac{3}{1}\right) + \frac{2}{3} \log\left(\frac{3}{2}\right) \right] = 0.551[\text{bits}]$$

$$\bullet \quad IG(X|Y) = 0.971 - 0.551 = 0.420[\text{bits}]$$

$$x_2 \leq 2, 3 \quad H(X_L|Y) = \frac{3}{5} \left[ \frac{3}{3} \log\left(\frac{3}{3}\right) \right] + \frac{2}{5} \left[ \frac{2}{2} \log\left(\frac{2}{2}\right) \right] = 0[\text{bits}]$$

$$\bullet \quad \boxed{IG(X_L|Y) = 0.971 - 0 = 0.971[\text{bits}]}$$

**Right Split**

$$x_1 \leq 4, 5 \quad H(X_R|Y) = \frac{3}{5} \left[ \frac{2}{3} \log\left(\frac{3}{2}\right) + \frac{1}{3} \log\left(\frac{3}{1}\right) \right] + \frac{2}{5} \left[ \frac{2}{2} \log\left(\frac{2}{2}\right) \right] = 0.551[\text{bits}]$$

$$\bullet \quad \boxed{IG(X_L|Y) = 0.971 - 0.551 = 0.420[\text{bits}]}$$

$$x_2 \leq 1 \quad H(X_R|Y) = \frac{2}{5} \left[ \frac{1}{2} \log\left(\frac{2}{1}\right) + \frac{1}{2} \log\left(\frac{2}{1}\right) \right] + \frac{3}{5} \left[ \frac{2}{3} \log\left(\frac{3}{2}\right) + \frac{1}{3} \log\left(\frac{3}{1}\right) \right] = 0.951[\text{bits}]$$

$$\bullet \quad IG(X_L|Y) = 0.971 - 0.951 = 0.020[\text{bits}]$$

$$x_2 \leq 2, 3 \quad H(X_R|Y) = \frac{4}{5} \left[ \frac{2}{4} \log\left(\frac{4}{2}\right) + \frac{2}{4} \log\left(\frac{4}{2}\right) \right] + \frac{1}{5} \left[ \frac{1}{1} \log\left(\frac{1}{1}\right) \right] = 0.8[\text{bits}]$$

$$\bullet \quad IG(X_L|Y) = 0.971 - 0.8 = 0.171[\text{bits}]$$

**Example: Decision Tree Continued:**

5. **Solution (Level 3):**  $x_2 \leq 2, 3$  and  $x_1 \leq 4, 5$  has the highest information gain. For clarity, choose  $x_2 \leq 3$  as the question for the left split and choose  $x_1 \leq 5$  as the question for the right split.

(a) Since 3 are pure splits already, therefore, look at right-left side only.

(b) Entropy before the split for the right-left side

$$\text{i. } H(X_{RL}) = \frac{2}{3} \log \left( \frac{3}{2} \right) + \frac{1}{3} \log \left( \frac{3}{1} \right) = 0.918[\text{bits}]$$

(c) Entropy after the split and information gain (everything in base 2 since 2-ary).

Split	Entropy
$x_2 \leq 1$	$H(X_{RL} Y) = \frac{1}{3} \left[ \frac{1}{1} \log \left( \frac{1}{1} \right) \right] + \frac{2}{3} \left[ \frac{1}{2} \log \left( \frac{2}{1} \right) + \frac{1}{2} \log \left( \frac{2}{1} \right) \right] = 0.667[\text{bits}]$ $\bullet IG(X Y) = 0.971 - 0.667 = 0.304[\text{bits}]$
$x_2 \leq 2, 3$	$H(X_{RL} Y) = \frac{1}{3} \left[ \frac{1}{1} \log \left( \frac{1}{1} \right) \right] + \frac{2}{3} \left[ \frac{2}{2} \log \left( \frac{2}{2} \right) \right] = 0[\text{bits}]$ $\bullet IG(X Y) = 0.971 - 0 = 0.971[\text{bits}]$

6. Now all regions in our graph contain a pure set (one class). Note this took more questions than needed, but IG is a heuristic so its not perfect.

