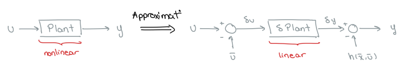
Modelling CS u: control input, y: plant output
State variable CS is in state variable form if
$\dot{x}_1 = f_1(t, x_1, \dots, x_n, u), \dots, \dot{x}_n = f_n(t, x_1, \dots, x_n, u)$
$y = h(t, x_1, \dots, x_n, u)$ is a collection of n 1st order ODEs.
Time-Invariant (TI) CS is TI if $f_i(\cdot)$ does not depend on t.
State space (SS) TI CS is in SS form if $\dot{x} = f(x, u)$, $y = h(x, u)$ where $x(t) \in \mathbb{R}^n$ is called the state.
Single-input-single-output (SISO) CS is SISO if $u(t), y(t) \in \mathbb{R}$.
LTI CS in SS form is LTI if $\dot{x} = Ax + Bu,\ y = Cx + Du$
$A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}$
where $x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m,\ y(t) \in \mathbb{R}^p$.
Input-Output (IO) LTI CS is in IO form if
$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdot + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^m} + \cdot + b_1\frac{du}{dt} + b_0 u$
where $m \le n$ (causality)

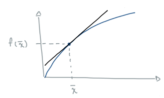
IO to SS Model 1. Define x s.t. highest order derivative in $\dot{x}$
2. Write $\dot{x} = Ax + Bu = f(x, u)$ by isolating for components of $\dot{x}$
3. Write $y = Cx + Du = h(x, u)$ by setting measurement output y to component of x
Equilibria y_d (steady state) b/c if $y(0) = y_d$ at $t = 0$, then $y(t) = y_d\ \forall t \ge 0$.

Equilibrium pair Consider the system $\dot{x} = f(x, u)$. The pair $(\bar{x}, \bar{u})$ is an equilibrium pair if $f(\bar{x}, \bar{u}) = 0$.
Equilibrium point $\bar{x}$ is an equilibrium point w/ control $u = \bar{u}$.
*If $u = \bar{u}$ and $x(0) = \bar{x}$ then $x(t) = \bar{x}\ \forall t \ge 0$ (i.e. a system that starts at equilibrium remains at equilibrium).
Find Equilibrium Pair/Point 1. Set $f(x, u) = 0$
2. Solve $f(x, u) = 0$ to find $(x, u) = (\bar{x}, \bar{u})$.
3. If specific $u = \bar{u}$, then find $x = \bar{x}$ by solving $f(x, \bar{u}) = 0$.

Linearization of Nonlinear System Consider system $\dot{x} = f(x, u)$ w/ equ. pair $(\bar{x}, \bar{u})$, then error coordinates around equ. pair $\delta x = x - \bar{x}$, $\delta u = u - \bar{u}$, $\delta y = y - h(\bar{x}, \bar{u})$ w/
$\dot{\delta x} = A\delta x + B\delta u$, $A = \frac{\partial f(\bar{x},\bar{u})}{\partial x} \in \mathbb{R}^{n_1 \times n_1}$, $B = \frac{\partial f(\bar{x},\bar{u})}{\partial u} \in \mathbb{R}^{n_1}$,
$\delta y = C\delta x + D\delta u$, $C = \frac{\partial h}{\partial x}(\bar{x}, \bar{u}) \in \mathbb{R}^{1\times n_1}$, $D = \frac{\partial h(\bar{x},\bar{u})}{\partial u} \in \mathbb{R}$
*Only valid at equ. pairs.

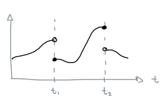


Linear Approx. Given a diff. fcn. $f : \mathbb{R} \to \mathbb{R}$, its linear approx. at $\bar{x}$ is $f_{\text{lin}} = f(\bar{x}) + f'(\bar{x})(x - \bar{x})$.
*Remainder Thm: $f(x) = f_{\text{lin}} + r(x)$ where $\lim_{x\to\bar{x}} \frac{r(x)}{x - \bar{x}} = 0$.

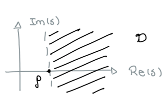


*Note: Can provide a good approx. near $\bar{x}$ but not globally.
*Gen. $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$, $f(x) = f(\bar{x}) + \frac{\partial f}{\partial x}(\bar{x})(x - \bar{x}) + R(x)$
*Jacobian: $\frac{\partial f}{\partial x}(\bar{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial f}{\partial x_{n_1}}(\bar{x}) \end{bmatrix} \in \mathbb{R}^{n_2 \times n_1}$
Linearization Steps 1. Find equ. pair $(\bar{x}, \bar{u})$
2. Derive A, B, C, D and then evaluate at $(\bar{x}, \bar{u})$
3. Write $\dot{\delta x} = A\delta x + B\delta u$ and $\delta y = C\delta x + D\delta u$

Laplace Transform Given a fcn $f : \mathbb{R}_+ \to \mathbb{R}^n$, its Laplace transform is $F(s) = \mathcal{L}\{f(t)\} := \int_{0^-}^{\infty} f(t)e^{-st}\,dt$, $s \in \mathbb{C}$.
*$\mathcal{L} : f(t) \mapsto F(s)$, $t \in \mathbb{R}_+$ (time domain) & $s \in \mathbb{C}$ (Laplace domain).
P.W. CTS: A fcn $f : \mathbb{R}_+ \to \mathbb{R}^n$ is **p.w. cts** if on every finite interval of $\mathbb{R}$, $f(\cdot)$ has at most a finite # of discontinuity points (t_i) and the limits $\lim_{t\to t_i^+} f(t)$, $\lim_{t\to t_i^-} f(t)$ are finite.



Exp. Order A function $f : \mathbb{R}_+ \to \mathbb{R}^n$ is of **exp. order** if $\exists$ constants $K, \rho, T > 0$ s.t. $\|f(t)\| \le Ke^{\rho t}$, $\forall t \ge T$.
Existence of LT Thm If $f(\cdot)$ is p.w. cts and of exp. order w/ constants $K, \rho, T > 0$, then $F(\cdot)$ exists and is defined $\forall\, s \in D := \{s \in \mathbb{C} : \text{Re}(s) > \rho\}$ and $F(\cdot)$ is analytic on D.
*D: Region of convergence (ROC), open half plane.



Unit Step $\mathbf{1}(t) := \begin{cases} 1, & \text{if } t \ge 0 \\ 0, & \text{otherwise} \end{cases}$
Table of Common Laplace Transforms: $f(t)\ |\ F(s)$
$\mathbf{1}(t) \mapsto \frac{1}{s}$ $\quad t\mathbf{1}(t) \mapsto \frac{1}{s^2}$ $\quad t^k\,\mathbf{1}(t) \mapsto \frac{k!}{s^{k+1}}$ $\quad e^{at}\,\mathbf{1}(t) \mapsto \frac{1}{s-a}$
$t^n e^{at}\,\mathbf{1}(t) \mapsto \frac{n!}{(s-a)^{n+1}}$ $\quad \sin(at)\,\mathbf{1}(t) \mapsto \frac{a}{s^2+a^2}$
$\cos(at)\,\mathbf{1}(t) \mapsto \frac{s}{s^2+a^2}$
Prop. of Laplace Transform **Linearity:**
$\mathcal{L}\{cf(t) + g(t)\} = c\mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$, $c \sim$ constant.
Differentiation: If the Laplace transform of $f'(t)$ exists, then $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^-)$.
If the Laplace transform of $f^{(n)}(t) := \frac{d^n f}{dt^n}(t)$ exists, then $\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - \sum_{i=1}^n s^{n-i} f^{(i-1)}(0^-)$.
Integration: $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$.
Convolution: Let $(f * g)(t) := \int_0^t f(\tau)g(t - \tau)d\tau$, then $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$.
Time Delay: $\mathcal{L}\{f(t - T)\mathbf{1}(t - T)\} = e^{-Ts}\mathcal{L}\{f(t)\}$, $t \ge 0$.
Multiplication by t: $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}[\mathcal{L}\{f(t)\}]$.
Shift in s: $\mathcal{L}\{e^{at}f(t)\} = F(s - a) = \mathcal{L}\{f(t)\}\big|_{s\to s-a}$, where $F(s) = \mathcal{L}\{f(t)\}$ & a const.

Cauchy's Residue THM