Intro: **Random Experiment:** An outcome for each run.
Sample Space Ω: Set of all possible outcomes.
Event: Subsets of Ω.
Prob. of Event A: $P(A) = \frac{\text{Number of outcomes in A}}{\text{Number of outcomes in } \Omega}$
Axioms: $P(A) \geq 0 \; \forall A \in \Omega$, $P(\Omega) = 1$,
If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B) \; \forall A, B \in \Omega$
Cond. Prob. $P(A|B) = \frac{P(A \cap B)}{P(B)}$
$* \, P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$
Independence: $P(A|B) = P(A) \Leftrightarrow P(A \cap B) = P(A)P(B)$
Total Prob. Thm: If $H_1, H_2, \ldots, H_n$ form a partition of Ω, then $P(A) = \sum_{i=1}^{n} P(A|H_i)P(H_i)$.
Bayes' Rule: $P(H_k|A) = \frac{P(H_k \cap A)}{P(A)} = \frac{P(A|H_k)P(H_k)}{\sum_{i=1}^{n} P(A|H_i)P(H_i)}$
*Posteriori: $P(H_k|A)$, Likelihood: $P(A|H_k)$, Prior: $P(H_k)$
1 RV: **CDF:** $F_X(x) = P[X \leq x]$
PMF: $P_X(x_j) = P[X = x_j] \; j = 1, 2, \ldots$
PDF: $f_X(x) = \frac{d}{dx} F_X(x)$
*$P[a \leq X \leq b] = \int_a^b f_X(x)\, dx$ IS THIS CORRECT?
Cond. PMF: $P_X(x|A) = P[X = x|A] = \frac{P[X=x, A]}{P[A]}$ IS THIS CORRECT?
Cond. PDF: $f_X(x|A) = \frac{f_{X,A}(x,a)}{f_A(a)}$ IS THIS CORRECT?
Exp.: $E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x) dx \mid \sum_{k=-\infty}^{\infty} h(k) P_X(x_i = k)$

Variance: $\sigma_X^2 = \text{Var}[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$
Cond. Exp.: $E[X|A] = \int_{-\infty}^{\infty} x f_X(x|A)\, dx$
2 RVs: **Joint PMF:** $P_{X,Y}(x, y) = P[X = x, Y = y]$
Joint PDF: $f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$
*$P[(X, Y) \in A] = \int \int_{(x,y) \in A} f_{X,Y}(x, y)\, dx\, dy$
Exp.: $E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dx\, dy$
Correlation (Corr.): $E[XY]$
Covar.: $\text{Cov}[X, Y] = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - E[X]E[Y]$
Corr. Coeff.: $\rho_{X,Y} = E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)\left(\frac{Y - \mu_Y}{\sigma_Y}\right)\right] = \frac{\text{Cov}[X,Y]}{\sigma_X \sigma_Y}$
Marginal PMF: $P_X(x) = \sum_{j=1}^{\infty} P_{X,Y}(x, y_j)$
Marginal PDF: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$
Cond. PMF: $P_{X|Y}(x|Y) = P[X = x|Y = y] = \frac{P_{X,Y}(x,y)}{P_Y(y)}$
Cond. PDF: $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Bayes' Rule
$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_{X|Y}(x|y) f_Y(y)}{\int_{-\infty}^{\infty} f_{X|Y}(x|y') f_Y(y')\, dy'}$
*$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)} = \frac{P_{X|Y}(x|y) P_Y(y)}{\sum_{j=1}^{\infty} P_{X|Y}(x|y_j) P_Y(y_j)}$
Ind.: $f_{X|Y}(x|y) = f_X(x) \; \forall y \Leftrightarrow f_{X,Y}(x, y) = f_X(x) f_Y(y)$
*If independent, then uncorrelated.
Uncorrelated: $\text{Cov}[X, Y] = 0 \Leftrightarrow \rho_{X,Y} = 0$
Orthogonal: $E[XY] = 0$
Cond. Exp.: $E[Y] = E[E[Y|X]]$ or $E[E[h(Y)|X]]$
*$E[E[Y|X]]$ w.r.t. X | $E[Y|X]$ w.r.t. Y.
Estimation: Estimate unknown parameter θ from n i.i.d. measurements $X_1, X_2, \ldots, X_n$, $\hat{\Theta}(\underline{X}) = g(X_1, X_2, \ldots, X_n)$
Estimation Error: $\hat{\Theta}(\underline{X}) - \theta$.
Unbiased: $\hat{\Theta}(\underline{X})$ is unbiased if $E[\hat{\Theta}(\underline{X})] = \theta$.
***Asymptotically unbiased:** $\lim_{n \to \infty} E[\hat{\Theta}(\underline{X})] = \theta$.
Consistent: $\hat{\Theta}(\underline{X})$ is consistent if $\hat{\Theta}(\underline{X}) \to \theta$ as $n \to \infty$ or $\forall \epsilon > 0, \lim_{n \to \infty} P[|\hat{\Theta}(\underline{X}) - \theta| < \epsilon] \to 1$.
Sample Mean: $M_n = \frac{1}{n} S_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.
*Given a sequence of i.i.d. RVs, $X_1, X_2, \ldots, X_n$, M_n is unbiased and consistent.
Chebychev's Inequality: $P[|X - E[X]| \geq \epsilon] \leq \frac{\text{Var}[X]}{\epsilon^2}$
Weak Law of Large #s: $\lim_{n \to \infty} P[|M_n - \mu| < \epsilon] = 1 \; \forall \epsilon > 0$.
ML Estimation: Choose parameter θ that is most likely to generate the obs. $x_1, x_2, \ldots, x_n$.
*Disc: $\hat{\Theta} = \arg\max_\theta P_{\underline{X}}(\underline{x}|\theta) \overset{\log}{\to} \hat{\theta} = \arg\max_\theta \sum_{i=1}^{n} \log P_X(x_i|\theta)$
*Cont: $\hat{\Theta} = \arg\max_\theta f_{\underline{X}}(\underline{x}|\theta) \overset{\log}{\to} \hat{\theta} = \arg\max_\theta \sum_{i=1}^{n} \log f_X(x_i|\theta)$
Maximum A Posteriori (MAP) Estimation:
*Disc: $\hat{\theta} = \arg\max_\theta P_{\Theta|\underline{X}}(\theta|\underline{x}) = \arg\max_\theta P_{\underline{X}|\Theta}(\underline{x}|\theta) P_\Theta(\theta)$
*Cont: $\hat{\theta} = \arg\max_\theta f_{\Theta|\underline{X}}(\theta|\underline{x}) = \arg\max_\theta f_{\underline{X}|\Theta}(\underline{x}|\theta) f_\Theta(\theta)$
*$f_{\Theta|\underline{X}}(\theta|\underline{x})$: Posteriori, $f_{\underline{X}|\Theta}(\underline{x}|\theta)$: Likelihood, $f_\Theta(\theta)$: Prior

Bayes' Rule: $P_{\Theta|\underline{X}}(\theta|\underline{x}) = \begin{cases} \frac{P_{\underline{X}|\Theta}(\underline{x}|\theta) P_\Theta(\theta)}{P_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ disc.} \\ \frac{f_{\underline{X}|\Theta}(\underline{x}|\theta) P_\Theta(\theta)}{f_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ cont.} \end{cases}$

$f_{\Theta|\underline{X}}(\theta|\underline{x}) = \begin{cases} \frac{P_{\underline{X}|\Theta}(\underline{x}|\theta) f_\Theta(\theta)}{P_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ disc.} \\ \frac{f_{\underline{X}|\Theta}(\underline{x}|\theta) f_\Theta(\theta)}{f_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ cont.} \end{cases}$

*Independent of θ: $f_{\underline{X}}(\underline{x}) = \int_{-\infty}^{\infty} f_{\underline{X}|\Theta}(\underline{x}|\theta) f_\Theta(\theta)\, d\theta$

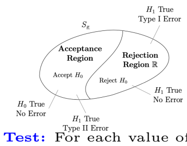
Beta Prior Θ is a Beta R.V. w/ $\alpha, \beta > 0$
$f_\Theta(\theta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1 - \theta)^{\beta-1} & \text{if } 0 < \theta < 1 \\ 0 & \text{otherwise} \end{cases}$
*$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt$
Prop.: 1. $\Gamma(x + 1) = x\Gamma(x)$. For $m \in \mathbb{Z}^+$, $\Gamma(m + 1) = m!$.
2. $\beta(\alpha, \beta) = \frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!} = \beta\binom{\alpha+\beta-1}{\alpha-1}$
3. Expected Value: $E[\Theta] = \frac{\alpha}{\alpha+\beta}$ for $\alpha, \beta > 0$
4. Mode (max of PDF): $\frac{\alpha-1}{\alpha+\beta-2}$ for $\alpha, \beta > 1$

Drawing Beta Dist. 1. Label x-axis from 0 to 1. 2. Identify mode.
3. Determine shape based on α and β: $\alpha = \beta = 1$ (uniform), $\alpha = \beta > 1$ (bell-shaped, peak at 0.5), $\alpha = \beta < 1$ (U-shaped w/ high density near 0 and 1), $\alpha > \beta$ (left-skewed), $\alpha < \beta$ (right-skewed).
Least Mean Squares (LMS) Estimation: Assume prior $P_\Theta(\theta)$ or $f_\Theta(\theta)$ w/ obs. $\underline{X} = \underline{x}$.
*$\hat{\theta} = g(\underline{x}) = \mathbb{E}[\Theta|\underline{X} = \underline{x}]$ | $\hat{\Theta} = g(\underline{X}) = \mathbb{E}[\Theta|\underline{X}]$

Uniform PDF $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$
*$E[X] = \frac{a+b}{2}$, $\text{Var}[X] = \frac{(b-a)^2}{12}$
Conditional Exp. $E[X|Y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx$

Binary Hyp. Testing: H_0: Null Hyp., H_1: Alt. Hyp.

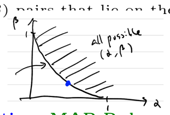
TI Err. (False Rejection): Reject H_0 when H_0 is true.
*$\alpha(R) = P[\underline{X} \in R \mid H_0]$
TII Err. (False Accept.): Accept H_0 when H_1 is true.
*$\beta(R) = P[\underline{X} \in R^c \mid H_1]$



Likelihood Ratio Test: For each value of $\underline{x}$,

*$L(\underline{x}) = \dfrac{P_{\underline{X}}(\underline{x}|H_1)}{P_{\underline{X}}(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$ or ξ

***MLT:** 1, **LRT:** ξ
Neyman-Pearson Lemma: Given a false rejection prob. (α), the LRT offers the smallest possible false accept. prob. (β), and vice versa.
*LRT produces (α, β) pairs that lie on the efficient frontier.



Bayesian Hyp. Testing: MAP Rule:

$L(\underline{x}) = \dfrac{p_{\underline{X}}(\underline{x}|H_1)}{p_{\underline{X}}(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \dfrac{P[H_0]}{P[H_1]}$

Min. Cost Bayes' Dec. Rule: C_{ij} is cost of choosing H_j when H_i is true. Given obs. $\underline{X} = \underline{x}$, the exp. cost of choosing H_j is $A_j(\underline{x}) = \sum_{i=0}^{1} C_{ij}\, P[H_i|\underline{X} = \underline{x}]$.

Min. Cost Dec. Rule: $L(\underline{x}) = \dfrac{P_{\underline{X}}(\underline{x}|H_1)}{P_{\underline{X}}(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \dfrac{(C_{01}-C_{00})P[H_0]}{(C_{10}-C_{11})P[H_1]}$.

*C_{01}: False accept. cost, C_{10}: False reject. cost.
Naive Bayes Assumption: Assume $X_1 \dots, X_n$ (features) are ind., then $p_{\underline{X}|\Theta}(\underline{x} \mid \theta)\Pi_{i=1}^{n} p_X(x_i \mid \theta)$.
Notation: $P_{\underline{X}|\Theta}(\underline{x}|\theta)$, only put RVs in subscript, not values.
$P_{\underline{X}}(\underline{x}|H_i)$, didn't put H in subscript b/c it's not a RV.
Binomial # of successes in n trials, each w/ prob. p
$b(x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, 2, \dots$
*$E[X] = \mu = np \;\mid\; Var(X) = \sigma^2 = np(1-p)$
Multinomial # of x_i successes in n trials, each w/ prob. p_i
$f(x_i \mid p_i \forall i, n) = \frac{n!}{x_1! \dots x_m!} p_1^{x_1} \dots p_m^{x_m}$
*$\sum_i x_i = n$, and $\sum_{i=1}^{m} p_i = 1$
*$E[X_i] = \mu_i = np_i \;\mid\; Var(X_i) = \sigma_i^2 = np_i(1-p_i)$
Hypergeometric # of successes in n draws from N items, k of which are successes

$h(x \mid N, n, k) = \dfrac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$

*$\max\{0, n-(N-k)\} \le x \le \min\{n, k\}$
*$E[X] = \mu = \frac{nk}{N} \;\mid\; Var(X) = \sigma^2 = \frac{N-n}{N-1} \cdot n \cdot \frac{k}{N} \cdot \left(1 - \frac{k}{N}\right)$
Negative Binomial # of trials until k successes, each w/ prob. p
$b^*(x \mid k, p) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$
*$x \ge k, x = k, k+1, \dots$
*$E[X] = \mu = \frac{k}{p} \;\mid\; Var(X) = \sigma^2 = \frac{k(1-p)}{p^2}$
Geometric # of trials until 1st success, each w/ prob. p
$g(x \mid p) = p(1-p)^{x-1}$
*$x \ge 1, x = 1, 2, 3, \dots$
*$E[X] = \mu = \frac{1}{p} \;\mid\; Var(X) = \sigma^2 = \frac{1-p}{p^2}$
Poisson # of events in a fixed interval w/ rate λ
$p(x \mid \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$
*$x \ge 0, x = 0, 1, 2, \dots$
*$E[X] = \mu = \lambda t \;\mid\; Var(X) = \sigma^2 = \lambda t$
Gaussian to Q Fcn: 1. Find $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt$.
2. Use table to find $Q(x)$ for $x \ge 0$.
Random Vector: $\underline{X} = [X_1, \dots, X_n]^T$
Mean Vector: $\underline{m}_{\underline{X}} = E[\underline{X}] = [\mu_1, \dots, \mu_n]^T$

Corr. Mat.: $R_{\underline{X}} = \begin{bmatrix} E[X_1^2] & E[X_1 X_2] & \cdots & E[X_1 X_n] \\ E[X_2 X_1] & E[X_2^2] & \cdots & E[X_2 X_n] \\ \vdots & \vdots & \ddots & \vdots \\ E[X_n X_1] & E[X_n X_2] & \cdots & E[X_n^2] \end{bmatrix}$

*R is real, symmetric, and PSD ($\forall \underline{a}, \underline{a}^T R \underline{a} \ge 0$).

Covar. Mat.: $K_{\underline{X}} = \begin{bmatrix} \mathrm{Var}[X_1] & \cdots & \mathrm{Cov}[X_1, X_n] \\ \mathrm{Cov}[X_2, X_1] & \cdots & \mathrm{Cov}[X_2, X_n] \\ \vdots & \ddots & \vdots \\ \mathrm{Cov}[X_n, X_1] & \cdots & \mathrm{Var}[X_n] \end{bmatrix}$

*$K_{\underline{X}} = R_{\underline{X}} - \underline{m}_{\underline{X}} = R_{\underline{X}} - \underline{m}\underline{m}^T$
*Diagonal $K_{\underline{X}} \iff X_1, \dots, X_n$ are (mutually) uncorrelated.
Lin. Trans. $\underline{Y} = A\underline{X}$ (A rotates and stretches $\underline{X}$)
Mean: $E[\underline{Y}] = A\underline{m}_{\underline{X}}$
Covar. Mat.: $K_{\underline{Y}} = AK_{\underline{X}}A^T$
Diag. Covar. Mat.: For any $\underline{X}$, if $\underline{Y} = P^T \underline{X}$, then $K_{\underline{Y}} = P^T K_{\underline{X}} P = \Lambda$ (i.e. $\underline{Y}$ is uncorrelated)
*$K_{\underline{X}} = P \Lambda P^T$
Find $K_{\underline{Y}}$ 1. Find eigenvalues, norm. eigenvectors of $K_{\underline{X}}$.
2. Set $\underline{Y} = P^T \underline{X}$, $K_{\underline{Y}} = \Lambda$.
PDF of L.T. If $\underline{Y} = A\underline{X}$ w/ A not singular, then
$f_{\underline{Y}}(\underline{y}) = \frac{f_{\underline{X}}(\underline{x})}{|\det A|}\Big|_{\underline{x} = A^{-1}\underline{y}}$
Find $f_{\underline{Y}}(\underline{y})$ 1. Given $f_{\underline{X}}(\underline{x})$, define transformation A
2. Determine $|\det A|$, A^{-1}, then $f_{\underline{Y}}(\underline{y})$.
Gaussian RVs: Analytic Tractability: PDF of jointly Gaussian $X_1, \dots, X_n$ is Guassian vector.
*$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2} |\det \Sigma|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \Sigma^{-1} (\underline{x}-\underline{\mu})}$
*$\underline{\mu} = \underline{m}_{\underline{X}}$, $\Sigma = K_{\underline{X}}$ (if Σ is not singular)

Properties of Guassian Vector: $\underline{X} \sim \mathcal{N}(\underline{\mu}, \Sigma)$
1. PDF is completely determined by $\underline{\mu}$, Σ.
2. $\underline{X}$ uncorrelated $\implies$ $\underline{X}$ independent.
3. Any L.T. $\underline{Y} = A\underline{X} + \underline{b}$ is Gaussian vector w/ $\underline{\mu}_{\underline{Y}} = A\underline{\mu}_{\underline{X}}$, $\Sigma_{\underline{Y}} = A\Sigma_{\underline{X}} A^T$.
4. Any subset of $\{X_i\}$ are jointly Gaussian.
5. Any cond. PDF of a subset of $\{X_i\}$ given the other elements is Gaussian.

Diag. of Guassian Covar. Eigen decomp. of $\Sigma_{\underline{X}}$: $\{\lambda_i\}, \{e_i\}$
$A = [\underline{e}_1, \ldots, \underline{e}_n]^T$, then $\underline{Y} = A\underline{X}$ has $\Sigma_{\underline{Y}} = \Lambda$.
*$\underline{Y}$: Vector of indep. Gaussian RVs.
Guassian Discriminant Analysis: Obs: $\underline{X} = \underline{x} = (x_1, \ldots, x_D)$
Hyp: $\underline{x}$ is gen. by $\mathcal{N}(\underline{\mu}_c, \Sigma_c)$, $c \in C$
Dec: Which "Guassian bump" generated $\underline{x}$?
Prior: $P[C = c] = \pi_c$ (Gaussian Mixture Model)

MAP Rule: $\hat{c} = \arg\max_c P_C[c|\underline{X} = \underline{x}]$
$\hat{c} = \arg\max_c f_{\underline{X}|C}(\underline{x} \mid c)\pi_c$

LGD: $\Sigma_c = \Sigma \ \forall c$, find c w/ best $\underline{\mu}_c$
$\hat{c} = \arg\max_c \underline{\beta}_c^T \underline{x} + \gamma_c$
*$\underline{\beta}_c^T = \underline{\mu}_c^T \Sigma^{-1} \underline{x}$ | $\gamma_c = \log \pi_c - \frac{1}{2}\underline{\mu}_c^T \Sigma^{-1} \underline{\mu}_c$
*Bin. hyp. dec. boundary: $\underline{\beta}_0^T \underline{x} + \gamma_0 = \underline{\beta}_1^T \underline{x} + \gamma_1$ (lin. in space of $\underline{x}$)
QGD: Σ_c are diff., find c w/ best $\underline{\mu}_c$, Σ_c
*Bin. hyp. dec. boundary: Quadratic in space of $\underline{x}$
How to find $\pi_c, \underline{\mu}_c, \Sigma_c$: Given n points gen. by GMM, then n_c points $\{\underline{x}_1^c, \ldots, \underline{x}_{n_c}^c\}$ come from $\mathcal{N}(\underline{\mu}_c, \Sigma_c)$
$\hat{\pi}_c = \frac{n_c}{n}$, $\hat{\mu}_c = \frac{1}{n_c}\sum_{i=1}^{n} \underline{x}_i^c$,
$\Sigma_c = \frac{1}{n_c}\sum_{i=1}^{n_c}(x_i^c - \hat{\mu}_c)(x_i^c - \hat{\mu}_c)^T$