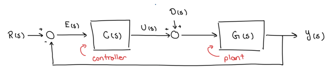
Standard Feedback Control Loop



$R(s)$: Ref., $E(s) = R(s) - y(s)$: Err., $C(s)$: Controller, $U(s)$: Control input, $D(s)$: Dist., $G(s)$: Plant, $y(s)$: Plant output.
***Assume:** $R(s)$ and $D(s)$ are strictly proper rational fcns w/ a fixed set of poles but arbitrary zeros & gain.
*$\mathcal{R}, \mathcal{D}$: Classes of ref. and dist. satisfying the above assumption.
Basic Control Prob.: Design $C(s)$ s.t. 3 spec. are met:
1. **Stability:** $\forall$ bdd $r(t), d(t)$, we have $u(t), e(t)$ bdd.
2. **Asymptotic Tracking:** When $d(t) = 0\ \forall t \geq 0$, then $\forall r(t) \in \mathcal{R}$, $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} r(t) - y(t) = 0$.
3. **Disturbance Rejection:** When $r(t) = 0\ \forall t \geq 0$, then $\forall d(t) \in \mathcal{D}$, $\lim_{t\to\infty} y(t) = 0$.
Open-Loop Control: 1. Design $u(t)$ s.t. $y(t)$ tracks ref. $y_r \in \mathbb{R}$, i.e. $\lim_{t\to\infty} y(t) = y_r$.
2. Set $u(t) = \gamma y_r \mathbf{1}(t)$ w/ $\gamma \in \mathbb{R}$ (const. scaling factor)
3. Apply FVT to find γ s.t. $\lim_{t\to\infty} y(t) = y_r$. 4. Determine $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} y_r - y(t)$
Limitations: 1. Req. perfect knowledge of plant paramters.
2. Not robust against parameter var./(unknown) dist.
3. Does not allow us to speed up convergence.
Feedback Control: 1. Design $u(t)$ s.t. $y(t)$ tracks ref. $y_r \in \mathbb{R}$, i.e. $\lim_{t\to\infty} y(t) = y_r$.
2. Set $u(t) = Ke(t) = K(y_r - y(t))$ w/ $K > 0$ (const. gain).
3. Use block mani. to find $y(s)$ in terms of input and $G(s)$.
4. Apply FVT to find K s.t. $\lim_{t\to\infty} y(t) = y_r$.
5. Determine $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} y_r - y(t)$
Advantages: 1. Doesn't req. perfect knowledge of plant param.
2. Robust against param. var./dist. by $\uparrow K$.
3. Allows us to speed up the rate of convergence by $\uparrow K$.
Disadvantages: 1. Feedback can introduce instability.
2. High-gain amplifies noise.
3. Asymptotic tracking doesn't occur.
Integral Control: 1. Design $u(t)$ s.t. $y(t)$ tracks ref. $y_r \in \mathbb{R}$, i.e. $\lim_{t\to\infty} y(t) = y_r$.
2. Set $u(t) = \mathcal{L}^{-1}\{C(s)E(s)\} = Ke(t) + KT_I \int_0^t e(\tau)d\tau$ (prop. int. (PI) controller) w/ $K, T_I > 0$ (const. gains).
*$C(s) = K\left(1 + \frac{T_I}{s}\right)$
3. Use block mani. to find $y(s)$ in terms of input and $G(s)$.
4. Apply FVT to find $\lim_{t\to\infty} y(t) = y_r$ as desired.
BIBO Stability of Closed-Loop System: **Gang of 4 TF:**

$$\begin{bmatrix} E(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{1+C(s)G(s)} & \frac{-G(s)}{1+C(s)G(s)} \\ \frac{C(s)}{1+C(s)G(s)} & \frac{-C(s)G(s)}{1+C(s)G(s)} \end{bmatrix} \begin{bmatrix} R(s) \\ D(s) \end{bmatrix}$$

BIBO Stable of CLS: The std. feedback control loop (CLS) is BIBO stable if all the Gang of 4 TFs are BIBO stable.
CLS is BIBO Stable THM: The CLS is BIBO stable iff
1. Poles of $\frac{1}{1+C(s)G(s)} \subseteq \mathbb{C}^-$
2. $C(s)G(s)$ has no pole-zero cancel. in $\bar{\mathbb{C}}^+ = \{s \in \mathbb{C} : \mathrm{Re}(s) \geq 0\}$.
Practical Considerations:
1. Don't cancel an unstable 0 of $G(s)$ w/ an unstable pole in $C(s)$.
2. Don't cancel an unstable pole of $G(s)$ w/ an unstable 0 in $C(s)$.
Asymp. Tracking of Poly. Suppose $d(t) = 0$ & want to track a <u>poly. ref. signal</u> of the form: $r(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$, that is: $R(s) = \frac{N_R(s)}{s^k}$, w/ $N_R(0) \neq 0$ and $\deg(N_R(s)) \leq k - 1$.
***GOAL:** Design $C(s)$ to achieve $\lim_{t\to\infty} e(t) = 0$.
Prop: Suppose $C(s)$ is designed so that:
1. $\frac{1}{1+C(s)G(s)}$ is BIBO stable
2. $C(s)G(s) = \frac{C'(s)G'(s)}{s^k}$ with $C'(0)G'(0) \neq 0$.
Then $\frac{1}{s^k + C'(s)G'(s)}$ is BIBO stable.
Asymp. Tracking of Poly. Thm Suppose $C(s)$ satisfies CLS is BIBO stable THM and $d(t) = 0\ \forall t \geq 0$. For any poly. ref. signal $r(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$, the following hold:
a. If $C(s)G(s)$ has k or more poles at $s = 0$, then $\lim_{t\to\infty} e(t) = 0$.
b. If $C(s)G(s)$ has $k - 1$ poles at $s = 0$, then:

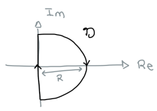
$$\lim_{t\to\infty} e(t) = \begin{cases} \frac{N_R(0)}{1+C'(0)G'(0)}, & \text{if } k = 1 \\ \frac{N_R(0)}{C'(0)G'(0)}, & \text{if } k \geq 2 \end{cases}$$

c. If $C(s)G(s)$ has k-2 or fewer poles at $s = 0$, then $\lim_{t\to\infty} |e(t)| = \infty$.
Type k: The TF $C(s)G(s)$ is of type k if it has k poles at $s = 0$.
Dist. Rejection: Suppose $r(t) = 0\ \forall t \geq 0$ and $d(t)$ is a poly. dist. signal of the form: $d(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$, that is: $D(s) = \frac{N_D(s)}{s^k}$, with $N_D(0) \neq 0$ and $\deg(N_D(s)) \leq k - 1$.
***GOAL:** Design $C(s)$ to achieve $\lim_{t\to\infty} e(t) = 0$.
Dist. Rejection Thm: Suppose $C(s)$ satisfies CLS is BIBO stable THM and $r(t) = 0\ \forall t \geq 0$. For any poly. dist. signal $d(t) = \sum_{i=0}^{k-1} c_i t^i 1(t)$, the following hold:
a. If $C(s)$ has k or more poles at $s = 0$, then $\lim_{t\to\infty} e(t) = 0$.
b. If $C(s)$ has $k - 1$ poles at $s = 0$, then $\lim_{t\to\infty} e(t) \neq 0$ exists.
c. If $C(s)$ has k-2 or fewer poles at $s = 0$, then $\lim_{t\to\infty} |e(t)| = \infty$.
General Thm (Internal Model Principle): Suppose $R(s)$ and $D(s)$ are strictly proper rational fns w/ poles in $\overline{\mathbb{C}^+}$. $C(s)$ solves the Basic Control Problem iff:
1) $C(s)$ makes the CLS BIBO stable;
2) $C(s)G(s)$ has the poles($R(s)$) w/ at least same multiplicities;
3) $C(s)$ has the poles($D(s)$) w/ at least same multiplicities.
Corollary: If $G(s)$ has zeros that are also poles of $R(s)$ or $D(s)$, then the Basic Control Problem is unsolvable.
Internal Model: The IMP states if $G(s)$ does not contain the poles of $R(s)$ and $D(s)$, then $C(s)$ must contain these poles. Since these poles enable $C(s)$ to reproduce $r(t)$ and $d(t)$, we say $C(s)$ must contain an <u>internal model</u> of $r(t)$ and $d(t)$.
Proposition: Suppose $G(s)$ is BIBO stable. Let $Y(s) = G(s)U(s)$, where $Y(s) = \mathcal{L}\{y(t)\}$ and $U(s) = \mathcal{L}\{u(t)\}$. If $\lim_{t\to\infty} u(t) = 0$, then $\lim_{t\to\infty} y(t) = 0$.
*Decaying input $\implies$ decaying output so don't worry in IMP.
General Controller Design Procedure: Given $R(s) = \mathcal{L}\{r(t)\}$ and $D(s) = \mathcal{L}\{d(t)\}$:
1. **Feasibility:** Verify no zero of $G(s)$ is an unstable pole of $R(s)$ or $D(s)$.
2. **Internal Model:** Let $p_1, \ldots, p_k$ denote the unstable poles of $R(s)$ or $D(s)$ not in $G(s)$, accounting for multiplicities. Construct:

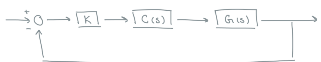
$$C(s) = C'(s) \cdot \frac{1}{(s - p_1) \ldots (s - p_k)}$$

3. **Stability:** Design $C'(s)$ so that the CLS is BIBO stable.
4. **Performance:** Tune controller parameters to achieve the desired performance specifications.

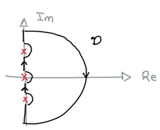
Argument Principle Let $\mathcal{D}$ be a simple (no self-intersections) closed (no endpoints) path in $\mathcal{C}$ oriented CCW. Suppose $F(s)$ has no poles or zeros on $\mathcal{D}$ & isolated poles inside $\mathcal{D}$. Let $\gamma(\theta)$ be a parametrization of $\mathcal{D}$, i.e. $\mathcal{D} = \{\gamma(\theta) : \theta \in \mathbb{R}\}$ and $\mathcal{F} = \{F(\gamma(\theta)) : \theta \in \mathbb{R}\}$. Then $\mathcal{F}$ encircles the origin $n_e = n_z - n_p$ times CCW.
*n_z: # of zeros of $F(s)$ inside $\mathcal{D}$
*n_p: # of poles of $F(s)$ inside $\mathcal{D}$
Notes:
1. A -ve CCW encirlement is the same as a +ve CW encirlement.
2. If $\mathcal{D}$ is oriented CW, the Argument Principle still holds by replacing $CCW \to CW$ everywhere.
Application to Feedback Loops: To stabilize the CLS, it suffices to consider the FB loop where we require:
-**Zeros of** $1 + C(s)G(s) \subseteq \mathbb{C}^-$ (focus on this)
-$C(s)G(s)$ has no unstable pole-zero cancellations.
See if $\exists$ zeros in $\mathbb{C}^+$. So consider the contour:
$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 = \{j\omega : \omega \in [-R, R]\} \cup \{Re^{j\theta} : \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$
*If $R \to \infty$, then more of $\mathbb{C}^+$ is contained inside $\mathcal{D}$.
*By the **Argument Principle**, if we:
-Count the number of encirclements of $1 + C(s)G(s)$ (n_e).
-Know the number of unstable poles of $1 + C(s)G(s)$ (n_p).
*Then, we can compute the number of zeros in $\mathbb{C}^+$.
Nyquist Contour: The path $\mathcal{D}$ above w/ $R \to \infty$.



Nyquist Stablity Criterion: Suppose $L(s) = C(s)G(s)$ is a strictly proper rational fn and has no poles on the Im axis. Also let $K \in \mathbb{R}$. Then the TF $\frac{1}{1+KL(s)}$ is BIBO stable iff
(1) $\mathcal{L} = \{L(j\omega) : \omega \in \mathbb{R}\}$ does not pass through the pt. $\frac{-1}{k}$
(2) $\mathcal{L}$ encircles $\frac{-1}{k}$ a total of n_p times CCW.
*n_p is the # of poles of $L(s)$ in $\mathbb{C}^+$.



Nyquist Plot of $L(s)$ $\mathcal{L} = \{L(j\omega) : \omega \in \mathbb{R}\}$
Obs. of Nyquist Plot: $L(s) = \frac{b_m s^m + \cdots + b_0}{s^n + \cdots + a_1 s + a_0}$ w/ $m < n$.
(1) When $\omega = 0$, $L(j\omega) = \frac{b_0}{a_0} \in \mathbb{R}$ is always on the Re-axis.
(2) As $\omega \to \infty$, $L(j\omega) \to 0$ and $\angle L(j\omega) = \begin{cases} -\frac{\pi}{2}(n-m) & \text{if } b_m > 0 \\ \pi - \frac{\pi}{2}(n-m) & \text{if } b_m < 0 \end{cases}$
(3) Since $\overline{L(j\omega)} = L(\overline{j\omega}) = L(-j\omega)$, $\mathcal{L}$ is symmetric w.r.t the Re-axis.
(4) $\mathcal{L}$ intersects:
*Re-axis when $\angle L(j\omega) \in \{0, \pi\} \mod 2\pi$
*Im-axis when $\angle L(j\omega) \in \{\pm\frac{\pi}{2}\} \mod 2\pi$
Nyquist Stability Criterion w/ Imaginary Poles: Suppose $L(s)$ satisfies all the requirements of the Nyquist Stability Criterion except it has poles on the imaginary axis. Then the conclusion of the Nyquist Stability Criterion hold true provided we use the indented Nyquist Contour.



*X: poles of $L(s)$ on the Im-axis.
*Radius of each semi-circle around each pole on the Im-axis is $\epsilon > 0$ and we consider the case when $\epsilon \to 0^+$.
Gain and Phase Margin:
Frequency Response: Given the TF $G(s)$, its frequency response is $G(j\omega)$ where $\omega \in \mathbb{R}$.
Proposition:
Gain Margin (GM):
Phase Margin (PM):
Crossover Frequency: