ECE368 Cheatsheet

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Summary: On second thoughts, the lecture notes he posts are good, so I think I'll just do the cheatsheet.

W1 (LG-IPPR 1.1, 1.2; Murphy 2.1 – 2.3)

1 L1: Probability Review

Summary:

FAQ:
- How to study? Practice, practice.
- What textbooks? Use 2024 version of Murphy, Leon Garcia as main reference, Bishop, 4th textbook is intro.
- How is HW graded? Effort, and tutorials are used to explain soln.

1.1 Sample Space

Motivation: If you have 4 sheeps and a flea, the probability that starting from sheep 1, the flea will jump to sheep 4 in 10 steps is 0.2.
- Ambigious as there are 2 different interpretations for the sample space (i.e. space of probability is not clear):
  - Set of sheeps
  - Set of number of steps

1.2 Probability Definitions

Definition:
- **Random Experiment:** An outcome (realization) for each run.
- **Sample Space Ω:** Set of all possible outcomes.
- **Events:** (measurable) subsets of Ω.
- **Probability of Event A:** $P[A] \equiv P[\text{'outcome is in A'}]$.

Example: Roll Fair Die
- $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- $P[\text{'even number'}] = \frac{1}{2}$.

1.3 Axioms of Probability

Definition:
1. $P[A] \geq 0$ for all $A \in \Omega$.
2. $P[\Omega] = 1$.
3. If $A \cap B = \emptyset$, then $P[A \cup B] = P[A] + P[B]$ for all $A, B \in \Omega$.

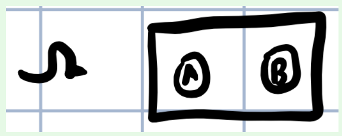

Figure 1: 3rd Axiom

1.4 Conditional Probability

Definition:

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \tag{1}$$

- $|$: Given event (data/obs.).

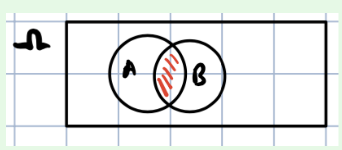

Figure 2: Conditional Probability

Notes:
- Changing sample space to B.
- Conditional probability satisfy the 3 axioms (i.e. are probabilities), can be viewed as probability measure on new sample space B.

1.4.1 Consequences of Conditional Probability

Definition:

$$P[A \cap B] = P[A|B]P[B] = P[B|A]P[A] \tag{2}$$

1.4.2 Independence

Definition: A and B are independent iff

$$P[A \cap B] = P[A]P[B] \iff P[A|B] = P[A] \iff P[B|A] = P[B] \tag{3}$$

1.4.3 Importance of Labelling

Example: **Toss 2 Fair Coins**

1. **Given:** Given that one of the coins is heads, what is the probability that the other coin is tails?
2. **Wrong Solution:** $\frac{1}{2}$ since $\{HH, HT, TH, TT\}$, so $P[T|H] = \frac{1}{2}$, which assumes that the coins are distinguishable (i.e. coin #1 is heads)
3. **Correct Solution:** $\frac{2}{3}$ since $\{HH, HT, TH\}$ as we didn't specify which coin was heads, so $P[T|H] = \frac{2}{3}$, which assumes that the coins are indistinguishable.

2 L2: Probability Review

2.1 Total Probability

Definition: If $H_1, \dots, H_n$ form a partition of Ω, then

$$P[A] = \sum_{i=1}^{n} P[A|H_i]P[H_i] \quad (4)$$

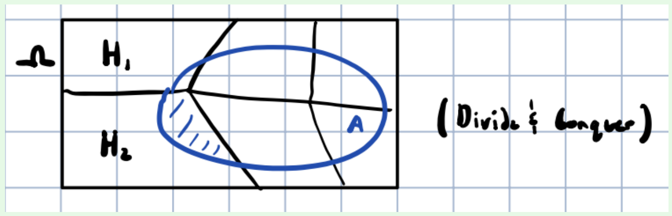

Figure 3: Total Probability

2.2 Bayes' Rule

Definition:

$$P[H_k|A] = \frac{P[H_k \cap A]}{P[A]} = \frac{P[A|H_k]P[H_k]}{\sum_{i=1}^{n} P[A|H_i]P[H_i]} \quad (5)$$

2.2.1 Posteriori Probability, Priori Probability (Prior), Likelihood

Definition:

- **Posteriori:** $P[H_k|A]$.
- **Priori:** $P[H_k]$.
- **Likelihood:** $P[A|H_k]$.

Example: Suppose a lie detector is 95% accurate, i.e. $P[\text{'out=truth'}|\text{'in=truth'}] = 0.95$ and $P[\text{'out=lie'}|\text{'in=lie'}] = 0.95$. It says that Mr. Ernst is lying. What is the probability Mr. Ernst is actually lying.

- **Observation:** $A = \text{'out=lie'}$.
- **Hypothesis:** $H_0 = \text{'in=lie'}$ and $H_1 = \text{'in=truth'}$.
- **Solution:** $P[H_0|A] = \dfrac{P[A|H_0]P[H_0]}{P[A|H_0]P[H_0] + P[A|H_1]P[H_1]} = \dfrac{0.95 \times P[H_0]}{0.95 \times P[H_0] + 0.05 \times (1 - P[H_0])}$.
- $H_0 = 0.01$**:** i.e. 1% of the population are liars, then $P[H_0|A] = \dfrac{0.95 \times 0.01}{0.95 \times 0.01 + 0.05 \times 0.99} = 0.16$.

Warning: Need to know priori probability.

2.2.2 Interpretation of Bayes' Rule

Notes: Taking one component of the total probability and normalizing it by the sum of all components.

2.3 Random Variables

Motivation: **Coin Toss** Mapping of each outcome to a real number

- $w \in \Omega$ is the outcome of a coin toss, and X is the RV, so $H \to 0$ and $T \to 1$.

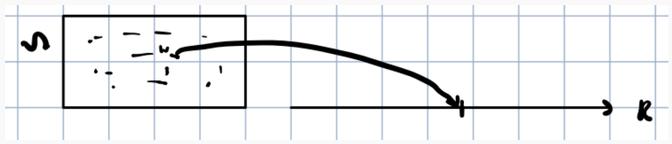

Figure 4: Random Variables

- Mapping is deterministic function. RV is not random or variable.

Definition: Mapping from Ω to $\mathbb{R}$.

2.4 Distribution of RV

2.4.1 Cumulative Distribution Function (CDF) of RV

Definition:

$$F_X(x) \equiv P[X \leq x] \tag{6}$$

2.4.2 Discrete RV Probability Mass Function (PMF)

Definition:

$$P_X(x_j) \equiv P[X = x_j] \quad j = 1, 2, 3, \ldots \tag{7}$$

Example: Binonmial RV w/ (n, p)

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{8}$$

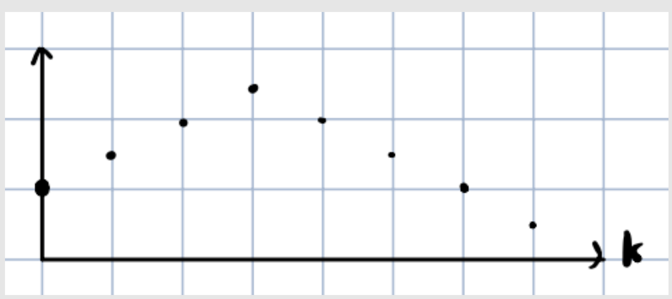

Figure 5: Binomial RV

2.4.3 Continuous RV Probability Density Function (PDF)

Definition:

$$f_X(x) \equiv \frac{d}{dx} F_X(x) \tag{9}$$

$$P[x < X < x + dx] = f_X(x)dx \tag{10}$$

Example: **Gaussian RV w/ (μ, σ^2)**

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{11}$$

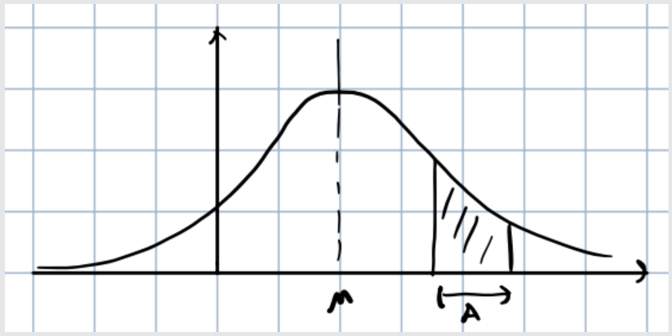

Figure 6: Gaussian RV

- $P[X \in A] = \int_A f_X(x)dx$.

Notes: Discrete RV has pdf w/ δ functions.

2.4.4 Conditional PMF/PDF

Definition:

$$P_X(x|A) \tag{12}$$

$$f_X(x|A) \tag{13}$$

Example: **Continuous**

$$f(x|X > a) = \begin{cases} \frac{f_X(x)}{P[X > a]} & \text{if } x > a \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

Example: **Geometric RV** Geometric RV X w/ success probability p

$$P_X(k) = (1-p)^{k-1}p \tag{15}$$

- **Memoryless Property:** $P_X[k|X > m] = \frac{p(1-p)^{k-1}}{(1-p)^m} = p(1-p)^{k-m-1}$.
  - So it only cares about the additional trials (i.e. same as resetting after m trials).

2.5 Expected Values

Definition:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx \overset{\text{If int. values}}{=} \sum_{k=-\infty}^{\infty} k f_X(k) \tag{16}$$

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x)dx \overset{\text{If int. values}}{=} \sum_{k=-\infty}^{\infty} h(k) f_X(k) \tag{17}$$

$$\text{Var}[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2 \tag{18}$$

$$E[X|A] = \int_{-\infty}^{\infty} x f_X(x|A) dx \tag{19}$$

Example: **Lottery Ticket (Geometric RV)**

1. **Given:** Buying one lottery ticket per week
   - Each ticket has $10^{-7} = p$ chance of winning the jackpot.
   - $X =$ '# of weeks to win jackpot'.
2. **Problem:** What is the expected number of weeks to win the jackpot?
3. **Solution:** $E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = \ldots = \frac{1}{p} = 10^7$ weeks.
4. **Extension (Memoryless Property):** If I have already played for 999999 weeks, what is the expected number of weeks to win the jackpot? $E[X - 999999|X > 999999] = E[X] = 10^7$ weeks.

3 L3: Probability Review

3.1 2 RVs

Notes: RVs are neither random nor a variable.

$$\underline{Z} = (X, Y)$$

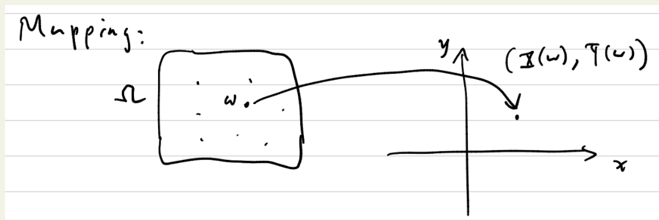

Figure 7: Mapping of RVs

3.2 Joint PMF/PDF

Definition:

$$P_{X,Y}(x, y) = P[X = x, Y = y] \tag{20}$$

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) \tag{21}$$

$$P[(X, Y) \in A] = \int\int_{(x,y)\in A} f_{X,Y}(x, y)\, dx\, dy \tag{22}$$

Example: Jointly Gaussian RVs X and Y with $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

3.3 Expectations

Definition:

$$E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dx\, dy$$

Notes:
- $g(X, Y)$ is also an RV, but inside the integral or sum, you use x and y as dummy variables to vary through the values of the RVs.

3.3.1 Correlation

Definition:

$$E[XY] \tag{23}$$

3.3.2 Covariance

Definition:

$$\text{Cov}[X,Y] = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - \mu_X\mu_Y = E[XY] - E[X]E[Y] \tag{24}$$

Notes:
- Mean shifted to 0.

3.3.3 Correlation Coefficient

Definition:

$$\rho_{X,Y} = E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right] = \frac{\text{Cov}[X,Y]}{\sigma_X\sigma_Y} \tag{25}$$

- $|\rho_{X,Y}| \leq 1$

Notes:
- Mean shifted to 0 and normalized by the standard deviation.

3.4 Marginal PMF/PDF

Definition:

$$P_X(x) = \sum_{j=1}^{\infty} P_{X,Y}(x,y_j), \quad P_Y(y) = \sum_{j=1}^{\infty} P_{X,Y}(x_j,y) \tag{26}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \tag{27}$$

Notes:
- Total probability theorem is being used here.

Example: Jointly Gaussian X and Y:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \\ &= \dots \quad \text{(completing the square)} \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}, \quad \text{marginally Gaussian} \end{aligned}$$

- Gaussian RVs has a property that the PDF of a single variable is equal to the marginal Gaussian of two variables.

3.5 Conditional PMF/PDF

Definition:

$$P_{X|Y}(x|y) \triangleq P[X=x|Y=y] = \frac{P_{X,Y}(x,y)}{P_Y(y)} \tag{28}$$

$$f_{X|Y}(x|y) \triangleq \frac{f_{X,Y}(x,y)}{f_Y(y)} \tag{29}$$

3.6 Bayes' Rule

Definition:

$$P_{Y|X}(x|y) = \frac{P_{X,Y}(x,y)}{P_X(x)} = \frac{P_{X|Y}(x|y)P_Y(y)}{\sum_{j=1}^{\infty} P_{X,Y}(x,y_j)P_Y(y_j)} \tag{30}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_{X|Y}(x|y)f_Y(y)}{\int_{-\infty}^{\infty} f_{X|Y}(x|y')f_Y(y')\,dy'} \tag{31}$$

3.7 Independent vs. Uncorrelated vs. Orthogonal

Definition:

1. Independent:
$$f_{X|Y}(x|y) = f_X(x)\ \forall y \Leftrightarrow f_{X,Y}(x,y) = f_X(x)f_Y(y) \tag{32}$$
2. Uncorrelated:
$$\text{Cov}[X,Y] = 0 \quad \Leftrightarrow \quad \rho_{X,Y} = 0 \tag{33}$$
3. Orthogonal:
$$E[XY] = 0 \tag{34}$$

Theorem: If independent, then uncorrelated.

Derivation:

$$\begin{aligned}
\text{Independent} \implies E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X(x)f_Y(y)\,dx\,dy \\
&= \left(\int_{-\infty}^{\infty} x f_X(x)\,dx\right)\left(\int_{-\infty}^{\infty} y f_Y(y)\,dy\right) \\
&\implies E[XY] = E[X]E[Y] \\
&\implies \text{Cov}[X,Y] = 0, \quad \text{uncorrelated} \\
&\not\Leftarrow \text{in general.}
\end{aligned}$$

Example: Jointly Gaussian RVs X and Y: If uncorrelated, i.e. $\rho_{X,Y} = 0$, then X and Y are independent.

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left\{-\frac{1}{2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}} \\
&= f_X(x)f_Y(y) \quad \text{independent}
\end{aligned}$$

3.8 Conditional Expectation

Definition:

$$E[Y] = E[E[Y|X]] \tag{35}$$

$$E[h(Y)] = E[E[h(Y)|X]] \tag{36}$$

Notes:
- $E[E[Y|X]]$ is w.r.t. X.
- $E[Y|X]$ is w.r.t. Y.

Derivation:

$$\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{Y|X}(y|x) f_X(x)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} y f_{Y|X}(y|x)\,dy\right) f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} E[Y|X=x] f_X(x)\,dx \quad \text{(using the total probability theorem)} \\
&= \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \\
&= E[g(X)] \\
&= E[E[Y|X]].
\end{aligned}$$

Example:
1. **Given:** An unknown voltage. $X \sim \text{Uniform}(0,1)$. Measurement from a (bad) voltmeter: $Y \sim \text{Uniform}(0,X)$.

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x}, & 0 < y < x \\ 0, & \text{otherwise} \end{cases}$$

- **Note:** Area under PDF is 1.

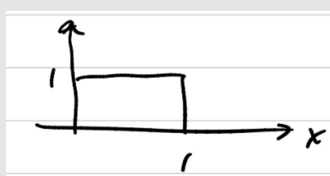

Figure 8: Uniform Distribution of X

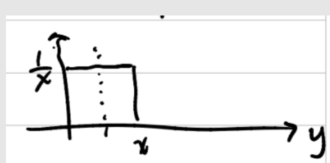

Figure 9: Uniform Distribution of Y

2. **Expected Value (Average Reading of Bad Voltmeter):**

$$\begin{aligned} E[Y] &= E[E[Y|X]] \\ &= E\left[\frac{X}{2}\right] \quad \text{Since in the middle of 0 and x} \\ &= \frac{1}{2} \cdot E[X] \\ &= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \quad \text{Since } E[X] \text{ (i.e. mean) is 0.5} \end{aligned}$$

3. **The Long Way:**

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)\, dx \\ &= \int_y^1 f_{Y|X}(y|x) f_X(x)\, dx \\ &= \int_y^1 \frac{1}{x} \cdot 1\, dx \\ &= -\ln y. \\ E[Y] &= \int_0^1 y \cdot (-\ln y)\, dy = \cdots = \frac{1}{4} \end{aligned}$$

4. **Question:** Suppose $Y = \frac{1}{8}$. What is "best" given X? This will be the quesiton for the rest of the course.

4 L4: Estimation of Sample Mean

4.1 Parameter Estimation:

Motivation: The readout of a sensor is $X = \theta + N$ volts

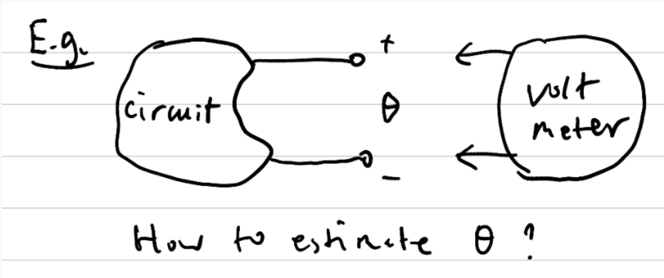

Figure 10:

- There is some noise N in the sensor, so we want to estimate the true value of θ (unknown parameter to be estimated)
  - e.g. Mean and/or variance of X.

4.2 Estimator:

Definition: Perform n independent and identically distributed (i.i.d.) measurements/observations of X: $X_1, X_2, \ldots, X_n$.

$$\hat{\Theta} = \hat{\Theta}(\underline{X}) = g(X_1, X_2, \ldots, X_n) \tag{37}$$

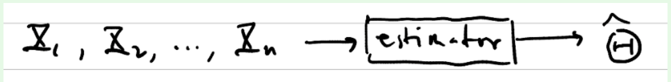

Figure 11:

4.2.1 Estimation Error:

Definition:

$$\hat{\Theta}(\underline{X}) - \theta \tag{38}$$

4.2.2 Unbiased

Definition: The estimator $\hat{\Theta}$ is unbiased if

$$\mathbb{E}[\hat{\Theta}(\underline{X})] = \theta \tag{39}$$

- **Asymptotically Unbiased:** $\lim_{n\to\infty} \mathbb{E}[\hat{\Theta}(\underline{X})] = \theta$ (big data)

4.2.3 Consistent

Definition: The estimator $\hat{\Theta}$ is consistent if $\hat{\Theta}(\underline{X}) \to \theta$ as $n \to \infty$, in probability, i.e., $\forall \epsilon > 0$,

$$\lim_{n\to\infty} P(|\hat{\Theta}(\underline{X}) - \theta| < \epsilon) \to 1 \tag{40}$$

as $n \to \infty$.

4.3 Sample Mean & Law of Large Numbers

Definition: Given a sequence of i.i.d. random variables (RVs), $X_1, X_2, \ldots, X_n$, w/ unknown mean μ, estimate μ. Let $S_n = X_1 + X_2 + \cdots + X_n$. The *sample mean* is

$$M_n = \frac{1}{n} S_n$$

- How good is M_n as an estimator of μ?
  - Use unbiased and consistent to evaluate M_n.

Example: Previous voltage measurement, e.g.,

$$X_i = \mu + N_i$$

where μ is the true value and N_i is the noise.

If we assume N_i are i.i.d. with zero mean,

$$E[X_i] = E[\mu + N_i] = E[\mu] + E[N_i] = \mu + 0 = \mu, \quad \forall i$$

4.3.1 Digression for Sum of RVs (not necessarily independent or identically distributed)

Derivation:

$$\begin{aligned} E[S_n] &= E[X_1 + \cdots + X_n] \\ &= E[X_1] + \cdots + E[X_n] \end{aligned}$$

Derivation:

$$\begin{aligned} \text{Var}[S_n] &= E\left[(S_n - E[S_n])^2\right] \\ &= E\left[\left(\sum_{i=1}^{n} X_i - E[X_i]\right)^2\right] \\ &= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i - E[X_i])(X_j - E[X_j])\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[(X_i - E[X_i])(X_j - E[X_j])\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \text{Cov}[X_i, X_j] \\ &= \sum_{i=1}^{n} \text{Var}[X_i] + \sum\sum_{i \neq j} \text{Cov}[X_i, X_j] \end{aligned}$$

4.3.2 Unbiased (i.i.d.)

Derivation:

$$\begin{aligned}
E[M_n] &= E\left[\frac{1}{n}S_n\right] \\
&= \frac{1}{n}\left(E[X_1] + \cdots + E[X_n]\right) \\
&= \frac{1}{n}(n\mu) \quad \text{since } X_i \text{ are i.i.d. so same expectation} \\
&= \mu \Rightarrow \text{Unbiased!}
\end{aligned}$$

4.3.3 Consistent (i.i.d.)

Derivation:

$$\begin{aligned}
\mathrm{Var}[M_n] &= \mathrm{Var}\left[\frac{1}{n}S_n\right] \\
&= \frac{1}{n^2}\mathrm{Var}[S_n] \quad \text{taking out constant requires squaring} \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{Var}[X_i] + \sum\sum_{i\neq j}\mathrm{Cov}[X_i, X_j]\right) \\
&= \frac{1}{n^2}(n\sigma^2) \quad \sigma^2 \triangleq \mathrm{Var}[X_i] \text{ and } X_i \text{ are i.i.d. so covariance is } 0 \\
&= \frac{\sigma^2}{n} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

- This means that there is no variance in the sample mean as n approaches infinity, so it converges to the true mean.

Recall the Chebyshev Inequality:

$$P\left[|X - E[X]| \geq \epsilon\right] \leq \frac{\mathrm{Var}[X]}{\epsilon^2}, \quad \forall \epsilon > 0.$$

Substitute in M_n:

$$\begin{aligned}
P\left[|M_n - E[M_n]| \geq \epsilon\right] &\leq \frac{\mathrm{Var}[M_n]}{\epsilon^2} \\
P\left[|M_n - \mu| \geq \epsilon\right] &\leq \frac{\sigma^2}{n\epsilon^2} \\
\Rightarrow P\left[|M_n - \mu| < \epsilon\right] &\geq 1 - \frac{\sigma^2}{n\epsilon^2} \to 1 \text{ as } n \to \infty \text{ then it is consistent}
\end{aligned}$$

Warning: Cov $= 0$ because independence implies uncorrelated.

4.3.4 Weak Law of Large Numbers

Definition: Even if σ is infinite, then $\forall \epsilon > 0$,

$$\lim_{n\to\infty} P\left[|M_n - \mu| < \epsilon\right] = 1$$

4.3.5 Confidence Interval: Finding n

Example: Measure an unknown voltage θ for n times and obtain independent measurements:

$$X_i = \theta + N_i,$$

where N_i are i.i.d. random variables with mean 0 and variance 1.

- We want to determine how many measurements n are sufficient so that

$$P\left(|M_n - \theta| < 0.1\right) \geq 0.95,$$

  where 0.1 is the desired precision and 0.95 is the confidence level.
- The sample mean is given by:

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i = \theta + \frac{1}{n}\sum_{i=1}^{n} N_i.$$

- The variance of X_i is:

$$\sigma^2 = \mathrm{Var}[X_i] = \mathrm{Var}[\theta + N_i] = \mathrm{Var}[N_i] = 1.$$

  - $\mathrm{Var}[aX + b] = a^2\mathrm{Var}[X]$, where $a = 1$ and $b = \theta$.
- Using Chebyshev's inequality:

$$1 - \frac{\sigma^2}{n\epsilon^2} \geq 0.95,$$

  where $\epsilon = 0.1$ (precision).
- Solving for n:

$$\begin{aligned} 1 - \frac{1}{n(0.1)^2} &\geq 0.95, \\ \frac{1}{n(0.1)^2} &\leq 0.05, \\ n &\geq 2000. \end{aligned}$$

Thus, at least 2000 measurements are needed to achieve the desired precision and confidence level.

5 L5: Sample Mean and Maximum Likelihood Estimation

FAQ:
- Why can we say that it is consistent for the last example?

5.1 Maximum Likelihood Estimation

Motivation: Choose parameter θ that is most likely to generate the observation $x_1, x_2, \ldots, x_n$.

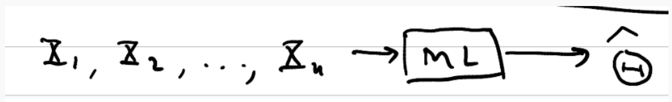

Figure 12:

Definition:

$$\hat{\Theta} = \arg\max_{\theta} P_{\underline{X}}(\underline{x}|\theta), \text{ discrete } X. \tag{41}$$

$$\hat{\Theta} = \arg\max_{\theta} f_{\underline{X}}(\underline{x}|\theta), \text{ continuous } X. \tag{42}$$

5.1.1 Log-Likelihood

Definition:

$$\hat{\theta} = \arg\max_{\theta} \sum_{i=1}^{n} \log P_X(x_i|\theta) \tag{43}$$

$$\hat{\theta} = \arg\max_{\theta} \sum_{i=1}^{n} \log f_X(x_i|\theta). \tag{44}$$

Warning: Can only go from argmax of fcn to argmax of log fcn, if it is i.i.d.

Derivation:

$$\begin{aligned}
&\text{i.i.d. } X_1, X_2, \ldots, X_n \implies \\
&\quad p_{\underline{X}}(\underline{x}|\theta) = \prod_{i=1}^{n} p_{X_i}(x_i|\theta) \\
&\quad = \prod_{i=1}^{n} p_X(x_i|\theta) \quad \text{drop the i due to i.i.d. assumption} \\
&\quad \log p_{\underline{X}}(\underline{x}|\theta) = \sum_{i=1}^{n} \log p_X(x_i|\theta).
\end{aligned}$$

Example:
1. **Model and Observations:**
   - Assume a biased coin with probability θ of showing heads. Find ML estimator for θ.
   - Toss the coin n times and obtain Bernoulli random variables $X_1, \ldots, X_n$ such that:

$$\text{"heads"} \to 1, \quad \text{"tails"} \to 0.$$

- Total number of heads is:

$$k = \sum_{i=1}^{n} X_i.$$

For example:

$$\underline{x} = (1, 0, 0, 0, 1, 1, 0, 1, 1, 1), \quad k = 6.$$

- Probability of observations $x_1, \ldots, x_n$ corresponding to parameter θ is:

$$p_{\underline{X}}(\underline{x}|\theta) = \theta^k (1-\theta)^{n-k}.$$

  - It is sufficient to know only k.
  - Note: Don't need the $\binom{n}{k}$ term because we are given the specific sequence of heads and tails.

2. **Log-Likelihood and Maximization:**
   - The log-likelihood function is:

$$\log p_{\underline{X}}(\underline{x}|\theta) = k \log(\theta) + (n-k)\log(1-\theta).$$

   - To maximize the log-likelihood over θ, set:

$$\begin{aligned} 0 &= \frac{\partial}{\partial\theta} \log p_{\underline{X}}(\underline{x}|\theta), \\ 0 &= \frac{k}{\theta} - \frac{n-k}{1-\theta}, \\ \theta &= \frac{k}{n}. \end{aligned}$$

   - Thus, the Maximum Likelihood Estimator (MLE) is:

$$\hat{\Theta} = \frac{k}{n}, \quad \text{where } k = \sum_{i=1}^{n} X_i.$$

   This corresponds to the observed frequency of heads, which is intuitive b/c the more heads we see, the more likely the coin is biased towards heads.

3. **Examples:**
   - For $\underline{x} = (1, 0, 0, 0, 1, 1, 0, 1, 1, 1)$:

$$\begin{aligned} p_X(\underline{x}|\theta) &= \theta^6(1-\theta)^4, \\ \hat{\theta} &= \frac{6}{10} = 0.6. \end{aligned}$$

   - For $\underline{x} = (0, 1, 1, 1, 0, 0, 1, 0, 1, 0)$:

$$\begin{aligned} p_X(\underline{x}|\theta) &= \theta^5(1-\theta)^5, \\ \hat{\theta} &= \frac{5}{10} = 0.5. \end{aligned}$$

Notes:

1. k is a sufficient statistic for this Maximum Likelihood (ML) estimator.

2. The expectation of the estimator $\hat{\theta}$ is:

$$\begin{aligned} E[\hat{\Theta}] &= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} E[X_i] \\ &= \frac{1}{n}(n\theta) \\ &= \theta \quad \text{(Unbiased)}. \end{aligned}$$

   - $E[X_i] = (1)\theta + (0)(1-\theta) = \theta$

3. In fact, $\hat{\Theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean, and θ is the true mean. Therefore, $\hat{\theta} \to \theta$ in probability, which implies that $\hat{\theta}$ is *consistent*.

6 L6: Maximum Likelihood and Laplace

6.1 MLE for Categorical Random Variables

Example:

1. We say that $X \sim \text{Cat}(\underline{\theta})$ if
$$P[X = m] = \theta_m, \quad m = 1, 2, \dots, M.$$
   - Going from 2 to M categories is a generalization of the Bernoulli distribution.

   The parameter $\underline{\theta}$ is a vector:
$$\underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_M \end{bmatrix},$$
   such that $\theta_m \geq 0$ and $\displaystyle\sum_{m=1}^{M} \theta_m = 1$.
2. Given n i.i.d. observations $X_1, \dots, X_n$, we aim to find the maximum likelihood estimator (MLE) of $\underline{\theta}$.
3. Define n_m as the number of observations that equal m:
$$n_m = \sum_{i=1}^{n} 1(x_i = m),$$
   where $1(x_i = m)$ is the indicator function. Note that $\displaystyle\sum_{m=1}^{M} n_m = n$.
4. The likelihood function is:
$$p_{\underline{X}}(\underline{x} \mid \underline{\theta}) = \prod_{m=1}^{M} \theta_m^{n_m}.$$
   - Similar to the Bernoulli distribution, but with M categories.

   Taking the log, we get:
$$\log p_{\underline{X}}(\underline{x} \mid \underline{\theta}) = \sum_{m=1}^{M} n_m \log \theta_m.$$
5. To find the optimal $\underline{\theta}$, we minimize the negative log-likelihood:
$$\min_{\underline{\theta}} - \sum_{m=1}^{M} n_m \log \theta_m,$$
   subject to the constraints $\theta_m \geq 0$ for $1 \leq m \leq M$ and $\displaystyle\sum_{m=1}^{M} \theta_m = 1$.
6. Solving this optimization problem, the MLE is:
$$\hat{\Theta}_m = \frac{N_m}{n} = \frac{\sum_{i=1}^{n} 1(X_i = n)}{n}, \quad \underline{\hat{\Theta}} = \begin{bmatrix} \frac{N_1}{n} \\ \vdots \\ \frac{N_m}{n} \end{bmatrix}.$$

6.2 MLE for Gaussian Random Variables

Example:

1. Given n i.i.d. observations $X_1, \dots, X_n$ of a Gaussian random variable with parameters (μ, σ^2), we aim to find the maximum likelihood estimators (MLEs) of μ and σ^2.

$$f_{\underline{X}}(\underline{x}|\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right),$$
$$\log f_{\underline{X}}(\underline{x}|\mu, \sigma^2) = \sum_{i=1}^{n} \left(-\frac{(x_i - \mu)^2}{2\sigma^2} - \log \sigma - \log \sqrt{2\pi}\right).$$

2. To find μ, take the derivative of the log-likelihood with respect to μ and set it to zero:

$$0 = \frac{\partial}{\partial \mu} \sum_{i=1}^{n} \left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right),$$
$$0 = \frac{1}{n} \sum_{i=1}^{n} x_i - \mu,$$
$$\mu = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

3. To find σ^2, take the derivative of the log-likelihood with respect to σ^2 and set it to zero:

$$0 = \frac{\partial}{\partial \sigma^2} \sum_{i=1}^{n} \left(-\frac{(x_i - \mu)^2}{2\sigma^2} - \frac{1}{2}\log \sigma^2\right),$$
$$0 = -\frac{1}{2} \sum_{i=1}^{n} \left(\frac{(x_i - \mu)^2}{\sigma^4}\right) + \frac{1}{2\sigma^2},$$
$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2.$$

4. Thus, the MLEs are:

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad \text{(sample mean)}$$
$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \hat{\mu})^2. \quad \text{(sample variance)}$$

   - Note: The sample variance is biased, so we often use $\frac{1}{n-1}$ instead of $\frac{1}{n}$ to make it unbiased.
   - Note: The sample mean is unbiased.

6.3 Will the Sun Rise Tomorrow? (Laplace's Problem)

Example:

- Observation: The Sun has risen for n consecutive days. Estimate the probability that it will rise tomorrow.
- Model: Assume n i.i.d. Bernoulli random variables $X_1, \dots, X_n$ with $P[X_i = 1] = \theta$.

6.3.1 Frequentist Approach

Example:

1. The Maximum Likelihood Estimator (MLE) is:
$$\hat{\theta} = \frac{K}{n} = \frac{\sum_{i=1}^{n} X_i}{n}.$$
2. If $K = n$ (i.e., the Sun has risen every day so far), then:
$$\hat{\theta} = \frac{n}{n} = 1.$$
3. Conclusion: The Sun will rise tomorrow with probability 1, regardless of what n is, based on the Frequentist approach.
   - This doesn't make sense b/c if $n = 1$ then we are assuming 100% it will rise based on one observation.

6.3.2 Bayesian Approach

Example:

1. Assume that θ is not fixed but drawn from a uniform distribution in $[0, 1]$. This means that the probability of the Sun rising is based on a uniform distribution.
2. We want to find the probability that the sun will rise tomorrow given that it has risen for n consecutive days:
$$P[X_{n+1} = 1 | X_1 = 1, \ldots, X_n = 1].$$
   Using Bayes' Theorem:
$$P[X_{n+1} = 1 | X_1 = 1, \ldots, X_n = 1] = \frac{P[X_1 = 1, \ldots, X_{n+1} = 1]}{P[X_1 = 1, \ldots, X_n = 1]}.$$
3. Compute $P[X_1 = 1, \ldots, X_n = 1]$:
$$P[X_1 = 1, \ldots, X_n = 1] = \int_0^1 P[X_1 = 1, \ldots, X_n = 1 | \Theta = \theta] f_\Theta(\theta)\, d\theta.$$
   - The joint probability is calculated by integrating the product of the likelihood and the prior (i.e. marginalizing over θ).
   - The likilihood becomes θ^n because the observations are i.i.d.
   - The prior is uniform, so $f_\Theta(\theta) = 1$.

   Since $f_\Theta(\theta) = 1$ (uniform prior) and $P[X_1 = 1, \ldots, X_n = 1 | \Theta = \theta] = \theta^n$, we have:
$$P[X_1 = 1, \ldots, X_n = 1] = \int_0^1 \theta^n \, d\theta = \frac{1}{n+1}.$$
4. Compute $P[X_1 = 1, \ldots, X_{n+1} = 1]$ similarly:
$$P[X_1 = 1, \ldots, X_{n+1} = 1] = \int_0^1 \theta^{n+1} \, d\theta = \frac{1}{n+2}.$$
5. Combine results:
$$P[X_{n+1} = 1 | X_1 = 1, \ldots, X_n = 1] = \frac{\frac{1}{n+2}}{\frac{1}{n+1}} = \frac{n+1}{n+2}.$$
6. Conclusion: As n increases, the probability approaches 1, providing more certainty with more data.

6.4 Sample Mean is Not Always an ML Estimator

Example: Given an unknown voltage x, we measure it using a voltmeter that outputs a random reading Y that is uniform in $[0, x]$. Suppose we make n i.i.d. measurements $Y_1, \dots, Y_n$ and wish to estimate $\mu = \frac{x}{2} = \mathbb{E}[Y]$.

1. PDF of Y:

$$f_Y(y \mid \mu) = \begin{cases} \frac{1}{2\mu} & \text{if } 0 \leq y \leq 2\mu, \\ 0 & \text{otherwise.} \end{cases}$$

   - This is a uniform distribution, where $x = 2\mu$.

2. **Sample Mean:**

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} Y_i \quad \cdots (1)$$

3. **To Find the ML Estimator:**

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y} \mid \mu) &= \prod_{i=1}^{n} f_Y(y_i \mid \mu) \\ &= \prod_{i=1}^{n} \frac{1}{2\mu} \cdot 1(0 \leq y_i \leq 2\mu) \\ &= \frac{1}{(2\mu)^n} \prod_{i=1}^{n} 1(0 \leq y_i \leq 2\mu). \end{aligned}$$

   - The indicator function ensures that all measurements are within the range $[0, 2\mu]$ for the likelihood to be non-zero. This is done because we assume that Y is uniformly distributed in $[0, 2\mu]$.

   The likelihood is maximized for:

$$\arg\max_{\mu} f_{\mathbf{Y}}(\mathbf{y} \mid \mu) = \max_{1 \leq i \leq n} \frac{1}{2} Y_i.$$

   Therefore:

$$\hat{\mu} = \max_{1 \leq i \leq n} \frac{1}{2} Y_i \quad \cdots (2)$$

   - The likelihood is non-zero only if μ is greater than or equal to the maximum of the measurements because all data points must lie within the range $[0, 2\mu]$.
   - i.e. $2\mu \geq \max_{1 \leq i \leq n} Y_i$, which is to ensure that ALL measurements are within the range $[0, 2\mu]$, so this must be μ.

4. Clearly, (1) $\neq$ (2).

7 L7: Maximum A Posteriori (MAP) Estimation

7.1 ML Estimator (Frequentist Approach)

Definition:

- Assume θ is **fixed** and find the "best" θ:

$$\hat{\theta} = \arg\max_{\theta} P_{\underline{X}}(\underline{x}|\theta) \quad \text{or} \quad \hat{\theta} = \arg\max_{\theta} f_{\underline{X}}(\underline{x}|\theta).$$

- Every θ value specifies a **different probability space** (e.g., coin, universe, etc.) for our experiment.

7.2 MAP Estimator (Bayesian Approach)

Definition:

1. Assume **random** parameter Θ in the **same sample space** as our experiment.
2. Assume we have a **prior** pmf/pdf for Θ: $P_\Theta(\theta)$ or $f_\Theta(\theta)$.
3. Find the most probable θ given the observations $\underline{X} = \underline{x}$:

$$\hat{\theta} = \arg\max_{\theta} P_{\Theta|\underline{X}}(\theta|\underline{x}), \quad \text{if } \theta \text{ discrete},$$

$$\hat{\theta} = \arg\max_{\theta} f_{\Theta|\underline{X}}(\theta|\underline{x}), \quad \text{if } \theta \text{ continuous}.$$

- **Posterior Distribution:** $f_{\Theta|\underline{X}}(\theta|\underline{x})$.

7.2.1 Four Cases of Bayes' Rule

Definition:

$$P_{\Theta|\underline{X}}(\theta|\underline{x}) = \begin{cases} \dfrac{P_{\underline{X}|\Theta}(\underline{x}|\theta)P_\Theta(\theta)}{P_{\underline{X}}(\underline{x})}, & \text{if } \underline{X} \text{ discrete}, \\ \dfrac{f_{\underline{X}|\Theta}(\underline{x}|\theta)P_\Theta(\theta)}{f_{\underline{X}}(\underline{x})}, & \text{if } \underline{X} \text{ continuous}. \end{cases}$$

$$f_{\Theta|\underline{X}}(\theta|\underline{x}) = \begin{cases} \dfrac{P_{\underline{X}|\Theta}(\underline{x}|\theta)f_\Theta(\theta)}{P_{\underline{X}}(\underline{x})}, & \text{if } \underline{X} \text{ discrete}, \\ \dfrac{f_{\underline{X}|\Theta}(\underline{x}|\theta)f_\Theta(\theta)}{f_{\underline{X}}(\underline{x})}, & \text{if } \underline{X} \text{ continuous}. \end{cases}$$

Notes: The denominator $f_{\underline{X}}(\underline{x})$ is independent of θ:

$$f_{\underline{X}}(\underline{x}) = \int_{-\infty}^{\infty} f_{\underline{X}|\Theta}(\underline{x}|\theta) f_\Theta(\theta) d\theta.$$

7.3 Comparison: ML vs MAP

Definition:

$$\hat{\theta}_{ML} = \arg\max_{\theta} P_{\underline{X}}(\underline{x}|\theta),$$

$$\hat{\theta}_{MAP} = \arg\max_{\theta} P_{\underline{X}|\Theta}(\underline{x}|\theta)P_\Theta(\theta).$$

The expressions are algebraically similar, but the philosophies differ.

7.4 Example: Unknown Voltage

Example:

1. For an unknown voltage, we have some prior knowledge that θ is uniformly distributed in $[0, 1]$:

$$f_\Theta(\theta) = \begin{cases} 1, & 0 \leq \theta \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

2. We measure it using a voltmeter that outputs random readings Y that is uniformly distributed in $[0, \theta]$. Suppose we make n measurements $\underline{Y} = (Y_1, Y_2, \ldots, Y_n)$.
3. How to estimate θ?

To Warm Up: ML Estimation (Ignoring the Prior)

$$\begin{aligned} f_{\underline{Y}|\Theta}(\underline{y}|\theta) &= \prod_{i=1}^{n} f_{Y|\Theta}(y_i|\theta) \\ &= \frac{1}{\theta^n} \prod_{i=1}^{n} 1(0 \leq y_i \leq \theta). \end{aligned}$$

$$\hat{\theta}_{ML} = \arg\max_\theta \frac{1}{\theta^n} 1(\theta \geq \max_{1 \leq i \leq n} y_i) = \max_{1 \leq i \leq n} y_i.$$

Solution: MAP Estimation

$$\begin{aligned} f_{\Theta|\underline{Y}}(\theta|\underline{y}) &= \frac{f_{\underline{Y}|\Theta}(\underline{y}|\theta) f_\Theta(\theta)}{f_{\underline{Y}}(\underline{y})} \\ &\propto f_{\underline{Y}|\Theta}(\underline{y}|\theta) f_\Theta(\theta) \\ &= \frac{1}{\theta^n} 1(\theta \geq \max_{1 \leq i \leq n} y_i) 1(0 \leq \theta \leq 1). \end{aligned}$$

$$\hat{\theta}_{MAP} = \arg\max_\theta \frac{1}{\theta^n} 1(\theta \geq \max_{1 \leq i \leq n} y_i) 1(0 \leq \theta \leq 1) = \max_{1 \leq i \leq n} y_i.$$

What If the Prior Is Different? Suppose the prior is:

$$f_\Theta(\theta) = \begin{cases} 2, & \frac{1}{2} \leq \theta \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

This means "The voltage is at least $\frac{1}{2}$ but no more than 1." Then:

$$f_{\Theta|\underline{Y}}(\theta|\underline{y}) \propto \frac{1}{\theta^n} 1(\theta \geq \max_{1 \leq i \leq n} y_i) 1\left(\frac{1}{2} \leq \theta \leq 1\right),$$

$$\hat{\theta}_{MAP} = \max\left\{\max_{1 \leq i \leq n} y_i, \frac{1}{2}\right\}.$$

Notes:

1. $\max_{1 \leq i \leq n} y_i$ is a sufficient statistic for ML and MAP in this scenario.
2. $\hat{\theta}_{MAP} \to \max_{1 \leq i \leq n} y_i$ as $n \to \infty$, regardless of the prior.
3. MAP optimization typically requires more computational effort than ML estimation.

8 L8: MAP Conjugate Prior

9 L9: Least Mean Squares (LMS) Estimation

Definition: Assume prior: $P_\Theta(\theta)$ or $f_\Theta(\theta)$ w/ observations: $\bar{X} = x$.

$$\hat{\theta} = g(\underline{x}) = \mathbb{E}[\Theta \mid \underline{X} = \underline{x}]$$
$$\text{or } \hat{\Theta} = g(\underline{X}) = \mathbb{E}[\Theta \mid \underline{X}].$$

Notes:

1. **MAP vs. LMS Estimators:**
   - **MAP:** Use the most probable θ given x.
   - **LMS:** Use the expected value (conditional on $\bar{X} = x$) of Θ, i.e., the "Conditional Expectation Estimator."
2. **Unbiasedness of LMS Estimator:**

$$\mathbb{E}[\hat{\Theta}] = \mathbb{E}[\mathbb{E}[\Theta \mid \underline{X}]] = \mathbb{E}[\Theta],$$
$$\implies \mathbb{E}[\hat{\Theta} - \Theta] = 0.$$

3. **LMS Estimator Minimizes Conditional MSE:**

$$\mathbb{E}\big[(\Theta - \hat{\Theta})^2 \mid \underline{X} = \underline{x}\big].$$

Proof:

(a) First, suppose no observations: $\hat{\Theta}$ is a constant. So we want:

$$\begin{aligned}\hat{\Theta} &= \arg\min_c \mathbb{E}\big[(\Theta - c)^2\big],\\ 0 &= \frac{d}{dc}\big[-2\mathbb{E}[\Theta] + 2c\big],\\ c &= \mathbb{E}[\Theta].\end{aligned}$$

(b) Alternate view:

$$\begin{aligned}\mathbb{E}\big[(\Theta - c)^2\big] &= \mathrm{Var}[\Theta - c] + \mathbb{E}[\Theta - c]^2,\\ &= \mathrm{Var}[\Theta] + (\mathbb{E}[\Theta] - c)^2.\end{aligned}$$

To minimize: Set bias $\mathbb{E}[\Theta] - c$ to zero, while for variance, we have no control.

(c) Now, with observations $\underline{X} = \underline{x}$ (i.e. given): $\hat{\theta} = g(\underline{x})$

$$\mathbb{E}\big[(\Theta - g(\underline{x}))^2 \mid \underline{X} = \underline{x}\big] = \mathrm{Var}[\Theta \mid \underline{X} = \underline{x}] + (\mathbb{E}[\Theta \mid \bar{X} = \underline{x}] - g(\underline{x}))^2.$$

To minimize: Set $(g(x) - \mathbb{E}[\Theta \mid \underline{X} = \underline{x}])^2 = 0$ since we have no control over the variance.

(d) Conclusion:

$$\hat{\theta} = g(x) = \mathbb{E}[\Theta \mid \underline{X} = \underline{x}].$$

9.1 Example: Prior Coin Toss Problem

Example:

$$\begin{aligned}\hat{\theta}_{\text{LMS}} &= \mathbb{E}[\Theta \mid X = k]\\ &= \frac{k + \alpha}{n + \alpha + \beta}.\end{aligned}$$

9.2 Example: Prior Voltage Problem

Example:

1. **Setup:**
   - Unknown voltage Θ.
   - Prior: $\Theta \sim \text{Uniform}[0, 1]$.
   - Volt meter reading Y given $\Theta = \theta$: $Y \sim \text{Uniform}[0, \Theta]$.
   - Independent measurements: $Y_1, \ldots, Y_n$ given θ.
2. **Likelihood:**

$$\begin{aligned} f_{\underline{Y}|\Theta}(\underline{y} \mid \theta) &= \prod_{i=1}^{n} f_{Y|\Theta}(y_i \mid \theta) \\ &= \frac{1}{\theta^n} \cdot 1(\theta \geq \max_{1 \leq i \leq n} y_i). \end{aligned}$$

3. **Posterior:**

$$f_{\Theta|\underline{Y}}(\theta \mid \underline{y}) = \frac{\frac{1}{\theta^n} \cdot \mathbf{1}(\theta \geq \max_{1 \leq i \leq n} y_i)\mathbf{1}(0 \leq \theta \leq 1)}{f_{\underline{Y}}(\underline{y})}.$$

4. **Estimators:**
   - ML/MAP:

$$\hat{\theta} = \max_{1 \leq i \leq n} y_i.$$

   - LMS:

$$\hat{\theta} = \mathbb{E}[\Theta \mid \underline{Y} = \underline{y}] = \int_{-\infty}^{\infty} \theta f_{\Theta|\underline{Y}}(\theta \mid \underline{y}) d\theta.$$

5. **Derivation for LMS:**
   (a) We need $f_{\underline{Y}}(\underline{y}) = \displaystyle\int_{-\infty}^{\infty} \frac{1}{\theta^n} \cdot \mathbf{1}(\theta \geq \max_{1 \leq i \leq n} y_i)\mathbf{1}(0 \leq \theta \leq 1) d\theta$

   (b) Compute $f_{\underline{Y}}(\underline{y})$ for $n = 1$:

$$\begin{aligned} f_{\underline{Y}}(\underline{y}) &= \int_{y}^{1} \frac{1}{\theta} d\theta, \quad 0 \leq y \leq 1 \\ &= \ln(\theta)\Big|_{y}^{1} \\ &= -\ln(y). \end{aligned}$$

   (c) Compute $\hat{\theta}$ for $n = 1$:

$$\begin{aligned} \hat{\theta} &= \int_{-\infty}^{\infty} \theta \frac{\frac{1}{\theta}\mathbf{1}(\theta \geq y) \cdot \mathbf{1}(0 \leq \theta \leq 1)}{-\ln y} d\theta \\ &= \frac{1}{-\ln(y)} \int_{y}^{1} d\theta \\ &= \frac{y-1}{ln(y)} \end{aligned}$$

6. **Graphical Interpretation:**

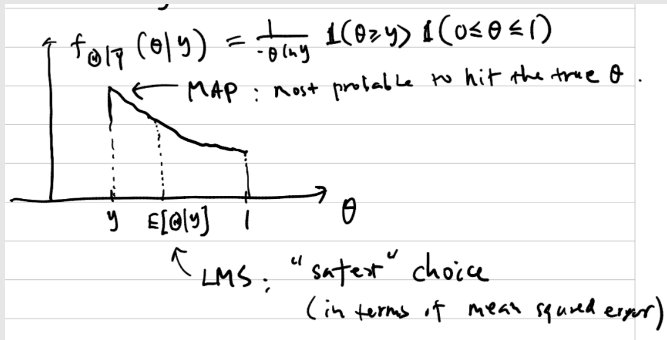
$f_{\Theta|Y}(\theta|y) = \frac{1}{-\theta \ln y} \mathbb{1}(\theta \geq y) \mathbb{1}(0 \leq \theta \leq 1)$
MAP : most probable to hit the true θ.
y
E[Θ|y]
1
θ
LMS : "safest" choice
(in terms of mean squared error)

Figure 13: