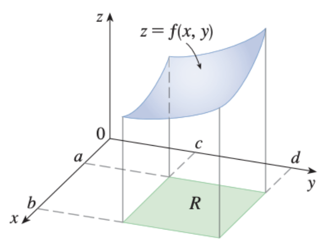
AER210: Stewart Textbook

15. Multiple Integrals

15.0 Double Integrals over Rectangles:

15.0.0 Important Formulas for Riemann Sums to Find Volume of S:

1. If $R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | \, a \le x \le b, \ c \le y \le d\}$ defining a closed rectangle.
2. Suppose $f(x, y) \ge 0$, and let S be the solid that lies above R and under the graph of f, $S = \{(x, y, z) \in \mathbb{R}^3 \,|\, 0 \le z \le f(x, y), \ (x, y) \in \mathbb{R}\}$.



3. Divide the rectangle R into subrectangles.

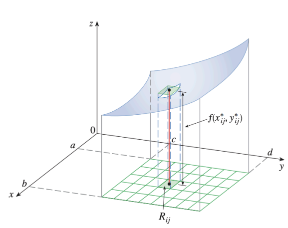
$$[a, b] \ into \ m \ subintervals \ [x_{i-1}, \ x_i], \ Width: \ \Delta x = \frac{b-a}{m}$$

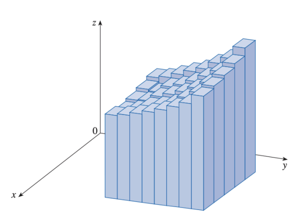
$$[c, d] \ into \ n \ subintervals \ [y_{i-1}, y_i], \ Width: \ \Delta y = \frac{d-c}{n}$$

4. Form the subrectangles:

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \,|\, x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$$

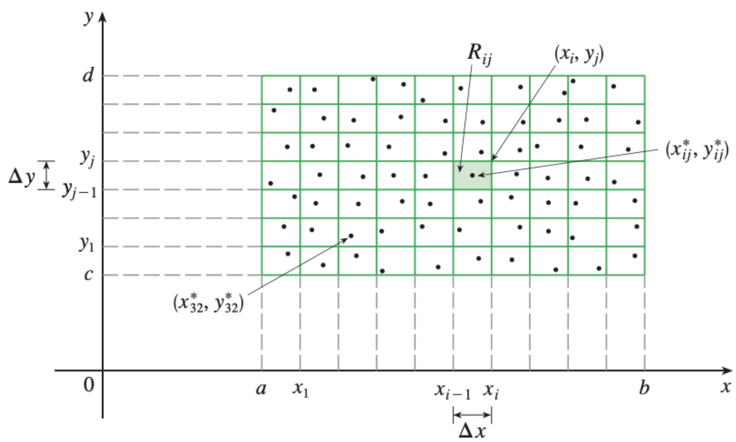
5. Each of these subrectangles has area of $\Delta A = \Delta x \Delta y$
6. We can choose a **sample point** (x_{ij}^*, y_{ij}^*) in each subrectangle R_{ij} then we can approximate all rectangular columns with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ to get the volume.





$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \Delta A$$

- For the double Riemann sum, treat the inner sum as the one you have to iterate through all the bigger ones.
  - The double sum is that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle and add the results.
- Choose the **sample point in the top right corner of R_{ij}** (i.e. (x_i, y_j)).



15.0.1 Definition of Double Integral:

The **double integral** of f over the rectangle R is

$$V = \iint_R f(x, y)dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \Delta A$$

If this limit exists.

- This evaluates the **volume** of the solid if $f(x, y) \geq 0$ that lies under the graph of f and above the rectangle R.
- Work from the inside out.

15.0.3 Midpoint Rule for Double Integrals:

$$\iint_R f(x, y)dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j) \Delta A$$

Where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$

15.0.4 Fubini's Theorem:

If f is continuous on the rectangle

$$R = \{(x, y)|\ a \le x \le b,\ c \le y \le d\}$$

Then

$$\iint_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dy\ dx = \int_c^d \int_a^b f(x, y)dx\ dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

- When choosing which to integrate first, choose the simpler one.

15.0.5 Special Case:

Suppose $f(x, y) = g(x)h(y)$ and $R = [a, b] \times [c, d]$

$$\iint_R g(x)h(y)dA = \int_a^b g(x)dx \int_c^d h(y)dy \text{ where } R = [a, b] \times [c, d]$$

15.0.6 Average Value

A function f of two variables defined on a rectangle R to be

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y)dA$$

Where A(R) is the area of R.

If $f(x, y) \ge 0$, the equation

$$A(R) \times f_{avg} = \iint_R f(x, y)dA$$

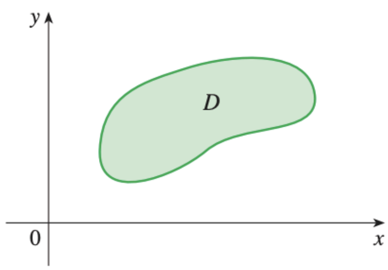
15.2 Double Integrals over General Regions:

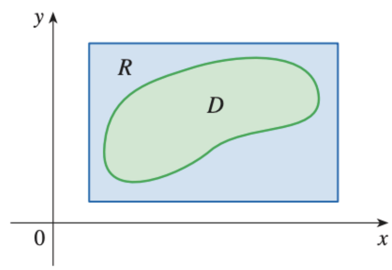
15.2.0 Double Integral over a General Region

If F is integrable over R, then we define the **double integral of f over D as**

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$





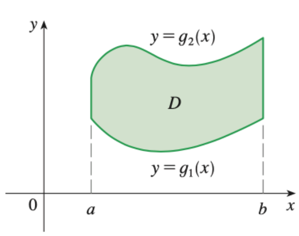
- Choose any rectangle R which contains D.

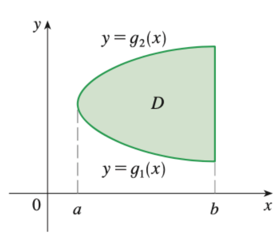
15.2.1 Type I Region

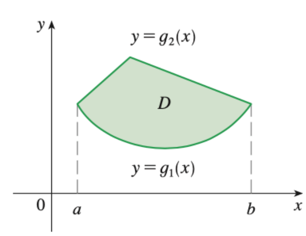
A plane region D is **type I** if it lies between the graphs of two continuous functions of x:

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

Where g_1 and g_2 are continuous on $[a, b]$.







- g_1 and g_2 must be continuous but doesn't have to be defined by a single formula.

15.2.2 Type I Integral

If f is continuous on a type I region D described by

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

Then

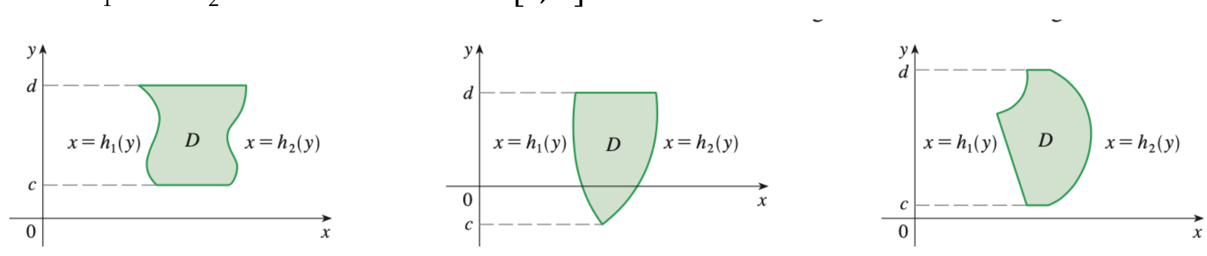
$$\int\int_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$$

15.2.3 Type II Region

A plane region D is **type II** if it lies between the graphs of two continuous functions of y:

$$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

Where h_1 and h_2 are continuous on $[c, d]$.



15.2.4 Type II Integral

If f is continuous on a type II region D described by

$$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

Then

$$\int\int_D f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy$$

15.2.5 Properties of Double Integrals

Property 5

$$\iint_D [f(x,y) + g(x,y)]dA = \iint_D f(x,y)dA + \iint_D g(x,y)dA$$

Property 6

$\iint_D cf(x,y)dA = c\iint_D f(x,y)dA$ where c is a constant

Property 7

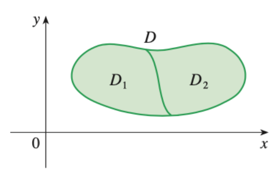
If $f(x,y) \geq g(x,y)$ $for\ all\ (x,y)\ in\ D$, then

$$\iint_D f(x,y)dA \geq \iint_D g(x,y)dA$$

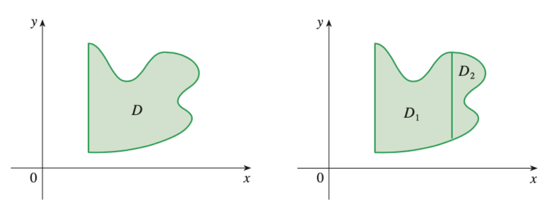
Property 8

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$



- Evaluate double integrals over regions D that are **not type I or type II,** but as a **union of regions of type I or type II.**



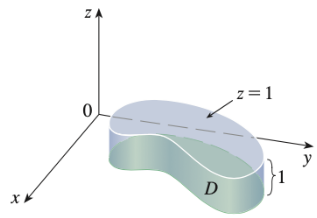
(a) D is neither type I nor type II.

(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

Property 9

Integrating $f(x,y) = 1$ $over\ a\ region\ D,\ we\ get\ the\ area\ of\ D$:

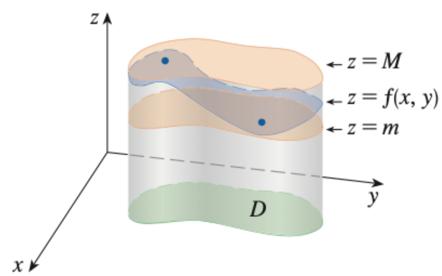
$$\iint_D 1dA = A(D)$$



Property 10

If $m \le f(x, y) \le M$ for all (x, y) $in\ D,\ then$

$$m \cdot A(D) \le \iint_D f(x, y)dA \le M \cdot A(D)$$



$Case:\ m > 0$

15.3 Double Integrals in Polar Coordinates

15.3.0 Relationship between Polar and Cartesian Coordinates:

$$x = rcos\theta,\ y = rsin\theta$$
$$r^2 = x^2 + y^2,\ tan\theta = \frac{y}{x}$$

- Be careful about the quadrant by making sure θ is correct.

15.3.1 Change to Polar Coordinates in a Double Integral:

If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_D f(x, y)dA = \int_\alpha^\beta \int_a^b f(rcos\theta, rsin\theta)rdrd\theta$$

- DONT forget the r on the right side of the formula.

- To convert from rectangular to polar, write x=rcostheta and y=rsintheta
- SPECIAL CASE WORKS WELL THESE.

15.3.2 Similar to Type II Regions

If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

Then

$$\iint_D f(x, y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(rcos\theta, rsin\theta)rdrd\theta$$

15.4 Applications of Double Integrals

15.4.0 Total Mass of the Lamina

$$m = \lim_{k,l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D \rho(x, y)dA$$

15.4.1 Total Electric Charge

If an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ at a point (x,y) in D, then the total electric charge Q is given by

$$Q = \iint_D \sigma(x, y)dA$$

15.4.2 Moment of the Entire Lamina About the X-Axis:

$$M_x = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y\rho(x, y)dA$$

15.4.3 Moment of the Entire Lamina About the X-Axis:

$$M_y = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x\rho(x, y)dA$$

15.4.4 Centre of Mass of a Lamina

The coordinates $(\overline{x}, \overline{y})$ of the centre of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)dA \qquad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)dA$$

Where the mass m is given by

$$m = \iint_D \rho(x, y)dA$$

- The lamina behaves as if its entire mass is concentrated at its centre of mass. Thus the lamina balances horizontally when supported at its centre of mass.

15.4.5 Moment of Inertia of the Lamina About the X-Axis:

$$I_x = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y^2 \rho(x, y)dA$$

15.4.6 Moment of Inertia of the Lamina About the Y-Axis:

$$I_y = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x^2 \rho(x, y)dA$$

15.4.7 Moment of Inertia of the Lamina About the Origin (Polar Moment of Inertia):

$$I_0 = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} [(x_{ij}^*)^2 + (y_{ij}^*)^2] \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D (x^2 + y^2) \rho(x, y)dA$$

- $I_0 = I_x + I_y$

15.4.8 Radius of Gyration of a Lamina About an Axis

$$mR^2 = I$$

Where R is the radius of gyration, m is the mass of the lamina and I is the moment of inertia about the given axis.

- If the mass of the lamina were concentrated at a distance R from the axis, then the moment of inertia of this "point mass" would be the same as the moment of inertia of the lamina.

15.4.9 Radius of Gyration $\bar{y}$ w.r.t X-Axis and Radius of Gyration $\bar{x}$ w.r.t. Y-Axis:

$$m\bar{y}^2 = I_x \quad m\bar{x}^2 = I_y$$

- There should be a DOUBLE BAR above x and y.
- $(\bar{x}, \bar{y})$ (DOUBLE BAR) is the point at which the mass of the lamina can be concentrated without changing the moments of inertia w.r.t the coordinate axes.

15.4.10 Probability Density Function f of a continuous random variable X

$f(x)\geq 0 \ for\ all\ x,\ \int_{-\infty}^{\infty} f(x)dx = 1,\ and\ the\ probability\ that\ X\ lies\ between\ a\ and\ b\ is$

$$P(a\leq X\leq b) = \int_a^b f(x)dx$$

15.4.11 Joint Density Function of X and Y

Function f of two variables such that the probability that (X,Y) lies in a region D is

$$P(a\leq X\leq b, c\leq Y\leq d) = \int_a^b \int_c^d f(x, y)dydx$$

- Joint density functions have the following properties:
  - $f(x, y)\geq 0$
  - $\iint_{\mathbb{R}^2} f(x, y)dA = 1$

15.4.12 Independent Random Variables

Suppose X is a random variable with probability density function $f_1(x)$ and Y is a random variable with density function $f_2(y)$. X and Y are independent random variables if their joint density function is

$$f(x, y) = f_1(x)f_2(y)$$

15.4.13 Mean

If X is a random variable with probability density function f, then its mean is

$$\mu = \int_{-\infty}^{\infty} xf(x)dx$$

15.4.14 X-Mean and Y-Mean (Expected Values of X and Y)

If X and Y are random variables with joint density function f, then the expected values of X and Y is

$$\mu_1 = \iint_{\mathbb{R}^2} xf(x, y)dA \qquad \mu_2 = \iint_{\mathbb{R}^2} yf(x, y)dA$$

15.5 Surface Area

The area of the surface with equation $z = f(x, y), (x, y) \in D$, where $f_x \, and \, f_y$ are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}dA$$

$$A(S) = \iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + [\frac{\partial z}{\partial y}]^2}dA$$

15.6 Triple Integrals

- Limits of integration in the inner integral contain at most 2 variables.
- Limits of integration in the middle integral contain at most 1 variable.
- Limits of integration in the outer integral must be constants.

15.6.0 Definition of Triple Integral

The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$

- Triple integral always exists if f is continuous.
- If we choose the point to be (x_i, y_j, z_k), we get:

$$\iiint_B f(x, y, z)dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta V$$

15.6.1 Fubini's Theorem for Triple Integral

If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)dV = \int_r^s \int_c^d \int_a^b f(x, y, z)dxdydz$$

- There are **6 total possible ways to integrate,** which all GIVE THE SAME VALUE.
    - Allows for simple integrals to be evaluated than others.
- The triple integral has the same properties as the double integral (properties 5-8).

15.6.2.0 Type I Region

A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) \mid (x, y) \in D, \ u_1(x, y) \leq z \leq u_2(x, y)\}$$

Where D is the projection of E onto the xy-plane.

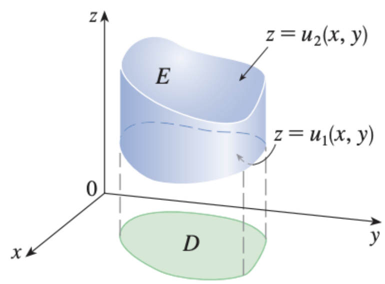


FIGURE 2
A type 1 solid region

15.6.2.1 General Type I Integral

If E is a type I region, then

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right]dA$$

- The inner integral has x and y held as constants, therefore, u_1 and u_2 are constants as well, while f is integrated w.r.t. to z.
- The upper surface is $z = u_2(x, y)$, the lower surface is $z = u_1(x, y)$

15.6.2.2 Type I Solid Region with a Type I Plane Region:

If the projection D of E onto the xy-plane is a type I plane region, then

$$E = \{(x, y, z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x, y) \le z \le u_2(x, y)\}$$

Which becomes

$$\iiint_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdydx$$

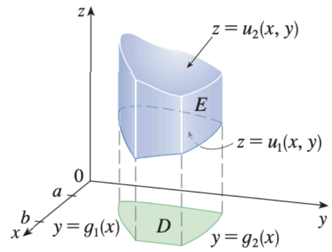


FIGURE 3
A type 1 solid region where the projection D is a type I plane region

15.6.2.3 Type I Solid Region with a Type II Plane Region:

If the projection D of E onto the xy-plane is a type II plane region, then

$$E = \{(x, y, z) \mid c \le y \le d, h_1(y) \le x \le h_2(y), u_1(x, y) \le z \le u_2(x, y)\}$$

Which becomes

$$\iiint_E f(x, y, z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dzdxdy$$

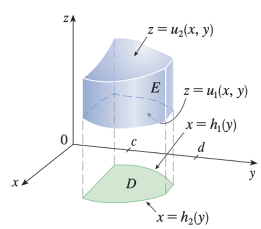


FIGURE 4
A type 1 solid region with a type II projection

15.6.2.4 Type II Region

A solid region E is of **type 2** of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \le z \le u_2(y, z)\}$$

Where, D is the projection of E onto the yz-plane.

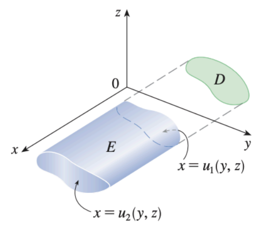


FIGURE 8
A type 2 region

15.6.2.5 General Type II Integral

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)dx\right]dA$$

- The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$
- D being a type I or type II extensions

15.6.2.6 Type III Region

A **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \le z \le u_2(x, z)\}$$

Where D is the projection of E onto the xz-plane.

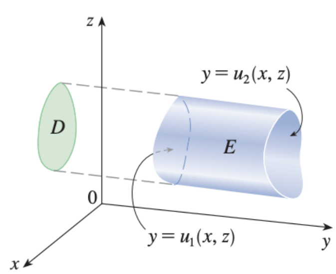


FIGURE 9
A type 3 region

15.6.2.7 General Type III Integral

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)dy\right]dA$$

- The left surface is $y = u_1(x, z)$, the right surface is $y = u_2(x, z)$

- D being a type I or type II extensions

15.6.3 Special Case

$If\ f(x, y, z) = 1\ for\ all\ points\ in\ E.\ Then\ the\ triple\ integral\ does\ represent\ the$ volume of E:

$$V(E) = \iiint_E dV$$

15.6.4 Total Mass of E

$$m = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} \rho(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*})\Delta V = \iiint_E \rho(x, y, z)dV$$

15.6.5 Moments of E About the 3 Coordinate Planes

$$M_{yz} = \iiint_E x\rho(x, y, z)dV \qquad M_{xz} = \iiint_E y\rho(x, y, z)dV$$

$$M_{xy} = \iiint_E z\rho(x, y, z)dV$$

15.6.6 Centre of Mass

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m}$$

$$\bar{z} = \frac{M_{xy}}{m}$$

- If the density is constant, the centre of mass of the solid is called the **centroid of E.**

15.6.7 Moments of Inertia About the Three Coordinates Axes:

$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)dV \qquad I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)dV$$

$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)dV$$

15.6.8 Total Electric Charge

The total electric charge on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)dV$$

15.6.9 Joint Density Function

If we have three continuous random variables X, Y, and Z, their joint density function is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$P((X, Y, Z) \in E) = \iiint_E f(x, y, z)dV$$

In particular,

$$P(a \leq X \leq b, c \leq Y \leq d, r \leq Z \leq s) = \int_a^b \int_c^d \int_r^s f(x, y, z)dzdydx$$

The joint density function satisfies

$$f(x, y, z) \geq 0 \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)dzdydx = 1$$

15.7 Triple Integrals in Cylindrical Coordinates

- In the cylindrical coordinate system, a point P is (r, θ, z), where r *and* θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.

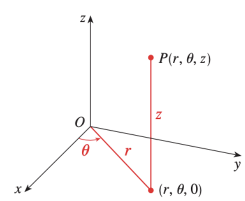


FIGURE 2
The cylindrical coordinates of a point

- **Cylindrical useful INVOLVING SYMMETRY ABOUT AN AXIS**
  - Z-axis is chosen to coincide with the axis of symmetry.

15.7.0 Cylindrical to Rectangular Coordinates

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

15.7.1 Rectangular to Cylindrical Coordinates

$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x} \quad z = z$$

15.7.2 Common Shapes in Cylindrical Coordinates

1. $r = c \text{ is a cylinder}$
2. $\theta = c \text{ is a vertical plane through the origin}$
3. $z = c \text{ is a horizontal plane}$

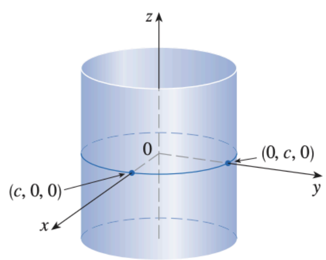


FIGURE 4
$r = c$, a cylinder

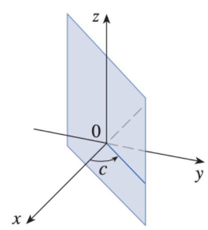


FIGURE 5
$\theta = c$, a vertical plane

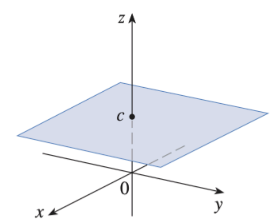


FIGURE 6
$z = c$, a horizontal plane

15.7.3 Triple Integrals in Cylindrical Coordinates

$$\iiint_E f(x, y, z)\,dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

- Use this when the function has the expression $x^2 + y^2$

15.8 Triple Integrals in Spherical Coordinates

- The **spherical coordinates** (ρ, θ, ϕ) of a point P in space, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP.

$$\rho \geq 0 \quad 0 \leq \phi \leq \pi$$

15.8.0 Common Shapes in Spherical Coordinates

- Useful where there is symmetry about a point, and the origin is placed at this point.

1. $\rho = c \; is \; a \; sphere$
2. $\theta = c \; is \; a \; half - plane$
3. $\phi = c \; is \; a \; half \; cone$

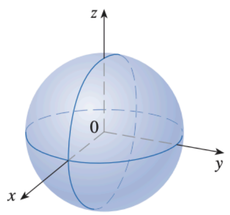


FIGURE 2 $\rho = c$, a sphere

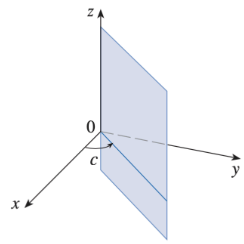


FIGURE 3 $\theta = c$, a half-plane

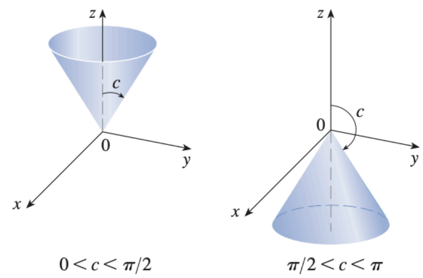


FIGURE 4 $\phi = c$, a half-cone

15.8.1 Spherical to Rectangular Coordinates

$$x = \rho sin\phi cos\theta \qquad y = \rho sin\phi sin\theta \qquad z = \rho cos\phi$$

15.8.2 Rectangular to Spherical Coordinates

$$\rho^2 = x^2 + y^2 + z^2 \qquad z = \rho cos\phi$$

15.8.3 Formula for Triple Integration in Spherical Coordinates

$$\iiint_E f(x, y, z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho sin\phi cos\theta, \rho sin\phi sin\theta, \rho cos\phi)\rho^2 sin\phi d\rho d\theta d\phi$$

Where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \le \rho \le b, \; \alpha \le \theta \le \beta, \; c \le \phi \le d\}$$

- Convert from rectangular to spherical in the function by writing x,y,z using 15.8.1-2
- DONT FORGET $\rho^2 sin\phi$ at the end

15.8.3 Formula for General Spherical Regions

$$E = \{(\rho, \theta, \phi) \mid \alpha \le \theta \le \beta, \; c \le \phi \le d, \; g_1(\theta, \phi) \le \rho \le g_2(\theta, \phi)\}$$

- Same thing however, the limits of integration for ρ are different.

15.9 Change of Variables in Multiple Integrals

- If we aren't given a transformation, then the **first step** is to think of an appropriate change of variables. If $f(x, y)$ is difficult to integrate, **then the form of f(x,y) may suggest a transformation.**

15.9.0 Definition of Jacobian of the Transformation for a 2×2

7 Definitio The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

15.9.1 Change of Variables in a Double Integral

> **9 Change of Variables in a Double Integral** Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then
>
> $$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

15.9.2 Definition of Jacobian of the Transformation for a 3×3

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

15.9.3 Change of Variables in a Double Integral

$$\int\int\int_R f(x, y, z) dV = \int\int\int_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$$

16. Vector Calculus

16.1 Vector Fields

- If $\mathbf{x} = < x, y, z >$, we write $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x, y, z)$, then $\mathbf{F}$ becomes a function that assigns a vector $\mathbf{F}(\mathbf{x})$ to a vector $\mathbf{x}$.

16.1.0 Definition of a Vector in $\mathbb{R}^2$

Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function **F** that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$**.**

16.1.1 Definition of a Vector in $\mathbb{R}^3$

Let E be a subset of $\mathbb{R}^3$**.** A **vector field on** $\mathbb{R}^3$ is a function **F** that assigns to each point (x,y,z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

16.1.2 Drawing $\mathbb{R}^2$/$\mathbb{R}^3$ Vector Field

Draw the arrow representing the vector $\mathbf{F}(x, y)$ & **F**(x,y,z) starting at the point (x,y) & (x,y,z) respectively.

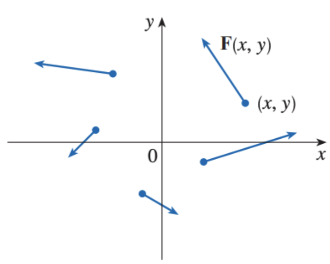


FIGURE 3
Vector field on $\mathbb{R}^2$

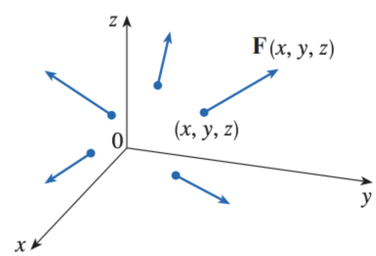


FIGURE 4
Vector field on $\mathbb{R}^3$

16.1.3 Component Functions P and Q of F(x,y) & P, Q, and R of F(x,y,z):

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = < P(x, y), Q(x, y) >$$
$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$$

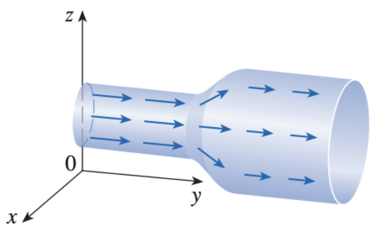
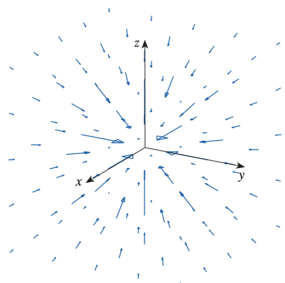
$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

- P, Q & R are scalar functions of two/three variables and are sometimes called **scalar fields.**

16.1.4 Continuity of Vector Fields

F is continuous if and only if its component functions P, Q, and R are continuous.

16.1.5 Different Types of Vector Fields

Vector Field	
Velocity Fields 	The speed at any given point is indicated by the length of the arrow, where **V**(x,y,z) is the velocity vector at point (x,y,z)
Gravitational Fields 	$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{\|x\|^3}\mathrm{x}$ The force **F**(**x**) at any given point **x** in space is given by the length of the arrow.
Electric Fields	$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{\|x\|^3}\mathrm{x}$
Gradient Fields	$\nabla f(x) = < f_x, f_y, f_z >$ ∇f is a vector field in $\mathbb{R}^2/\mathbb{R}^3$
Conservative Vector Fields	If there exists a function f such that $\mathbf{F} = \nabla f$ Where **F** is the gradient of some scalar function. f is called a **potential function** for **F.**

16.2 Line Integrals

- We are integrating over a curve C.

- The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b

16.2.0 Line Integral of f along C in $\mathbb{R}^2/\mathbb{R}^3$

$$\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$$

$\mathbb{R}^2$:

If f is defined on a smooth curve C given by

$$x = x(t) \mid y = y(t) \mid a \le t \le b \text{ or equivalently, } \mathrm{r}(t) = x(t)\mathrm{i} + y(t)\mathrm{j}$$

Then the **line integral of f along C** (w.r.t arc length) is

$$\int_C f(x,y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i = \int_a^b f(x(t), y(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$$

If this limit exists

$\mathbb{R}^3$:

If f is defined on a smooth space curve C given by

$$x = x(t),\ y = y(t),\ z = z(t),\ a \le t \le b \mid \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

Then the **line integral of f along C** (w.r.t arc length) is

$$\int_C f(x,y,z)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i = \int_a^b f(x(t), y(t), z(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}dt$$

If this limit exists

- If s(t) is the length of C between **r**(a) and **r**(t), then
  $\frac{ds}{dt} = |r'(t)| = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$
- Express everything in terms of the parameter t: **use the parametric equations to express x and y in terms of t and write ds alone.**

16.2.1 Special Case for Line Integral (Single Integral)

If C is the line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, the parametric equations of C:

$$x = x,\ y = 0,\ a \leq x \leq b$$

Then

$$\int_C f(x, y)ds = \int_a^b f(x, 0)dx$$

16.2.2 How to Interpret a Line Integral

We can interpret the line integral of a positive function as an area. If

$f(x, y) \geq 0$, $\int_C f(x, y)ds$ represents the area of one side of the "fence" whose base is C and whose height above the point (x,y) is f(x,y)

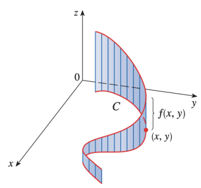
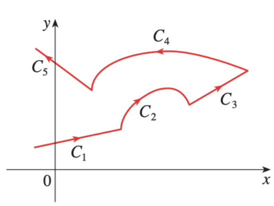


FIGURE 2

16.2.3 Piecewise-Smooth Curve Line Integral of f along C:

If C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where the initial point of C_{i+1} is the terminal point of C_i. Then

$$\int_C f(x, y, z)ds = \int_{C_1} f(x, y, z)ds + \int_{C_2} f(x, y, z)ds + \ldots + \int_{C_n} f(x, y, z)ds$$



16.2.4 Mass of the Wire

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\Delta s_i = \int_C \rho(x, y)ds$$

16.2.5 Centre of Mass of the Wire

The **center of mass** of the wire with density function ρ is located at the point $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{1}{m}\int_C x\rho(x, y)ds \mid \overline{y} = \frac{1}{m}\int_C y\rho(x, y)ds$$

16.2.6 Line Integrals w.r.t. x, y, and z

Line integrals w.r.t. x, y, and z can be evaluated by expressing everything in terms of t:

$$x = x(t),\ y = y(t),\ z = z(t),\ dx = x'(t)dt,\ dy = y'(t)dt,\ dz = z'(t)dt$$

Then

$$\int_C f(x, y, z)dx = \int_a^b f(x(t), y(t), z(t))x'(t)dt$$

$$\int_C f(x, y, z)dy = \int_a^b f(x(t), y(t), z(t))y'(t)dt$$

$$\int_C f(x, y, z)dz = \int_a^b f(x(t), y(t), z(t))z'(t)dt$$

OR

$$\int_C P(x, y, z)dx + \int_C Q(x, y, z)dy + \int_C R(x, y, z)dz = \int_C P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$$

- Make sure to express everything in terms of t.
- Just remember if it is with two variable functions, just remove the z.

16.2.7 Vector Representation of a Line Segment

A vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

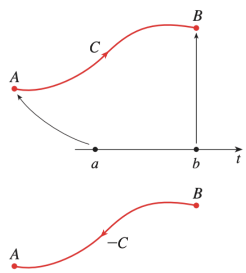
$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \mid 0 \le t \le 1$$

- We often need to parametrize a line segment to find a **parametric representation** for a curve.

16.2.8 Path/Orientation

In general, the value of a line integral depends not just on the endpoints of the curve but also on the path.

In general, a given parametrization $x = x(t),\ y = y(t),\ a \le t \le b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t. If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation, then



Line Integral w.r.t x or y:

$$\int_{-C} f(x,y)dx = -\int_{C} f(x,y)dx \mid \int_{-C} f(x,y)dy = -\int_{C} f(x,y)dy$$

Line Integral w.r.t arc length:

$$\int_{-C} f(x,y)ds = \int_{C} f(x,y)ds$$

- Since $\Delta s_i > 0$, whereas Δx_i *and* Δy_i change sign when we reverse the orientation of C.

16.2.9 Special Case f(x,y,z)=1:

$$\int_{C} ds = \int_{a}^{b} |r'(t)|dt = L$$

Where L is the length of the curve C.

16.2.10 Work Done by the Force Field F

$$W = \int_C F(x, y, z) \cdot T(x, y, z)ds = \int_C F \cdot Tds$$

- Work is the line integral w.r.t arc length of the tangential component of the force.
- T is the unit tangent vector at (x,y,z) on C

16.2.11.0 Definition of Line Integral of F along C

Let **F** be a continuous vector field defined on a smooth curve C given by a vector function **r**(t), $a \leq t \leq b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_C \mathbf{F} \cdot \mathbf{T}ds$$

- $\mathbf{r}(t) = < x(t), y(t), z(t) >$
- $d\mathbf{r} = \mathbf{r}'(t)dt$
- This helps to find the work done by the force field.

16.2.11.1 Note

Even though $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}ds$ and integrals w.r.t arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

Because the unit tangent vector **T** is replaced by its negative when C is replaced by $-C$.

16.2.12 Connection Between Line Integrals of Vector Fields and Line Integrals of Scalar Fields in $\mathbb{R}^2/\mathbb{R}^3$

Suppose the vector field **F** on $\mathbb{R}^2/\mathbb{R}^3$ is given in component form by the equation

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} \mid \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

Then

$$\int_C \mathbf{F}\cdot\, d\mathbf{r} = \int_C Pdx + Qdy \mid \int_C \mathbf{F}\cdot\, d\mathbf{r} = \int_C Pdx + Qdy + Rdz$$

16.3 The Fundamental Theorem for Line Integrals

16.3.0 Theorem 2

Let C be a smooth curve given by the vector function **r**(t), $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

16.3.1 Notes for Theorem 2

Note 1:

- We can evaluate the line integral of a conservative vector field (the gradient vector field of the potential function f) by **having the value of f at the endpoints on C.**
- The line integral of ∇f is the **net change in f.**

If f is a function of two variables and C is a plane curve with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$

If f is a function of three variables and C is a space curve with initial point $A(x_1, y_1, z_1)$ and terminal point $B(x_2, y_2, z_2)$, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Note 2:

If C_1 and C_2 are smooth curves with the same initial points and the same terminal points, then we can conclude that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

- Note: Since **r**(t), therefore d**r**=**r**′(t)dt
- The line integral of a **conservative vector field** depends only on the **end points of a curve.**

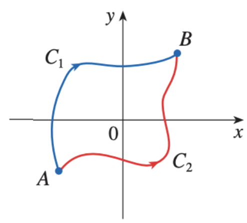


FIGURE 2
$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$

Note 3:

- Theorem 2 is also true for piecewise-smooth curves.

16.3.2 Definition of Independent of Path

In general, if **F** is a continuous vector field with domain D, we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial points and same endpoints.

- Therefore, line integrals of conservative vector fields are independent of path.

16.3.2 Definition of Closed

A curve is called **closed** if its terminal point coincides with its initial point, that is,

$$\mathbf{r}(b) = \mathbf{r}(a)$$

FIGURE 3
A closed curve

16.3.3 Theorem 3

$\int_C \mathbf{F} \cdot \, d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot \, d\mathbf{r} = 0$ for every closed path C in D.

- Since we know that the line integral of any **conservative vector field F** is independent of path, it follows that $\int_C \mathbf{F} \cdot \, d\mathbf{r} = 0$ for any closed path.

16.3.4 Definition of Open

D is **open**, which means that for every point P in D there is a disk with center P that lies entirely in D (So D doesn't contain any of its boundary points).

16.3.5 Definition of Connected

D is **connected**, which means that any two points in D can be joined by a path that lies in D.

16.3.6 Theorem 4 - The Only Vector Fields that are Independent of Path are Conservative:

Suppose **F** is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot \, d\mathbf{r}$ is independent of path in D, then **F** is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

16.3.7 Theorem 5

If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

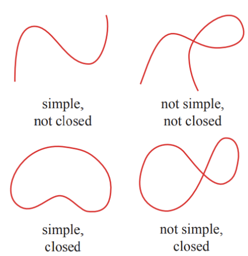
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{dx}$$

- P and Q have first-order partial derivatives on D because a conservative vector field means that $\nabla f = \mathbf{F}$.

16.3.8 Definition of Simple Curve

A curve that doesn't intersect itself anywhere between its endpoints.

$\mathbf{r}(a) = \mathbf{r}(b)$ for a simple closed curve but $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ $when\ a < t_1 < t_2 < b$



16.3.9 Definition of Simply-Connected Region

A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D.

- These regions **contain no holes and cannot have separate pieces.**

16.3.10 Theorem 6 (Verifying a Vector Field on $\mathbb{R}^2$ is Conservative):

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

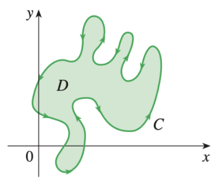
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{dx} \text{ throughout D}$$

Then **F** is conservative.

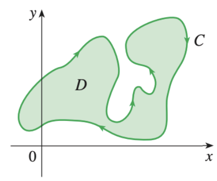
16.4 Green's Theorem

16.4.0 Definition of Positive Orientation

Positive orientation of a simple closed curve C refers to a *single counterclockwise* traversal of C. Thus if C is given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, then the region D is always on the left as the point $\mathbf{r}$(t) traverses C.



(a) Positive orientation



(b) Negative orientation

16.4.1 Green's Theorem (Counterpart of the Fundamental Theorem of Calculus for Double Integrals):

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int\limits_{D}\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{dy}\right) dA = \int\limits_{C} P dx + Q dy$$

- Note: The RS is $\int\limits_{C} \mathbf{F} \cdot\, d\mathbf{r}$, where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$
- There is an integral involving derivatives on the LS, and the RS involves the values of the original functions only on the **boundary** of the domain.

16.4.2 Notation

The following notation is used to indicate that the line integral is calculated using the positive orientation of the closed curve C:

$$\oint\limits_{C} P dx + Q dy \text{ or } \oint_{C} P dx + Q dy$$

Another notation for the positively oriented boundary curve of D is ∂D, which can be used for Green's theorem.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{dy}\right) dA = \int_{\partial D} P dx + Q dy$$

16.4.3 Finding Areas with Green's Theorem

Since the area of D is $\iint_D 1 dA$, therefore we want P and Q so that

$$\frac{\partial Q}{\partial y} - \frac{\partial P}{dx} = 1$$

There are several possibilities:

$$P(x, y) = 0 \mid P(x, y) = - y \mid P(x, y) = - \frac{1}{2} y$$

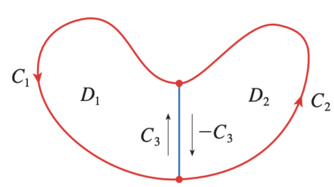
$$Q(x, y) = x \mid Q(x, y) = 0 \mid Q(x, y) = \frac{1}{2} x$$

Then Green's Theorem gives the following formulas for the area of D:

$$A = \oint_C x dy = - \oint_C y dx = \frac{1}{2} \oint_C x dy - y dx$$

16.4.4 Extended Version of Green's Theorem where D is a Finite Union of Simple Regions:

If D is the region:



Then we can write $D = D_1 \cup D_2$, *where* D_1 *and* D_2 are both simple. The boundary of D_1 *is* $C_1 \cup C_3$ and the boundary of D_2 *is* $C_2 \cup (- C_3)$ so, applying Green's to D_1 and D_2 separately:

$$\int_{C_1 \cup C_3} P dx + Q dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{dy}\right) dA$$

$$\int_{C_2 \cup (-C_3)} Pdx + Qdy = \int\int_{D_2} (\frac{\partial Q}{\partial x} - \frac{\partial P}{dy})dA$$

If we add these equations, the line integrals along C_3 and $-C_3$ cancel, so we get

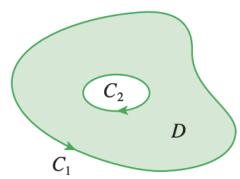
$$\int_{C_1 \cup C_2} Pdx + Qdy = \int\int_{D} (\frac{\partial Q}{\partial x} - \frac{\partial P}{dy})dA$$

Which is Green's Theorem for $D = D_1 \cup D_2$, *since its boundary is* $C = C_1 \cup C_2$.

- This same **argument can be applied for any finite union of nonoverlapping simple regions.**

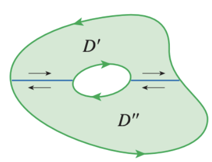
16.4.5 Extended Version of Green's Theorem where D is Not Simply-Connected (Holes):

The boundary C of the region D consists of two simple closed curves C_1 and C_2:



These boundary curves are oriented so that the region D is always on the left as the curve C is traversed. Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2.

If we divide D into two regions D′ and D′′:



Then by applying Green's Theorem to D′ and D′′:

$$\int\int_{D} (\frac{\partial Q}{\partial x} - \frac{\partial P}{dy})dA = \int\int_{D'} (\frac{\partial Q}{\partial x} - \frac{\partial P}{dy})dA + \int\int_{D''} (\frac{\partial Q}{\partial x} - \frac{\partial P}{dy})dA$$

$$= \int_{\partial D'} Pdx + Qdy + \int_{\partial D''} Pdx + Qdy$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get:

$$\iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{dx}\right) dA = \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy = \int_C Pdx + Qdy$$

Which is Green's Theorem for the region D.

16.5 Curl and Divergence

16.5.0 Curl

If **F** is a vector field on $\mathbb{R}^3$ (**F**= P**i**+ Q**j**+ R**k**) and the partial derivatives of P, Q, and R all exist, then the curl of **F** is the vector field on $\mathbb{R}^3$ defined by

$$curl\,\mathbf{F} = \nabla \times \mathbf{F}$$

Where ∇ (“del”) $=\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ is a vector differential operator.

- curl **F** is a vector field.

16.5.1 Theorem 3 (The curl of a gradient vector field is 0)

If f is a function of three variables that has continuous second-order partial derivatives, then

$$curl(\nabla f) = \mathbf{0}$$

Since a conservative vector field is one for which **F**= ∇f**:**

If **F** is conservative, then $curl$ **F**=**0**

- This gives us a way of verifying that a vector field is *not* conservative.
- The converse of Theorem 3 is not true in general.

16.5.2 Theorem 4

If **F** is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $curl$ **F**=**0**, then **F** is a conservative vector field.

- Theorem 4 is the three-dimensional version of Theorem 6 on 16.3.10.

16.5.3.0 What does the Curl Represent Physically?

The curl vector is associated with rotations. In section 16.8, we show that particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of $curl\ \mathbf{F}(x, y, z)$, following the right-hand rule, and the length of this curl vector is a measure of how quickly the particles move around the axis.

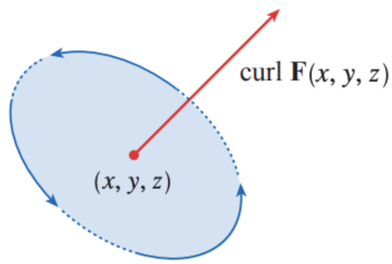


FIGURE 1

16.5.3.1 Definition of Irrotational

If $curl\ \mathbf{F}=\mathbf{0}$ at a point P, then the fluid is free from rotations at P and **F** is called **irrotational** at P.

- In this case, a tiny paddle wheel moves with fluid but doesn't rotate about its axis.

If $curl\ \mathbf{F}\neq\mathbf{0}$, the paddle wheel rotates about its axis.

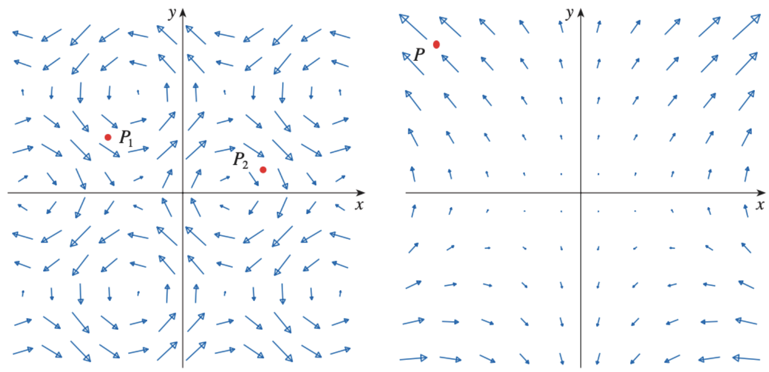
16.5.3.2 Illustration of Curl

In Figure 2(a), $curl\ \mathbf{F}\neq\mathbf{0}$ at most points, including P_1 and P_2.

A tiny paddle wheel placed at P_1 would rotate CCW about its axis (the fluid near P_1 flows roughly in the same direction but with greater velocity on one side of the point than on the other), so the curl vector at P_1 points in the direction of **k**.

A paddle wheel at P_2 would rotate CW and the curl vector there points in the direction of -**k**.

In Figure 2(b), *curl* **F**=**0** everywhere. A paddle wheel placed at P moves with the fluid but doesn't rotate about its axis.



(a) $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + \cos x\,\mathbf{j}$
curl $\mathbf{F}(x, y, z) = -(\sin x + \cos y)\,\mathbf{k}$

(b) $\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + y)\,\mathbf{j}$
curl $\mathbf{F}(x, y, z) = \mathbf{0}$

FIGURE 2 Velocity fields in fluid flow. (Only the part of **F** in the *xy*-plane is shown; the vector field looks the same in all horizontal planes because **F** is independent of *z* and the *z*-component is 0.)

16.5.4 Divergence

If **F**= P**i**+ Q**j**+ R**k** is a vector field on $\mathbb{R}^3$ and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist, then the **divergence of F** is the function of three variables defined by

$$div\ \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \mid div\ \mathbf{F} = \nabla \cdot \mathbf{F}$$

- If **F** is a vector field on $\mathbb{R}^2$, then div **F** is a function of two variables with a similar definition.
- div **F** is a scalar field.
- div **F**(x, y, z) measures the tendency of the fluid to diverge from the point (x,y,z).

16.5.5 Theorem 11

If **F**= P**i**+ Q**j**+ R**k** is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$div\ curl\ \mathbf{F} = 0$$

16.5.6.0 What does the Curl Represent Physically?

$div\,\mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z)

16.5.6.1 Definition of Incompressible

If $div\,\mathbf{F} = 0$, then **F** is said to be **incompressible.**

16.5.6.2 Illustration of Divergence

In Figure 3(a), $div\,\mathbf{F} \neq 0$ in general.

At the point P_1, $div\,\mathbf{F}$ is negative (the vectors that start near P_1 are shorter than those that end near P_1, so the net flow is inward there).

At the point P_2, $div\,\mathbf{F}$ is positive (the vectors that start near P_2 are longer than those that end near P_2, so the net flow is outward there).

In Figure 3(b), $div\,\mathbf{F} = 0$ everywhere (the vectors that start and end near any point P are about the same length).

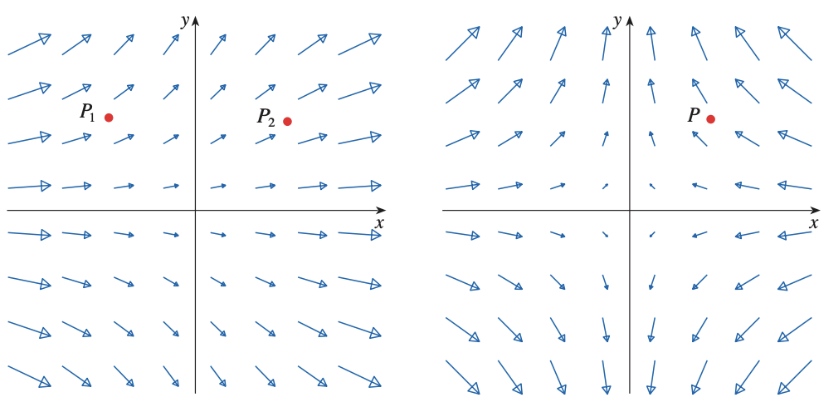


FIGURE 3
Velocity fields in fluid flow. (Only the part of **F** in the *xy*-plane is shown; the vector field looks the same in all horizontal planes because **F** is independent of *z* and the *z*-component is 0.)

(a) $\mathbf{F}(x, y, z) = (1 + x^2)\,\mathbf{i} + y\,\mathbf{j}$
div $\mathbf{F}(x, y, z) = 2x + 1$

(b) $\mathbf{F}(x, y, z) = -x\,\mathbf{i} + y\,\mathbf{j}$
div $\mathbf{F}(x, y, z) = 0$

16.5.7.0 Laplace Operator

$$\nabla^2 = \nabla \cdot \nabla$$

16.5.7.1 Laplace's Equation

$$\bigtriangledown^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

16.5.7.2 Laplace Operator to a Vector Field

$$\bigtriangledown^2 \mathbf{F} = \bigtriangledown^2 P\mathbf{i} + \bigtriangledown^2 Q\mathbf{j} + \bigtriangledown^2 R\mathbf{k}$$

16.5.8.0 1st Vector Form of Green's Theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \mathbf{F}\cdot\mathbf{T}ds = \int\int_D (curl\,\mathbf{F})\cdot\mathbf{k}dA$$

- Expresses the line integral of the tangential component of **F** along C as the double integral of the vertical component of curl **F** over the region D enclosed by C.

16.5.8.1 2nd Vector Form of Green's Theorem

$$\oint_C \mathbf{F}\cdot\mathbf{n}ds = \int\int_D div\,\mathbf{F}(x, y)dA$$

- The line integral of the normal component of **F** along C is equal to the double integral of the divergence of **F** over the region D enclosed by C.

16.6 Parametric Surfaces and Their Areas

16.6.0 Parametric Surface / Equations

We can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v. We suppose that

$$\mathbf{r}(u,v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

Is a vector-valued function defined on a region D in the uv-plane. So x, y, and z, the component functions of **r**, are functions of the two variables u and v with domain D.

The set of all points (x, y, z) in $\mathbb{R}^3$ such that

$$x = x(u, v) \mid y = y(u, v) \mid z = z(u, v)$$

And (u, v) varies throughout D, is called a **parametric surface** S and the equations are called **parametric equations** of S

- Each choice of u and v gives a point on S; by making all choices, we get all of S.
  - The surface S is traced out by the tip of the position vector $\mathbf{r}(u, v)$ *as* (u,v) moves throughout the region D.

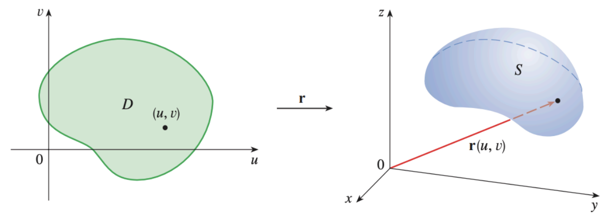
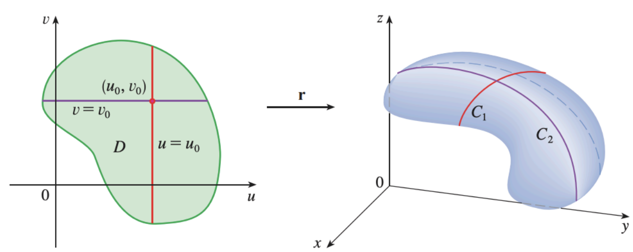


FIGURE 1
A parametric surface

16.6.1 Grid Curves

If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$, then there are two useful families of curves that lie on S, corresponding to **vertical and horizontal lines in the *uv*-plane.**



Keep either $u = u_0$ or $v = v_0$ a constant, then $\mathbf{r}(u_0, v)$ or $\mathbf{r}(u, v_0)$ becomes a vector function of a single parameter and defines a curve lying on S.

- C_1 is a curve lying on S from keeping u a constant.
- C_2 is a curve lying on S from keeping v a constant.

16.6.2.0 Note from Example 4

For a general parametric surface we are making a map and the grid curves are similar to lines of latitude and longitude. Describing a point on a parametric surface by giving specific values of u and v is like giving the latitude and longitude of a point.

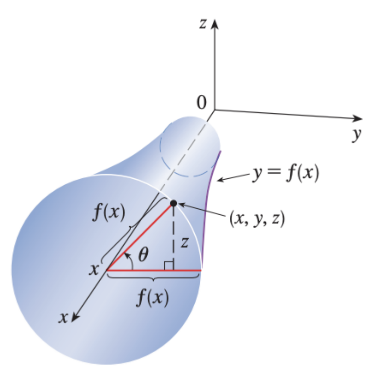
16.6.2.1 How to Find a Vector Function to Represent a Given Surface Tips

- Use cylindrical or spherical coordinates whenever you see fit with the two other variables not being used being the parameters.
- Make sure you are plugging in the value of the one you don't know.
- Use the plane representation with two non-parallel direction vectors and an initial point if not on the origin.
- In general, a surface given as the graph of a function of x and y (i.e. $z = f(x, y)$, can always be regarded as a parametric surface by taking **x and y parameters** (i.e. $x = x,\ y = y,\ z = f(x, y)$)
- **Parametric representations of surfaces are not unique.**

16.6.3 Surfaces of Revolution

Let's consider the surface S obtained by rotating the curve $y = f(x)$, $a \le x \le b$, **about the x-axis**, where $f(x) \ge 0$. Let θ be the angle of rotation as shown in Figure 12. If (x, y, z) is a point on S, then

$$x = x \mid y = f(x)cos\theta \mid z = f(x)sin\theta$$



- x, and θ are parameters and parameter domain is $a \le x \le b$, $0 \le \theta \le 2\pi$.
- This can be changed to being about the y, or z axis.

16.6.4.0 Tangent Planes

Tangent plane to a parametric surface S traced out by a vector function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

At a point P_0 with position vector $\mathbf{r}(u_0, v_0)$.

For a smooth surface, the **tangent plane** is the plane that contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$
$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

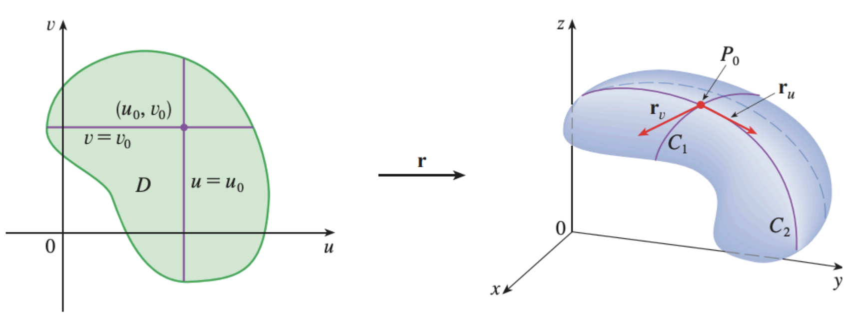
1. Find tangent vectors
2. Take the cross product to get the normal vector.
3. Find the values of u and v by subbing in the point P_0 into parametric equations (i.e. x,y,z)
4. Use $ax + by + cz = d$, where (a, b, c) *is the normal vector* and sub in P_0 to get d.

16.6.4.1 Tangent Vectors to Grid Curves

If $u = u_0 \mid v = v_0$, we get a grid curve $C_1 \mid C_2$ given by $\mathbf{r}(u_0, v) \mid \mathbf{r}(u, v_0)$ (ie. a vector function of single parameter $v \mid u$) that lies on S.

Its tangent vector to C_1 | C_2 at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ w.r.t v | u

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}$$
$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$$



16.6.3.1 Definition of Smooth

If $\mathbf{r}_u \times \mathbf{r}_v$ is never **0,** then the surface S is called **smooth** (it has no "corners").

16.6.4 Definition of Surface Area

If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \mid (u,v) \in D$$

And S is covered just once as (u,v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA$$

Where

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} \mid \mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

- If the surface S is $z = f(x,y)$, where (x,y) lies in D, then its surface area is the same as 15.5 and it's a consequence of this formula.

16.7 Surface Integrals

16.7.0.0 Definition of Surface Integral of f Over the Surface S

If the components are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in the interior of D, then

$$\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v| dA$$

- Remember that f(**r**(u,v)) is evaluated with the parametric equations for x,y, and z (eg. $x = x(u,v)$) in the formula for $f(x,y,z)$
- Remember that **D** is in the coordinates of the **parameters.**
- $dS = |\mathbf{r}_u \times \mathbf{r}_v| dA$
- We assume that the surface is covered only once as (u,v) ranges throughout D. The value of the surface integral does not depend on the parametrization that is used.

16.7.0.1 Special Case

If $f(x, y, z) = 1$, then

$$\iint_S 1dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA = A(S)$$

16.7.1 Total Mass of the Sheet

If a thin sheet has the shape of a surface S and the density at the point (x, y, z) is $\rho(x, y, z)$, then the total **mass** of the sheet is

$$m = \iint_S \rho(x, y, z)dS$$

16.7.2 The Center of Mass of a Sheet

$$\bar{x} = \frac{1}{m}\iint_S x\rho(x, y, z)dS \mid \bar{y} = \frac{1}{m}\iint_S y\rho(x, y, z)dS \mid \bar{z} = \frac{1}{m}\iint_S z\rho(x, y, z)dS$$

16.7.3 Surface Integrals of Graphs of Functions

If a parametric surface is defined as

$$x = x \mid y = y \mid z = g(x, y)$$

Then

$$\iint_S f(x, y, z)dS = \iint_D f(x, y, g(x, y))\sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}dA$$

- This can also work to **project S onto the yz-plane or xz-plane, just CHANGE THE FORMULA ACCORDINGLY.**

16.7.4 Piecewise-Smooth Surface

If S is a piecewise-smooth surface, that is, a finite union of smooth surfaces $S_1, S_2, \ldots, S_n$ that intersect only along their boundaries, then the surface integral of f over S is

$$\iint_S f(x, y, z)dS = \iint_{S_1} f(x, y, z)dS + \ldots + \iint_{S_n} f(x, y, z)dS$$

16.7.5 Definition of Oriented Surface

If it is possible to choose a unit normal vector **n** at every such point (x, y, z) so that **n** varies continuously over S, then S is called an **oriented surface** and the given choice of **n** provides S with an **orientation.**

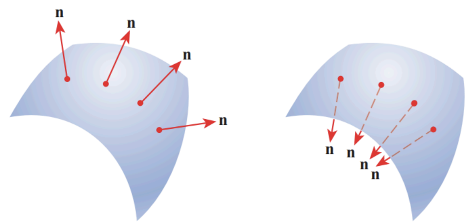


FIGURE 7
The two orientations of an orientable surface

- There are two possible orientations for any **orientable (two-sided) surface.**

16.7.6 Unit Normal Vector

If S is a smooth orientable surface given in parametric form by a vector equation **r** (u, v), then it is automatically supplied with the orientation of the unit normal vector:

$$\mathbf{n} = \frac{\bar{r}_u \times \bar{r}_v}{|\bar{r}_u \times \bar{r}_v|}$$

And the opposite orientation is given by -**n**.

16.7.7 Definition of Closed Surface and Positive/Negative Orientation Convention

A **closed surface** is a surface that is the boundary of a solid region E, the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from E, and inward-pointing normals give the **negative orientation.**

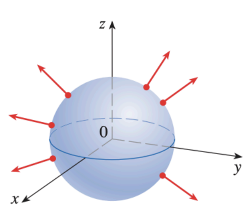


FIGURE 8
Positive orientation

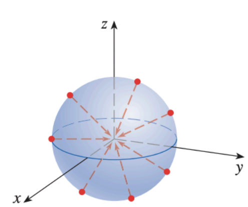


FIGURE 9
Negative orientation

16.7.8.0 Definition of Surface Integral (or Flux Integral) of F over S:

If **F** is a continuous vector field defined on an oriented surface S with unit normal vector **n,** then the **surface integral of F over S** is

$$\int_S\int \mathbf{F}\cdot\, d\mathbf{S} = \int_S\int \mathbf{F}\cdot\mathbf{n}\, dS$$

This integral is also called the **flux** of **F** across S.

- The surface integral of a vector field over S is equal to the surface integral of its normal component over S (as previously defined in 16.7.0.0).

$$\int_S\int \mathbf{F}\cdot\, d\mathbf{S} = \int_D\int \mathbf{F}\cdot\, (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

- This formula assumes the orientation of S induced by $\mathbf{r}_u \times \mathbf{r}_v$. For the opposite orientation, multiply by -1.
- D is the parameter domain
- Don't forget to have **F**(**r**(u,v)) for example by subbing in parametric equations of **r** into **F**.

16.7.8.1 Flux Integrals of z=g(x,y)

A surface S given by a graph $z = g(x, y)$, we can think of x and y as parameters then,

$$\int_S\int \mathbf{F}\cdot\, d\mathbf{S} = \int_D\int \left(-\, P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

- This formula assumes **upward orientation** of S; for **downward orientation**, multiply by -1.
- D is the parameter domain
- Similar formulas if S is given by $y = h(x, z)$ or $x = k(y, z)$

16.7.8.2 Electric Flux of E

If **E** is an electric field, then the surface integral

$$\int_S\int \mathbf{E}\cdot\, d\mathbf{S}$$

is called the **electric flux** of **E** through the surface S.

16.7.8.3 Gauss's Law

The net charge enclosed by a closed surface S is

$$Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

Where ε_0 is a constant.

16.7.9.4.0 Heat Flow

Suppose the temperature at a point (x,y,z) in a body is u(x,y,z). Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K \nabla u$$

Where K is an experimentally determined constant called the **conductivity** of the substance.

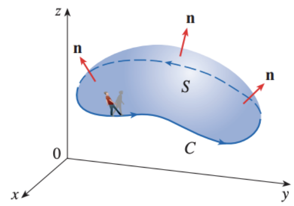
16.7.9.4.1 Rate of Heat Flow

The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

16.8 Stokes' Theorem

- **Higher-dimensional version of Green's Theorem:** Relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve)
- The orientation of S induces the **positive orientation of the boundary curve C:**



- If you walk in the positive direction around C with your head pointing in the direction of **n,** then the surface will always be on your left.

16.8.0 Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let **F** be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F}\cdot\, d\mathbf{r} = \int\int_S curl\,\mathbf{F}\cdot\, d\mathbf{S}$$

- Since $\int_C \mathbf{F}\cdot\, d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}ds$ and $\int\int_S curl\,\mathbf{F}\cdot\, d\mathbf{S} = \int\int_S curl\,\mathbf{F}\cdot\mathbf{n}dS$**,** therefore, the line integral around the boundary curve of S of the tangential component of **F** is equal to the surface integral over S of the normal component of the curl of **F**.

16.8.1 Notation

The positively oriented boundary curve of the oriented surface S is often written as ∂S, so Stokes' Theorem can be expressed as

$$\int\int_S curl\,\mathbf{F}\cdot\, d\mathbf{S} = \int_{\partial S} \mathbf{F}\cdot\, d\mathbf{r}$$

16.8.2 Green's Theorem Being a Special Case of Stokes' Theorem

If S is flat and lies in the xy-plane with upward, orientation, the unit normal is **k**, the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \mathbf{F}\cdot\, d\mathbf{r} = \int\int_S curl\,\mathbf{F}\cdot\, d\mathbf{S} = \int\int_S (curl\,\mathbf{F})\cdot\mathbf{k}dA$$

- Vector form of Green's theorem.

16.8.3 Note

In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

$$\int\int_{S_1} curl\, \mathbf{F}\cdot\ d\mathbf{S} = \int_C \mathbf{F}\cdot\ d\mathbf{r} = \int\int_{S_2} curl\, \mathbf{F}\cdot\ d\mathbf{S}$$

- Useful when it is difficult to integrate over one surface but easy to integrate over the other.

16.8.4.0 Definition of Circulation of v around C.

Suppose that C is an oriented closed curve and **v** represents the velocity field in fluid flow. Consider the line integral

$$\int_C \mathbf{v}\cdot\ d\mathbf{r} = \int_C \mathbf{v}\cdot\mathbf{T}ds$$

This means that the closer the direction of **v** is to the direction of **T**, the larger the value of **v·T (**Recall that if **v** and **T** point in generally opposite directions, then **v·T** is negative).

Thus $\int_C \mathbf{v}\cdot\ d\mathbf{r}$ is a **measure of the tendency of the fluid to move around C in the same direction as the orientation of C,** which is called the **circulation** of **v** around C.

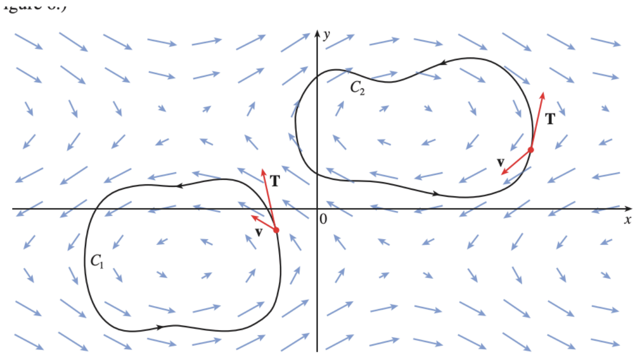


FIGURE 6
$\int_{C_1} \mathbf{v}\cdot d\mathbf{r} > 0$, positive circulation
$\int_{C_2} \mathbf{v}\cdot d\mathbf{r} < 0$, negative circulation

16.8.4.1 Relationship Between the Curl and the Circulation:

$$\mathit{curl}\,\mathbf{v}(P_0)\ \cdot\mathbf{n}(P_0)\ =\ \lim_{a\to 0}\ \frac{1}{\pi a^2}\int_{C_a}\mathbf{v}\cdot\ d\mathbf{r}$$

- P_0 is a point in the fluid.
- C_a is the boundary curve around the small disk S_a with radius a and center P_0.
- curl **v**·**n** is a measure of the rotating effect of the fluid about the axis **n.** The curling effect is greatest about the axis parallel to curl **v.**

Imagine a tiny paddle wheel placed in the fluid at a point P, as in Figure 7; the paddle wheel rotates fastest when its axis is parallel to curl **v**.

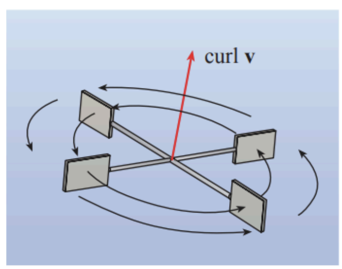


FIGURE 7

16.9 The Divergence Theorem

16.8.0 Simple Solid Regions

Regions E that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**.

16.8.1 The Divergence Theorem

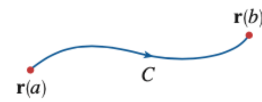
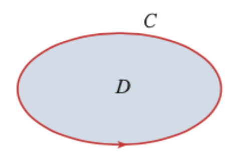
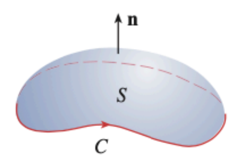
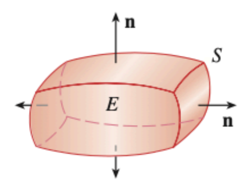
Let *E* be a simple solid region and let *S* be the boundary surface of *E*, given with positive (outward) orientation. Let **F** be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E div\, \mathbf{F}\, dV$$

- The flux of **F** across the boundary surface of E is equal to the triple integral of the divergence of **F** over E.

16.10 Summary

- In each case: An integral of a “derivative” over a region on the left side, and the right side involves the values of the original function only on the **boundary** of the region.

Curves and their boundaries (endpoints)		
Fundamental Theorem of Calculus	$\int_a^b F'(x)\,dx = F(b) - F(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$	
Surfaces and their boundaries		
Green's Theorem	$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$	
Stokes' Theorem	$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$	
Solids and their boundaries		
Divergence Theorem	$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$	

$\mathbb{R}^2$**:**

If f is defined on a smooth curve C given by

$$x = x(t) \mid y = y(t) \mid a \leq t \leq b \text{ or equivalently, } \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

Then the **line integral of f along C** (w.r.t arc length)

$$\int_C f(x, y)ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

If this limit exists