# General

# 1 Math

## 1.1 Euler's Formula and Trigonometric Identities

**Definition**: For $\theta \in \mathbb{R}$,

$$e^{j\theta} = \cos\theta + j\sin\theta,$$
$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right),$$
$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right).$$

## 1.2 Trigonometric Phase Shifts

**Definition**:

$$\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right),$$
$$\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right),$$
$$-\sin(\theta) = \cos\left(\theta + \frac{\pi}{2}\right),$$
$$-\cos(\theta) = \sin\left(\theta - \frac{\pi}{2}\right).$$

## 1.3 Sinc Function

**Definition**:

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

## 1.4 Geometric Sums

**Definition**: For $\alpha \in \mathbb{C}, n \in \mathbb{N}$,

$$\sum_{i=0}^{n} \alpha^i = \begin{cases} \dfrac{1-\alpha^{n+1}}{1-\alpha}, & \text{if } \alpha \neq 1, \\ n+1, & \text{if } \alpha = 1; \end{cases}$$

If $|\alpha| < 1$, then

$$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}.$$

## 1.5 Complex Number Properties

**Definition**: Magnitude:

$$|z|, \quad |z|^2$$

Complex conjugate:

$$z^* = |z|\cos(\theta) - j|z|\sin(\theta),$$
$$z = |z|\cos(\theta) + j|z|\sin(\theta),$$
$$z^* = |z|e^{-j\theta},$$
$$z = |z|e^{j\theta}.$$

Polar form:

$$|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2},$$
$$|z|^2 = zz^*,$$

where $z = x + jy$ and $z^* = x - jy$.
Angle:

$$\tan(\theta) = \frac{y}{x}$$

# 2 Signal Basics

## 2.1 Even and Odd

**Definition**:

$$x_{\text{even}} = \frac{1}{2}(x + \tilde{x})$$

$$x_{\text{odd}} = \frac{1}{2}(x - \tilde{x})$$

$$x = x_{\text{even}} + x_{\text{odd}}$$

## 2.2 Real and Odd

**Definition**:

$$\text{Re}(x) = \frac{1}{2}(x + x^*)$$

$$\text{Im}(x) = \frac{1}{2j}(x - x^*)$$

# 3 Systems

**Definition**:

## Discrete Time

## 3.1 Support

**Definition**: $x \in \mathbb{C}^{\mathbb{Z}}, x[n] \neq$ zero is the smallest interval $\{a, a+1, \ldots, b\}$ s.t.

$$x[n] = 0 \text{ for } n \notin \{a, a+1, \ldots, b\}$$

## 3.2 Power

**Definition**:

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

## Continuous Time

## 3.1 Support

**Definition**: $x \in \mathbb{C}^{\mathbb{R}}, x(t) \neq$ zero is the smallest interval $[a, b]$ s.t.

$$x(t) = 0 \text{ for } t \notin [a, b]$$

## 3.2 Power

**Definition**:

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 \, dt$$

## 3.3 Energy

**Definition**:
$$E_x = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

## 3.4 $N$-Periodic

**Definition**: A signal $x$ is $N$-periodic for a positive integer $N$ if
$$x[n] = x[n+N]$$
for all $n \in \mathbb{Z}$.

### 3.4.1 Fundamental Period

**Definition**: $N_0$ of $x$ is the smallest positive value of $N$ such that $x$ is $N_0$-periodic.

## 3.5 Inner Product

**Definition**: Let $x, y \in \mathbb{C}^{\mathbb{Z}}$ be $N$-periodic, then
$$\langle x, y\rangle = \frac{1}{N} \sum_{n \in [N]} x[n] y^*[n]$$

## 3.3 Energy

**Definition**:
$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

## 3.4 $T$-Periodic

**Definition**: A signal $x$ is $T$-periodic for a positive real number $T$ if
$$x(t) = x(t+T)$$
for all $t \in \mathbb{R}$.

### 3.4.1 Fundamental Period

**Definition**: $T_0$ (if it exists) is the smallest positive value of $T$ such that $x$ is $T_0$-periodic.

## 3.5 Inner Product

**Definition**: Let $x, y \in \mathbb{C}^{\mathbb{R}}$ be $T$-periodic signals, then
$$\langle x, y\rangle = \frac{1}{T} \int_T x(t) y^*(t) \, dt \in \mathbb{C}$$

## 3.6 FS Properties

**Definition**: Given $x \overset{\text{FS}}{\underset{N}{\longleftrightarrow}} c_k$:

1. Time Reversal: $\tilde{x}[n] \overset{\text{FS}}{\underset{N}{\longleftrightarrow}} c_{-k}$
2. Conjugation: $x^*[n] \overset{\text{FS}}{\underset{N}{\longleftrightarrow}} c^*_{-k}$
3. Time Shifting: $x[n-n_0] \overset{\text{FS}}{\underset{N}{\longleftrightarrow}} e^{-j2\pi\frac{k}{N}n_0}c_k$
   - $e^{-j2\pi\frac{k}{N}n_0}$: Phase factor.
4. Parseval's Relation:
$$\frac{1}{N}\sum_{n\in[N]}|x[n]|^2 = \sum_{k\in[N]}|c_k|^2$$
5. Modulation (Frequency-Shifting):
$$x[n]e^{j2\pi\frac{m}{N}n} \overset{\text{FS}}{\underset{N}{\longleftrightarrow}} c_{k-m}$$
   - $(k-m)\%N$: Cyclic indexing

Given $x \overset{\text{FS}}{\underset{N}{\longleftrightarrow}} a_k$ and $y \overset{\text{FS}}{\underset{N}{\longleftrightarrow}} b_k$, with $\alpha_1, \alpha_2 \in \mathbb{C}$:

1. Linearity: $\alpha_1 x + \alpha_2 y \overset{\text{FS}}{\underset{N}{\longleftrightarrow}} \alpha_1 a_k + \alpha_2 b_k$
2. Multiplication: $(x[n]y[n]) \overset{\text{FS}}{\underset{N}{\longleftrightarrow}} \sum_{\ell\in[N]} a_\ell b_{k-\ell}$

## 3.6 FS Properties

**Definition**: Given $x \overset{\text{FS}}{\underset{T}{\longleftrightarrow}} c_k$:

1. Time Reversal: $\tilde{x}(t) \overset{\text{FS}}{\underset{T}{\longleftrightarrow}} c_{-k}$
2. Conjugation: $x^*(t) \overset{\text{FS}}{\underset{T}{\longleftrightarrow}} c^*_{-k}$
3. Time Scaling: For $a > 0$:
$$x(at) \overset{\text{FS}}{\underset{T/a}{\longleftrightarrow}} c_{\frac{k}{a}}$$
   For $a < 0$:
$$x(at) \overset{\text{FS}}{\underset{T/(-a)}{\longleftrightarrow}} c_{-k}$$
4. Time Shifting: $x(t-t_0) \overset{\text{FS}}{\underset{T}{\longleftrightarrow}} e^{-j2\pi\frac{k}{T}t_0}c_k$
   - $e^{-j2\pi\frac{k}{T}t_0}$: Phase factor.
5. Parseval's Relation: $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k\in\mathbb{Z}}|c_k|^2$
6. Modulation (Frequency-Shifting): $\forall m \in \mathbb{Z}$, then $x(t)e^{j2\pi\frac{m}{T}t} \overset{\text{FS}}{\underset{T}{\longleftrightarrow}} c_{k-m}$
7. Differentiation: $x'(t) \overset{\text{FS}}{\underset{T}{\longleftrightarrow}} j\frac{2\pi k}{T}c_k$

Given $x \overset{\text{FS}}{\underset{T}{\longleftrightarrow}} a_k$ and $y \overset{\text{FS}}{\underset{T}{\longleftrightarrow}} b_k$, $\forall \alpha_1, \alpha_2 \in \mathbb{C}$:

1. Linearity: $\alpha_1 x + \alpha_2 y \overset{\text{FS}}{\underset{T}{\longleftrightarrow}} \alpha_1 a_k + \alpha_2 b_k$
2. Multiplication: $(x(t)y(t)) \overset{\text{FS}}{\underset{T}{\longleftrightarrow}} \sum_{n\in\mathbb{Z}} a_n b_{k-n}$