ECE367 Cheatsheet

Hanhee Lee

September 11, 2024

Contents

1 Vectors, Norms, Inner Products (Ch. 2.1-2.2) 3

- 1.1 Linear transformation . . . 3
  - 1.1.1 Matrix representation of a linear transformation . . . 3
- 1.2 Vectors . . . 3
- 1.3 Vector spaces . . . 3
  - 1.3.1 How to prove or disprove a vector space? . . . 4
- 1.4 Subspace . . . 5
- 1.5 Span . . . 5
  - 1.5.1 How to draw the span? . . . 5
- 1.6 Linear independent (LI) set . . . 6
  - 1.6.1 How to determine if a set is linearly independent . . . 6
- 1.7 Basis . . . 6
  - 1.7.1 Dimension . . . 7
- 1.8 Norms (Notion of distance) . . . 7
  - 1.8.1 Norm balls . . . 8
  - 1.8.2 Motivation for Norms . . . 8
  - 1.8.3 Distance metric . . . 8
- 1.9 Inner product (Notion of angle) . . . 9
  - 1.9.1 Examples of inner products . . . 9
  - 1.9.2 Connection of inner product to angle . . . 9
  - 1.9.3 Cauchy-Schwartz inequality and its generalization . . . 10
  - 1.9.4 Inner product induces a norm . . . 10
- 1.10 Orthogonal decomposition . . . 10
  - 1.10.1 Mutually orthogonal . . . 10
  - 1.10.2 Orthonormal basis . . . 11
  - 1.10.3 Orthogonal . . . 11
  - 1.10.4 Orthogonal complement . . . 11

2 Orthogonal Decomposition, Projecting onto Subspaces, Gram-Schmidt, QR Decomposition, Hyperplanes and Half-Spaces (Ch. 2.2-2.3) 11

- 2.1 Projection onto subspaces . . . 11
  - 2.1.1 Basic problem . . . 11
- 2.2 Gram-Schmidt and QR decomposition . . . 15
- 2.3 Hyperplanes and half-spaces . . . 15
- 2.4 Non-euclidean projection . . . 15

3 Non-Euclidean Projection, Projection onto Affine Sets, Functions, Gradients and Hessians (Ch. 2.3-2.4) 15

- 3.1 Projection onto affine sets . . . 15
- 3.2 Functions . . . 15
- 3.3 Gradients . . . 15
- 3.4 Hessians . . . 15

4 Matrices, Range, Null Space, Eigenvalues, Eigenvectors, Matrix Diagonalization (Ch. 3.1-3.5) 15

- 4.1 Matrices . . . 15

4.2 Range . . . 15
4.3 Null Space . . . 15
4.4 Eigenvalues and eigenvectors . . . 15
4.5 Matrices diagonalization . . . 15

5 Symmetric Matrices, Orthogonal Matrices, Spectral Decomposition, Positive Semidefinite Matrices, Ellipsoids (Ch. 4.1-4.4) 15
5.1 Symmetric matrices . . . 15
5.2 Orthogonal matrices . . . 15
5.3 Spectral decomposition . . . 15
5.4 Positive semidefinite matrices . . . 15
5.5 Ellipsoids . . . 15

6 Singular Value Decomposition, Principal Component Analysis (Ch. 5.1, 5.3.2) 15
6.1 Singular value decomposition . . . 15
6.2 Principle component analysis . . . 15

7 Interpretations of SVD, Low-Rank Approximation (Ch. 5.2-5.3.1) 15
7.1 Interpretation of SVD . . . 15
7.2 Low-rank approximation . . . 15

8 Least Squares, Overdetermined and Underdetermined Linear Equations (Ch. 6.1-6.4) 15
8.1 Least squares . . . 15
8.2 Overdetermined linear equation . . . 15
8.3 Underdetermined linear equation . . . 15

9 Regularized Least-Squares, Convex Sets and Convex Functions (Ch. 6.7.3, 8.1-8.4) 15
9.1 Regularized least-squares . . . 15
9.2 Convex sets and convex functions . . . 15

10 Lagrangian Method for Constrained Optimization, Linear Programming and Quadratic Programming (Ch. 8.5, 9.1-9.6) 15
10.1 Lagrangian method for constrained optimization . . . 15
10.2 Linear programming and quadratic programming . . . 15

11 Numerical Algorithms for Unconstrained and Constrained Optimization (Ch. 12.1-12.3) 15
11.1 Numerical algorithms for unconstrained optimization . . . 15
11.2 Numerical algorithms for constrained optimization . . . 15

List of Figures

1 Vector addition and scalar multiplication. . . . 4
2 Norm balls of different p values. . . . 8
3 Ordering of the vector spaces. . . . 10
4 Drawing any x. . . . 11
5 Visual representation of the projection problem. . . . 12
6 Generalization of projection. . . . 13

List of Tables

1 Vectors, Norms, Inner Products (Ch. 2.1-2.2)

1.1 Linear transformation

Definition: $T: X \to Y$ that satisfies

1. **Additivity:** $T(x_1 + x_2) = T(x_1) + T(x_2)$
2. **Homogeneity:** $T(\alpha x) = \alpha T(x)$

- **Note:** Linear algebra is the study of linear transformations over vector spaces.

1.1.1 Matrix representation of a linear transformation

Definition: Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation. When $\mathcal{V} = \mathbb{R}^n$ (or $\mathbb{C}^n$) and $\mathcal{W} = \mathbb{R}^m$ (or $\mathbb{C}^m$), then T can be uniquely represented as a matrix $A \in \mathbb{R}^{m\times n}$ such that:

$$T(\mathbf{x}) = A\mathbf{x}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- **Key:** Any linear transformation is a matrix multiplication. Any matrix multiplication is a linear transformation.

1.2 Vectors

Definition: Ordered collection of numbers, where $x_i \in \mathbb{R}$ or $\mathbb{C}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{x}^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

- n: Dimension of $\mathbf{x}$
- $\mathbf{x}$: Column vector
- $\mathbf{x}^T$: Transpose of x (row vector)
- T: Transpose
- x_i: i-th element of x.

1.3 Vector spaces

Definition: A vector space over a field $\mathbb{F}$ (e.g. $\mathbb{R}/\mathbb{C}$) consists of:

1. A set of vectors $\mathcal{V}$
2. A vector addition operator $+$: $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$ s.t. $\forall x, y \in \mathcal{V} \to x + y \in \mathcal{V}$ (i.e. closed under VA)
3. A scalar multiplication operator $\cdot$: $\mathbb{F} \times \mathcal{V} \to \mathcal{V}$ s.t. $\forall \alpha \in \mathbb{F}$, $\forall x \in \mathcal{V} \to \alpha x \in \mathcal{V}$ (i.e. closed under SM)

- $\times$ is not scalar multiplication.

For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{F}$. The following properties are satisfied:

- **Vector addition** satisfies (i.e., Abelian group):
  1. **Commutativity:** $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
  2. **Associativity:** $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$.
  3. **Additive identity:** $\exists\, \mathbf{0} \in \mathcal{V}$ s.t. $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$.
  4. **Additive inverse:** $\forall\, \mathbf{x}$, $\exists\, \mathbf{y}$ s.t. $\mathbf{x} + \mathbf{y} = \mathbf{0}$ (i.e. $\mathbf{y} = -\mathbf{x}$).
- **Scalar multiplication** satisfies:

1. **Associativity:** $\alpha \cdot (\beta \cdot \mathbf{x}) = (\alpha \cdot \beta) \cdot \mathbf{x}$.
2. **Multiplicative Identity:** $\exists\, 1 \in \mathbb{F}$ s.t. $1 \cdot \mathbf{x} = \mathbf{x}$.
3. **Right Distributivity:** $\alpha \cdot (\mathbf{x} + \mathbf{y}) = \alpha \cdot \mathbf{x} + \alpha \cdot \mathbf{y}$.
4. **Left Distributivity:** $(\alpha + \beta) \cdot \mathbf{x} = \alpha \cdot \mathbf{x} + \beta \cdot \mathbf{x}$.

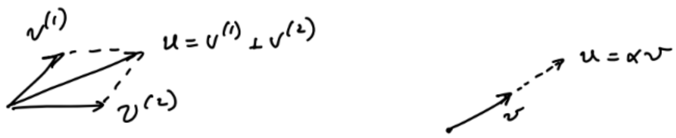

Figure 1: Vector addition and scalar multiplication.

1.3.1 How to prove or disprove a vector space?

Process:
Prove:
1. Prove that $\mathcal{V}$ is closed under VA and SM.
2. Prove all the properties under VA and SM.

Disprove:
1. Disprove one of the properties or that it isn't closed under VA and SM.

Warning: If standard addition and multiplication then, closed under VA and SM properties is enough to prove it's a vector space.

Example:
- Let $\mathcal{V} = \mathbb{R}^n$ and $\mathbb{F} = \mathbb{R}$: This represents vectors of dimension n where each element belongs to $\mathbb{R}$.

$$\mathcal{V} = \mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \right\}$$

For $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$:

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

For $\alpha, \beta \in \mathbb{R}$:

$$\alpha \cdot \mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

- Let $\mathcal{V} = \mathbb{C}^n$ and $\mathbb{F} = \mathbb{C}$: This represents vectors of dimension n with complex components.

$$\mathcal{V} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{C} \right\}$$

$\mathcal{V}$ is a vector space over $\mathbb{C}$ under element-wise addition and scalar multiplication.

- Let $\mathcal{V} = \{$set of all continuous functions $f : \mathbb{R} \to \mathbb{R}^n\}$ and $\mathbb{F} = \mathbb{R}$:
  Let $f_1, f_2 \in \mathcal{V}$, and for $t \in \mathbb{R}$:
  $$(f_1 + f_2)(t) = f_1(t) + f_2(t) \quad \Rightarrow \quad f_1 + f_2 \in \mathcal{V}$$
  For $\alpha \in \mathbb{R}$:
  $$(\alpha f)(t) = \alpha f(t) \quad \Rightarrow \quad \alpha f \in \mathcal{V}$$
  - f is the vector, $\mathbb{R} \to \mathbb{R}^n$ is the input-output relationship. For 2D, $f(x) = [x_1, x_2]^T$, where x is the input, the vector is the output in 2D, and the vector is f.
- Let $\mathcal{V} = \mathcal{P}_n$, the set of all polynomials with real coefficients and degree $\leq n$:
  $$\mathcal{V} = \mathcal{P}_n = \left\{p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{R}\right\}$$
  $\mathcal{V}$ is a vector space over $\mathbb{R}$ under standard addition and scalar multiplication.

1.4 Subspace

Definition: A **subspace** is a subset of a vector space $\mathcal{V}$ that is a vector space by itself.
- **Test:** To check whether a subset is a subspace, check that it is closed under VA & SM.

Example:
- Let $\mathcal{V} = \mathbb{R}^3$, and consider the set:
  $$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$
  This set S is a subspace of $\mathbb{R}^3$.
- Let $\mathcal{V} = \mathbb{R}^3$, and consider the set:
  $$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$
  This set S is **not** a subspace of $\mathbb{R}^3$ because adding two vectors will make the last component 2.
- Let $\mathcal{V} = \mathbb{R}^n$, and consider the set:
  $$S = \{\mathbf{0}\}$$
  This set S is a subspace of $\mathbb{R}^n$.

1.5 Span

Definition: Given a finite set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in the same vector space $\mathcal{V}$ over some field $\mathbb{F}$ then,
$$\text{Span}(S) = \left\{ \sum_{i=1}^{m} \alpha_i \mathbf{v}_i \mid \alpha_i \in \mathbb{F} \right\}$$
- **Note:** Span(S) is always a subspace of V.

1.5.1 How to draw the span?

Process:
1. Identify the vectors.
2. Plot the vectors: Plot each vector on a coordinate plane starting at the origin.
3. Draw the span: Extend the vectors in both directions to show the line or plane formed by their span. If they span the entire plane, draw dashed lines extending their direction.

Example:

- Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$:
$$\text{span}(S) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$
This set span(S) forms a plane in $\mathbb{R}^3$. The vectors span the xy-plane with the z-coordinate fixed at zero.
- Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right\}$:
$$\text{span}(S) = \left\{ x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$$
In this case, span(S) is a line in $\mathbb{R}^3$ along the x-axis with y and z coordinates fixed at zero.

1.6 Linear independent (LI) set

Definition: A set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is LI if no vector in S can be written as a LC of other vectors in S. In other words, the only α_i's that makes $\sum_{i=1}^{m} \alpha_i \mathbf{v}_i = 0$ is $\alpha_i = 0$, $\forall i$.

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is LI, then $\forall$ $\mathbf{u} \in$ span(S), there is a **unique** set of α_i's s.t. $\mathbf{u} = \sum_{i=1}^{k} \alpha_i \mathbf{v}_i$ (i.e. there is no redundancies in representation)
  - **Coordinates:** $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ of $\mathbf{u}$ w.r.t. S.
- If S is linearly dependent, then one of the vectors can be written as a LC of the other vectors. In this case, we can remove that vector and continue this process until the remaining set is LI.
  - **Note:** Such an irreducible linearly independent set is called a **basis** of span(S).

1.6.1 How to determine if a set is linearly independent

Process:
1. Write a linear combination with coefficients $\alpha_1, \ldots, \alpha_k$.
2. Set the linear combination equal to 0.
3. Solve for $\alpha_1, \ldots, \alpha_k$ by solving the set of equations (i.e. each component is one equation).
4. If $\alpha_1 = \ldots = \alpha_k = 0$, then it is linearly independent.
5. Else, linearly dependent by finding a counter example, where the linear combination is 0 for $\alpha_1, \ldots, \alpha_k$ not all equal to 0.

1.7 Basis

Definition: A set of vectors B is a basis of a vector space $\mathcal{V}$ if
- B is LI
- $Span(B) = \mathcal{V}$

Example: What is the standard basis for $\mathcal{V} = \mathbb{R}^n$?

$$\mathbf{e}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots \quad , \quad \mathbf{e}^{(n)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad \text{for } \mathbb{R}^n$$

If $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, then:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

1.7.1 Dimension

Definition: The dimension is the number of basis vectors.

- **Note:** Basis is not unique. But $dim(\mathcal{V})$ is well-defined.

Example:

- $\dim\left(\text{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}\right)\right) = 2$
- $\dim\left(\text{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}\right)\right) = 1$
- $\dim(\{\mathbf{0}\}) = 0$
- The dimension for $\mathcal{V} = \mathbb{R}^n$ of the standard basis is n

1.8 Norms (Notion of distance)

Definition: Let $\mathcal{V}$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A norm is a function $\|\cdot\|\colon \mathcal{V} \to \mathbb{R}$ that satisfies

1. **Non-negativity:** $\|\mathbf{x}\| \geq 0, \ \forall \mathbf{x} \in \mathcal{V}, \text{ and } \|\mathbf{x}\| = 0 \text{ iff } \mathbf{x} = \mathbf{0}$
2. **Homogeneity:** $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\| \ \forall \mathbf{x} \in \mathcal{V}, \ \alpha \in \mathbb{F}$
3. **Triangle inequality:** $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|, \ \forall \mathbf{x}, \ \mathbf{y} \in \mathcal{V}$ (triangular inequality)

Example: ℓ_p norms:

$$\|\mathbf{x}\|_p \equiv \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}, \quad 1 \leq p < \infty.$$

- **Note:** For $p < 1$, triangular inequality doesn't hold.

1. **Sum-of-absolute-values length** $p = 1$: $\|\mathbf{x}\|_1 \equiv \sum_{k=1}^{n} |x_k|$
2. **Euclidean length** $p = 2$: $\|\mathbf{x}\|_2 \equiv \sqrt{\sum_{k=1}^{n} x_k^2}$
3. **Max absolute value norm** $p = \infty$: $\|\mathbf{x}\|_\infty \equiv \max_{k=1,\ldots,n} |x_k|$
   - Largest term will dominate as if we common factor out the largest term, each of the other terms will go to 0 as noted in the lp norm.
4. **Cardinality** $p = 0$: The number of non-zero vectors in x is

$$\|\mathbf{x}\|_0 = \text{card}(\mathbf{x}) \equiv \sum_{k=1}^{n} \mathbb{I}(x_k \neq 0), \quad \text{where} \quad \mathbb{I}(x_k \neq 0) \equiv \begin{cases} 1 & \text{if } x_k \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

   - **Key:** Not a norm since $\|\alpha\mathbf{x}\|_0 = \|\mathbf{x}\|_0 \neq |\alpha| \cdot \|\mathbf{x}\|_0$ (e.g. if $\alpha = 2$ then this would double the count of number of non-zero vectors for the RS)

1.8.1 Norm balls

Definition: The set of all vectors with ℓ_p norm less than or equal to one,

$$B_p = \{\mathbf{x} : \|\mathbf{x}\|_p \leq 1\} \quad (1)$$

Example: For 2D, the norm balls are as follows:

- $\ell_2 : \; B_2 = \left\{\mathbf{x} \;\middle|\; \sqrt{x_1^2 + x_2^2} \leq 1\right\}$
- $\ell_1 : \; B_1 = \{\mathbf{x} \mid |x_1| + |x_2| \leq 1\}$
- $\ell_\infty : \; B_\infty = \{\mathbf{x} \mid \max |x_i| \leq 1 \text{ or } |x_1| \leq 1, |x_2| \leq 1\}$
- $\ell_0 : \; B_0 = \{\mathbf{x} \mid \operatorname{card}(\mathbf{x}) \leq 1\}$

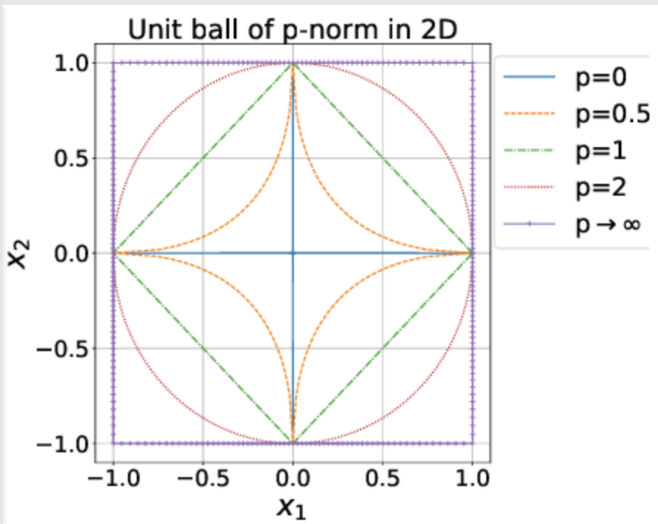

Figure 2: Norm balls of different p values.

1.8.2 Motivation for Norms

Example: In optimization problems, different norms are used to achieve various goals. Suppose we are trying to solve an optimal control problem, where $x = (x_1, \ldots, x_n)$ are some action variables.

- $\min \|\mathbf{x}\|_2^2 = x_1^2 + \ldots + x_n^2$ (i.e. minimizing the total energy (power) in $\mathbf{x}$)
- $\min \|\mathbf{x}\|_\infty$ (i.e. minimizing the peak energy in $\mathbf{x}$).
- $\min \|\mathbf{x}\|_1$ (i.e. minimizing the sum of action variables).
- $\min \|\mathbf{x}\|_0$ (i.e. find sparse solution)

1.8.3 Distance metric

Definition: A norm induces a distance metric between two vectors x and y in $\mathbb{V}$ as

$$d(x, y) = \|x - y\|$$

- **Note:** The ℓ_2-norm induces the Euclidean distance

$$\|x - y\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

1.9 Inner product (Notion of angle)

Definition: An inner product on a vector space $\mathcal{V}$ is a function $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathcal{F}$ such that:

1. **Positive definiteness:** $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0\ \forall \mathbf{x} \in \mathcal{V}$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ iff $\mathbf{x} = 0$
2. **Conjugate Symmetry:** $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$
   - $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ in $\mathbb{R}^n$
   - $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ in $\mathbb{C}^n$.
3. **Linearity in first argument:** $\langle \alpha \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V},\ \alpha \in \mathbb{F}$

Example: How to use the properties of inner products?

$$\begin{aligned} \langle x, \alpha y + z \rangle &\overset{(2)}{=} \overline{\langle \alpha y + z, x \rangle} \\ &\overset{(3)}{=} \overline{\alpha}\, \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} \quad \text{also by conjugate prop.} \\ &\overset{(2)}{=} \overline{\alpha} \langle x, y \rangle + \langle x, z \rangle \end{aligned}$$

1.9.1 Examples of inner products

Example:

- In $\mathbb{R}^n$ (Dot product): $\langle \mathbf{x}, \mathbf{y} \rangle = \displaystyle\sum_{i=1}^{n} x_i y_i = \mathbf{x}^\top \mathbf{y} = \mathbf{y}^\top \mathbf{x}$
  - **Key:** $\langle \mathbf{x}, \mathbf{x} \rangle = \displaystyle\sum_{i=1}^{n} x_i^2 = \mathbf{x}^\top \mathbf{x} = \|\mathbf{x}\|_2^2$
- In $\mathbb{C}^n$: $\langle \mathbf{x}, \mathbf{y} \rangle = \displaystyle\sum_{i=1}^{n} \overline{x_i} y_i = \mathbf{x}^H \mathbf{y} = \overline{\mathbf{y}^H \mathbf{x}}$
  - $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{x}^H = \begin{bmatrix} \overline{x_1} & \cdots & \overline{x_n} \end{bmatrix}$
- $\mathcal{V} = \left\{ f : \mathbb{R} \to \mathbb{R}\, ;\, \displaystyle\int_{-\infty}^{+\infty} f^2(t)\, dt < \infty \right\}$ (i.e. the set of square integrable functions)

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t) g(t)\, dt$$

1.9.2 Connection of inner product to angle

In $\mathbb{R}^n$, the notion of inner product has a geometric interpretation, and is closely related to the notion of angle between vectors.

Definition:

$$\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2} = \left\langle \frac{\mathbf{x}}{\|\mathbf{x}\|_2}, \frac{\mathbf{y}}{\|\mathbf{y}\|_2} \right\rangle \tag{2}$$

- $\langle \mathbf{x}, \mathbf{y} \rangle = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ (i.e. perpendicular)
- $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$ (i.e. $\mathbf{x}$ and $\mathbf{y}$ are aligned)
- $\langle \mathbf{x}, \mathbf{y} \rangle = -\|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$ (i.e $\mathbf{x}$ and $\mathbf{y}$ are in opposite directions)
- $\langle \mathbf{x}, \mathbf{y} \rangle > 0 \Rightarrow \cos\theta > 0 \Rightarrow$ angle is acute
- $\langle \mathbf{x}, \mathbf{y} \rangle < 0 \Rightarrow \cos\theta < 0 \Rightarrow$ angle is obtuse

Derivation: L3: Inner products and orthogonality.

1.9.3 Cauchy-Schwartz inequality and its generalization

Definition:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \tag{3}$$

Hölder's Inequality (generalization):

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q \quad \text{where } 1 \leq p, q < \infty \text{ and } \frac{1}{p} + \frac{1}{q} = 1 \tag{4}$$

Example: For $p = 1$ and $q = \infty$, we have:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|_1 \cdot \|\mathbf{y}\|_\infty$$

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \left(\sum_{i=1}^{n} |x_i| \right) \cdot \max_i |y_i|$$

1.9.4 Inner product induces a norm

Definition: Any inner product induces a norm, but not all norms are induced by an inner product.

- **Key:** If given an inner product, take the square root of the inner product to get the norm.
  - e.g. $\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$

Figure 3: Ordering of the vector spaces.

1.10 Orthogonal decomposition

1.10.1 Mutually orthogonal

Definition: A set of non-zero vectors $S = \left\{ \mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(d)} \right\}$ is **mutually orthogonal** if $\langle \mathbf{v}^{(i)}, \mathbf{v}^{(j)} \rangle = 0 \; \forall \; i \neq j$.

- **Fact:** Orthogonal set of vectors $S = \left\{ \mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(d)} \right\}$ is linearly independent.
  - **Proof:** In L3.

1.10.2 Orthonormal basis

Definition: Set of orthogonal basis vectors that have unit norm.

If $S = \left\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \dots, \mathbf{v}^{(d)}\right\}$ is a set of mutually orthogonal vectors, then $\left\{\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \dots, \frac{\mathbf{v}_d}{\|\mathbf{v}_d\|}\right\}$ is an orthonormal basis for $span(S)$

Example: Standard basis is an orthonormal basis for $\mathbb{R}^n$

1.10.3 Orthogonal

Definition: Consider $\mathbf{x} \in \mathcal{V}$, and let S be a subspace of $\mathcal{V}$. We say $\mathbf{x}$ is orthogonal to S if:

$$\langle \mathbf{x}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in S.$$

We write: $\mathbf{x} \perp S$.

1.10.4 Orthogonal complement

Definition: The **orthogonal complement** of S, denoted $S^\perp$, is the set of all orthogonal vectors to S:

$$S^\perp = \{\mathbf{x} \in \mathcal{V} : \mathbf{x} \perp S\}$$

- $S^\perp$ is a subspace. (Closed under addition and scalar multiplication)
- $S \cap S^\perp = \{\mathbf{0}\}$
- **Orthogonal decomposition:** Any $\mathbf{x} \in \mathcal{V}$ can be uniquely written as: $\mathbf{x} = \mathbf{x}_S + \mathbf{x}_{S^\perp}$ where $\mathbf{x}_S \in S$ and $\mathbf{x}_{S^\perp} \in S^\perp$

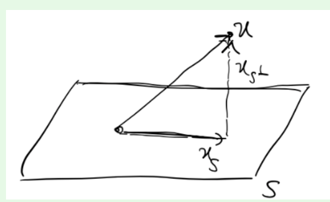

Figure 4: Drawing any x.

- $\mathcal{V} = S + S^\perp = \left\{\mathbf{u} + \mathbf{v} : \mathbf{u} \in S, \mathbf{v} \in S^\perp\right\}$

2 Orthogonal Decomposition, Projecting onto Subspaces, Gram-Schmidt, QR Decomposition, Hyperplanes and Half-Spaces (Ch. 2.2-2.3)

2.1 Projection onto subspaces

2.1.1 Basic problem

Definition: Given $x \in \mathcal{V}$ and a subspace S. Find the closest point (in norm) in S to x:

$$\text{Proj}_S(x) = \arg\min_{y \in S} \|y - x\| \tag{5}$$

- $\|y - x\|$: Some norm.
- **Subspace:** S doesn't have to be a subspace.
- $\arg\min$: Vector y that minimizes $\|x - y\|$

Example: Projection onto a 1-dimensional subspace.
Let $S = \text{span}(\mathbf{v})$, and we denote the projection of $\mathbf{x}$ onto S as:

$$\text{Proj}_S(\mathbf{x}) = \mathbf{x}^*$$

Under the Euclidean norm, we have nice geometry: we should have

$$\langle \mathbf{x} - \mathbf{x}^*, \mathbf{v} \rangle = 0$$

Since $\mathbf{x}^* \in S$, $\mathbf{x}^* = \alpha \mathbf{v}$ for some scalar α.
We need to find α.
So,

$$\langle \mathbf{x} - \alpha\mathbf{v}, \mathbf{v} \rangle = 0$$
$$\Rightarrow \langle \mathbf{x}, \mathbf{v} \rangle - \alpha \langle \mathbf{v}, \mathbf{v} \rangle = 0$$
$$\Rightarrow \alpha = \frac{\langle \mathbf{x}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Thus,

$$\mathbf{x}^* = \alpha \mathbf{v} = \frac{\langle \mathbf{x}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$$

which simplifies to:

$$\mathbf{x}^* = \frac{\mathbf{x}^\top \mathbf{v}}{\|\mathbf{v}\|_2^2} \mathbf{v} = \left\langle \mathbf{x}, \frac{\mathbf{v}}{\|\mathbf{v}\|_2} \right\rangle \frac{\mathbf{v}}{\|\mathbf{v}\|_2}$$

- **Note:** $\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|_2} \right\|_2 = 1$, so we can think of $\left\{ \frac{\mathbf{v}}{\|\mathbf{v}\|_2} \right\}$ as an orthonormal basis for S.
- **Note:** x^* is the point we are looking for in the projection problem.

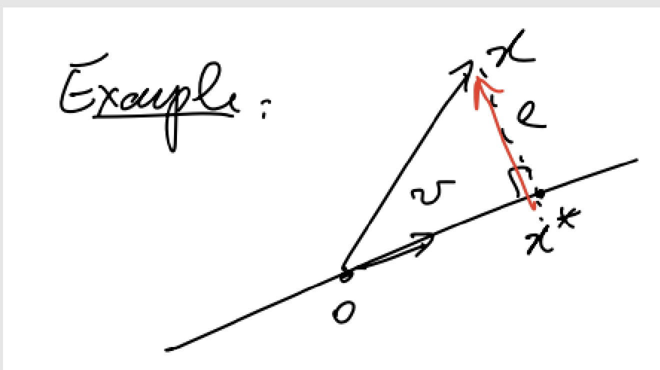

Figure 5: Visual representation of the projection problem.

Example: This can be generalized to higher dimensions. Let S be a subspace of $\mathcal{V}$, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ be an orthonormal basis of S.
Let

$$\mathbf{x}^* = \sum_{i=1}^{d} \alpha_i \mathbf{v}_i$$

Goal: Find $\alpha_1, \dots, \alpha_d$ so as to minimize the norm $\|\mathbf{x} - \mathbf{x}^*\|_2$.
By geometry, we require that

$$\langle \mathbf{e}, \mathbf{v}_j \rangle = 0 \quad \forall\, j = 1, \dots, d$$

which implies:

$$\langle \mathbf{x} - \mathbf{x}^*, \mathbf{v}_j \rangle = 0 \quad \forall\, j$$
$$\Rightarrow \langle \mathbf{x} - \sum_{i=1}^{d} \alpha_i \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \quad \forall\, j$$

Using linearity of the inner product:

$$\Rightarrow \langle \mathbf{x}, \mathbf{v}_j \rangle = \sum_{i=1}^{d} \alpha_i \langle \mathbf{v}_i, \mathbf{v}_j \rangle$$

Since $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if $i \neq j$ and 1 if $i = j$, this simplifies to:

$$\alpha_j = \langle \mathbf{x}, \mathbf{v}_j \rangle \quad \text{b/c only the i=j term survives}$$

Thus,

$$\mathbf{x}^* = \sum_{i=1}^{d} \alpha_i \mathbf{v}_i = \sum_{i=1}^{d} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i$$
$$= \sum_{i=1}^{d} (\mathbf{x}^\top \mathbf{v}_i) \mathbf{v}_i$$

where $\mathbf{v}_i \in \mathbb{R}^n$.

$$\mathbf{e} = \mathbf{x} - \mathbf{x}^* \in S^\perp, \quad \mathbf{x}^* \in S$$

So,

$$\mathbf{x} = \mathbf{x}^* + \mathbf{e}, \quad \text{where} \quad \mathbf{x}^* \in S, \quad \mathbf{e} \in S^\perp$$

This is an example of orthogonal decomposition.

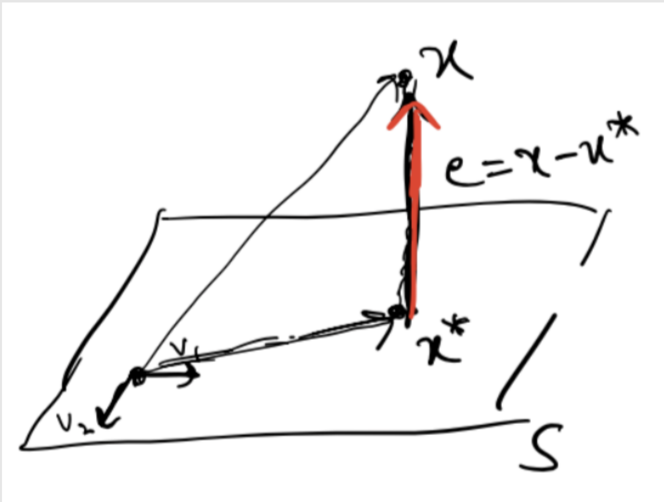

Figure 6: Generalization of projection.

Example: Fourier series in L3.

2.2 Gram-Schmidt and QR decomposition

2.3 Hyperplanes and half-spaces

2.4 Non-euclidean projection

3 Non-Euclidean Projection, Projection onto Affine Sets, Functions, Gradients and Hessians (Ch. 2.3-2.4)

3.1 Projection onto affine sets

3.2 Functions

3.3 Gradients

3.4 Hessians

4 Matrices, Range, Null Space, Eigenvalues, Eigenvectors, Matrix Diagonalization (Ch. 3.1-3.5)

4.1 Matrices

4.2 Range

4.3 Null Space

4.4 Eigenvalues and eigenvectors

4.5 Matrices diagonalization

5 Symmetric Matrices, Orthogonal Matrices, Spectral Decomposition, Positive Semidefinite Matrices, Ellipsoids (Ch. 4.1-4.4)

5.1 Symmetric matrices

5.2 Orthogonal matrices

5.3 Spectral decomposition

5.4 Positive semidefinite matrices

5.5 Ellipsoids

6 Singular Value Decomposition, Principal Component Analysis (Ch. 5.1, 5.3.2)

6.1 Singular value decomposition

6.2 Principle component analysis

7 Interpretations of SVD, Low-Rank Approximation (Ch. 5.2-5.3.1)

7.1 Interpretation of SVD

7.2 Low-rank approximation

8 Least Squares, Overdetermined and Underdetermined Linear Equations (Ch. 6.1-6.4)

8.1 Least squares

8.2 Overdetermined linear equation

8.3 Underdetermined linear equation