ECE355 Cheatsheet

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Signals and General Systems

1 Continuous and discrete-time signals (Ch. 1.1)

1.1 Signal energy and power

1.1.1 Total energy

Definition:
Continuous: Total energy from $t_1 \leq t \leq t_2$ is

$$E_{[t_1,t_2]} = \int_{t_1}^{t_2} |x(t)|^2 dt \tag{1}$$

- $x(t)$: Continuous-time signal
- $|x|$: Magnitude of the number

Discrete: Total energy from $n_1 \leq n \leq n_2$ is

$$E_{[t_1,t_2]} = \sum_{n=n_1}^{n_2} |x[n]|^2 \tag{2}$$

- $x[t]$: Discrete-time signal

1.1.2 Average power

Definition:
Continuous: Average power from $t_1 \leq t \leq t_2$ is

$$P_{[t_1,t_2]} = \frac{E_{[t_1,t_2]}}{t_2 - t_1} \tag{3}$$

Discrete: Average power from $n_1 \leq n \leq n_2$ is

$$P_{[t_1,t_2]} = \frac{E_{[t_1,t_2]}}{n_2 - n_1 + 1} \tag{4}$$

1.1.3 Total energy over infinite time interval

Definition:

Continuous:

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \tag{5}$$

Discrete:

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{6}$$

1.1.4 Average power over infinite time interval

Definition:
Continuous:

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \tag{7}$$

Discrete:

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2 \tag{8}$$

2 Time dilation, shifting (Ch. 1.2)

2.1 Time shifting

Definition:
CT: $x(t) = x(t - t_0),\ t_0 \in \mathbb{R}$

DT: $x[n] = x[n - n_0],\ n_0 \in \mathbb{Z}$

2.2 Time scaling

Definition:
CT: $x(t) = x(at),\ a \in \mathbb{R}$
- $|a| > 1$: Contraction
- $|a| < 1$: Expansion
- $a < 0$: Time reversal (reflect across y-axis)

DT: $y[n] = x[an],\ an \in \mathbb{Z}$

2.3 Scaling and shifting (CT)

Definition:

$$y(t) = x(at + b),\ a, b \in \mathbb{R} \tag{9}$$

- **Order:** Apply the shift, then the scaling.

2.4 Periodic signals

Definition:
CT: $x(t)$ is periodic with T iff $\exists\, T > 0$, s.t. $x(t) = x(t + T)$, $\forall t$.

DT: $x[n]$ is periodic with N iff $\exists\, N > 0$, s.t. $x[n] = x[n + N]$, $\forall n$.

2.4.1 Fundamental period:

Definition:
CT: T_0 of $x(t)$ is the smallest period.

DT: N_0 of $x[t]$ is the smallest period.

2.5 Even and odd signals

Definition:
CT:
- **Even:** $x(-t) = x(t)$ (symmetrical about y-axis)
- **Odd:** $x(-t) = -x(t)$ (symmetrical about origin)

DT:
- **Even:** $x[-n] = x[n]$
- **Odd:** $x[-n] = -x[n]$

- **Note:** Odd signal must be 0 at $t = 0$ or $n = 0$.

2.6 Fact:

Definition: Any signal can be broken into a sum of an even and odd signal.

3 Complex exponential signals (Ch. 1.3)

3.1 Review of complex numbers

Definition:
- $z = a + jb, \quad j \triangleq \sqrt{-1}$
- $r = |z| = \sqrt{a^2 + b^2}$
- $\theta \triangleq \angle z = \arctan\left(\frac{b}{a}\right)$
- $z = re^{j\theta} = r(\cos(\theta) + j\sin(\theta))$
- $\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

3.2 CT Complex exponential and sinusoidal signals

3.2.1 Complex exponential signal

Definition: The complex exponential signal is of the form

$$x(t) = Ce^{at},\ C, a \in \mathbb{C} \tag{10}$$

3.2.2 Periodic complex exponential and sinusoidal signal

Definition:

$$x(t) = Ce^{j\omega_0 t} = |C|e^{j(\omega_0 t + \phi)} = |C|cos(\omega_0 t + \phi) + jsin(\omega_0 t + \phi) \tag{11}$$

- $\omega_0 = 2\pi f_0$: Fundamental frequency
- $C = |C|e^{j\phi}$
- $T_0 = \frac{2\pi}{|\omega_0|}$: Fundamental period

3.2.3 General complex exponential signal

Definition:
$$x(t) = |C|e^{rt}e^{j(\omega_0 t+\phi)} = |C|e^{rt}\cos(\omega_0 t+\phi) + j|C|e^{rt}\sin(\omega_0 t+\phi) \tag{12}$$

- $Re\{x(t)\} = |C|e^{rt}\cos(\omega_0 t+\phi)$
- $Im\{x(t)\} = |C|e^{rt}\sin(\omega_0 t+\phi)$

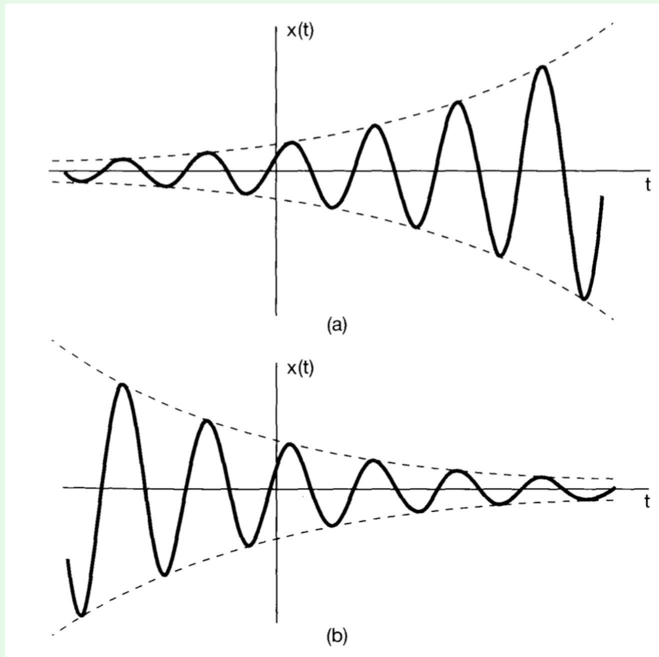

Figure 1: Real part of the general form. (a) $r > 0$ and (b) $r < 0$

3.2.4 Properties

Definition:
1. The larger the magnitude of ω_0, the higher is the rate of oscillation in the signal.
2. $e^{j\omega_0 t}$ is periodic for any value of ω_0.

3.3 DT Complex exponential and sinusoidal signals

3.3.1 Complex exponential sequence

Definition: The complex exponential sequence is of the form
$$x[n] = Ce^{\beta n} \tag{13}$$

- $C, \beta \in \mathbb{C}$

3.3.2 Sinusoidal signals

Definition:
$$x[n] = e^{j\omega_0 n} = cos(\omega_0 n) + jsin(\omega_0 n) \tag{14}$$

3.3.3 General complex exponential signal

Definition:

$$x[n] = C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \phi) + j|C||\alpha|^n \sin(\omega_0 n + \phi) \tag{15}$$

- $C = |C|e^{j\phi}$
- $\alpha = |\alpha|e^{j\omega_0}$
  - For $|\alpha| = 1$, the real and imaginary parts of a complex exponential sequence are sinusoidal.
  - For $|\alpha| < 1$, they correspond to sinusoidal sequences multiplied by a decaying exponential.
  - For $|\alpha| > 1$, they correspond to sinusoidal sequences multiplied by a growing exponential.

3.4 Comparison between CT and DT signals

Intuition: Assumes that m and N do not have any factors in common.
$e^{j\omega_0 t}$:

- Distinct signals for distinct values of ω_0
- Periodic for any choice of ω_0
- Fundamental frequency ω_0
- Fundamental period
  - $\omega_0 = 0$: undefined
  - $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$

$e^{j\omega_0 n}$:

- Identical signals for values of ω_0 separated by multiples of 2π
- Periodic only if $\omega_0 = \frac{2\pi m}{N}$ for some integers $N > 0$ and m
- Fundamental frequency* $\frac{\omega_0}{m}$
- Fundamental period*
  - $\omega_0 = 0$: undefined
  - $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

4 Step and impulse functions (Ch. 1.4)

4.1 DT Unit impulse and step sequences

4.1.1 Unit impulse

Definition:

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \tag{16}$$

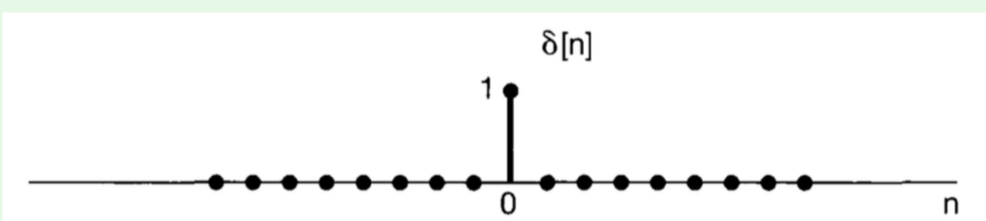

Figure 2: DT unit impulse

4.1.2 Unit step

Definition:

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \tag{17}$$

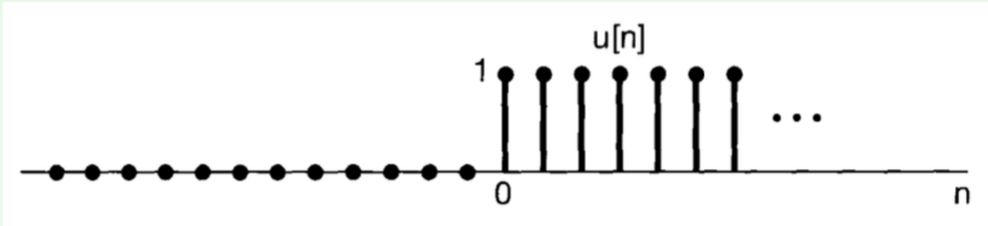

Figure 3: DT unit step

4.1.3 Relationship between impulse and step

Definition:

1. **First Difference:**
$$\delta[n] = u[n] - u[n-1]$$
2. **Running Sum:**
$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$
Let $k = n - m$:
$$u[n] = \sum_{k=\infty}^{0} \delta[n-k] = \sum_{k=0}^{\infty} \delta[n-k]$$

4.1.4 Sampling

Definition: If we consider a unit impulse $\delta[n-n_0]$ at $n = n_0$, then

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0] \tag{18}$$

4.2 CT Unit impulse and step functions

4.2.1 Impulse

Definition:

$$\delta(t) = \frac{du(t)}{dt} \tag{19}$$

4.2.2 Step

Definition:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{20}$$

- **Note:** Step is discontinuous at $t = 0$.

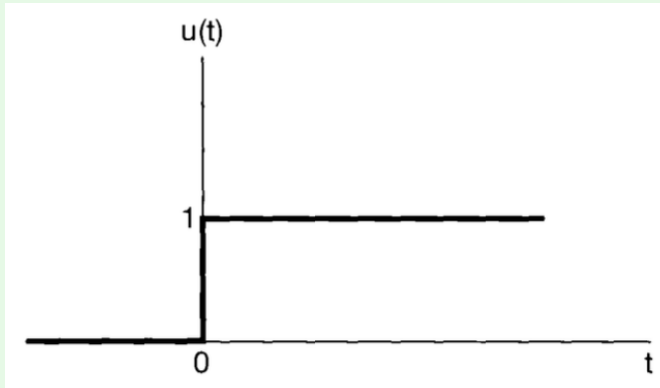

Figure 4: CT unit step function

4.2.3 Running integral

Definition:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \int_{0}^{x} \delta(t-\sigma)\, d\sigma \quad (21)$$

4.2.4 Sampling

Definition: For an impulse at t_0 then,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0) \quad (22)$$

5 General systems and basic properties (Ch. 1.5-6)

5.1 Systems

Definition: A process that transforms input signals into output signals.

- **Note:** Signals can be discrete or continuous.
  - **CT:** $x(t) \to y(t)$
  - **DT:** $x[n] \to y[n]$

5.2 Interconnection of systems

Definition:

- **Series:**

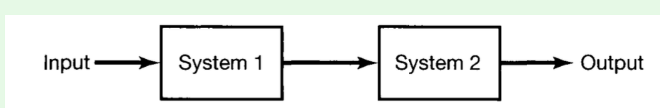

Figure 5: Series system.

- **Parallel:**

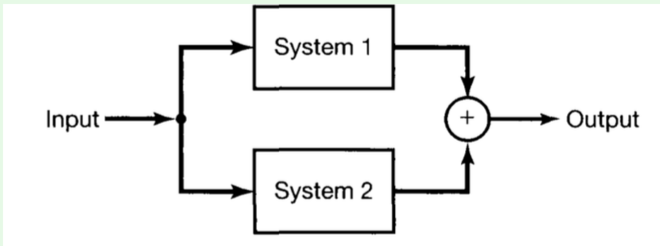

Figure 6: Parallel system.

- **Series and parallel:**

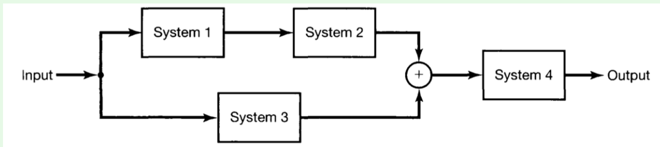

Figure 7: Series and parallel system.

- **Feedback:**

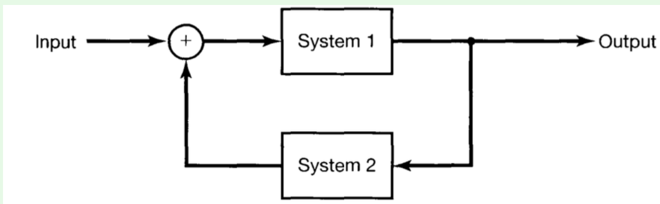

Figure 8: Feedback system.

5.3 Basic properties

5.3.1 Memory

Definition: A system is said to be *memoryless* if $y(t_0)$ depends only on $x(t_0)$, $\forall t_0$

5.3.2 Invertibility

Definition: A system is invertible if distinct input leads to distinct output.

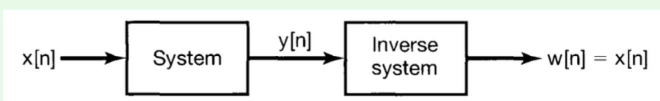

Figure 9: Inverse system concept.

5.3.3 Causality

Definition: A system is causal if the output at any time depends only on values of the input at the present time and in the past.

DT: $y[n_0]$ does not depend on values of $x[n]$ for $n > n_0$.

CT: $y(t_0)$ does not depend on values of $x(t)$ for $t > t_0$.

5.3.4 Stability

Definition: A system is bounded-input-bounded-output (BIBO) stable if bounded input signals (i.e. $|x(t)| < \infty \ \forall t$), leads to bounded output signals (i.e. $|y(t)| < \infty \ \forall t$)

5.3.5 Time invariance

Definition: A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal.

- **DT:** If $y[n]$ is the output of a time-invariant system when $x[n]$ is the input, then $y[n-n_0]$ is the output when $x[n-n_0]$ is applied.
- **CT:** If $y(t)$ is the output of a time-invariant system when $x(t)$ is the input, then $y(t-t_0)$ is the output when $x(t-t_0)$ is applied.

5.3.6 Linearity

Definition:
Suppose for inputs x_1, x_2 correspond to outputs y_1, y_2 respectively, then a system is linear if:

1. **Additivity:** The response to $x_1 + x_2$ is $y_1 + y_2$.
2. **Homogeneity:** The response to ax_1 is ay_1, where $a \in \mathbb{C}$.

Superposition: If x_k, $k = 1, 2, 3, \ldots$, are a set of inputs to a linear system with corresponding outputs y_k, $k = 1, 2, 3, \ldots$, then the response to a linear combination of these inputs given by

$$x = \sum_k a_k x_k \text{ is } y = \sum_k a_k y_k$$

- **Consequence:** $x = 0 \to y = 0$

Linear Time-Invariant Systems

6 Impulse response (Ch. 2.1)

6.1 DT Sifting property

Definition: Any DT signal can be written as

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \tag{23}$$

- **Key:** $\delta[n-k]$ is nonzero only when $k = n$, so it preserves only that value.

6.2 DT Unit impulse response

Definition: $h[n]$ is the output of the LTI system when $\delta[n]$ is the input.

$$h[n] = h_0[n] \tag{24}$$

Intuition: $\delta[n] \to \text{LTI} \to h[n]$

6.3 CT Sifting property

Definition: Any CT signal can be written as

$$x(t) = \int_{-x}^{+x} x(\tau)\delta(t-\tau)\, d\tau \tag{25}$$

6.4 CT Unit impulse response

Definition: $h(t)$ is the output of the LTI system when $\delta(t)$ is the input.

$$h(t) = h_0(t) \tag{26}$$

Intuition: $\delta(t) \rightarrow \text{LTI} \rightarrow h(t)$

7 Convolution in discrete time (Ch. 2.1)

7.1 Convolution sum

Definition: For an LTI system, the convolution of the sequences $x[n]$ and $h[n]$ (i.e. the sliding of $h[n-k]$ past $x[k]$ is

$$y[n] = x[n] \star h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \tag{27}$$

- $x[k]$: Input applied at time k.
- $x[k]h[n-k]$: Response due to the input (i.e. it is a shifted and scaled version of $h[n]$).
- $y[n]$:
  - Output from the superposition of all these time-shifted impulse responses.
  - Output from the sum over k of functions $x[k]h[n-k]$ in n.

8 Convolution in continuous time (Ch. 2.2)

8.1 Convolution integral

Definition: The convolution of two signals $x(t)$ and $h(t)$ is

$$y(t) = x(t) \star h(t) = \int_{-x}^{+x} x(\tau)h(t-\tau)\, d\tau \tag{28}$$

- $x(\tau)$: Input
- $h(t-\tau)$: Weight
- $y(t)$: Weighted integral of the input

Process:
1. Flip $h(\tau)$ about the y-axis for $h(-\tau)$
2. Slide $h(-\tau)$ from left to right for $h(t-\tau)$
3. For any t, multiply $x(\tau)$ and $h(t-\tau)$
4. Integrate from $\tau = -\infty$ to $\tau = +\infty$ to obtain $y(t)$

8.2 Consequence of convolution

Intuition: For LTI systems, the characteristics of the system is completely determined by its impulse response.

9 Properties of LTI systems (Ch. 2.3)

9.1 Commutative

Definition:
DT:

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-x}^{+x} h[k]x[n-k] \tag{29}$$

CT:

$$x(t) * h(t) = h(t) * x(t) = \int_{-x}^{+x} h(\tau)x(t-\tau)\, d\tau \tag{30}$$

9.2 Distributive

Definition:
DT:

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n] \tag{31}$$

CT:

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t) \tag{32}$$

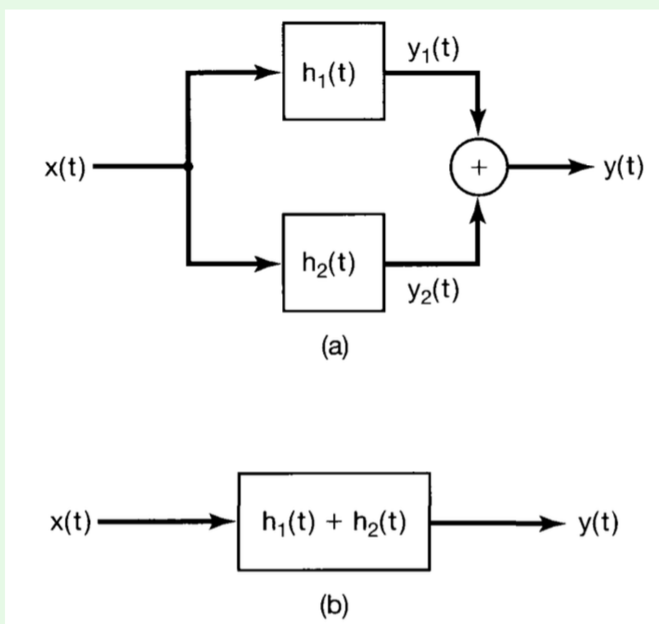

Figure 10: Distributive property for a parallel interconnection of LTI systems

9.3 Consequence of commutative and distributive

Definition: The response of an LTI system to the sum of two inputs must equal the sum of the responses to these signals individually.
DT:

$$[x_1[n] + x_2[n]] * h[n] = x_1[n] * h[n] + x_2[n] * h[n] \tag{33}$$

CT:

$$[x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t) \tag{34}$$

9.4 Associative

Definition:
DT:

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n] \tag{35}$$

CT:

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t) \tag{36}$$

- **Key:** It does not matter in which order we convolve these signals.

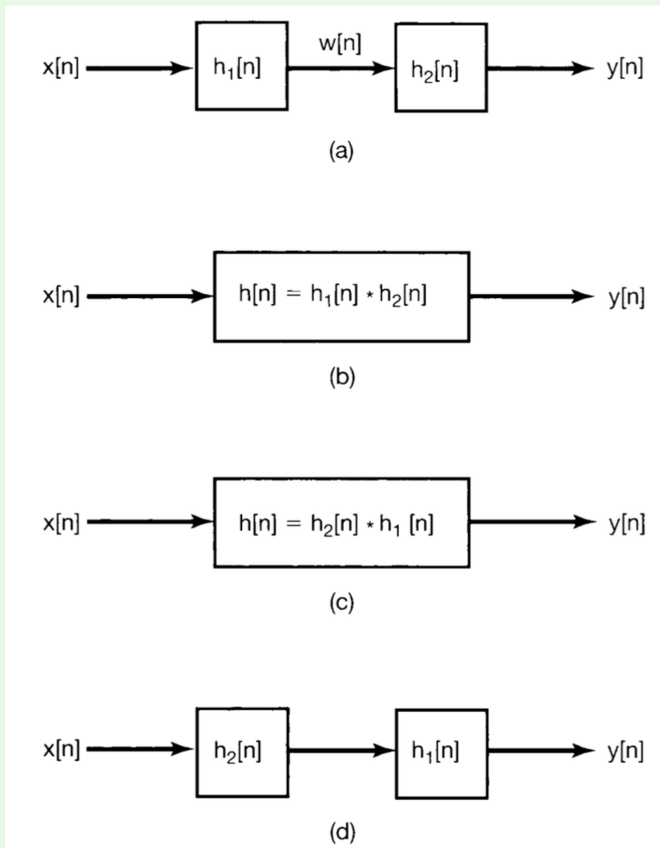

Figure 11: Associative property of convolution and the implication of this and the commutative property for the series interconnection of LTI systems.

9.5 Memory

Definition:
DT: If $h[n] = 0$ for $n \neq 0$, then an LTI system is memoryless iff $h[n] = K\delta[n]$, where $K = h[0]$ is a constant, so the convolution sum reduces to $y[n] = Kx[n]$

CT: If $h(t) = 0$ for $t \neq 0$, then an LTI system is memoryless iff $h(t) = K\delta(t)$, where K is a constant, so the convolution integral reduces to $y(t) = Kx(t)$

9.6 Invertibility

Definition: If an LTI system is invertible, then it has an LTI inverse.

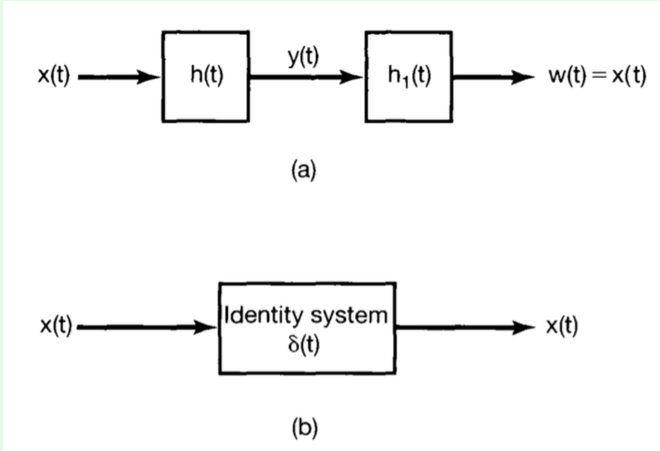

Figure 12: Inverse system for CT LTI systems. The system with impulse response $h_1(t)$ is the inverse of the system with impulse response $h(t)$

- **DT:** $h[n] \star h_1[n] = \delta[n]$
- **CT:** $h(t) \star h_1(t) = \delta(t)$

9.7 Causality

Definition:

DT: A LTI system is causal if $h[n] = 0$ for $n < 0$, so the convolution sum becomes $y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$

CT: A LTI system is causal if $h(t) = 0$ for $t < 0$, so the convolution integral becomes $y(t) = \int_0^{\infty} h(\tau)x(t-\tau)d\tau$

- **Initial rest:** Equivalent to the initial rest condition if $x(t) = 0$ for $t < t_0$, then $y(t) = 0$ for $t < t_0$
- **Signal causality:** Causality of an LTI system is equivalent to its impulse response being a causal signal.

9.8 Stability

Definition:

DT: The LTI system is stable if the impulse response is *absolutely summable*, that is, if $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$.

CT: The LTI system is stable if the impulse response is *absolutely integrable*, that is, if $\int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty$.

Fourier Series and Fourier Transform Representations

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