ECE367 Cheatsheet

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1 Week 1 (Ch. 2.1-2.2)

1.1 Vectors

Definition: A single collection of numbers, where $x_i \in \mathbb{R}$ or $\mathbb{C}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{x}^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

- n: Dimension of $\mathbf{x}$
- $\mathbf{x}$: Column vector
- $\mathbf{x}^T$: Row vector
- T: Transpose

1.1.1 Vector spaces

Definition: A set of vectors $\mathcal{V}$ that are closed under addition and scalar multiplication.

Components:
- $\mathbf{v}^{(1)} + \mathbf{v}^{(2)}$ is the sum of the corresponding components: $v_i^{(1)} + v_i^{(2)}$.
- $\alpha\mathbf{v}$ is multiplying each component by α: αv_i.

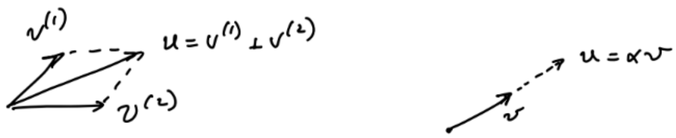

Figure 1: Vector addition and scalar multiplication.

1.1.2 Properties of vector spaces

Definition:
- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
- $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
- $\exists\ \mathbf{0} \in \mathbb{V}$ s.t. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- $\exists\ -\mathbf{u} \in \mathbb{V}$ s.t. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- $\exists\ \alpha$ s.t. $\alpha\mathbf{u} = \mathbf{u} \quad (\alpha = 1)$

1.1.3 Span and subspace

Definition: Let $S = \left\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(m)}\right\}$, then the **span** is

$$\text{Span}(S) = \left\{\sum_{i=1}^{m} \alpha_i \mathbf{v}^{(i)} \mid \alpha_i \in \mathbb{R}\right\}$$

- **Note:** The span of a set of vectors is always a subspace.

Definition: A **subspace** is a subset of a vector space that is a vector space by itself.
- **Note:** 0 is always in a subspace.

1.1.4 Linear independent (LI) set

Definition:
- $S = \left\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(m)}\right\}$ is LI if no element of S can be expressed as a LC of other elements in S (i.e. The only α_i's that makes $\sum_{i=1}^{m} \alpha_i \mathbf{v}^{(i)} = 0$ is $\alpha_i = 0,\ \forall i$).
- If S is a LI set, then for any $u \in \mathit{span}(S)$, there is a unique set of α_i's s.t. $u = \sum_{i=1}^{m} \alpha_i \mathbf{v}^{(i)}$ (i.e. there is no redundancies in representation)
- If a set of $\left\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(m)}\right\}$ has redundancy, then we can remove the vector that can be represented as a LC of other vectors until all the redundancies are removed.
  - **Note:** Such an irreducible LI set can serve as a basis for $\mathit{Span}\left\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(m)}\right\}$.

1.1.5 Basis

Definition: A set of vectors $\mathcal{B}$ is a basis of a vector space $\mathcal{V}$ if (i) $\mathcal{B}$ is LI, (ii) $Span(\mathcal{B}) = \mathcal{V}$

Example: What is the standard basis?

$$\mathbf{e}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots \quad , \quad \mathbf{e}^{(n)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad \text{for } \mathbb{R}^n$$

1.1.6 Cardinality

Definition: The dimension of vector spaces $\mathcal{V}$ is the **cardinality** of $\mathcal{B}$.

- **Cardinality:** A set refers to the number of elements in the set.
- **Note:** Basis is not unique. But $dim(\mathcal{V})$ is well-defined.

1.2 Norms

In optimization problems, different norms are used to achieve various goals.

Definition: Notion of distance, where $\|\mathbf{v}\|$ is a function that maps $\mathcal{V} \to \mathbb{R}$ that satisfies

1. $\|\mathbf{v}\| \geq 0,\ \forall \mathbf{v} \in \mathcal{V},$ and $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = \mathbf{0}$
2. $\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|\ \forall \mathbf{v} \in \mathcal{V},\ \alpha \in \mathbb{R}$
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|,\ \forall x,\ y \in \mathcal{V}$ (triangular inequality)

Example: ℓ_p norms:

$$\|\mathbf{x}\|_p \equiv \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}, \quad 1 \leq p < \infty.$$

- **Sum-of-absolute-values length** $p = 1$: $\|\mathbf{x}\|_1 \equiv \sum_{k=1}^{n} |x_k|$
  - For $p < 1$, triangular inequality doesn't hold.
- **Euclidean length** $p = 2$: $\|\mathbf{x}\|_2 \equiv \sqrt{\sum_{k=1}^{n} x_k^2}$
- **Max absolute value norm** $p = \infty$: $\|\mathbf{x}\|_\infty \equiv \max_{k=1,\dots,n} |x_k|$
  - Longest term will dominate.
- **Cardinality** $p = 0$: The number of non-zero vectors in x is

$$\|\mathbf{x}\|_0 = \text{card}(\mathbf{x}) \equiv \sum_{k=1}^{n} \mathbb{I}(x_k \neq 0), \quad \text{where} \quad \mathbb{I}(x_k \neq 0) \equiv \begin{cases} 1 & \text{if } x_k \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

  - Not a norm since $\|\alpha\mathbf{x}\|_0 = \|\mathbf{x}\|_0 \neq |\alpha| \cdot \|\mathbf{x}\|_0$

1.2.1 Norm balls

Definition: The set of all vectors with ℓ_p norm less than or equal to one,

$$B_p = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_p \leq 1\} \tag{1}$$

Example:

- $\ell_2: \; B_2 = \left\{ \mathbf{x} \;\middle|\; \sqrt{x_1^2 + x_2^2} \le 1 \right\}$
- $\ell_1: \; B_1 = \{ \mathbf{x} \mid |x_1| + |x_2| \le 1 \}$
- $\ell_\infty: \; B_\infty = \{ \mathbf{x} \mid \max |x_i| \le 1 \}$
- $\ell_0: \; B_0 = \{ \mathbf{x} \mid \text{card}(\mathbf{x}) \le 1 \}$

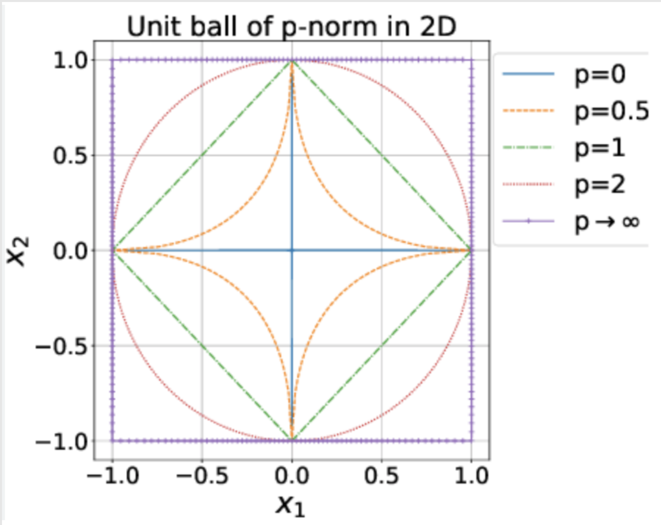

Figure 2: Norm balls of different p values.

1.3 Inner products

Definition: $\mathbf{x}, \mathbf{y} \in \mathcal{V}$: $\mathcal{V} \to \mathbb{R}$ into a scalar denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$. The inner product satisfies the following axioms: for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and scalar α,

- $\langle \mathbf{x}, \mathbf{x} \rangle \ge 0$
- $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ iff $\mathbf{x} = 0$
- $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
- $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ in $\mathbb{R}^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ in $\mathbb{C}^n$.

1.3.1 Standard inner product

Definition: Notion of angle between two vectors in $\mathbb{R}^n$, defined as the row-column product of two vectors:

- In $\mathbb{R}^n$: $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i = \mathbf{x}^\top \mathbf{y}$
- In $\mathbb{C}^n$: $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} \overline{x_i} y_i = \mathbf{x}^H \mathbf{y}$

1.3.2 Connect inner product with angle

Definition:

$$\cos\theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2} = \left\langle \frac{\mathbf{x}}{\|\mathbf{x}\|_2}, \frac{\mathbf{y}}{\|\mathbf{y}\|_2} \right\rangle \tag{2}$$

- **Orthogonal:** $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, the angle between vectors $\mathbf{x}$ and $\mathbf{y}$ is $\theta = \pm 90^\circ$.
- **Parallel:** When the angle θ is 0° or $\pm 180^\circ$, vectors $\mathbf{x}$ and $\mathbf{y}$ are aligned, meaning $\mathbf{y} = \alpha \mathbf{x}$ for some scalar α.

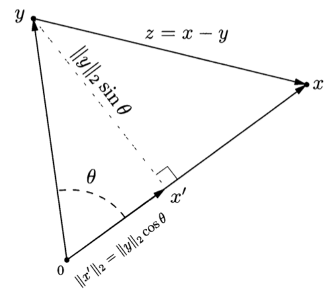

Figure 3: Visual representation of angle between vectors.

1.3.3 Cauchy-Schwartz inequality and its generalization

Definition:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \tag{3}$$

Hölder's Inequality (generalization):

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q \quad \text{where } 1 \leq p, q < \infty \text{ and } \frac{1}{p} + \frac{1}{q} = 1 \tag{4}$$

1.4 Orthogonal decomposition

1.4.1 Orthogonality

Definition: A set of non-zero vectors $S = \left\{ \mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(d)} \right\}$ is **mutually orthogonal** if $\langle \mathbf{v}^{(i)}, \mathbf{v}^{(j)} \rangle = 0 \; \forall \; i \neq j$.

- **Fact:** Orthogonal set of vectors form a basis for $Span = \left\{ \mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(d)} \right\}$

1.4.2 Orthonormal basis

Definition: Set of orthogonal basis vectors that have unit norm:

$$\left\{ \frac{\mathbf{v}^{(1)}}{\|\mathbf{v}^{(1)}\|}, \ldots, \frac{\mathbf{v}^{(d)}}{\|\mathbf{v}^{(d)}\|} \right\}$$

1.4.3 Orthogonal component

Definition: Given $S \subseteq \mathcal{V}$ is a subspace of $\mathcal{V}$, a vector $\mathbf{x} \in \mathcal{V}$ is said to be **orthogonal** to S if $\forall \mathbf{v} \in S$, we have $\langle \mathbf{x}, \mathbf{v} \rangle = 0$.

1.4.4 Orthogonal complement

Definition: A vector $\mathbf{x} \in \mathcal{V}$ is orthogonal to a subset S of an inner product space $\mathcal{V}$ if $\mathbf{x} \perp s$ for all $s \in S$. The set of vectors in $\mathcal{V}$ that are orthogonal to S is called the *orthogonal complement* of S:

$$S^{\perp} = \{ \mathbf{x} \in \mathcal{V} \mid \mathbf{x} \perp S \} \tag{5}$$

- $dim(S) + dim(S^{\perp}) = dim(\mathcal{V})$

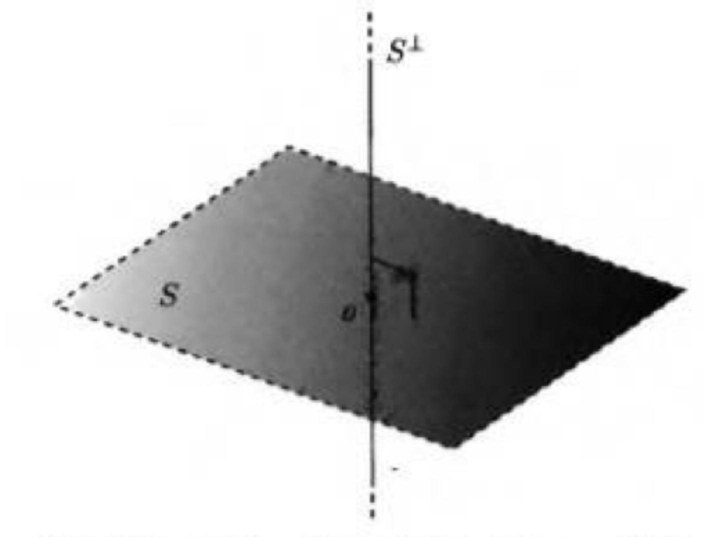

Figure 4: Orthogonal complement.

1.4.5 Orthogonal decomposition

Definition: Any $\mathbf{x} \in \mathcal{V}$ can be expressed as

- $\mathbf{x} = \mathbf{x}_S + \mathbf{x}_{S^\perp}$
- $\mathcal{V} = S \oplus S^\perp$

2 Week 2 (Ch. 2.3)

2.1 Projection onto subspaces

2.1.1 Problem that projections solve

Definition: Given a vector $\mathbf{x} \in \mathcal{V}$. Find the closest point in S to x:

$$\text{Proj}_S(x) = \arg\min_{y \in S} \|y - x\| \tag{6}$$

- $\|y - x\|$: Some norm.

2.1.2 Higher dimensional S

Definition:

2.2 Fourier series

2.3 Gram-Schmidt and QR decomposition

2.4 Hyperplanes and half-spaces

2.5 Non-euclidean projection

3 Week 3 (Ch. 2.3-2.4)

3.1 Projection onto affine sets

3.2 Functions

3.3 Gradients

3.4 Hessians

4 Week 4 (Ch. 3.1-3.5)

4.1 Matrices

4.2 Range

4.3 Null Space

4.4 Eigenvalues and eigenvectors

4.5 Matrices diagonalization

5 Week 5 (Ch. 4.1-4.4)

5.1 Symmetric matrices

5.2 Orthogonal matrices

5.3 Spectral decomposition

5.4 Positive semidefinite matrices

5.5 Ellipsoids

6 Week 6 (Ch. 5.1, 5.3.2)

6.1 Singular value decomposition

6.2 Principle component analysis

7 Week 7 (Ch. 5.2-5.3.1)

7.1 Interpretation of SVD

7.2 Low-rank approximation

8 Week 8 (Ch. 6.1-6.4)

8.1 Least squares

8.2 Overdetermined linear equation

8.3 Underdetermined linear equation

9 Week 9 (Ch. 6.7.3, 8.1-8.4)

9.1 Regularized least-squares

9.2 Convex sets and convex functions