ECE358 Cheatsheet

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Read the textbook for this course.

1   Asymptotics (Ch. 3.1-2 pg. 50-63, L2)

Asymptotic efficiency focuses on understanding how the running time of an algorithm grows with the input size, particularly for large inputs.

1.1   Big-O

Definition: $O(g(n)) = \{f(n): \exists$ positive constants c and n_0 such that $0 \leq f(n) \leq cg(n) \ \forall \ n \geq n_0\}$
  - **Asymptotic upper bound:** Grows **no faster** than a certain rate, based on the highest-order term.

Warning: Every function $f(n)$ in the set $O(g(n))$ must be **asymptotically nonnegative** (i.e. $f(n)$ must be positive whenever n is sufficiently large).

1.2   Big-Omega

Definition: $\Omega(g(n)) = \{f(n): \exists$ positive constants c and n_0 such that $0 \leq cg(n) \leq f(n) \ \forall \ n \geq n_0\}$
  - **Asymptotic lower bound:** Grows **at least as fast** as a certain rate, based on the highest-order term.

1.3   Big-Theta

Definition: $\Theta(g(n)) = \{f(n): \exists$ positive constants c_1, c_2, and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \ \forall \ n \geq n_0\}$
  - **Asymptotically tight bounds:** Grows **precisely** at a certain rate, based on the highest-order term.
  - **Constant factor:** Characterizes the rate of growth of the function to within a constant factor from above and below. These two constant factors need not be equal.
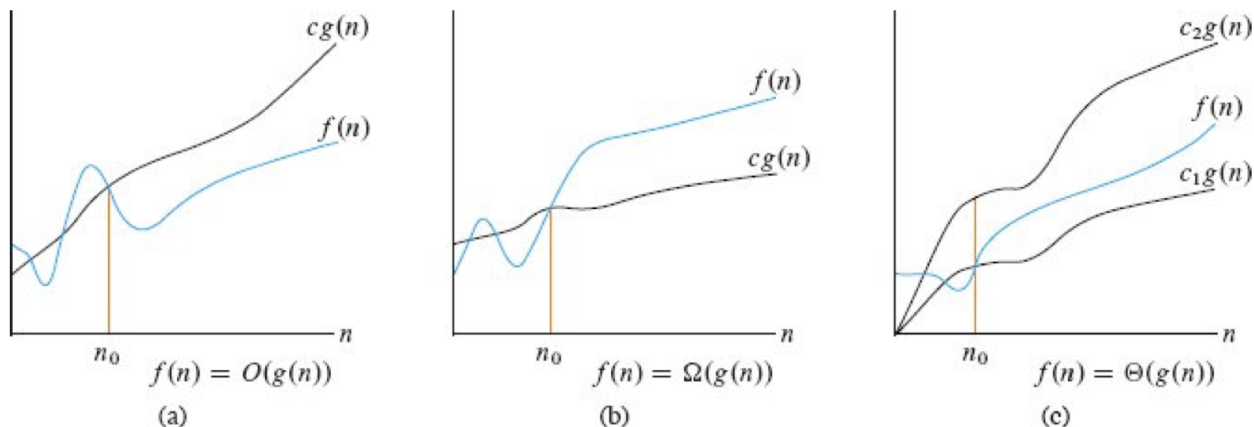
Figure 1: Graphical examples of the Big-O, Big-Omega, and Big-Theta.

Example: Find the Big-O, Big-Omega, and Big-Theta of the following function: $7n^3 + 100n^2 - 20n + 6$.
  1. **Highest-order term:** $7n^3$
  2. **Remove constants:** n^3
  3. **Big-O notation:** $O(n^3)$
      - In general, $O(n^c)$ for any constant $c \geq 3$ because the function grows no faster than this.

4. **Big-Omega notation:** $\Omega(n^3)$
   - In general, $\Omega(n^c)$ for any constant $c \leq 3$ because the function grows at least as fast than this.
5. **Big-Theta:** Since O and Ω are the same, therefore, $\Theta(n^3)$ (by theorem)

Intuition/Tips: In all asymptotic notations, you are trying to describe a function after n_0 (i.e. ignore all flunctuation before).

1.4   Theorem 3.1

Theorem: For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

1.5   Asymptotic notation and running times

Warning: Make sure that the asymptotic notation you use is as precise as possible without overstating which running time (i.e. worst-case, best-case, or any other-case) it applies to.

Example: What asymptotic notation should you use for insertion sort's worst-case, best-case, and general running time?

- **Worst-case:** $O(n^2)$, $\Omega(n^2)$, and $\Theta(n^2)$ can be used, but $\Theta(n^2)$ is the most precise.
- **Best-case:** $O(n)$, $\Omega(n)$, and $\Theta(n)$ can be used, but $\Theta(n)$ is the most precise.
- **General:** $O(n^2)$ because in all cases, its running time grows no faster than n^2. Or $\Omega(n)$ because in all cases, its running time is at least as fast as n.

1.6   Abuses of asymptotic notation

Intuition/Tips:
- **Equality:**
  - When asymptotic notation stands alone on the RS of an equation (or inequality), then $=$ means $\in$.
  - When asymptotic notation is in a formula, it is an anonymous function (AF) that we do not care to name.
  - When asymptotic notation appears on the LS of an equation: No matter how the AF is chosen on the LS, there is a way to choose the AF on the RS to make the equation valid.
- **Variable tending toward ∞ must be inferred from context:**
  - e.g. $O(g(n))$, then we are interested in the growth of $g(n)$ as n grows.
  - e.g. $f(n) = O(1)$, then $f(n)$ is bounded from above by a constant as n goes to ∞.
  - e.g. $T(n) = O(1)$ for $n < 3$ is that there exists a positive constant c such that $T(n) \leq c$ for $n < 3$.

1.7   Comparing function properties

Definition:
Transitivity:
- $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ imply $f(n) = \Theta(h(n))$
- $f(n) = O(g(n))$ and $g(n) = O(h(n))$ imply $f(n) = O(h(n))$
- $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ imply $f(n) = \Omega(h(n))$

Reflexivity:
- $f(n) = \Theta(f(n))$
- $f(n) = O(f(n))$
- $f(n) = \Omega(f(n))$

Symmetry:

- $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$.

Transpose symmetry:
- $f(n) = O(g(n))$ iff $g(n) = \Omega(f(n))$

Different functions:
- $n^a \in O(n^b)$, iff $a \leq b$.
- $\log_a(n) \in O(\log_b(n))$, for all a, b.
- $c^n \in O(d^n)$, iff $c \leq d$.
- If $f(n) \in O(f'(n))$ and $g(n) \in O(g'(n))$, then:
    - $f(n) \cdot g(n) \in O(f'(n) \cdot g'(n))$.
    - $f(n) + g(n) \in O(\max\{f'(n), g'(n)\})$.

Intuition/Tips:
- $f(n) = O(g(n))$ is like $a \leq b$
- $f(n) = \Omega(g(n))$ is like $a \geq b$
- $f(n) = \Theta(g(n))$ is like $a = b$

1.8   Polynomially-bounded

Definition: $f(n)$ is polynomially-bounded if $f(n) = O\left(n^k\right)$ for some real value of k.

Theorem: $f(n) = O\left(n^k\right)$ iff $lg(f(n)) = O(lg(n))$

1.9   Limit method

Definition: Find the asymptotic relationship between two functions for which you might not have any intuition about.

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \implies f(n) = \mathcal{O}(g(n)) \tag{1}$$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c,\ 0 < c < \infty \implies f(n) = \Theta(g(n)) \tag{2}$$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \implies f(n) = \Omega(g(n)) \tag{3}$$

1.9.1   L'Hôpital's rule

Definition: If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$, then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \tag{4}$$

1.9.2   Logs of limits and limits of logs

Definition:

$$lg\left(\lim_{x \to c} g(x)\right) = \lim_{x \to c} lg(g(x)) \tag{5}$$

2 Logarithms, Summations (L7)

2.1 Logarithms (Ch. 3.3 pg. 66-7)

2.1.1 Definition and notation

Definition:
$$a = b^c \iff \log_b a = c \tag{6}$$

Notation:

- $lg\ n = log_2 n$
- $ln\ n = log_e n$
- $lg^k\ n = (lg\ n)^k$
- $lg^{(2)} n = lg\ lg\ n = lg(lg\ n)$

2.1.2 Properties

Definition: $\forall$ real $a > 0$, $b > 0$, $c > 0$, and n, we have
1. $a = b^{\log_b a}$
2. $\log_c(ab) = \log_c a + \log_c b$
3. $\log_b a^n = n \log_b a$
4. $\log_b a = \dfrac{\log_c a}{\log_c b}$
5. $\log_b \left(\dfrac{1}{a}\right) = -\log_b a$
6. $\log_b a = \dfrac{1}{\log_a b}$
7. $a^{\log_b c} = c^{\log_b a}$
8. $\log_b \dfrac{a}{c} = \log_b a - \log_b c$

2.2 Functional iteration

Definition: $f(n)$ iteratively applied i times to an initial value of n.

$$f^{(i)}(n) = \begin{cases} n & \text{if } i = 0, \\ f\left(f^{(i-1)}(n)\right) & \text{if } i > 0. \end{cases} \tag{7}$$

2.3 Iterated logarithm function

Definition: The minimum number of times i that the logarithm function must be applied to n for the result to be less than or equal to 1:
$$lg^* n = \min\left\{ i \geq 0\ :\ lg^{(i)} n \leq 1 \right\} \tag{8}$$

Intuition/Tips:
- **Definition of** $lg^{(i)} n$**:** The expression $lg^{(i)} n$ denotes the logarithm function applied i times in succession.
  - If $i = 1$, then $lg^{(1)} n = lg\ n$. If $i = 2$, then $lg^{(2)} n = lg(lg\ n)$, and so on.
  - This is different from $lg^i n$, which would mean $(lg\ n)^i$, i.e., raising $lg\ n$ to the power i.
- **Conditions for Definition:** The iterated logarithm $lg^{(i)} n$ is only defined if $lg^{(i-1)} n > 0$. This constraint exists because the logarithm of a non-positive number is undefined in real numbers.
- **Useful formula:** $O(n lg^* n) \approx O(n)$

Example: The iterated logarithm is a *very* slowly growing function:
- $\lg^* 2 = 1$ because one application of the logarithm to 2 results in a value less than or equal to 1.
- $\lg^* 4 = 2$
- $\lg^* 16 = 3$ because three applications of the logarithm to reach a value less than or equal to 1.
- $\lg^* 65536 = 4$
- $\lg^*(2^{65536}) = 5$

2.4   Fibonacci Numbers (Ch. 3.3)

2.4.1   Definition

Definition:
$$F_i = \begin{cases} 0 & \text{if } i = 0, \\ 1 & \text{if } i = 1, \\ F_{i-1} + F_{i-2} & \text{if } i \geq 2. \end{cases} \tag{9}$$

2.4.2   Golden ratio and its conjugate

Definition:
$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.61803\ldots \tag{10}$$

and its conjugate, by

$$\hat{\phi} = \frac{1 - \sqrt{5}}{2} \approx -0.61803\ldots \tag{11}$$

Specifically, we have

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} \tag{12}$$

2.5   Summations (Ap. A.1 pg. 1140-51)

2.5.1   Arithmetic series

Definition:
$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2} = \Theta(n^2) \tag{13}$$

2.5.2   General arithmetic series

Definition: For $a \geq 0$ and $b > 0$,
$$\sum_{k=1}^{n} (a + bk) = \Theta(n^2) \tag{14}$$

2.5.3   Sums of squares and cubes

Definition:
Sums of squares:
$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{15}$$

Sums of cubes:

$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} \tag{16}$$

2.5.4 Finite geometric series

Definition: For $x \neq 1$,

$$\sum_{k=1}^{n} x^k = 1 + x + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} \tag{17}$$

2.5.5 Infinite decreasing geometric series

Definition: For $|x| < 1$,

$$\sum_{k=1}^{\infty} x^k = \frac{1}{1-x} \tag{18}$$

2.5.6 Harmonic series

Definition: For positive integers n, the nth harmonic number is

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1) \tag{19}$$

2.5.7 Telescoping series

Definition: For any sequence $a_0, a_1, \ldots, a_n$,

$$\sum_{k=1}^{n}(a_k - a_{k-1}) = a_n - a_0 \quad \text{OR} \quad \sum_{k=0}^{n-1}(a_k - a_{k+1}) = a_0 - a_n \tag{20}$$

- Each of the terms is added in exactly once and subtracted out exactly once.

2.5.8 Reindexing summations

Intuition/Tips:

$$\sum_{k=0}^{n} a_{n-k} = \sum_{j=0}^{n} a_j \tag{21}$$

- $j = n - k$
- If the summation index appears in the body of the sum with a minus sign, it's worth thinking about reindexing.

2.5.9 Products

Definition: The finite product $a_1 a_2 \cdots a_n$ can be expressed as:

$$\prod_{k=1}^{n} a_k \tag{22}$$

2.5.10   Product to summation

> **Definition**:
> $$lg\left(\prod_{k=1}^{n} a_k\right) = lg\left(a_1 \cdot a_2 \cdots a_n\right) = lg(a_1) + lg(a_2) + \ldots + lg(a_n) = \sum_{k=1}^{n} lg(a_k) \tag{23}$$

3   Induction, Contradiction (L3)

3.1   Induction (Ap. A.2)

Motivation: The most basic way to evaluate a series is to use induction.

> **Process**: Given proposition $P(n)$
>   1. Basis: Prove the base case $P(1)$
>   2. Inductive hypothesis: Assume true for $P(n)$
>   3. Inductive step: Use the hypothesis to show its true for $P(n) \to P(n+1)$
> Therefore, $\forall\, n P(n)$.

> **Intuition/Tips**: You don't always need to guess the exact value of a summation in order to use induction. Instead, use induction to prove an upper or lower bound on a summation.

> **Example**: Prove $\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$
>   1. **Basis:** $n = 1$, $\dfrac{1(1+1)}{2} = 1$
>   2. **Inductive hypothesis:** Assume true for n, $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$
>   3. **Inductive step:** Prove for $n+1$: $1 + 2 + \ldots + n + (n+1) = \dfrac{n(n+1)}{2} + (n+1) = \dfrac{(n+1)(n+2)}{2}$
>
> Therefore, we proved by induction that the formula, $\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$ for $n+1$ is true for $n+1$.

> **Example**: Prove the asymptotic upper bound $\sum_{k=0}^{n} 3^k = O\left(3^n\right)$ or $\sum_{k=0}^{n} 3^k \le c3^n$ for some constant c.
>   1. **Basis:** $n = 0$: $\sum_{k=0}^{0} 3^k = 1 \le c$ as long as $c \ge 1$
>   2. **Inductive hypothesis:** Assume that the bound holds for n.
>   3. **Inductive step:** Prove for $n+1$:
>
> $$\sum_{k=0}^{n+1} 3^k = \sum_{k=0}^{n} 3^k + 3^{n+1}$$
> $$\le c3^n + 3^{n+1} \text{ by the inductive hypothesis}$$
> $$= \left(\frac{1}{3} + \frac{1}{c}\right) c3^{n+1} \text{ by factoring out } c3^{n+1}$$
> $$\le c3^{n+1} \text{ since we are using the inequality it still holds true}$$
>
> Therefore, as long as $\left(\dfrac{1}{3} + \dfrac{1}{c}\right) \le 1$ or $c \ge \dfrac{3}{2}$. Thus, $\sum_{k=0}^{n} 3^k = O(3^n)$.

Warning: Consider the following fallacious proof that $\sum_{k=1}^{n} k = O(n)$.

1. **Basis:** $\sum_{k=1}^{1} k = O(1)$
2. **Inductive hypothesis:** Assume that the bound holds for n.
3. **Inductive step:** Prove for $n + 1$:

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1)$$
$$= O(n) + (n+1)$$
$$= O(n+1) \quad \text{(wrong!)}$$

The bug in the argument is that the "constant" hidden by the "big-O" grows with n and thus is not constant. We have not shown that the same constant works for all n.

3.2   Contradiction

Process: Property $P(n)$ which you want to prove true, and it can be true or false.
1. If want to prove true, assume $\neg P(n)$.
2. Work towards a contradiction by working with the expression $\neg P(n)$ and prove this to be false.
3. If this resulted in a false statement then $P(n)$ is true.

Example: Prove that if $x^2 - 5x + 4 < 0,$ then $x > 0$
1. **ATaC:** Assume towards a contradiction (ATaC) that $x^2 - 5x + 4 < 0$ but $x \leq 0$.

2. Analyze the quadratic expression:
$$x^2 - 5x + 4 = (x-1)(x-4)$$

Thus, the inequality becomes:
$$(x-1)(x-4) < 0$$

3. This inequality implies that x must lie between the roots 1 and 4, i.e., $1 < x < 4$.

4. **Contradiction:** However, the assumption $x \leq 0$ contradicts this because there are no values of $x \leq 0$ that satisfy $1 < x < 4$.

Therefore, the contradiction shows that the assumption $x \leq 0$ cannot be true if $x^2 - 5x + 4 < 0$. Hence, if $x^2 - 5x + 4 < 0$, it must be that $x > 0$.

Example: Prove $\sqrt{2}$ is irrational.
1. **ATaC:** Suppose $\sqrt{2}$ is rational. Then we can write:

$$\sqrt{2} = \frac{a}{b}$$

where a and b are integers with no common divisors other than 1 (i.e., the fraction is in its simplest form).

2. **Square Both Sides:**

$$2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2$$

This implies that a^2 is even (since it is twice an integer). Therefore, a must also be even (by a lemma which states that if a^2 is even, then a is even).

3. **Express a as an Even Number:**

$$a = 2k \text{ for some integer } k$$

Substitute $a = 2k$ into the equation:

$$(2k)^2 = 2b^2 \Rightarrow 4k^2 = 2b^2 \Rightarrow 2k^2 = b^2$$

This implies that b^2 is even, and thus b must also be even.

4. **Contradiction:** Since both a and b are even, they have a common factor of 2. This contradicts our initial assumption that $\dfrac{a}{b}$ is in its simplest form.

The contradiction shows that our assumption that $\sqrt{2}$ is rational is false. Therefore, $\sqrt{2}$ is irrational.

4   Recurrences (Ch. 2.3, L4)

4.1   Recurrences introduction (Ch. 4.1)

Divide-and-conquer method is useful to solve recurrences, which has three steps:

1. **Divide** the problem into one or more subproblems that are smaller instances of the same problem.

2. **Conquer** the subproblems by solving them recursively.

3. **Combine** the subproblem solutions to form a solution to the original problem.

> **Definition**: A **recurrence** is an equation (or inequality) that describes a function in terms of its value on other, typically smaller, arguments.
> - **Inequality:** You will use Ω (i.e. lower bound) or O (i.e. upper bound).
>
> A recurrence $T(n)$ is **algorithmic** if, for every sufficiently large **threshold** constant $n_0 > 0$, the following two properties hold:
>   1. $\forall\ n < n_0,\ T(n) = \Theta(1)$ (i.e. $\exists\ c_1,\ c_2 \in \mathbb{R}$ s.t. $0 < c_1 \leq T(n) \leq c_2$ for $n < n_0$)
>   2. $\forall\ n \geq n_0$, every path of recursion terminates in a defined base case within a finite number of recursive invocations (prevents infinite recursive loop or failure to compute a solution).

> **Intuition/Tips**: Whenever a recurrence is stated without an explicit base case, we assume that the recurrence is algorithmic.
> - **Implication:** This means we can pick any sufficiently large threshold constant n_0.

4.2   Mergesort

> **Definition**: The Merge Sort algorithm is defined as:
>
> $$\text{mergesort}(A, p, r) \to O(n \log n) \tag{24}$$
>
> where A is the array to be sorted, p is the starting index, and r is the ending index.

```python
def merge_sort(A, p, r):
    if p >= r:                    # zero or one element?
        return

    q = (p + r) // 2              # midpoint of A[p : r]
    merge_sort(A, p, q)           # recursively sort A[p : q] --> T(n/2)
    merge_sort(A, q + 1, r)       # recursively sort A[q + 1 : r] --> T(n/2)
    # Merge A[p : q] and A[q + 1 : r] into A[p : r]
    merge(A, p, q, r) # --> O(n)
```

Listing 1: Merge Sort Pseudocode

The time complexity of mergesort is

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n) \tag{25}$$

- $2T(n/2)$ is the recursive time complexity of handling a subproblem half the size.
- $O(n)$ is the linear time required to merge the results.
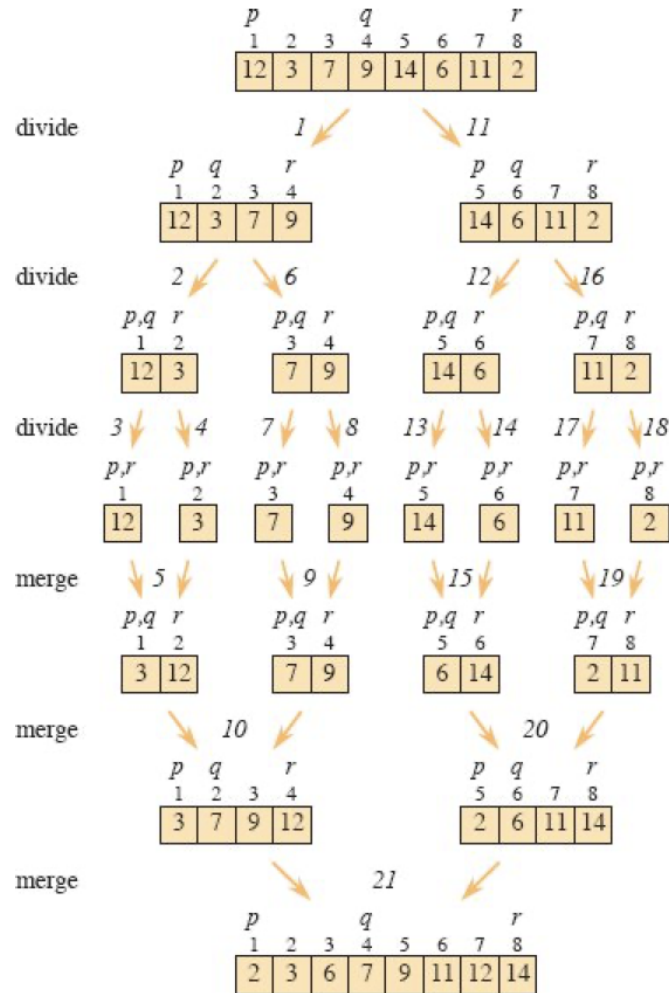
Figure 2: Merge sort visualization.

4.3   Master theorem (Ch. 4.5 pg. 101-6)

Theorem: Let $a \geq 1$, $b > 1$, and $f(n)$ be a function, so that the recurrence is

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \tag{26}$$

Then the asymptotic behavior of $T(n)$ is

1. If $f(n) = O\left(n^{\log_b(a)-\epsilon}\right)$ for $\epsilon > 0$, then $T(n) = \Theta\left(n^{\log_b a}\right)$.
2. If $f(n) = \Theta\left(n^{\log_b(a)}\right)$, then $T(n) = \Theta\left(n^{\log_b a}\log n\right)$.
3. If $f(n) = \Omega\left(n^{\log_b(a)+\epsilon}\right)$ for $\epsilon > 0$ and $af\left(\frac{n}{b}\right) \leq cf(n)$ for $0 < c < 1$, then $T(n) = \Theta(f(n))$.

Process:
1. Identify the recurrence relationship.
2. State a, b, and $f(n)$. Make sure the conditions are met.
3. Calculate $n^{\log_b a}$.
4. Compare $f(n)$ with $n^{\log_b a}$ to see which case the function applies too.
   (a) If ϵ case is used, then apply an abitrary value to see (usually natural numbers work well).
5. Write down the answer by applying the Master Theorem.

Example: Find the time complexity of Merge Sort using Master Theorem.
1. **Identify the Recurrence Relation:**
   - The recurrence relation for the merge sort algorithm is given by:
   $$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$
   - This represents dividing the problem into two subproblems of half the size and then merging the results in linear time.
2. **State Parameters:**
   - Compare the recurrence relation with the general form:
   $$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$
   - For the given problem:
     - $a = 2$: The number of subproblems.
     - $b = 2$: The factor by which the problem size is divided.
     - $f(n) = O(n)$: The cost of dividing and merging the results.
3. **Calculate $n^{\log_b a}$:**
   - Compute $\log_b a$:
   $$\log_b a = \log_2 2 = 1$$
   - Thus, $n^{\log_b a} = n^1 = n$.
4. **Compare $f(n)$ with $n^{\log_b a}$:**
   - $f(n) = O(n)$ and $n^{\log_b a} = n$.
   - Since $f(n) = O(n)$ and $f(n) = \Theta(n^{\log_b a}) = \Theta(n)$, this fits Case 2 of the Master Theorem.
5. **Apply the Master Theorem - Case 2:**
   - Case 2 states: If $f(n) = \Theta(n^{\log_b a})$, then:
   $$T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n \log n)$$
   - Therefore, the time complexity of merge sort is:
   $$T(n) = \Theta(n \log n)$$

Example: Find the time complexity of this recurrence using Master Theorem.
1. **Identify the Recurrence Relation:**
   - The recurrence relation is:
   $$T(n) = 9T\left(\frac{n}{3}\right) + n$$
2. **State Parameters:**
   - Comparing with the general form $T(n) = aT\left(\frac{n}{b}\right) + f(n)$, we have:
     - $a = 9$: Number of subproblems.
     - $b = 3$: Factor by which the problem size is divided.
     - $f(n) = n$: The cost of the work done outside the recursive calls.
3. **Calculate $n^{\log_b a}$:**
   - Compute $\log_b a$:
   $$\log_3 9 = 2$$
   - Thus, $n^{\log_b a} = n^2$.
4. **Compare $f(n)$ with $n^{\log_b a}$:**

- Given $f(n) = n$, we have:

$$f(n) = n = O\left(n^{\log_b a - \epsilon}\right) = O\left(n^{2-1}\right) = O(n)$$

- Since $f(n) = O\left(n^{\log_b a - \epsilon}\right)$, this fits Case 1 of the Master Theorem.
5. **Apply the Master Theorem - Case 1:**
   - Case 1 states: If $f(n) = O\left(n^{\log_b a - \epsilon}\right)$ for some $\epsilon > 0$, then:

$$T(n) = \Theta\left(n^{\log_b a}\right) = \Theta(n^2)$$

   - Hence, the time complexity is:
$$T(n) = \Theta(n^2)$$

Example: Find the time complexity of this recurrence using Master Theorem.
1. **Identify the Recurrence Relation:**
   - The recurrence relation is:
$$T(n) = 3T\left(\frac{n}{4}\right) + n\log(n)$$

2. **State Parameters:**
   - Comparing with the general form $T(n) = aT\left(\frac{n}{b}\right) + f(n)$, we have:
     - $a = 3$: Number of subproblems.
     - $b = 4$: Factor by which the problem size is divided.
     - $f(n) = n\log(n)$: The cost of the work done outside the recursive calls.
3. **Calculate $n^{\log_b a}$:**
   - Compute $\log_b a$:
$$\log_4 3 \approx 0.793$$

   - Thus, $n^{\log_b a} = n^{0.793}$.
4. **Compare $f(n)$ with $n^{\log_b a}$:**
   - $f(n) = n\log(n)$ is compared with $\Omega\left(n^{0.793+0.2}\right)$, which implies:

$$n\log(n) = \Omega\left(n^{0.993}\right)$$

   - This indicates that $f(n)$ dominates $n^{\log_b a}$ with a polynomial difference, which suggests considering Case 3 of the Master Theorem.
5. **Verify Condition for Case 3 of the Master Theorem:**
   - Check if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some $c < 1$:

$$3\left(\frac{n}{4}\right)\log\left(\frac{n}{4}\right) \leq \frac{3}{4}n\log(n)$$

   - This inequality is true for the chosen constants, satisfying Case 3.
6. **Apply the Master Theorem - Case 3:**
   - Since $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$ and the regularity condition is satisfied, we conclude:

$$T(n) = \Theta(n\log n)$$

4.4   Substitution (Ch. 4.3 pg. 90-4)

Process:
1. Guess the form of the solution for $T(n) = ?$
2. Use induction to show that the solution works.
   (a) Basis: Find the base case using values of n that correspond (i.e. make sense) with the guessed solution.
   (b) Inductive hypothesis:
   (c) Inductive step:
3. Find the constants.

Intuition/Tips:
- **Bounds:** Rather than trying to prove Θ-bound directly, first prove an O-bound, and then prove an Ω-bound, then use Theorem 3.1.
- **Making a good guess:**
  - See if the recurrence is similar to one you've seen before, then guessing a similar solution.
  - Determine loose upper and lower bounds on the recurrence and then reduce your range of uncertainty.
- **Trick:** Subtract a lower-order term when the math fails to work out in the induction proof.
- **Avoid:**
  - Don't use asymptotic notation in the inductive hypothesis for the sub-method.
  - You must be careful that the constants hidden by any asymptotic notation are the same constants throughout the proof.

Example: Find the time complexity of the recurrence using sub-method.
1. **Guess the Form of the Solution:**
   - Given recurrence relation:
   $$T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n$$
   - Guess: $T(n) = cn \log n$.
2. **Basis:**
   - Check the base cases:
   $$T(2) = 4, \quad T(3) = 5$$
   - Both satisfy $T(n) = cn \log n$, verifying the base cases.
3. **Inductive Hypothesis:**
   - Assume $T(k) \le ck \log k$ for all $k < n$.
   - Specifically, assume:
   $$T\left(\frac{n}{2}\right) \le c \cdot \frac{n}{2} \log\left(\frac{n}{2}\right)$$
4. **Inductive Step:**
   - Show it holds for $T(n)$:
   $$T(n) = 2T\left(\frac{n}{2}\right) + n$$
   - Substitute the inductive hypothesis:
   $$T(n) \le 2\left(c \cdot \frac{n}{2} \log\left(\frac{n}{2}\right)\right) + n$$
   - Simplify:
   $$= cn \log\left(\frac{n}{2}\right) + n$$
   - Use the logarithm property $\log(ab) = \log a + \log b$:
   $$= cn(\log n - \log 2) + n$$
   $$= cn \log n - cn \log 2 + n$$
   - Factor n:
   $$= n(c \log n - c \log 2 + 1)$$
5. **Find the Constant c:**
   - To keep $T(n) \le cn \log n$, ensure:
   $$c \log n - c \log 2 + 1 \le c \log n$$
   - Simplifying, we need:
   $$-c \log 2 + 1 \le 0$$
   - This implies:
   $$c \ge \frac{1}{\log 2}$$
   - Choose $c \ge 2$ (since $\log 2 \approx 0.693$), which satisfies the inequality $2 \ge 1.44$.

4.5   Recursion tree method (Ch. 4.4 pg. 95-101)

Definition: In a recursion tree, each node represents the cost of a single subproblem somewhere in the set of recursive function invocations.

Process:
1. Sum the costs within each level of the tree to obtain the per-level costs.
2. Sum all the per-level costs to determine the total cost of all levels of the recursion.
3. (1) Generate a good-guess, then verify using sub-method. (2) Use as a direct solution.

Intuition/Tips: How to make the recursion tree based on the recurrence.

Example:
1. **Given Recurrence Relation:**
   - The recurrence relation is:
   $$T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{2n}{3}\right) + n$$

2. **Building the Recursion Tree:**
   - The tree starts with $T(n)$ at the root.
   - Each node $T(n)$ branches into two child nodes:
   $$T\left(\frac{n}{4}\right) \quad \text{and} \quad T\left(\frac{2n}{3}\right)$$
   - This branching continues recursively until the problem size becomes small (base case).
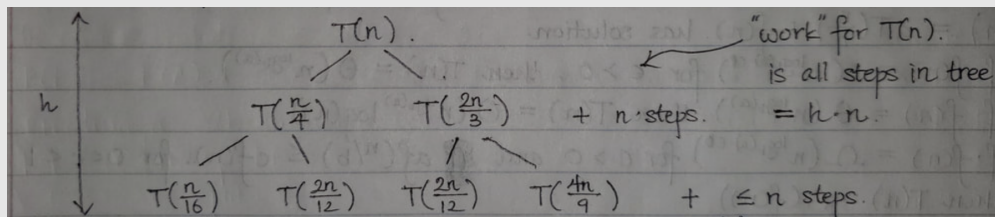

Figure 3: Recursion tree that is made by subbing in the $T(\#)$ into the recurrence relation to get the nodes below.

3. **Calculating Work Done at Each Level:**
   - At the root, the work done is n.
   - At the next level, the work is divided between:
   $$T\left(\frac{n}{4}\right) \quad \text{and} \quad T\left(\frac{2n}{3}\right)$$
   - This pattern continues, and the work at each level is the sum of the work done by each subproblem.
   - The total work at each level is n, as shown by the distribution of the work across the nodes.
4. **Height of the Tree:**
   - The longest path (height h) of the tree is determined by the rightmost path since $\frac{2}{3}$ is larger than $\frac{1}{4}$.
   - The height h can be calculated using the formula:
   $$\left(\frac{2}{3}\right)^h \cdot n = 1$$
   - Solving for h:
   $$h = \log_{3/2}(n)$$

5. **Total Work Done in the Tree:**

- The total work is the sum of the work done at each level times the height of the tree:

$$h \cdot n$$

- Substituting the value of h:

$$h \cdot n = \log_{3/2}(n) \cdot n$$

- This expression simplifies to:

$$O(n \log n)$$

- Therefore, the total work done by the recursion tree is $O(n \log n)$.

5 Graphs, Trees (Ap. B.4-5, L6)

5.1 Graphs

5.1.1 Directed and undirected graphs

Definition:
- **Directed graph (digraph):** G is a pair (V, E), which are vertices V and edges E.
    - **Self-loop:** Edges from a vertex to itself.
- **Undirected graph:** $G = (V, E)$, where E consists of *unordered* pairs of vertices (i.e. direction doesn't matter)
    - **Self-loop:** Forbidden.
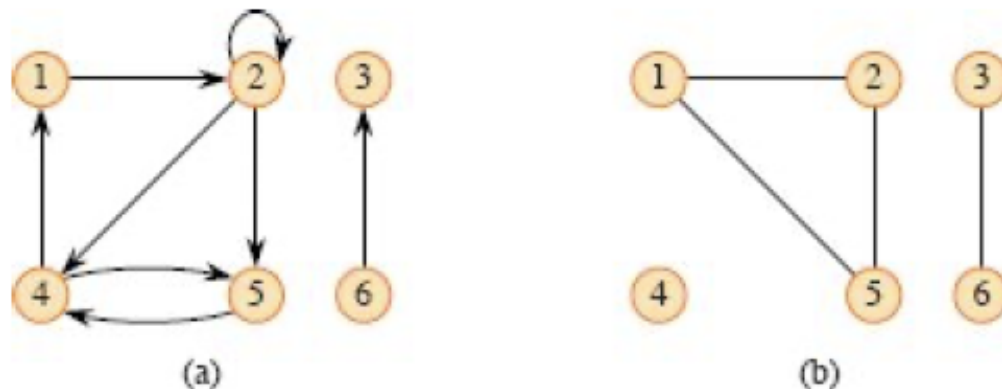
Figure 4: (a) Directed graph, (b) Undirected graph.

5.1.2 Terminology

Terminology:
- **Weighted G:** e.g. distance, cost, etc.
- **Path:** A sequence of vertices in which each vertex is adjacent to the next one.
- **Simple Path:** A path with no repetition of vertices.
- **Simple Cycle:** Simple path with same start/end vertex.
- **Acyclic Graph:** A graph with no cycles is an acyclic graph.
- **Directed Acyclic Graph:** A DAG is a directed acyclic graph.
- **Connected:** Two vertices are connected if there is a path between them.
- **Connected graph:** $\exists$ path between $\forall$ 2 vertices.
- **Degree of V (Undirected G):** Number of edges incident on it.
- **In/Out Degree of V (Directed G):** Out-degree is the # of edges leaving it, while in-degree is the # of edges entering it.

- **Degree of V (Directed G):** In-degree plus out-degree.
- **Degrees of all V**: $2E$
- **Bipartite Gs:** V can be partitioned into 2 sets V_1 and V_2 s.t. $V_1 \cap V_2 = $ and $V_1 \cup V_2 = V$ and adjacencies only between elements of V_1 and V_2.
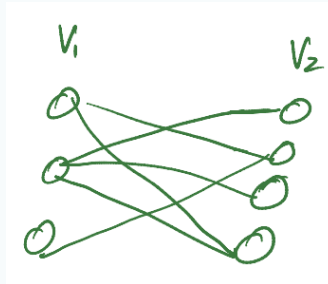

Figure 5: Bipartite graph.

- **Induced subgraph:** Subset of G and the associated edges.
- **Complete G (clique):** $\exists$ edge between $\forall$ 2 vertices.

5.1.3   Graph representation

Definition:
Adjacency matrix (AM): An $n \times n$ matrix where $M[i][j] = 1$ if there is an edge between v_i and v_j, and 0 otherwise.

Adjacency list (AL): For $n = |V|$ vertices, n linked lists. The ith linked list, $L[i]$ is a list of all the vertices that are adjacent to vertex i.

Is there an edge between v_i and v_j?
- **AM:** $O(1)$
- **AL:** $O(d)$ where d is the maximum degree in the graph.

Find all vertices adjacent to v_i:
- **AM:** $O(|V|)$ where $|V|$ is the number of vertices in the graph.
- **AL:** $O(d)$

Space requirements:
- **AM:** $O(|V|^2)$
- **AL:** $O(|V| + |E|)$
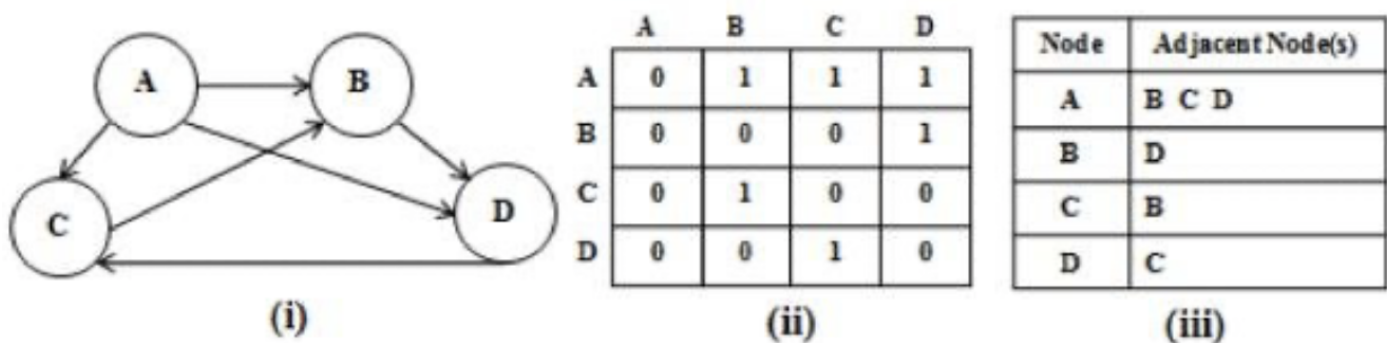   - AL is good for sparse G (i.e. $E << V^2$).

Node	Adjacent Node(s)
A	B C D
B	D
C	B
D	C

Figure 6: (i) Directed graph, (ii) Adjacency matrix, (iii) Adjacency list

5.1.4   Clique

> **Definition**: Every two vertices have an edge.
>
> $$\#edges = \frac{V(V-1)}{2}$$

5.2   Free trees

> **Definition**: A free tree is a connected, acyclic, undirected graph.

5.2.1   Properties

> **Definition**: Let $G = (V, E)$ be an undirected graph. The following statements are equivalent:
>   1. G is a free tree.
>   2. Any two vertices in G are connected by a unique simple path.
>   3. G is connected, but if any edge is removed from E, the resulting graph is disconnected.
>   4. G is connected, and $|E| = |V| - 1$.
>   5. G is acyclic, and $|E| = |V| - 1$.
>   6. G is acyclic, but if any edge is added to E, the resulting graph contains a cycle.
>   • **Note:** There's a proof to show each of these statements are equivalent.

5.3   Forest

> **Definition**: An undirected graph is acyclic but possibly disconnected.
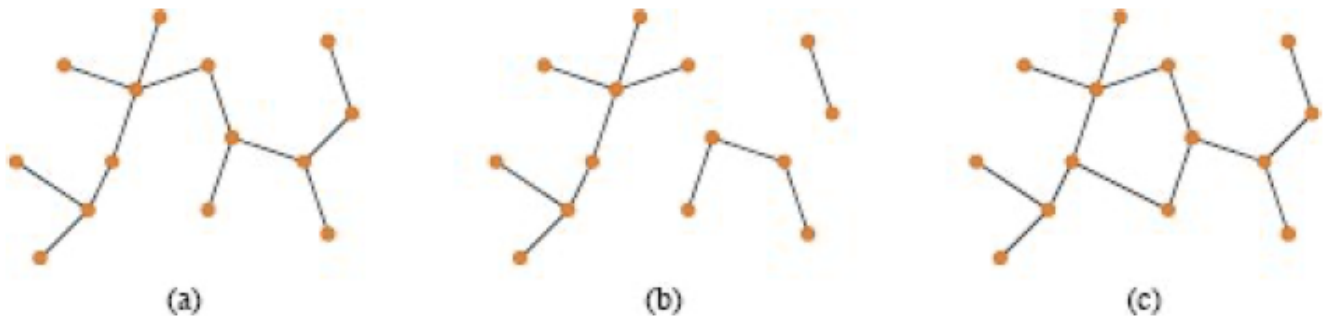
Figure 7: (a) A free tree, (b) A forest, (c) a graph that contains a cycle and is therefore neither a tree nor a forest.

5.4   Rooted and ordered trees

5.4.1   Rooted trees

> **Definition**: A **rooted tree** is a free tree in which one of the vertices is distinguished from the others.
>   • **Root:** Distinguished vertex of the tree.
>   • **Node:** Vertex of a rooted tree.

5.4.2   Ordered trees

> **Definition**: An **ordered tree** is a rooted tree in which the children of each node are ordered.

Figure 8: Rooted and ordered trees. If the tree is ordered, the relative left-to-right order of the children of a node matters, but if rooted, then they are the same tree.

5.4.3   Terminology

> **Terminology**:
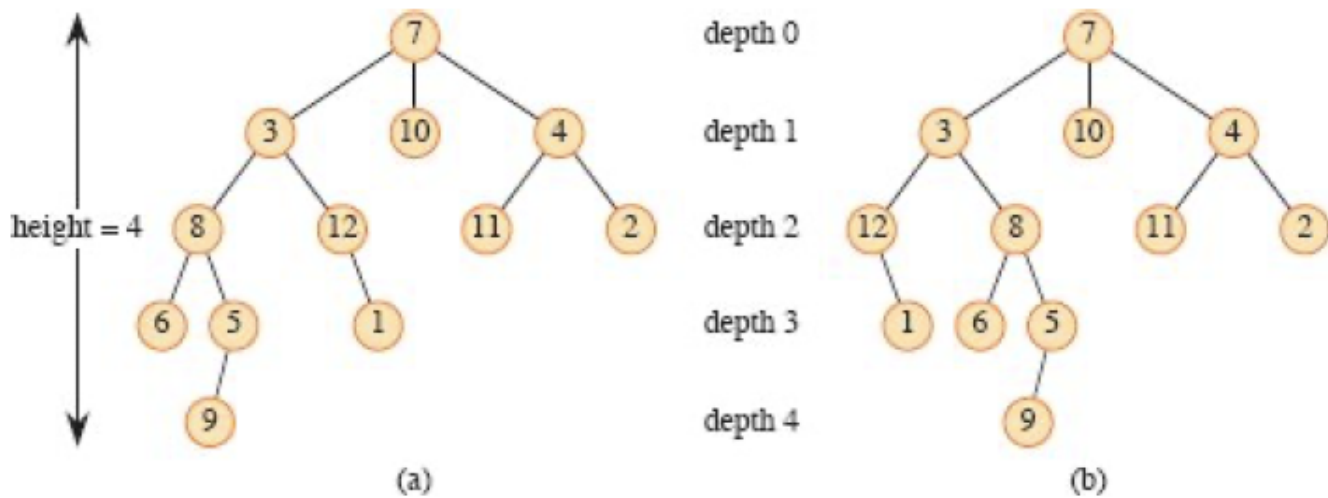> - **Parent:** A node y is the parent of node x if y is directly connected to x on the path from the root.
> - **Child:** A node x is a child of node y if y is the parent of x.
> - **Siblings:** Nodes are siblings if they share the same parent.
> - **Leaf (or External Node):** A leaf is a node with no children.
> - **Internal Node:** An internal node is a nonleaf node, which means it has at least one child.
> - **Degree:** The degree of a node x is the number of children it has.
> - **Depth:** The depth of a node x is the length of the path from the root to x.
> - **Level:** A level of a tree consists of all nodes at the same depth.
> - **Height:** The height of a node is the number of edges in the longest path from that node to a leaf.
>   - **Height of tree:** From root to any leaf.

5.5   Binary and positional trees

5.5.1   Binary trees

> **Definition**: A **binary tree** T is a structure defined on a finite set of nodes that either
> - contains no nodes, or
> - is composed of three disjoint sets of nodes: a **root** node, a **left subtree**, and a **right subtree**.

5.5.2   Terminology

> **Terminology**:
> - **Empty Tree:** A binary tree with no nodes.
> - **Left and Right Child:** The roots of the non-empty left and right subtrees of the root.
> - **Full Binary Tree:** Every node is either a leaf or has exactly two children.
> - **Position Matters:** The distinction between left and right children is crucial in a binary tree, unlike in general ordered trees.

5.5.3   Positional trees

Definition: The children of a node are labeled with distinct positive integers.

5.5.4   K-ary trees

Definition: A positional tree in which each node $\leq k$ children. A binary tree has $k = 2$.

A **complete k-ary tree** is a k-ary tree in which all leaves have the same depth, and every internal node has exactly k children.

6   Permutations, Combinations (Ap. C.1, L5)

6.1   Rule of sum and product

Definition: If there are m-ways for event A to happen and n-ways for event B to happen then...

Rule of product: $\exists\, m \times n$ ways for A *and* B to happen.

Rule of sum: $\exists\, m + n$ ways for A *or* B to happen.

6.2   Permutations

Definition: Number of ways to pick r distinct objects out of n where *order matters* and *repetition isn't allowed*.

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!} \tag{27}$$

- n: total number of elements in the set.
- r: number of elements taken from the set.

6.3   Permutations with identical items

Definition: If there are m kinds of items and q_k, $k = 1,\ldots,m$ of each kind, then total number of permutations where *order matters* is

$$\binom{n}{q_1,\ldots,q_m} = \frac{n!}{q_1!\, q_2! \,\cdots\, q_m!} \tag{28}$$

- $\displaystyle\sum_{k=1}^{m} q_k = n$

6.4   Permutations with repetitions

Definition: Number of ways to arrange r-objects out of n objects with unlimited repetition is given by: n^r.

6.5   Combinations

Definition: Number of ways to choose r objects from n where *order doesn't matter*.

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} \tag{29}$$

6.6　Binomial theorem

Definition:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{30}$$

- $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$

7　Probability (Ap. C.2, L8)

7.1　Sample space

Definition: The set of *all possible outcomes* of a statistical experiment, denoted by S.

7.2　Event

Definition: A subset of a sample space S. An event is any outcome or combination of outcomes.

7.3　Probability axioms

Definition: For A, $B \subseteq S$
1. $0 \le P(A) \le 1$
2. $P(S) = 1$
3. $P(A \cup B) = P(A) + P(B)$ for two mutually exclusive events A and B.

7.4　Additive rule

Definition:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{31}$$

7.5　Uniform distribution

Definition: If $\forall s \in S$ has probability $P(s) = \dfrac{1}{|S|}$, then it is a uniform distribution.

7.6　Independence

Definition: $P(A \cap B) = P(A)P(B)$ if A, B independent.

7.7　Bayes theorem

Definition: For events with $P(A) > 0$ and $P(B) > 0$, the probability A happens given B happens is:

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A|B)P(B)}{P(A)} \tag{32}$$

7.8   Bayes' rule with total probability

Definition: Suppose $C_1, \ldots, C_k$ is a partition. Then

$$P(B|A) = \frac{P(B)P(A|B)}{\sum_{i=1}^{k} P(C_i) P(A|C_i)} \tag{33}$$

Often B is an element of $C_1, \ldots, C_k$, say $B = C_n$. Then

$$P(C_n|A) = \frac{P(C_n) P(A|C_n)}{\sum_{i=1}^{k} P(C_i) P(A|C_i)} \tag{34}$$

Process:
1. Write down all the probabilities.
2. Try solving the problem directly using definitions.

Intuition/Tips: If given P(A|B) and want P(B|A), then automatically use Bayes' Rule.

7.9   Discrete random variable

Definition: An RV is a function that associates a real number with each element of the sample space. Denote RVs with capital letters.

7.10   Probability mass function

Definition: The set of ordered pairs $(x, f(x))$ of the discrete RV X if, for each possible outcome x,
1. $f(x) \geq 0$ for each outcome $X = x$
2. $\sum_{x} f(x) = 1$ (i.e. total probability sums to 1)
3. $f(x) = P(X = x)$ (i.e. probability of each outcome)

7.11   Expectation

Definition: Let X be an RV with distribution $f(x)$, then

$$E[X] = \sum_{x \in X} x f(x) \tag{35}$$

where the sum is taken over all possible values of X.

7.12   Properties of expectation

Definition:
1. $E[X + Y] = E[X] + E[Y]$ (linearity)
2. $E[\alpha X] = \alpha E[X]$ (linearity)
3. $E[XY] = E[X]E[Y]$ if independent

8   Heaps, Heapsort (Ch. 6)

8.1   Intro to heapsort

8.1.1   In-place Sorting

Definition: Given an array A to sort the numbers, sorts within the array and uses a constant number of variables to do bookkeeping.
- **Time Complexity**: $O(n \log n)$.
- **Explanation:** Describe in terms of a tree.
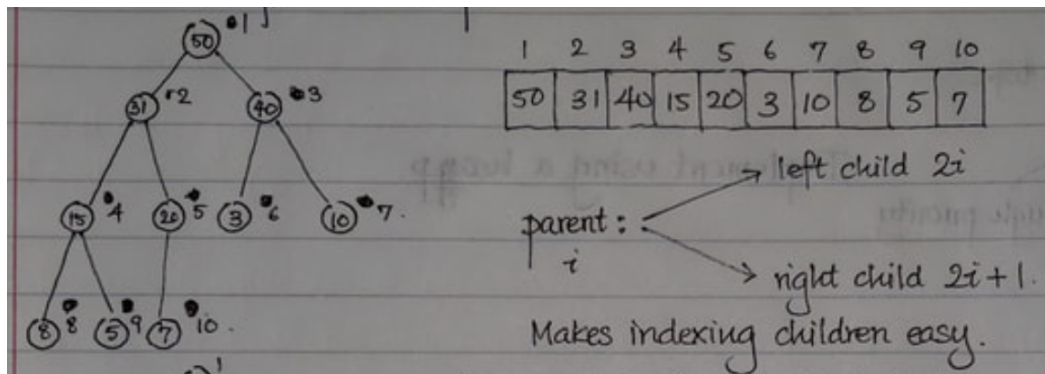- **Pseudo-code:** Uses the array representation.

Figure 9: (Left) Tree format heap. (Right) Array format heap with formulas for parent and children.

8.1.2   Indexing

Definition:
1. **Parent:** i
2. **Left child:** $2i$
3. **Right child:** $2i + 1$

8.1.3   Heap-tree: (2 properties)

Definition:
- **Heap Shape:** A complete binary tree where the last level is not filled, but leaves are pushed to the left.
- **Heap Order (maxheap):** Key(parent) > Key(children)
- **Heap Order (minheap):** Key(parent) < Key(children)

8.2   Heap operations

8.2.1   Bubble up

8.2.2   Extract max

8.2.3   Bubble down

8.3   Heapsort

8.4   Heap runtime and priority queue

8.4.1   Tight bound for build heap

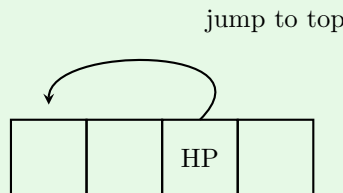**Definition**: There are *at most* $\left\lceil \dfrac{n}{2^{h+1}} \right\rceil$ nodes of height h in a heap.

The tight bound is

$$\sum_{h=0}^{\log(n)} \# \text{ nodes at height h} \cdot \text{height of the node} = \sum_{h=0}^{\log(n)} \left\lceil \frac{n}{2^{h+1}} \right\rceil \cdot h$$

$$\leq O\left(\frac{n}{2} \cdot \sum_{h=0}^{\infty} h \cdot \frac{1}{2^h} \right)$$

$$= O(n) \cdot \frac{\frac{1}{2}}{(1 - \frac{1}{2})^2}$$

$$= O(n) \cdot 2 = O(n)$$

8.4.2   Priority queue

Definition: A queue where the first element dequeued is the one with the highest priority.
  • Implement a PQ using a heap for efficient priority management.

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9.1　Intro

9.2　QS basic analysis

9.3　QS worst case

9.4　Randomized QS

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10.1　Lower bound on sorting and counting sort

10.2　Radix sort

11　Selection sort, Binary search trees (Ch. 12)

11.1　Selection sort

11.2　Binary search trees

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