ECE355 Cheatsheet

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1 Tips

Intuition:
- May diverge from textbook, but only responsible for lecture content.
- Tutorials: Review of last week's topics and assigned problems.
- Piazza for asking questions.
- ISM: Investigate topic of interest that uses signals or systems with 10 pages that are reference, explain concepts in your own way.
- Quiz every week except for term tests.
- 30 minutes, appears Tuesday morning and ends Tuesday night.
- Easier than usual questions that tests understanding.
- Open book with MC, numerical answer.

2 Mathematical Review

2.1 Sets

Definition: An unordered collection of objects (i.e. elements or members)
- A set *contains* its elements or elements of a set are *contained in* that set.

2.1.1 Set notation

Terminology:
- $\ldots$ mean "and so on"
- : mean "such that"
- $\in$ mean "contained"
- $\notin$ mean "not contained"
- $\emptyset$ mean "empty set (i.e. a set contains no elements")
- $A \subseteq B$ mean "Only if every element of A is also an element of B"
- $B \supseteq A$ mean "B is a superset of A to mean A is a subset of B"
- Normally, elements of a set are listed just once.

Example:
Sets:
- $E = \{0, 2, 4, 6, 8\}$, where $2 \in E$ and $1 \notin E$
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $P = \{0, 1, ..., 255\}$
- $O = \{x \in \mathbb{Z} : \ x = 2k + 1 \text{ for some } k \in \mathbb{Z}\}$
- $\{\emptyset, \{\emptyset\}\}$ (i.e. A set that has other sets as elements).

Subset:
- $E \subseteq \mathbb{Z}$

Theorem: $A = B$ means $A \subseteq B$ and $B \subseteq A$.
- **Note:** Have to prove in both directions.

Example: $\{1, 2, 3\} = \{3, 2, 1, 1, 2\}$

2.1.2 Important sets

Definition:
1. **Natural:** $\mathbb{N} = \{0, 1, 2, 3, \dots\}$:
2. **Integers:** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$:
3. **Rational:** $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$:
4. **Real:** $\mathbb{R}$:
5. **Complex:** $\mathbb{C} = \{a + bj : a, b \in \mathbb{R}\}$
    - j: imaginary unit, where $j^2 = -1$ and $j = \sqrt{-1}$

- **Note:** $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

2.2 Ordered n-tuples

Definition: An ordered collection of n elements, where n is a positive integer, denoted as $(a_1, a_2, \dots, a_n)$, where a_1 is the first element, and so on, up to a_n.

2.2.1 How are two tuples equal?

Definition: Unlike sets, both the order of elements and the repetition of values are significant. Therefore, two ordered n-tuples are considered equal (i.e. $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$) iff:

$$a_1 = b_1, a_2 = b_2, \dots, a_n = b_n.$$

2.2.2 Cartesian product

Definition: **Two sets:** The *Cartesian product* of two sets A and B (in that order), denoted as $A \times B$, is the set of all *ordered pairs* or *ordered 2-tuples* (a, b) where $a \in A$ and $b \in B$. Thus

$$A \times B = \{(a, b) : a \in A, b \in B\}. \tag{1}$$

- **General:** $B \times A \neq A \times B$
- **2-fold Cartesian product:** $A \times A$ is denoted as A^2

More than two sets: The Cartesian product of sets $A_1, A_2, \dots, A_n$, denoted as $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \dots, a_n)$, where $a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n$. Thus

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) : a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}. \tag{2}$$

- **n-fold Cartesian product:** $A \times A \times \cdots \times A$ is denoted as A^n

2.3 Functions

Definition: A function $f : A \to B$ from a set A (the domain of f) to a set B (the codomain of f) assigns to each element $a \in A$ exactly one element $b \in B$, usually denoted as $b = f(a)$.

2.3.1 Range/Image

Definition: **The range or image** of f is the subset of the codomain B given as

$$\text{Im}_f(A) = \{b \in B : \exists a \in A(f(a) = b)\}.$$

- **English:** Set of values "hit" by f as its argument ranges over the set A.

2.3.2 Inverse Image

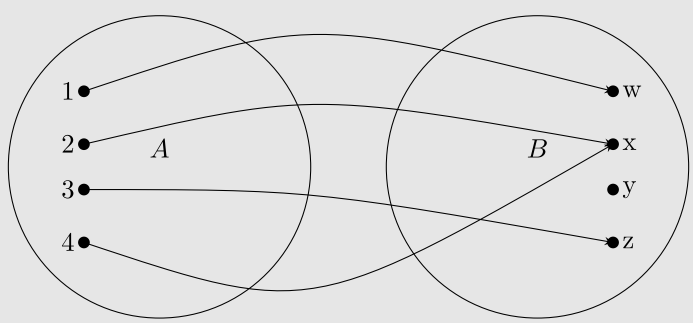
Definition: **The inverse image or pre-image** of any element $b \in B$ under the mapping by f is the set

$$f^{-1}(b) = \{a \in A : f(a) = b\}.$$

- **English:** Set of elements of the domain that map to b under transformation by f.
- **Key:** If b is an element of the codomain that is not in the range of f, then $f^{-1}(b) = \emptyset$

Example:
- Domain of g: $A = \{1, 2, 3, 4\}$
- Codomain of g: $B = \{w, x, y, z\}$
- Image of A: $\text{Im}_g(A) = \{w, x, z\} \subseteq B$
- Inverse Image

$$g^{-1}(w) = \{1\}$$
$$g^{-1}(x) = \{2, 4\}$$
$$g^{-1}(y) = \emptyset$$
$$g^{-1}(z) = \{3\}$$



2.3.3 Injective

Definition: A function $f : A \to B$ is called injective (or an injection or one-to-one) if $\forall a_1 \forall a_2$

$$a_1 \neq a_2 \to f(a_1) \neq f(a_2).$$

$$(f(a_1) = f(a_2) \to a_1 = a_2)$$

- **English:** Maps distinct elemetns of the domain to distinct elements of the codomain.

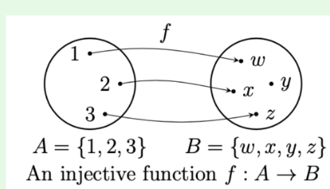


Figure 1: Injective function.

Process: Show a function is injective:
1. Set $f(x_1) = f(x_2)$
2. Prove $x_1 = x_2$ from step 1.

Show a function is not injective:
1. Find a counterexample where $f(a_1) = f(a_2)$.

2.3.4 Surjective

Definition: A function $f : A \to B$ is called surjective (or a surjection or onto) if

$$\forall b(f^{-1}(b) \neq \emptyset), \quad \text{or} \quad \forall b \exists a(f(a) = b),$$

- **English:** Every element in the codomain has a mapping back to the domain.

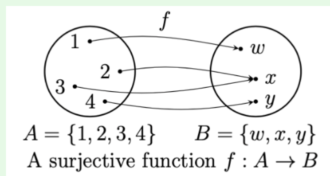


Figure 2: Surjective function.

Process: Show a function is surjective:
1. Find the inverse of $f(x) = y$ by writing x in terms of y denoted f^{-1}
2. See if the inverse satisfies the codomain, and there is no empty set.

Show a function is not surjective:
1. Find a counterexample, where you get the empty set for $b \in B$

Warning: Any nonsurjective function is a surjective function obtained from the original function by having the codomain match the range.

2.3.5 Bijective

Definition: A function $f : A \to B$ that is both injective and surjective is called bijective (or a bijection or a one-to-one correspondence).
- **Correspondence:** Inverse exists

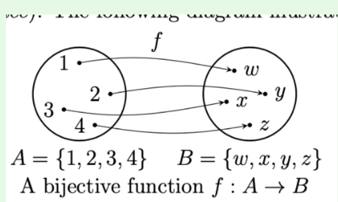


Figure 3: Bijective function.

2.3.6 Composition of g with f

Definition: If $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$ s.t. $a \to g(f(a))$ (i.e. first apply f, then apply g)

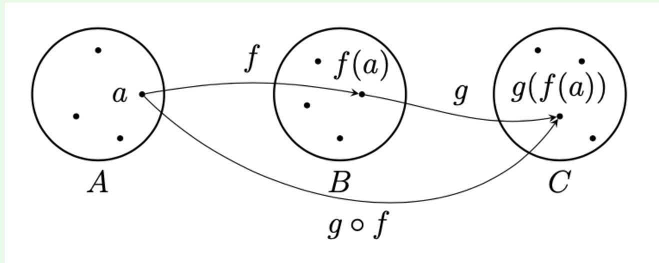


Figure 4: Composition example

- **Order is important:** $f(g(a)) \neq g(f(a))$

2.3.7 Identity map

Definition:

$$\mathrm{id}_A : A \to A \quad \mathrm{id}(a) = a \ \forall a \in A$$

2.3.8 Bijective property

Definition: Let $f : A \to B$, then iff f is bijective, $\exists$ a function $f^{-1} : B \to A$ s.t. $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

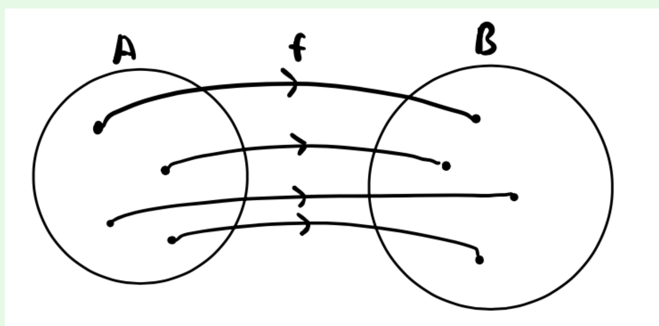


Figure 5: Illustration of bijective function

2.3.9 Set of all functions with domain and codomain

Definition: The set of all fcns with domain A and codomain B is itself a set denoted B^A.

Example: If $A = \{1, 2\}$ and $B = \{x, y, z\}$, then B^A has $3^2 = 9$ elements (i.e., B^A).

$$f = \begin{pmatrix} 1 & 2 \\ f(1) & f(2) \end{pmatrix}$$

The set B^A is:

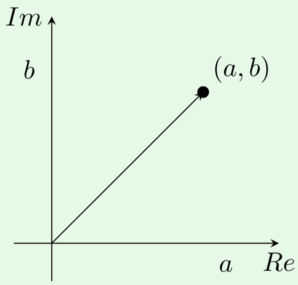
$$B^A = \left\{ \begin{pmatrix} 1 & 2 \\ x & x \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ x & y \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ x & z \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ y & x \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ y & y \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ y & z \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ z & x \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ z & y \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ z & z \end{pmatrix} \right\}$$

2.4 Complex math

2.4.1 Complex number basics

Definition:

- $z = a + bj$, where $a, b \in \mathbb{R}$
  - $\text{Re}(z) = a$
  - $\text{Im}(z) = b$
- **Complex conjugate:** If $z = a + bj$, then $z^* = a - bj$.
- **Magnitude:** $|z| = \sqrt{z \cdot z^*} = \sqrt{a^2 + b^2}$.



Example: Expand the following function:

$$\begin{aligned}(a+bj)(c+dj) &= ac + (bc+ad)j + bdj^2 \\ &= ac + (bc+ad)j - bd \quad \text{since } j^2 = -1.\end{aligned}$$

2.4.2 Complex exponential function

Definition:

$$\exp : \mathbb{C} \to \mathbb{C} \text{ via } \exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{3}$$

- **Entire function:** Convergent no matter the values of z.

Let $\theta \in \mathbb{R}$, the expansion of $\exp(j\theta)$ is:

$$\exp(j\theta) = \cos\theta + j\sin\theta \tag{4}$$

2.4.3 Complex plane with radius r

Intuition:

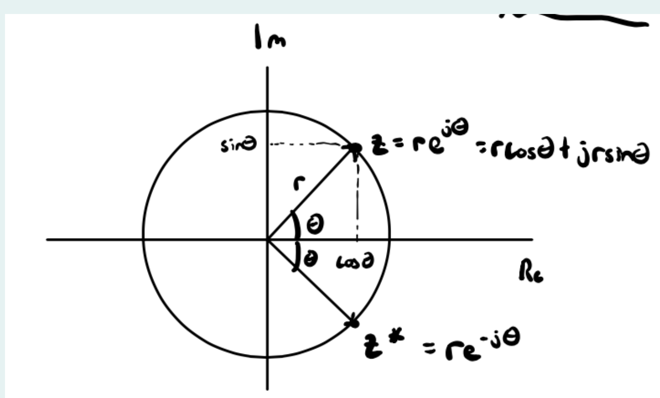


Figure 6: Complex plane in general with radius r.

- **Bounds:** $r \geq 0$ and $-\pi < \theta \leq \pi$

- **Polar:** Multiplication
- **Rectangular:** Additive

2.4.4 Complex conjugate

Definition:

$$z^* = re^{-j\theta} \tag{5}$$

2.4.5 Converting between polar and rectangular form

Process:

Polar to rectangular: $e^{j\theta}$

1. Find r and θ from $re^{j\theta}$
2. Write in rectangular form: $z = rcos\theta + jrsin\theta$

Rectangular to polar: $a + bj$

1. Find r using Pythagorean theorem: $r = \sqrt{a^2 + b^2}$
2. Find θ using trigonometry: $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, where b is the opposite and a is adjacent.
3. Write in polar form: $z = re^{j\theta}$

- **Note:** Both forms can be found intuitively through a drawing of the complex plane.

2.5 Propositional logic

2.5.1 Proposition

Definition: A declarative statement that can be either *true* or *false*, denoted by a symbol (e.g. p or q).

2.5.2 Compound proposition

Definition: Formed from existing propositions via negation and logical connectives.

2.5.3 Logical negation (logical not)

Definition: An operation that takes a proposition p to another proposition "not p", denoted $\neg p$ or p.

p	$\neg p$
F	T
T	F

Figure 7: Truth table for negation.

Example: What is the truth value of the double negation?
It is not the case that it is not the case that p is the same as that of p.

- i.e. $\neg\neg p$ and p to be *logically equivalent*.

2.5.4 Logical conjunction (logical AND)

Definition: Two propositions p and q can be connected with a logical conjunction, denoted $\land$.

p	q	$p \land q$
F	F	F
F	T	F
T	F	F
T	T	T

Figure 8: Truth table of AND, where truth value T only when p and q are truth.

2.5.5 Logical disjunction (logical OR)

Definition: Two propositions p and q can be connected with a logical disjunction, denoted $\lor$.

p	q	$p \lor q$
F	F	F
F	T	T
T	F	T
T	T	T

Figure 9: Truth table of OR, where truth value F only when both p and q are F and truth value T when either of p or q or both are true.

2.5.6 De Morgan's Laws

Definition:

$$\neg(p \land q) \equiv (\neg p) \lor (\neg q) \quad \text{and} \quad \neg(p \lor q) \equiv (\neg p) \land (\neg q) \tag{6}$$

2.5.7 Logical implication

Definition: Two propositions p and q can be connected with a logical *implication* denoted $\to$ or "implies," to form the logical proposition $p \to q$.

- **Antecedent:** p.
- **Consequent:** q.
- **English:** The proposition $p \to q$ can be translated into English as "if p then q," or "q if p."
- **Logically equivalent:** $p \to q$ and $\neg p \lor q$

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

Figure 10: Truth table of logical implication, where truth value F only when p is true and q is false

Warning: The following all mean the same thing:
- $p \to q$
- p implies q
- if p, then q
- q if p
- p is a sufficient condition for q
- p only if q (i.e. $p \to q \equiv \neg q \to \neg p$ i.e. implication is logically equivalent to its contrapositive)
- q is a necessary condition for p

2.5.8 Converse, inverse, contrapositive

Definition: Let $p \to q$ be a proposition. The following are the related forms of this proposition:
- The *converse* of $p \to q$ is the proposition $q \to p$.
- The *inverse* of $p \to q$ is the proposition $\neg p \to \neg q$.
- The *contrapositive* of $p \to q$ is the proposition $\neg q \to \neg p$.

antecedent p	consequent q	implication $p \to q$	converse $q \to p$	inverse $\neg p \to \neg q$	contrapositive $\neg q \to \neg p$
F	F	T	T	T	T
F	T	T	F	F	T
T	F	F	T	T	F
T	T	T	T	T	T

Figure 11: Truth table

Warning: The converse of an implication is *not* logically equivalent to the implication.

2.5.9 Biconditional

Definition: Two propositions p and q can be connected with a logical *biconditional*, denoted $\leftrightarrow$ or "iff' to form the logical proposition $p \leftrightarrow q$.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

Figure 12: Truth table of biconditional, where having truth value "true" whenever p and q have the same truth value, and "false" whenever p and q have different truth values.

- **Logically equivalent:** The biconditional is logically equivalent to the conjunction $(p \to q) \wedge (q \to p)$ of an implication and its converse.

2.5.10 Rules of inference

Logic is used to deduce truth of certain propositions from the truth of other propositions.

Definition:

1. **Modus ponens (MP):**
$$\frac{p \to q,\ p}{\therefore q}$$
(If $p \to q$ and p are both true, then q.)
2. **Modus tollens (MT):**
$$\frac{p \to q,\ \neg q}{\therefore \neg p}$$
(If $p \to q$ and $\neg q$ are both true, then $\neg p$.)
3. **Modus tellendo ponens (MTP):**
$$\frac{p \vee q,\ \neg p}{\therefore q}$$
(If $p \vee q$ and $\neg p$ are both true, then q.)
4. **Modus ponendo tollens (MPT):**
$$\frac{\neg(p \wedge q),\ p}{\therefore \neg q}$$
(If $\neg(p \wedge q)$ and p are both true, then $\neg q$.)

2.6 Predicate logic

Definition: Defined via *predicates*, which are prototypes for propositions involving *predicate variables* (i.e. placeholder variables), each associated with a specific set (i.e. *domain of discourse* for that variable)

- **Key:** When specific values from the domains of discourse are substituted for each of the predicate variables in a predicate, a specific proposition with a truth value is obtained.

2.6.1 Quantifiers

Definition:

1. **Universal quantifier**, denoted $\forall$. When applied to a predicate $P(x)$, it asserts that the proposition $P(x)$ is true for every x in the domain of discourse. Formally, it is written as $\forall x(P(x))$.
   - Effects the conjunction (AND)
2. **Existential quantifier**, denoted $\exists$. When applied to a predicate $P(x)$, it asserts that the proposition $P(x)$ is true for at least one x in the domain of discourse. Formally, it is written as $\exists x(P(x))$.
   - Effects the disjunction (OR)
   - $\exists x \in A(P(x)) \equiv \exists x(x \in A \wedge P(x))$

2.6.2 De Morgan's Law

Definition:

$$\neg(\forall x(P(x))) \equiv \exists x(\neg P(x)) \tag{7}$$

- **English:** Failure of P to hold universally is equivalent to the existence of at least one element in the domain of discourse for which P fails to hold.

$$\neg(\exists x(P(x))) \equiv \forall x(\neg P(x)) \tag{8}$$

- **English:** Failure of the existence of an element for which P holds is equivalent to P failing to hold for all elements in the domain of discourse

2.7 Geometric series

Definition:
Finite:

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \text{if } \alpha = 1, \\ \dfrac{1-\alpha^N}{1-\alpha}, & \text{for any complex number } \alpha \neq 1 \end{cases} \tag{9}$$

Infinite: If $|\alpha| < 1$,

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \tag{10}$$

For any integer k, assuming $|\alpha| < 1$,

$$\sum_{n=k}^{\infty} \alpha^n = \alpha^k \sum_{n=0}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha}. \tag{11}$$

Intuition: Useful for DT since those are in terms of sums.

Signals and General Systems

3 Continuous and discrete-time signals (Ch. 1.1)

3.1 4 main classes of signals

Definition:
1. $\mathbb{R}^{\mathbb{Z}}$ (i.e. real-valued, discrete time)
2. $\mathbb{C}^{\mathbb{Z}}$ (i.e. complex-valued, discrete time)
3. $\mathbb{R}^{\mathbb{R}}$ (i.e. real-valued, continuous time)
4. $\mathbb{C}^{\mathbb{R}}$ (i.e. complex-valued, continuous time)
- **Assumption:** Complex unless told otherwise.

Intuition:
- () is continuous time.
- [] is discrete time.

3.2 Support

Definition: The support of a CT signal $x \in \mathbb{C}^{\mathbb{R}}$, $x(t) \neq$ zero is the smallest interval $[a, b]$ s.t.:

$$x(t) = 0 \text{ for } t \notin [a, b]$$

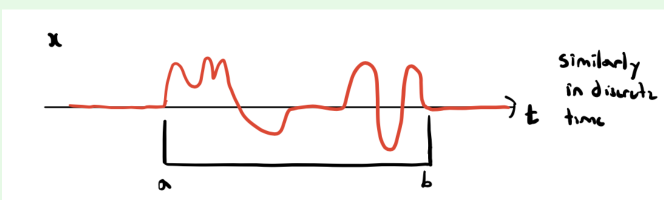


Figure 13: Support of a nonzero signal.

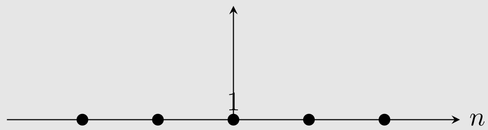
The support of a DT signal $x \in \mathbb{C}^{\mathbb{Z}}$, $x[n] \neq$ zero is the smallest interval $\{a, a+1, \ldots, b\}$ s.t.:

$$x[n] = 0 \text{ for } n \notin \{a, a+1, \ldots, b\}$$

Example:

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

has support $\{0\}$.



Process: DT:

1. Understand the support of the original signal: Support of $x[n] = n_1, \ldots, n_k$
2. Time shift by k:
   (a) Right shift: Support of $x[n-k] = \{n_1 + k, \ldots, n_k + k\}$
   (b) Left shift: Support of $x[n+k] = \{n_1 - k, \ldots, n_k - k\}$
3. Time reversal: Reflects the signal across the vertical axis s.t. Support of $x[-n] = \{-n_1, \ldots, -n_k\}$
4. Time scaling: Scaling by a (keep only integers) s.t. Support of $x[an] = \left\{\frac{n_1}{a}, \ldots, \frac{n_k}{a}\right\}$
   (a) If $a > 1$, then compression
   (b) If $0 < a < 1$, then expanded

CT:

1. Understand the support of the original signal:
   - Identify the range of t for which the signal $x(t) \neq 0$. This range is known as the support of the signal.
2. Set the argument (e.g. if $x(1-t)$, then the argument is $1-t$) as an inequality to the support.
3. Solve for t.
4. If it is a product or a sum, then you must use logic to see which function will take priority to include all cases.
   (a) Product: The lowest bound should take priority b/c the product will be zero as soon as either signal is zero (i.e. only non-zero when both signals are non-zero)
   (b) Sum: The highest bound should take priority b/c a sum will be zero when both signals are zero.

Warning: You might look for the values s.t. it is guaranteed to be 0.

3.2.1 How to sketch CT signals?

Process:

1. **Factor Out Scaling and Shifting:** If the transformation is of the form $x(at + b)$, factor out the scaling term to rewrite it as $x\left(a\left(t + \frac{b}{a}\right)\right)$.
2. **Time Scaling:** If the transformation involves a factor a (e.g., $x(at)$), first scale the time axis.
   - Compress the signal if $|a| > 1$ or stretch it if $0 < |a| < 1$.
   - Adjust the support accordingly: $[t_1, t_2] \to \left[\frac{t_1}{a}, \frac{t_2}{a}\right]$.
3. **Time Reversal:** If the transformation involves $-t$ (e.g., $x(-t)$), apply the reversal after scaling.
   - Reflect the signal across the vertical axis.
   - Reverse the support: $[t_1, t_2] \to [-t_2, -t_1]$.
4. **Time Shifting:** If the transformation involves a shift t_0 (e.g., $x(t \pm t_0)$), apply the shift last.
   - Move the signal to the right for $-t_0$ or to the left for $+t_0$.
   - **Right shift:** Adjust the support by adding t_0 to both limits: $[t_1, t_2] \to [t_1 + t_0, t_2 + t_0]$.
   - **Left shift:** Adjust the support by adding t_0 to both limits: $[t_1, t_2] \to [t_1 - t_0, t_2 - t_0]$.
5. **Sketch:** Sketch the signal.
6. **Label the axes and key points:** Max/min values, supports, etc.

3.3 Signal energy and power

3.3.1 Energy

Definition: Energy of a signal (if it exists) as

1. **CT:** $E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt \quad \in \mathbb{R} \geq 0$
2. **DT:** $E_x = \sum_{n=-\infty}^{+\infty} |x[n]|^2 \quad \in \mathbb{R} \geq 0$

- **Energy signal:** A signal of finite energy (i.e. zero average power) is called an **energy signal**.
- **Negative:** No negative energies.

3.3.2 Power

Definition: The **average power** is defined (if it exists) as:

1. **CT:** $x \in \mathbb{C}^{\mathbb{R}}$ then $P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 \, dt$
2. **DT:** $x \in \mathbb{C}^{\mathbb{Z}}$ then $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

- **Power signal:** A signal of finite average power is called a **power signal**.

Warning:

- **Zero average power:** Every energy signal has zero average power. This is because the energy is finite and spread out over an infinite time, causing the power to approach zero.
- **Infinite energy:** Power signal only when there is infinite energy.

3.3.3 Examples of energy and power signals

Example:

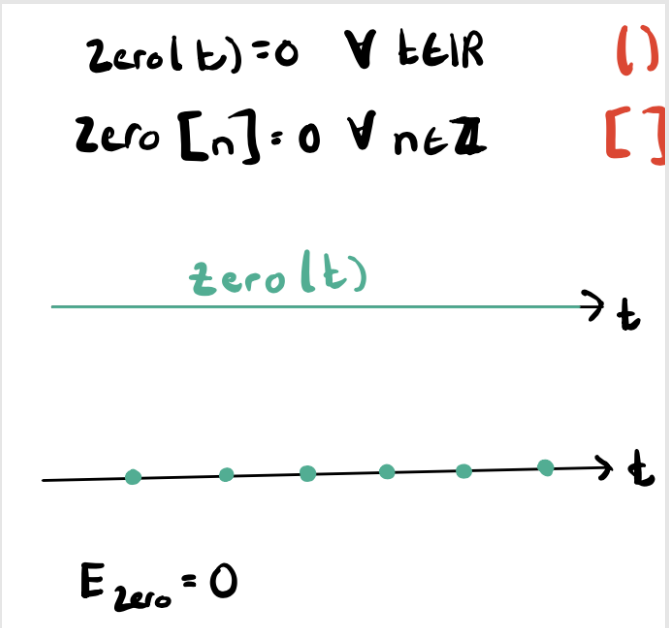


Figure 14: Zero

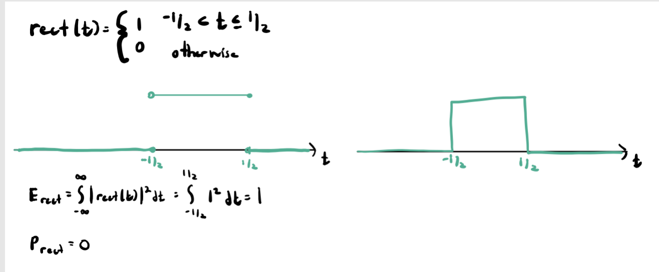


Figure 15: Rectangular

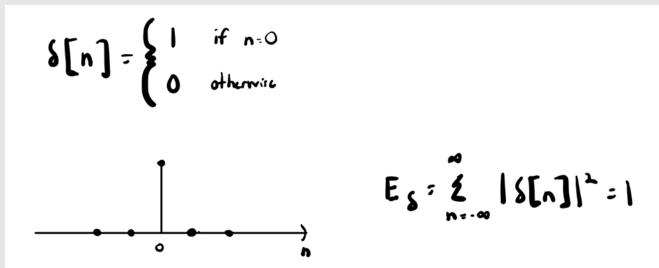


Figure 16: Impulse

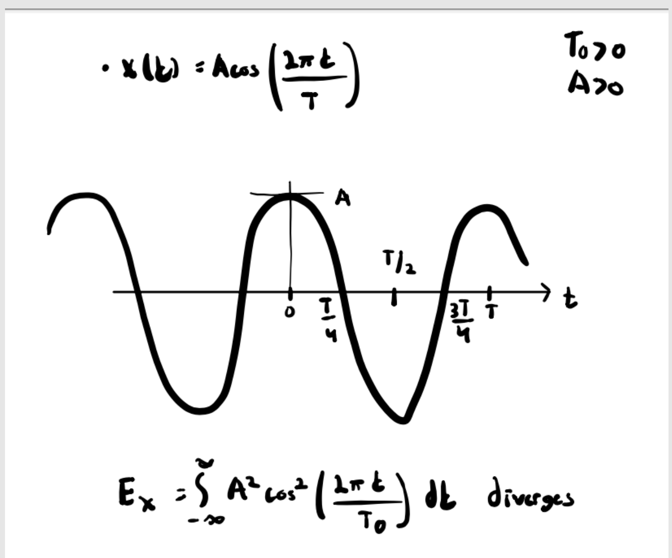


Figure 17: Cosine

Let $x(t) = A\cos\left(\frac{2\pi t}{T_0}\right)$ for some $T_0 > 0$ and some $A > 0$. (Here I've replaced the T from class with T_0.) Let's compute the power of x.

$$\begin{aligned}
\text{We have } P_x &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 \, \mathrm{d}t \qquad , \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2\cos^2\left(\frac{2\pi t}{T_0}\right) \mathrm{d}t \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{A^2}{2}\left(1+\cos\left(\frac{4\pi t}{T_0}\right)\right) \mathrm{d}t \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{A^2}{2}\,\mathrm{d}t + \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{A^2}{2}\cos\left(\frac{4\pi t}{T_0}\right) \mathrm{d}t \\
&= \lim_{T\to\infty} \frac{1}{2T}\cdot\frac{2TA^2}{2} + \lim_{T\to\infty} \frac{1}{2T}\frac{T_0}{4\pi}\sin\left(\frac{4\pi t}{T_0}\right)\Bigg|_{-T}^{T} \\
&= \frac{A^2}{2} + \lim_{T\to\infty} \frac{1}{2T}\frac{2T_0}{4\pi}\sin\left(\frac{4\pi T}{T_0}\right) \\
&= \frac{A^2}{2}
\end{aligned}$$

The final limit in the last expression converges to zero since the function $\sin()$ is bounded between -1 and 1 and T_0 is a constant.

In conclusion, a cosine wave of amplitude A has power $\frac{A^2}{2}$. The period T_0 doesn't play a role, i.e., this result is true for *any* period $T_0 > 0$

Figure 18: Cosine

Intuition: Sketch the signal whenever possible.

3.4 Zero-energy signals

Definition:

DT: zero$[n]$ has zero energy.

- **Are there others?** No, if $x[i] \neq 0$ for some i. $E_x \geq |x[i]|^2 > 0$

CT: zero(t) has zero energy.

- **Are there others?** Yes, examples are below.

3.4.1 Examples of CT zero energy signals

Example:

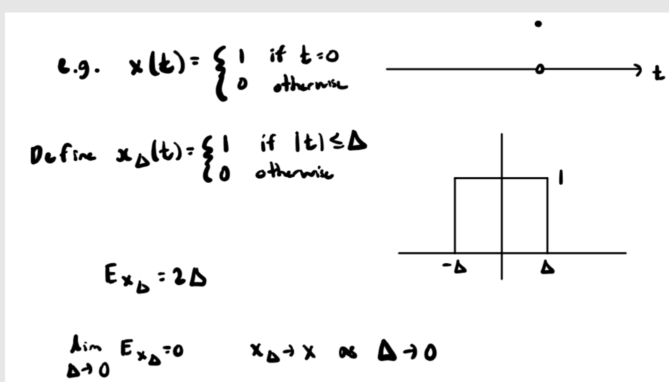


Figure 19: Impulse function with zero energy. (Top): Showing the extreme case. Bottom: Showing the delta case as it goes to 0.

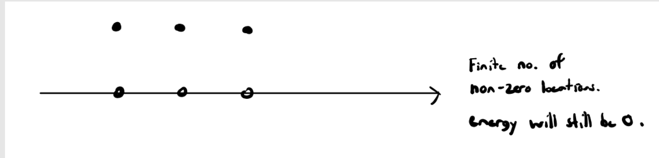


Figure 20: Finite number of locations will have zero energy.

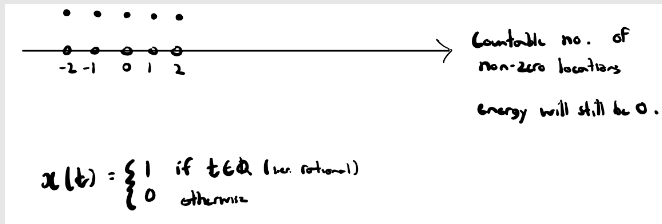


Figure 21: Countable number of locations will have zero energy.

3.4.2 Almost everywhere

Definition: If $x(t)$ has zero energy, we will say $x(t) = \text{zero}(t)$ almost everywhere.

$$x \overset{\text{a.e.}}{=} \text{zero} \tag{12}$$

$x(t) = y(t)$ almost everywhere (i.e. $x \overset{\text{a.e.}}{=} y$) if $x - y \overset{\text{a.e.}}{=} \text{zero}$

- **English:** Physically indistinguishable, where signals that are equal almost everywhere are treated as equivalent because discrepancies occur in regions.
- **Implication:** On exams, if they are equal almost everywhere, then it be given leeway in marking to be the same.

3.5 Signal spaces are vector spaces

This holders for all 4 main classes of signals.

3.5.1 Signal addition

Definition: Given two signals $x, y \in \mathbb{R}^{\mathbb{R}}$, we can form a new signal $x + y$

$$(x + y)(t) = x(t) + y(t) \quad \text{by superposition} \tag{13}$$

$\mathbb{R}^{\mathbb{R}}$ is closed under VA. $\forall x, y, z \in \mathbb{R}^{\mathbb{R}}$:

1. **Commutative:** $x + y = y + x$
2. **Associative:** $x + (y + z) = (x + y) + z$
3. **Additive identity:** $\text{zero}(t)$ is the identity fcn.
4. **Additive inverse:** Every signal x has an additive inverse $-x$, s.t. $x + (-x) = \text{zero}$

3.5.2 Scalar multiplication

Definition: Given any scalar $a \in \mathbb{R}$, and any signal $x \in \mathbb{R}^{\mathbb{R}}$ we can form a new signal $ax \in \mathbb{R}^{\mathbb{R}}$

$$(ax)(t) = ax(t) \tag{14}$$

- **Amplify:** $|a| > 1$

- **Attentuate:** $|a| < 1$

$\mathbb{R}^{\mathbb{R}}$ is closed under SM. $\forall a, b \in \mathbb{R}, \forall x, y \in \mathbb{R}^{\mathbb{R}}$:

1. **Distributivity of signals:** $a(x + y) = (ax) + (ay)$
2. **Associativity:** $a(bx) = (ab)x$
3. **Scalar identity:** $1x = x$
4. **Distributivity of scalars:** $(a + b)x = ax + bx$

4 Time dilation, shifting (Ch. 1.2)

4.1 Affine transformations of the Independent Variable

In general, $y(t) = x(at + b)$ for any $a, b \in \mathbb{R}$ (and usually $a \neq 0$)

4.1.1 Time dilation

Definition: $x(t) \to x\left(\frac{t}{a}\right)$ then

1. **Speed up:** If $a > 1$ (i.e. compressed)
2. **Slow down:** If $0 < a < 1$ (i.e. stretched)

Example:

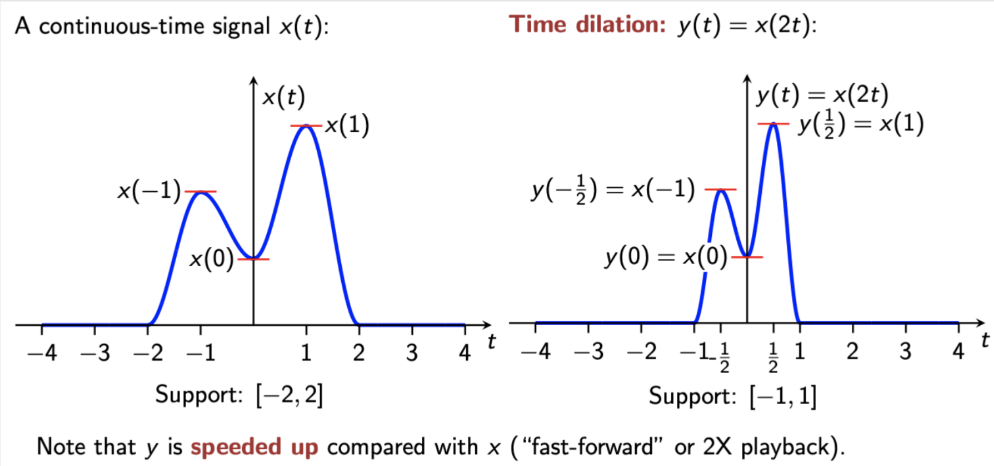


Figure 22: Time dilation, which sped up compared to x

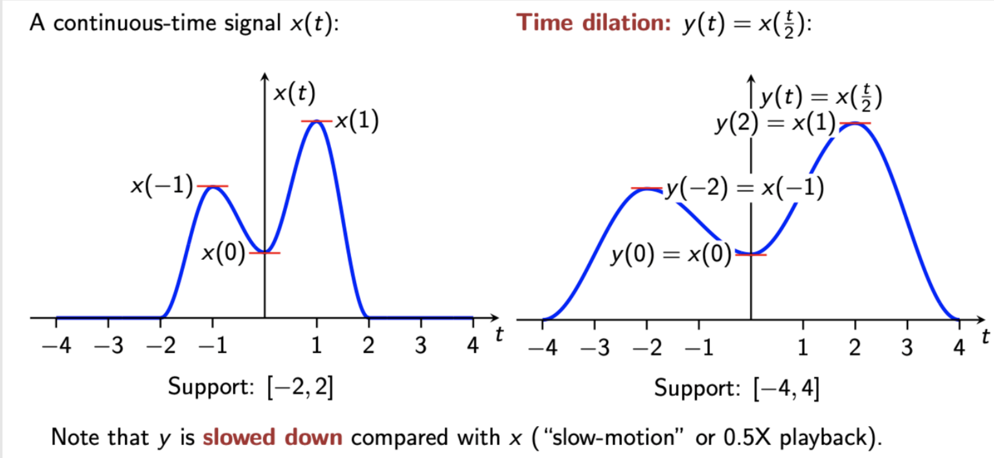


Figure 23: Time dilation, which slowed down compared to x

4.1.2 Time reversal

Definition: $x(t) \to x(-t) = \tilde{x}(t)$ (i.e. reflect across y-axis)

Example:

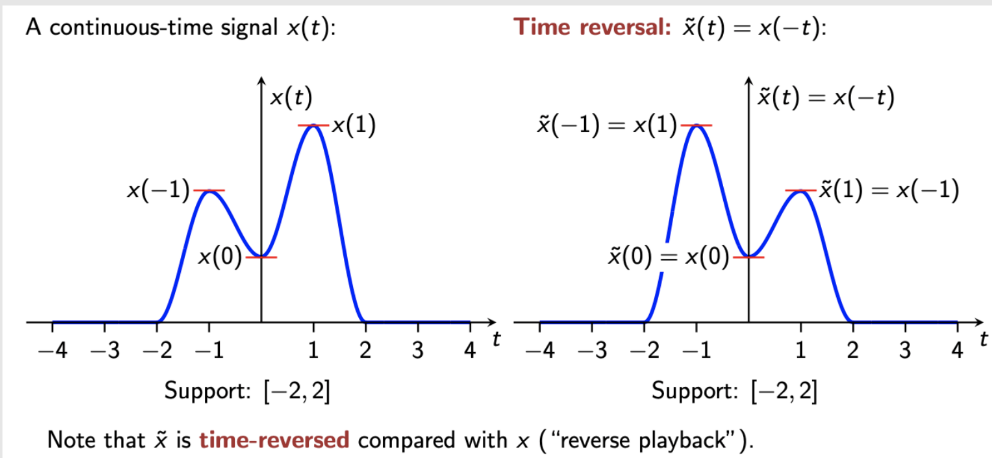


Figure 24: Time reversal, which reverses time compared to x

4.1.3 Time delay

Definition: $x(t) \to x(t - a)$ for $a > 0$ (i.e. right shift)

Example:

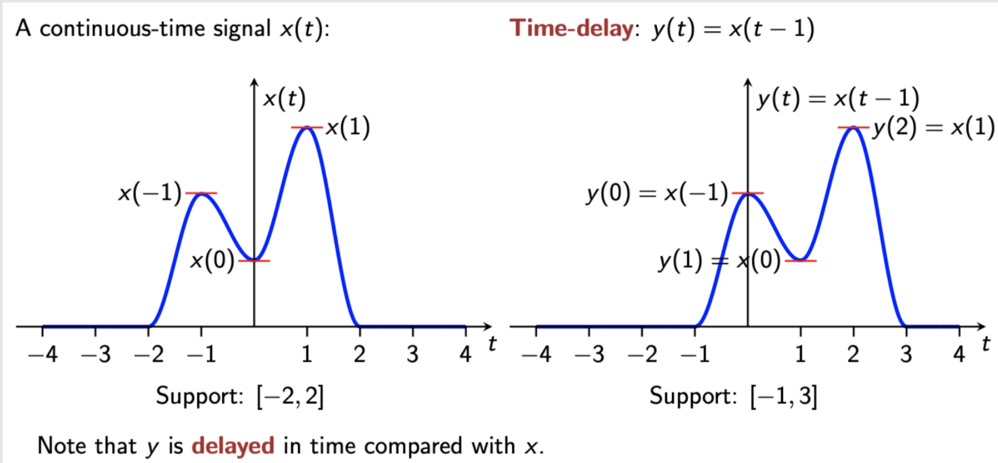


Figure 25: Time delay, which delays time compared to x

4.1.4 Time advance

Definition: $x(t) \to x(t+a)$ for $a > 0$ (i.e. left shift)

Example:

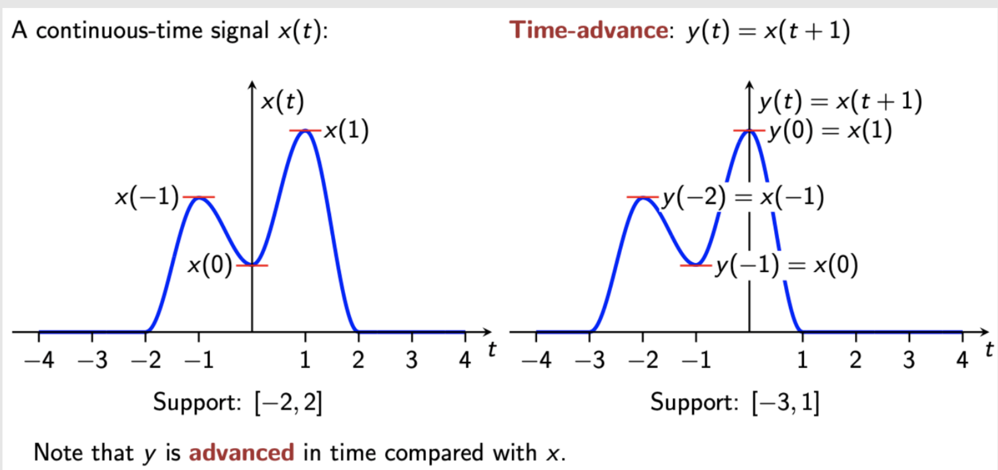


Figure 26: Time advance, which advances time compared to x

4.1.5 Combined transformations

Example:
1. Time delay and shift

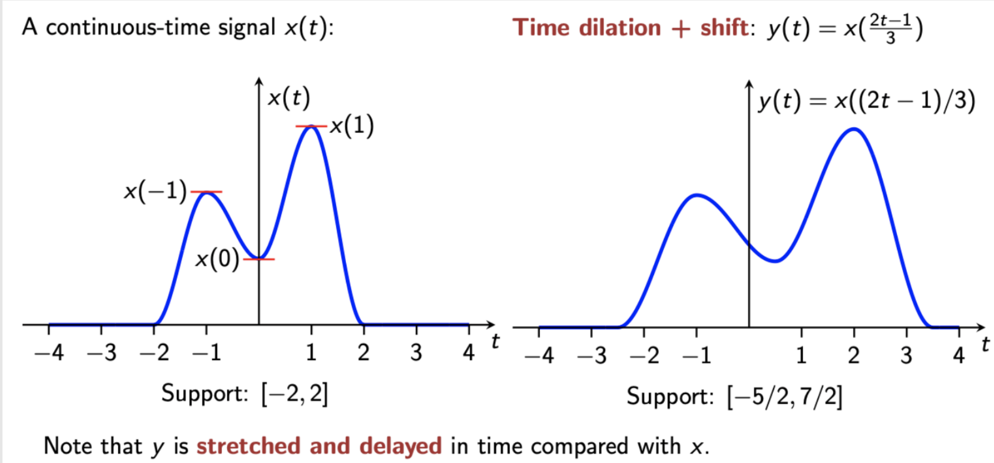


Figure 27: Time is stretched and delayed in time compared to x

2. Time reversal, dilation, and shift

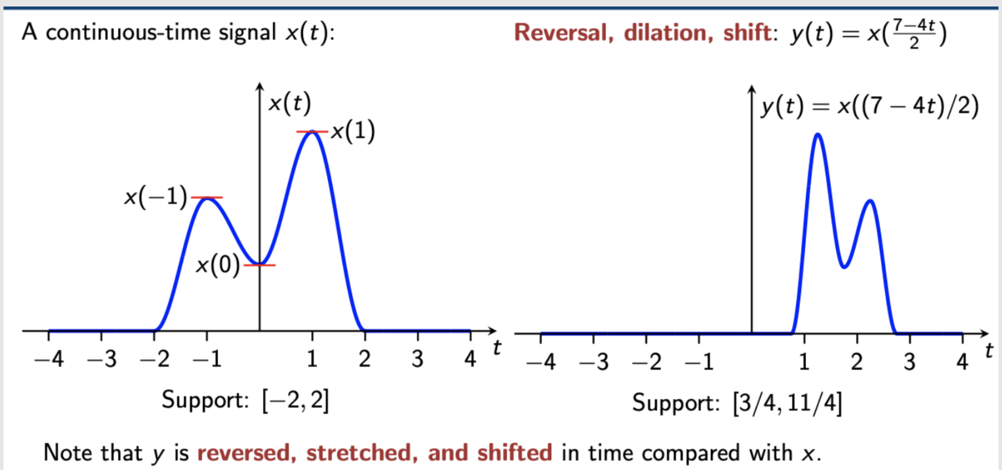


Figure 28: Time is reversal, dilated, and shifted compared to x

4.2 Transformations of Discrete Time

In general, $y[n] = x[an + b]$ for any $a, b \in \mathbb{Z}$ (and usually $a \neq 0$)

Example:

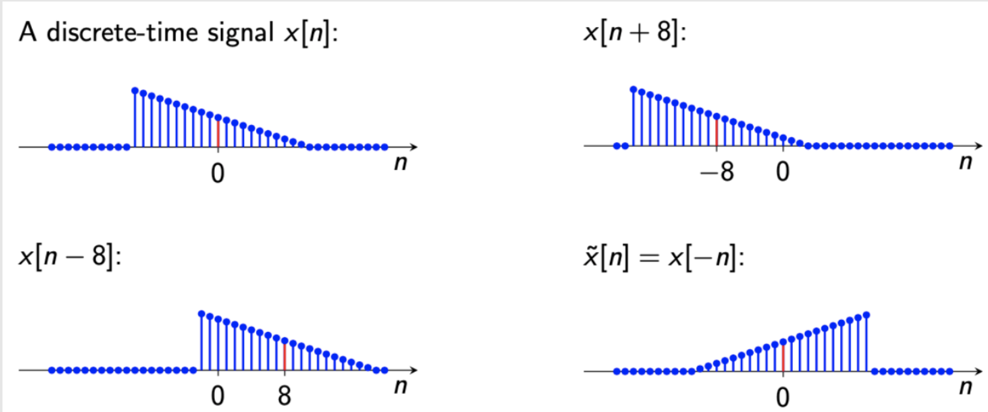


Figure 29: Transformation of DT signal.

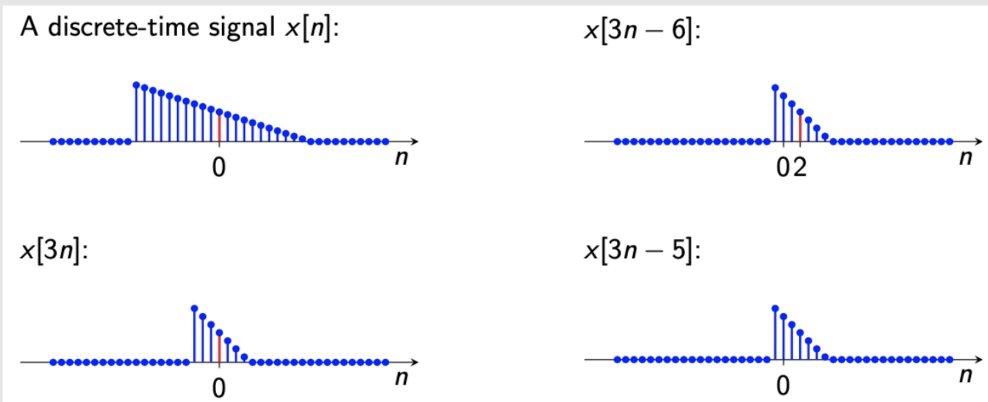


Figure 30: Transformation of DT signal.

Warning: The same transformations in CT hold for DT, but we need to be careful.

- When $|a| > 1$, only one in every $|a|$ samples from x is retained.
  - For $y[n] = x[an]$, the points of y at any n correspond to x evaluated at intervals of a. If $a = 3$, then:

$$y[0] = x[0], \quad y[1] = x[3], \quad y[2] = x[6], \quad \ldots$$

    This demonstrates how only every third sample is retained, compressing the original signal.
- Defining $y[n] = x[n/2]$ does not make sense, since $x[-1/2], x[1/2], x[3/2], \ldots$ are undefined.

4.3 Periodic Signals

4.3.1 CT: T-periodic

Definition: A CT signal x is T-periodic for some positive real number T if

$$x(t+T) = x(t) \quad \text{for all} \quad t \in \mathbb{R}. \tag{15}$$

- If x is T-periodic, then $x(t+kT) = x(t)$ for all $k \in \mathbb{Z}$ and all $t \in \mathbb{R}$. (i.e. if x is T-periodic, then x is also kT-periodic)
- Let $y(t) = x(t+T)$, then x is T-periodic if $y \overset{a.e.}{=} x$.

4.3.2 CT: Fundamental period

Definition: The **fundamental period** (if it exists) of a CT periodic signal x is the smallest positive real number T_0 such that x is T_0-periodic.

- **Fundamental frequency:** $T_0 = \dfrac{1}{f_0}$

Warning: A constant signal $x(t) = C$ is T-periodic for all $T \in (0, \infty)$. Such a signal has no fundamental period since the set $(0, \infty)$ does not have a smallest element.

4.3.3 DT: N-Periodic

Definition: A DT signal x is N-periodic for some positive integer N if

$$x[n+N] = x[n] \quad \text{for all} \quad n \in \mathbb{Z} \tag{16}$$

- If x is N-periodic, then $x[n+kN] = x[n]$ for all $k, n \in \mathbb{Z}$ (i.e. If x is N-periodic, then x is also kN-periodic).

Warning: A 1-periodic signal must be constant.

4.3.4 DT: Fundamental Period

Definition: The **fundamental period** of a DT periodic signal x is the smallest positive integer N_0 such that x is N_0-periodic.

Example:

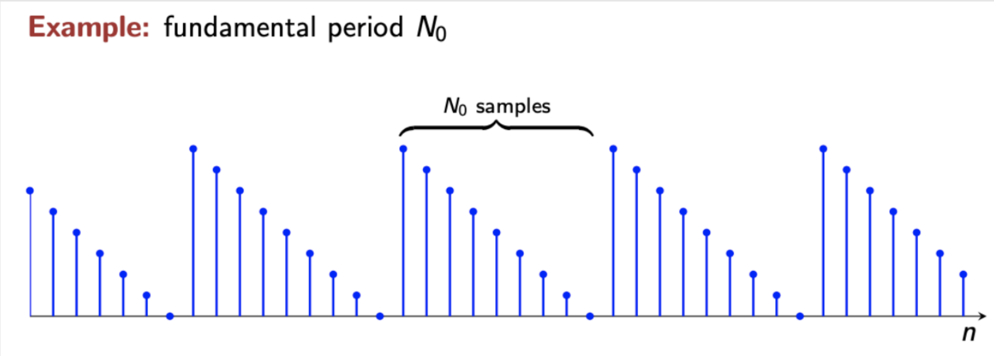


Figure 31: Fundamental period of a DT signal

Warning: The fundamental period cannot include the same sample twice (i.e. don't pick the range inclusive of two peaks). However, this is fine in CT signals.

4.4 Even and Odd Signals

Definition:
A signal x is said to be **even** if $x = \tilde{x}$.

- An even signal has mirror-image symmetry about the time origin.

A signal x is said to be **odd** if $x = -\tilde{x}$.

- An odd signal has reversed mirror-image symmetry about the time origin.
  - Therefore an odd signal must have value 0 at the time origin.

Example:

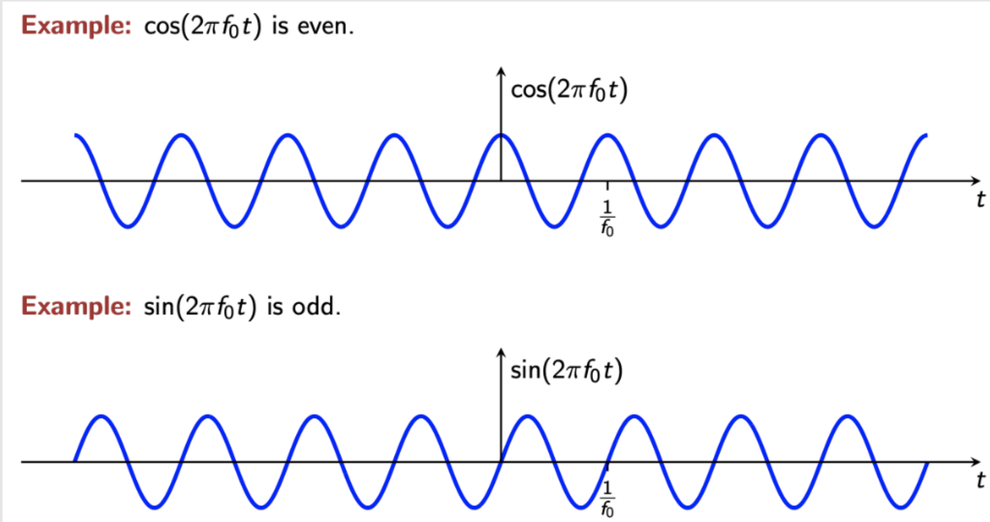


Figure 32: Even and odd examples.

4.4.1 Even and odd parts of a signal

Definition:
The **even part** of a signal x is the signal

$$x_{\text{even}} = \frac{1}{2}(x + \tilde{x}) \tag{17}$$

The **odd part** of a signal x is the signal

$$x_{\text{odd}} = \frac{1}{2}(x - \tilde{x}) \tag{18}$$

- $x_{even} + x_{odd} = x$

Example: Prove $x_{even}(-t) = x_{even}(t)$ and prove $x_{odd}(-t) = -x_{odd}(t)$

$$x_{\text{even}}(-t) = \frac{1}{2}(x(-t) + \tilde{x}(-t)) = \frac{1}{2}(\tilde{x}(t) + x(t)) = x_{\text{even}}(t)$$

$$x_{\text{odd}}(-t) = \frac{1}{2}(x(-t) - \tilde{x}(-t)) = \frac{1}{2}(\tilde{x}(t) - x(t)) = -x_{\text{odd}}(t)$$

Example:

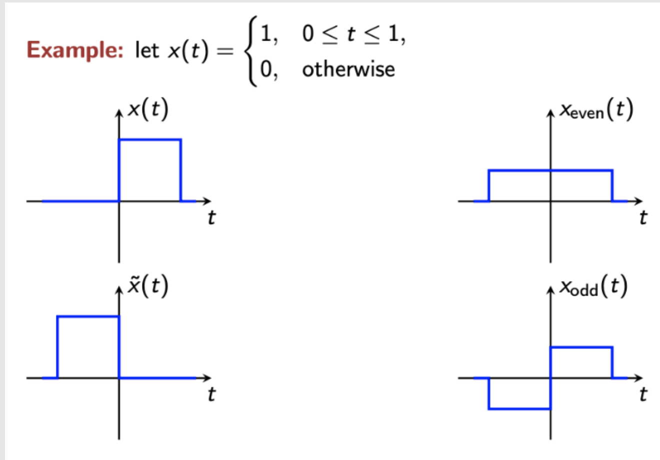


Figure 33: Even and odd decomposition example.

5 Complex exponential signals (Ch. 1.3)

5.1 CT: Complex exponential signals

Definition: A **complex exponential** signal x in CT is a signal of the form

$$x(t) = Ae^{st} \in \mathbb{C}^{\mathbb{R}} \tag{19}$$

where A and s are arbitrary complex-valued constants.

- A: A scalar (affecting the magnitude and phase x), so only consider the special case when $A = 1$.
- $s = \alpha + j\omega$ for $\alpha, \omega \in \mathbb{R}$: These parameters control the shape of the complex exponential signal x.
- ω: Angular frequency (if t is measured in seconds, ω is measured in radians per second).
- $f \in \mathbb{R}$: Frequency s.t. $\omega = 2\pi f$ (if t is measured in seconds, f is measured in hertz (Hz)).

5.2 CT: Real-valued exponential signals

Definition: If $\omega = 0$ (equivalently, $f = 0$), then $s = \alpha$ is purely real, and we get a purely-real signal:

$$x(t) = e^{\alpha t}, \quad \alpha \in \mathbb{R}. \tag{20}$$

Three different general behaviours are possible:

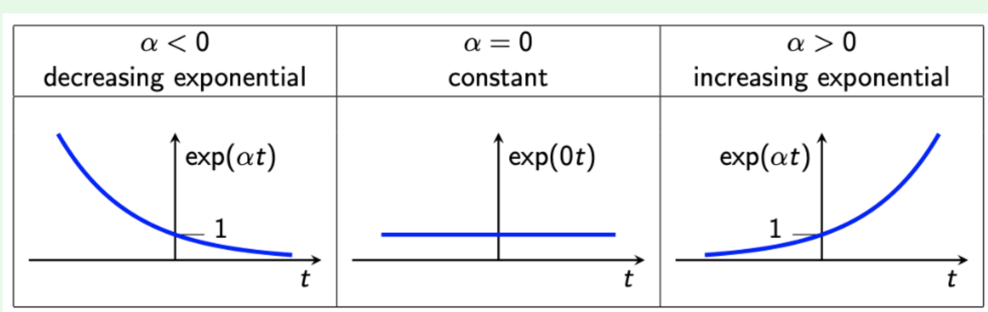


Figure 34: The three different general behaviours when the complex part is 0.

5.3 CT: Sinusoidal complex exponential signals

Definition: If $\alpha = 0$, then $s = j\omega = j2\pi f$ is purely imaginary, and we get

$$x(t) = e^{j\omega t} = e^{j2\pi f t} \tag{21}$$

- $x(t) = e^{j\omega t}$: **Rotating unit-magnitude phasor** in the complex plane
  - Rotating *counter-clockwise* if $\omega > 0$
  - Rotating *clockwise* if $\omega < 0$.
- If t is measured in seconds, the phasor performs $|f|$ revolutions (cycles) per second.

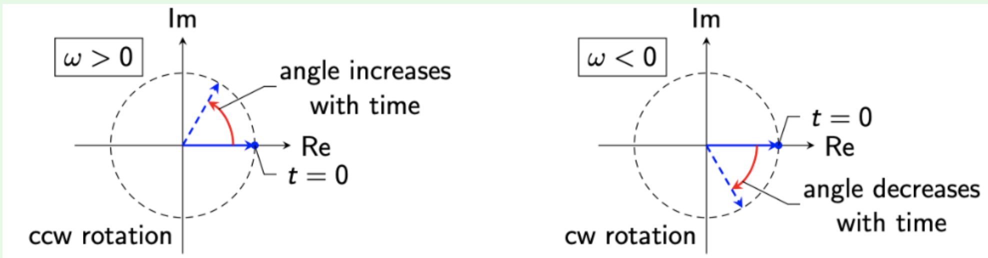


Figure 35: CCW and CW being illustrated depending on the value of the angular frequency.

5.3.1 CT: Rotating unit-magnitude phasor

Definition:
For $x(t) = e^{j\omega t} = e^{j2\pi f t}$, the graphs can be illustrated as

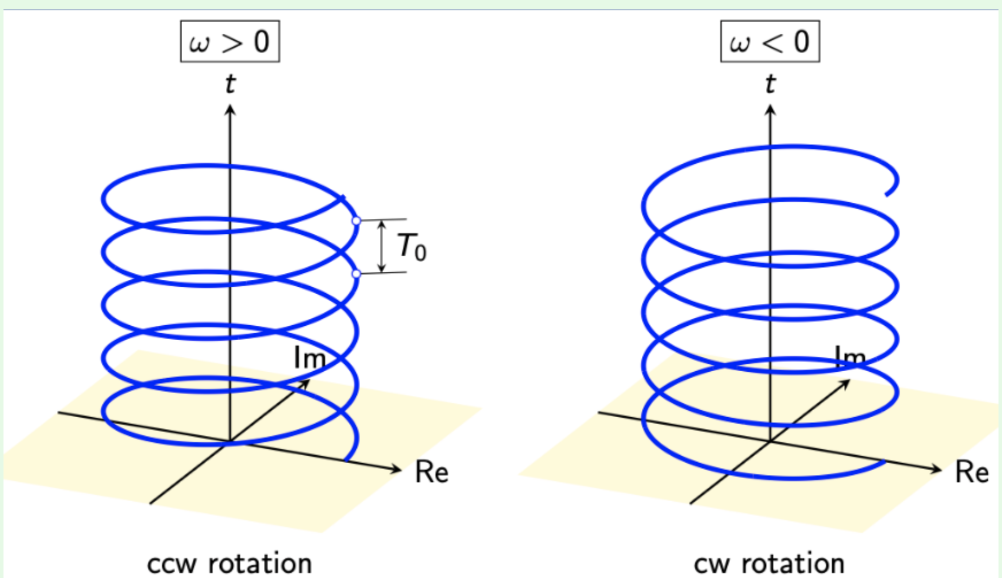


Figure 36: Rotating unit-magnitude phasor for both general cases of omega.

- **Fun. Period:** $T_0 = \dfrac{1}{f} = \dfrac{2\pi}{\omega}$

5.3.2 CT: Real and imaginary parts

Definition: For $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$, then

$$\text{Re}(e^{j\omega t}) = \cos(\omega t) \quad \text{and} \quad \text{Im}(e^{j\omega t}) = \sin(\omega t) \tag{22}$$

For $e^{j2\pi ft} = \cos(2\pi ft) + j\sin(2\pi ft)$, then

$$\text{Re}(e^{j2\pi ft}) = \cos(2\pi ft) \quad \text{and} \quad \text{Im}(e^{j2\pi ft}) = \sin(2\pi ft) \tag{23}$$

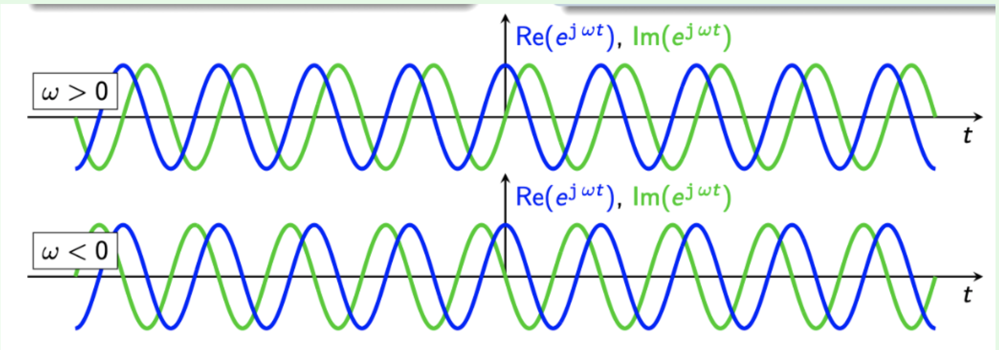


Figure 37: Real and imaginary components for both cases of omega.

5.4 The general case

Definition: If $s = \alpha + j\omega = \alpha + j2\pi f$, with $\alpha \neq 0$ and $\omega \neq 0$, we obtain $e^{(\alpha+j\omega)t}$, a **rotating phasor** in the complex plane with a **time-varying magnitude**.

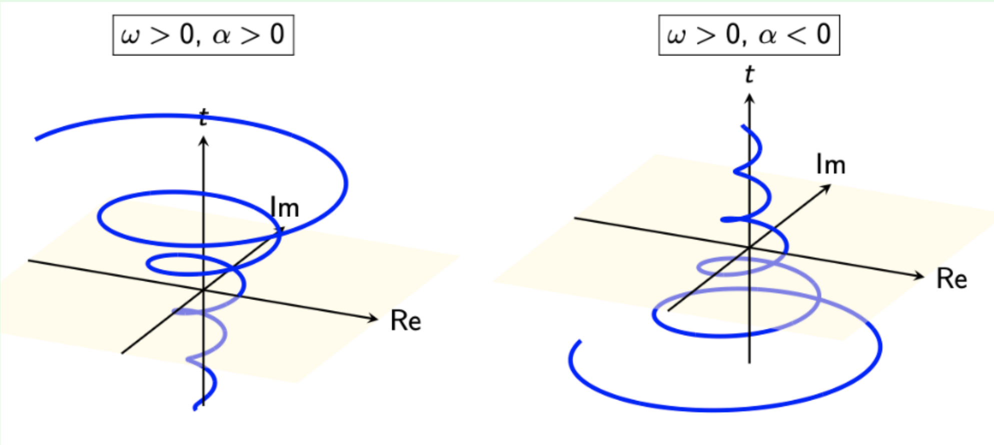


Figure 38: The general case for the CT complex exponential signal

5.4.1 Real and imaginary parts

Definition:

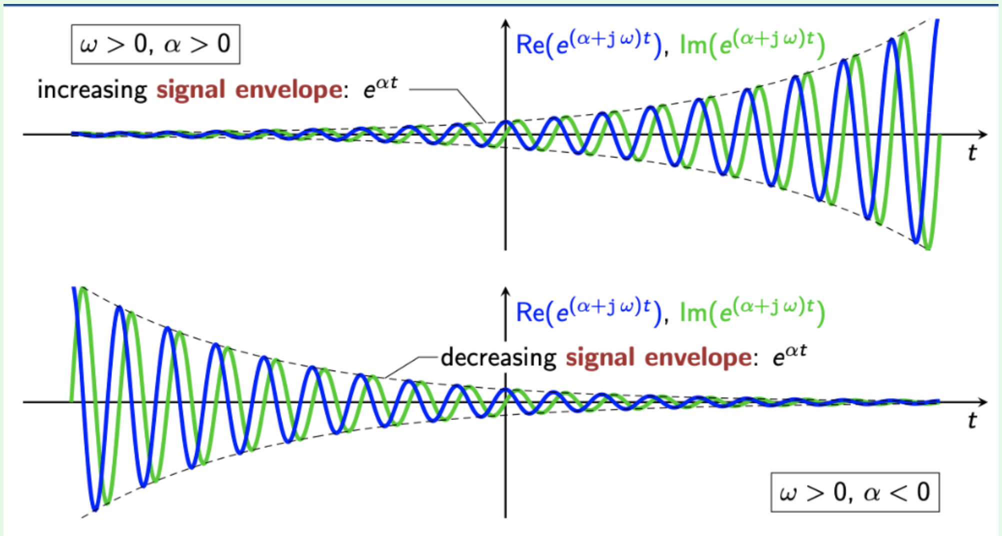


Figure 39: Real and imaginary parts of the general case for two cases of omega and alpha.

5.5 DT: Complex exponential signals

Definition: A **complex exponential** signal x in DT is a signal of the form

$$x[n] = Ae^{sn} \in \mathbb{C}^{\mathbb{Z}} \tag{24}$$

where A and s are arbitrary complex-valued constants.

- A: A scalar (affecting the magnitude and phase x), so only consider the special case when $A = 1$.
- $s = \alpha + j\omega = \alpha + j2\pi f \quad \text{for} \quad \alpha, \omega = 2\pi f \in \mathbb{R}$.
  - If $\alpha \neq 0$, we obtain an increasing or decreasing **signal envelope**, just as in CT, so we will only consider the special case when $\alpha = 0$.
- ω: Natural frequency (If time n is measured in samples, then ω has units of radians per sample).
- f: Frequency has units of cycles per sample (since a "sample" is a dimensionless quantity, frequency is dimensionless in DT).

5.5.1 Oscillatory vs. Periodic

Intuition: Depending on the value of ω, we expect $e^{j\omega n}$ to be **oscillatory** (though not necessarily **periodic**):

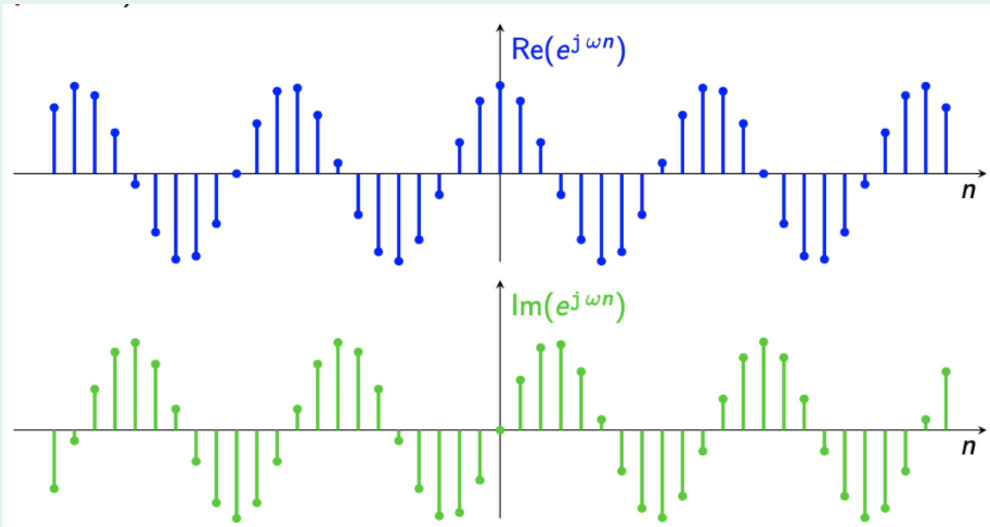


Figure 40: Real and imaginary components of a DT signal.

1. A discrete-time complex exponential signal is given by:
$$x[n] = e^{j\omega n} = \cos(\omega n) + j\sin(\omega n).$$
2. The signal is always **oscillatory** because the sine and cosine functions cause continuous wave-like oscillations for any value of ω.
3. The signal is **periodic** if there exists an integer N such that:
$$x[n+N] = x[n] \quad \text{for all} \quad n.$$
This leads to the condition:
$$e^{j\omega N} = 1 \quad \text{or} \quad \omega N = 2\pi k, \quad k \in \mathbb{Z}.$$
4. The signal is periodic if and only if $\omega/2\pi$ is a **rational number**, i.e., $\omega = 2\pi\frac{k}{N}$ for integers k and N.
5. If $\omega/2\pi$ is an **irrational number**, no such N exists, and the signal will be oscillatory but **not periodic**, as the signal never repeats exactly.

5.5.2 Equivalent frequencies

Definition: Natural frequencies ω_1 and ω_2 are said to be **complex-exponential equivalent**, written $\omega_1 \equiv \omega_2$, if $e^{j\omega_1 n} = e^{j\omega_2 n}$ for all $n \in \mathbb{Z}$.
- I.e. $\omega_1 \equiv \omega_2$ if the complex exponential signals $e^{j\omega_1 n}$ and $e^{j\omega_2 n}$ are **identical**.

Theorem: Complex logarithms of unity: For $z \in \mathbb{C}$, $e^z = 1$ if and only if $z = j2\pi m$ for some $m \in \mathbb{Z}$.
- **Key:** Help us to determine when $\omega_1 \equiv \omega_2$:

Derivation: Let $z = a + jb$, where $a, b \in \mathbb{R}$. Then,
$$e^z = e^a e^{jb}.$$
1. For e^z to have *unit magnitude*, we require $a = 0$, since $e^a = 1$ only if $a = 0$.
2. Now, we consider the term e^{jb}. The only purely real values that e^{jb} can achieve are $+1$ and -1. This is because e^{jb} lies on the unit circle in the complex plane, and for it to be purely real, it must lie at one of the two real-axis points on the circle.
3. The value $e^{jb} = +1$ is achieved if and only if $b = j2\pi m$ for some $m \in \mathbb{Z}$.

Thus, $z = j2\pi m$ for some $m \in \mathbb{Z}$ is necessary and sufficient for $e^z = 1$.

Theorem: Equivalent Frequencies: Natural frequencies ω_1 and ω_2 are complex-exponential equivalent if and only if $\omega_1 - \omega_2 = 2\pi m$ for some $m \in \mathbb{Z}$.

Frequencies f_1 and f_2 are complex-exponential equivalent if and only if $f_1 - f_2 = m$ for some $m \in \mathbb{Z}$.

Derivation: We have $e^{j\omega_1 n} = e^{j\omega_2 n}$ for all $n \in \mathbb{Z}$ if and only if $e^{j(\omega_1-\omega_2)n} = 1$ for all $n \in \mathbb{Z}$.

1. For $n = 1$, the previous theorem implies that it is necessary for $\omega_1 - \omega_2 = j2\pi m$ for some $m \in \mathbb{Z}$. This ensures that $e^{j(\omega_1-\omega_2)n} = 1$ holds when $n = 1$.
2. However, the condition $\omega_1 - \omega_2 = j2\pi m$ is also sufficient to guarantee that $e^{j(\omega_1-\omega_2)n} = 1$ holds for all $n \in \mathbb{Z}$. This shows that the condition works for all integer values of n.
3. Thus, the condition $\omega_1 - \omega_2 = j2\pi m$ is both necessary and sufficient for ω_1 and ω_2 to be equivalent.

In conclusion, $\omega_1 \equiv \omega_2$ if and only if $\omega_1 - \omega_2 = j2\pi m$ for some $m \in \mathbb{Z}$.

Intuition:

- Because $\omega \equiv \omega + 2\pi m$ for any integer m, it is useful to select ω to satisfy

$$-\pi < \omega \leq \pi.$$

  - Natural frequencies outside of this range can be reduced to this range by adding or subtracting a suitable integer multiple of 2π.
- Because $f \equiv f + m$ for any integer m, it is useful to select f to satisfy

$$-\frac{1}{2} < f \leq \frac{1}{2}.$$

Example: The **highest frequency** discrete-time complex exponential signal, with $\omega = \pi$ (rad/sample) or $f = \frac{1}{2}$ (cycles/sample), is

$$x[n] = e^{j\pi n} = (-1)^n.$$

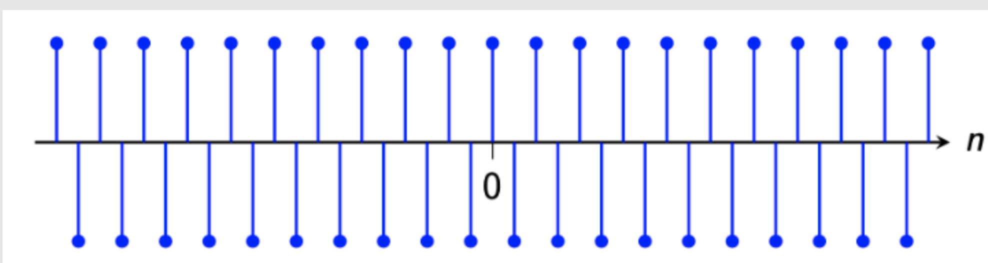


Figure 41: Example of a DT signal with its highest frequency.

1. **Angular Frequency $\omega = \pi$:** Represents the highest possible angular frequency in DT.
   - Therefore, $\omega = \pi$ is the midpoint of the frequency range $-\pi < \omega \leq \pi$, and beyond this, frequencies wrap around (i.e., $\omega + 2\pi m$ for integer m).
2. **Oscillatory Behavior:** At $\omega = \pi$, the signal alternates between 1 and -1 with every sample.

$$x[n] = (-1)^n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

   This fast alternation between 1 and -1 represents the maximum rate of oscillation that can be captured in a DT system.
3. **Frequency:** $f = \frac{1}{2}$ cycles/sample. This is because the signal completes one full oscillation (from 1 to -1 and back to 1) every two samples. Therefore, the frequency $f = \frac{1}{2}$ is the highest possible frequency in terms of cycles per sample.

4. **Effect on Sampling:** Any frequency higher than this would be indistinguishable from a lower frequency due to aliasing effects (i.e. already represented in lower signals).

5.5.3 When is a DT complex exponential signal periodic?

Theorem: The DT complex exponential signal $e^{j2\pi fn}$ is periodic if and only if $f \in \mathbb{Q}$.

- **Note:** This was shown in oscillatory vs. periodic.

5.5.4 Computing the fundamental period

Definition: Let $x[n] = e^{j2\pi fn} = e^{j2\pi\left(\frac{a}{b}\right)n}$

- $f = \dfrac{a}{b}$: Rational frequency
  - a and b are integers, with $b \neq 0$ and with $b = 1$ if $a = 0$.
  - a and b to have no common factors, (i.e. $\dfrac{a}{b}$ is reduced to lowest terms).

Then the **fundamental period** is

$$N_0 = b.$$

- i.e. The smallest positive integer N_0 such that fN_0 is $N_0 = b$ since no smaller multiple of f clears the denominator.

6 Step and impulse functions (Ch. 1.4)

7 General systems and basic properties (Ch. 1.5-6)

Linear Time-Invariant Systems

8 Impulse response (Ch. 2.1)

9 Convolution in discrete time (Ch. 2.1)

10 Convolution in continuous time (Ch. 2.2)

11 Properties of LTI systems (Ch. 2.3)

Fourier Series and Fourier Transform Representations

12 Periodic signals and Fourier series

13 Properties of Fourier series

14 Response of LTI systems to periodic signals

15 Aperiodic signals and Fourier transform

16 Fourier transform properties; time-frequency duality

Sampling

17 Bandlimited signals

18 The sampling theorem (Ch. 7.1)

19 Reconstruction (Ch. 7.2)

Communication Systems

20 Amplitude modulation systems

21 Envelope detection, coherent detection

22 Single-sideband modulation

23 Angle modulation

24 Concepts of digital communication