1. Sets and Counting

1.1 Operation with Sets

$A \cap \emptyset = \emptyset \mid A \cup \emptyset = A \mid A \cap A' = \emptyset \mid A \cup A' = S \mid S' = \emptyset \mid \emptyset' = S \mid (A')' = A \mid (A \cap B)' = A' \cup B' \mid (A \cup B)' = A' \cap B' \mid A \cap S = A \mid A \cap B \cap S = A \cap B$

1.2 Fundamental Principle of Counting: $m \times n$

1.3 Permutation: $nPr = \frac{n!}{(n-r)!}$

1.4 Permutation with Identical Items and Partitions

$$\binom{n}{n_1, \dots, n_m} = \frac{n!}{n_1! n_2! \cdots n_m!},\ n = \sum_{k=1}^{m} n_k$$

1.5 Combination: $nCr = \frac{n!}{r!(n-r)!}$

1.6 Mutually Exclusive: $A \cap B = \emptyset$

2. Definitions of Probability

2.1 Probability of an Event: $P = \frac{\text{Number of favorable outcomes}}{\text{Total outcomes}}$

2.2 Probability of Union of Mutually Exclusive Events:

$$P(A_1 \cup \dots \cup A_N) = P(A_1) + \dots + P(A_N)$$

2.3 Additive Rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = \sum_{i=1}^{3} P(X_i) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

2.4 Conditional Probability

$$P(B|A) = \frac{P(B \cap A)}{P(A)};\ P(A) > 0$$

- $P(B'|A) = 1 - P(B|A) \mid P(B'|A') = 1 - P(B|A')$
- $P(B' \cap C) = P(A \cap B' \cap C) + P(A' \cap B \cap C)$

2.5 Product Rule

$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B) = P(B \cap A);\ P(A), P(B) > 0$

2.6 Independence of Events

$P(A|B) = P(A)$ or $P(B|A) = P(B)$ or $P(A \cap B) = P(A)P(B)$

- **Generalization:** $P(A_1 \dots A_N) = P(A_1) \cdots P(A_N)$

3. Bayes' Rule

3.1 Bayes' Rule: $P(B|A) = \frac{P(A|B)P(B)}{P(A)};\ P(B), P(A) > 0$

3.2 Partition: For $B_1, \dots, B_k$: $B_i \cap B_j = \emptyset$ & $B_1 \cup \dots \cup B_k = S$.

3.3 Total Probability Theorem

$P(A) = \sum_{i=1}^{k} P(A \cap B_i) = \sum_{i=1}^{k} P(A|B_i)P(B_i)$ for the partition $B_1, \dots, B_k$

3.4 Bayes' Rule with Total Probability Theorem

$P(B|A) = \frac{P(B)P(A|B)}{\sum_{i=1}^{k} P(C_i)P(A|C_i)}$ for the partition $C_1, \dots, C_k$ or $P(C_n|A) = \frac{P(C_n)P(A|C_n)}{\sum_{i=1}^{k} P(C_i)P(A|C_i)}$

4. RVs & Distributions

4.0 Probability Mass Function (PMF - Discrete)

- $f(x) \geq 0$ for each outcome $X = x$
- $\sum_x f(x) = 1$
- $f(x) = P(X = x)$

4.1 Cumulative Distribution Function (CDF - Discrete)

$$P(X \leq x) = F(x) = \sum_{t \leq x} f(t) \text{ for } x \in \mathbb{R}$$

- $P(a \leq X \leq b) = F(b) - F(a-1) \mid P(X < a) = F(x \leq a - 1)$
- $P(x > a) = 1 - F(x \leq a) \mid P(x \geq a) = 1 - F(x \leq a - 1)$
- $P(a < X < b) = P(b - 1) - P(a)$

4.2 Probability Density Function (PDF - Continuous)

- $f(x) \geq 0$ for each possible value $X = x$
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- $\int_a^b f(x)dx = P(a < X < b)$
- $P(a \leq x \leq b) = P(a \leq x < b) = P(a < x \leq b) = P(a < x < b)$

4.3 Cumulative Distribution Function (CDF - Continuous)

- $P(X \leq x) = F(x) = \int_{-\infty}^{x} f(t)dt$ for $x \in \mathbb{R}$
- $F(\infty) = P(X \leq \infty) = \int_{-\infty}^{\infty} f(t)dt = 1$
- $P(a < X \leq b) = F(b) - F(a)$

4.4 Joint PMF

- $f(x, y) \geq 0$ for all $(x, y) \in S$
- $\sum_x \sum_y f(x, y) = 1$
- For $A \subset S$: $P((X, Y) \in A) = \sum_{(x,y) \in A} f(x, y)$
- $P(X = x, Y = y) = f(x, y)$

4.5 Joint PDF

- $f(x, y) \geq 0$ for all $(x, y) \in S$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1$
- For $A \subset S$: $P((X, Y) \in A) = \int_{(x,y) \in A} f(x, y)dxdy$
- $P(X < Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} f(x, y)dxdy$
- $P(a < X < b, c < Y < d) = \int_a^b \int_c^d f(x, y)dxdy$

4.6 Marginal Distribution

Discrete: $g(x) = \sum_y f(x, y) \mid h(y) = \sum_x f(x, y)$

Continuous: $g(x) = \int_{-\infty}^{\infty} f(x, y)dy \mid h(y) = \int_{-\infty}^{\infty} f(x, y)dx$

4.7 Conditional Distributions:

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

Discrete Probability: $P(a \leq X \leq b \mid Y = \alpha) = \sum_{a \leq x \leq b} f(x|y = \alpha)$

Continuous Probability: $P(a \leq X \leq b \mid Y = \alpha) = \int_a^b f(x|y = \alpha)dx$

4.8 Independence of RVs of Marginals g(x), h(y) and Joint f(x,y)

$$f(x, y) = g(x)h(y)$$

5. Expectations, Variance & Covariance

5.1 Expectation of a Function of One RV

Discrete: $E[g(X)] = \sum_x g(x)f(x)$

Continuous: $E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$

5.2 Expectation of a Function of Two RVs

Discrete: $E[g(X, Y)] = \sum_y \sum_x g(x, y)f(x, y)$

Continuous: $E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f(x, y)dxdy$

5.3 Expectation of Linear Combinations of RVs

- $E[aX + bY] = aE[X] + bE[Y]$
- $E[aX + b] = aE[X] + b$
- $E[g(X, Y) + h(X, Y)] = E[g(X, Y)] + E[h(X, Y)]$

5.4 Independence of Expectation of RVs

$E[XY] = E[X]E[Y]$ (**only if** X and Y are independent)

5.4 Variance of RV: $\sigma^2 = var(X) = E[(X - \mu)^2]$

Discrete: $\sigma^2 = \sum_x (x - \mu)^2 f(x)$

Continuous: $\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx$

Useful Formula: $\sigma^2 = E[X^2] - \mu^2$

5.5 Covariance of RVs: $\sigma_{XY} = cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$

Discrete: $\sigma_{XY} = \sum_x \sum_y (x - \mu_X)(y - \mu_Y)f(x, y)$

Continuous: $\sigma_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y)f(x, y)dxdy$

Useful Formula: $\sigma_{XY} = E[XY] - E[X]E[Y] = E[XY] - \mu_X\mu_Y$

5.6 Variance of Linear Combinations of RVs (Not Independent)

$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma^2_{XY}$

$\sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y - 2ab\sigma^2_{XY}$

$\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma^2_{XY}$

$\sigma^2_{aX+bY-cZ} = a^2\sigma^2_X + b^2\sigma^2_Y + c^2\sigma^2_Z + 2ab\sigma^2_{XY} - 2ac\sigma^2_{XZ} - 2bc\sigma^2_{YZ}$

5.7 Variance of Linear Combinations of RVs (Independent): $\sigma^2_{XY} = 0$

5.8 Correlation Coeff. of RVs: $\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{E[XY]-E[X]E[Y]}{\sigma_X\sigma_Y};\ -1 \leq \rho_{XY} \leq 1$

6. Common Discrete Distributions

6.1 Binomial

PMF: $b(x; n, p) = [nCx] \cdot p^x(1 - p)^{n-x};\ x = 0, 1, 2, \dots$

Mean: $E[X] = \mu = np$ | **Variance:** $Var(X) = \sigma^2 = np(1 - p)$

6.2 Multinomial

PMF: $f(x_1, \dots, x_m; p_1, \dots, p_m, n) = [nC(x_1, \dots, x_m)]p_1^{x_1} \cdots p_m^{x_m};\ \sum_i x_i = n;\ \sum_{i=1}^{m} p_i = 1$

Mean: $E[X_i] = \mu_i = np_i$ | **Variance:** $Var(X_i) = \sigma^2_i = np_i(1 - p_i)$

6.3 Hypergeometric

PMF: $h(x; N, n, k) = \frac{[kCx][(N-k)C(n-x)]}{[NCn]};\ max\{0, n - (N - k)\} \leq x \leq min\{n, k\}$

Mean: $E[X] = \mu = \frac{nk}{N}$ | **Variance:** $Var(X) = \sigma^2 = \frac{N-n}{N-1} \cdot n \cdot \frac{k}{N}\left(1 - \frac{k}{N}\right)$

6.4 Negative Binomial

PMF: $b^*(x; k, p) = [(x - 1)C(k - 1)]p^k(1 - p)^{x-k};\ x \geq k,\ x = k, k + 1, \dots$

Mean: $E[X] = \mu = \frac{k}{p}$ | **Variance:** $Var(X) = \sigma^2 = \frac{k(1-p)}{p^2}$

6.5 Geometric

PMF: $g(x; p) = p(1 - p)^{x-1};\ x \geq 1,\ x = 1, 2, 3, \dots$

Mean: $E[X] = \mu = \frac{1}{p}$ | **Variance:** $Var(X) = \sigma^2 = \frac{1-p}{p^2}$

6.6 Poisson

PMF: $p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!},\ x \geq 0,\ x = 0, 1, 2, \dots$

Mean: $E[X] = \mu = \lambda t$ | **Variance:** $Var(X) = \sigma^2 = \lambda t$

6.7 Poisson Approximation for Binomial

$$\lim_{n \to \infty, p \to 0} b(x; n, p) = p(x; \lambda t = np)$$

6.8 Chebyshev's Theorem (Discrete or Continuous RVs)

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2},\ k > 0$$

7. Common Continuous Distributions

7.1 Uniform

PDF: $f(x; A, B) = \begin{cases} \frac{1}{B-A}; & A \leq x \leq B \\ 0, & otherwise \end{cases}$

Mean: $E[X] = \mu = \frac{A+B}{2}$ | **Variance:** $Var(X) = \sigma^2 = \frac{(B-A)^2}{12}$

7.2 Normal (Gaussian)

PDF: $n(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$ **for** $-\infty < x < \infty$

Mean: $E[X] = \mu$ | **Variance:** $E\left[(X-\mu)^2\right] = \sigma^2$

7.3 Standard Normal

PDF: $n(x;\mu=0,\sigma=1)$

CDF: $P(X\le x) = \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} exp\left\{-\frac{t^2}{2}\right\}dt$

Probability of X: $P(A\le X\le B) = \Phi(B)-\Phi(A)$

7.35 Standardized Variable Transformation

Standardized Normal Variable: $Z = \frac{X-\mu}{\sigma}$

Transformation of CDF from X to Z: $P(X\le x) = P\left(Z\le\frac{x-\mu}{\sigma}\right)$

Probability of X: $P(A\le X\le B) = P\left(\frac{A-\mu}{\sigma}\le Z\le\frac{B-\mu}{\sigma}\right) = \Phi\left(\frac{B-\mu}{\sigma}\right) - \Phi\left(\frac{A-\mu}{\sigma}\right)$

Relationship b/w PDFs of X and Z: $n(x;\mu,\sigma) = \frac{n\left(\frac{x-\mu}{\sigma};0,1\right)}{\sigma}$

7.4 Normal Approximation of Binomial PMF

$P(X\le x) \approx P\left(Z\le\frac{x+0.5-np}{\sqrt{np(1-p)}}\right)$ | $P(x_1\le X\le x_2) \approx P\left(\frac{x_1-0.5-np}{\sqrt{np(1-p)}}\le Z\le\frac{x_2+0.5-np}{\sqrt{np(1-p)}}\right)$

$P(x_1 < X < x_2) \approx P\left(\frac{x_1+0.5-np}{\sqrt{np(1-p)}}\le Z\le\frac{x_2-0.5-np}{\sqrt{np(1-p)}}\right)$

Mean: $\mu = np$ s.t. $np\ge5$ | **Variance:** $\sigma^2 = np(1-p)$ $s.t.$ $n(1-p)\ge5$

7.5 Gamma

Gamma Function: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx;\ \alpha>0$

- $\Gamma\left(\alpha=\frac{1}{2}\right) = \sqrt{\pi}$ | $\Gamma(n) = (n-1)!\ for\ n\in N$

PDF: $f(x;\alpha,\beta) = \begin{cases}\frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}, x>0\\ 0, otherwise\end{cases}$ $; x,\alpha,\beta>0$

Mean: $E[X]=\mu=\alpha\beta$ | **Variance:** $Var(X)=\sigma^2=\alpha\beta^2$

7.6 Exponential

PDF: $f(x;\beta) = \frac{1}{\beta}exp\left(-\frac{x}{\beta}\right);\ x\ge0$

Mean: $E[X]=\mu=\beta$ | **Variance:** $Var(X)=\sigma^2=\beta^2$

Memoryless Nature: $P(X\ge t_0+t\mid X\ge t_0) = P(X\ge t)$

Relationship to Poisson: $\beta = 1/\lambda$ (i.e. $rate=\lambda=1/\beta$)

7.6 Chi-Squared

PDF: $f(x;\alpha=\nu/2,\beta=2) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)}x^{\frac{\nu}{2}-1}exp\left(-\frac{x}{2}\right);\ x>0$

Mean: $E[X]=\mu=\nu$ | **Variance:** $Var(X)=\sigma^2=2\nu$

8. Functions of RVs

8.1 Transformation of Discrete RVs from X to Y for a PMF

Transformation (Discrete/Cont.): $Y=u(X)\Rightarrow X=u^{-1}(Y)$ given $f(x)$

PMF of X: $f(x)$

PMF of Y: $g(y) = f\left(u^{-1}(y)\right)$

8.2 Transformation of Continuous RVs from X to Y for a PDF

PDF of X: $f(x)$

CDF of Y: $G(y) = P(Y\le y) = \int_{-\infty}^{x=u^{-1}(y)} f(t)dt$

PDF of Y: $g(y) = \frac{d}{dy}G(y) = f\left(u^{-1}(y)\right)\cdot\left|\frac{du^{-1}(y)}{dy}\right|$

8.3 Transformation of Continuous RVs for Partitioned Sets X To Y

Transformation: $Y=u(X)\Rightarrow X_1=u_1^{-1}(Y),..., X_k=u_k^{-1}(Y)$ given $f(x_i)$

PDF: $g(y) = \sum_{i=1}^{k} f(u_i^{-1}(y))\cdot|J_i|$, where $J_i = \frac{\partial x_i}{\partial y}$

8.4 Linear Combination of RVs from X to Y

Transformation: $Y=aX\Rightarrow X=\frac{Y}{a}$ given $f(x)$

PMF (Discrete X): $h(y) = f\left(\frac{y}{a}\right)$

PDF (Continuous X): $h(y) = \frac{1}{|a|}f\left(\frac{y}{a}\right)$

8.5 PMFs and PDFs for Z=X+Y of Independent RVs X and Y with Distributions f(x) and g(y)

Transformation: $X=W,\ Y=Z-W$

Discrete: $h(z) = \sum_{w=-\infty}^{\infty} f(w)g(z-w)$

Continuous: $h(z) = \int_{-\infty}^{\infty} f(w)g(z-w)dw$

9. Moments

9.1 rth Moment: $\mu_r' = E\left[X^r\right]$

-Discrete X: $\mu_r' = \sum x^r f(x)$

-Continuous X: $\mu_r' = \int_{-\infty}^{\infty} x^r f(x)dx$

9.2 Relationship of Mean and Variance to rth moment

Mean: $\mu=\mu_1'$ | **Variance:** $\sigma^2 = \mu_2' - \mu^2$

9.3 Moment Generating Function (MGF): $M_X(t) = E\left[e^{tX}\right]$

-Discrete X: $M_X(t) = \sum_x e^{tx}f(x)$

-Continuous X: $M_X(t) = \int_{-\infty}^{\infty} e^{tx}f(x)dx$

9.4 Relationship between rth moment about the origin and MGF (Continuous & Discrete)

$$\mu_r' = \frac{d^rM_X(t)}{dt^r}\Big|_{t=0}$$

9.5 MGF of a Normal RV: $M_X(t) = e^{\mu t+\frac{t^2\sigma^2}{2}}$

9.6 MGF of Geometric: $M_X(t) = \frac{pe^t}{1-qe^t};\ t<ln(q)$

9.7 MGF of Poisson: $M_X = exp\left(\mu\left(e^t-1\right)\right)$

9.8 MGFs for Linear Combinations of RVs from MGF of X to MGF of Y=aX (Continuous & Discrete)

$$M_{Y=aX}(t) = M_X(at)$$

9.9 MGF of a Z=X+Y and Z=aX+bY of Two Independent RVs X and Y:

$$M_{Z=X+Y}(t) = M_X(t)M_Y(t) \mid M_{Z=aX+bY}(t) = M_X(at)M_Y(bt)$$

10. Sampling

10.1 Random Sample

Sample Data: $x_1,..., x_n$

10.2 Sample Mean

Empirical Value of the Mean: $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$

RV Representing the Sample Mean: $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$

10.3 Median

Data in increasing order $x_{(1)},..., x_{(n)}$

$$\text{Median} = \begin{cases}\frac{x_{(n/2)}+x_{(n/2+1)}}{2} & \text{if } n \text{ is even,}\\ x_{((n+1)/2)} & \text{if } n \text{ is odd.}\end{cases}$$

10.4 Sample Variance for s or S

$$s^2 = \frac{1}{n-1}\sum_{i=1}^n\left(x_i-\bar{x}\right)^2 = \frac{1}{n(n-1)}\left[n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2\right]$$

10.5 Sample Standard Deviation for s or S

$$s=\sqrt{s^2} = \sqrt{\frac{1}{n-1}\sum_{i=1}^n\left(x_i-\bar{x}\right)^2} = \sqrt{\frac{1}{n(n-1)}\left[n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2\right]}$$

10.6 Sample Range

$$Range = max(x_i) - min(x_i)$$

10.7 Box-and-Whisker Plots (Box Plot): Data in increasing order.

Location of Q_i: $(n+1)\times\frac{i}{4},\ for\ i=1,2,3.$

Interquartile Range: $IQR = Q_3 - Q_1$

Median: Q_2

Lower Whisker: $Q_1 - 1.5\cdot IQR$ | **Upper Whisker:** $Q_3+1.5\cdot IQR$

11. Sampling Distributions

11.1 Central Limit Theorem for $\bar X$ with known σ:

Average Standardized Variable: $Z_n = \frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}}$

Transformation: $P(\bar X\le\#); P(\bar X\ge\#); P(\#\le\bar X\le\#) \Rightarrow \frac{\#-\mu}{\sigma/\sqrt{n}}$

Statistics: $\sigma_{\bar X} = \frac{\sigma}{\sqrt n}$ | $\mu_{\bar X}=\mu$ | $\sigma_{\bar X} = s$

11.2 Comparing the Mean of 2 Independent Samples with known σ

Statistic: $Z = \frac{\left(\bar X_1-\bar X_2\right)-\left(\mu_1-\mu_2\right)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$

Trans.: $P\left(\bar X_1-\bar X_2\le\#\right); P\left(\bar X_1-\bar X_2\ge\#\right); P\left(\#\le\bar X_1-\bar X_2\le\#\right)\Rightarrow\frac{\#-\left(\mu_1-\mu_2\right)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$

11.3 Chi-Squared Distribution

Statistic: $\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^n\left(X_i-\bar X\right)^2$ with $\nu=n-1$

Transformation: $P(s^2\le\#); P(s^2\ge\#); P(\#\le s^2\le\#)\Rightarrow\frac{\#(n-1)}{\sigma^2}$

Useful Formula: $\chi^2\sigma^2 = sum(x_i-\bar x)^2$

11.4 T-Distribution with n≤30:

Statistic: $T=\frac{\bar X-\mu}{S/\sqrt n}$ with $\nu=n-1$

Transformation: $P(\bar X\le\#); P(\bar X\ge\#); P(\#\le\bar X\le\#) \Rightarrow \frac{\#-\mu}{S/\sqrt n}$

11.5 F-Distribution

Statistic: $F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$ with $\nu_1=n_1-1$ & $\nu_2=n_2-1$

Transformation: $P\left(\frac{S_1^2}{S_2^2}\le\#\right); P\left(\frac{S_1^2}{S_2^2}\ge\#\right); P\left(\#\le\frac{S_1^2}{S_2^2}\le\#\right)\Rightarrow\#\cdot\frac{\sigma_1^2}{\sigma_2^2}$

12. Quantiles

12.2 Points for Q-f plot:

Data in increasing order $x_{(1)},..., x_{(n)}$: $\left(f_i=\frac{i-3/8}{n+1/4},\ x_i\right); i=1,2,...,n$

12.3 Relation of Quantile Function to CDF

$$q(f) = F^{-1}$$

12.4 Normal Quantile $q_{\mu,\sigma}(f)$:

$$q_{\mu,\sigma}(f) = \mu+\sigma\left\{4.91\left[f^{0.14}-(1-f)^{0.14}\right]\right\}$$

$$q_{0,1}(f) = 4.91\left[f^{0.14}-(1-f)^{0.14}\right]$$

12.6 Points for a Normal Q-Q Plot:

$$\left(q_{0,1}(f_i),\ x_i\right)$$

13. Point Estimates

13.0 Notation: $\theta=\mu/\sigma$ | $\hat\theta = \bar x/s$ | $\hat\Theta = \bar X/S$

13.1 Unbiased Estimator: $E[\hat\Theta]=\theta$

- **Mean:** $E[\bar X]=\mu$ | **Variance:** $E[S^2]=\sigma^2$

13.2 Efficient Unbiased Estimators: $Var\left(\hat\Theta_{efficient/unbiased}\right) < Var\left(\hat\Theta_i\right)$

14. Types of Intervals

14.1 Two-Sided CIs for Mean with Known σ^2

Statistic: $Z=\frac{\bar X-\mu}{\sigma/\sqrt n}$

Confidence Level: $1 - \alpha$

Probability:

$$P\left(-z_{\alpha/2} \le Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right) = 1 - \alpha$$

$$P\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

Two-Sided Intervals:

$$\hat{\Theta}_L = \bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = \hat{\Theta}_U$$

Standard Error: $SE = \frac{\sigma}{\sqrt{n}}$

Margin Error: $ME \le z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = z_{\alpha/2}SE$

Sample Size: $n = \left(\frac{z_{\alpha/2}\sigma}{ME}\right)^2$

14.2 One-Sided Confidence Intervals with Known σ^2

Statistic: $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$

Probability for Upper Bound Only:

$$P\left(Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \ge -z_{\alpha}\right) = 1 - \alpha$$

$$P\left(\mu \le \bar{X} + z_{\alpha}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

Lower and Upper Bound for One-Sided Interval:

$$\hat{\Theta}_L = \bar{x} - z_{\alpha}\frac{\sigma}{\sqrt{n}} \mid \hat{\Theta}_U = \bar{x} + z_{\alpha}\frac{\sigma}{\sqrt{n}}$$

14.3 Two-Sided CIs for Mean with Unknown σ^2

Statistic: $T = \frac{\bar{X}-\mu}{S/\sqrt{n}}$ with $\nu = n - 1$

Probability:

$$P\left(-t_{\alpha/2} \le T = \frac{\bar{X}-\mu}{S/\sqrt{n}} \le t_{\alpha/2}\right) = 1 - \alpha$$

$$P\left(\bar{X} - t_{\alpha/2}\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{\alpha/2}\frac{S}{\sqrt{n}}\right) = 1 - \alpha$$

Two-Sided Interval:

$$\hat{\Theta}_L = \bar{X} - t_{\alpha/2}\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{\alpha/2}\frac{S}{\sqrt{n}} = \hat{\Theta}_U$$

14.4 Prediction Intervals of Next Data Point with Known Variance σ^2

Statistic: $Z = \frac{X_0 - \bar{X}}{\sigma\sqrt{1+\frac{1}{n}}}$

Probability:

$$P\left(-z_{\alpha/2} < Z = \frac{x_0 - \bar{x}}{\sigma\sqrt{1+\frac{1}{n}}} < z_{\alpha/2}\right) = 1 - \alpha$$

$$P\left(\bar{x} - z_{\alpha/2}\sigma\sqrt{1 + \frac{1}{n}} \le x_0 \le \bar{x} + z_{\alpha/2}\sigma\sqrt{1 + \frac{1}{n}}\right) = 1 - \alpha$$

Two-Sided Interval:

$$\hat{\Theta}_L = \bar{x} - z_{\alpha/2}\sigma\sqrt{1 + \frac{1}{n}} \le x_0 \le \bar{x} + z_{\alpha/2}\sigma\sqrt{1 + \frac{1}{n}} = \hat{\Theta}_U$$

14.5 Two-Sided Tolerance Intervals

Confidence Level: $100(1 - \gamma)\%$

Coverage Level: $100(1 - \alpha)\%$

Probability: $P(\bar{x} \pm ks) = 1 - \gamma$

Two-Sided Interval: $\bar{x} \pm ks$

14.6 Two-Sided Confidence Intervals with Two Samples

C1: Difference between two means with known σ

Probability:

$$P\left(-z_{\alpha/2} \le Z = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \le z_{\alpha/2}\right) = 1 - \alpha$$

$$P\left(\left(\bar{x}_1 - \bar{x}_2\right) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \left(\mu_1 - \mu_2\right) < \left(\bar{x}_1 - \bar{x}_2\right) + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right) = 1 - \alpha$$

Two-Sided Interval:

$$\left(\bar{x}_1 - \bar{x}_2\right) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \left(\mu_1 - \mu_2\right) < \left(\bar{x}_1 - \bar{x}_2\right) + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

C2: Difference between two means with unknown (but equal) σ

Statistic: $T = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$ with $\nu = n_1 + n_2 - 2$

Sample Size-Weighted Average: $S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$

Probability:

$$P\left(-t_{\alpha/2} \le T = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \le t_{\alpha/2}\right) = 1 - \alpha$$

$$P\left(\left(\bar{x}_1 - \bar{x}_2\right) - t_{\alpha/2}s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \le \mu_1 - \mu_2 \le \left(\bar{x}_1 - \bar{x}_2\right) + t_{\alpha/2}s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right) = 1 - \alpha$$

Two-Sided Interval:

$$\left(\bar{x}_1 - \bar{x}_2\right) - t_{\alpha/2}s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} < \mu_1 - \mu_2 < \left(\bar{x}_1 - \bar{x}_2\right) + t_{\alpha/2}s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

C3: Difference between two means with unknown (and different) σ.

Statistic: $T' = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$ with $\nu \approx \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\left(s_1^2/n_1\right)^2/(n_1 - 1) + \left(s_2^2/n_2\right)^2/(n_2 - 1)}$

Probability:

$$P\left(-t_{\alpha/2} \le T' = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \le t_{\alpha/2}\right) \approx 1 - \alpha$$

$$P\left(\left(\bar{x}_1 - \bar{x}_2\right) - t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \le \mu_1 - \mu_2 \le \left(\bar{x}_1 - \bar{x}_2\right) + t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right) \approx 1 - \alpha$$

Two-Sided Interval:

$$\left(\bar{x}_1 - \bar{x}_2\right) - t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \le \mu_1 - \mu_2 \le \left(\bar{x}_1 - \bar{x}_2\right) + t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

14.7 Confidence Intervals for Paired Observations:

Statistic: $T = \frac{\bar{d} - \mu_d}{s_d/\sqrt{n}}$ with $\nu = n - 1$

Difference: $D_i = X_{1,i} - X_{2,i}$

Population Mean: $\mu_D = \mu_1 - \mu_2$ | **Sample Mean:** $\bar{d} = \bar{x}_1 - \bar{x}_2$

Variance: $Var(D_i) = \sigma^2_{X_{1,i}} + \sigma^2_{X_{2,i}} - 2Cov(X_{1,i}, X_{2,i})$

Sample Variance: $S_D^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(D_i - \bar{D}\right)^2$

Probability:

$$P\left(t_{\alpha/2} < T = \frac{\bar{d} - \mu_d}{s_d/\sqrt{n}} < t_{\alpha/2}\right) = 1 - \alpha$$

$$P\left(\bar{d} - t_{\alpha/2}\frac{s_d}{\sqrt{n}} < \mu_d < \bar{d} + t_{\alpha/2}\frac{s_d}{\sqrt{n}}\right) = 1 - \alpha$$

Two-Sided Interval:

$$\bar{d} - t_{\alpha/2}\frac{s_d}{\sqrt{n}} < \mu_D < \bar{d} + t_{\alpha/2}\frac{s_d}{\sqrt{n}}$$

14.8 Confidence Intervals Estimating a Proportion:

Statistic: $Z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}$

Point Estimator of p (proportion): $\hat{P} = \frac{X}{n}$

Sample Proportion: $\hat{p} = \frac{x}{n}$

Mean: $\mu_{\hat{P}} = p$

Variance: $\sigma^2_{\hat{P}} = \frac{p(1-p)}{n}$

Probability:

$$P\left(-z_{\alpha/2} < Z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} < z_{\alpha/2}\right) = 1 - \alpha$$

For n is large:

$$P\left(\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = 1 - \alpha$$

Two-Sided Interval:

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

14.85 Difference in Proportion Interval (Extracurricular):

$$\left(\hat{p}_1 - \hat{p}_2\right) - z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} \le p_1 - p_2 \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

14.9 Choice of Sample Size

1st Method (Given $\hat{p}$):

$$n = \frac{z^2_{\alpha/2}\hat{p}(1-\hat{p})}{\delta^2}$$

- **Margin Error:** $\delta = z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

2nd Method (Not given $\hat{p}$):

$$n \ge \frac{z^2_{\alpha/2}}{4\delta^2}$$

14.9 Two-Sided Confidence Intervals for Variance

Statistic: $\chi^2 = \frac{(n-1)S^2}{\sigma^2}$ *with* $\nu = n - 1$

Probability:

$$P\left(\chi^2_{1-\alpha/2} < \chi^2 = \frac{(n-1)S^2}{\sigma^2} < \chi^2_{\alpha/2}\right) = 1 - \alpha$$

$$P\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}\right) = 1 - \alpha$$

Two-Sided Interval:

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

15. Maximum Likelihood Estimation

15.0 Likelihood Function:

$$L\left(x_1, ..., x_n; \theta\right) = f\left(x_1, \theta\right) \cdots f(x_n, \theta) = \prod_{i=1}^{n} f\left(x_i; \theta\right)$$

15.1 Maximum Likelihood Estimator: $\hat{\theta}: \frac{\partial[ln(L)]}{\partial\theta}\Big|_{\theta=\hat{\theta}} = 0$

15.2 Common Product Notation Rules

1. $\prod_{i=1}^{n} a = a \cdot ... \cdot a = a^n$
2. $\prod_{i=1}^{n} ab = \prod_{i=1}^{n} a \cdot \prod_{i=1}^{n} b = a^n \cdot b^n$
3. $\prod_{i=1}^{n} e^{x_i} = e^{x_1} \cdots e^{x_n} = e^{\sum_{i=1}^{n} x_i}$
4. $\prod_{i=1}^{n} x_i^b = \left(\prod_{i=1}^{n} x_i\right)^b$
5. $ln\left(\prod_{i=1}^{n} x_i\right) = ln\left(x_1 \cdots x_n\right) = \sum_{i=1}^{n} ln\left(x_i\right)$

15.3 Common Summation Notation Rules

1. $\sum_{i=1}^{n} a = an$
2. $\sum_{i=1}^{n} a_i + b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

15.4 Logarithm Rules

1. $ln(ab) = ln(a) + ln(b)$
2. $ln\left(\frac{a}{b}\right) = ln(a) - ln(b)$
3. $ln\left(a^n\right) = nln(a)$

16. Type 1 & 2 Error (Two-Sided)

16.0 Probability of Type I Error (False Positive) [i.e. Level of Sig.]: μ_{H_0}

$$\alpha = 1 - P\left(X_L < \bar{x} < X_U\right) \Rightarrow \alpha = 1 - P\left(Z_L < Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < Z_U\right)$$

Lower: $\alpha = P\left(Z < Z_L\right)$ | **Upper:** $\alpha = P\left(Z > Z_U\right) = 1 - P\left(Z < Z_U\right)$

16.1 Probability of a Type II Error (False Negative): $\mu = \mu_{H_1}$

$$\beta = P\left(X_L < \bar{x} < X_U\right) \Rightarrow \beta = P\left(Z_L < Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < Z_U\right)$$

Lower: $\beta = P\left(Z > Z_L\right)$ | **Upper:** $\beta = P\left(Z < Z_U\right)$

16.2 Statistical Power: $1 - \beta$

17. Hypothesis Testing Basis

17.1 Different Type of Hypotheses:

Null Hypothesis: H_0: $\theta = \theta_0$

Alternative Hypothesis (2-sided): H_1: $\theta \neq \theta_0$

Alternative Hypothesis (Upper 1-sided): H_1: $\theta > \theta_0$

Alternative Hypothesis (lower 1-sided): H_1: $\theta < \theta_0$

17.2.1 Critical Reject Region for Hypothesis (Symmetric): $X = Z, T$

2-Sided: $\alpha = Crit.\ Region$: $\{X: X \leq -x_{\alpha/2}\ and\ X \geq x_{\alpha/2}\}$

Lower 1-Sided: $\alpha = Crit.\ Region$: $\{X: X \leq -x_{\alpha}\}$

Upper 1-Sided: $\alpha = Crit.\ Region$: $\{X: X \geq x_{\alpha}\}$

17.2.2 P-Value (Symmetric): $X = Z, T,\ x = z, t$

2-Sided: $P\ Value = 2P(X > |x|)$

Upper/Lower 1-Sided: $P\ Value = P(X > |x|)$

17.2.3 Determining Outcome of Hypothesis (Symmetric): $x = z, t$

2-Sided:

Reject H_0: (1) $P\ Value \leq \alpha$; (2) $\frac{P\ Value}{2} \leq \frac{\alpha}{2}$; (3) $\left|x_{p-value}\right| > \left|x_{\alpha/2}\right|$

Fail to Reject H_0: (1) $P\ Value > \alpha$; (2) $\frac{P\ Value}{2} > \frac{\alpha}{2}$; (3) $\left|x_{p-value}\right| < \left|x_{\alpha/2}\right|$

1-Sided:

Reject the Null: (1) $P\ Value \leq \alpha$; (2) $\left|x_{p-value}\right| > \left|x_{\alpha}\right|$

Fail to Reject H_0: (1) $P\ Value > \alpha$; (2) $\left|x_{p-value}\right| < \left|x_{\alpha}\right|$

17.3.1 Critical Reject Region for Hypothesis (Asymmetric): $X = \chi^2, f$

2-Sided: $\alpha = Crit.\ Region$: $\{X: X \leq x_{1-\alpha/2}\ and\ X \geq x_{\alpha/2}\}$

Lower 1-Sided: $\alpha = Crit.\ Region$: $\{X: X \leq x_{1-\alpha}\}$

Upper 1-Sided: $\alpha = Crit.\ Region$: $\{X: X \geq x_{\alpha}\}$

17.3.2 P-Value (Asymmetric): $X = \chi^2, f,\ x = \chi^2, f$

2-Sided: $\frac{P\ Value}{2} = P(X > x)$; $\frac{P\ Value}{2} = P(X < x) = 1 - P(X > x)$

Lower 1-Sided: $P\ Value = P(X < x) = 1 - P(X > x)$

Upper 1-Sided: $P\ Value = P(X > x)$

17.3.3 Determining Outcome of Hypothesis (Asymmetric): $x = \chi^2, f$

2-Sided:

Reject H_0: (1) $\frac{P\ Value}{2} \leq \frac{\alpha}{2}$; (2) $\left[x_{1-\alpha/2} < x < x_{\alpha/2}\right]^{-1}$

Fail to Reject H_0: (1) $\frac{P\ Value}{2} > \frac{\alpha}{2}$; $x_{1-\alpha/2} < x < x_{\alpha/2}$

1-Sided:

Reject H_0: (1) $P\ Value \leq \alpha$; $(UP)\ x_{R-tail} > x_{\alpha}$; $(LOW)\ x_{L-tail} < x_{1-\alpha}$

Fail to Reject H_0: (1) $P\ Value > \alpha$; $(UP)\ x_{R-tail} < x_{\alpha}$;

$(LOW)\ x_{L-tail} > x_{1-\alpha}$

18. Hypothesis Testing

18.0 Hypothesis Testing for One Mean with known σ: $\theta = \mu \mid \theta_0 = \mu_0$

Test Statistic (if CLT holds): $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$

18.1 Hypothesis Testing for One Mean with unknown σ**:**

$\theta = \mu \mid \theta_0 = \mu_0$

Test Statistic: $T = \frac{\bar{X}-\mu_0}{s/\sqrt{n}}$ with $\nu = n - 1$

18.2 Hypothesis Testing for Two Means: $\theta = \mu_1 - \mu_2 \mid \theta_0 = d_0$

C1: Known σ_1 **and** σ_2

Test Statistic: $Z = \frac{\left(\bar{x}_1-\bar{x}_2\right)-d_0}{\sqrt{\sigma_1^2/n_1+\sigma_2^2/n_2}}$

C2: Unknown and $\sigma_1 = \sigma_2$

Test Statistic: $t = \frac{\left(\bar{x}_1-\bar{x}_2\right)-d_0}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$ with $\nu = n_1 + n_2 - 2$

- $s_p^2 = \frac{s_1^2(n_1-1)+s_2^2(n_2-1)}{n_1+n_2-2}$

C3: Unknown and $\sigma_1 \neq \sigma_2$

Test Statistic: $t' = \frac{\left(\bar{x}_1-\bar{x}_2\right)-d_0}{\sqrt{s_1^2/n_1+s_2^2/n_2}}$ with $\nu \approx \frac{\left(s_1^2/n_1+s_2^2/n_2\right)^2}{\left(s_1^2/n_1\right)^2/\left(n_1-1\right)+\left(s_2^2/n_2\right)^2/\left(n_2-1\right)}$

18.3 H-Testing for Paired Observations: $\theta = \bar{d} = \mu_1 - \mu_2 \mid \theta_0 = d_0$

Null Hypothesis: H_0: $\mu_d = \mu_1 - \mu_2 = 0\ (usually)$

Test Statistic: $t = \frac{\bar{d}-d_0}{s_d/\sqrt{n}}$ with $\nu = n - 1$

- $s_d^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(d_i - \bar{d}\right)^2$

18.4 Hypothesis Testing for One Variance: $\theta = \sigma^2 \mid \theta_0 = \sigma_0^2$

Test Statistic: $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$ with $\nu = n - 1$

18.5 Hypothesis Testing for Two Variances:

Null Hypothesis: H_0: $\sigma_1^2 = \sigma_2^2$

Alternative Hypothesis (two-sided): H_1: $\sigma_1^2 \neq \sigma_2^2$

Alternative Hypothesis (one-sided): H_1: $\sigma_1^2 > \sigma_2^2\ (UP)\ or\ \sigma_1^2 < \sigma_2^2\ (LOW)$

Test Statistic: $f = \frac{s_1^2}{s_2^2}$ with $\nu_1 = n_1 - 1$ & $\nu_2 = n_2 - 1$

Reciprocal Relationship for Left Tail: $f_{1-\alpha}\left(v_1, v_2\right) = \frac{1}{f_{\alpha}\left(v_2, v_1\right)}$

19. Goodness of Fit

$$\chi^2 = \sum_{i=1}^{k}\frac{\left(o_i-e_i\right)^2}{e_i} \text{ with } \nu = k - 1 \text{ \& } e_i = nP(i) \text{ \& } o_i \geq 5\ \forall\ i$$

P-Value (upper 1-sided): $P\ value = P\left(\chi^2 > \chi_0^2\right)$

20. Linear Regression

20.0 True Regression Line: $\mu_{Y|x} = \beta_0 + \beta_1 x$

20.1 Simple Linear Regression Model (RV):

$$Y = \beta_0 + \beta_1 x + \epsilon$$

- $E[\epsilon] = 0 \mid var(\epsilon) = \sigma^2$

20.2 Fitted Regression Line: $\hat{y} = b_0 + b_1 x$

20.3 Residual Error in Regression

Given $\left\{\left(x_i, y_i\right);\ i = 1, 2, ..., n\right\}$:

$$e_i = y_i - \hat{y}_i;\ i = 1, 2, ..., n$$

20.4 Useful Relationship for a Single Data Point:

$$y_i = b_0 + b_1 x_i + e_i$$

20.5 Estimating the Regression Coefficients b_1 & b_0:

$$b_0 = \frac{\sum_{i=1}^{n} y_i - b_1 \sum_{i=1}^{n} x_i}{n} = \bar{y} - b_1 \bar{x}$$

$$b_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)}{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}$$

20.6 Mean and Variance of Estimator B_1:

$$\mu_{B_1} = E\left[B_1\right] = \beta_1 \mid \sigma_{B_1}^2 = \frac{\sigma^2}{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}$$

20.7 Mean and Variance of Estimator B_0:

$$\mu_{B_0} = E\left[B_0\right] = \beta_0 \mid \sigma_{B_0}^2 = \left(\frac{\sum_{i=1}^{n} x_i^2}{n\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}\right)\sigma^2$$

21. Analysis of Linear Regression

21.0 Sums of Error

$$S_{xx} = \sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2 = \sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n} = \left(\sum_{i=1}^{n} x_i^2\right) - n\left(\bar{x}\right)^2$$

$$S_{yy} = \sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2 = \sum_{i=1}^{n} y_i^2 - \frac{\left(\sum_{i=1}^{n} y_i\right)^2}{n} = \left(\sum_{i=1}^{n} y_i^2\right) - n\left(\bar{y}\right)^2$$

$$S_{xy} = \sum_{i=1}^{n}\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right) = \sum_{i=1}^{n} x_i y_i - \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n} = \left(\sum_{i=1}^{n} x_i y_i\right) - n\bar{x}\bar{y}$$

21.1 Sum of Squared Errors (SSE)

$$SSE = S_{yy} - b_1 S_{xy}$$

21.2 Unbiased Estimator for σ^2

$$s^2 = \frac{SSE}{n-2} = \frac{\sum_{i=1}^{n}\left(y_i-\hat{y}_i\right)^2}{n-2} = \frac{S_{yy}-b_1 S_{xy}}{n-2}$$

21.3 Confidence Interval for the Slope

Statistic: $T = \frac{B_1-\beta_1}{S/\sqrt{S_{xx}}} = \frac{B_1-\beta_1}{\sqrt{\frac{S_{yy}-b_1 S_{xy}}{(n-2)S_{xx}}}}$ with $\nu = n - 2$

Two-Sided Interval:

$$b_1 - t_{\alpha/2}\frac{s}{\sqrt{S_{xx}}} \leq \beta_1 \leq b_1 + t_{\alpha/2}\frac{s}{\sqrt{S_{xx}}}$$

21.4 Confidence Interval for the Intercept

Statistic: $T = \frac{B_0-\beta_0}{S\cdot\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{nS_{xx}}}}$ with $\nu = n - 2$

Two-Sided Interval:

$$b_0 - t_{\alpha/2}\frac{s}{\sqrt{nS_{xx}}}\sqrt{\sum_{i=1}^{n} x_i^2} \leq \beta_0 \leq b_0 + t_{\alpha/2}\frac{s}{\sqrt{nS_{xx}}}\sqrt{\sum_{i=1}^{n} x_i^2}$$

21.5 Hypothesis Testing for the Slope: $\theta = \beta_1 \mid \theta_0 = \beta_{1_0}$

Statistic: $t = \frac{b_1-\beta_{1_0}}{s/\sqrt{S_{xx}}}$ with $\nu = n - 2$

21.6 Hypothesis Testing for the Intercept: $\theta = \beta_0 \mid \theta_0 = \beta_{0_0}$

Statistic: $t = \frac{b_0-\beta_{0_0}}{s\cdot\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{nS_{xx}}}}$ with $\nu = n - 2$

21.7 Coefficient of Determination

$$R^2 = 1 - \frac{SSE}{SST};\ 0 \leq R^2 \leq 1$$

- **Error Sum of Squares:** $SSE = \sum_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2$
- **Total Corrected Sum of Squares:** $SST = \sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2$

21.8 Interpolation

$$\frac{Y_{unknown}-Y_1}{X_{unknown}-X_1} = \frac{Y_2-Y_1}{X_2-X_1}$$