ESC195: Cheat Sheet

0. General:

$$ln(a \cdot b) = ln(a) + ln(b)$$

$$ln(\frac{a}{b}) = ln(a) - ln(b)$$

Tips for Finite Series:

If the series is finite and you wanna manipulate it:

$$\sum_{i=1}^{n} a_n = \sum_{i=0}^{n-1} a_{n+1}$$

1. Trigonometric Identities:

1.1 Pythagorean Theorem Identities:

$$sin^2\theta + cos^2\theta = 1$$

$$tan^2\theta + 1 = sec^2\theta$$

$$1 + cot^2\theta = csc^2\theta$$

1.5 Double-Angle Identities:

$$sin\ 2x = 2sin\ x\ cos\ x \text{ or } \frac{1}{2}sin2x = sinxcosx$$

$$cos\ 2x = cos^2x - sin^2x \mid cos\ 2x = 2cos^2x - 1 \mid cos\ 2x = 1 - 2sin^2x$$

1.6 Half-Angle Identities:

$$cos^2x = \frac{1+cos\ 2x}{2}$$

$$sin^2x = \frac{1-cos\ 2x}{2}$$

1.7 Product Identities:

$$sin\ x\ cos\ y = \frac{1}{2}[sin(x - y) + sin(x + y)]$$

$$cos\ x\ cos\ y = \frac{1}{2}[cos(x - y) + cos(x + y)]$$

$$sin\,x\,sin\,y = \frac{1}{2}[cos(x - y) - cos(x + y)]$$

2. Hyperbolic Functions:

2.1 Definition of Hyperbolic Functions:

$$sinh\,x = \frac{e^x - e^{-x}}{2} \mid csch\,x = \frac{1}{sinh\,x}$$

$$cosh\,x = \frac{e^x + e^{-x}}{2} \mid sech\,x = \frac{1}{cosh\,x}$$

$$tanh\,x = \frac{sinh\,x}{cosh\,x} \mid coth\,x = \frac{cosh\,x}{sinh\,x}$$

2.2 Hyperbolic Identities:

$$sinh(-x) = -sinh(x)$$

$$cosh^2(x) - sinh^2(x) = 1$$

$$sinh(x + y) = sinh(x)cosh(y) + sinh(y)cosh(x)$$

$$cosh(x + y) = cosh(x)cosh(y) + sinh(x)sinh(y)$$

2.3 Hyperbolic Derivatives:

$$\frac{d}{dx}(sinh(x)) = cosh(x) \mid \frac{d}{dx}(csch(x)) = -csch(x)coth(x)$$

$$\frac{d}{dx}(cosh(x)) = sinh(x) \mid \frac{d}{dx}(sech(x)) = -sech(x)tanh(x)$$

$$\frac{d}{dx}(tanh(x)) = sech^2(x) \mid \frac{d}{dx}(coth(x)) = -csch^2(x)$$

2.4 Inverse Hyperbolic Functions:

$$sinh^{-1}(x) = ln(x + \sqrt{x^2 + 1}),\ x \epsilon R$$

$$cosh^{-1}(x) = ln(x + \sqrt{x^2 - 1}),\ x \geq 1$$

$$tanh^{-1}(x) = \frac{1}{2}ln(\frac{1+x}{1-x}),\ -1 < x < 1$$

3. L'Hôpital's Rule:

Transform to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ to use L'Hôpital's Rule:

Indeterminate form	Conditions	Transformation to $0/0$	Transformation to ∞/∞
$\frac{0}{0}$	$\lim_{x\to c} f(x) = 0,\ \lim_{x\to c} g(x) = 0$	—	$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{1/g(x)}{1/f(x)}$
$\frac{\infty}{\infty}$	$\lim_{x\to c} f(x) = \infty,\ \lim_{x\to c} g(x) = \infty$	$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{1/g(x)}{1/f(x)}$	—
$0 \cdot \infty$	$\lim_{x\to c} f(x) = 0,\ \lim_{x\to c} g(x) = \infty$	$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} \frac{f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} \frac{g(x)}{1/f(x)}$
$\infty - \infty$	$\lim_{x\to c} f(x) = \infty,\ \lim_{x\to c} g(x) = \infty$	$\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} \frac{1/g(x) - 1/f(x)}{1/(f(x)g(x))}$	$\lim_{x\to c}(f(x) - g(x)) = \ln \lim_{x\to c} \frac{e^{f(x)}}{e^{g(x)}}$
0^0	$\lim_{x\to c} f(x) = 0^+, \lim_{x\to c} g(x) = 0$	$\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{\ln f(x)}{1/g(x)}$
1^∞	$\lim_{x\to c} f(x) = 1,\ \lim_{x\to c} g(x) = \infty$	$\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{\ln f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{g(x)}{1/\ln f(x)}$
∞^0	$\lim_{x\to c} f(x) = \infty,\ \lim_{x\to c} g(x) = 0$	$\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{\ln f(x)}{1/g(x)}$

4. Evaluation Techniques for Limits:

- **Continuous Function** → Plug in the a, where x → a
- **Continuous Functions and Composition** →
  $f(x)$ *is continuous at* b *and* $\lim_{x \to a} g(x) = b$ *then* $\lim_{x \to a} f(g(x)) = f(b)$
- **Factor and cancel**
- **Rationalize numerator/denominator**
- **Combine rational expressions**
- **L'Hospital's Rule** → If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ then take the derivative of $f(x)$ *and* $g(x)$ and plug in a.
- **Polynomials at Infinity** → $\lim_{x \to \pm\infty} \frac{p(x)}{q(x)}$. Factor largest power of x in $q(x)$ out of both $p(x)$ and $q(x)$ then use the **property:** $\lim_{x \to \pm\infty} \frac{1}{x^r} = 0$
- **Piecewise Function** → Compute the two one sided limits and see if they equal each other.

5. Techniques of Integration:

5.1 Symmetry:

Odd: $If\ f\ is\ odd\ on\ [-\ a, a]\ then\ \int_{-a}^{a} f(x)dx\ =\ 0$

Even: $If\ f\ is\ even\ on\ [-\ a, a]\ then\ \int_{-a}^{a} f(x)dx\ =\ 2\int_{0}^{a} f(x)dx$

5.2 Completing the Square:

$$ax^2 + bx + c \rightarrow a(x + d)^2 + e$$

$$d = \frac{b}{2a} \mid e = c - \frac{b^2}{4a}$$

5.4 Useful Common Integrals and Derivatives (*+C*):

$$\frac{d}{dx}(x) = 1$$
$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$
$$\frac{d}{dx}(\mathbf{e}^x) = \mathbf{e}^x$$
$$\frac{d}{dx}(\ln(x)) = \frac{1}{x},\ x > 0$$
$$\frac{d}{dx}(\ln|x|) = \frac{1}{x},\ x \neq 0$$
$$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a},\ x > 0$$

$$\int k\,dx = k\,x + c$$
$$\int x^n\,dx = \tfrac{1}{n+1}x^{n+1} + c, n \neq -1$$
$$\int x^{-1}\,dx = \int \tfrac{1}{x}\,dx = \ln|x| + c$$
$$\int \tfrac{1}{ax+b}\,dx = \tfrac{1}{a}\ln|ax+b| + c$$
$$\int \ln u\,du = u\ln(u) - u + c$$
$$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$$

$$\int \cos u\,du = \sin u + c$$
$$\int \sin u\,du = -\cos u + c$$
$$\int \sec^2 u\,du = \tan u + c$$
$$\int \sec u \tan u\,du = \sec u + c$$
$$\int \csc u \cot u du = -\csc u + c$$
$$\int \csc^2 u\,du = -\cot u + c$$

$$\int \tan u\,du = \ln|\sec u| + c$$
$$\int \sec u\,du = \ln|\sec u + \tan u| + c$$
$$\int \tfrac{1}{a^2+u^2}\,du = \tfrac{1}{a}\tan^{-1}\left(\tfrac{u}{a}\right) + c$$
$$\int \tfrac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\tfrac{u}{a}\right) + c$$

$$\int usin(u)du = sin(u) - ucos(u) + C$$

$$\int ucos(u)du = cos(u) + usin(u) + C$$

$$\int u^2 sin(u)du = 2usin(u) - (u^2 - 2)cos(u) + C$$

$$\int u^2 cos(u)du = 2ucos(u) + (u^2 - 2)sin(u) + C$$

$$\int sin^n x = \frac{-1}{n} sin^{n-1} xcosx + \frac{n-1}{n}\int sin^{n-2} xdx$$

$$\int cos^n x = \frac{1}{n} cos^{n-1} xsinx + \frac{n-1}{n}\int cos^{n-2} xdx$$

5.5 Integration Using Trigonometric Identities:

Strategy for Evaluating $\int sin^m xcos^n xdx$

a. If the power of cos is odd, save one cos factor and use $sin^2 x + cos^2 x = 1$, then substitute.
b. If the power of sine is odd, save one sine factor and use $sin^2 x + cos^2 x = 1$, then substitute.
c. If powers of both sine and cosine are even, use half-angle identities.

Strategy for Evaluating $\int tan^m xsec^n xdx$

d. If the power of sec is even, save a factor of $sec^2 x$ and use $tan^2 x + 1 = sec^2 x$, then substitute.
e. If the power of tangent is odd, save a factor of $secxtanx$ and use $tan^2 x + 1 = sec^2 x$, then substitute.

Strategy for Evaluating $\int cot^m(x)csc^n(x)dx$

If the power of cotangent is odd, save a factor of $cotxcscx$, and use Pythagorean Identity, then substitute $u = cscx$

5.6 Improper Integrals:

Tips:

- Be extremely careful if there is a discontinuity in between the bounds, then you have to separate it.
  - **Eg.** $\int_{-1}^{3} \frac{dx}{x^2} \neq [\frac{-1}{x}]_{-1}^{3}$. The integrals equals $\int_{-1}^{0} \frac{dx}{x^2}$ & $\int_{0}^{3} \frac{dx}{x^2}$
- Know what the graphs look like

Important Example:

$$\int_a^{\infty} \frac{dx}{x^p} = \lim_{b \to \infty} \int_a^b \frac{dx}{x^p} = \lim_{b \to \infty} [\frac{1}{1-p} b^{-p+1} - \frac{a^{-p+1}}{1-p}]$$

- $\frac{a^{1-p}}{1-p}$ for $p > 1$ converges
- Diverges for $p < 1$

Comparison Test:

Let f, g be continuous functions and $0 \leq f(x) \leq g(x)$ where $x \in [a, \infty)$,.

- If $\int_a^{\infty} g(x)dx\ converges$, so does $\int_a^{\infty} f(x)dx$
- If $\int_a^{\infty} f(x)dx$ diverges, so does $\int_a^{\infty} g(x)dx$

6. Applications of Integration:

6.1 Arc Length of a Curve:

$$s = \int_a^b \sqrt{1 + [\frac{dy}{dx}]^2} dx$$

6.2 Surface Area of a Surface of Revolution:

6.2.1 X-Axis:

$$A = \int_a^b 2\pi f(x)\sqrt{1 + [\frac{dy}{dx}]^2} dx$$

$$A = \int_a^b 2\pi y\sqrt{1 + [\frac{dx}{dy}]^2} dy$$

6.2.2 Y-Axis:

$$A = \int_a^b 2\pi f(y)\sqrt{1 + [\frac{dx}{dy}]^2} dy$$

$$A = \int_a^b 2\pi x\sqrt{1 + [\frac{dy}{dx}]^2} dx$$

6.3 Force a Fluid Exerts on the Flat Wall of a Container:

$$F = \rho g \int_a^b h(y)L(y)dy$$

6.4 Moment of R about the y-axis and x-axis respectively:

$$M_y = \rho \int_a^b xf(x)dx \text{ and } M_x = \rho \int_a^b \frac{1}{2}[f(x)]^2 dx$$

6.5 Centroid of a Curve:

$$\overline{x} = \frac{1}{A}\int_a^b xf(x)dx$$

$$\bar{y} = \frac{1}{2A}\int_a^b (f(x))^2 dx$$

6.6 Centroid of Two Curves:

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]dx$$

$$\bar{y} = \frac{1}{2A}\int_a^b [(f(x))^2 - (g(x))^2]dx$$

6.7 Volume of Revolution using Pappus's Centroid Theorem:

$$V = 2\pi RA$$

Where R ($\bar{x}$ or $\bar{y}$) is the distance from the centroid to the axis of rotation and A is the area of the rotated region.

6.8 Surface Area of a Surface of Revolution:

$$A = 2\pi Rd$$

Where d is the arclength of the curve, and R ($\bar{x}$ or $\bar{y}$) is the distance from the centroid to the axis of rotation.

7. Parametric Equations

7.1 Derivative of Parametric Equations:

$$\frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} \; if \; \frac{dx}{dt} \neq 0$$

- When $\frac{dy}{dt} = 0$, there is a horizontal tangent.
- When $\frac{dx}{dt} = 0$, there is a vertical tangent.

7.2 Second Derivative of Parametric Equations:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$$

7.3 Area of Parametric Equations:

If $x = x(t)$ *and* $y = y(t)$, $t_1 \leq t \leq t_2$, then

$$A = \int_{t_1}^{t_2} y(t)x'(t)dt$$

Def'n: A curve is traversed in the +ve sense as t increases, if the enclosed area is on the left (CWW).

7.4 Arc Length

If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' *and* g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α *to* β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

7.5 Relation to Motion and Velocity:

$$s(t) = \int_{\alpha}^{t} \sqrt{(\frac{dx}{du})^2 + (\frac{dy}{du})^2} du = distance\ travelled$$

$$\frac{ds}{dt} = \sqrt{(\frac{dx}{du})^2 + (\frac{dy}{du})^2} = speed$$

7.6 Surface Area

X-Axis:

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

Y-Axis:

$$S = \int_{\alpha}^{\beta} 2\pi x\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$$

7.7 Graphing Parametric Curves:

1. Try converting to a cartesian equation and plot it normally (if it makes it easier)
2. Check for potential vertical tangents by setting $x'(t) = 0$
3. Check for potential horizontal tangents by setting $y'(t) = 0$
4. Find x and y intercepts by setting $x(t) = y(t) = 0$ and then using the t value you get for the other coordinate.
5. Look for periodicity in either x, y, or both.
6. Find the coordinate and the slope $\frac{dy}{dx}$ $at\ t = 0$ and at the endpoint $t = t_f$

7.8 Common Parametric Curves:

Cycloid:

$$x(\theta) = a(\theta - sin\theta) \,||\, y(\theta) = a(1 - cos\theta)$$

8. Polar Curves:

8.1 Relationship between Polar and Cartesian Coordinates:

$$x = rcos\theta, y = rsin\theta$$
$$r^2 = x^2 + y^2, tan\theta = \frac{y}{x}$$

- Be careful about the quadrant by making sure θ is correct.

8.2 Symmetry:

1. **Symmetry About X-Axis**

$$r(-\theta) = r(\theta)$$

2. **Symmetry About Y-Axis**

$$r(\pi - \theta) = r(\theta)$$

3. **Symmetry About Origin**

$$r(\pi + \theta) = r(\theta)$$

8.3 Intersection of Polar Curves:

Tips:

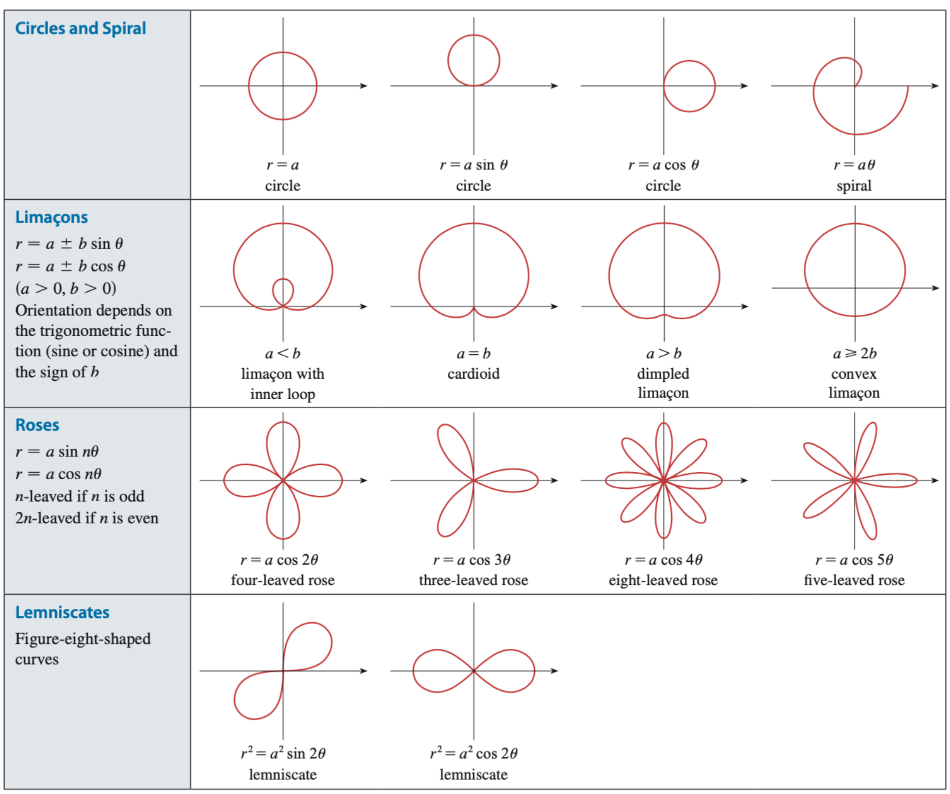
- Draw a picture
- Watch out for intersection of 0
- Watch out for intersections that are two times.
- 90/number with theta.

8.4 Tangents in Polar Coordinates:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

- When $\frac{dy}{d\theta} = 0$, horizontal tangent.
- When $\frac{dy}{d\theta} = 0$, vertical tangent.

8.5 Common Polar Curves:



<table>
<tr><th>Types</th><th>How to Graph</th></tr>
<tr><td>Circles
$r = acos\theta$
$r = asin\theta$
r=1</td><td>$a < 0$ indicates that the circle will directed towards the negative x or y-axis
$cos\theta \rightarrow$ x-axis, while $sin\theta \rightarrow$ y-axis
a is the diameter
For graphs with just a number, it indicates the radius not the diameter</td></tr>
<tr><td>Limacons
$r = a \pm bsin\theta$
$r = a \pm bcos\theta$</td><td>+ → oriented towards positive axis
- → oriented towards negative axis
$cos\theta \rightarrow$ x-axis, while $sin\theta \rightarrow$ y-axis
Plot two x and y intercepts.
a indicates the intercepts that are the same
$b - a$ is the closer intercept
$b + a$ is the farther intercept</td></tr>
<tr><td>Roses</td><td>$acosn\theta \rightarrow$ The first leaf is on the x-axis.
Even = 2n leaves
Odd = n leaves
a is the length of leaf
$cos\theta \rightarrow 0$ for first leaf, $\frac{2\pi}{n}$ for interval (n is number of leaves)
$sin\theta \rightarrow \frac{\pi}{2n}$ for first leaf, $\frac{2\pi}{n}$ for interval (n is number of leaves)</td></tr>
<tr><td>Lemniscates
$r^2 = a^2cos^2\theta$
$r^2 = a^2sin^2\theta$</td><td>$cos\theta \rightarrow$ x-axis, while $sin\theta \rightarrow$ angled between the x and y axis.
a indicates left and right bounds.</td></tr>
<tr><td>General</td><td>To find where r=0, just set the equation to 0 so you can know the angles
Think of r and theta as a vector with theta being the direction and r being the magnitude.</td></tr>
</table>

8.6 Area of a Polar Region

$$A = \int_{a}^{b} \frac{1}{2} [r(\theta)]^2 d\theta$$

8.7 Area Between Polar Curves:

$$A = \frac{1}{2}\int_{\alpha}^{\beta}[{r_1}^2 - {r_2}^2]d\theta$$

- Be careful because it has to be an outer function (r_1)and inner function (r_2), not the addition of two areas.
- Whichever curve is closer to the origin is r_2.

8.8 Arc Length of a Curve with Polar Equation:

$$L = \int_{a}^{b} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$$

9. Theorems:

9.1 Cauchy's Mean Value Theorem:

Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a,b), and $g'(x) \neq 0$ for all x in (a,b). Then there is a number c in (a,b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

10. Sequences:

10.1.1 Definition of Limit of a Sequence:

If $\lim_{n \to \infty} a_n = L$ exists, then the sequence **converges**. Otherwise, the sequence **diverges.**

10.1.2 Precise Definition of Limit of a Sequence:

A sequence of $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L$$

iff for every $\varepsilon > 0$, there exists an integer N such that if $n > N$ then $|a_n - L| < \varepsilon$

10.2 Definition $\{a_n\}$ is (monotonic):

- Increasing iff $a_n < a_{n+1}$
- Non-decreasing $a_n \leq a_{n+1}$
- Decreasing $a_n > a_{n+1}$
- Non-decreasing $a_n \geq a_{n+1}$

10.3 Definition of Bounded Sequence:

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \text{ for all } n \geq 1$$

A sequence is **bounded below** if there is a number m such that

$$m \leq a_n \; for \; all \; n \geq 1$$

If a sequence is bounded above and below, then it is called a **bounded sequence.**

10.4 Definition of Convergent & Divergent:

If a sequence has a limit it is said to be **convergent** otherwise **divergent.**

1) If a sequence is convergent, it is bounded.
2) If a sequence is unbounded, it is divergent.
3) A bounded sequence is not necessarily convergent.

10.5 Theorem for Sequences

10.5.1 Theorem:

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

10.5.2 Monotonic Sequence Theorem (for large n):

Every bounded, monotonic sequence is convergent. A bounded non-decreasing sequence converges to its least upper bound. A bounded non-increasing sequence converges to its greatest lower bound.

10.5.3 Properties of Sequences

Given $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$

1) $\lim_{n\to\infty} (a_n + b_n) = L + M$
2) $\lim_{n\to\infty} \alpha a_n = \alpha L, \alpha \in R$
3) $\lim_{n\to\infty} a_n \cdot b_n = L \cdot M$
4) $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}, b \neq 0, M \neq 0$
5) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$

10.5.4 Pinching Theorem for Sequences

If for large n, $a_n \le b_n \le c_n$ and if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$

10.5.5 Theorem

Given $\lim_{n\to\infty} c_n = c$. If f is continuous at c, then: $\lim_{n\to\infty} f(c_n) = f(c)$

10.6 Important Limits:

1. For $x > 0$, $\lim_{n \to \infty} x^{\frac{1}{n}} = 1$
2. If $\|x\| < 1$ then $\lim_{n \to \infty} x^{n} = 0$
3. $\lim_{n \to \infty} \frac{1}{n^{\alpha}} = 0,\ \alpha > 0$
4. a) $\lim_{n \to \infty} \frac{x^{n}}{n!} = 0,\ x \in R$ b) $\lim_{n \to \infty} \frac{n!}{n^{n}} = 0$
5. $\lim_{n \to \infty} \frac{\ln n}{n} = 0$
6. $\lim_{n \to \infty} n^{\frac{1}{n}} = 1$
7. $\lim_{n \to \infty} (1 + \frac{x}{n})^{n} = e^{x}$

11. Series:

11.1 Partial Sum:

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ...$ Let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + ... + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n \to \infty} s_n = s$ exists as a real number, then

the series $\sum a_n$ is called **convergent** and we write

$$\sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series.

If the sequence $\{s_n\}$ is divergent, then the series is called **divergent.**

11.2 Types of Series:

11.2.1 Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 = \frac{a}{1-r}$$

- $|r| < 1\ converges$
- $|r| \geq 1\ diverges$
- $s_n = \frac{a(1-r^n)}{1-r}$

11.2.2 P-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

- $p > 1\ converges$
- $0 < p \leq 1\ divergent$
- p=1 is the **harmonic series**

11.2.3 Binomial Series

If k is any real number and $|x| < 1$, then

$$(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots$$

Where $\binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}, n = 1, 2, \ldots, k$

11.3 Absolutely Convergent & Conditionally Convergent

If $\sum |a_n|$ converges, then $\sum a_n$ is **absolutely convergent.**

If $\sum a_n$ converges, but $\sum |a_n|$ does not, then $\sum a_n$ is **conditionally convergent.**

11.4 Theorems for Series

11.4.1 Theorem:

If $\sum_{k=0}^{\infty} a_k = L$ and $\sum_{k=0}^{\infty} b_k = M$ then $\sum_{k=0}^{\infty} (a_k + b_k) = L + M$

If $\sum_{k=0}^{\infty} a_k = L$ then $\sum_{k=0}^{\infty} \alpha a_k = \alpha L, \alpha \in R$

11.4.2 Theorem:

$\sum_{n=0}^{\infty} a_n$ converges iff $\sum_{n=j}^{\infty} a_n$ converges, $j =+ \ ve \ integer.$

$\sum_{n=j}^{\infty} a_n = L - (a_0 + a_1 + a_2 +... + a_{j-1})$

11.4.3 Theorem:

If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$

11.4.4 Theorem:

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

11.5 Estimates:

Estimate	Info

Remainder Estimate for IT	Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for x≥n and $\sum a_n$ is convergent. If $R_n = s - s_n$, then $$\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_{n}^{\infty} f(x)dx$$ **Lower and Upper Bound:** $$s_n + \int_{n+1}^{\infty} f(x)dx \le s \le s_n + \int_{n}^{\infty} f(x)dx$$
AS Estimate Theorem	If $s = \sum(-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies 1) $b_{n+1} \le b_n$ 2) $\lim_{n\to\infty} b_n = 0$ Then, $$\vert R_n \vert = \vert s - s_n \vert \le b_{n+1}$$

11.6 Convergence Tests

Test	Formula/Conditions
Divergence Test	If $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=0}^{\infty} a_n$ diverges.
Integral Test	If f is continuous, decreasing, and +ve on $[k, \infty]$, then: $\sum_{n=k}^{\infty} f(n)$ converges iff $\int_{k}^{\infty} f(x)dx$ converges
DCT	Given $\sum a_n$, $\sum b_n$; $a_n > 0$, $b_n > 0$

	1) If $\sum b_n$ is converged, and if $a_n \leq b_n$ for all n sufficiently large, then $\sum a_n$ converges. 2) If $\sum b_n$ is divergent, and if $a_n \geq b_n$ for all n sufficiently large, then $\sum a_n$ diverges
LCT	Given $\sum a_n$, $\sum b_n$; $a_n > 0$, $b_n > 0$ 1) If $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$ then both series converge or diverge. 2) If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and if $\sum b_n$ converges, then $\sum a_n$ diverges. 3) If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and if $\sum b_n$ diverges, then $\sum a_n$ diverges
AST	Let $\{a_n\}$ be a sequence of +ve numbers If $a_{n+1} < a_n$ (decreasing) for all n and $\lim_{n \to \infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.
Ratio Test	Given $\sum a_n$, $a_n \geq 0$. If $(a_n)^{\frac{1}{n}} \to p$ as $n \to \infty$, then: 1) If $p < 1$ then $\sum a_n$ converges 2) If $p > 1$ then $\sum a_n$ diverges 3) If $p = 1$ inconclusive
Root Test	Given $\sum a_n$, $a_n > 0$. If $\lim_{n \to \infty} \lvert\frac{a_{n+1}}{a_n}\rvert = \lambda$ then

	1) If $\lambda < 1$ $\sum a_n$ converges 2) If $\lambda > 1$ $\sum a_n$ diverges 3) If $\lambda = 1$ inconclusive

11.7 Power Series

11.7.1 Definition of Power Series:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$$

11.7.2 Theorem for Power Series Convergence

For a power series $\sum_{n=0}^{\infty} c_n (x - a)^n$, there are 3 possibilities wrt convergence:

1. The series converges only when $x = a$
2. The series converges for all x
3. The series converges in some interval $|x - a| < R$
   - $R = radius\ of\ convergence$
   - Intervals of Convergence (Test endpoints)

11.7.3 Representation of Functions as Power Series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ... = \frac{1}{1-x} = f(x) \text{ for } |x| < 1$$

11.7.4 Theorem: Term by Term Differentiation & Integration:

Consider the power series $\sum c_n (x - a)^n$ w/ $R = R_0 > 0$, then

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + ... = \sum_{n=0}^{\infty} c_n (x - a)^n$$

is differentiable (and therefore continuous) on $(a - R_0, a + R_0)$, and:

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x - a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n(x - a)^n\right]$$

$$\int\left[\sum_{n=0}^{\infty} c_n(x - a)^n\right] = \sum_{n=0}^{\infty} \int c_n(x - a)^n dx$$

11.7.5 Multiplication and Division of Power Series

Multiplication:

- Take each term in one series and multiply it by every term in the other series.
- Then adding like terms together

Example: $\frac{e^x}{1-x} = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} +...)(1 + x + x^2 + x^3 +...)$

- We are going to multiply **each term** in the geometric series by **every term in e^x.**

1: $= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} +...$

x: $+ x + x^2 + \frac{x^3}{2} +...$

x^2: $+ x^2 + x^3 +...$

x^3: $+ x^3 +...$

$= 1 + 2x + \frac{5}{2}x^2 + \frac{16}{6}x^3 +...$

Division:

- Use long division with the expanded form of each series.

Example: $tanx = \frac{sinx}{cosx} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} +...}{1 - \frac{x^2}{2} + \frac{x^4}{4!} +...}$

Then perform long division: $1 - \frac{x^2}{2} + \frac{x^4}{4!} +... \overline{)\, x - \frac{x^3}{3!} + \frac{x^5}{5!}}$

11.6 Taylor and Maclaurin Series

11.6.1 Theorem:

If $f(x)$ has a power series representation about a:

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n, \ |x - a| < R$$

then the coefficients of the series are

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Taylor Series of f about a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n = f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \ldots$$

Maclaurin Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \ldots$$

11.6.2 Theorem:

If $f(x) = T_n(x) + R_n(x)$ and

$$\lim_{n \to \infty} R_n(x) = 0$$

for $|x - a| < R$ then f is equal to the sum of its Taylor series on $|x - a| < R$.

11.6.3 Taylor's Inequality:

If $|f^{(n+1)}(x)| \leq M \ for \ |x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!} \text{ for } |x - a| \leq d$$

11.6.4 Taylor's Theorem:

Given that f has $n + 1$ continuous derivatives on an open interval I containing a, then for all $x \in I$:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n(x)$$

Where $R_n(x) = \int_a^x f^{(n+1)}(t)(x - t)^n dt$

11.6.5 Error Estimation:

1. Alternating series $|R_n(x)| < |a_{n+1}|$
2. Taylor's formula $|R_n| < |\frac{M(x-a)^{n+1}}{(n+1)!}|$

11.6.6 Important Maclaurin Series and Their Radii of Convergence:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

11.7.0 Fourier Series

11.7.1 Definition of Fourier Series

If f is a piecewise continuous function on $[- L, L]$. Then the **Fourier series** of f is the series

$$a_0 + \sum_{n=1}^{\infty}(a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L}))$$

Where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

And, for n≥1,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos(\frac{n\pi x}{L})dx \mid b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin(\frac{n\pi x}{L})dx$$

- If $f(x)$ is odd, then $a_n = 0$
- If $f(x)$ is even, then $b_n = 0$

11.7.2 Fourier Convergence Theorem:

If f is a periodic function with period 2π and f and f' are piecewise continuous on $[- \pi, \pi]$, then the Fourier series is convergent. The sum of the Fourier series is equal to $f(x)$ at all numbers x where f is continuous. At the numbers x where f is discontinuous, the sum of the Fourier series is the average of the right and left limits, that is

$$\frac{1}{2}[f(x^+) + f(x^-)]$$

12. Vectors and the Geometry of Space

12.1 Vector Equation:

$$r = r_0 + tv$$

12.2 Parametric Equations for a Line

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

12.3 Symmetric Equation for a Line:

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- Note: If any of $a, b\ and/or\ c = 0$, then $x = x_0, y = y_0, z = z_0$ separate from the symmetric equations.

12.3 Scalar Equation of the Plane:

$$ax + by + cz + d = \text{where } d = -(ax_0 + by_0 + cz_0)$$

- $\hat{n} = (a, b, c)$
- $P_0(x_0, y_0, z_0)$ is a point in the plane.

12.4 Distance:

Between any point $P_1(x_1, y_1, z_1)$ in space and a plane given by $ax + by + cz + d = 0$

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

12.4 Quadric Surfaces Steps

1) Domain/Range
2) Intercepts - w/ coordinate axes
3) Traces - intersection with coordinate planes
4) Sections - intersection with other planes
5) Centre
6) Symmetry

7) Bounded/Unbounded

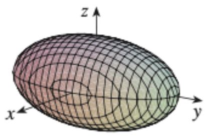
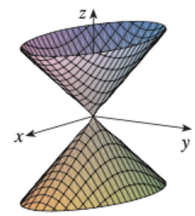
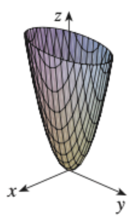
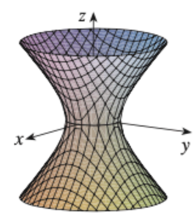
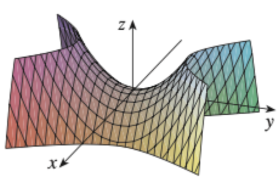
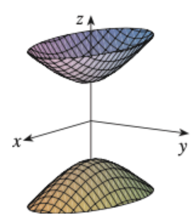
12.5 Projections - Curves of Intersections

C: (x, y, z) s.t. $z = f(x, y)$ and $z = g(x, y)$

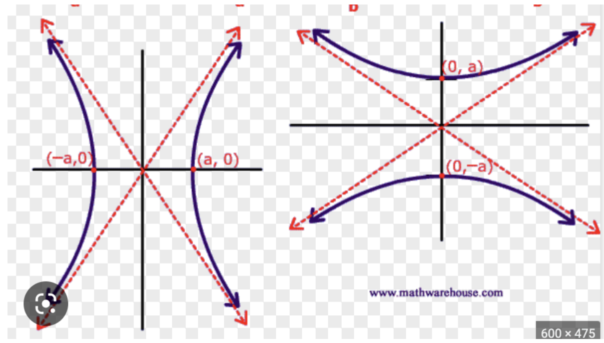
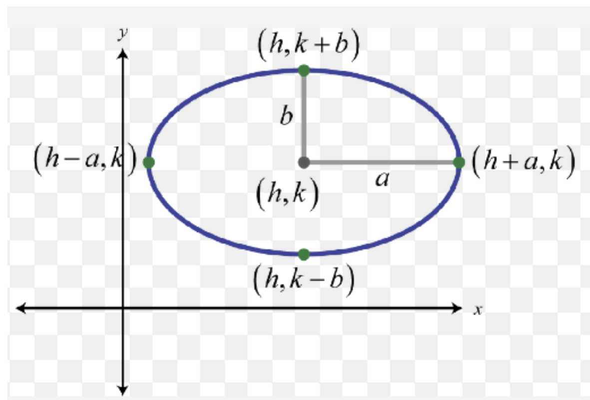
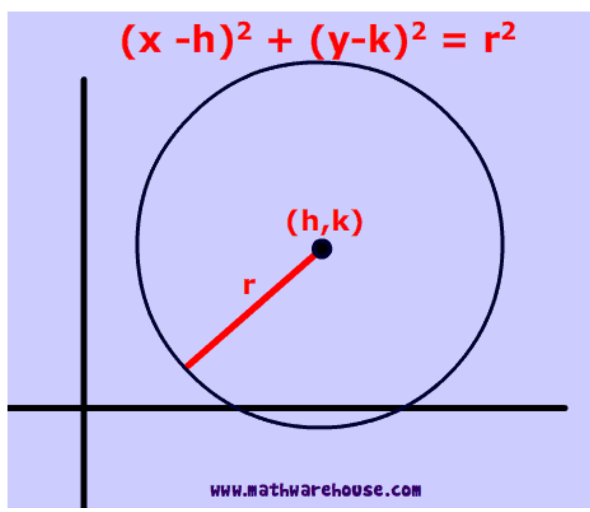
(x, y, z): $f(x, y) = g(x, y)$
$\Rightarrow$ *vertical cylinder*

$(x, y, z = 0)$: $f(x, y) = g(x, y)$
$\Rightarrow$ *projection*

12.6 Graphs of Quadric Surfaces:

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	**Cone** 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	**Hyperboloid of One Sheet** 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	**Hyperboloid of Two Sheets** 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

12.6.1 Graphs of Common 2D Shapes and Equations

Types:	Graphs:
Hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ (Hor. Trans. Axis) $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ (Ver. Trans Axis)	
Ellipses $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	
Circle $(x - h)^2 + (y - k)^2 = r^2$	

13. Vector Functions

13.1 Limits

Given $\overline{f}(t) \to \overline{L}$, $\overline{g}(t) \to \overline{M}$, $u(t) \to A$ as $t \to t_0$

Then:

1) $\overline{f}(t) + \overline{g}(t) \to \overline{L} + \overline{M}$
2) $\alpha\overline{f}(t) \to \alpha\overline{L}$
3) $u(t) \cdot \overline{f}(t) \to A \cdot \overline{L}$
4) $\overline{f}(t) \cdot \overline{g}(t) \to \overline{L} \cdot \overline{M}$
5) $\overline{f}(t) \times \overline{g}(t) \to \overline{L} \times \overline{M}$

Note: $\overline{f}(t)$ is continuous at t_0 if $\lim_{t \to t_0} \overline{f}(t) = \overline{f}(t_0)$

13.2 Differentiable Formulas:

3 Theorem Suppose **u** and **v** are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

13.3 Theorem:

If $|r(t)| = c$ $(a\ constant)$, then $r'(t)$ is orthogonal to $r(t)$ for all t.

13.4 Arc Length of a Vector Function

Suppose the curve has the vector equation $r(t) =< f(t), g(t), h(t) >$, $a \leq t \leq b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f', g', h' are continuous. If the curve is traversed exactly once as t increases from a to b, then the length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt$$

$$L = \int_a^b \sqrt{[\frac{dx}{dt}]^2 + [\frac{dy}{dt}]^2 + [\frac{dz}{dt}]^2}dt$$

$$L = \int_a^b |r'(t)|dt$$

13.5 Parameterizing a Curve with Respect to Arc Length:

Solve for t as a function of arc length: $t = t(s)$. Then the curve can be **re-parameterized** in terms of s by substituting for t: $\bar{r}(t) = \bar{r}(t(s))$.

13.6 Curvature for 2-D space curves (How quickly the curve changes):

$$\kappa = |\frac{d\phi}{ds}| = \frac{|\bar{T}'(t)|}{|\bar{r}'(t)|} = \frac{|\frac{d^2y}{dx^2}|}{(1+(\frac{dy}{dx})^2)^{\frac{3}{2}}} = \frac{|x'y''-y'x''|}{[(x')^2+(y')^2]^{\frac{3}{2}}}$$

13.7 Radius of Curvature:

The circle that best describes how C behaves near P

$$\rho = \frac{1}{\kappa}$$

13.8 Curvature for 3-D space curves:

$$\kappa = \left|\frac{d\overline{T}}{ds}\right| = \left|\left|\frac{\overline{T}'(t)}{\overline{r}'(t)}\right|\right| = \frac{||\overline{r}'(t) \times \overline{r}''(t)||}{||\overline{r}'(t)||^3}$$

13.9 Unit Tangent:

$$T(t) = \frac{\overline{r}'(t)}{||\overline{r}'(t)||}$$

13.10 Unit Normal:

$$\overline{N}(t) = \frac{\overline{T}'(t)}{||\overline{T}'(t)||}$$

13.11 Binormal Vector:

$$\overline{B}(t) = \overline{T} \times \overline{N}$$

- Helps to find the **osculating plane** which best contains a curve at a given point.

13.12 Normal Plane:

- **Normal vector is T**

13.13 Osculating Plane

The plane that comes closest to containing the part of the curve near the point P.

- **Normal vector is B**

13.14 Physics

13.14.1 Newton's 2nd Law of Motion

$$\overline{F}(t) = m\overline{r}''(t) = \overline{p}'(t)$$

13.14.2 Parametric Equation of Projectile Motion:

$$x = (v_0 cos\alpha)t \text{ and } y = (v_0 sin\alpha)t - \frac{1}{2}gt^2$$

13.14.3 Angular Momentum:

$$\overline{L} = \overline{r} \times \overline{p} = m\overline{r} \times \overline{v}$$
$$||\overline{L}|| = mrv$$

13.14.4 Definition of Torque:

$$\overline{\tau} = \overline{r} \times \overline{F}$$

13.14.5 Definition of Central Force:

$\overline{F}$ is a central force if $\overline{F}(t)$ is always parallel to $\overline{r}$.

13.14.6 Acceleration:

$$\overline{a} = a_T\widehat{T} + a_N\widehat{N}$$
$$a_T = \frac{r'(t)\cdot r''(t)}{|r'(t)|}$$
$$a_N = \frac{|r'(t)\times r''(t)|}{|r'(t)|}$$

14. Partial Derivatives

14.1 Level Curves:

The **level curves** of a function f of two variables are the curves with equations

$$f(x, y) = k$$

Where k is a constant (in the range of f).

14.2 Multivariable Limits and Continuity:

14.2.0 Tips for Multivariable Limits:

- ALWAYS TRY DIRECT SUBSTITUTION FIRST
- DRAW A PICTURE SECOND
- Multiply by conjugate
- Find different paths to find the limit to not exist.
- Squeeze theorem (need to be tightly bound)

14.2.1 Definition of the Limit of a Function of 2 Variables:

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b).

$\lim_{(x,y) \to (a,b)} f(x, y) = L$ iff for each $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that if $(x, y) \in D$ and

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \text{ then } |f(x, y) - L| < \varepsilon$$

14.2.2 Definition of the Limit of a Function of Several Variables:

Let f be a function whose domain includes the region arbitrarily close to, but not necessarily including $\overline{x}_0$.

$\lim_{\overline{x} \to \overline{x}_0} f(\overline{x}) = L$ iff for each $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t. if $0 < ||\overline{x} - \overline{x}_0|| < \delta$ then

$$|f(\overline{x}) - L| < \varepsilon$$

14.2.3 Showing That a Limit Does Not Exist:

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then

$$\lim_{(x,y) \to (a,b)} f(x, y) \text{ does not exist.}$$

14.2.4 Delta-Epsilon Steps

General Steps:

Prove $\lim\limits_{(x,y) \to (a,b)} f(x, y) = L$

1. **Starting:**

   $Given\ \varepsilon > 0,\ there\ exists\ a\ \delta > 0\ s.t.\ 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta,$
   then $|f(x, y) - L| < \varepsilon,$
2. Manipulate the ε equation to make it simpler and then see how you can manipulate delta to get epsilon.
3. Then write delta in terms of epsilon.
4. **Final:** Given $\varepsilon > 0$, choose $\delta =?$, then when $0 < |x - c| < \delta$, we have proved that $|f(x) - L| < \varepsilon$, therefore
5. Starting from the delta equation, go to the epsilon equation to check your answer.

14.2.5 Continuity

$If\ \lim\limits_{\bar{x} \to \bar{x}_0} f(\bar{x}) = f(\bar{x}_0)$, then f is continuous at $\bar{x}_0$.

$\Rightarrow \lim\limits_{x \to x_0} f(x, y_0) = f(x_0, y_0)$ and $\lim\limits_{y \to y_0} f(x_0, y) = f(x_0, y_0)$

14.2.6 Theorem: The Continuity of Composite Functions:

If g is continuous $\bar{x}_0$, and f is continuous at the number g($\bar{x}_0$), then $f(g(\bar{x}_0))$ is continuous at $\bar{x}_0$.

14.3 Partial Derivatives:

14.3.1 Definition of Partial Derivatives of f(x,y,z):

$$f_x = \lim_{h \to 0} \frac{f(x+h,y,z)-f(x,y,z)}{h}$$

$$f_y = \lim_{h \to 0} \frac{f(x,y+h,z)-f(x,y,z)}{h}$$

$$f_z = \lim_{h \to 0} \frac{f(x,y,z+h)-f(x,y,z)}{h}$$

14.3.2 Finding Partial Derivatives:

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

14.3.3 Clairaut's Theorem:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

On every open set on which f and its partials $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ are continuous.

$\Rightarrow$ *Three variables*:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y};\ \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z};\ \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}$$

14.4 Tangent Planes and Linear Approximations:

14.4.1 Linearization:

The equation for a tangent plane at the point $(a, b, f(a, b))$ represents the linearization of f at that point:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

14.4.2 Theorem:

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b)

14.4.3 Total Differential Equation:

For $z = f(x, y)$, the total differential, dz, in terms of the independent differentials dx and dy is:

$$dz = \nabla f(x, y) \cdot < dx, dy >$$

14.4.4 Equation for Tangent Line to Curve f(x,y)=C at (x_0,y_0):

$$\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0$$

- Gradient is perpendicular to tangent line

14.4.7 Equation of Tangent Plane to Level Surfaces:

Since $\nabla f(\hat{x})$ is perpendicular to the level surface at $\hat{x}_0$. If $f(x, y, z) = C$, then $the\ tangent\ plane\ to\ the\ level\ surfaces\ at\ \hat{x}_0$ is

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

If $f(x, y) = z,\ then\ f(x, y, z) = 0 = f(x, y) - z$. Therefore $\nabla f = < f_x, f_y, - 1 >$:

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

- Gradient is normal to the tangent plane.

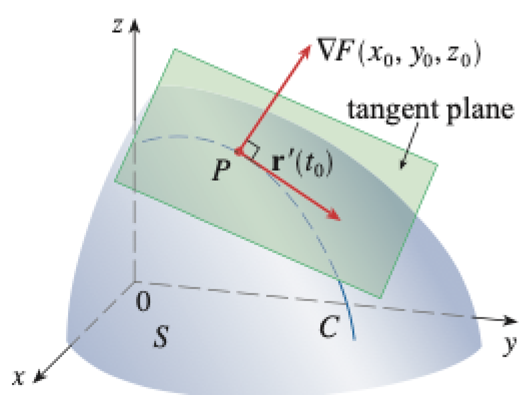


FIGURE 10

14.4.8 Equation of Normal Line:

If $\hat{r}(q) = \hat{x}_0 + \nabla f(\hat{x}_0)t$ where $\hat{x}_0 = (x_0, y_0, z_0)$

$$x = x_0 + tf_x$$
$$y = y_0 + tf_y$$
$$z = z_0 + tf_z$$

- Normal line is parallel to the gradient.

14.5 Chain Rule:

14.5.1 Chain Rule:

Suppose that u is a differentiable function of the n variables $x_1, x_2, ..., x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, ..., t_m$. Then u is a function of $t_1, t_2, ..., t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + ... + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

For each $i = 1, 2, ..., m$.

- **Tree diagrams can help**

14.5.2 Implicit Differentiation:

Suppose $F(x, y) = 0$ defines y implicitly as a differentiable function of x.

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

- You can manipulate the equation to move everything to one side.

Suppose z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

14.6 Directional Derivatives and the Gradient Vector:

14.6.1 Definition of Differentiability:

f is differentiable at $\widehat{x}$ iff there exists a vector $\triangledown f(\widehat{x})$ such that:

$$f(\widehat{x} + \widehat{h}) - f(\widehat{x}) = \triangledown f(\widehat{x}) \cdot \widehat{h} + o(\widehat{h})$$

$g(\widehat{h}) = o(\widehat{h})$ if $\lim_{\widehat{h} \to \widehat{0}} \frac{g(\widehat{h})}{|h|} = 0$ then $\triangledown f(\widehat{x})$ exists.

- $g(\widehat{h})$ *goes to* 0 *quicker than* $||\widehat{h}||$, therefore it's called $o(\widehat{h})$.
    - You have to get $g(\widehat{h})$ into a form that is a vector.

14.6.2 Definition of Gradient for Three Variables:

The **gradient** of f is the vector function $\triangledown f$ defined by

$$\triangledown f(x, y, z) = < f_x, f_y, f_z >$$

14.6.3 Theorem for Directional Derivative for Three Variables:

If f is a differentiable function, then f has a directional derivative at $\widehat{x}_0$ in the direction of any unit vector $\widehat{u}$ and

$$D_{\widehat{u}} f(\widehat{x}_0) = \triangledown f(\widehat{x}_0) \cdot \widehat{u}$$

- $||\widehat{u}||$ has magnitude of 1.

14.6.4 Theorem:

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\widehat{u}} f(\widehat{x})$ is

$$|\triangledown f(\widehat{x})|$$

and it occurs when $\widehat{u}$ has the same direction as the gradient vector $\triangledown f(\widehat{x})$

14.6.5 Properties of the Gradient Vector:

Let f be a differentiable function of two or three variables and suppose that $\nabla f(\widehat{x}) \neq \widehat{0}$.

- The directional derivative of f at $\widehat{x}$ in the direction of a unit vector $\widehat{u}$ is given by $D_u f(\widehat{x}) = \nabla f(\widehat{x}) \cdot \widehat{u}$
- $\nabla f(\widehat{x})$ points in the direction of maximum rate of increase of f at $\widehat{x}$, and that maximum rate of change is $|\nabla f(\widehat{x})|$.
- $\nabla f(x)$ is perpendicular to the level curve or level surface of f through $\widehat{x}$.

14.7 Maximum and Minimum Values:

14.7.1 Definition of Local Maximum and Minimum:

f has a local maximum at $\widehat{x}_0$ iff $f(\widehat{x}_0) \geq f(\widehat{x})$ for $\widehat{x}$ in some neighbourhood of $\widehat{x}_0$. f has a local minimum at $\widehat{x}_0$ iff $f(\widehat{x}_0) \leq f(\widehat{x})$ for $\widehat{x}$ in some neighbourhood of $\widehat{x}_0$.

14.7.2 Theorem:

If f has a local extreme value at $\widehat{x}_0$, then $\nabla f(\widehat{x}_0) = \widehat{0}$ or $\nabla f(\widehat{x}_0)$ DNE.

14.7.3 Definitions of Critical Points, Stationary Points, and Saddle Points:

- Points where $\nabla f = \widehat{0}$ or DNE are called critical points.
- Points where $\nabla f = \widehat{0}$ are called stationary points.
- Stationary points which are not local extrema are called saddle points.

14.7.4 Second Derivative Test:

For $f(x, y)$ with continuous 2nd order partials, and $\nabla f(x_0, y_0) = \widehat{0}$, then (x_0, y_0) is a critical point:

$$D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2$$

1) If $D < 0$, then (x_0, y_0) is called a saddle point.
2) If $D > 0$, $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum.
3) If $D > 0$, $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum.
4) If $D = 0$, then inconclusive.

14.7.5 Theorem:

If f is continuous on a bounded, closed set, then f takes on both an absolute minimum and an absolute maximum on that set.

14.7.6 Process for Finding Minimums and Maximums:

$\boxed{9}$ To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.

2. Find the extreme values of f on the boundary of D.

3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

- **Use single variable optimization when looking at the boundaries by subbing in the bound into f and doing the derivative and finding the max or min.**

14.8 Lagrange Multipliers:

- You still have to determine if its a maximum or minimum
- You don't have to find λ. At the bare minimum, you must find x_0, y_0, z_0.
- When you have to do the bounds to check, just do **single variable optimization.**

14.8.2 Lagrange Multipliers for Three Variables

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$]:

1. Find all values of $x, y, z,\ and\ \lambda$ such that
    1. Constraint: $g(x, y, z) = k$
    2. $f_x(x, y, z) = \lambda g_x(x, y, z)$
    3. $f_y(x, y, z) = \lambda g_y(x, y, z)$
    4. $f_z(x, y, z) = \lambda g_z(x, y, z)$
2. Evaluate f at all the points (x,y,z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

14.8.3 Lagrange Multipliers for Two Constraints

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ and $h(x, y, z) = c$

1. $g(\widehat{x_0}) = k$
2. $h(\widehat{x_0}) = c$
3. $f_x(\widehat{x_0}) = \lambda g_x(\widehat{x_0}) + \mu h_x(\widehat{x_0})$
4. $f_y(\widehat{x_0}) = \lambda g_y(\widehat{x_0}) + \mu h_y(\widehat{x_0})$
5. $f_z(\widehat{x_0}) = \lambda g_z(\widehat{x_0}) + \mu h_z(\widehat{x_0})$

14.9.1 Theorem:

Let f_x and f_y be functions of two variables, each continuously differentiable. The linear combination: $f_x \hat{i} + f_y \hat{j}$ is a gradient iff

$$f_{xy} = f_{yx}.$$

Let $f_x, f_y,\ and\ f_z$ be functions of three variables, each continuously differentiable.

The linear combination: $f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$ is a gradient iff,

$$f_{xy} = f_{yx}\ and\ f_{xz} = f_{zx}\ and\ f_{yz} = f_{zy}$$

14.9.2 Finding the Function From its Gradient.

1. Make sure thats its a gradient
2. Take the integral of each partial derivative treating all the other variables as constants (i.e. if $f_x \Rightarrow f(x, y)$ then $y\ and\ z\ is\ a\ constant$ so add $+\ \phi(y, z)$)
3. Combine all the terms that appear **ONCE.**
4. Add constant integration.

14.10 Theorem:

If, in the closed rectangle $x \in [a, b]$ and $y \in [c, d]$, the function $f(x, y)$ has a continuous derivative with respect to x, then for $x \in [a, b]$:

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\int_c^d f(x, y)dy = \int_c^d \frac{\partial f}{\partial x}dy$$

14.11 Theorem:

If

$$A(t) = \int_{x_1(t)}^{x_2(t)} f(x)dx,\ f(x)\geq 0$$

Then

$$\frac{dA}{dt} = f(x_2)\frac{dx_2}{dt} - f(x_1)\frac{dx_1}{dt}$$

14.12 Theorem: Leibnitz's Rule:

Given a region R in the x-y plane in which the functions $\phi_1(x)$ and $\phi_2(x)$ have continuous derivatives with respect to x, and in which $f(x, y)$ is continuously differentiable. If $F(x) = \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x, y)dy$ then

$$\frac{\partial F}{\partial x} = \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial f}{\partial x}dy + f(x, y = \phi_2(x))\frac{d\phi_2}{dx} - f(x, y = \phi_1(x))\frac{d\phi_1}{dx}$$