Uppal MAT185: Cheat Sheet

C0: A Postcard from Vector Space

0.1 Definition of Vector Space:

A **real vector space** is a set V together with two operations called *vector addition* and *scalar multiplication* such that the following axioms hold.

AI. For all vectors **x**,**y**$\in V$, $x + y \in V$
AII. For all vectors $x, y, z \in V$, $(x + y) + z = x + (y + z)$
AIII. There exists a unique vector $0 \in V$ with the property that $x + 0 = x$ for all vectors $x \in V$
AIV. For each vector $x \in V$, there exists a unique vector $- x \in V$ with the property that $x + (- x) = 0$

MI. For all vectors $x \in V$, and scalars $c \in \mathbb{R}$, $cx \in V$
MII. For all vectors $x \in V$, and scalars $c, d \in \mathbb{R}$, $(cd)x = c(dx)$
MIII. For all vectors $x \in V$, and scalars $c, d \in \mathbb{R}$, $(c + d)x = cx + dx$; and for all vectors $x, y \in V$, and scalars $c \in \mathbb{R}$, $c(x + y) = cx + cy$
MIV. For all vectors $x \in V$, $1x = x$.

0.2 Cancellation Theorem:

Let V be a vector space, and let $x, y, z \in V$. If

$$x + z = y + z$$

then

$$x = y$$

0.3 Proposition I:

Let V be a vector space, and let $x \in V$. Then $0x = 0$

0.4 Proposition II:

Let V be a vector space, and let $x \in V$. Then $(- 1)x =- x$

0.5 Proposition III:

Let V be a vector space, and let $x \in V$. Then $-x + x = 0$

0.6 Proposition IV:

Let V be a vector space, and let $x \in V$. Then $0 + x = x$

0.7 Commutativity:

For all vectors, $x, y \in V$

$$x + y = y + x$$

C1: The Subspace Homesick Blues

1.0 Definition of Subspace:

A **subspace** of a vector space V is a subset $W \subseteq V$ that is itself a vector space with the same operations of vector addition and scalar multiplication as in V.

1.1 Theorem, Subspace Test:

A non-empty subset W of a vector space V is subspace of V if and only if $cx + y \in W$ whenever $x, y \in W$, and $c \in \mathbb{R}$

1.2 Definition of Column Space of A:

For $A \in {}^m\mathbb{R}^n$, the column space of A

$$col\ A = \{Ax \in {}^m\mathbb{R} \mid x \in {}^n\mathbb{R}\}$$

is a subspace of ${}^m\mathbb{R}$

1.3.0 Definition of Intersection:

The **intersection** of two sets U and W is the set

$$U \cap W = \{x \mid x \in U \text{ and } x \in W\}$$

1.3.1 Theorem:

If U and W are subspaces of a vector space V, then $U \cap W$ is also a subspace of V.

1.4.0 Definition of Sum:

The **sum** of two sets U and W is the set

$$U + W = \{u + w \mid u \in U \text{ and } w \in W\}$$

1.4.1 Theorem:

If U and W are subspaces of a vector space V, then $U + W$ is also a subspace of V.

1.5 Definition of Linear Combination:

Let S be a non-empty subset of a vector space V. A **linear combination** of vectors in S is an expression of the form

$$c_1 s_1 + c_2 s_2 + \ldots + c_k s_k$$

where $s_1, s_2, \ldots, s_k \in S$, and $c_1, c_2, \ldots, c_k \in \mathbb{R}$

Note: A linear combination is **trivial** if $c_1 = c_2 = \ldots = c_k = 0$; otherwise it is **non-trivial.**

1.6 Definition of Span

Let S be a subset of a vector space V.

If S is non-empty, then **span S** is the set of all linear combinations of vectors in S.

We define $span\ \emptyset = \{0\}$ where $\emptyset$ denotes the empty set.

1.7 Theorem:

If S is a subset of a vector space V, then span S is a subspace of V.

C2: Covering All The Bases

2.1 Definition of Linearly Dependence:

A list of vectors $x_1, x_2, \dots, x_k$ in a vector space V is **linearly dependent** if there are scalars $c_1, c_2, \dots, c_k$, **not all zero,** such that $c_1x_1 + c_2x_2 + \dots + c_kx_k = 0$.

In other words, $x_1, x_2, \dots, x_k$ is linearly dependent if and only if the zero vector can be represented as a non-trivial linear combination of $x_1, x_2, \dots, x_k$.

2.2 Definition of Linearly Independent:

A list of vectors $x_1, x_2, \dots, x_k$ in a vector space V is **linearly independent** if the only scalars $c_1, c_2, \dots, c_k$ such that $c_1x_1 + c_2x_2 + \dots + c_kx_k = 0$ are $c_1 = \dots = c_k = 0$.

In other words, $x_1, x_2, \dots, x_k$ is linearly independent iff the **only representation** of the zero vector as a linear combination of $x_1, x_2, \dots, x_k$ is the **trivial one.**

2.2.1 Theorem:

Let $x_1, x_2, \dots, x_k$ be a linearly independent list of vectors in a vector space V. Then

$$a_1x_1 + a_2x_2 + \dots + a_kx_k = b_1x_1 + b_2x_2 + \dots + b_kx_k$$

iff $a_j = b_j$ for each $j = 1, 2, \dots, k$

In other words, $x_1, x_2, \dots, x_k$ is linearly independent if and only if each vector in $span\{x_1, x_2, \dots, x_3\}$ has a **unique representation** as a linear combination of the vectors $x_1, x_2, \dots, x_k$.

2.3 Lemma:

Let $k \geq 2$ and let $x_1, x_2, ..., x_k$ be a list of linearly independent vectors in a vector space V. Then the list $x_1, x_2, ..., \hat{x}_j, ..., x_k$ is linearly independent for any $j = 1, 2, ..., k$

2.4 Theorem 7.1:

Let $x_1, x_2, ..., x_k$ be a list of vectors in a non-zero vector space V.

a) Suppose $x_1, x_2, ..., x_k$ is linearly independent and does not span V. If $x \in V$ and $x \notin span\{x_1, x_2, ..., x_k\}$, then the list $x_1, x_2, ..., x_k$, x is linearly independent.

In other words, under certain conditions, we can extend a linearly independent list to a longer linearly independent list.

b) Suppose $x_1, x_2, ..., x_k$ is linearly dependent and spans V. If $c_1x_1 + c_2x_2 + ... + c_kx_k = 0$ is a non-trivial linear combination and $c_j \neq 0$ for some $j = 1, 2, ..., k$, then $x_1, x_2, ..., \hat{x}_j, ..., x_k$ spans V.

In other words, under certain conditions, we can **reduce** a linearly dependent list that spans V and still span V.

2.5 Replacement Lemma:

Let V be a non-zero vector space and suppose the list $x_1, x_2, \ldots, x_k$ spans V. Let **x** be a nonzero vector in V, and suppose

$$x = c_1 x_1 + c_2 x_2 + \ldots + c_k x_k$$

If $c_j \neq 0$ for some $j = 1, 2, \ldots, k$ then the list $x_1, x_2, \ldots, \hat{x}_j, \ldots, x_k, x$ spans V.

In other words, under certain conditions, we can replace a vector in the list $x_1, x_2, \ldots, x_k$ with x and still span V.

2.6 Theorem: Fundamental Theorem of Linear Algebra:

Let V be a vector space, and suppose that the list $x_1, x_2, \ldots, x_k$ spans V. If $y_1, y_2, \ldots, y_l$ is linearly independent list in V, then $l \leq k$.

In other words, the number of vectors in any linearly independent list of vectors in V cannot exceed the number of vectors in any spanning set for V.

2.7 Definition of Basis:

A list of vectors $x_1, x_2, \ldots, x_k$ in a vector space V is a **basis** for V if

1. $V = span\{x_1, x_2, \ldots, x_k\}$,
2. $x_1, x_2, \ldots, x_k$ are linearly independent.

2.7.1 Fundamental Characteristic of Bases

If $x_1, x_2, \ldots, x_k$ is a basis for a vector space V then

- Each vector in V is a linear combination of $x_1, x_2, \ldots, x_k$ since $x_1, x_2, \ldots, x_k$ spans V.
- This linear combination is unique since $x_1, x_2, \ldots, x_k$ is linearly independent.

2.8 Definition of Dimension:

Let V be a vector space and let k be a positive integer.

If there is a list of vectors $x_1, x_2, ..., x_k$ of vectors that is a basis for V, then V has **dimension** k.

2.9 Theorem: 7.1 Redux

Let V be a finite dimensional vector space, and let $x_1, x_2, ..., x_k \in V$.

a) If $dim\, V > k$, and $x_1, x_2, ..., x_k$ are linearly independent, then there is a basis for V that includes the list $x_1, x_2, ..., x_k$.

In other words, any linearly independent list of vectors can be extended to a basis for V.

b) If $span\{x_1, x_2, ..., x_k\} = V$, then $dim\, V \leq k$ and there is a sublist of $x_1, x_2, ..., x_k$ that is a basis for V.

In other words, any list of vectors that span V can be shortened to a list that is a basis for V.

2.10 Theorem:

Let U be a subspace of a k-dimensional vector space V. Then U is finite dimensional and $dim\, U \leq k$. Furthermore, $dim\, U = k$ if and only if $U = V$.

C3: Rank and File

3.1 Definition of Column Space:

$Let\ a_1, a_2, ..., a_n$ be the columns of $A \in {}^k\mathbb{R}^n$. The **column space** of A is

$$\begin{aligned} col\ A &= \{Ax | x \in {}^n\mathbb{R}\} \\ &= \{x_1 a_1 + x_2 a_2 + ... + x_n a_n \mid x_1, x_2, ..., x_n \in \mathbb{R}\} \\ &= span\{a_1, a_2, ..., a_n\} \subseteq {}^k\mathbb{R} \end{aligned}$$

3.2 Definition of Rank:

Let $A \in {}^k\mathbb{R}^n$. The dimension of the column space of A is the **rank** of A.

$$dim\ col\ A = rank\ A$$

In other words, rank A equals the number of linearly independent columns of A.

3.3 Lemma 11.1

If each of the columns of $C \in {}^m\mathbb{R}^n$ is a linear combination of the columns of $A \in {}^m\mathbb{R}^k$, then there exists a matrix $B \in {}^k\mathbb{R}^n$ such that $C = AB$

3.4 Definition of Invertibility:

$A \in {}^n\mathbb{R}^n$ is **invertible** if there exists a matrix $B \in {}^n\mathbb{R}^n$ such that $AB = I_n$, where I_n is the $n \times n$ identity matrix.

3.5 Theorem:

Let $A \in {}^n\mathbb{R}^n$. Then,

1. A is invertible if and only if $col\ A$ has n linearly independent columns.
2. A is invertible if and only if $rank\ A = n$

3.6 Definition of Row Space:

Let $A \in {}^k\mathbb{R}^n$. The **row space** of A is

$$row\ A = \{A^T x \mid x \in {}^k\mathbb{R}\}$$
$$= col\ A^T \subseteq {}^n\mathbb{R}$$

3.7 Rank Theorem:

For any matrix A,

$$dim\ col\ A = dim\ row\ A = rank\ A$$

In other words, the number of linearly independent columns of A equals the number of linearly independent rows of A, and this number is the rank of A.

3.8 Theorem:

Let $A \in {}^k\mathbb{R}^n$, and $B \in {}^n\mathbb{R}^r$. Then

$$rank\ AB \leq min\{rank\ A,\ rank\ B\}$$

3.9 Theorem:

Let $A \in {}^k\mathbb{R}^n$, and $b \in {}^k\mathbb{R}$. Then

$$rank\ A \leq rank[A \mid b].$$

Furthermore, the system $Ax = b$ has a solution if and only if $rank\ A = rank[A|b]$

3.10 Definition of Null Space:

Let $A \in {}^k\mathbb{R}^n$. The **null space** of A is

$$null\ A = the\ set\ of\ solutions\ to\ the\ homogeneous\ system\ Ax = 0$$
$$= \{x \mid Ax = 0\} \subseteq {}^n\mathbb{R}$$

3.10.1 Definition of Nullity:

The **nullity** of A is the dimension of the null space of A. That is,

$$nullity\ A = dim\ null\ A$$

3.11 Rank-Nullity Theorem:

For any matrix A,

$$rank\ A + nullity\ A = the\ number\ of\ columns\ in\ A$$

3.12 Definition of Full Row Rank and Full Column Rank:

Let $A \in {}^{k}\mathbb{R}^{n}$. If $rank\ A = k$, then A has **full row rank;** if $rank\ A = n$, then A has **full column rank.**

3.13 Theorem:

$Let\ A \in {}^{k}\mathbb{R}^{n}$.

If $B \in {}^{r}\mathbb{R}^{k}$ has full column rank, then

$$rank\ A = rank\ BA$$

If $C \in {}^{n}\mathbb{R}^{m}$ has full row rank, then

$$rank\ A = rank\ AC$$

In other words, multiplying a matrix A on the left by a matrix with full column rank does not change the rank of A. Additionally, multiplying a matrix A on the right by a matrix with full row rank does not change the rank of A.

3.14 Lemma:

Let $A \in {}^{k}\mathbb{R}^{n}$**.** Then

$$rank\ A = rank\ A^{T}A = rank AA^{T}$$

3.15 Theorem:

Let $A \in {}^{k}\mathbb{R}^{n}$. Then,

1) A has full column rank if and only if $A^{T}A$ is invertible.
2) A has full row rank if and only if AA^{T} is invertible.

C4: Coordination Plans

4.1 Definition of Linear Transformation:

Let V and W be vector spaces. A function $T\colon V \to W$ is a **linear transformation if**

$$T(x + y) = Tx + Ty$$
$$T(cx) = cTx$$

For all $x, y \in V$, and all $c \in \mathbb{R}$

- $x + y$ is **vector addition** in V
- $Tx + Ty$ is **vector addition** in W.
- cx is **scalar multiplication** in V.
- cTx is **scalar multiplication** in W.

4.2 Definition of Image:

Let V and W be vector spaces and let $T\colon V \to W$ be a linear transformation.

Let $A \subseteq V$. The **image** of the set A under T is

$$T(A) = \{Tx \mid x \in A\}$$

4.3 Theorem:

Let V and W be vector spaces and let $T\colon V \to W$ be a linear transformation. If A is a subspace of V, its image $T(A)$ is a subspace of W.

4.4 Definition of Kernel and Image:

Let V and W be vector spaces and let $T\colon\ V \to W$ be a linear transformation.

The **kernel** of T is

$$ker\, T\ =\ \{x\ \in\ V\ |\ Tx\ =\ 0\}$$

The **image** of T is

$$im\, T\ =\ T(V)\ =\ \{Tx\ |\ x\ \in\ V\}$$

Note: im T is a **subspace of W.** ker T is a **subspace of V.**

Furthermore, if V and W are finite dimensional, then so too are $ker\ T\ and\ im\ T$, and $dim\ ker\ T \leq dim\ V$, and $dim\ im\ T \leq dim\ W$ by **2.10 Theorem.**

4.5 Lemma 16.1:

Let V and W be vector spaces. Suppose that V is finite dimensional and that $v_1, v_2, ..., v_n$ is a basis for V. Then $im\ T\ =\ span\{Tv_1, Tv_2, ..., Tv_n\}$.

Note: As a consequence of the Lemma, $dim\ im\ T \leq n\ =\ dim\ V$. We can conclude

$$dim\ im\ T \leq min\{dim\ V,\ dim\ W\}$$

4.6 Definition of Injective, Surjective, and Bijective:

Let V and W be vector spaces, and let $T\colon\ V \to W$ be a linear transformation.

T is **injective** if $Tx\ =\ Ty$ then $x\ =\ y$. (one-to-one)
T is **surjective** if for all $y\ \in\ W$, there exists an $x\ \in\ V$ such that $y\ =\ Tx$. (onto)
T is **bijective** if it is both injective and surjective. (invertible)

4.7 Dimension Theorem:

Let V and W be vector spaces. Suppose V is finite dimensional and let $T\colon\ V \to W$ be a linear transformation. Then

$$dim\ im\, T\ +\ dim\ ker\, T\ =\ dim\, V$$

4.8.0 Definition of Coordinates of x with respect to the basis α:

Let $\alpha\ = v_1, v_2, ..., v_n$ be a basis for an n-dimensional vector space V. Write any $x\ \in\ V$ as a unique linear combination

$$x\ = c_1 v_1\ +\ c_2 v_2\ + ... +\ c_n v_n$$

Of the basis vectors $v_1, v_2, ..., v_n$. The scalars $c_1, c_2, ..., c_n$ are the **coordinates of x with respect to the basis α.**

4.8.1 Definition of Coordinate Vector of x with respect to the basis α:

The vector:

$$[x]_\alpha\ =\ [c_1\, c_2\, ...\ c_n]^T \in {}^n\mathbb{R}$$

Is the **coordinate vector of x with respect to the basis** α.

4.9 Theorem:

Let $\alpha\ = v_1, v_2, ..., v_n$ be a basis for an n-dimensional vector space V. The function $T\colon\ V \to {}^n\mathbb{R}$ be defined by

$$Tx\ =\ [x]_\alpha$$

is a linear transformation.

4.10 Definition of matrix T with respect to the bases *α* and β:

Let V and W be non-zero finite dimensional vector spaces. Let $\alpha = v_1, v_2, \ldots, v_n$ be a basis for V, let $\beta = w_1, w_2, \ldots, w_m$ be a basis for W, and let $T: V \to W$ be a linear transformation. The **matrix representation of T with respect to the bases *α* and β** is denoted $[T]_\alpha^\beta$ and is the matrix such that

$$[Tx]_\beta = [T]_\alpha^\beta [x]_\alpha$$

For every $x \in V$

- $]_\alpha^\beta$, where α is where you are starting and β is where you are ending.

Note: $[T]_\alpha^\beta$ is the matrix whose columns are $[Tv_1]_\beta, [Tv_2]_\beta, \ldots, [Tv_n]_\beta$ (**in that order**)

4.11 Definition of Change of Basis Matrix from α to β:

Let $\alpha = v_1, v_2, \ldots, v_n$ and $\beta = w_1, w_2, \ldots, w_n$ be two bases for an n-dimensional vector space V. The $n \times n$ matrix $P_{\beta\alpha} = [[v_1]_\beta \, [v_2]_\beta \cdots [v_n]_\beta]$ is called the **change of basis matrix from α to β** (or the **matrix of transition from α to β),** and is the matrix such that

$$[x]_\beta = P_{\beta\alpha}[x]_\alpha$$

For every $x \in V$.

4.12 Theorem:

Let $\alpha = v_1, v_2, \ldots, v_n$ be a basis for an n-dimensional vector space V. If $\beta = w_1, w_2, \ldots, w_n$ is another basis for V, then the change of basis matrix $P_{\beta\alpha}$ is invertible, and its inverse is $P_{\alpha\beta}$.

4.13 Theorem:

Let α and β be two different bases for a finite dimensional vector space V, and let $T: V \rightarrow V$ be a linear transformation. If $S = P_{\alpha\beta}$, then S is invertible and

$$[T]_\alpha^\alpha = P_{\alpha\beta}[T]_\beta^\beta P_{\beta\alpha}$$
$$= S[T]_\beta^\beta S^{-1}$$

C5: Great Determinations

5.1 Definition of Determinant:

The **determinant** of a 2×2 matrix A is the unique function $det: A \in {}^2\mathbb{R}^2 \rightarrow \mathbb{R}$ defined on the rows of A that satisfies

(i) $\det \begin{bmatrix} b\mathbf{a}_1 + c\mathbf{a'}_1 \\ \mathbf{a}_2 \end{bmatrix} = b \det \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} + c \det \begin{bmatrix} \mathbf{a'}_1 \\ \mathbf{a}_2 \end{bmatrix}$ for all $b, c \in \mathbb{R}$.

(ii) $\det \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} = -\det \begin{bmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \end{bmatrix}$

(iii) $\det \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = 1$. In other words $\det I_2 = 1$, where I_2 is the 2×2 identity matrix.

Note: Determinant of a matrix A is the **unique, alternating, multilinear** function on the rows of A whose value on the identity is 1.

5.2 Theorem:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

5.3 Theorem:

Theorem: *Any* real-valued function f defined on the rows of a 2×2 matrix $A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}$ such that

(i) $f(\begin{bmatrix} b\mathbf{a}_1 + c\mathbf{a'}_1 \\ \mathbf{a}_2 \end{bmatrix}) = bf(\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}) + cf(\begin{bmatrix} \mathbf{a'}_1 \\ \mathbf{a}_2 \end{bmatrix})$ for all $b, c \in \mathbb{R}$.

(ii) $f(\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}) = -f(\begin{bmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \end{bmatrix})$

satisfies $f(A) = (\det A)f(I)$.

Note a non-zero function f that satisfies (i) and (ii) is called **a** determinant function. If $f(I) = 1$, then f is **the** determinant (i.e. $f(A) = det(A)$)

5.4 Definition of Multilinear:

Definition: A function f on the rows of a matrix A is called *multilinear* if, for each $j = 1, 2, \ldots, n$, and for all $b, c \in \mathbb{R}$,

$$f(\begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ b\mathbf{a}_j + c\mathbf{a}'_j \\ \vdots \\ \mathbf{a}_n \end{bmatrix}) = bf(\begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_n \end{bmatrix}) + cf(\begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}'_j \\ \vdots \\ \mathbf{a}_n \end{bmatrix})$$

In other words, f is multilinear if it's a linear function of each of its rows while the remaining rows are held fixed.

5.5 Definition of Alternating:

Definition: A function f on the rows of a matrix A is called *alternating* if for all $j \neq k$

$$f(\begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_k \\ \vdots \\ \mathbf{a}_n \end{bmatrix}) = -f(\begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_k \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_n \end{bmatrix})$$

In other words, f is alternating if whenever any two rows of A are interchanged, f changes sign.

5.6 Definition of ij Minor:

The **ij minor** of an $n \times n$ matrix A is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column of A. The **ij** minor is denoted A_{ij}.

5.8 Theorem:

There exists a unique alternating multilinear function $f: {}^n\mathbb{R}^n \rightarrow \mathbb{R}$ satisfying $f(I_n) = 1$ which is called the determinant function. We write $f(A) = det\, A$ and

$$det\, A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} det(A_{ij}) \;\; for\; i = 1, 2, ..., n$$

In addition, ***any*** alternating multilinear function f satisfies $f(A) = (det\, A)f(I)$

5.9 Theorem:

Let $a_1, a_2, ..., a_n$ denote the rows of $n \times n$ matrix A. Then

$$\det \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_i + c\mathbf{a}_j \\ \vdots \\ \mathbf{a}_n \end{bmatrix} = \det A$$

In other words, adding a multiple of one row to another does not change the value of det A.

Note: We also know that

- By the **alternating property** of the determinant, interchanging any two rows changes the sign of the determinant.
- By the **multilinear property** of the determinant, multiplying any row by a scalar c changes the value of the determinant by a factor of c.

5.10 Product Theorem:

Let $A, B \in {}^n\mathbb{R}^n$. Then

$$det\ AB = det(A)det(B)$$

5.11 Theorem:

If $A \in {}^n\mathbb{R}^n$ is invertible then $det\ A^{-1} = (det\ A)^{-1}$

5.12 Definition of Adjoint:

Let $A \in {}^n\mathbb{R}^n$. The **adjoint** of A, denoted adj A, is the $n \times n$ matrix whose ij entry is $(-1)^{i+j} det\ A_{ji}$.

5.13 Theorem:

Let $A \in {}^n\mathbb{R}^n$.

a) $A(adj\ A) = (det\ A)I_n$
b) If A is invertible, $A^{-1} = \frac{1}{det\ A} adj\ A$

5.14 Theorem:

Let $A \in {}^n\mathbb{R}^n$. For any fixed $j = 1, 2, ..., n$,

$$det\ A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} det\ A_{ij}$$

- Expand along any **column**

5.15 Corollary:

Let $A \in {}^n\mathbb{R}^n$. Then,

$$det\ A^T = det\ A$$

5.16 Cramer's Rule:

Suppose A is invertible. Then the unique solution to the system of equations $Ax = b$ is

$$x = A^{-1}b$$
$$= \frac{1}{det\ A}(adj\ A)b$$

But the ith entry of the product $(adj\ A)b$ is

$$\sum_{j=1}^{n}(-1)^{i+j}b_j det\ A_{ji}.$$

But this is the determinant of the matrix B_i whose *columns* are $a_1, a_2, ..., a_{i-1}, b, a_{i+1}, ..., a_n$ where the $a_1, a_2, ..., \hat{a}_i, ..., a_n$ are the columns of A. In other words,

$$x_i = \frac{det\ B_i}{det\ A} \text{ for each } i = 1, 2, ..., n$$

C6: Eigenthis and Eigenthat

6.1 Definition of Eigenvector

Let $A \in {}^n\mathbb{R}^n$. A vector $x \in {}^n\mathbb{R}$ is an **eigenvector** of A if $x \neq 0$ and

$$Ax = \lambda x$$

For some scalar λ

6.2 Definition of Eigenvalue:

The scalar λ is called an **eigenvalue** of A corresponding to x.

6.3 Definition of Eigenspace:

Let $A \in {}^n\mathbb{R}^n$. For a given eigenvalue λ, the **eigenspace** of A corresponding to eigenvalue λ is

$$E_\lambda(A) = \{x \in {}^n\mathbb{R} \mid Ax = \lambda x\}$$

In other words, $E_\lambda(A)$ is the set of all eigenvectors of A corresponding to eigenvalue λ, together with the **zero vector**.

- WHEN YOU ARE CALCULATING THE EIGENSPACE, DON'T FORGET TO MULTIPLY A BY -1!!!

6.4 Definition of Characteristic Polynomial:

Let $A \in {}^n\mathbb{R}^n$. The **characteristic polynomial** of A is

$$c_A(\lambda) = det(\lambda I - A)$$

6.5 Theorem:

Let $A \in {}^n\mathbb{R}^n$. The characteristic polynomial of A has the form

$$c_A(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + ... + c_1\lambda + c_0$$

Where $c_{n-1} = -\, trA$ and $c_0 = (-\,1)^n det\, A$.

As an immediate consequence of this theorem, we can conclude that an $n \times n$ matrix A can have at most n distinct eigenvalues.

6.6 Definition of Similar:

Let A, B$\in {}^{n}\mathbb{R}^{n}$.

A and B are **similar** if A and B represent the same linear transformation with respect to (possibly) different bases.

Equivalently, A and B are **similar** if there is an invertible $S\in {}^{n}\mathbb{R}^{n}$ such that $A = SBS^{-1}$

Similar matrices have the **same determinant, rank, characteristic polynomial, eigenvalues, and trace. The converse, however, is not true!**

6.7 Definition of Diagonal:

An $n \times n$ matrix $D = [d_{ij}]$ is **diagonal** if $d_{ij} = 0$ whenever $i \neq j$.

$diag(\lambda_1, \lambda_2, ..., \lambda_n)$ is used to denote an $n \times n$ diagonal matrix whose entries are $\lambda_1, \lambda_2, ..., \lambda_n$, in that order.

6.8 Definition of Diagonalizable:

$A\in {}^{n}\mathbb{R}^{n}$ is said to be **diagonalizable** if it is similar to $D\in {}^{n}\mathbb{R}^{n}$.

6.9 Theorem 26.1:

Let $A \in {}^n\mathbb{R}^n$. Then A is diagonalizable if and only if there exists a basis for ${}^n\mathbb{R}$ consisting of eigenvectors of A.

Note: The proof of this theorem illustrates that:

1. A is similar to a diagonal matrix D if and only if the columns of S are eigenvectors of A.
2. The diagonal entries of D are the corresponding eigenvalues.
3. The matrix S is the change of basis matrix from the basis consisting of eigenvectors to the standard basis.

6.10 Definition of Algebraic Multiplicity, Geometric Multiplicity:

Let λ be an eigenvalue of $A \in {}^n\mathbb{R}^n$.

The **algebraic multiplicity** of λ is the number of times λ appears as a root of $c_A(\lambda)$
The **geometric multiplicity** of λ is the dimension of the associated eigenspace $E_\lambda(A)$.

6.11 Theorem 28.1

Let $A \in {}^n\mathbb{R}^n$ and suppose that

$$c_A(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \dots (\lambda - \lambda_k)^{m_k}$$

where $\lambda_1, \lambda_2, \dots, \lambda_k$ are the distinct eigenvalues of A.

If for each eigenvalue λ of A, the algebraic and geometric multiplicities of λ are equal, then A is diagonalizable.

6.12 Theorem 28.2

Let $A \in {}^n\mathbb{R}^n$. If A is diagonalizable then for each eigenvalue λ of A, the algebraic and geometric multiplicities of λ are equal.

6.13 Definition of a System of Linear Differential Equations with Constant Coefficients:

A system of the form

$$x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t)$$
$$x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t)$$

is called a **system of linear differential equations with constant coefficients.** We can represent the system as

$$x' = Ax$$

where $A = [a_{ij}]$ is the 2×2 matrix of the coefficients of the system.

6.14 Theorem:

Let $A \in {}^2\mathbb{R}^2$ be a diagonalizable matrix with eigenvalues λ_1, λ_2 (not necessarily distinct). Let v_1, v_2 be a basis for $\mathbb{R}^2$ consisting of eigenvectors of A: If $x_0 = c_1v_1 + c_2v_2$, then the system $x' = Ax$ subject to the initial condition $x(0) = x_0$ has solution

$$x(t) = e^{\lambda_1 t}(c_1v_1) + e^{\lambda_2 t}(c_2v_2)$$

6.15 Definition of a System of Linear Differential Equations with Constant Coefficients:

A system of the form

$$x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + ... + a_{1n}x_n(t)$$
$$x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + ... + a_{2n}x_n(t)$$
$$\cdots$$
$$x_n'(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + ... + a_{nn}x_n(t)$$

Where each $x_i(t)$ is real-valued function of a real variable, is called a **system of linear differential equations with constant coefficients.**

6.16 Lemma:

Let $A \in {}^n\mathbb{R}^n$. If x_0 is an eigenvector of A with eigenvalue λ, then the system $x' = Ax$, $x(0) = x_0$ has solution $x(t) = e^{\lambda t}x_0$.

6.17 Theorem 30.1

Let $A \in {}^n\mathbb{R}^n$ be a diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct). Let $v_1, v_2, \ldots, v_n$ be a basis for $\mathbb{R}^n$ consisting of eigenvectors of A. If $x_0 = c_1v_1 + c_2v_2 + \ldots + c_nv_n$, then the system $x' = Ax$, $x(0) = x_0$ has solution

$$x(t) = e^{\lambda_1 t}(c_1v_1) + e^{\lambda_2 t}(c_2v_2) + \ldots + e^{\lambda_n t}(c_nv_n)$$

6.18 Theorem 30.2:

Let A be a diagonalizable matrix, and $A = SDS^{-1}$, where D is diagonal. If $y(t)$ is a solution to the initial value problem

$$y' = Dy,\ y(0) = S^{-1}x_0$$

then $x(t) = Sy(t)$ is a solution to the initial value problem

$$x' = Ax,\ x(0) = x_0$$