CHE260 (Heat Transfer): Cheat Sheet

1. *Find:* Read the problem and state what you are asked to find in your own words. This step may seem obvious, but it is surprising how often people misunderstand the question or miss essential information.
2. *Known:* List all the information and properties provided, as well as information about properties that remain constant (e.g. the volume in a isochoric process or the temperature in an isothermal process).
3. *Diagrams:* Draw a schematic diagram of the system. Mark the system boundary and decide whether the system is a control mass or volume. Show energy and mass transfers between the system and surroundings by arrows. Draw a process diagram if necessary.
4. *Assumptions:* Decide how you are going to model the system. List all assumptions that you make.
5. *Governing Equations:* Which conservation law are you going to apply? Depending on what you are trying to find, you may use principles of conservation of mass or energy. Write down the governing equations.
6. *Properties:* List all property values that are not given in the problem statement. This includes information extracted from tables or other sources.
7. *Solution:* Substitute known values of variables in the governing equations and solve them to find unknowns.
8. *Answer:* State the answer and confirm that it is what was asked for.
9. *Discussion:* Are your results reasonable? Can you draw any conclusions from them?

- Correct amount of sig figs from question statement on answers and round off answers at the end.
- Think intuitively about the problem to see if the numbers make sense.
- Figure out what type of question it is and the necessary equations.
- For questions that you need to find the surface area, only include the exposed surface areas. (eg. you probably only need the sides for the cylinder and not the top and bottom since they aren't getting convected).
- Secondly, you can choose an arbitrary deepness and keep the length the same.
  - You can assume L and A can be 1 m and 1 m^2.
- You can either do the calculation with the surface area times the ratio, or just take the specific area.
- If the heat transfer processes are happening in the same medium, across the same delta T, then they would be in parallel. If they are occurring across different media, one after the other, they are in series.

Miscellaneous

Surface Area/Volume of Sphere

$$A = 4\pi r^2 = \pi d^2$$

$$V = \frac{4}{3}\pi r^3$$

Surface Area/Volume of Cylinder

$$V = \pi r^2 h$$

$$A = 2\pi r h + 2\pi r^2$$

Mass Flow Rate

$$\dot{m} = \frac{\dot{Q}}{h}$$

- heat of fusion, heat of vaporization is h

Mass

$$m = \dot{m}t$$

Q

$$\dot{Q} = \dot{m}c(T_2 - T_1)$$

$$\dot{Q}_{gen} = \dot{e}_{gen}V$$

- $\dot{e}_{gen}\left[\frac{W}{m^3}\right]$

Efficiency

$$\eta = \frac{\dot{q}_{usable}}{\dot{q}_{solar}}$$

Notes:

- $T_{\infty} \neq T_{surr}$. These can mean different things for radiation and convection.
- A is the cross-sectional area for conduction, L is the length at which conduction is being conducted.

Introduction

L1: Introduction to Heat Transfer ✅

Learning Objectives:

- Identify whether a system is at equilibrium and/or steady state and describe the differences
- Explain the difference between kinetics and thermodynamics in determining the state of a system
- Describe the impact of surface area on heat transfer rate

1.0 Equilibrium vs. Steady State

Equilibrium: No net driving force for change.

Steady State: No change with time.

- **Note:** Equilibrium includes steady state, but steady state doesn't have to include equilibrium.

1.1 Heat Transfer: Rate vs. Flux

Rate of Heat Transfer:

$$\dot{Q} = rate\ of\ heat\ transfer$$

- **Note:** $\dot{Q}\ proportional\ to\ A$

Heat Flux:

$$\dot{q} = \frac{\dot{Q}}{A} \left[\frac{W}{m^2}\right]$$

- **Note:** Heat transfer rate per unit area

L2: Heat Transfer Mechanisms: Conduction, Convection (16) ✅

Learning Objectives:

- Name the 3 main mechanisms of heat transfer
- Provide a molecular description of conductive heat transfer of gas, liquid and solid
- Calculate conductive heat flux using Fourier's Law
- Explain the physical significance of each term in Fourier's Law
- Calculate convective heat flux using Newton's Law of Cooling
- Explain the physical significance of each term in Newton's Law of Cooling
- Give real-life examples of conductive and convective heat transfer

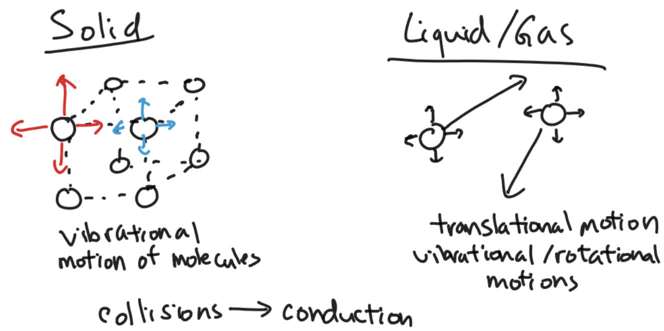
2.0 Conduction

Heat transfer through a stationary medium as a result of temperature difference/gradient.

- **Note:** Can occur in solid, liquid, or gas but without **bulk** motion of the medium.
- **Note:** @ SS: $\dot{q}_{in} = \dot{q}_{out}$

2.1 How/Why Does Conduction Occur in Solids and Liquids/Gas?

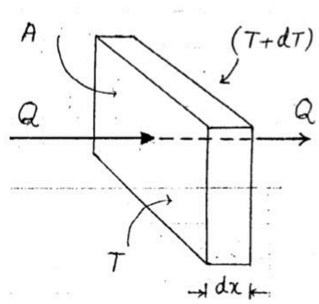
Temperature is a proxy for KE in molecules. Higher T, higher KE, faster molecular motions.



2.2 Fourier's Law (ie. Governing Equation for Conductive Heat Transfer)

$$\dot{Q}_{cond} = \dot{q}A = -kA\frac{dT}{dx}\ [W]$$

- k is the heat conductivity ($k = n\bar{v}mc\left[\frac{W}{mK}\right]$).
- A is cross-sectional area
- $\frac{dT}{dx}$ is temperature gradient as heat transfers.



2.3 Convection

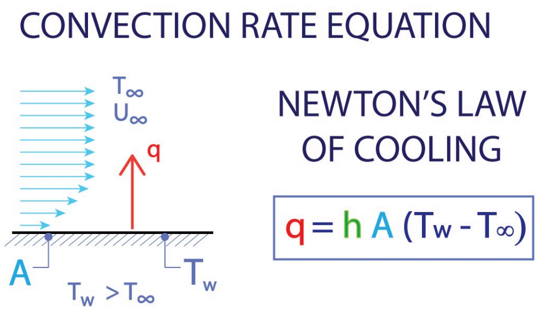
Heat transfer between a stationary surface and a moving fluid.

- **Note:** Convection = Conduction (right at the interface) + Advection (E transfer due to bulk motion of the fluid)

2.4 Heat Transfer via Convection

$$\dot{q}_{conv.} = h(T_s - T_\infty)\ [\frac{W}{m^2}]$$

- h is heat transfer coefficient $\left[\frac{W}{m^2K}\right]$
- T_s is temperature of surface
- T_∞ is temperature way outside of the boundary.
- Multiply by surface area in which convection is coming out to get $\dot{Q}$



L3: Radiation, 1D Heat Conduction Equation (16) ✅

Learning Objectives

- Apply Stefan-Boltzmann Law to calculate radiative heat transfer
- Describe how blackbody radiation is used as a model for maximum rate of radiative heat transfer
- Apply Kirchhoff's law to calculate radiation emitted from real surfaces
- Solve heat transfer problems where multiple mechanisms are taking place simultaneously

3.0 Terminology

Terms	Definition
Radiation	Energy emitted by all matter in the form of EM waves. • Consider opaque object (from surface only)

Blackbody	The idealized surface that emits radiation at this maximum rate from Stefan-Boltzmann Law (blackbody radiation). • $\varepsilon = 1$ and $\alpha = 1$
Graybody	If ε is independent of wavelength, then gray body.

3.1 Stefan-Boltzmann Law (Maximum Rate of Radiation for Blackbody)

$$\dot{Q}_{emit,max} = \sigma A_s T_s^4 \; [W]$$

- $\sigma = 5.670 \times 10^{-8} \left[\frac{W}{m^2 K^4}\right]$ (Stefan Boltzmann Constant)

3.2 Radiation Emitted by Real Surfaces

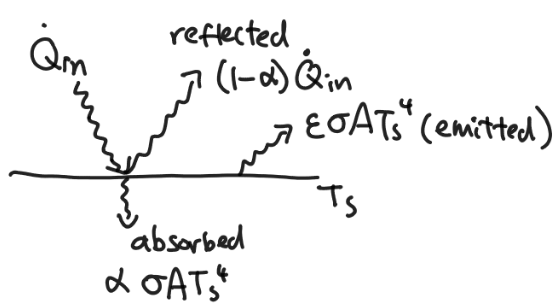
$$\dot{Q}_{emit} = \varepsilon \sigma A_s T_s^4 \; [W]$$

- Note: All real surfaces are less than this blackbody radiation at the same temperature.
- ε is the emissivity of the surface, which is a measure of how closely a surface approximates a blackbody for which $\varepsilon = 1$ ($0 \le \varepsilon \le 1$)

3.3 Kirchoff's Law

Relates absorptivity and emissivity of real surfaces.

$$\alpha = \varepsilon$$

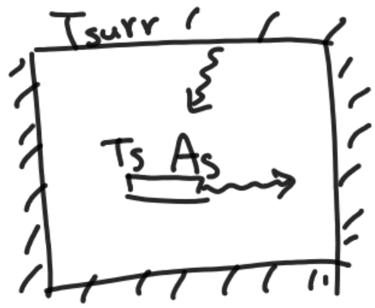


- α is the absorptivity, which is the fraction of the radiation energy incident on a surface that is absorbed by the surface (0≤α≤1)

3.4 Special Case:

Small surface is completely surrounded by a much larger surface.

$$\dot{Q}_{net} = \varepsilon\sigma A_s(T_s^4 - T_{surr}^4)$$



3.5 Useful Heat Transfer Balance

$$\dot{Q}_{gen} = \dot{Q}_{rad} - \dot{Q}_{absorbed}$$

If $\dot{Q}_{gen} = 0$

$$\dot{Q}_{absorbed} = \dot{Q}_{rad}$$

- Note: The radiation emitted with an absorptivity of the main source is usually $\dot{Q}_{rad} = \dot{Q}_{absorbed} = \alpha\dot{Q}_{source}$

Conduction

L4: 1D Heat Conduction (16) ✅

Learning Objectives

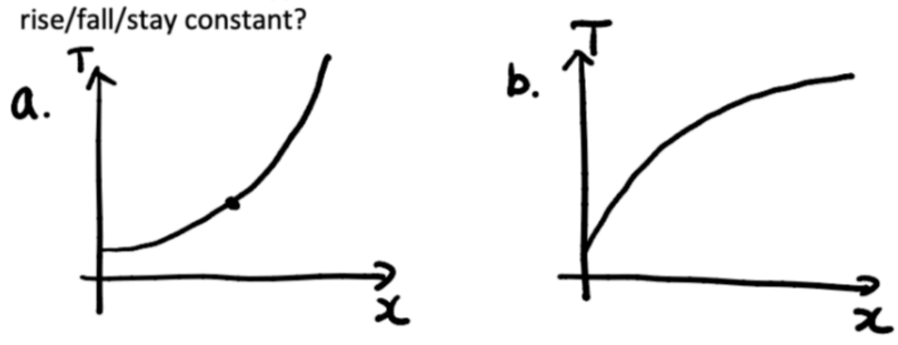
- Explain the physical meaning of each term in the 1D Heat Conduction equation.
- Solve the temperature profile of an object under heat conduction at steady state.
- Explain the different temperature profiles between Cartesian, cylindrical and spherical coordinates.

4.0 1D Heat Conduction:

$$\frac{\partial T}{\partial t} = \left(\frac{k}{\rho c_p}\right)\frac{\partial^2 T}{\partial x^2}$$

- At SS, $\frac{\partial^2 T}{\partial x^2} = 0$
- $\alpha = \frac{k}{\rho c_p}\left[\frac{m^2}{s}\right]$ (Thermal diffusivity)
  - High k: material conducts heat well.
  - High ρc_p: material stores heat well.

4.1 What is the sign of $\frac{\partial T}{\partial t}$? Explain intuitively why temperature would rise/fall/stay constant?



- **Left:** $\frac{\partial^2 T}{\partial x^2} > 0$ ($ie.$ $concave\ up$), therefore, $\frac{\partial T}{\partial t} > 0$
- **Right:** $\frac{\partial^2 T}{\partial x^2} < 0$ ($ie.$ $concave\ down$), therefore, $\frac{\partial T}{\partial t} < 0$

4.2 3D Cartesian Coordinates:

$$\frac{\partial T}{\partial t} = \left(\frac{k}{\rho c_p}\right)\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

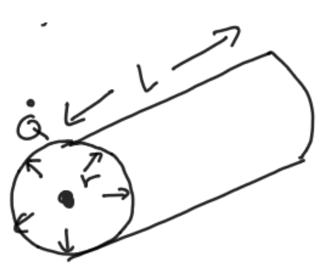
$$\frac{\partial T}{\partial t} = \left(\frac{k}{\rho c_p}\right)\nabla^2 T$$

- At SS, $\nabla^2 T = 0$

4.3 1D Cylindrical Coordinates:

$$\rho c_p \frac{\partial T}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

- Only change is r, keeping $z\ and\ \theta\ constant$.



4.4 1D Spherical Coordinates:

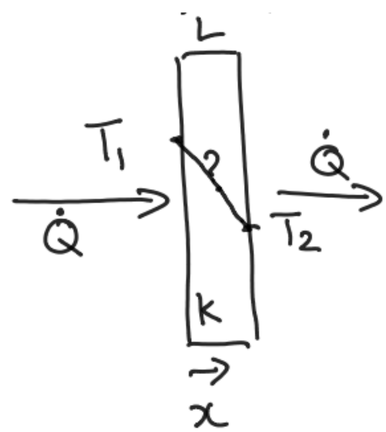
$$\rho c_p \frac{\partial T}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)$$

- Only change is r, keeping ϕ *and* θ *constant.*

4.5 Steady State, 1D Conductive Heat Transfer

$$\dot{Q} = \frac{kA}{L}(T_1 - T_2)\ [W]$$

- **Note:** Look in 17.1 for derivation assuming $\frac{dT}{dx}$ *is constant* from Fourier's law
- **Note:** Notice that L is the thickness, but for surface area, it becomes the length of the tube, so BE CAREFUL.



L5: Thermal Resistance (17.1) ✅

Learning Objectives

- Model conductive heat transfer using thermal resistances

- Draw parallels between electrical circuits and heat transfer
- Calculate thermal resistances for 1D conduction and convection in Cartesian, cylindrical and spherical coordinates
- Solve for heat transfer rate when multiple processes are in series

5.0 Conduction Resistance

$$R_{cond.} = \frac{L}{kA}\left[\frac{K}{W}\right]$$

- A is the surface area.

5.1 Convection Resistance

$$R_{conv.} = \frac{1}{hA}\left[\frac{K}{W}\right]$$

- A is the surface area
- **Note:** As $h \to \infty$ the surface offers no resistance to convection, so it doesn't slow down the heat transfer process.

5.2 Radiation Heat Transfer Coefficient

$$h_{rad} = \frac{\dot{Q}_{rad}}{A(T_s - T_{surr})} = \varepsilon\sigma(T_s^2 + T_{surr}^2)(T_s + T_{surr})\left[\frac{W}{m^2 \cdot K}\right]$$

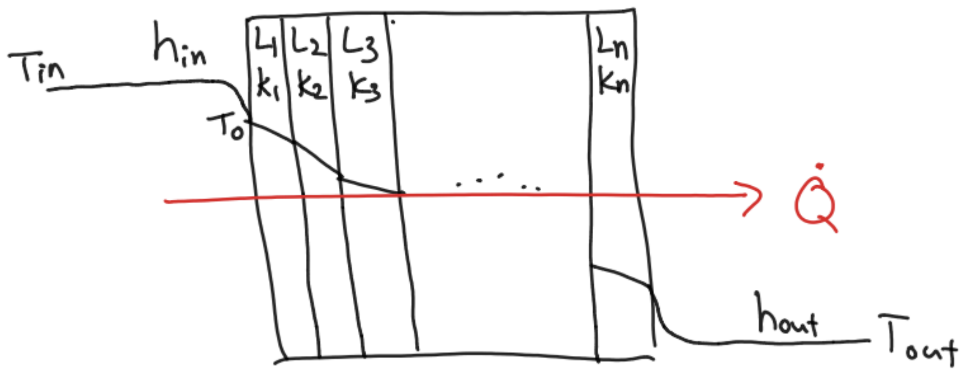
- **Note:** T_s and T_{surr} must be in K.

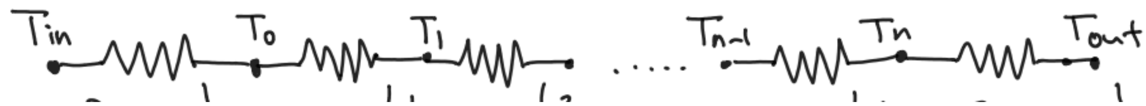
5.3 Radiation Resistance

$$R_{rad} = \frac{1}{h_{rad}A}\left[\frac{K}{W}\right]$$

- A is the surface area.

5.4 Heat Resistances to Model Heat Transfer in Series





$$R_{in} = \frac{1}{h_{in}A} \mid R_1 = \frac{L_1}{k_1 A} \mid R_2 = \frac{L_2}{k_2 A} \mid \dots \mid R_n = \frac{L_n}{k_n A} \mid R_{out} = \frac{1}{h_{out}A}$$

$$\dot{Q} = \frac{T_{in}-T_0}{R_{in}} = \frac{T_0-T_1}{R_1} = \frac{T_1-T_2}{R_2} =\dots= \frac{T_{n-1}-T_n}{R_n} = \frac{T_n-T_{out}}{R_{out}}$$

Rearranging all these equations (Assuming higher temperatures from left to right):

$$T_{in} - T_0 = \dot{Q}R_{in}$$

$$T_0 - T_1 = \dot{Q}R_1$$

$$T_1 - T_2 = \dot{Q}R_2$$

$$\dots$$

$$T_{n-1} - T_n = \dot{Q}R_n$$

$$T_n - T_{out} = \dot{Q}R_{out}$$

Adding these equations:

$$\dot{Q} = \frac{T_{in}-T_{out}}{R_{in}+\sum_{i=1}^{n} R_i + R_{out}} = UA(T_{in} - T_{out})$$

- U is the overall heat transfer coefficient.

- $UA = \frac{1}{R_{total}}$

5.5 Process:

- Always create a resistance diagram sketch, then find the heat transfer.
- When determining the temperature difference, since we know that heat flows from higher to lower temperature, just assume this convention of doing higher temp. - lower temp.
    - Be careful about the sign, if you have to find different stuff such as the temperature.
- **Note:** You can always derive the resistances by thinking of $T_1 - T_2$ is ΔV, and $\dot{Q}$ is like I.

L6: Thermal Resistance Networks, Contact Resistance (17.2-3) ✅

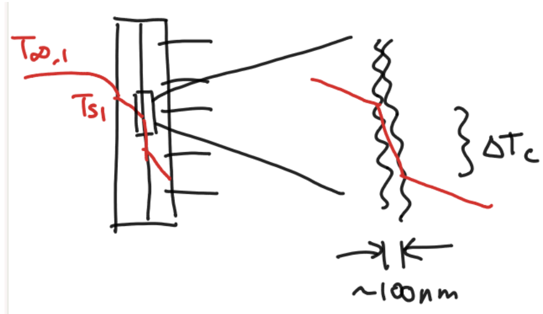
Learning Objectives

- Account for temperature drop across rough surfaces using contact resistance and conductance
- Solve 1 dimensional heat transfer problems involving heat transfer in series and in parallel in Cartesian coordinates

6.0 Thermal Contact Resistance

$$R_c = \frac{\Delta T_c}{\dot{q}} \left[\frac{m^2 K}{W}\right]$$

- **Note:** Units are different from R_{cond}, R_{conv}.
    - In order to get the same units divided by the area.
    - **Question:** What area? Surface area or cross sectional area?
- **Note:** Imperfect contact adds a "contact resistance" to heat transfer in series.



6.1 Thermal Contact Conductance

$$h_c = \frac{1}{R_c} = \frac{\dot{q}}{\Delta T_c} \left[\frac{W}{m^2 K}\right]$$

- **Note:** Just like heat transfer coefficient
- **Note:** We want the contact surface to be closer for higher thermal contact conductance
  - Note: $\uparrow$ $Malleable, \ \uparrow h_c$
  - Note: $\uparrow$ $Pressure, \ \uparrow h_c$

6.2 Generalized Thermal Resistance Networks

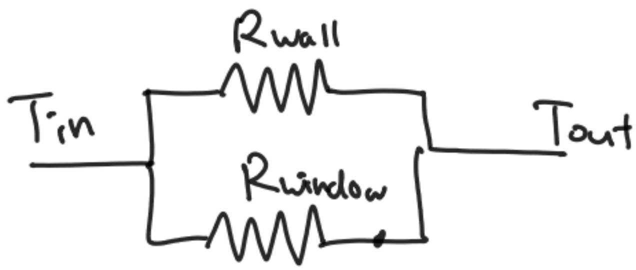
- **Note:** In parallel, it's the sum of the heat transfers, but in series, the heat transfers are equal.

Eg. Heat Transfer Through Wall and Window Occur in Parallel

$$\dot{Q}_{total} = \dot{Q}_{wall} + \dot{Q}_{window}$$

$$\dot{Q}_{total} = \frac{\Delta T}{R_{total}} = \frac{\Delta T}{R_{wall}} + \frac{\Delta T}{R_{window}}$$

$$\frac{1}{R_{total}} = \frac{1}{R_{wall}} + \frac{1}{R_{window}} \Rightarrow R_{total} = \left(\frac{1}{R_{wall}} + \frac{1}{R_{window}}\right)^{-1}$$



- **Note:** Just think about electrical circuits.

L7: Cylinders and Spheres, Insulation (17.4-5) ✅

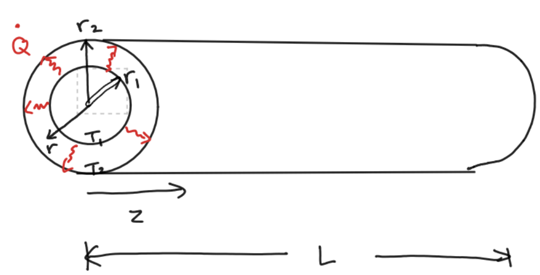
Learning Objectives:

1. Calculate thermal resistances for 1D conduction and convection in cylindrical and spherical coordinates
2. Solve for steady heat transfer rate when multiple processes are in series in cylindrical and spherical coordinates
3. Identify scenarios in which insulation may increase or decrease heat transfer rate

7.0 1D Heat Transfer via Conduction for Cylinders

$$\dot{Q}_{cond} = \frac{2\pi L k (T_1 - T_2)}{ln\left(\frac{r_2}{r_1}\right)}$$

- **Assumption:** 1D conduction in radial direction only $\frac{\partial T}{\partial z} << \frac{\partial T}{\partial r}$

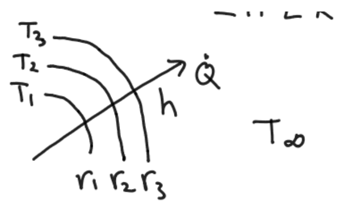


7.1 Conduction Resistance for Cylinder

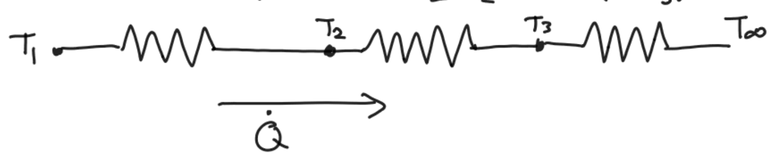
$$R_{cond,cyl} = \frac{ln(r_2/r_1)}{2\pi Lk}$$

- **Note:** Approach is the same but the resistances look different

Eg. Resistance Network in Cylindrical Coordinates



$$R_{total} = \frac{\ln(r_2/r_1)}{2\pi L k_1} + \frac{\ln(r_3/r_2)}{2\pi L k_2} + \frac{1}{h(2\pi r_3)L}$$



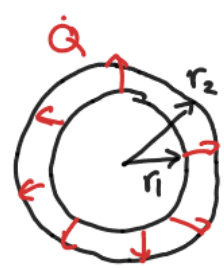
7.2 Convection Resistance for Cylinder

$$R_{conv,cyl} = \frac{1}{h(2\pi r)L}$$

- $A_s = 2\pi r L$

7.3 1D Heat Transfer in Spherical Coordinates

$$\dot{Q} = \frac{4\pi k r_1 r_2}{r_2 - r_1}(T_1 - T_2)$$

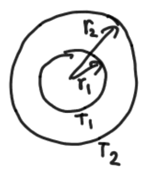


7.4 Conduction Resistance

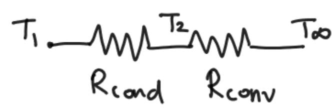
$$R_{cond,sph} = \frac{r_2 - r_1}{4\pi k r_1 r_2}$$

7.5 Does Making Insulation Thicker Increase/Decrease Heat Transfer Rate?

1. Find the total resistance.
2. Find the critical point (maximum/minimum) of the resistance with respect to the variable that varies depending on which term you are looking at.
3. Then graph it to see where the insulation increases/decreases heat transfer rate.



T_∞



$$\dot{Q} = \frac{T_1 - T_\infty}{R_{cond} + R_{conv}}$$

$$R_{total} = \frac{\ln(r_2/r_1)}{2\pi k L} + \frac{1}{h(2\pi r_2)L}$$

$\uparrow r_2 \uparrow R_{cond}$ — increasing thickness of insulation increases R_{cond}

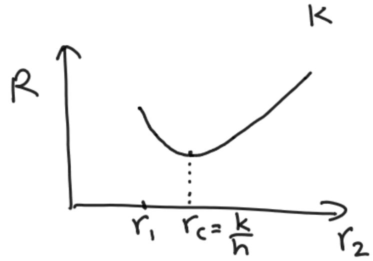
$\uparrow r_2 \downarrow R_{conv}$ — increases the area across which convective heat transfer occurs

Need to find r_c : $\frac{dR}{dr_2} = 0$

$$\frac{dR}{dr_2} = \frac{1}{2\pi k L}\frac{1}{r_2} - \frac{1}{h(2\pi r_2^2)L} = 0$$

$$\frac{1}{k} - \frac{1}{h r_2} = 0$$

$$r_c = \frac{k}{h}$$



Critical Radius:

$$r_c = \frac{k}{h}$$

- **Note:** When $r > \frac{k}{h}$, heat transfer rate decreases (ie. overall resistance increases when insulation is added) and when $r < \frac{k}{h} = r_c$, heat transfer rate increases (ie. overall resistance decreases when insulate is added).

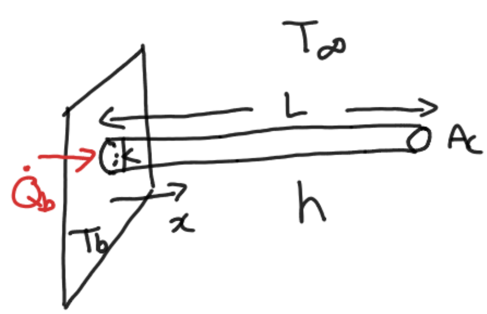
L1-7 Process for Resistive Networks

L8: Heat Transfer from Finned Surfaces (17.6) ✅

Learning Objectives:

1. Explain the heat transfer equation from finned surfaces for each boundary condition
2. Calculate heat transfer rate for (1) infinitely long fin and (2) adiabatic fin tip and (3) corrected length approximation

8.0 Heat Fins



- A_c: Cross-sectional area (assumed to be constant) (eg. the area of the circle for cylindrical fin).
- L: Length
- p: Perimeter of fin (eg. $2\pi r$ for cylindrical fin)
- k: Heat conductivity of material

- h: Heat transfer coefficient for convection across fin
- T_b: Temperature at base
- T_∞: Temperature of bulk fluid
- **Note:** The shape of the fin doesn't have to be cylindrical.
- **Note:** Convection happens across the fin, so the area is the shell $perimeter \cdot L$.

$$\frac{d^2\Theta}{dx^2} - \left(\frac{hp}{kA_c}\right)\theta = 0$$

$$\theta(x) = c_1 e^{-ax} + c_2 e^{ax}$$

- $a = \sqrt{\frac{hp}{kA_c}}$
- $\theta = T - T_\infty$ (Excess temperature)

First Boundary Condition:

$$\theta(0) = \theta_b \mid \theta(0) = c_1 + c_2 = \theta_b$$

- **Note:** 2nd Boundary conditions are based on the cases of different assumptions.

8.1 Case a: Infinitely long fin

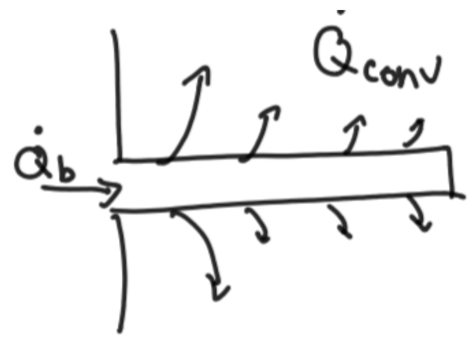
1. **2nd Boundary Condition:** $\theta(x) \to 0,\ x \to \infty$
2. **Temperature Profile:**

$$\theta(x) = \theta_b e^{-ax}$$

3. **Heat Transfer Rate from the Base:**

$$\dot{Q}_b = \sqrt{hpkA_c}(T_b - T_\infty)$$

- **Note:** Convection is the same $\dot{Q}_{conv} = \sqrt{hpkA_c}(T_b - T_\infty)$ because the heat transfer from the base goes 100% into convection.



8.2 Case b: Finite length, no heat transfer at tip (aka. adiabatic tip)

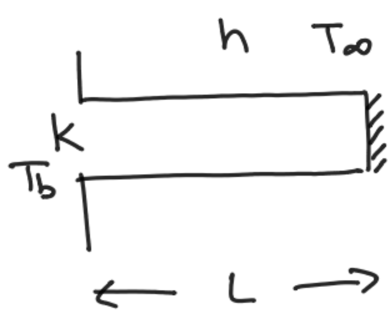
1. **2nd Boundary Condition:** $\dot{q}(x = L) = 0$
2. **Temperature Profile:**

$$\theta(x) = \theta_b \frac{cosh[a(L-x)]}{cosh(aL)}$$

3. **Heat Transfer Rate from the Base:**

$$\dot{Q}_b = \sqrt{hpkA_C}(T_b - T_\infty)tanh(aL)$$

- **Note:** As $L \rightarrow \infty,\ tanh(aL) \rightarrow 1,\ so\ \dot{Q}_b \rightarrow \sqrt{hpkA_c}(T_b - T_\infty)$, so it's just a special case of the first case.



8.3 Case c: Finite Length, Heat Transfer at Fin Tip

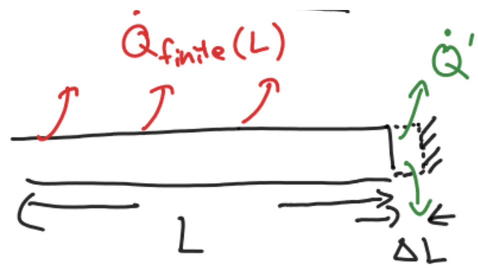
1. **Temperature Profile:**

$$\theta(x) = \theta_b \frac{cosh[a(L_c-x)]}{cosh(aL_c)}$$

2. **Heat Transfer Rate from the Base:**

$$\dot{Q}_b = \sqrt{hpkA_C}(T_b - T_\infty)tanh[a(L_c)]$$

- $\Delta L = \frac{A_c}{p}$
- $L_c = L + \Delta L = L + \frac{A_c}{p}$



L9: Heat Transfer from Finned Surfaces (17.6) ✅

Learning Objectives

- Evaluate heat fins as effective means to enhance heat transfer by examining fin efficiency and fin effectiveness.

9.0 Two Questions When Designing Heat Fins

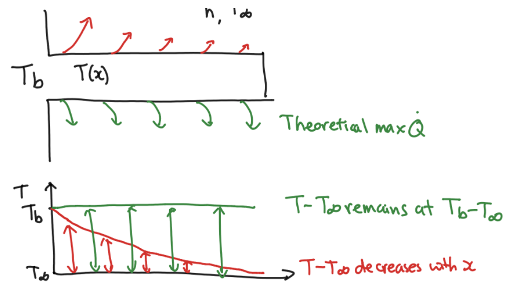
1. How does heat transfer rate compare to no fin at all? **(Effectiveness)**
2. How does heat transfer rate compare to the theoretical maximum? **(Efficiency)**

9.1 Fin Efficiency

$$\eta = \frac{\dot{Q}_{fin}}{\dot{Q}_{max}} = \frac{Actual\ heat\ transfer\ rate\ from\ the\ fin}{Ideal\ heat\ transfer\ rate\ from\ the\ fin\ if\ the\ entire\ fin\ were\ at\ base\ temperature}$$

$$\dot{Q}_{fin} = \eta_{fin}\dot{Q}_{max} = \eta_{fin}hA_{fin}(T_b - T_\infty)$$

- $A_{fin} = pL$ is the area of fin perimeter (ie. surface area)
- $\dot{Q}_{max} = hpL(T_b - T_L)$



1. **Case a (infinitely long fin):**

$$\eta = \frac{1}{aL}$$

- $a = \sqrt{\frac{hp}{kA_c}}$

2. **Case b (finite length, adiabatic tip):**

$$\eta = \frac{tanh(aL)}{aL}$$

- $a = \sqrt{\frac{hp}{kA_c}}$
- **Note:** This is the general case as mentioned in the effectiveness section.

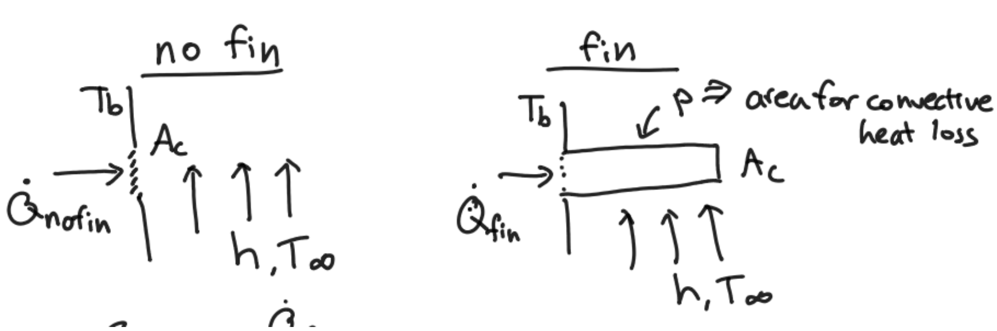
3. **Case c (finite length):**

$$\eta = \frac{tanh(aL_c)}{aL_c}$$

- $a = \sqrt{\frac{hp}{kA_c}}$

9.2 Fin Effectiveness

$$\varepsilon_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{nofin}}$$



1. **Case a (infinitely long fin):**

When $L \to \infty$, $tanh(aL) \to 1$.

$$\varepsilon = \sqrt{\frac{kp}{hA_C}}$$

2. **Case b (finite length, adiabatic tip):**

$$\varepsilon_{fin} = \sqrt{\frac{kp}{hA_C}} tanh(aL)$$

- A_c is the cross-sectional area of the fin at the base
- $\dot{Q}_{no\ fin} = hA_C(T_b - T_\infty)$ (Heat transfer rate if no fins are attached to the surface)
- $\dot{Q}_{fin} = \sqrt{hpkA_C}(T_b - T_\infty)tanh(\sqrt{\frac{hp}{kA_c}}L)$ (Heat transfer rate if fins are attached to the surface)
- **Note:** $tanh(aL)$ is between 0 and 1.
- **Note:** This is the general case, if you want infinitely long fin, then as $L \to \infty$ then $tanh(aL) \to 1$, and finite length has the correction length.

3. **Case c (finite length):**

When needing the correction factor to case b)

$$\varepsilon_{fin} = \sqrt{\frac{kp}{hA_C}} tanh(aL_c)$$

9.3 What Does Effectiveness Indicate?

- $\varepsilon_{fin} = 1$ indicates that the addition of fins to the surface does not affect heat transfer at all.
- $\varepsilon_{fin} < 1$ indicates that the fin actually acts as **insulation**, slowing down the heat transfer from the surface.
- $\varepsilon_{fin} > 1$ indicates that fins are **enhancing** heat transfer from the surface, and is viable when (ie. want ε≥2):

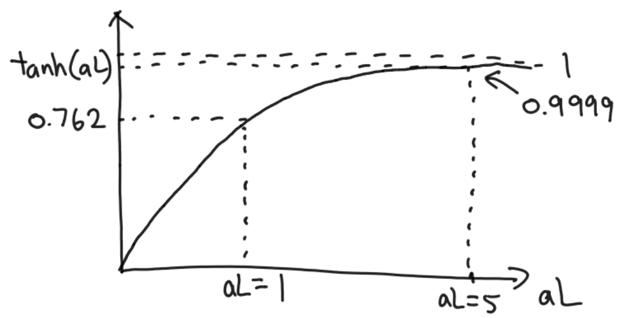
9.4 When is Effectiveness High?

ε_{fin} is high when

- k is high => use good conductors (eg. Cu, Al).
- $\frac{p}{A_c}$ ⇒ maximize specific surface area (eg. for cylinder, $\frac{p}{A_c} = \frac{2\pi r}{\pi r^2} = \frac{2}{r}$).
- h is low => works better in gasses (lower h) than in liquids.

9.5 When can we assume fins are infinitely long?

$$\frac{\dot{Q}_{finite}}{\dot{Q}_{inf.\,long}} = tanh(aL)$$



- $aL \geq 5 \Rightarrow infinitely\ long$

9.6 Incorporating Heat Fins in Heat Transfer Problems

$$\dot{Q}_{total} = \dot{Q}_{fin} + \dot{Q}_{nofin} = h(A_{nofin} + \eta_{fin}A_{fin})(T_b - T_\infty)$$

- $\dot{Q}_{nofin} = hA_{nofin}(T_b - T_\infty)$
- $\dot{Q}_{fin} = \eta_{fin}hA_{fin}(T_b - T_\infty)$

$$R_{heat\ sink} = \frac{T_b - T_\infty}{\dot{Q}_{total}} = \frac{1}{h(A_{nofin} + \eta_{fin}A_{fin})}$$

- **Note:** Table 17-3 gives $\eta_{fin}\ for\ many\ different\ shapes.$
- **Note:** This allows us to incorporate it as another resistance in the network, which has a object with both the area of the fin and no fin.

9.7 Relationship between fin efficiency and fin effectiveness

$$\varepsilon_{fin} = \frac{A_{fin}}{A_{nofin}} \eta_{fin}$$

- $A_{no\,fin}$ is the cross sectional area at the base.

9.8 Other Shapes

- **Note:** The other shapes give you a way to calculate the efficiency, which allows you to calculate $\dot{Q}$ using $\eta_{fin} \dot{Q}_{max} = \eta_{fin} h A_{fin} (T_b - T_\infty)$.
- **Note:** This also allows you to incorporate into resistance problems using what we learnt for resistive networks of heat fins in 9.6

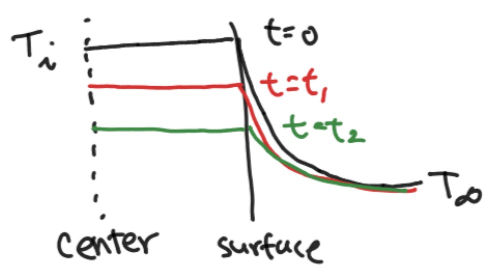
L10: Transient Heat Conduction Part 1 (18.1)

Learning Objectives

- Explain the criteria for lumped capacitance assumption
- Determine temperature as a function of time for transient heat conduction
- Explain the physical meaning of the Biot number

10.0 Physical Meaning of Lumped Capacitance Approximation

The object has a constant temperature (ie. no temperature gradient in the object).

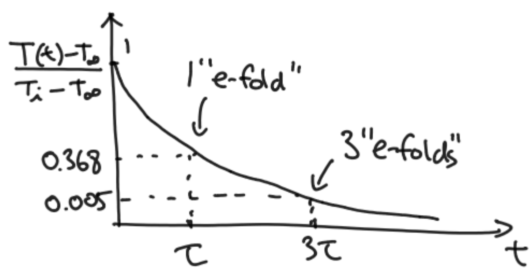


10.1 Temperature:

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-\frac{t}{\tau}}$$

- $\tau = \frac{\rho c_p V}{hA_s} = \frac{\rho c_p L_c}{h}$ $[s]$ (Time constant for temperature change)
  - V is the volume, A_s is the surface area.
- T_i is the initial temperature $T(t = 0)$
- T_∞ is the surrounding temperature
- **Note:** Assumes lumped capacitance approximation

10.2 Analysis of Time Constant for Temperature Change:



$t > 3\tau$ $T - T_\infty$ is within 0.5% of $T_i - T_\infty$

- **Note:** The smaller the value of τ, the higher the rate of decay in temperature to the environment.
- **Note:** Every τ is an e-fold.
  - **Shorter τ:** ↑ $\frac{A_s}{V}$, h: Easier to transfer heat per unit volume
    - **Eg.** Sphere: $\frac{A}{V} = \frac{4\pi r^2}{\frac{4}{3}\pi r^3} = \frac{3}{r}$
  - **Longer τ:** ↑ ρ, c_p: Easier to store energy.

10.3 Characteristic Length:

Characteristic length scale in the direction of conduction:

$$L_c = \frac{V}{A_s}$$

- A_s is the surface area

10.4 Biot Number:

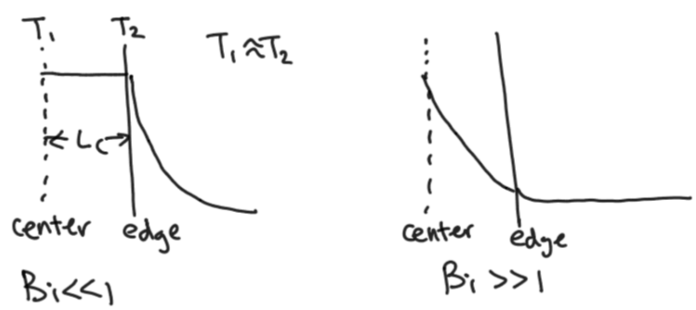
$$Bi = \frac{R_{cond}}{R_{conv}} = \frac{h_{fluid} L_c}{k_{solid}}$$

- **Note:** The Biot number is the ratio of the internal resistance of a body to heat conduction to its external resistance to heat convection.

10.5 When is Lumped Capacitance Assumption Valid?

$When\ Bi < 0.1\text{: } R_{cond} << R_{conv}\ lumped\ capacitance\ assumption\ holds$

10.6 Difference when Bi<<1 and Bi>>1:



10.7 Heat Transfer:

Convection Heat Transfer B/W Body and Environment

$$\dot{Q}(t) = hA_s[T_\infty - T(t)]\ [W]$$

10.8 Process

1. List knowns
2. Calculate the characteristic length
3. Calculate $Bi = \frac{V}{A_s} < 0.1$ to check lumped capacitance assumption.
   a. **Rectangular Wall:** $L_C = \frac{wall\ thickness}{2}$
   b. **Cylinder:** $L_c = \frac{r}{2}$

c. **Sphere:** $L_c = \frac{r}{3}$

4. If the lumped capacitance assumption doesn't hold, go to 11.5 process.
5. Calculate τ
6. Use the main equation

- **Note:** Sometimes you will have to assume LC and check after.
- **Note:** This is the **IDEAL** process, but normally, start with the governing equation and look at the unknowns and break the problem from a top-down approach.

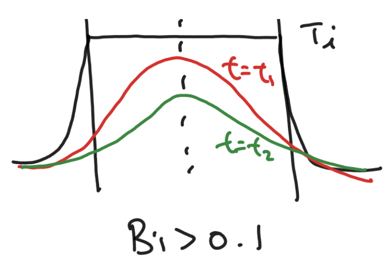
L11: Transient Heat Conduction Part 2 - Multidimensional Problems (18.2)

Learning Objectives:

- Calculate transient temperature profile of a wall, cylinder or sphere using one-term approximation of the heat equation.
- Analyze heat transfer problems using dimensionless numbers.
- Use appropriate scales to normalize time, position, and temperature in transient heat problems.
- Calculate the change in temperature at the centerline and at other positions in a material undergoing transient heat transfer.

11.0 Physical Meaning of Lumped Capacitance NOT Holding

- Assuming that $Bi > 0.1$, so our lumped sum analysis doesn't hold.
  - **Note:** Cannot neglect temperature gradient inside body.



11.1 Non-Dimensionalization (ie. Scaling)

1. **Dimensionalized Problem for a plane wall:**

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

- $\alpha = \frac{k}{\rho c_p}\left[\frac{m^2}{s}\right]$ (Thermal diffusivity)
- **IC:** $t = 0,\ T(x, 0) = T_i$
- **BC's:** $-k\frac{\partial T(x,t)}{\partial x}\Big|_{x=L} = h(T - T_\infty)\ (surface)$ and $\frac{\partial T}{\partial x}\Big|_{x=0} = 0$

2. **Scaling the PDE (ie. Nondimensionalization Exercise)**

Goal: Normalizing each variable by this quantity such that x is b/w 0 and 1 (ie. $x \sim o(1)$).

$T - T_\infty \sim T_i - T_\infty$: $$\theta(X, Fo) = \frac{T(x,t) - T_\infty}{T_i - T_\infty}$$ (dimensionless T)
$x \sim L$: $$X = \frac{x}{L}$$ (dimensionless x)
$$Bi = \frac{hL}{k_{solid}} = \frac{R_{cond}}{R_{conv}}$$ (dimensionless h)
$$Fo = \frac{\alpha t}{L_c^2}$$ (dimensionless time) • **Note:** Think Fo as "time relative to time needed to conduct heat across L." • **Note:** $Fo = \frac{time}{characteristic\ timescale\ for\ heat\ diffusion\ (ie.\ conduction)}$

- **Note:** In cylindrical or spherical coordinates, they can be non-dimensionalized in a similar way.
    - **Note:** Replace the space variable x with r and the half-thickness L with the outer radius r_o (these are treated as the characteristic length instead of V/A for the lumped system analysis).
- **Note:** All these are on the order of 1 magnitude.

Dimensionless DE:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} \Rightarrow \frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial Fo}$$

Dimensionless BC's: $\frac{\partial \theta(0,Fo)}{\partial X} = 0$ and $\frac{\partial \theta(1,Fo)}{\partial X} =- Bi\theta(1, Fo)$

Dimensionless IC: $\theta(X, 0) = 1$

Solution:

$$\theta = \theta(X, Bi, Fo)$$

- **Note:** X gives position, Bi gives heat transfer properties, and Fo is time.

11.2 Special Case: Lumped Capacitance (Bi<0.1)

$$\theta = e^{-\frac{t}{\tau}} = exp\left(-\frac{hA_s t}{\rho V c_p}\right) = e^{-Bi\cdot Fo} = f(Fo, Bi)$$

- Note: The lumped system analysis is a special case of this with no space variables.

11.3 One-Term Approximation (Not Expected to Know Series Table 18-1)

For $Fo > 0.2$ ($or\ t > \frac{0.2L^2}{\alpha}$), one-term approximation is within 2% (Table 18-2).

- **Note:** A_1, λ_1 are functions of Bi.

Plane Wall

Temperature

$$\theta_{wall}(X, Fo) = \frac{T(x,t)-T_\infty}{T_i-T_\infty} = A_1 e^{-\lambda_1^2 Fo} cos\left(\lambda_1 X\right),\ Fo > 0.2$$

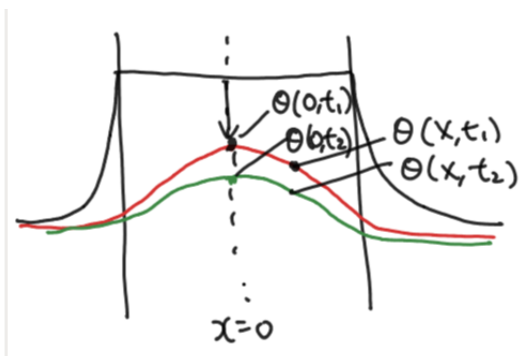
Center (X=0):

$$\theta_{0,wall}(0, Fo) = \frac{T_0-T_\infty}{T_i-T_\infty} = A_1 e^{-\lambda_1^2 Fo}$$

Ratio:

$$\frac{\theta_{wall}(X,Fo)}{\theta_{0,wall}(0,Fo)} = cos\left(\lambda_1 X\right)$$

- **Note:** $cos\left(\lambda_1 X\right)$ is constant w.r.t time, so the ratios will be the same at all times and only depend on the position.



Cylinder

Temperature

$$\theta_{cyl}(X, Fo) = \frac{T(r,t)-T_\infty}{T_i-T_\infty} = A_1 e^{-\lambda_1^2 Fo} J_0\left(\frac{\lambda_1 r}{r_0}\right),\ Fo > 0.2$$

Center (r=0):

$$\theta_{0,cyl}(0, Fo) = \frac{T_0-T_\infty}{T_i-T_\infty} = A_1 e^{-\lambda_1^2 Fo}$$

Ratio: $$\frac{\theta_{cyl}(X,Fo)}{\theta_{0,cyl}(0,Fo)} = J_0\left(\frac{\lambda_1 r}{r_0}\right)$$
Sphere
Temperature $$\theta_{sph} = \frac{T(r,t)-T_\infty}{T_i-T_\infty} = A_1 e^{-\lambda_1^2 Fo}\frac{sin\left(\frac{\lambda_1 r}{r_0}\right)}{\left(\frac{\lambda_1 r}{r_0}\right)},\ Fo > 0.2$$ • $X = \frac{r}{r_0}$
Center (r=0): $$\theta_{0,sph} = \frac{T_0-T_\infty}{T_i-T_\infty} = A_1 e^{-\lambda_1^2 Fo}$$
Ratio: $$\frac{\theta_{sph}(X,Fo)}{\theta_{0,sph}(0,Fo)} = \frac{sin\left(\frac{\lambda_1 r}{r_0}\right)}{\left(\frac{\lambda_1 r}{r_0}\right)}$$

11.4 Table 18-2:

TABLE 18–2

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres (Bi = hL/k for a plane wall of thickness $2L$, and Bi = hr_o/k for a cylinder or sphere of radius r_o)

	Plane wall		*Cylinder*		*Sphere*	
Bi	λ_1(rad)	A_1	λ_1(rad)	A_1	λ_1(rad)	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

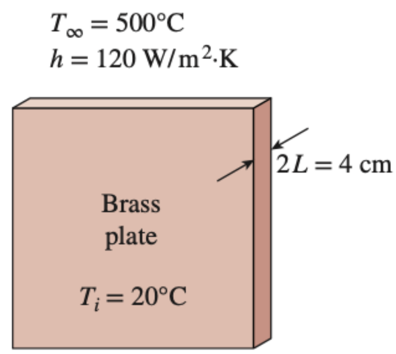
TABLE 18–3

The zeroth- and first-order Bessel functions of the first kind

η	$J_0(\eta)$	$J_1(\eta)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	0.4708
2.8	−0.1850	0.4097
3.0	−0.2601	0.3391
3.2	−0.3202	0.2613

11.5 Process Assuming Lumped Capacitance has Been Checked

1. List knowns
2. Calculate characteristic length scale depending on geometry.
   a. **Careful:** Table 18-2, $L_c = \frac{d_0}{2} = r_0$ should be used for cylinders and spheres
      i. **NOT** $L_c = \frac{r}{2}$ *or* $L_c = \frac{r}{3}$. So the "real" Bi number uses $\frac{V}{A_s}$, while in Table 18-2, it's different.
      ii. **Note:** $L_c = \frac{thickness}{2}$ for plane walls in both cases.
      iii. This also applies to L_c in Fo.



3. Get values from the table for the appropriate values of λ_1 and A_1 corresponding to the Bi.
    a. Interpolation may be needed: $\frac{Y_{avg}-Y_1}{X_{avg}-X_1} = \frac{Y_2-Y_1}{X_2-X_1}$
    b. Find J_0 if applicable to the geometry.
4. Calculate $\alpha = \frac{k}{\rho c_p} \left[\frac{m^2}{s}\right]$
5. Use one term approximation formulas and Fo to calculate what you need.
6. Check Fo>0.2 assumption.

- **Note:** Use $L_c = \frac{V}{A_s}$ to determine if $Bi < 0.1$. If so, use that L_c as is to determine τ. If $Bi = \frac{V}{A_s} > 0.1$ or if the question asks to use 1TA, then recompute Bi using $L_c = \frac{thickness}{2}$ or $L_c = r_0$
    - **Note (E18-4):** Either definition can be used in determining the applicability of the lumped system analysis unless $Bi \approx 0.1$.
- **Note Q61:** The lowest temperature will occur when $\frac{r_0}{r} = 1$.
- **Note:** This is the **IDEAL** process, but normally, start with the governing equation and look at the unknowns and break the problem from a top-down approach.

L12: Transient Heat Conduction Part 3 - Semi-Infinite Body (18.3)

Learning Objectives:

- Describe the implications of idealization of a semi-infinite body
- Calculate the transient temperature profile and heat flux across a semi-infinite body with a specified surface temperature.

12.0 Definition of Semi-Infinite Solid

Infinite in all directions except for one, which has a finite position that it starts at.

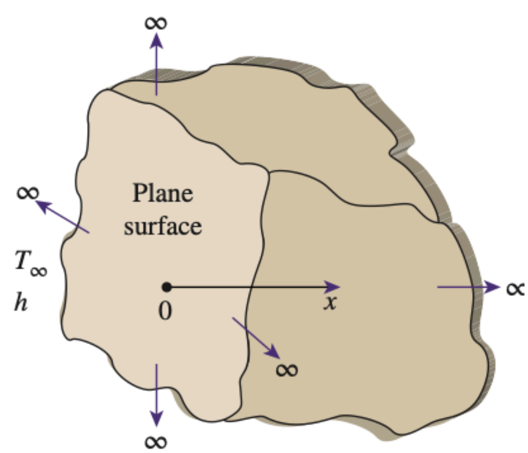


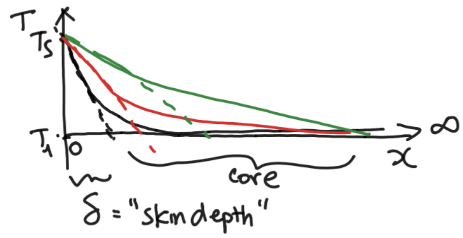
FIGURE 18–19
Schematic of a semi-infinite body.

12.1 Temperature vs. Position of Semi-Infinite Body

Skin Depth: Sharp temperature gradient, $\delta = \delta(t) \sim \sqrt{\alpha t}$ $(ie.\ function\ of\ time)$, which gives an idea how the "steep part" of the temperature profile changes with time.

- **Note:** This is the part that has the most dramatic change in the temp..

Core: Temperature gradient is "flat"



12.2 Similarity Variable

$$\eta = \frac{x}{2\delta} = \frac{x}{2\sqrt{\alpha t}}$$

- α is the thermal diffusivity.
- **Note:** Similarity variable is the x-position relative to where the steep temperature gradient is at that particular point in time.

12.3 Error Function and Complementary Error Function

$$erf(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-u^2} du$$

$$erfc(\eta) = 1 - erf(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-u^2} du$$

TABLE 18–4

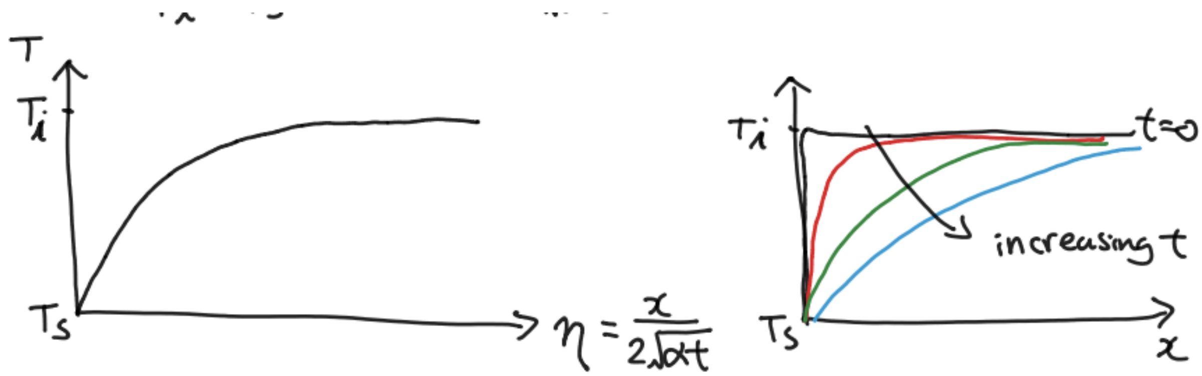
The complementary error function

η	erfc (η)	η	erfc (η)	η	erfc (η)	η	erfc (η)	η	erfc (η)	η	erfc (η)
0.00	1.00000	0.38	0.5910	0.76	0.2825	1.14	0.1069	1.52	0.03159	1.90	0.00721
0.02	0.9774	0.40	0.5716	0.78	0.2700	1.16	0.10090	1.54	0.02941	1.92	0.00662
0.04	0.9549	0.42	0.5525	0.80	0.2579	1.18	0.09516	1.56	0.02737	1.94	0.00608
0.06	0.9324	0.44	0.5338	0.82	0.2462	1.20	0.08969	1.58	0.02545	1.96	0.00557
0.08	0.9099	0.46	0.5153	0.84	0.2349	1.22	0.08447	1.60	0.02365	1.98	0.00511
0.10	0.8875	0.48	0.4973	0.86	0.2239	1.24	0.07950	1.62	0.02196	2.00	0.00468
0.12	0.8652	0.50	0.4795	0.88	0.2133	1.26	0.07476	1.64	0.02038	2.10	0.00298
0.14	0.8431	0.52	0.4621	0.90	0.2031	1.28	0.07027	1.66	0.01890	2.20	0.00186
0.16	0.8210	0.54	0.4451	0.92	0.1932	1.30	0.06599	1.68	0.01751	2.30	0.00114
0.18	0.7991	0.56	0.4284	0.94	0.1837	1.32	0.06194	1.70	0.01612	2.40	0.00069
0.20	0.7773	0.58	0.4121	0.96	0.1746	1.34	0.05809	1.72	0.01500	2.50	0.00041
0.22	0.7557	0.60	0.3961	0.98	0.1658	1.36	0.05444	1.74	0.01387	2.60	0.00024
0.24	0.7343	0.62	0.3806	1.00	0.1573	1.38	0.05098	1.76	0.01281	2.70	0.00013
0.26	0.7131	0.64	0.3654	1.02	0.1492	1.40	0.04772	1.78	0.01183	2.80	7.5E-05
0.28	0.6921	0.66	0.3506	1.04	0.1413	1.42	0.04462	1.80	0.01091	2.90	4.1E-05
0.30	0.6714	0.68	0.3362	1.06	0.1339	1.44	0.04170	1.82	0.01006	3.00	2.2E-05
0.32	0.6509	0.70	0.3222	1.08	0.1267	1.46	0.03895	1.84	0.00926	3.20	6.0E-06
0.34	0.6306	0.72	0.3086	1.10	0.1198	1.48	0.03635	1.86	0.00853	3.40	1.5E-06
0.36	0.6107	0.74	0.2953	1.12	0.1132	1.50	0.03390	1.88	0.00784	3.60	3.6E-07

12.3 Case 1: Specified Surface Temperature, $T_s = constant$

$$\frac{T(x,t)-T_s}{T_i-T_s} = erf\left(\frac{x}{2\sqrt{\alpha t}}\right) \mid \frac{T(x,t)-T_i}{T_s-T_i} = erfc\left(\frac{x}{2\sqrt{\alpha t}}\right) = 1 - erf\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$\dot{q}_s(t) = \frac{k(T_s-T_i)}{\sqrt{\pi\alpha t}}$$



- **Note for the right picture:** Assume the surface body is cold, then the temperature at time=0 becomes hot at T_i, so it spikes up.
  - Over time the heat flux goes from right to left (ie. from hot to cold), making the surface gradually hotter over time.
  - So once an earlier surface becomes hot, then the subsequent layer will become hot, and it will propagate through.
  - As a result, the skin depth increases (ie. the stretching of the function) such that it remains constant, so the skin depth grows over time.
  - So the skin depth is how the heat penetrates into the surface over time.
- **Note:** The similarity variables take in both time and position.

https://www.geogebra.org/m/fsvkzhr7

12.4 Process

1. Calculate $\alpha = \frac{k}{\rho c_p} \left[\frac{m^2}{s}\right]$

2. Calculate $\left(\frac{x}{2\sqrt{\alpha t}}\right)$ then find the appropriate erf.
    a. **Note:** May need to interpolate
3. Use the equations above to find what you are looking for to find the temperature or heat transfer rate.

Convection

L13: Forced Convection, Velocity and Thermal Boundary Layers (19.1-2)

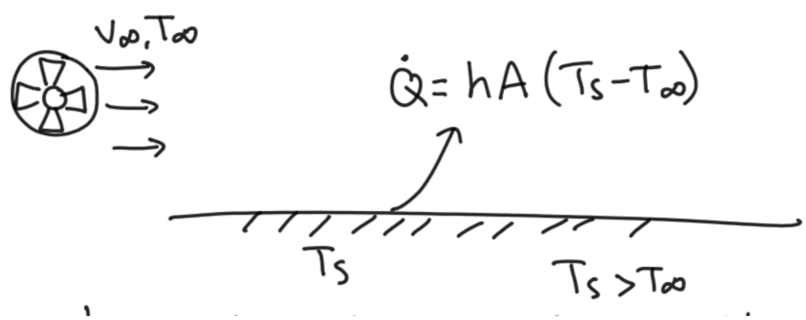
Learning Objectives

- Define the momentum and thermal boundary layers, and explain their evolution over a flat plate.
- Explain how shear stress and heat transfer coefficients are calculated from velocity and temperature profiles in the boundary layer, respectively.
- Differentiate between local and average heat transfer coefficient.
- Describe the change of local heat transfer coefficient over a flat plate and provide physical reasoning behinds its trend
- State the criteria for laminar and turbulent flows over a flat plate.

13.0 Convection:

Heat transfers from a stationary surface to a moving fluid.

- **Note:** Forced convection is motion imposed by external means.



13.1 What is h in general?

$$h = f(geometry, fluid\ properties,\ flow\ properties)$$

Relevant Properties:

- **Fluid Properties:**
  - k is conductivity
  - c_p is heat capacity
  - ρ is density
  - μ is viscosity
- **Flow Properties:**
  - v_∞ is velocity
  - *Re* is Reynold's number (laminar/turbulent)
- **Geometry:**
  - Shape (plate, cylinder, pipe, sphere,...)
  - Size
  - Characteristic length scale.

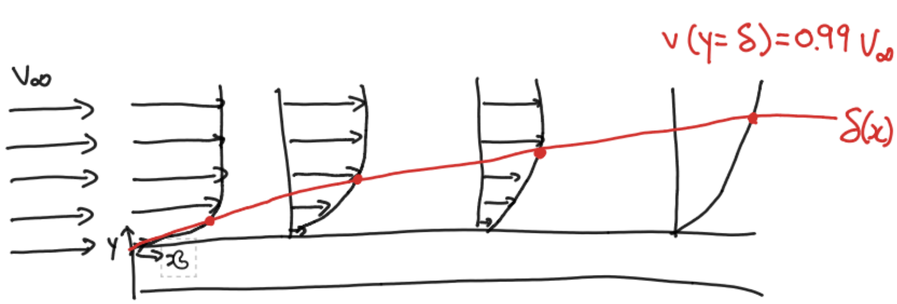
13.2 Boundary Layer: Momentum

The flow region adjacent to the wall in which the viscous effects (and thus the velocity gradients) are significant

Shear Stress:

$$\tau = \mu \frac{\partial v}{\partial y}\Big|_{y=0}$$

- μ (Fluid viscosity)
  - **Note:** Fluid viscosity is the fluid property responsible for the no-slip condition and the development of the boundary layer.



- **Intuition:**
  - The more viscous, the more shear stress (harder it pulls)
  - The bottommost fluid will pull from the bottom up, as it starts a shear stress that opposes the velocity gradient.
    - So as you move through the channel, there will be more shear stress (horizontal gradient), making the velocity gradient less.
  - The propagation of shear stress is represented by the boundary layer.
  - The shear stress diffuses up the velocity.
  - Fluid exerts a drag force on the plane.
- **Redline:** Where the velocity is 99% of the free stream velocity as the boundary moves across in the x direction. So anything above the redline, is basically the free stream velocity.

13.3 No-Slip Condition

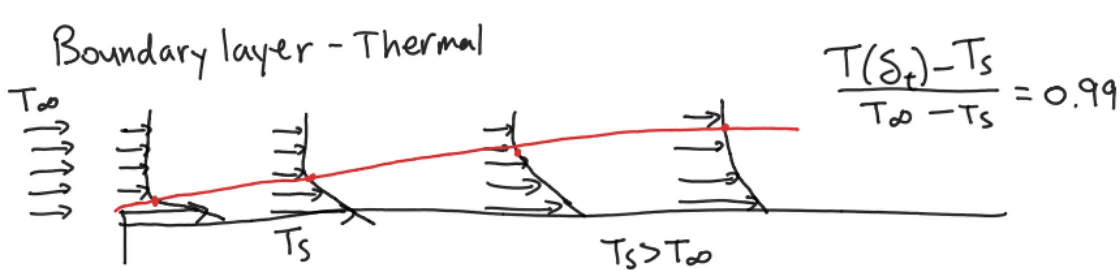
A fluid in direct contact with a solid "sticks" to the surface due to viscous effects, and there is no slip.

- **Note:** The no-slip condition is responsible for the development of the velocity profile, since there is 0 velocity at the solid wall.

13.4 Boundary Layer - Thermal

The flow region over the surface in which the temperature variation in the direction normal to the surface is significant.

$$\dot{q} = -k_{fluid} \frac{\partial T}{\partial y}\Big|_{y=0}$$



- **Intuition:**
  - In the beginning, the surface below is at T_s, but as the fluid moves, the temperature propagates upwards.
  - The arrow length is a temperature.
- **Redline:** Assuming the surface is hotter than the free stream temperature. As the flow increases, the heat conducts upwards so the next little part will be at 0.99. The part at which you reach 99% of the temperature is different as you move through x, so the redline demonstrates the temperature difference being almost the same, so anything above it will be essentially the free stream temperature.
- **Note:** x is the horizontal direction and y is the vertical direction.
  - Therefore, the rate of temperature is the sharpest at the surface, and decreases as y increases.

13.5 Heat Transfer Coefficient

$$h = \frac{-k_{fluid} \frac{\partial T}{\partial y}|_{y=0}}{T_s - T_\infty} \Rightarrow h\ is\ a\ function\ of\ x$$

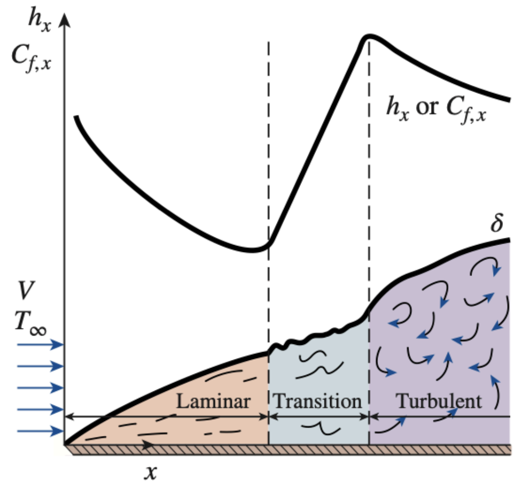
- **Derivation:** $\dot{q} = -k_{fluid} \frac{\partial T}{\partial y}|_{y=0}$ due to the no-slip condition ($v = 0$) at $y = 0$. This is equal to $h(T_s - T_\infty) = -k_{fluid} \frac{\partial T}{\partial y}|_{y=0}$.

13.6 Average h:

$$\bar{h} = \frac{1}{L}\int_0^L h dx$$

- **Note:** Since $\frac{\partial T}{\partial y}$ changes with x, average h is wanted.

13.7 How does h change with x?



- **Beginning - Laminar Region:** The velocity flow is laminar flow, so the momentum is moderated.
- **Transition:** As the position increases, the boundary layer becomes thicker, so the boundary layer becomes turbulent.
- **Ending - Turbulent Region:** With the turbulent flow, the heat transfer increases since the boundary can spin onto the surface.
- **Importance:** Need to see if laminar or turbulent before calculating heat transfer.

13.8 Reynold's Number (When does the flow become turbulent?)

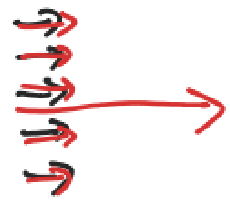
$$Re = \frac{\rho v_\infty x}{\mu}$$

- **Note:** Re is the ratio of inertial stress to viscous (shear stress)
  - $inertia\ stress \sim change\ in\ kinetic\ energy\ of\ fluid \sim \rho v_\infty^2$
  - $shear\ stress: \tau = \mu \frac{\partial v}{\partial y}\ ,\ where\ v \sim v_\infty\ and\ y \sim L$
- **Note:** Critical Reynold's Number for a Flat Plate is $Re_{cr} = 10^5 \sim 3 \times 10^6$

Intuitive Explanation of Reynold's Number:

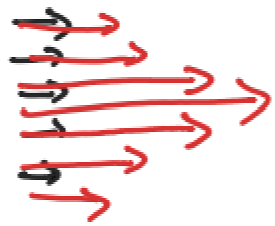
1. **High Re**

- Shear not enough to keep consistent flow
- Any small perturbation will stay, allowing for bigger differences in velocity.



2. **Low Re**

- Shear "moderates" velocities in adjacent layers.
- Easier for shear stress to "diffuse" through adjacent layers to counter the velocity gradient.



13.9 Momentum Transfer vs. Heat Transfer (Closely Linked)

$$\tau = \mu \frac{\partial v}{\partial y} = \frac{\mu}{\rho} \frac{\partial(\rho v)}{\partial y} \mid \dot{q} = - k \frac{\partial T}{\partial y} = - \frac{k}{\rho c_p} \frac{\partial(\rho c_p T)}{\partial y}$$

- **Momentum Transfer:**
  - $\nu = \frac{\mu}{\rho} \left[\frac{m^2}{s}\right] (kinematic\ viscosity)$
  - $\rho v\ (concentration\ of\ momentum)$
- **Heat Transfer:**
  - $\alpha = \frac{k}{\rho c_p} \left[\frac{m^2}{s}\right] (thermal\ diffusivity)$
  - $\rho c_p T\ (concentration\ of\ thermal\ energy)$

L14: Dimensionless Numbers - Reynolds, Prandtl and Nusselt Numbers (19.3)

Learning Objectives

- Explain the physical meaning of each term in the full heat balance equation.
- Render a governing equation dimensionless and identify dimensionless numbers important for the problem.
- Explain the physical meaning of Prandtl number
- Explain the physical meaning of Nusselt number, and explain why correlations for heat transfer problems are expressed as: $Nu = f(Pr, Re)$

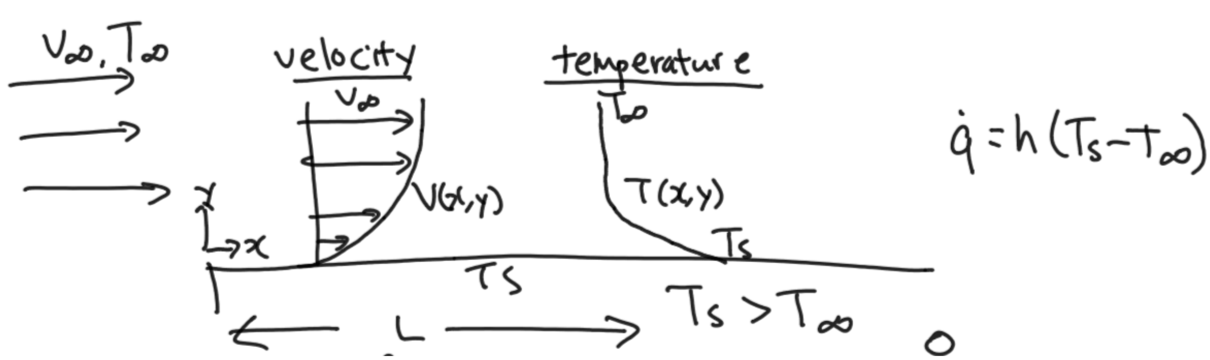
14.0 Non-Dimensionalization

1. **Full Heat Balance Equation:**

$$\rho c_p\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + \vec{v}\cdot\nabla T\right) = k\nabla^2 T$$

2. **Physical Intuition:**



3. **Simplifying Heat Balance Equation:**

$$\rho c_p\left(v_x \frac{\partial T}{\partial x}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

- **Assumptions:**
  - Steady state (any change in time-dependent terms are 0).
  - $v_z = 0,\ \frac{\partial}{\partial z} = 0,\ v_y = 0$

4. **Scaling Analysis**

- Objective: Try to find $\dot{q} = h(T_s - T_\infty)$ using $\rho c_p\left(v_x \frac{\partial T}{\partial x}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$
- Note: find the characteristic scale for each variable to ensure o(1).

x	$x \sim L:\ x^* = \frac{x}{L}$
y	$y \sim L:\ y^* = \frac{y}{L}$
T	$(T_s - T) \sim (T_s - T_\infty):\ \theta^* = \frac{T_s - T}{T_s - T_\infty}$ $\frac{\partial \theta^*}{\partial x} = \frac{1}{T_\infty - T_s}\frac{\partial T}{\partial x}\ \vert\ \frac{\partial \theta^*}{\partial x^*} = L\frac{\partial \theta^*}{\partial x} = \frac{L}{T_\infty - T_s}\frac{\partial T}{\partial x}$ $\frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{L^2}{T_\infty - T_s}\frac{\partial^2 T}{\partial x^2}\ \vert\ \frac{\partial^2 \theta^*}{\partial y^{*2}} = \frac{L^2}{T_\infty - T_s}\frac{\partial^2 T}{\partial y^2}$
v_x	$v_x \sim v_\infty:\ v^* = \frac{v_x}{v_\infty}$ • $v_\infty\ (free\ stream\ velocity)$

5. **Subbing in dimensionless scaling factors into the heat balance equation:**

$$v^* \frac{\partial \theta^*}{\partial x^*} = \frac{k}{\rho c_p v_\infty L}\left(\frac{\partial^2 \theta^*}{\partial x^{*2}} + \frac{\partial^2 \theta^*}{\partial y^{*2}}\right)$$

6. **Finding the critical dimensionless numbers**

$$\frac{k}{\rho c_p v_\infty L} = \frac{k}{\rho c_p} \cdot \frac{\rho}{\mu} \cdot \left(\frac{\mu}{\rho v_\infty L}\right) = \frac{(k/\rho) c_p}{\mu/\rho}\frac{1}{Re} = \frac{\alpha}{\nu}\frac{1}{Re} = \frac{1}{Pr}\frac{1}{Re}$$

7. **Find h:**

$$h = \frac{-k_f \frac{\partial T}{\partial y}|_{y=0}}{T_s - T_\infty}$$

- **Note:** $h = f(L, v_\infty, \rho, \mu, c_p, k_f, geometry)$

Use the non-dimensional factors to isolate $\frac{\partial\theta}{\partial y^*}|_{y^*=0}$

$$\frac{\partial\theta}{\partial y^*}|_{y^*=0} = \frac{hL}{k_{fluid}} = Nu$$

14.1 Nusselt Number (ie. Dimensionless Temperature Gradient or Convection Heat Transfer Coefficient)

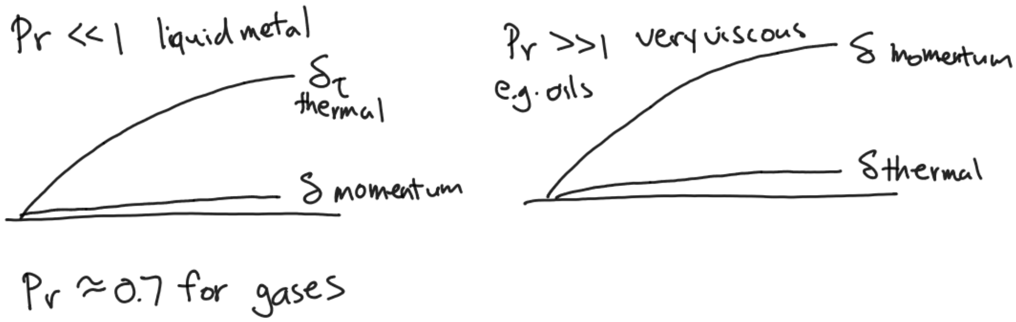
$$Nu = \frac{hL_c}{k_{fluid}}$$

- L_c is the characteristic length
- k is the thermal conductivity.
- **Note:** the Nusselt number represents the enhancement of heat transfer through a fluid layer.
    - $\uparrow Nu \Rightarrow \uparrow Effective\ Convection$
    - $Nu = 1 \Rightarrow heat\ transfer\ across\ the\ layer\ by\ conduction\ only$
- **Note:** Nu is the ratio of convection to conduction in a fluid.
- **Note:** $Nu = f(Re, Pr, geometry)$

14.2 Prandtl Number

$$Pr = \frac{\nu}{\alpha} = \frac{\mu c_p}{k}$$

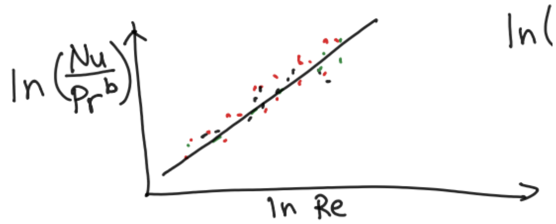
- **Note:** Pr is the ratio of momentum diffusivity to thermal diffusivity.



- **Note:** $\delta\ is\ diffusivity$

14.3 How to Find h (or dimensionless h) Experimentally?

- **Note:** This is determined experimentally by plotting $ln\left(\frac{Nu}{Pr^b}\right)$ with $ln(Re)$



- Therefore, all Nu numbers are of the form

$$Nu = c_1 Re^a Pr^b$$

- **Note:** $\forall\ experiments,\ where\ c_1,\ a,\ and\ b\ will\ depend\ on\ conditions$

L15: Flow Over Flat Plates, Cylinders and Spheres (19.4)

Learning Objectives

- Calculate heat transfer rates for external flows
- Identify the appropriate empirical correlation for the geometry and flow regime.
- Relate the heat transfer rate to changes in flow around circular objectives.

15.0 Empirical Relationships

$$Nu = c_1 Re^a Pr^b$$

- **Note:** Laminar on left and turbulent on right.
- **Note:** Re is specific to the flow

Average Nusselt Number Over the Entire Plate

- **Note:** The local Nusselt Number is at location x, while the average Nusselt number is over the entire plate L.

Laminar	Turbulent
$Nu = \frac{hL}{k_{fluid}} = 0.664 Re_L^{0.5} Pr^{1/3}$ • $Re_L < 5 \times 10^5$ • $Re_L = \frac{\rho v_\infty L}{\mu}$	$Nu = \frac{hL}{k_{fluid}} = 0.037 Re_L^{0.8} Pr^{1/3}$ • $0.6 \le Pr \le 60,\ 5 \times 10^5 \le Re_L \le 10^7$ • **Note:** This assumes that the laminar region is very small and that flow over most of the plate is turbulent.

Combined Laminar and Turbulence

$$Nu = \frac{hL}{k_{fluid}} = \left(0.037 Re_L^{0.8} - 871\right) Pr^{1/3}$$

- $0.6 \le Pr \le 60,\ 5 \times 10^5 \le Re_L \le 10^7$
- **Note:** Turbulent boundary layer ($Re_{cr} = 0$), simplifies to the equation for turbulent flow.
- **Note** A flat plate is sufficiently long for the flow to become turbulent, but not long enough to disregard the laminar flow region.

Churchill and Ozoe (ie. For laminar flow over an isothermal flat plate for all Pr)

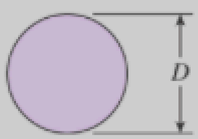
$$Nu_x = \frac{h_x x}{k} = \frac{0.3387 Pr^{1/3} Re_x^{1/2}}{\left[1 + (0.0468/Pr)^{2/3}\right]^{1/4}}$$

- $Re_x Pr \geq 100$

Option 1: Average Nusselt Number for External Flow Over Cylinders

Main content E 19▯2

Empirical correlations for the average Nusselt number for forced convection over circular and noncircular cylinders in crossflow

Cross section of the cylinder	Fluid	Range of Re	Nusselt number
Circle 	Gas or liquid	0.4 4	$Nu = 0.989\,Re^{0.330}\,Pr^{1/3}$
		4 40	$Nu = 0.911\,Re^{0.385}\,Pr^{1/3}$
		40 4000	$Nu = 0.683\,Re^{0.466}\,Pr^{1/3}$
		4000 40,000	$Nu = 0.193\,Re^{0.618}\,Pr^{1/3}$
		40,000 400,000	$Nu = 0.027\,Re^{0.805}\,Pr^{1/3}$

Option 2: Average Nusselt Number for External Flow Over Cylinder

$$Nu_{cyl} = \frac{hD}{k_{fluid}} = 0.3 + \frac{0.62Re^{1/2}Pr^{1/3}}{\left[1+(0.4/Pr)^{2/3}\right]^{1/4}}\left[1 + \left(\frac{Re}{282,000}\right)^{5/8}\right]^{4/5}$$

- All properties are evaluated @ $T_{film} = \frac{T_s + T_\infty}{2}$

Average Nusselt Number for External Flow Over Sphere

$$Nu_{sph} = \frac{hD}{k_{fluid}} = 2 + \left[0.4Re^{1/2} + 0.06Re^{2/3}\right]Pr^{0.4}\left(\frac{\mu_\infty}{\mu_s}\right)^{1/4}$$

- $3.5 \le Re \le 8 \times 10^4$, $0.7 \le Pr \le 380$, and $1.0 \le \left(\frac{\mu_\infty}{\mu_s}\right) \le 3.2$
- **Note:** Fluid properties are evaluated at the free-stream temperature T_∞.
- **Note:** μ_s is evaluated at the surface temperature T_s.
- **Note:** The results obtained from them can be off by as much as 30%.

15.1 Table A-22 (Use this to find appropriate properties)

Temp. T, °C	Density ρ, kg/m³	Specific heat c_p, J/kg · K	Thermal conductivity k, W/m · K	Thermal diffusivity α, m²/s	Dynamic viscosity μ, kg/m · s	Kinematic viscosity ν, m²/s
50	1.092	1007	0.02735	2.487×10^{-5}	1.963×10^{-5}	1.798×10^{-5}
60	1.059	1007	0.02808	2.632×10^{-5}	2.008×10^{-5}	1.896×10^{-5}
70	1.028	1007	0.02881	2.780×10^{-5}	2.052×10^{-5}	1.995×10^{-5}
80	0.9994	1008	0.02953	2.931×10^{-5}	2.096×10^{-5}	2.097×10^{-5}
90	0.9718	1008	0.03024	3.086×10^{-5}	2.139×10^{-5}	2.201×10^{-5}
100	0.9458	1009	0.03095	3.243×10^{-5}	2.181×10^{-5}	2.306×10^{-5}
120	0.8977	1011	0.03235	3.565×10^{-5}	2.264×10^{-5}	2.522×10^{-5}
140	0.8542	1013	0.03374	3.898×10^{-5}	2.345×10^{-5}	2.745×10^{-5}
160	0.8148	1016	0.03511	4.241×10^{-5}	2.420×10^{-5}	2.975×10^{-5}
180	0.7788	1019	0.03646	4.593×10^{-5}	2.504×10^{-5}	3.212×10^{-5}
200	0.7459	1023	0.03779	4.954×10^{-5}	2.577×10^{-5}	3.455×10^{-5}
250	0.6746	1033	0.04104	5.890×10^{-5}	2.760×10^{-5}	4.091×10^{-5}
300	0.6158	1044	0.04418	6.871×10^{-5}	2.934×10^{-5}	4.765×10^{-5}

15.2 Steps to Solve Problems

1. Find properties: ρ_{air}, k_{air}, μ_{air}, α, ν, Pr using Table A-22 by finding the appropriate temperature based on the flow over geometry (ie. flat plate, cylinder, or sphere) for Table A-22:
   a. **Careful:** Flow over cylinder and flat plate use $T_{film} = \frac{T_s + T_\infty}{2}\ [K]$
   b. **Careful:** Flow over sphere uses $T_\infty\ except\ for\ \mu_s,\ which$ is T_s.
   c. **Careful:** If at a different pressure, note that ν is **inversely proportional to the pressure, therefore:** $\nu_P = \frac{\nu_{@1atm}}{P\ [atm]}$.
   d. **Note:** Sometimes you will need to guess this value of temperature to get the values and check later.
   e. **Note:** Interpolate values in Table A-22.
   f. **Note:** Use $\nu\ instead\ of\ \frac{\rho}{\mu}$ for Reynold's Numbers
   g. **Note:** $101.325\ [kPa] = 1\ [atm]$
2. Choose correct empirical correlation for Nu
   a. Flow regime (ie. Calculate Re) to determine whether laminar or turbulent.

$$Re_L = \frac{\rho v_\infty L}{\mu} = \frac{v_\infty L}{\nu}$$

- **Note:** L is the length at which the boundary layer is formed. Another way to think about it is that it's parallel to the boundary layer across the whole plate.

3. Calculate the Nu using the found empirical correlation from 2)
4. Calculate h using Nu relationship:

$$Nu = \frac{hL}{k_{fluid}}$$

   a. **Careful:** Nu is similar to Bi, but we are using k_{fluid}, while Bi uses k_{solid}
5. Calculate $\dot{Q}$:

$$\dot{Q} = hA(T_s - T_\infty)$$

- **Note:** Be careful of the correlations you use.
- **Note:** This is the **IDEAL** process, but normally, start with the governing equation and look at the unknowns and break the problem from a top-down approach.

L17: Non-dimensionalization

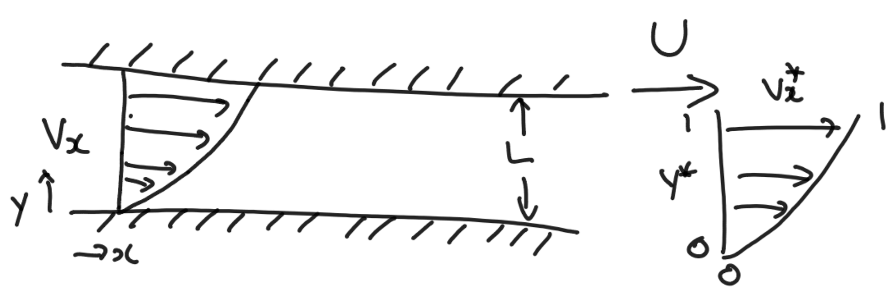
17.0 Navier Stokes Non-Dimensionalization

Motivation: Taking a complex differential equation, and have the dimensionless numbers pop out.

Navier Stokes Equation (F=ma)

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla\vec{v}\right) = -\nabla p + \mu\nabla^2\vec{v} + \rho\vec{g}$$

- $\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla\vec{v}\right)$ is the acceleration
- $-\nabla p$ is the pressure drop.
- $\rho\vec{g}$ is the gravitational acceleration
- $\mu\nabla^2\vec{v}$ is the shear force



In 1D:

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_x$$

1. **Simplify**

- Steady state: $\frac{\partial}{\partial t} = 0$.
- Ignore gravity.
- Ignore z direction: $\frac{\partial}{\partial z} = 0$

$$\rho v_x \frac{\partial v_x}{\partial x} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right)$$

2. **Determine all variables with a scale associated with it**

- $v_x,\ x,\ and\ y$

3. **Find the characteristic scale and normalize them w.r.t to the scale:**

$v_x \sim u$	$v_x^* = \frac{v_x}{u} \sim o(1) \Rightarrow v_x = v_x^* u$
$y \sim L$	$y^* = \frac{y}{L} \sim o(1) \Rightarrow x = x^* L$
$x \sim L$ • **Note:** Assume this is on the same length as y since we don't know the scale of this since we have no choice.	$x^* = \frac{x}{L} \sim o(1) \Rightarrow y = y^* L$

4. **Rewrite the equation, render equation dimensionless by subbing the equations in:**

$$\rho \frac{u^2}{L} v_x^* \frac{\partial v_x^*}{\partial x^*} = -\frac{1}{L}\frac{\partial p}{\partial x^*} + \mu \frac{u}{L^2}\left(\frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{\partial^2 v_x^*}{\partial y^{*2}}\right)$$

Multiply $by\ L^2\ and\ divide\ by\ u\ and\ divide\ by\ \mu$:

$$\frac{\rho u L}{\mu} v_x^* \frac{\partial v_x^*}{\partial x^*} = -\frac{L}{\mu u}\frac{\partial p}{\partial x^*} + \left(\frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{\partial^2 v_x^*}{\partial y^{*2}}\right)$$

$\partial p \sim \Delta p$:

$$\frac{\rho u L}{\mu} v_x^* \frac{\partial v_x^*}{\partial x^*} = -\frac{\rho u L}{\mu}\left(\frac{\Delta p}{\rho u^2}\right)\frac{\partial p^*}{\partial x^*} + \left(\frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{\partial^2 v_x^*}{\partial y^{*2}}\right)$$

5. **Analyze the problem by looking at dimensionless numbers.**

- $\frac{\partial v_x^*}{\partial x^*}$ is the acceleration (inertial)
- $\left(\frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{\partial^2 v_x^*}{\partial y^{*2}}\right)$ is the viscous term
- $\frac{\rho u L}{\mu}$ is Reynold's number
- $\frac{\Delta p}{\rho u^2}$ is related to friction coefficient

6. **Conclusion:**

Therefore, Re, and friction coefficient are key determinants of flow profile.

17.1 Process for Non Dimensionalization

1. Simplify the equation using the assumptions given.
2. Determine all variables that have a scale associated with it.
3. Find the characteristic scale and normalize the variable with the characteristic scale to make a new variable.
4. Rewrite the equation in terms of the new variable subbing the equations in, rendering the equation dimensionless
5. Analyze the problem by looking for dimensionless numbers
6. Make a conclusion.

17.2 Ed

#265