Medici MAT185: Cheat Sheet

Chapter 4: A Postcard from Vector Space

Vector Space:

A vector space V over a field Γ of elements {α, β, γ...} called scalars, is a **set** of elements $\{u, v, w...\}$ called vectors, such that the following **axioms are satisfied**.

These axioms involve two operations:

1. **Vector Addition**, denoted as $u + w$, such that for all $u, v, w \in V$

Ai. Closure: $u + v \in V$
Aii. Associativity: $(u + v) + w = u + (v + w)$
Aiii. Zero: ∃ a zero or null vector $0 \in V$ such that $u + 0 = u$
Aiv. Negative: ∃ a negative $-u \in V$ such that $u + (-u) = 0$

2. **Scalar Multiplication**, denoted as αu, such that for all $u, v \in V$ and all $\alpha, \beta \in \Gamma$

Mi. Closure: $\alpha u \in V$
Mii. Associativity: $\alpha(\beta u) = (\alpha\beta)u$
Miii. Distributivity:

a) $(\alpha + \beta)u = \alpha u + \beta u$
b) $\alpha(u + v) = \alpha u + \alpha v$

Be careful: One is the rule of adding real numbers and the other is adding vectors.
Miv. Unitary: For the identity element $1 \in \Gamma$, $1u = u$

Proposition I:

For every $u, -u \in V$, $\; -u + u = 0$

Proposition II:

For every, $u \in V,\ 0 + u = u$

Theorem I: Cancellation Theorem

If $u + w = v + w$ then $u = v$ for any $u, v, w \in V$

Proposition III:

Let $u \in V$, then…

1. The zero vector $0 \in V$ is unique.
2. The negative $-u$ of u is unique.
3. $-(-u) = u$

Definition of Subtraction:

If $u, v \in V$, then the subtraction of v from u, denoted by $u - v$, is

Proposition IV (Commutative Property):

For $u, v \in V,\ u + v = v + u$

Proposition V (Properties of Zero):

$\forall\, v \in V$ and all $\alpha \in \Gamma$,

1. $0v = 0$
2. $\alpha 0 = 0$
3. If $\alpha v = 0$, then either $\alpha = 0$ *or* $v = 0$

Proposition VI:

$\forall\, v \in V$ and $\alpha \in \Gamma$, $(-\alpha)v = -(\alpha v) = \alpha(-v)$

Chapter 5: The Subspace Homesick Blues

Definition of Subset:

A subset U of a vector space V is a subspace of V iff U is itself a vector space over the same field Γ with the same vector addition and scalar multiplication of V

Theorem I: Subspace Test

Let U be a subset of a vector space V. Then U is a subspace of V, over the same field Γ with the same vector addition and scalar multiplication as V, iff for all $u, v \in V$ and all $\alpha \in \Gamma$,

S1. Zero: $\exists$ a zero or null vector $0 \in U$
S2. Closure under VA: $u + v \in U$
S3. Closure under SM: $\alpha u \in U$

Definition of Linear Combination:

A vector $v \in V$ is linear combination of $\{v_1, v_2 ... v_n\} \subset V$ iff it can be written as

$$v = \sum_{j=1}^{n} \lambda_j v_j = \lambda_1 v_1 ... + \lambda_n v_n \text{ for some } \lambda_j \in \Gamma$$

Definition of Span:

The span of $\{v_1, v_2 ... v_n\} \subset V$, denoted $span\{v_1, v_2 ... v_n\}$, is

$$span\{v_1, v_2 ... v_n\} = \{v | v = \sum_{j=1}^{n} \lambda_j v_j, \forall \lambda_j \in \Gamma\}$$

Proposition I:

The span of $\{v_1, v_2 ... v_n\} \subset V$ is a subspace of the vector space V

Proposition II:

Let $U = span\{v_1, v_2 ... v_n\} \sqsubseteq V$. If W is subspace of V containing the vectors $\{v_1, v_2 ... v_n\}$, then $U \sqsubseteq W$.

Chapter 6: Covering All the Bases

Definition of Linear Independence:

A set of vectors $\{v_1, v_2 ... v_n\} \subset V$ is linearly independent iff

$$\sum_{j=1}^{n} \lambda_j v_j = \lambda_1 v_1 + ... + \lambda_n v_n = 0$$

implies that all $\lambda_j = 0$

Proposition I:

If $\{v_1, v_2 ... v_n\} \subset V$ is linearly independent and $v = \sum_{j=1}^{n} \lambda_j v_j$ for all $v \in V$, then λ_j are uniquely determined.

Theorem I:

Let $\{v_1, v_2 ... v_n\} \subset V$, a vector space. For every $v_k (k = 1 ... n)$, $span\{v_1 ... v_{k-1}, v_{k+1} ... v_n\} \subsetneq span\{v_1 ... v_n\}$ iff $\{v_1, v_2 ... v_n\}$ is linearly independent.

Corollary:

Let $\{v_1, v_2 ... v_n\} \subset V$, a vector space. For at least one $v_k (1 \le k \le n)$, $span\{v_1 ... v_{k-1}, v_{k+1} ... v_n\} = span\{v_1 ... v_n\}$ iff $\{v_1, v_2 ... v_n\}$ is linearly dependent.

Theorem II Fundamental Theorem of Algebra:

Let V be a vector space spanned by n vectors. If a set of m vectors from V is linearly independent, then $m \le n$.

Definition of Bases:

A set of vectors $\{e_1, e_2 ... e_n\} \in V$ is a basis for the vector space V iff

1. $\{e_1, e_2 ... e_n\}$ is linearly independent
2. $\{e_1, e_2 ... e_n\}$ spans V.

Theorem III:

Every basis for a given vector space contains the same number of vectors.

Definition of Dimensions:

The dimension of a vector space V, denoted $dim\ V$, is the number of vectors in any of its bases.

Proposition II:

Let V be a finite-dimensional vector space w/ $dim\ V = n$. Then,

1. A linearly independent set of vectors in V can at most contain n vectors.
2. A spanning set for V must at least contain n vectors.

Theorem IV:

Let $\{v_1, v_2 ... v_n\} \subset V$ be linearly independent. Then for a vector $v \in V$, $\{v, v_1, v_2 ... v_n\}$ is linearly independent iff $v \notin span\{v_1, v_2 ... v_n\}$

Theorem V Existence of Bases:

Let V be a vector space spanned by a finite set of vectors. Then every linearly independent set of vectors in V can be extended to a basis for V.

Theorem VI:

Let U & W be subspaces of a finite-dimensional vector space V. It follows that

1. U is finite-dimensional and $dim\, U \leq dim\, V$
2. If $U \subseteq W$, then $dim\, U \leq dim\, W$
3. If $U \subseteq W$ and $dim\, U = dim\, W$, then $U = W$

Theorem VII:

Any spanning set for a vector space V contains a basis for V.

Theorem VIII:

Let V be a vector space and $dim\, V = n$. Then,

1. Any set $\{v_1 ... v_n\} \subset V$ that's linearly independent is a basis for V; and
2. Any set $\{v_1 ... v_n\} \subset V$ that spans V is a basis for V.

Chapter 7: Rank and File

Definition of Row Space:

The row space of $A \in {}^{m}R^{n}$, denoted row A, is $row\, A \triangleq span\{r_1, r_2 ... r_m\}$ where $r_i \in R^n$ are the rows of A.

Definition of Column Space:

The column space of $A \in {}^{m}R^{n}$, denoted col A, is $col\, A \triangleq span\{c_1, c_2 ... c_n\}$ where $c_j \in {}^{m}R$ are the columns of A.

Proposition I:

Let $A \in {}^{m}R^{n}$, $U \in {}^{m}R^{m}$ and $V \in {}^{n}R^{n}$. Then $row\, UA \leq row\, A$ with equality holding if U is invertible. Furthermore, $col\, AV \subseteq col\, A$ with equality holding if V is invertible.

Proposition II:

Let $\{x_1, x_2 ... x_r\} \subset {}^m R$ and let $U \in {}^m R^m$ be invertible. Then $\{x_1, x_2 ... x_r\}$ is linearly independent iff $\{U_{x_1}, U_{x_2} ... U_{x_r}\}$ is linearly independent.

Lemma I:

Let $A \in {}^m R^n$. Then $row\ \tilde{A} = row\ A$, where Ã is the RREF of A, and hence $dim\ row\ \tilde{A} = dim\ row\ A$. Moreover, the non-zero rows of Ã constitute a basis for $row\ A$.

Lemma II:

Let $A \in {}^m R^n$. Then

1. The set of columns with leading "1"s $\{ç_{j_1}, ç_{j_2} ... ç_{j_r}\}$ of Ã, the RREF of A, constitutes a basis for $col\ \tilde{A}$.
2. The set of corresponding columns $\{c_{j_1}, c_{j_2} ... c_{j_r}\}$ of A constitutes a basis for col A.

As such $dim\ col\ \tilde{A} = dim\ col\ A$

Theorem I:

Let $A \in {}^m R^n$. Then $dim\ row\ A = dim\ col\ A$

Definition of Rank:

Let $A \in {}^m R^n$. The rank of A, denoted rank A, is the common dimension of row A and col A.

Properties of Rank:

Property I: If $A \in {}^{m}R^{n}$, then $rank\ A = rank\ \tilde{A}$

Property II: If $A \in {}^{m}R^{n}$, then $rank\ A = rank\ A^{T}$

Property III: If $A \in {}^{m}R^{n}$, $U \in {}^{m}R^{m}$ and $V \in {}^{n}R^{n}$, then $rank\ UA \leq rank\ A$ and $rank\ AV \leq rank\ A$ with equality holding if U and V are, respectively, invertible.

Theorem II Dimension Formula:

Let $A \in {}^{m}R^{n}$. Then $dim\ null\ A = n - rank\ A$

Theorem III: (Square Matrices)

Let $A \in {}^{n}R^{n}$. Then the following statements are equivalent.

1. A is invertible
2. A has full rank n
3. The rows of A are linearly independent.
4. The columns of A are linearly independent.
5. For $x \in {}^{n}R$, $Ax = 0$ implies $x = 0$
6. For $z \in {}^{n}R$, $z^{T}A = 0$ implies $z = 0$

Theorem IV: (Column Version)

Let $A \in {}^{m}R^{n}$. Then the following statements are equivalent.

1. $rank\ A = n$
2. The columns of A are linearly independent.
3. For $x \in {}^{n}R$, $Ax = 0$ implies $x = 0$.
4. $A^{T}A$ is invertible.
5. A has a left inverse, i.e. $BA = I$ for some $B \in {}^{n}R^{m}$

Lemma III:

Let $s \in {}^{n}R$. Then, if $s^{T}s = 0$, $s = 0$

Theorem V: (Row Version)

Let $A \in {}^{m}R^{n}$. Then the following statements are equivalent.

1. $rank\ A = m$
2. The rows of A are linearly independent.
3. For $z \in {}^{m}R$, $z^{T}A = 0$ implies $z = 0$.
4. AA^{T} is invertible.
5. A has a right inverse, i.e. $AB = I$ for some $B \in {}^{n}R^{m}$

Chapter 8: Coordination Plans

Definition of Linear Transformation:

Proposition I:

Definition of Image:

Definition of Kernel

Proposition II:

Theorem I Dimension Formula:

Definition of

Chapter 9: Great Determinations

Determinant of a 2x2 Matrix:

Sarrus's Rule:

Definition of Determinant Function:

Theorem I Properties of a Determinant Function:

Proposition I:

Proposition II:

Lemma I:

Theorem II:

Definition of Determinant:

Definition of Minor Matrix:

Definition of Laplace Expansion:

Theorem III:

Determinants of Elementary Matrices:

Theorem IV Cauchy-Binet Product Theorem:

Theorem V Transpose Theorem:

Theorem VI Invertibility Theorem:

Corollary:

Theorem VII Maclaurin-Cramer Rule:

Definition of Cofactor:

Definition of Adjoint:

Theorem VIII: