ROB311 Quiz 3

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Contents

1 Overview

Summary:
- MCTS
- $\alpha - \beta$ pruning
- Game Theory

- Exam problems are more difficult, but use past exams and over prepare.

Partially Observable Probabilistic Decision Problems

2 Reinforcement Learning

Summary: In a RL problem, $p(\cdot \mid \cdot, \cdot)$ and/or $r(\cdot\cdot, \cdot)$ unknown, so we have to estimate q-star empirically.

Equation

$$q^*(s,a) = \lim_{K \to \infty} \bar{R}_K$$

- $\bar{R}_K = \dfrac{1}{K}\displaystyle\sum_{k=1}^{K} r_k$: empirical average reward.
- r_k: reward obtained in the k^{th} simulation.
- K: # of times action a taken in state s (# of simulations)
- $\gamma = 0$

$$q^*(s,a) \leftarrow q^*(s,a) + \frac{1}{N(s,a)}\left(r(s,a,s') - q^*(s,a)\right)$$

- $N(s,a)$: # of times action a taken in state s.
- $\gamma = 0$

$$q^*(s,a) \leftarrow q^*(s,a) + \frac{1}{N(s,a)}\left(\left[r(s,a,s') + \gamma \max_{a'} q^*(s',a')\right] - q^*(s,a)\right)$$

- Using old q^* values to estimate.
- $\gamma \neq 0$

$$\pi(a \mid s) = \begin{cases} 1 & a = \arg\max_{a'} q^*(s,a) \\ 0 & \text{otherwise} \end{cases}$$

2.1 Running Average Update Rule

Definition:

$$\bar{x} \leftarrow \bar{x} + \alpha(x_{\text{new}} - \bar{x}).$$

- α: learning rate

2.2 Q-Learning Algorithm

Algorithm:

```
procedure Q_LEARNING():
    for each episode do
        set initial state $s \leftarrow s_0$
        while $s \notin \mathcal{T}$ do # $\mathcal{T}$: terminal states
            randomly choose an action in $\mathcal{A}(s)$
            get next state, $s'$, and reward $r$
            update $N(s,a)$ and $q^*(s,a)$ as follows:

                $$q^*(s,a) \leftarrow q^*(s,a) + \frac{1}{N(s,a)} \left( r(s,a,s') + \gamma \max_{a'} q^*(s',a') - q^*(s,a) \right)$$

                $N(s,a) \leftarrow N(s,a) + 1$

            $s \leftarrow s'$
        end while
    end for
```

- **Note:** Possible infinite while loop if $\mathcal{T}$ is not reached.

2.3 Modified Q-Learning Algorithm

Algorithm:

```
procedure Q_LEARNING():
    for each episode do
        $l \leftarrow 0$
        set initial state $s \leftarrow s_0$
        while $s \notin \mathcal{T}$ and $l < l_{\max}$ do
            randomly choose an action in $\mathcal{A}(s)$
            get next state, $s'$, and reward $r$
            update $N(s,a)$ and $q^*(s,a)$ as follows:

                $$q^*(s,a) \leftarrow q^*(s,a) + \frac{1}{N(s,a)} \left( r(s,a,s') + \gamma \max_{a'} q^*(s',a') - q^*(s,a) \right)$$

                $N(s,a) \leftarrow N(s,a) + 1$

            $l \leftarrow l + 1$
            $s \leftarrow s'$
        end while
    end for
```

Notes: Choice of γ and $l_{\max}$ are coupled:

- $\gamma \approx 1$ requires large $l_{\max}$
- $\gamma \approx 0$ requires small $l_{\max}$

2.4 Training vs. Testing

Notes: Episodes are classified as either:
- training (sim): reward accumulated during episode does not count
- testing (test): reward accumulated during episode counts

2.4.1 K Sims, 1 Test

Notes:
1. select actions randomly during K simulations
2. extract optimal policy, π^*
3. use π^* during test

2.4.2 K Tests

Notes:
- maximize average reward over K tests
- must balance between exploration and exploitation
- Common ways to balance exploration and exploitation: ε-greedy strategy, UCB algorithm

Strategy	Description
ε-greedy	choose optimal action with probability $\varepsilon(k)$

- In episode k, choose the optimal action with probability $\varepsilon(k)$, where:
  - $\varepsilon(0) \approx 0$
  - $\varepsilon(k)$ is increasing as you keep exploring.
  - $\varepsilon(k) \to 1$ as $k \to \infty$
- Common choice for $\varepsilon(k)$ is $1 - \frac{1}{k}$.

UCB algorithm	choose action that maximizes $\text{UCB}(\cdot)$

$$\text{UCB}(s,a) = \begin{cases} q^*(s,a) + C\sqrt{\frac{\log k}{N(s,a)}}, & \text{if } N(s,a) > 0 \\ \infty, & \text{otherwise} \end{cases}$$

- In episode k, choose the action that maximizes $\text{UCB}(\cdot)$.
- C: exploration parameter
- $N(s,a)$: # of times a taken from s.

3 Partially Observable MDPs (POMDPs)

Summary: In a **POMDPs**, we assume that:

- environment modelled using state space, $\mathcal{S}$
- single agent
- S_t = state after transition t
- A_t = action inducing transition t
- stochastic state transitions with memoryless property:

$$S_T \perp S_0, A_1, \dots, A_{T-1}, S_{T-2} \mid S_{T-1}, A_T$$

- R_t = reward for transition t, i.e., (S_{T-1}, A_T, S_T)
- O_t = observation of S_t
  - Measurement of a state (i.e. appproximation, so may not be exact)
  - **Key:** Since actual state is unknown, so are legal actions.

Name	**Function:**
Initial state distribution	$p_0(s) := \mathbb{P}[S_0 = s]$
Transition distribution	$p(s'\|s,a) := \mathbb{P}[S_t = s'\|A_t = a, S_{t-1} = s]$

- Assume $\mathcal{A}(s) = \mathcal{A}(s') := \mathcal{A}\ \forall s, s'$ (i.e. since actual state is unknown, so are legal actions, so assume all actions are legal):
  - if $a \notin \mathcal{A}(s)$, then $p(s'|s,a) = 0$ for all $s' \neq s$

Reward function	$r(s,a,s') :=$ reward for transition (s,a,s')

- Assume $\mathcal{A}(s) = \mathcal{A}(s') := \mathcal{A}\ \forall s, s'$ (i.e. since actual state is unknown, so are legal actions, so assume all actions are legal):
  - if $a \notin \mathcal{A}(s)$, then $r(s,a,s') = 0$ for all s'

Policy for choosing actions	$\pi_t(a\|o_0, \dots, o_t) := \mathbb{P}[A_t = a\|O_0 = o_0, \dots, O_t = o_t]$

- Observe that policy is now time-dependent.
- **Special Case:** If we assume the agent cannot use past observations, $A_t \perp O_0, \dots, O_{t-1} \mid O_t$, policy becomes time-independent,

$$\pi_t(a|o_0, \dots, o_t) = \pi_0(a|o_t).$$

  - Only need to specify π_0.

Measurement model	$m(o\|s) := \mathbb{P}[O_t = o\|S_t = s]$
Belief after t observations	$b_t(s_t\|a_{1:t}, o_{0:t}) = \mathbb{P}[S_t = s_t\|A_t = a_t, O_{0:t} = o_{0:t}]$ $b_t(s_t\|a_{1:t}, o_{0:t}) = m(o_t\|s_t) \sum_{s_{t-1}} p(s_t\|s_{t-1}, a_t) b_{t-1}(s_{t-1}\|a_{1:t-1}, o_{0:t-1})$

- b_t: Probability distribution
- $b_0(s_0) = \mathbb{P}[S_0 = s_0]$: Initial belief distribution
- Only holds for $t \geq 1$.
  - @t: Measurement before and after action for the belief is the same except at $t = 0$ b/c of initial belief.
- For $t = 0$ (assuming uniform prior): $b_0(s_0|o_0) = \dfrac{m(o_0|s_0)}{\sum_s m(o_0|s)}$.

3.1 Bayesian Network

Notes: $S_0, O_0, A_1, R_1, S_1, O_1, A_2, R_2, S_2, O_2, \ldots$ form a Bayesian network:

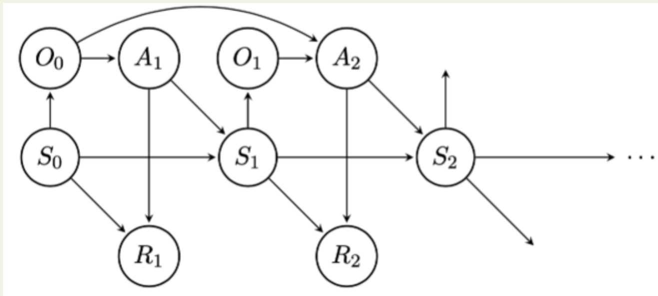

Figure 1

- Assuming $A_t \perp O_0, \ldots, O_{t-1} \mid O_t$. WHERE DOES THIS COME INTO PLAY.

3.2 Belief (Probability Distribution) Over the States:

Notes: Assume actual state is the most likely state.

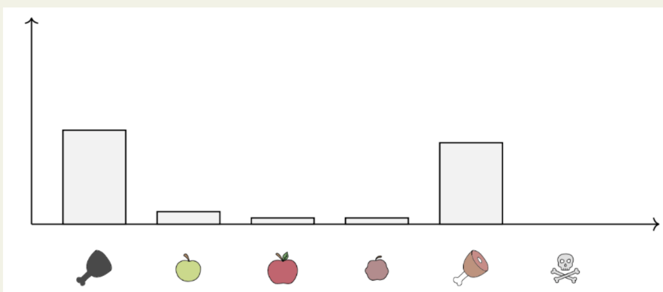

Figure 2

- Usually assume uniform distribution before you observe anything.
- **Flow:** Measurement $\rightarrow$ Take action $\rightarrow$ Update belief $\rightarrow$ Take action.

3.3 Examples

Example:

1. **Given:**
    - Now suppose Cavemen wants to feed child:
        - Cannot know satiety of child exactly.
        - Whether apple is edible or not must be inferred from senses.
    - Graph

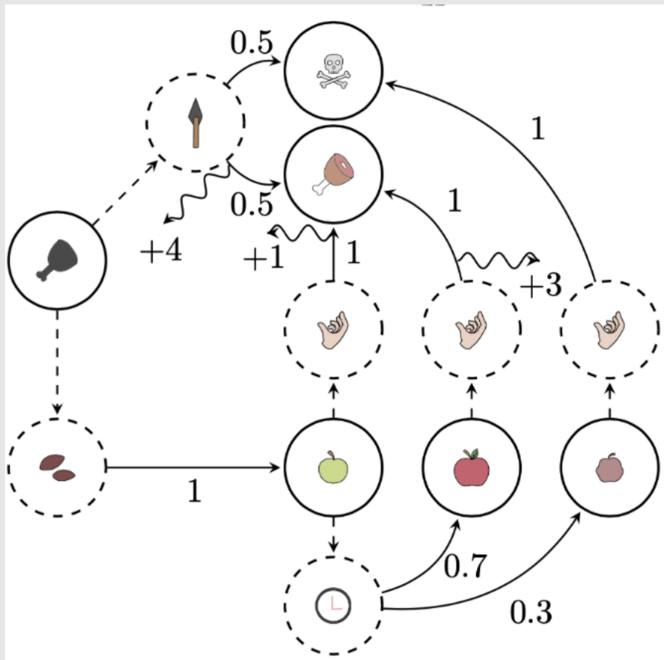

Figure 3

- Possible obsevations for the apple:

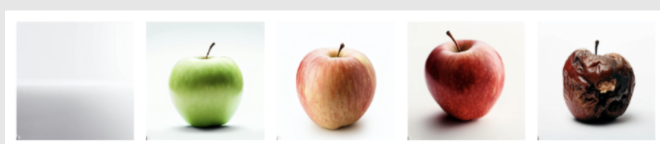

Figure 4

- Possible states for the child's satiety:

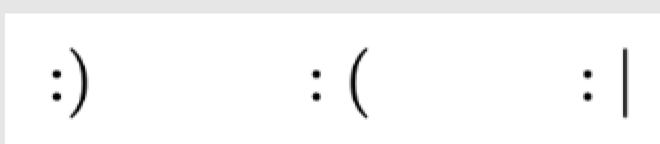

Figure 5

- Measurement distribution for the apple and child's satiety:

						:)	:(	:|
	1.0	0.0	0.0	0.0	0.0	0.0	0.8	0.2
	0.2	0.6	0.2	0.0	0.0	0.0	0.8	0.2
	0.0	0.3	0.4	0.3	0.0	0.0	0.8	0.2
	0.0	0.0	0.0	0.2	0.8	0.0	0.8	0.2
	1.0	0.0	0.0	0.0	0.0	0.0	0.8	0.2
	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

Figure 6: $m(o_1|s) = P(o_1|s)$ and $m(o_2|s) = P(o_2|s)$

  - $\sum = 1$ across the rows
  - o_1: What is the probability of observing a certain state of the apple given the true state?
  - o_2: What is the probability of observing a certain state of the child given the true state?
- **Key:** Assume independence between the observations of the child's satiety and the apple's edibility: $P(o|s) = P(o_1|s) \cdot P(o_2|s)$.

2. **Problem**
   - Initial distribution, $b_0(s_0)$ over states is uniform.
   - Action sequence is $\langle a_1, a_2, a_3 \rangle = \langle \text{seed}, \text{wait}, \text{wait} \rangle$.
   - Observation sequence is $\langle o_0, o_1, o_2, o_3 \rangle = \langle (\text{:(,no apple}), (\text{:(,ga}), (\text{:(,ra}), (\text{:|,ra}) \rangle$.
   - Find state distribution: $b_3(s_3 \mid a_{1:3}, o_{0:3})$.
3. **Solution:**

Example:
- Action sequence is $\langle a_1, a_2, a_3 \rangle = \langle \text{plant seed}, \text{wait}, \text{wait} \rangle$.
- Observation sequence is $\langle o_0, o_1, o_2, o_3 \rangle = \langle [\text{:(,no apple}], [\text{:(,ga}], [\text{:(,ra}], [\text{:|,ra}] \rangle$.
- Find state distribution: $b_3(s_3 \mid a_{1:3}, o_{0:3})$.

$$b_0(s_0 \mid o_0) = \frac{m(o_0 \mid s_0)}{\sum_s m(o_0 \mid s)} \tag{1}$$

$$b_t(s_t \mid a_{1:t}, o_{0:t}) = m(o_t \mid s_t) \sum_{s_{t-1}} p(s_t \mid s_{t-1}, a_t) b_{t-1}(s_{t-1} \mid a_{1:t-1}, o_{0:t-1}) \tag{2}$$

s	$b_0(s)$	$b_0(s_0 \mid o_0)$	$b_1(s_1 \mid o_{0:1}, a_1)$	$b_2(s_2 \mid o_{0:2}, a_{1:2})$	$b_3(s_3 \mid o_{0:3}, a_{1:3})$
No meat	1/6	0.4545			

- $\sum_s m([\text{:(,no apple}] \mid s) = (1.0)(0.8) + (0.2)(0.8) + (0.0)(0.8) + (0.0)(0.8) + (1.0)(0.8) + (1.0)(0.0) = 1.76$
- $b_0(\text{No meat} \mid [\text{:(,no apple}]) = \frac{(1.0)(0.8)}{1.76} = \frac{0.8}{1.76} = 0.4545$
- $b_1(\text{No meat} \mid \text{plant seed}, [\text{:(,no apple}], [\text{:(,ga}]) = (0.0)(0.8) \left[(0 \cdot 0.4545)\right]$

s	$b_0(s)$	$b_0(s_0 \mid o_0)$
Green apple	1/6	0.0909

- $b_0(\text{Green apple} \mid o_0) = \frac{(0.2)(0.8)}{1.76} = \frac{0.16}{1.76} = 0.0909$
- $b_1(\text{Green apple} \mid a_1, o_{0:1}) = (0.0)(0.8) \left[(0 \cdot 0.0909)\right]$

s	$b_0(s)$	$b_0(s_0 \mid o_0)$
Red apple	1/6	0

- $b_0(\text{Red apple} \mid o_0) = \frac{(0.0)(0.8)}{1.76} = \frac{0.0}{1.76} = 0$

s	$b_0(s)$	$b_0(s_0 \mid o_0)$
Rotten apple	1/6	0

- $b_0(\text{Rotten apple} \mid o_0) = \frac{(0.0)(0.8)}{1.76} = \frac{0.0}{1.76} = 0$

s	$b_0(s)$	$b_0(s_0 \mid o_0)$
Meat	1/6	0.4545

- $b_0(\text{Meat} \mid o_0) = \frac{(1.0)(0.8)}{1.76} = \frac{0.8}{1.76} = 0.4545$

s	$b_0(s)$	$b_0(s_0 \mid o_0)$
Dead	1/6	0

- $b_0(\text{Dead} \mid o_0) = \frac{(1.0)(0.0)}{1.76} = \frac{0.0}{1.76} = 0$

4 Estimating the Optimal Quality Function

4.1 Estimating the Optimal Quality Function

Motivation: The agent need not know the model of the environment. However, it must actually make moves, even when learning.

If the agent doesn't have a model, it must estimate q^*, $\mathcal{A}^*$, and π^*.

Definition: When the environment is in state s, the agent can take an action a and:
- **Update $\hat{q}$:** $\hat{q}(s,a;t) \leftarrow (1-\alpha)\hat{q}(s,a;t) + \alpha\left(r' + \gamma \max_{a'} \hat{q}(s',a';t+1)\right)$
  - $0 \le \alpha \le 1$: learning rate
- **Compute $\hat{\mathcal{A}}$:** $\hat{\mathcal{A}}(s;t) = \arg\max_{a' \in \mathcal{A}(s)} \hat{q}(s,a';t)$
- **Compute $\hat{\pi}$:** $\hat{\pi}(a' \mid s;t) = 0 \; \forall a' \notin \hat{\mathcal{A}}(s;t)$

4.2 Exploration versus Exploitation

Motivation: To ensure $\hat{q}$ converges to q^* and the agent's expected return is maximized, the agent must balance exploration and exploitation.

Definition:
- **Exploitation:** Choose the most promising actions based on current knowledge.
  - Use optimal policy: $\hat{\pi}(\cdot,\cdot;t)$
- **Exploration:** Choose the least tried actions to improve current knowledge.
  - Choose actions randomly

4.2.1 Simplified Case:

Example:
- **Given:** Assume the environment is stateless, but rewards are random.

Figure 7

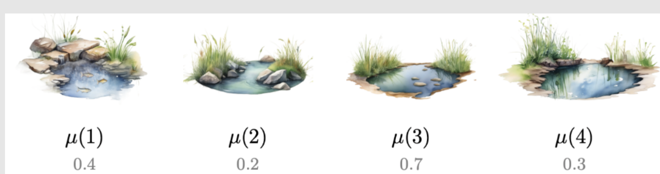

Figure 8

  - $\mu(a)$: expected reward for action a (unknown to the agent):
  - $0 \le \mu(a) \le 1$ for all a.
- **Best-case expected return:** (with $\gamma = 1$ under π^*) from transition t is:

$$u^*(t) := (T - t) \max_{a'} \mu(a')$$

where in this case:

$$\pi^*(a;t) = 0 \quad \text{if } a \notin \arg\max_{a'} \mu(a').$$

- **Estimation of $\mu(\cdot)$.** Since the agent does not have a model, it must estimate $\mu(\cdot)$.

  The agent can take an action a and:
  1. **Update** $n(\cdot)$ and $\hat{\mu}(\cdot)$:

  $$n(a) \leftarrow n(a) + 1$$

  $$\hat{\mu}(a) \leftarrow \left(1 - \frac{1}{n(a)}\right) \hat{\mu}(a) + \frac{1}{n(a)} r'$$

  2. **Compute** $\hat{\pi}$:

  $$\hat{\pi}(a;t) = 0 \text{ for all } a \notin \arg\max_{a'} \hat{\mu}(a').$$

- **Alternate Policies** We want to compare the expected return under various policies. The expected return from transition t under a policy ρ is:

  $$u^\rho(t) := \mathbb{E}^\pi[G_t] = \sum_{a'} \rho(a';t) \left(\mu(a') + u^\rho(t+1)\right).$$

4.3 Alternate Policies

Summary: To ensure the agent's expected return is maximized, the agent must strike still strike a balance exploration and exploitation.

In the following cases, the expected return from transition t is

$$u^{\text{avg}}(t) \equiv \frac{T-t}{|\mathcal{A}|} \sum_a \mu(a)$$

We want to choose ρ so that $u^\rho > u^{\text{avg}}$.

Policy	**Function:**
Exploitation only	Choose a random action, same for all transitions
Exploration only	Choose a random action, different for each transition
Softmax	Apply a soft-max over $\hat{u}$ $\rho(a;t) = \left[\sum_{a'} \exp\left(\frac{\hat{\mu}(a')}{\tau}\right)\right]^{-1} \exp\left(\frac{\hat{\mu}(a)}{\tau}\right)$ • Choose a temperature value decrease with t. • $\tau(t) \in [0, \infty), \tau \to 0$
ϵ-greedy	Use $\hat{\pi}$ w/ prob. $1-\epsilon$, otherwaise take a random action $\rho(a;t) = \epsilon \frac{1}{|\mathcal{A}|} + (1-\epsilon)\hat{\pi}(a;t)$ • Choose an exploration rate decrease w/ t. • $\epsilon(t) \in [0,1], \epsilon \to 0$
Upper confidence bound	Choose the action with the highest ucb$(\cdot)$ $\rho(a;t) = 0$ if $a \notin \arg\max_{a'} \text{ucb}(a';t)$ • Compute ucb$(\cdot)$ for each action. • $\text{ucb}(a;t) = \hat{\mu}(a) + \sqrt{\frac{\ln t}{n(a)}}$

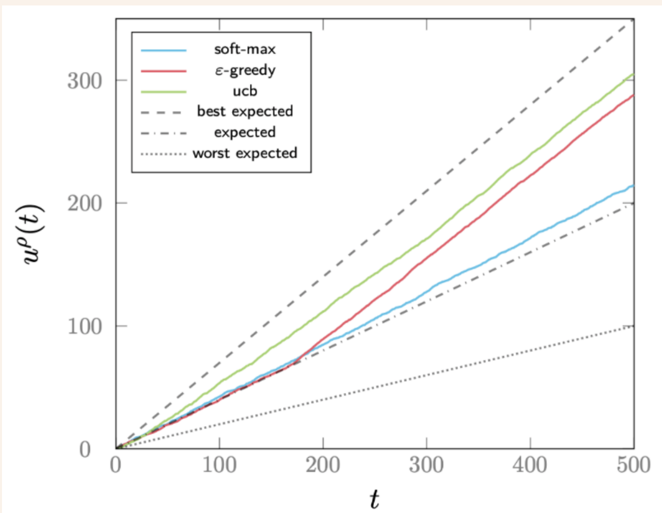

Figure 9