ROB311 Quiz 3

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Partially Observable Probabilistic Decision Problems

1 Reinforcement Learning

Summary: In a RL problem, $p(\cdot | \cdot, \cdot)$ and/or $r(\cdot, \cdot)$ unknown, so we have to estimate q-star empirically.

Equation

 $q^*(s,a) = \lim_{K \to \infty} \bar{R}_K$

- $\bar{R}_K = \frac{1}{K} \sum_{k=1}^K r_k$: empirical average reward.
- r_k : reward obtained in the k^{th} simulation.
- K: # of times action a taken in state s (# of simulations)
- $\bullet \ \gamma = 0$

 $q^*(s,a) \leftarrow q^*(s,a) + \frac{1}{N(s,a)} (r(s,a,s') - q^*(s,a))$

- N(s,a): # of times action a taken in state s.
- \bullet $\gamma = 0$

 $q^*(s, a) \leftarrow q^*(s, a) + \frac{1}{N(s, a)} \left(\left[r(s, a, s') + \gamma \max_{a'} q^*(s', a') \right] - q^*(s, a) \right)$

- Using old q^* values to estimate.
- $\gamma \neq 0$

 $\pi(a \mid s) = \begin{cases} 1 & a = \arg\max_{a'} q^*(s, a) \\ 0 & \text{otherwise} \end{cases}$

1.1 Running Average Update Rule

Definition:

$$\bar{x} \leftarrow \bar{x} + \alpha (x_{\text{new}} - \bar{x}).$$

• α : learning rate

1.2 Q-Learning Algorithm

```
Algorithm:
   procedure Q_LEARNING():
         for each episode do
               set initial state s \leftarrow s_0
               while s \notin \mathcal{T} do # \mathcal{T}: terminal states
                     randomly choose an action in \mathcal{A}(s)
                     get next state, s', and reward r
                     update N(s,a) and q^{st}(s,a) as follows:
                          q^*(s, a) \leftarrow q^*(s, a) + \frac{1}{N(s, a)} \left( r(s, a, s') + \gamma \max_{a'} q^*(s', a') - q^*(s, a) \right)
                          N(s,a) \leftarrow N(s,a) + 1
12
13
               end while
14
         end for
    • Note: Possible infinite while loop if \mathcal{T} is not reached.
```

1.3 Modified Q-Learning Algorithm

```
Algorithm:
   procedure Q_LEARNING():
         for each episode do
               l \leftarrow 0
                set initial state s \leftarrow s_0
                while s \notin \mathcal{T} and l < l_{\max} do
                      randomly choose an action in \mathcal{A}(s)
                      get next state, s^\prime, and reward r
                      update N(s,a) and q^*(s,a) as follows:
                            q^*(s, a) \leftarrow q^*(s, a) + \frac{1}{N(s, a)} \left( r(s, a, s') + \gamma \max_{a'} q^*(s', a') - q^*(s, a) \right)
10
                            N(s,a) \leftarrow N(s,a) + 1
12
13
                     l \leftarrow l+1
14
15
                end while
          end for
```

Notes: Choice of γ and l_{max} are coupled:

- $\gamma \approx 1$ requires large $l_{\rm max}$
- $\gamma \approx 0$ requires small l_{max}

1.4 Training vs. Testing

Notes: Episodes are classified as either:

- training (sim): reward accumulated during episode does not count
- testing (test): reward accumulated during episode counts

1.4.1 K Sims, 1 Test

Notes:

- 1. select actions randomly during K simulations
- 2. extract optimal policy, π^*
- 3. use π^* during test

1.4.2 K Tests

Notes:

- \bullet maximize average reward over K tests
- must balance between exploration and exploitation
- Common ways to balance exploration and exploitation: ε -greedy strategy, UCB algorithm

Strategy Description

 ε -greedy

choose optimal action with probability $\varepsilon(k)$

- In episode k, choose the optimal action with probability $\varepsilon(k)$, where:
 - $-\varepsilon(0)\approx 0$
 - $-\varepsilon(k)$ is increasing as you keep exploring.
 - $-\varepsilon(k) \to 1 \text{ as } k \to \infty$
- Common choice for $\varepsilon(k)$ is $1 \frac{1}{k}$.

UCB algorithm choose action that maximizes $UCB(\cdot)$

$$UCB(s, a) = \begin{cases} q^*(s, a) + C\sqrt{\frac{\log k}{N(s, a)}}, & \text{if } N(s, a) > 0\\ \infty, & \text{otherwise} \end{cases}$$

- In episode k, choose the action that maximizes UCB(·).
- C: exploration parameter
- N(s,a): # of times a taken from s.

1.5 Examples

Example:

1. **Given:** Consider the following pseudo-code for Q-learning (incomplete):

```
\begin{array}{c} 1 \\ N(s,a) \leftarrow 0, \quad \forall s,a \\ q^*(s,a) \leftarrow 0, \quad \forall s,a \\ \text{for each episode do} \\ 4 \\ 5 \\ l \leftarrow 0 \\ 6 \\ \text{while } s \notin \mathcal{T} \text{ and } l < l_{\max} \text{ do} \\ 7 \\ 8 \\ \text{get next state } s', \text{ and reward } r(s,a,s') \\ 9 \\ N(s,a) \leftarrow N(s,a) + 1 \\ 10 \\ q^*(s,a) \leftarrow \text{ update rule} \\ 11 \\ 12 \\ \text{end while} \\ \text{end for} \end{array}
```

2. **Problem 1:** Now consider replacing the update rule in line 10 with:

$$q^*(s, a) \leftarrow \left(1 - \frac{1}{N(s, a)}\right) q^*(s, a) + \frac{1}{N(s, a)} \left(r(s, a, s') + \gamma \max_{a'} q^*(s', a')\right)$$

Will the resulting algorithm be equivalent to vanilla Q-learning?

- 3. Solution 1: Yes
- 4. **Problem 2:** Which line in the pseudo-code should be modified to adjust the amount of exploration versus exploitation?
- 5. Solution 2: Line 7

Explanation: Exploration vs. exploitation is controlled by the action selection strategy. Modifying line 7 (e.g., from random to ϵ -greedy or softmax selection) directly adjusts this trade-off.

- 6. **Problem 3:** Suppose vanilla Q-learning is used in an infinite MDP (i.e., an MDP with infinitely long paths). Which of the following best describes how l_{max} and γ are related?
 - (A) l_{max} should be an increasing function of γ
 - (B) l_{max} should be a decreasing function of γ
 - (C) l_{max} and γ are often chosen independently
- 7. Solution 3: (A) l_{max} should be an increasing function of γ

Explanation: A higher γ implies longer-term reward importance, so the agent must simulate longer episodes to capture long-term effects accurately.

2 Partially Observable MDPs (POMDPs)

Summary: In a **POMDPs**, we assume that:

- \bullet environment modelled using state space, \mathcal{S}
- single agent
- S_t = state after transition t
- $A_t = action inducing transition t$
- stochastic state transitions with memoryless property:

$$S_T \perp S_0, A_1, \dots, A_{T-1}, S_{T-2} \mid S_{T-1}, A_T$$

- R_t = reward for transition t, i.e., (S_{T-1}, A_T, S_T)
- O_t = observation of S_t
 - Measurement of a state (i.e. appproximation, so may not be exact)
 - **Key:** Since actual state is unknown, so are legal actions.

Name	Function:
Initial state distribution	$p_0(s) := \mathbb{P}[S_0 = s]$
Transition distribution	$p(s' s,a) := \mathbb{P}[S_t = s' A_t = a, S_{t-1} = s]$

• Assume $A(s) = A(s') := A \forall s, s'$ (i.e. since actual state is unknown, so are legal actions, so assume all actions are legal):

- if
$$a \notin \mathcal{A}(s)$$
, then $p(s'|s,a) = 0$ for all $s' \neq s$

Reward function r(s, a, s') := reward for transition (s, a, s')

• Assume $A(s) = A(s') := A \forall s, s'$ (i.e. since actual state is unknown, so are legal actions, so assume all actions are legal):

- if
$$a \notin \mathcal{A}(s)$$
, then $r(s, a, s') = 0$ for all s'

Policy for choosing actions $\pi_t(a|o_0,\ldots,o_t) := \mathbb{P}[A_t = a|O_0 = o_0,\ldots,O_t = o_t]$

- Observe that policy is now time-dependent.
- Special Case: If we assume the agent cannot use past observations, $A_t \perp O_0, \ldots, O_{t-1} \mid O_t$, policy becomes time-independent,

$$\pi_t(a|o_0,\ldots,o_t)=\pi_0(a|o_t).$$

- Only need to specify π_0 .

$$\begin{aligned} \text{Measurement model} & & & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & \\ & & \\$$

- b_t : Probability distribution
- $b_0(s_0) = \mathbb{P}[S_0 = s_0]$: Initial belief distribution
- Only holds for $t \geq 1$.
 - @t: Measurement before and after action for the belief is the same except at t=0 b/c of initial belief.
- For t = 0 (assuming uniform prior): $b_0(s_0|o_0) = \frac{m(o_0|s_0)}{\sum_s m(o_0|s)}$

2.1 Bayesian Network

Notes: $S_0, O_0, A_1, R_1, S_1, O_1, A_2, R_2, S_2, O_2, \dots$ form a Bayesian network:

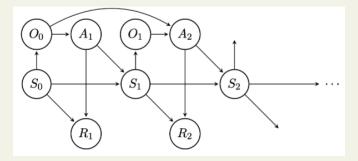


Figure 1

• Assuming $A_t \perp O_0, \dots, O_{t-1} \mid O_t$. WHERE DOES THIS COME INTO PLAY.

2.2 Belief (Probability Distribution) Over the States:

Notes: Assume actual state is the most likely state.

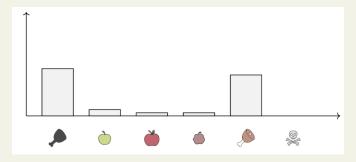


Figure 2

- Usually assume uniform distribution before you observe anything.
- \bullet Flow: Measurement \to Take action \to Update belief \to Take action.

2.3 Examples

Example:

- 1. Given:
 - Now suppose Cavemen wants to feed child:
 - Cannot know satiety of child exactly.
 - Whether apple is edible or not must be inferred from senses.
 - Graph

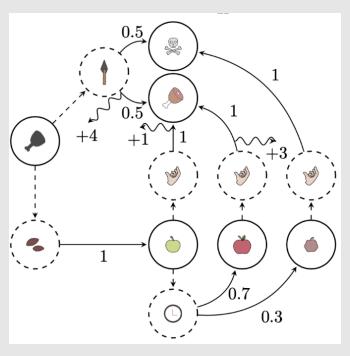


Figure 3

• Possible obsevations for the apple:



Figure 4

• Possible states for the child's satiety:



Figure 5

• Measurement distribution for the apple and child's satiety:

	**				:)	: (:
1.0 0.2 0.0 0.0 1.0	0.0	0.0	0.0	0.0	0.0	0.8	0.2
0.2	0.6	0.2	0.0	0.0	0.0	0.8	0.2
0.0	0.3	0.4	0.3	0.0	0.0	0.8	0.2
0.0	0.0	0.0	0.2	0.8	0.0	0.8	0.2
1.0	0.0	0.0	0.0	0.0	0.0	0.8	0.2
1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

Figure 6: $m(o_1|s) = P(o_1|s)$ and $m(o_2|s) = P(o_2|s)$

- $-\sum_{i=1}^{n} = 1$ across the rows $-o_1$: What is the probability of observing a certain state of the apple given the true state? $-o_2$: What is the probability of observing a certain state of the child given the true state?
- Key: Assume independence between the observations of the child's satiety and the apple's edibility: $P(o|s) = P(o_1|s) \cdot P(o_2|s).$

2. Problem

- Initial distribution, $b_0(s_0)$ over states is uniform.
- Action sequence is $\langle a_1, a_2, a_3 \rangle = \langle \text{seed, wait, wait} \rangle$.
- Observation sequence is $\langle o_0, o_1, o_2, o_3 \rangle = \langle (:(,no apple), (:(,ga)), (:(,ra)), (:|,ra)) \rangle$.
- Find state distribution: $b_3(s_3 \mid a_{1:3}, o_{0:3})$.

3. Solution:

Example:

$$b_0(s_0 \mid o_0) = \frac{m(o_0 \mid s_0)}{\sum_s m(o_0 \mid s)} \tag{1}$$

$$b_t(s_t \mid a_{1:t}, o_{0:t}) = m(o_t \mid s_t) \sum_{s_{t-1}} p(s_t \mid s_{t-1}, a_t) b_{t-1}(s_{t-1} \mid a_{1:t-1}, o_{0:t-1})$$
(2)

 $b_0(s)$ $b_0(s_0 \mid o_0)$ $b_1(s_1 \mid a_1, o_{0:1})$ $b_2(s_2 \mid a_{1:2}, o_{0:2})$ 1/6 0.4545 0 $b_3(s_3 \mid a_{1:3}, o_{0:3})$

No meat

- $\sum m([:(,\text{no apple}] \mid s) = (1.0)(0.8) + (0.2)(0.8) + (0.0)(0.8) + (0.0)(0.8) + (1.0)(0.8) + (1.0)(0.0) = 1.76$
- b_0 (No meat | [:(,no apple]) = $\frac{(1.0)(0.8)}{1.76} = \frac{0.8}{1.76} = 0.4545$
- b_1 (No meat | plant seed, [:(,no apple], [:(,ga]) = (0.0)(0.8) $\left| \underbrace{(0 \cdot 0)}_{c_0 \text{NA}} \underbrace{(0 \cdot 0)}_{a_1 = \text{plant seed}} \right| = 0$

1/6 Green apple 0.0909

- $b_0(\text{Green apple} \mid o_0) = \frac{(0.2)(0.8)}{1.76} = \frac{0.16}{1.76} = 0.0909$
- $b_1(\text{Green apple} \mid a_1, o_{0:1}) = (0.6)(0.8) \left[\underbrace{(1 \cdot 0.4545)}_{s_0 = \text{No meat, } a_1 = \text{plant seed}} \right] = 0.2182$

1/6 Red apple

- $b_0(\text{Red apple} \mid o_0) = \frac{(0.0)(0.8)}{1.76} = \frac{0.0}{1.76} = 0$
- $b_1(\text{Red apple} \mid a_1, o_{0:1}) = (0.3)(0.8) \left[\underbrace{0 \cdot 0}_{s_0 = \text{NA}, \ a_1 = \text{plant seed}} \right] = 0$

Rotten apple 1/6

- $b_0(\text{Rotten apple} \mid o_0) = \frac{(0.0)(0.8)}{1.76} = \frac{0.0}{1.76} = 0$
- $b_1(\text{Rotten apple} \mid a_1, o_{0:1}) = (0.0)(0.8)$ $\left| \underbrace{(0 \cdot 0)}_{s_0 = \text{NA, } a_1 = \text{plant seed}} \right| = 0$

1/6 Meat 0.4545

- $b_0(\text{Meat} \mid o_0) = \frac{(1.0)(0.8)}{1.76} = \frac{0.8}{1.76} = 0.4545$
- $b_1(\text{Meat} \mid a_1, o_{0:1}) = (0.0)(0.8) \left[\underbrace{0 \cdot 0}_{s_0 = \text{NA}, \ a_1 = \text{plant seed}} \right] = 0$

1/6 Dead

- $b_0(\text{Dead} \mid o_0) = \frac{(1.0)(0.0)}{1.76} = \frac{0.0}{1.76} = 0$
- $b_1(\text{Dead} \mid a_1, o_{0:1}) = (0.0)(0.0)$ 0 = 0

3 Estimating the Optimal Quality Function

3.1 Estimating the Optimal Quality Function

Motivation: The agent need not know the model of the environment. However, it must actually make moves, even when learning.

If the agent doesn't have a model, it must estimate q^* , \mathcal{A}^* , and π^* .

Definition: When the environment is in state s, the agent can take an action a and:

- Update \hat{q} : $\hat{q}(s, a; t) \leftarrow (1 \alpha)\hat{q}(s, a; t) + \alpha \left(r' + \gamma \max_{a'} \hat{q}(s', a'; t + 1)\right)$
 - $-0 \le \alpha \le 1$: learning rate
- Compute \hat{A} : $\hat{A}(s;t) = \arg \max_{a' \in A(s)} \hat{q}(s,a';t)$
- Compute $\hat{\pi}$: $\hat{\pi}(a' \mid s; t) = 0 \ \forall a' \notin \hat{\mathcal{A}}(s; t)$

3.2 Exploration versus Exploitation

Motivation: To ensure \hat{q} converges to q^* and the agent's expected return is maximized, the agent must balance exploration and exploitation.

Definition:

- Exploitation: Choose the most promising actions based on current knowledge.
 - Use optimal policy: $\hat{\pi}(\cdot,\cdot;t)$
- Exploration: Choose the least tried actions to improve current knowledge.
 - Choose actions randomly

3.2.1 Simplified Case:

Example:

• Given: Assume the environment is stateless, but rewards are random.



Figure 7



Figure 8

- $-\mu(a)$: expected reward for action a (unknown to the agent):
- $-0 \le \mu(a) \le 1$ for all a.
- Best-case expected return: (with $\gamma = 1$ under π^*) from transition t is:

$$u^*(t) := (T - t) \max_{a'} \mu(a')$$

where in this case:

$$\pi^*(a;t) = 0$$
 if $a \notin \arg \max_{a'} \mu(a')$.

• Estimation of $\mu(\cdot)$. Since the agent does not have a model, it must estimate $\mu(\cdot)$.

The agent can take an action a and:

1. **Update** $n(\cdot)$ and $\hat{\mu}(\cdot)$:

$$n(a) \leftarrow n(a) + 1$$

$$\hat{\mu}(a) \leftarrow \left(1 - \frac{1}{n(a)}\right)\hat{\mu}(a) + \frac{1}{n(a)}r'$$

2. Compute $\hat{\pi}$:

$$\hat{\pi}(a;t) = 0$$
 for all $a \notin \arg \max_{a'} \hat{\mu}(a')$.

• Alternate Policies We want to compare the expected return under various policies. The expected return from transition t under a policy ρ is:

$$u^{\rho}(t) := \mathbb{E}^{\pi}[G_t] = \sum_{a'} \rho(a';t) \left(\mu(a') + u^{\rho}(t+1)\right).$$

3.3 Alternate Policies

Summary: To ensure the agent's expected return is maximized, the agent must strike still strike a balance exploration and exploitation.

In the following cases, the expected return from transition t is

$$u^{\text{avg}}(t) \equiv \frac{T - t}{|\mathcal{A}|} \sum_{a} \mu(a)$$

We want to choose ρ so that $u^{\rho} > u^{\text{avg}}$.

Policy	Function:
Exploitation only	Choose a random action, same for all transitions
Exploration only	Choose a random action, different for each transition
Softmax	Apply a soft-max over \hat{u} $\rho(a;t) = \left[\sum_{a'} \exp\left(\frac{\hat{\mu}(a')}{\tau}\right)\right]^{-1} \exp\left(\frac{\hat{\mu}(a)}{\tau}\right)$

- ullet Choose a temperature value decrease with t.
- $\tau(t) \in [0, \infty), \tau \to 0$

e-greedy Use $\hat{\pi}$ w/ prob. $1 - \epsilon$, otherwaise take a random action $\rho(a;t) = \epsilon \frac{1}{|\mathcal{A}|} + (1 - \epsilon)\hat{\pi}(a;t)$

- Choose an exploration rate decrease w/t.
- $\epsilon(t) \in [0,1], \epsilon \to 0$

Upper confidence bound — Choose the action with the highest $\mathrm{ucb}(\cdot)$ $\rho(a;t)=0$ if $a\notin\arg\max_{a'}\mathrm{ucb}(a';t)$

- Compute $ucb(\cdot)$ for each action.
- $\operatorname{ucb}(a;t) = \hat{\mu}(a) + \sqrt{\frac{\ln t}{n(a)}}$

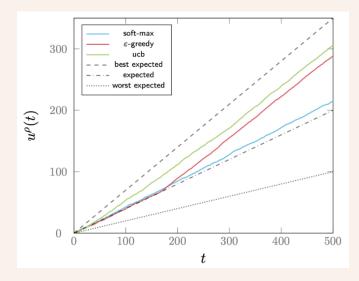


Figure 9

3.4 Examples

Example:

1. Problem 1:

Consider the policy:

$$\pi(a|s) = \begin{cases} 0.99 & \text{if } a = \arg\max_{a'} q(s, a') \\ 0.01 & \text{otherwise} \end{cases}$$

Which of the following best describes its behaviour?

- (A) It mostly exploits
- (B) It mostly explores
- (C) It mostly explores initially, but mostly exploits in the long-run
- (D) It mostly exploits initially, but mostly explores in the long-run
- (E) None of the above

2. Solution 1:

Correct answer: (A) It mostly exploits. *Explanation:* The policy heavily favors the action with the highest Q-value (with 99

3. Problem 2:

Consider the policy:

$$\pi(a|s) = \frac{\exp\left(\frac{q(s,a)}{\tau}\right)}{\sum\limits_{a' \in \mathcal{A}(s)} \exp\left(\frac{q(s,a')}{\tau}\right)}$$

where τ decreases each episode.

Which of the following best describes its behaviour?

- (A) It mostly exploits
- (B) It mostly explores
- (C) It mostly explores initially, but mostly exploits in the long-run
- (D) It mostly exploits initially, but mostly explores in the long-run
- (E) None of the above

4. Solution 2:

Correct answer: (C) It mostly explores initially, but mostly exploits in the long-run. Explanation: The softmax policy allows for exploration when τ is high. As $\tau \to 0$, it becomes increasingly greedy, converging to exploitation.

5. Problem 3:

Consider the policy:

$$\pi(a|s) = \begin{cases} 1 & \text{if } a = \arg\max_{a'} f(s, a') \\ 0 & \text{otherwise} \end{cases}$$

where

$$f(s,a) = \begin{cases} q(s,a) + C\sqrt{\frac{\log(\tau)}{N(s,a)}} & \text{if } N(s,a) > 0\\ \infty & \text{else} \end{cases}$$

for some constant C > 0 and increasing τ .

Which of the following best describes its behaviour?

- (A) It mostly exploits
- (B) It mostly explores
- (C) It mostly explores initially, but mostly exploits in the long-run
- (D) It mostly exploits initially, but mostly explores in the long-run
- (E) None of the above

6. Solution 3:

Correct answer: (C) It mostly explores initially, but mostly exploits in the long-run. Explanation: This is a UCB (Upper Confidence Bound) policy. The exploration bonus decreases as N(s, a) increases, so the agent prioritizes exploring under-visited actions early on but exploits later.

7. Problem 4:

Consider the policy:

$$\pi(a|s) = \begin{cases} 1 & \text{if } a = \arg\min_{a'} N(s, a') \\ 0 & \text{otherwise} \end{cases}$$

Which of the following best describes its behaviour?

- (A) It mostly exploits
- (B) It mostly explores
- (C) It mostly explores initially, but mostly exploits in the long-run
- (D) It mostly exploits initially, but mostly explores in the long-run
- (E) None of the above

8. Solution 4:

Correct answer: (B) It mostly explores. *Explanation:* This policy always chooses the least-visited action. It does not consider value (Q-function) and is purely driven by state-action visitation counts, thus ensuring exploration.