ROB311 Quiz 2

Hanhee Lee

April 13, 2025

Contents

Probabilistic Inference Problems

1   Bayesian Networks

Definition: Vertices represent random variables and edges represent dependencies between variables.

1.1   Junction

Definition: A **junction** $\mathcal{J}$ consists of three vertices, X_1, X_2, and X_3, connected by two edges, e_1 and e_2:

Figure 1

- X_1 and X_2 are not independent, X_2 and X_3 are not independent, but when is X_1 and X_3 independent?

1.1.1   Causal Chain

Definition: A causal chain is a junction $\mathcal{J}$ s.t.

Figure 2

- X_1 and X_3 are not independent (unconditionally), but are independent given X_2.

Notes:
- **Analogy:** Given X_2, X_1 and X_3 are independent. Why? X_2's door closes when you know X_2, so X_1 and X_3 are independent.
- **Distinction b/w Causal and Dependence:** X_1 and X_2 are dependent. However, from a causal perspective, X_1 is influencing X_2 (i.e. $X_1 \to X_2$).

1.1.2   Common Cause

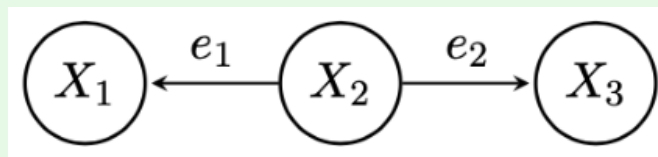**Definition**: A common cause is a junction $\mathcal{J}$ s.t.

Figure 3

- X_1 and X_3 are not independent (unconditionally), but are independent given X_2.

Notes:
- **Analogy:** Given X_2, X_1 and X_3 are independent. Why? Consider the following example:
  - Let X_2 represent whether a person smokes or not, X_1 represent whether they have yellow teeth, X_3 represent whether they have lung cancer.
- Without knowing X_2, observing X_1 provides information about X_3 because yellow teeth are associated with smoking, which in turn increases the likelihood of lung cancer.
- If X_2 is known, then knowing whether a person has yellow teeth provides no additional information about whether they have lung cancer beyond what is already known from smoking status.

1.1.3 Common Effect
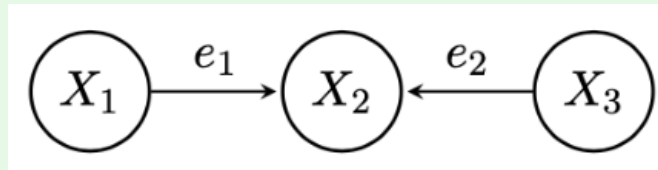
Definition: A common effect is a junction $\mathcal{J}$ s.t.

Figure 4

- X_1 and X_3 are independent (unconditionally), but are not independent given X_2 or any of X_2's descendents.

Notes:
- **Analogy:** Consider the following example:
  - Let X_2 represent whether the grass is wet, X_1 represent whether it rained, X_3 represent whether the sprinkler was on.
- Without knowing whether the grass is wet (X_2), the occurrence of rain (X_1) and the sprinkler being on (X_3) are independent events. The rain may occur regardless of the sprinkler, and vice versa.
- However, once we observe that the grass is wet (X_2), the two events become dependent:
  - If we learn that the sprinkler was not on, then the wet grass must have been caused by rain.
  - If we learn that it did not rain, then the wet grass must have been caused by the sprinkler.

2   Dependence Separation

2.1   Independence

Theorem: Any two variables, X_1 and X_2, in a Bayesian network, $\mathcal{B} = (\mathcal{V}, \mathcal{E})$, are independent given $\mathcal{K} \subseteq \mathcal{V}$ if every undirected path is blocked.

2.1.1   Blocked Undirected Path

Definition: An undirected path,

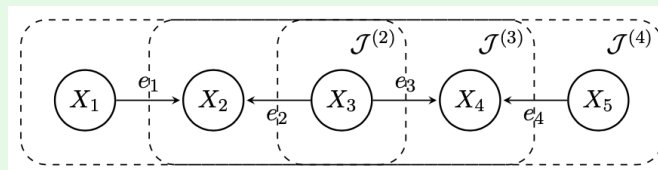$$p = \langle (X_1, e_1, X_2), \ldots, (X_{|p|-1}, e_{|p|-1,|p|}, X_{|p|}) \rangle,$$

Figure 5

is **blocked** given $\mathcal{K} \subseteq \mathcal{V}$ if any of its junctions,

$$\mathcal{J}^{(n)} = \{(X_{n-1}, X_n, X_{n+1}), (e_{n-1}, e_n)\},$$

is blocked given $\mathcal{K}$.

2.1.2   Blocked Junction

Definition: $\mathcal{J} = (\{X_1, X_2, X_3\}, \{e_1, e_2\})$ is **blocked** given $\mathcal{K} \subseteq \mathcal{V}$ if X_1 and X_3 are independent given $\mathcal{K}$.

2.2   Consequence of Dependence Separation

Theorem: For any variable, $X \in \mathcal{V}$, it can be shown that X is independent of X's non-descendants, $\mathcal{V} \setminus \mathrm{des}(X)$, given X's parents, $\mathrm{pts}(X)$.
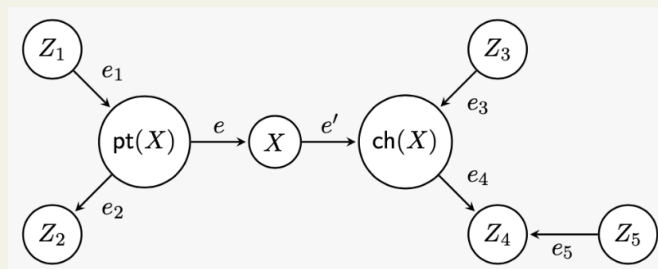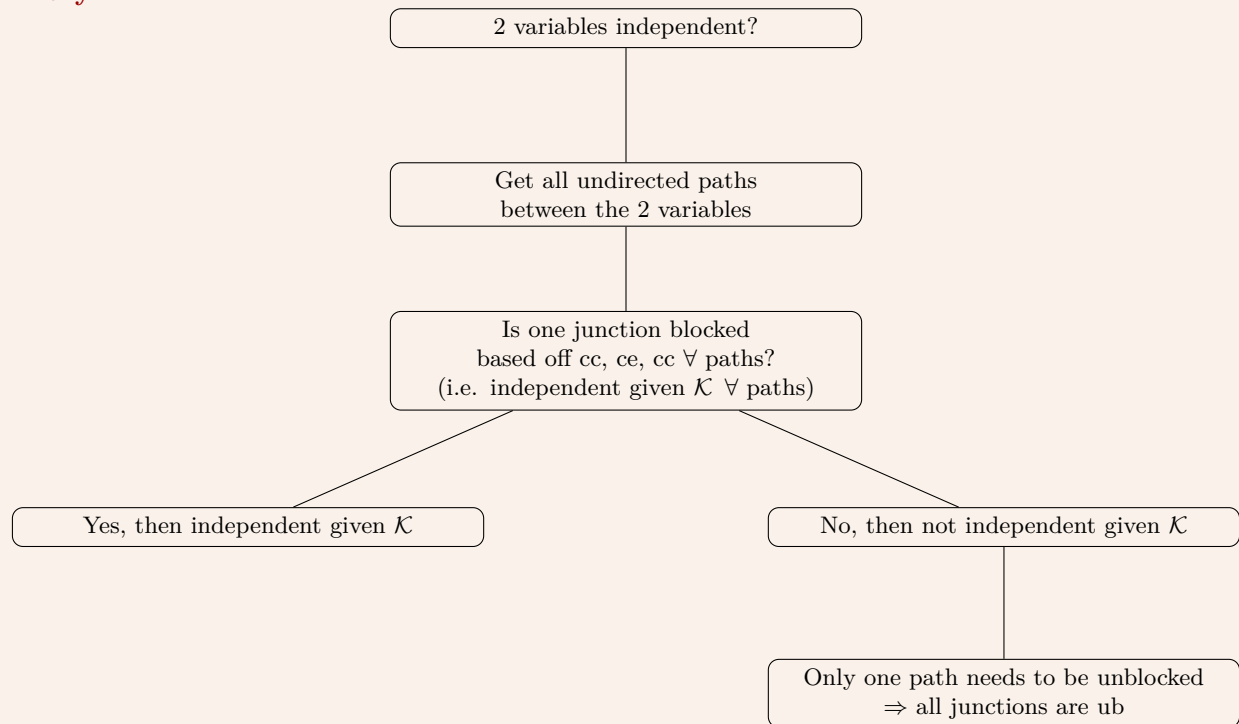
Notes:

Figure 6

- Given X's parent, apply junction rules to determine that X is independent of its non-descendants.
- $\mathcal{J} = \{(Z_1, \mathrm{pt}(X), X), (e_1, e)\}$ shows that Z_1 and X are independent given $\mathrm{pt}(X)$ (causal chain).
- $\mathcal{J} = \{(Z_2, \mathrm{pt}(X), X), (e_2, e)\}$ shows that Z_2 and X are independent given $\mathrm{pt}(X)$ (common cause).
- Given $\mathrm{ch}(X)$'s parent, apply junction rules to determine that $\mathrm{ch}(X)$ is independent of its non-descendants.
- $\mathcal{J} = \{\mathrm{pt}(X), X, \mathrm{ch}(X)), (e, e')\}$ shows that $\mathrm{pt}(X)$ and $\mathrm{ch}(X)$ are independent given X (causal chain).

- Given Z_4's parent, apply junction rules to determine that Z_4 is independent of its non-descendants.
- $\mathcal{J} = \{X, \text{ch}(X), Z_4, (e', e_4)\}$ shows that X and Z_4 are independent given $\text{ch}(X)$ (causal chain).

2.3   Flow Chart

Summary:

2 variables independent?

Get all undirected paths
between the 2 variables

Is one junction blocked
based off cc, ce, cc $\forall$ paths?
(i.e. independent given $\mathcal{K}$ $\forall$ paths)

Yes, then independent given $\mathcal{K}$

No, then not independent given $\mathcal{K}$

Only one path needs to be unblocked
$\Rightarrow$ all junctions are ub

2.4   Canonical Problems

2.4.1   Undirected Path Blocked?

Process:
1. **Given:** Undirected path p and $\mathcal{K}$
2. Check if any of the junctions on the undirected path are blocked given $\mathcal{K}$.
   - i.e. Check if X_1 and X_3 of the junction are independent given $\mathcal{K}$.

2.4.2   Independence

Process:
1. Given a Bayesian network w/ 2 variables to find independence.
2. Find all undirected paths between the 2 variables in the Bayesian network.
3. Identify a set of variables, $\mathcal{K}$, that block at least one junction in all undirected paths.
   - Test a junction by seeing junction given relationships.
4. If all undirected paths are blocked, then the 2 variables are independent given $\mathcal{K}$.

Warning:
- **Common effect:** Blocked by default.
  - **Decesdents: of a common effect:** Outward arrows from a common effect node as given it may not be blocked.
  - **Subset:** If $\mathcal{K}$ includes variables that cause all undirected paths to be blocked, then variables are independent. BUT be careful of common effect, in which it is blocked by default so YOU DONT INCLUDE IT IN $\mathcal{K}$.
- Cyclic paths are not blocked by default.

2.4.3   Find all Subsets

Notes:
- For each full independence set, add an OR b/w each subset. Be careful of common effect.

Example:
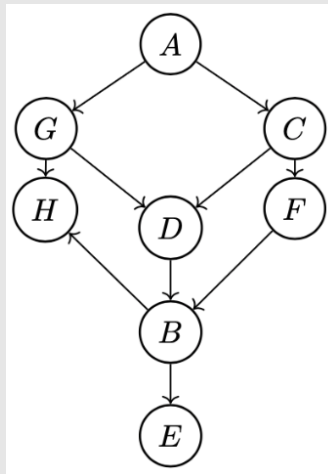1. **Given:** Bayesian network.

Figure 7

2. **Problem:** A and E are
    - independent if $\mathcal{K} =$
    - not necessarily independent for $\mathcal{K} =$
3. **Soln:**
    (a) **Undirected Paths:**
        - $A \to G \to H \to B \to E$
        - $A \to G \to D \to B \to E$
        - $A \to C \to F \to B \to E$
        - $A \to C \to D \to B \to E$

Example: **Independent**:

$\mathcal{K}$
$\{G, C\}$

- $A \iff G \iff H \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(A, G, H), (e_1, e_2)\}$ is blocked given G since A, H independent given G (causal chain)
- $A \iff G \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(A, G, D), (e_1, e_2)\}$ is blocked given G since A, D independent given G (causal chain)
- $A \iff C \iff F \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(A, C, F), (e_1, e_2)\}$ is blocked given C since A, F independent given C (causal chain)
- $A \iff C \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(A, C, D), (e_1, e_2)\}$ is blocked given C since A, D independent given C (causal chain)

$\{D, F\}$

- $A \iff G \iff H \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(G, H, B), (e_1, e_2)\}$ is blocked NOT given H since G, B independent NOT given H (common effect)
- $A \iff G \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(G, D, B), (e_1, e_2)\}$ is blocked given D since G, B independent given D (causal chain)
- $A \iff C \iff F \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(C, F, B), (e_1, e_2)\}$ is blocked given F since C, B independent given F (causal chain)
- $A \iff C \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(C, D, B), (e_1, e_2)\}$ is blocked given D since C, B independent given D (causal chain)

Not Necessarily Independent:

$\mathcal{K}$
$\{H, D, F\}$

- $A \iff G \iff H \iff B \iff E$ is unblocked given $\mathcal{K}$ since $\mathcal{J} = \{(G, H, B), (e_1, e_2)\}$ is unblocked given H since G, B not independent given H (common effect)
- $A \iff G \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(G, D, B), (e_1, e_2)\}$ is blocked given D (causal chain) since G, B independent given D (causal chain)
- $A \iff C \iff F \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(C, F, B), (e_1, e_2)\}$ is blocked given F since C, B independent given F (causal chain)
- $A \iff C \iff D \iff B \iff E$ is blocked given $\mathcal{K}$ since $\mathcal{J} = \{(C, D, B), (e_1, e_2)\}$ is blocked given D since C, B independent given D (causal chain)

Example: Determine all subsets of $\{B, C, D, F, G, H\}$ for which A and E are independent.
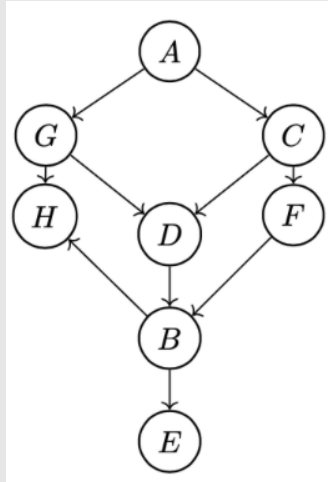

Figure 8

1. **Undirected Paths:**
   - $A \to G \to H \to B \to E$
   - $A \to G \to D \to B \to E$
   - $A \to C \to F \to B \to E$
   - $A \to C \to D \to B \to E$

$$\mathcal{K}$$

$\{B\}$ (Any subset that contains B will be independent)

- $AGHBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(H, B, E), (e_1, e_2)\}$ is b since H, E indep. given B (causal chain)
- $AGDBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(D, B, E), (e_1, e_2)\}$ is b since D, E indep. given B (causal chain)
- $ACFBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(F, B, E), (e_1, e_2)\}$ is b since F, E indep. given B (causal chain)
- $ACDBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(D, B, E), (e_1, e_2)\}$ is b since D, E indep. given B (causal chain)

$\{C\}$ (Not independent)

- $AGDBE$ is ub given $\mathcal{K}$ since $\mathcal{J} = \{(D, B, E), (e_1, e_2)\}$ is ub since D, E NOT indep. given $\mathcal{K}$ (causal chain)

$\{D\}$ (Not indepedent)

- $ACFBE$ is ub given $\mathcal{K}$ since $\mathcal{J} = \{(F, B, E), (e_1, e_2)\}$ is ub since F, E NOT indep. given $\mathcal{K}$ (causal chain)

$\{F\}$ (Not independent)

- $AGDBE$ is ub given $\mathcal{K}$ since $\mathcal{J} = \{(D, B, E), (e_1, e_2)\}$ is ub since D, E NOT indep. given $\mathcal{K}$ (causal chain)

$\{G\}$ (Not independent)

- $ACFBE$ is ub given $\mathcal{K}$ since $\mathcal{J} = \{(F, B, E), (e_1, e_2)\}$ is ub since F, E NOT indep. given $\mathcal{K}$ (causal chain)

$\{H\}$ (Not independent)

- $ACFBE$ is ub given $\mathcal{K}$ since $\mathcal{J} = \{(F, B, E), (e_1, e_2)\}$ is ub since F, E NOT indep. given $\mathcal{K}$ (common effect)

Example:

$$\mathcal{K}$$

$\{C, D\}$ (Any subset that contains C and D except H will be independent)

- $AGHBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(G, H, B), (e_1, e_2)\}$ is b since G, B indep. not given H (common effect)
- $AGDBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(G, D, B), (e_1, e_2)\}$ is b since G, B indep. given D (causal chain)
- $ACFBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(A, C, F), (e_1, e_2)\}$ is b since A, F indep. given C (causal chain)
- $ACDBE$ is b given $\mathcal{K}$ since $\mathcal{J} = \{(A, C, D), (e_1, e_2)\}$ is b since A, D indep. given C (causal chain)
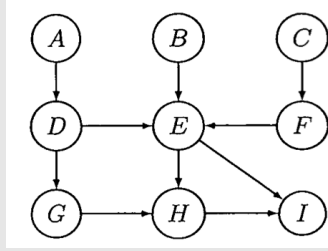

. . .

Example:
1. **Given:**

Figure 9

2. **Problem 1:** Is it gauranteed that $A \perp C$?
3. **Solution 1:** True b/c all undirected paths are blocked.
   (a) ADEFC is b since $\mathcal{J} = \{(D, E, F), (e_1, e_2)\}$ is b since D, F indep. NOT given E (common effect)
   (b) ADGHEFC is b since $\mathcal{J} = \{(G, H, E), (e_1, e_2)\}$ is b since G, E indep. NOT given H (common effect)
   (c) ADGHIEFC is b since $\mathcal{J} = \{(H, I, E), (e_1, e_2)\}$ is b since H, E indep. NOT given I (common effect)

4. **Problem 2:** Is it gauranteed that $B \perp C \mid I$?
5. **Solution 2:** False b/c BEFC is ub.
   (a) BEFC is ub since $\mathcal{J} = \{(B, E, F), (e_1, e_2)\}$ is ub since B, F NOT indep. given E's descendent, I (common effect)

6. **Problem 3:** Is it gauranteed that $D \perp I \mid \{E, G\}$?
7. **Solution 3:** True b/c all undirected paths are blocked.
   (a) DEI is b since $\mathcal{J} = \{(D, E, I), (e_1, e_2)\}$ is b since D, I indep. given E (causal chain)
   (b) DEHI is b since $\mathcal{J} = \{(D, E, H), (e_1, e_2)\}$ is b since D, H indep. given E (causal chain)
   (c) DGHI is b since $\mathcal{J} = \{(D, G, H), (e_1, e_2)\}$ is b since D, H indep. given G (causal chain)
   (d) DGHEI is b since $\mathcal{J} = \{(D, G, H), (e_1, e_2)\}$ is b since D, H indep. given G (causal chain)

8. **Problem 4:** Is it gauranteed that $C \perp H \mid G$?
9. **Solution 4:** False b/c CFEH is ub.
   (a) CFEH is ub since $\mathcal{J} = \{(C, F, E), (e_1, e_2)\}$ is ub since C, E NOT indep. given G (causal chain)

Example:
1. **Problem 5:** Suppose each variable is binary. What is the size of the domain of the joint distribution b/w the variables?
2. **Solution 5:**
   (a) Since 9 variables, each with 2 values, the size of the domain of the joint distribution is $2^9 = 512$.

3. **Problem 6:** Suppose each variable is binary. What is the min # of values that actually need to be stored to represent the joint distribution entirely based on the Bayesian network? Use the fact that probability distributions are normalized.
4. **Solution 6:** $1+1+1+2+8+2+2+4+4 = 25$ values need to be stored.
   (a) $P(A)$, $P(B)$, $P(C)$ has 1 value each
      - Since $P(\#)$ can represent 2 values, i.e. $P(0) = 1 - P(1)$, so only need to store 1 value.
   (b) $P(D \mid A)$, $P(F \mid C)$, $P(G \mid D)$ has 2 values each
      - Same idea, can take the complement of the other value for 4 values.
   (c) $P(H \mid G, E)$, $P(I \mid E, H)$ has 4 values each
      - Same idea, can take the complement of the other value for 8 values.
   (d) $P(E \mid D, B, F)$ has 8 values
      - Same idea, can take the complement of the other value for 16 values.