Notation: $P_{X|Y}(x \mid y) = P[X = x \mid Y = y]$
*Subscript indicates the RV, and the value indicates the realization.
Intro:
Random Experiment: An outcome for each run.
Sample Space Ω: Set of all possible outcomes.
Event: Measurable subsets of Ω.
Prob. of Event A: $P(A) = \frac{\text{Number of outcomes in A}}{\text{Number of outcomes in } \Omega}$
Axioms: (1) $P(A) \geq 0\ \forall A \in \Omega$, (2) $P(\Omega) = 1$,
(3) If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)\ \forall A, B \in \Omega$
Additive Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Cond. Prob. $P(A|B) = \frac{P(A \cap B)}{P(B)}$
*Prob. measured on new sample space B.
*$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$
Independence: $P(A|B) = P(A) \Leftrightarrow P(A \cap B) = P(A)P(B)$
Total Prob. Thm: If $H_1, H_2, \ldots, H_n$ form a partition of Ω, then $P(A) = \sum_{i=1}^{n} P(A|H_i)P(H_i)$.
Partition: $H_1, \ldots, H_n$ is a partition if
(1) $H_i \cap H_j = \emptyset$ for $i \neq j$, (2) $H_1 \cup H_2 \cup \ldots \cup H_n = \Omega$.
Bayes' Rule: $P(H_k|A) = \frac{P(H_k \cap A)}{P(A)} = \frac{P(A|H_k)P(H_k)}{\sum_{i=1}^{n} P(A|H_i)P(H_i)}$
*Posteriori: $P(H_k|A)$, Likelihood: $P(A|H_k)$, Prior: $P(H_k)$
1 RV:
Cumulative Distribution Fn (CDF): $F_X(x) = P[X \leq x]$
Prob. Mass Fn (PMF): $P_X(x_j) = P[X = x_j]\ j = 1, 2, \ldots$
Prob. Density Fn (PDF): $f_X(x) = \frac{d}{dx} F_X(x)$
*$P[a \leq X \leq b] = \int_a^b f_X(x)\, dx$
Exp.:
*$E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x) dx$
*$E[h(X)] = \sum_{k=-\infty}^{\infty} h(k) P_X(x_i = k)$
*$E[aX + b] = aE[X] + b$
Variance: $\sigma_X^2 = \mathrm{Var}[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$
*$\mathrm{Var}[aX + b] = a^2 \mathrm{Var}[X]$
*$\mathrm{Var}[aX + bY] = a^2 \mathrm{Var}[X] + b^2 \mathrm{Var}[Y] + 2ab\mathrm{Cov}[X, Y]$
2 RVs:
Joint PMF: $P_{X,Y}(x, y) = P[X = x, Y = y]$
Joint PDF: $f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$
*$P[(X, Y) \in A] = \int\int_{(x,y)\in A} f_{X,Y}(x, y)\, dx\, dy$
Exp.: $E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dx\, dy$
Correlation: $E[XY]$
***Indep:** $E[XY] = E[X]E[Y]$
Covar.: $\mathrm{Cov}[X, Y] = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - E[X]E[Y]$
Corr. Coeff.: $\rho_{X,Y} = E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right] = \frac{\mathrm{Cov}[X,Y]}{\sigma_X \sigma_Y}$
*$-1 \leq \rho_{X,Y} \leq 1$
Marginal PMF: $P_X(x) = \sum_{j=1}^{\infty} P_{X,Y}(x, y_j) \mid P_Y(y)$
Marginal PDF: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy \mid f_Y(y)$
Cond. PMF: $P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)} \mid P_{Y|X}(y|x)$
Cond. PDF: $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \mid f_{Y|X}(y|x)$
Bayes' Rule
*$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_{X|Y}(x|y) f_Y(y)}{\int_{-\infty}^{\infty} f_{X|Y}(x|y') f_Y(y')\, dy'}$
*$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)} = \frac{P_{X|Y}(x|y) P_Y(y)}{\sum_{j=1}^{\infty} P_{X|Y}(x|y_j) P_Y(y_j)}$

$$P_{\Theta|\underline{X}}(\theta|\underline{x}) = \begin{cases} \frac{P_{\underline{X}|\Theta}(\underline{x}|\theta) P_\Theta(\theta)}{P_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ disc.} \\ \frac{f_{\underline{X}|\Theta}(\underline{x}|\theta) P_\Theta(\theta)}{f_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ cont.} \end{cases}$$

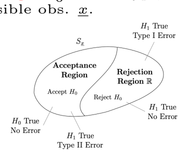
$$f_{\Theta|\underline{X}}(\theta|\underline{x}) = \begin{cases} \frac{P_{\underline{X}|\Theta}(\underline{x}|\theta) f_\Theta(\theta)}{P_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ disc.} \\ \frac{f_{\underline{X}|\Theta}(\underline{x}|\theta) f_\Theta(\theta)}{f_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ cont.} \end{cases}$$

*$f_{\underline{X}}(\underline{x}) = \int_{-\infty}^{\infty} f_{\underline{X}|\Theta}(\underline{x}|\theta) f_\Theta(\theta)\, d\theta$
*$P_{\underline{X}}(\underline{x}) = \sum_{j=1}^{\infty} P_{\underline{X}|\Theta}(\underline{x}|\theta_j) P_\Theta(\theta_j)$
Ind.: $f_{X|Y}(x|y) = f_X(x)\ \forall y \Leftrightarrow f_{X,Y}(x, y) = f_X(x) f_Y(y)$
Thm: If independent, then uncorrelated unless Guassian.
Uncorrelated: $\mathrm{Cov}[X, Y] = 0 \Leftrightarrow \rho_{X,Y} = 0$
Orthogonal: $E[XY] = 0$
Cond. Exp.: $E[Y] = E[E[Y|X]]$ or $E[E[h(Y)|X]]$
$E[Y] = \int_{-\infty}^{\infty} E[Y|X] f_X(x)\, dx$
*$E[Y \mid X] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, dy$
*$E[E[Y|X]]$ w.r.t. $X \mid E[Y|X]$ w.r.t. Y.
Q Fcn: $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt$
Counting:
Permutations: $P(n, k) = \frac{n!}{(n-k)!}$
*Order matters
Combinations: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
*Order doesn't matter
Multinomial Coeff.: $\binom{n}{n_1, \ldots, n_m} = \frac{n!}{n_1! \ldots n_m!}$

PMF/PDF: Make sure normalized to 1.
Bernoulli: Probability of success $x = 1$ or failure $x = 0$ in 1 trial, w/ success occurring w/ prob. p
Binomial: Probability of obtaining k successes in n i.i.d. trials, each w/ success prob. p.
Geometric: Probability that the 1st success occurs on the kth i.i.d. trials, each w/ success prob. p.
Negative Binomial: Probability that the rth success occurs on the kth i.i.d. trials, each w/ success prop. p.
Hypergeometric: Probability of drawing k successes in r draws from a population of N, which contains m successes, w/o replacement.
Poisson: Probability of observing k events in a fixed interval, given a constant rate λ, assuming events indep.
Multinomial: Probability of obtaining counts $(n_1, \ldots, n_r)$ in n trials, where each outcome belongs to one of r categories w/ probabilities $(p_1, \ldots, p_r)$.
Uniform: Any value in $[a, b]$ is equally likely, w/ constant density $\frac{1}{b-a}$.
Exponential: Probability of waiting time x until the 1st event in a Poisson process w/ rate λ, modelling time b/w indep. events.
Gamma: Probability of waiting time x until α events in a Poisson process w/ rate λ, generalizing the exponential distribution.
Gaussian: Probability of observing x in a normal distribution w/ mean μ and variance σ^2, modelling continuous data.
Beta: $[0, 1]$ w/ α, β, used as a prior in Bayesian inference.
Estimation: Estimate unknown parameter θ from n i.i.d. mea-

surements $X_1, X_2, \ldots, X_n$, $\hat{\Theta}(\underline{X}) = g(X_1, X_2, \ldots, X_n)$
Estimation Error: $\hat{\Theta}(\underline{X}) - \theta$.
Unbiased: $\hat{\Theta}(\underline{X})$ is unbiased if $E[\hat{\Theta}(\underline{X})] = \theta$.
***Asymptotically unbiased:** $\lim_{n\to\infty} E[\hat{\Theta}(\underline{X})] = \theta$.
Consistent: $\hat{\Theta}(\underline{X})$ is consistent if $\hat{\Theta}(\underline{X}) \to \theta$ as $n \to \infty$ or $\forall \epsilon > 0, \lim_{n\to\infty} P[|\hat{\Theta}(\underline{X}) - \theta| < \epsilon] \to 1$.
Sufficient: A statistic is sufficient if the expression depends only on the statistic, it should be made up of $x_1, x_2, \ldots, x_n$.
Sample Mean: $M_n = \frac{1}{n} S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.
*Given a sequence of i.i.d. RVs, $X_1, X_2, \ldots, X_n$, M_n is unbiased and consistent.
Sample Variance: $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - M_n)^2$.
*Given a sequence of i.i.d. RVs, $X_1, X_2, \ldots, X_n$, S_n^2 is biased and consistent.
*Use $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - M_n)^2$ for unbiased.
Chebychev's Inequality: $P[|X - E[X]| \geq \epsilon] \leq \frac{\mathrm{Var}[X]}{\epsilon^2}$
*$P[|X - E[X]| < \epsilon] \geq 1 - \frac{\mathrm{Var}[X]}{\epsilon^2}$
Weak Law of Large #s: $\lim_{n\to\infty} P[|M_n - \mu| < \epsilon] = 1 \ \forall \epsilon > 0$.
ML Estimation: Choose θ that is most likely to generate the obs. $x_1, x_2, \ldots, x_n$.
*Disc: $\hat{\Theta} = \arg\max_\theta P_{\underline{X}}(\underline{x}|\theta) \overset{\log}{\to} \hat{\theta} = \arg\max_\theta \sum_{i=1}^{n} \log P_X(x_i|\theta)$
*Cont: $\hat{\Theta} = \arg\max_\theta f_{\underline{X}}(\underline{x}|\theta) \overset{\log}{\to} \hat{\theta} = \arg\max_\theta \sum_{i=1}^{n} \log f_X(x_i|\theta)$
Maximum A Posteriori (MAP) Estimation:
*Disc: $\hat{\theta} = \arg\max_\theta P_{\Theta|\underline{X}}(\theta|\underline{x}) = \arg\max_\theta P_{\underline{X}|\Theta}(\underline{x}|\theta)P_\Theta(\theta)$
*Cont: $\hat{\theta} = \arg\max_\theta f_{\Theta|\underline{X}}(\theta|\underline{x}) = \arg\max_\theta f_{\underline{X}|\Theta}(\underline{x}|\theta)f_\Theta(\theta)$
*$f_{\Theta|\underline{X}}(\theta|\underline{x})$: Posteriori, $f_{\underline{X}|\Theta}(\underline{x}|\theta)$: Likelihood, $f_\Theta(\theta)$: Prior
Least Mean Squares (LMS) Estimation: Assume prior $P_\Theta(\theta)$ or $f_\Theta(\theta)$ w/ obs. $\underline{X} = \underline{x}$.
*$\hat{\theta} = g(\underline{x}) = \mathbb{E}[\Theta|\underline{X} = \underline{x}]$ | $\hat{\Theta} = g(\underline{X}) = \mathbb{E}[\Theta|\underline{X}]$

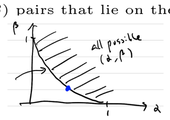
Conditional Exp. $E[X|Y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$
Binary Hyp. Testing: H_0: Null Hyp., H_1: Alt. Hyp.
$\Omega_{\underline{X}}$: Set of all possible obs. $\underline{x}$.



TI Err. (False Rejection): Reject H_0 when H_0 is true.
*$\alpha(R) = P[\underline{X} \in R \mid H_0]$ (false alarm)
TII Err. (False Accept.): Accept H_0 when H_1 is true.
*$\beta(R) = P[\underline{X} \in R^c \mid H_1]$ (missed detection)
Prob. of Error: $P = \alpha\pi_0 + \beta(1 - \pi_0)$

Likelihood Ratio Test: $\forall \underline{x}$ $L(\underline{x}) = \frac{P_{\underline{X}}(\underline{x}|H_1)}{P_{\underline{X}}(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$ or ξ

***Max. Likelihood Test:** 1, **Likelihood Ratio Test:** ξ
Neyman-Pearson Lemma: Given a false rejection prob. (α), the LRT offers the smallest possible false accept. prob. (β), and vice versa.
*LRT produces (α, β) pairs that lie on the efficient frontier.



Bayesian Hyp. Testing:
MAP Rule: $L(\underline{x}) = \frac{p_{\underline{X}}(\underline{x}|H_1)}{p_{\underline{X}}(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P[H_0]}{P[H_1]}$
Min. Cost Bayes' Dec. Rule: C_{ij} is cost of choosing H_j when H_i is true. Given obs. $\underline{X} = \underline{x}$, the expected cost of choosing H_j is $A_j(\underline{x}) = \sum_{i=0}^{1} C_{ij}\, P[H_i|\underline{X} = \underline{x}]$.
Min. Cost Dec. Rule: $L(\underline{x}) = \frac{P_{\underline{X}}(\underline{x}|H_1)}{P_{\underline{X}}(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(C_{01}-C_{00})P[H_0]}{(C_{10}-C_{11})P[H_1]}$.
*C_{01}: False accept. cost, C_{10}: False reject. cost.
Naive Bayes Assumption: Assume $X_1 \ldots, X_n$ (features) are ind., then $p_{\underline{X}|\Theta}(\underline{x} \mid \theta) = \Pi_{i=1}^{n} p_{X_i|\Theta}(x_i \mid \theta)$.
Notation: $P_{\underline{X}|\Theta}(\underline{x}|\theta)$, only put RVs in subscript, not values.
$P_{\underline{X}}(\underline{x}|H_i)$, didn't put H in subscript b/c it's not a RV.
Beta Prior Θ is a Beta R.V. w/ $\alpha, \beta > 0$
$$f_\Theta(\theta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} & \text{if } 0 < \theta < 1 \\ 0 & \text{otherwise} \end{cases}$$
*$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$
Prop.: 1. $\Gamma(x+1) = x\Gamma(x)$. For $m \in \mathbb{Z}^+$, $\Gamma(m+1) = m!$.
2. $\beta(\alpha, \beta) = \frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!} = \beta\binom{\alpha+\beta-1}{\alpha-1}$
3. Expected Value: $E[\Theta] = \frac{\alpha}{\alpha+\beta}$ for $\alpha, \beta > 0$
4. Mode (max of PDF): $\frac{\alpha-1}{\alpha+\beta-2}$ for $\alpha, \beta > 1$

Drawing Beta Dist. 1. Label x-axis from 0 to 1. 2. Identify mode.
3. Determine shape based on α and β: $\alpha = \beta = 1$ (uniform), $\alpha = \beta > 1$ (bell-shaped, peak at 0.5), $\alpha = \beta < 1$ (U-shaped w/ high density near 0 and 1), $\alpha > \beta$ (left-skewed), $\alpha < \beta$ (right-skewed).
Uniform PDF $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$
*$E[X] = \frac{a+b}{2}$, $\mathrm{Var}[X] = \frac{(b-a)^2}{12}$

Random Vector: $\underline{X} = (X_1, \ldots, X_n) = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = [X_1 \quad \cdots \quad X_n]^T$
Mean Vector: $\underline{m}_{\underline{X}} = E[\underline{X}] = [\mu_1, \ldots, \mu_n]^T$
Corr. Mat.: $R_{\underline{X}} = \begin{bmatrix} E[X_1^2] & \cdots & E[X_1X_n] \\ E[X_2X_1] & \cdots & E[X_2X_n] \\ \vdots & \ddots & \vdots \\ E[X_nX_1] & \cdots & E[X_n^2] \end{bmatrix}$
*Real, symmetric ($R = R^T$), and PSD ($\forall \underline{a}, \underline{a}^T R \underline{a} \geq 0$).

Covar. Mat.: $K_{\underline{X}} = \begin{bmatrix} \mathrm{Var}[X_1] & \cdots & \mathrm{Cov}[X_1, X_n] \\ \mathrm{Cov}[X_2, X_1] & \cdots & \mathrm{Cov}[X_2, X_n] \\ \vdots & \ddots & \vdots \\ \mathrm{Cov}[X_n, X_1] & \cdots & \mathrm{Var}[X_n] \end{bmatrix}$

*$K_{\underline{X}} = R_{\underline{X} - \underline{m}_{\underline{X}}} = R_{\underline{X}} - \underline{m}\underline{m}^T$

*Diagonal $K_{\underline{X}} \iff X_1, \ldots, X_n$ are (mutually) uncorrelated.

Lin. Trans. $\underline{Y} = A\underline{X}$ (A rotates and stretches $\underline{X}$)

Mean: $E[\underline{Y}] = A\underline{m}_{\underline{X}}$

Covar. Mat.: $K_{\underline{Y}} = AK_{\underline{X}}A^T$

Diagonalization of Covar. Mat. (Uncorrelated):

$\forall \underline{X}$, set $P = [\underline{e}_1, \ldots, \underline{e}_n]$ of $K_{\underline{X}}$, if $\underline{Y} = P^T\underline{X}$, then

$K_{\underline{Y}} = P^T K_{\underline{X}} P = \Lambda$

*$\underline{Y}$: Uncorrelated RVs, $K_{\underline{X}} = P\Lambda P^T$

Find an Uncorrelated $K_{\underline{Y}}$

1. Find eigenvalues, normalized eigenvectors of $K_{\underline{X}}$.

2. Set $K_{\underline{Y}} = \Lambda$, where $\underline{Y} = P^T\underline{X}$

PDF of L.T. If $\underline{Y} = A\underline{X}$ w/ A not singular, then

$f_{\underline{Y}}(\underline{y}) = \frac{f_{\underline{X}}(\underline{x})}{|\det A|}\Big|_{\underline{x} = A^{-1}\underline{y}}$

Find $f_{\underline{Y}}(\underline{y})$ 1. Given $f_{\underline{X}}(\underline{x})$ and RV relations, define A

2. Determine $|\det A|$, A^{-1}, then $f_{\underline{Y}}(\underline{y})$.

Gaussian RVs: $\underline{X} \sim \mathcal{N}(\underline{\mu}, \Sigma)$

PDF of jointly Gaus. $X_1, \ldots, X_n \equiv$ Guas. vector:

$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2}|\det \Sigma|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \Sigma^{-1}(\underline{x}-\underline{\mu})}$

*1D: $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

*$\underline{\mu} = \underline{m}_{\underline{X}}$, $\Sigma = K_{\underline{X}}$ (Σ not singular)

*Indep.: $f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2}\prod_{i=1}^n \sigma_i} e^{-\frac{1}{2}\sum_{i=1}^n \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2}$

*IID: $f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2}$

*Cond. PDF: $f_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y}) = \mathcal{N}(\underline{\mu}_{\underline{X}|\underline{Y}}, \Sigma_{\underline{X}|\underline{Y}})$

Properties of Guassian Vector:

1. PDF is completely determined by $\underline{\mu}$, Σ.
2. $\underline{X}$ uncorrelated $\iff$ $\underline{X}$ independent.
3. Any L.T. $\underline{Y} = A\underline{X}$ is Gaus. vector w/ $\underline{\mu}_{\underline{Y}} = A\underline{\mu}_{\underline{X}}$, $\Sigma_{\underline{Y}} = A\Sigma_{\underline{X}}A^T$.
4. Any subset of $\{X_i\}$ are jointly Gaus.
5. Any cond. PDF of a subset of $\{X_i\}$ given the other elements is Gaus.

Diagonalization of Guassian Covar. (Indep.)

$\forall \underline{X}$, set $P = [\underline{e}_1, \ldots, \underline{e}_n]$ of $\Sigma_{\underline{X}}$, if $\underline{Y} = P^T\underline{X}$, then

$\Sigma_{\underline{Y}} = P^T \Sigma_{\underline{X}} P = \Lambda$

*$\underline{Y}$: Indep. Gaussian RVs, $\Sigma_{\underline{X}} = P\Lambda P^T$

How to go from Y to X? 1. Given, $\underline{X} \sim \mathcal{N}(\underline{\mu}, \Sigma)$

2. $\underline{V} \sim \mathcal{N}(\underline{0}, I)$ 3. $\underline{W} = \sqrt{\Lambda}\underline{V}$ 4. $\underline{Y} = P\underline{W}$ 4. $\underline{X} = \underline{Y} + \underline{\mu}$

Guassian Discriminant Analysis:

Obs: $\underline{X} = \underline{x} = (x_1, \ldots, x_D)$

Hyp: $\underline{x}$ is generated by $\mathcal{N}(\underline{\mu}_c, \Sigma_c)$, $c \in C$

Dec: Which "Guassian bump" generated $\underline{x}$?

Prior: $P[C = c] = \pi_c$ (Gaussian Mixture Model)

MAP: $\hat{c} = \arg\max_c P_C[c|\underline{X} = \underline{x}] = \arg\max_c f_{\underline{X}|C}(\underline{x} \mid c)\pi_c$

LGD: Given $\Sigma_c = \Sigma\ \forall c$, find c w/ best $\underline{\mu}_c$

$\hat{c} = \arg\max_c \underline{\beta}_c^T \underline{x} + \gamma_c$

*$\underline{\beta}_c^T = \underline{\mu}_c^T \Sigma^{-1}$ | $\gamma_c = \log \pi_c - \frac{1}{2}\underline{\mu}_c^T \Sigma^{-1} \underline{\mu}_c$

Bin. Hyp. Decision Boundary $\underline{\beta}_0^T \underline{x} + \gamma_0 = \underline{\beta}_1^T \underline{x} + \gamma_1$

*Linear in space of $\underline{x}$

QGD: Given Σ_c are diff., find c w/ best $\underline{\mu}_c$, Σ_c

$\hat{c} = \arg\max_c -\frac{1}{2}\log|\Sigma_c| - \frac{1}{2}(\underline{x} - \underline{\mu}_c)^T \Sigma_c^{-1}(\underline{x} - \underline{\mu}_c) + \log \pi_c$

Bin. Hyp. Decision Boundary Quadratic in space of $\underline{x}$

How to find $\underline{\pi}_c, \underline{\mu}_c, \Sigma_c$: Given n points gen. by GMM, then

n_c points $\{\underline{x}_1^c, \ldots, \underline{x}_{n_c}^c\}$ come from $\mathcal{N}(\underline{\mu}_c, \Sigma_c)$

$\hat{\pi}_c = \frac{n_c}{n}$ (categorical RV)

$\hat{\mu}_c = \frac{1}{n_c}\sum_{i=1}^n \underline{x}_i^c$, (sample mean)

$\Sigma_c = \frac{1}{n_c}\sum_{i=1}^{n_c}(x_i^c - \hat{\mu}_c)(x_i^c - \hat{\mu}_c)^T$ (biased sampled var.)

Guassian Estimation:

MAP Estimator for $\underline{X}$ Given $\underline{Y}$ When $\underline{W} = (\underline{X}, \underline{Y}) \sim \mathcal{N}(\underline{\mu}, \Sigma)$

Given $\underline{X} = \{X_1, \ldots, X_n\}$, $\underline{Y} = \{Y_1, \ldots, Y_m\}$

$\hat{\underline{x}}_{\mathrm{MAP}}(\underline{y}) = \hat{\underline{x}}_{\mathrm{LMS}}(\underline{y}) = \underline{\mu}_{\underline{X}|\underline{Y}} = \underline{\mu}_{\underline{X}} + \Sigma_{\underline{XY}}\Sigma_{\underline{YY}}^{-1}(\underline{y} - \underline{\mu}_Y)$

*$\hat{\underline{x}}_{\mathrm{MAP/LMS}}$: Linear fcn of $\underline{y}$

Covar. Matrices: $\Sigma = \begin{bmatrix} \Sigma_{\underline{XX}} & \Sigma_{\underline{XY}} \\ \Sigma_{\underline{YX}} & \Sigma_{\underline{YY}} \end{bmatrix}$

*$\Sigma_{\underline{XX}} = \Sigma_{\underline{X}} = E\left[(\underline{X} - \underline{\mu}_{\underline{X}})(\underline{X} - \underline{\mu}_{\underline{X}})^T\right]$ | $\Sigma_{\underline{YY}} = \Sigma_{\underline{Y}}$

*$\Sigma_{\underline{XY}} = E\left[(\underline{X} - \underline{\mu}_{\underline{X}})(\underline{Y} - \underline{\mu}_{\underline{Y}})^T\right]$ | $\Sigma_{\underline{YX}} = \Sigma_{\underline{XY}}^T$

Prec. Matrices: $\Lambda = \Sigma^{-1}$

Mean and Covar. Mat. of $\underline{X}$ Given $\underline{Y}$:

*$\underline{\mu}_{\underline{X}|\underline{Y}} = \underline{\mu}_{\underline{X}} + \Sigma_{\underline{XY}}\Sigma_{\underline{YY}}^{-1}(\underline{y} - \underline{\mu}_Y)$

*$\Sigma_{\underline{X}|\underline{Y}} = \Sigma_{\underline{X}} - \Sigma_{\underline{XY}}\Sigma_{\underline{YY}}^{-1}\Sigma_{\underline{YX}}$

***Reducing Uncertainty:** 2nd term is PSD, so given $\underline{Y} = \underline{y}$, always reducing uncertainty in $\underline{X}$.

ML Estimator for θ w/ Indep. Guas:

Given $\underline{X} = \{X_1, \ldots, X_n\}$: $\hat{\theta}_{\mathrm{ML}} = \frac{\sum_{i=1}^n \frac{x_i}{\sigma_i^2}}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}$ (weighted avg. $\underline{x}$)

*$X_i = \theta + Z_i$: Measurement | $Z_i \sim \mathcal{N}(0, \sigma_i^2)$: Noise (indep.)

*$\frac{1}{\sigma_i^2}$: Precision of X_i (i.e. weight)

*Larger $\sigma_i^2 \implies$ less weight on X_i (less reliable measurement)

***SC:** If $\sigma_i^2 = \sigma^2\ \forall i$ (iid), then $\hat{\theta}_{\mathrm{ML}} = \frac{1}{n}\sum_{i=1}^n x_i$.

MAP Estimator for θ w/ Indep. Gaus., Gaus. Prior:

Given $\underline{X} = \{X_1, \ldots, X_n\}$, prior $\Theta \sim \mathcal{N}(x_0, \sigma_0^2)$

$$\hat{\theta}_{\mathrm{MAP}} = \frac{\sum_{i=0}^n \frac{x_i}{\sigma_i^2}}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} = \frac{\frac{1}{\sigma_0^2}}{\frac{1}{\sigma_0^2} + \sum_{i=1}^n \frac{1}{\sigma_i^2}} x_0 + \frac{\sum_{i=1}^n \frac{1}{\sigma_i^2}}{\frac{1}{\sigma_0^2} + \sum_{i=1}^n \frac{1}{\sigma_i^2}} \hat{\theta}_{\mathrm{ML}}$$

*$X_i = \theta + Z_i$: Measurement | $Z_i \sim \mathcal{N}(0, \sigma_i^2)$: Noise (indep.)

*f_Θ: Gaussian prior $\equiv$ prior meas. x_0 w/ σ_0^2.

*__SC:__ As $n \to \infty$, $\hat{\theta}_{\text{MAP}} \to \hat{\theta}_{\text{ML}}$. As $\sigma_0^2 \to \infty$, $\hat{\theta}_{\text{MAP}} \to \hat{\theta}_{\text{ML}}$

LMMSE Estimator for $\underline{X}$ Given $\underline{Y}$ w/ non-Guas. $\underline{X}$, $\underline{Y}$:

$\hat{\underline{x}}_{\text{LMMSE}}(\underline{y}) = \underline{\mu}_{\underline{X}} + \Sigma_{\underline{XY}}\Sigma_{\underline{YY}}^{-1}(\underline{y} - \underline{\mu}_Y)$

Linear Guassian System: Given $\underline{Y} = A\underline{X} + \underline{b} + \underline{Z}$

*$\underline{X} \sim \mathcal{N}(\underline{\mu}_{\underline{X}}, \Sigma_{\underline{X}})$, $\underline{Z} \sim \mathcal{N}(\underline{0}, \Sigma_{\underline{Z}})$: Noise (indep. of $\underline{x}$)

*$A\underline{X} + \underline{b}$: channel distortion, $\underline{Y}$: Observed sig.

MAP/LMS Estimator for $\underline{X}$ Given $\underline{Y}$ w/ $\underline{W} = (\underline{X}, \underline{Y})$

Given $\underline{W} = \begin{bmatrix} \underline{X} \\ A\underline{X} + \underline{b} + \underline{Z} \end{bmatrix} = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} \underline{X} \\ \underline{Z} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \underline{b} \end{bmatrix}$

$\hat{x}_{\text{MAP/LMS}} = \underline{\mu}_{\underline{X}} + \Sigma_{\underline{X}} A^T (A\Sigma_{\underline{X}} A^T + \Sigma_{\underline{Z}})^{-1}(\underline{y} - A\underline{\mu}_{\underline{X}} - \underline{b})$

*$\Sigma_{\underline{XY}} = \Sigma_{\underline{X}} A^T$, $\Sigma_{\underline{YY}} = A\Sigma_{\underline{X}} A^T + \Sigma_{\underline{Z}}$

$\hat{x}_{\text{MAP/LMS}} = \left(\Sigma_{\underline{X}}^{-1} + A^T\Sigma_{\underline{Z}}^{-1}A\right)^{-1}\left(A^T\Sigma_{\underline{Z}}^{-1}(\underline{y} - \underline{b}) + \Sigma_{\underline{X}}^{-1}\underline{\mu}_{\underline{X}}\right)$

*__Use:__ Good to use when $\underline{Z}$ is indep.

Covar. Mat of $\underline{X}$ Given $\underline{Y} = \underline{y}$: $\Sigma_{\underline{X}|\underline{y}} = \left(\Sigma_{\underline{X}}^{-1} + A^T\Sigma_{\underline{Z}}^{-1}A\right)^{-1}$

Linear Regression: Estimate unknown target fn $Y = g(\underline{X})$ w/ iid obs. $\{(\underline{x}_1, y_1), \ldots, (\underline{x}_n, y_n)\}$ (MLE/MAP)

*$\underline{y} = [y_1 \quad \cdots \quad y_n]^T$

*$X = \begin{bmatrix} \underline{x}_1^T \\ \vdots \\ \underline{x}_n^T \end{bmatrix} \in \mathbb{R}^{n \times D}$

ML Estimator: $Y = h(\underline{x}) = \underline{w}^T\underline{x} + Z \approx g(\underline{X})$, then $\hat{\underline{w}}_{\text{ML}} = (XX^T)^{-1}X^T\underline{y}$

*Assume X^TX has full rank (i.e. invertible) since $n \gg D$

*n: # of obs., D: # of features.

*$\underline{x} = \{x_1, \ldots, x_D\}$: Input features

*$\underline{w} = \{w_1, \ldots, w_D\}$: Weights (parameter)

*$Z \sim \mathcal{N}(0, \sigma^2)$: Noise (i.i.d.)

*Y: Target/observed output

*$X^\dagger = (X^TX)^{-1}X^T$: Pseudo-inverse of X (minimizes $||X\underline{w} - \underline{y}||_2^2 \iff$ maximizes the likelihood of training data)

Non-Linear Trans: $\hat{y} = \underline{w}^T\underline{\phi}(\underline{x}) + Z$ w/ same assumptions, then $\hat{\underline{w}}_{\text{ML}} = (XX^T)^{-1}X^T\underline{y}$

*$\underline{\phi}(\underline{x})$: Non-linear transformation of $\underline{x}$

-E.g. of 1 dim x: $\underline{\phi}(x) = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^M \end{bmatrix}$: Polynomial regression

*M: Degree of polynomial, $D = 1 + M$: # of features.

*$X = \begin{bmatrix} \underline{\phi}(\underline{x}_1)^T \\ \vdots \\ \underline{\phi}(\underline{x}_n)^T \end{bmatrix} \in \mathbb{R}^{n \times D}$

Underfitting vs. Overfitting:

*Underfitting: Model too simple, high bias, low variance.

-Results in high train/test error.

*Overfitting: Model too complex, low bias, high variance.

-Results in low train error, high test error.

MAP Estimator (Bayesian Linear Regression): Assume prior $w_i \sim \mathcal{N}(0, \tau^2)$ (i.i.d.) and $\hat{y} = \underline{w}^T\underline{x} + Z$, then $\hat{\underline{w}}_{\text{MAP}} = (X^TX + \lambda I)^{-1}X^T\underline{y}$

*$\lambda = \frac{\sigma^2}{\tau^2}$: Regularization parameter

*X: Can be linear or non-linear transformation of $\underline{x}$

*$\underline{x} = \{x_1, \ldots, x_D\}$: Input features

*$\underline{w} = \{w_1, \ldots, w_D\}$: Weights (parameter)

*$Z \sim \mathcal{N}(0, \sigma^2)$: Noise (i.i.d.)

*Y: Target/observed output

Notes:

1. Useful when training data set size is small i.e. $n \ll D$.
2. Regularization: Prevents overfitting by penalizing large weights.

*$\tau = \infty \implies \lambda = 0$: No regularization so $\hat{\underline{w}}_{\text{MAP}} = \hat{\underline{w}}_{\text{ML}}$

*$\tau = 0 \implies \lambda = \infty$: Infinite regularization so $\hat{\underline{w}}_{\text{MAP}} = \underline{0}$

*$\tau \downarrow \implies \lambda \uparrow$: More regularization, simpler model.

*$\tau \uparrow \implies \lambda \downarrow$: Less regularization, more complex model.

Guassian Linear System Given training data $\underline{Y} = X\underline{w} + \underline{Z}$

$\hat{\underline{w}}_{\text{MAP}} = \mu_{\underline{w}|\underline{Y}} = (X^TX + \lambda I)^{-1}X^T\underline{y}$

*$\underline{w} \sim \mathcal{N}(\underline{0}, \tau^2 I)$, $\underline{Z} \sim \mathcal{N}(\underline{0}, \sigma^2 I)$

*$E[\hat{\underline{w}}(\underline{Y})] \to \underline{w}$ as $n \to \infty$

*Note: Matching it to canonical form.

Covar. Mat: $\Sigma_{\underline{w}|\underline{y}} = \left(\frac{1}{\sigma^2}X^TX + \frac{1}{\tau^2}I\right)^{-1} \preceq \tau^2 I$

-Less uncertainty in $\underline{w}$ w/ more data. As $n \uparrow$, $\Sigma_{\underline{w}|\underline{y}} \downarrow$

Bayesian Prediction Given some new $\underline{x}'$ (test data sample), find its label y'

Plug-In Approx: $\hat{Y}' = \underline{x}'^T\hat{\underline{w}}_{\text{MAP}}(\mathcal{D}) + Z'$

*$\mathcal{D}$: Training data set, $Z' \sim \mathcal{N}(0, \sigma^2)$: Noise

Bayesian Prediction: Use $Y' = \underline{x}'^T\underline{w} + Z'$ and $f_{\underline{w}|\underline{Y}}(\underline{w} \mid \underline{y}) = \mathcal{N}(\mu_{\underline{w}|\underline{Y}}, \Sigma_{\underline{w}|\underline{Y}})$ to return $f_{Y'}(y' \mid \mathcal{D})$ where Y' is Gaussian given $\mathcal{D}$ w/

*$\mu_{Y'|\mathcal{D}} = \underline{x}'^T\mu_{\underline{w}|\underline{Y}}$

*$\sigma^2_{Y'|\mathcal{D}} = \underline{x}'^T\Sigma_{\underline{w}|\underline{Y}}\underline{x}' + \sigma^2$

Linear Classification (Hyp. Test):

Binary Logistic Regression: Estimate $\underline{w}$ s.t. it is a soft decision

$P_{Y|\underline{X}}(1 \mid \underline{x}) = \frac{P_{\underline{X}|Y}(\underline{x}|1)P_Y(1)}{P_{\underline{X}|Y}(\underline{x}|0)P_Y(0) + P_{\underline{X}|Y}(\underline{x}|1)P_Y(1)}$

$P_{Y|\underline{X}}(1 \mid \underline{x}) = \frac{1}{1+e^{-\alpha}} = \sigma(\alpha)$

*$P_{Y|\underline{X}}(0 \mid \underline{X}) = 1 - \sigma(\alpha) = \frac{1}{1+e^{\alpha}} = \sigma(-\alpha)$

*$\alpha = \log\frac{P_{\underline{X}|Y}(\underline{x}|1)P_Y(1)}{P_{\underline{X}|Y}(\underline{x}|0)P_Y(0)} = \underline{w}^T\underline{x}$

-$\alpha \to \infty \implies$ more likely to be in class 1

-$\alpha \to -\infty \implies$ more likely to be in class 0.

-$\alpha = 0 \implies$ equally likely to be in class 0 or 1.

Non-Linear Trans. Use $\sigma(\underline{w}^T\phi(\underline{x}))$

ML Estimator: Given $\mathcal{D} = \{(\underline{x}_i, y_i)\}, i = 1, \ldots, n$, then $\hat{\underline{w}}_{\text{ML}} = \arg\min_{\underline{w}} - \sum_{i=1}^n \log P_{Y|\underline{X}}(y_i \mid \underline{x}_i, \underline{w})$

Cross Entropy b/w actual y_i and $P_{Y|\underline{X}}(\cdot \mid \underline{x}_i, \underline{w})$ is

$P_{Y|\underline{X}}(y_i \mid \underline{x}_i, \underline{w}) = \sum_{i=1}^n -\left(y_i \log P(1 \mid \underline{x}_i, \underline{w}) + (1 - y_i)\log P(0 \mid \underline{x}_i, \underline{w})\right)$

*Note: Measures the distance between 2 distributions.

*Dropped the subscripts.

Gradient Descent: No closed-form soln. so use GD.
MAP Estimator: Given $\mathcal{D} = \{(\underline{x}_i, y_i)\}, i = 1, \dots, n$, then
$\hat{\underline{w}}_{\text{MAP}} = \arg\min_{\underline{w}} - \sum_{i=1}^{n} \log P_{Y|\underline{X}}(y_i \mid \underline{x}_i, \underline{w}) + \lambda ||\underline{w}||^2$
*$\underline{w} \sim \mathcal{N}(\underline{\mu}, \Sigma)$: Prior on $\underline{w}$
*Necessary: B/c same boundary $\underline{w}^T \underline{x} = 0$ for any scaling of $\underline{w}$.
Multiclass Logistic Regression: $Y \in \{1, 2, \dots, C\}$, then use
softmax fn $P_Y(k \mid \underline{x}, \underline{w}_1, \dots, \underline{w}_C) = \dfrac{e^{\underline{w}_k^T \underline{x}}}{\sum_{c=1}^{C} e^{\underline{w}_c^T \underline{x}}}$
*$W = [\underline{w}_1, \dots, \underline{w}_C] \in \mathbb{R}^{D \times C}$: Weights matrix
ML Estimator: Given $\mathcal{D} = \{(\underline{x}_i, y_i)\}, i = 1, \dots, n$, then
$\hat{W}_{\text{ML}} = \arg\min_W - \sum_{i=1}^{n} \log P(y_i \mid \underline{x}_i, W)$
MAP Estimator: Given $\mathcal{D} = \{(\underline{x}_i, y_i)\}, i = 1, \dots, n$, then
$\hat{W}_{\text{MAP}} = \arg\min_W - \sum_{i=1}^{n} \log P(y_i \mid \underline{x}_i, W) + \sum_{c=1}^{C} \lambda_c ||\underline{w}_c||^2$
Markov:
Notation:
*$P[X_n = x_n, \dots, X_0 = x_0] = P(x_n, \dots, x_0)$
*Index the possible values of X_n w/ integers $0, 1, 2, \dots$
Markov Chain (Memoryless/Markovian Property): A sequence of discrete-valued RVs $X_0, X_1, \dots$ is a (discrete-time) Markov chain if
$P[\underbrace{X_{k+1} = x_{k+1}}_{\text{Future}} \mid \underbrace{X_k = x_k}_{\text{Present}}, \underbrace{X_{k-1} = x_{k-1}, \dots, X_0 = x_0}_{\text{Past}}] =$
$P[X_{k+1} = x_{k+1} \mid X_k = x_k] \ \forall k, x_1, \dots, x_{k+1}$
*$\textbf{Markovian:}$ $P(x_n, \dots, x_0) = P(x_n \mid x_{n-1}) \cdots P(x_1 \mid x_0) P(x_0)$
***Equiv. Form:** $k+1 \to n_{k+1}$, $k \to n_k$ and so on
for any $n_{k+1} > n_k > \dots > n_0$ (i.e. farther in future/past)
State Distribution: State distribution of the MC at time n is
$P_j(n) \equiv P[X_n = j], j = 0, 1, \dots \ \mid \ \underline{P}(n) \equiv [P_0(n), P_1(n), \dots]$
*Subscript: Value of X_n, Argument: Time step
*Row vector NOT col vector.
Transition Probabilities:
$P_{ij}(n, n+1) \equiv P[X_{n+1} = j \mid X_n = i] \ \forall i, j, n$
Homogeneous MC: $P_{ij}(n, n+1) = P_{ij} \ \forall i, j, n$
*Time invariant, P_{ij} does not depend on n

Transition Probability Matrix: $P = \begin{bmatrix} P_{00} & P_{01} & \cdots \\ P_{10} & P_{11} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$

Notes: (1) **Stochastic Matrix:** (1) All entries of P are non-negative and (2) each row sums to 1: $\sum_j P_{ij} = 1 \ \forall i$
(2) State Dist. at time $n+1$: $\underline{P}(n) = \underline{P}(n-1)P$
*$\underline{P}(n) = \underline{P}(0)P^n$ in terms of initial distribution $\underline{P}(0)$
(3) State Dist. at time $n+m$: $\underline{P}(n+m) = \underline{P}(m)P^n \ \forall n, m$
n-step Transition Probabilities: Stochastic matrix P^n s.t. $P_{ij}^{(n)} \equiv P[X_{k+n} = j \mid X_k = i]$ for $n \geq 0$ are the entries of P^n
Limiting Distribution A MC has a limiting distribution $\underline{q}$ if for any initial distribution $\underline{P}(0)$
$\underline{P}(\infty) \equiv \lim_{n\to\infty} \underline{P}(n) = \underline{q}$ or
$\underline{P}(0)P^\infty \equiv \underline{P}(0) \lim_{n\to\infty} P^n = \underline{q}$

Theorem: A MC has a limiting distribution $\underline{q}$ iff
$q_i = \lim_{n\to\infty} P_{ij}^{(n)} \ \forall i, j$
*i.e. every row of P^∞ equals $\underline{q}$ (row vector)
Steady State (Stationary) Distribution $\underline{\pi}$ is a steady state distribution of a MC if $\underline{\pi} = \underline{\pi}P$
*$1 = \sum_j \pi_j$
Theorem: If a limiting dist. exists $\underline{q} = \underline{P}(\infty)$, then it is also a steady state dist.
Ergodic: For a finite-state, irreducible, and aperiodic MC, then
(1) Limiting dist. $\underline{q} = \lim_{n\to\infty} \underline{P}(n)$ exists and
$q_j = \lim_{n\to\infty} P_{ij}^{(n)} \ \forall i, j$
(2) Steady state dist. $\underline{\pi}$ is unique.
(3) $\underline{\pi} = \underline{q}$
How Fast Does $\underline{P}(n)$ Converge to $\underline{\pi}$? (1) $\underline{\pi}^T = \underline{\pi}^T P^T$
*$\underline{\pi}^T$ is an eigenvector of P^T w/ eigenvalue 1
(2) Suppose P^T has eigenvectors $U \equiv [\underline{\pi}^T, \underline{u}_2, \dots, \underline{u}_D]$ and eigenvalues $\Lambda \equiv \text{diag}[1, \lambda_2, \dots, \lambda_D]$, then
$P^T U = U\Lambda \implies P^T = U\Lambda U^{-1}$ so n times
$P^n = (P^T)^n = (U\Lambda U^{-1})^n = U\Lambda^n U^{-1}$
Therefore, $\Lambda^n = \text{diag}[1, \lambda_2^n, \dots, \lambda_D^n]$
(3) For ergodic MC, $P^n \to [\underline{\pi}, \dots, \underline{\pi}]^T$ (i.e. rank 1)
Therefore, # of non-zero eigenvalues is 1, so the rest of the eigenvalues must be $|\lambda_i| < 1 \ \forall i \geq 2$ s.t. $\Lambda^n = \text{diag}[1, 0, \dots, 0]$
Rate of Convergence: Depends on the 2nd largest eigenvalue of P^T i.e. $(\lambda_2)^n$ is the rate of convergence.
Bayesian Network (DAG): Network of RVs $X_1, \dots, X_n$ w/ directed edges
***Not State-Transition Diagram:** 1 RV w/ different values w/ different probabilities to each value.
***Fully Connected Graph (General):** No special dependency structure, so doesn't give additional info as we can write joint dist. from defn. (true for any graph).
***Non-Fully Connected Graph (Bayes' Net):** Conveys useful info about the dependency structure.
Factorization of Joint Dist. Suppose the dependencies among RVs can be represented by a DAG, then
$P(x_1, \dots, x_n) = \prod_{i=1}^{N} P(x_i \mid \text{pa}\{X_i\})$
***General:** $P(\underline{x}) = \prod_{i=1}^{N} P(x_i \mid x_{i-1}, x_{i-2}, \dots)$
-$P(x_1 \mid x_0) = P(x_1)$
Topological Ordering: Often index the RVs s.t. each child has an index greater than those of the parents.
Fact: Every DAG has at least one topological ordering.
Conditional Independence: $A \perp B \mid C$ if
(1) $P(a, b \mid c) = P(a \mid c)P(b \mid c) \ \forall a, b, c$ (i.e. A and B are indep. given C)
(2) $P(a \mid b, c) = P(a \mid c) \ \forall a, b, c$ (i.e. B gives no add. info about A given C)
Common Cause (T-T): $A \perp B \mid C$, o.w. $A \not\perp B$
Causal Chain (H-T/T-H): $A \perp B \mid C$, o.w. $A \not\perp B$
Common Effect (H-H): $A \perp B$, o.w. $A \not\perp B \mid C$ or its descendants
***Explaining Away:** If $A \to B \leftarrow C$, then if you observe B, then the other cause A is less likely to be the cause for the effect B.
Directed Separation (D-seperation): For non-overlapping subsets of RVs $\mathcal{A}, \mathcal{B}, \mathcal{C}$, if all undirected paths blocked, then $\mathcal{A}$ and $\mathcal{B}$ are **d-separated** by $\mathcal{C}$, i.e. $\mathcal{A} \perp \mathcal{B} \mid \mathcal{C}$
Blocked Path: An undirected path is blocked if it includes a node s.t.
1. The node is head-to-tail or tail-to-tail (Cases 1 and 2) and it is in set $\mathcal{C}$

2. The node is head-to-head, but neither itself nor any of its descendants are in set $\mathcal{C}$ (Case 3)

Markov Boundary (Blanket): Minimal set of RVs $\mathcal{M}$ that isolate X_i from all the remaining RVs, i.e.
$X_i \perp \mathcal{N} \setminus (\{X_i\} \cup \mathcal{M}) \mid \mathcal{M}$
*$\mathcal{N}$: Set of all RVs
*$\mathcal{M}$ = parents $\cup$ children $\cup$ co-parents: Blocks all paths b/w X_i and the remaining nodes.

Markov Random Field: Represent RVs as an **undirected** graph s.t. conditional independence $\mathcal{A} \perp \mathcal{B} \mid \mathcal{C}$ hold iff all paths b/w $\mathcal{A}$ and $\mathcal{B}$ so through $\mathcal{C}$.
***Markov blanket of** X_i : = set of neighbours of X_i
***No Order:** Simplifies, but no way to order the RVs, so lose directivity, lose info.

Independence: See if all paths b/w $\mathcal{A}$ and $\mathcal{B}$ are blocked by $\mathcal{C}$ (i.e. given $\mathcal{C}$)

Clique: A set of nodes s.t. there is link b/w any pair of them

Maximal Clique: A clique s.t. we cannot add another node in the set and maintain a clique.

Hammersley-Clifford Theorem: Let $\underline{x}_c$ denote the values of RVs in set C. Any strictly postiive dist. $P(\underline{x})$ that satisfies a Markov random field can be factorized as
$P(\underline{x}) = \frac{1}{Z} \prod_{c \in C} \psi_c(\underline{x}_c) \overset{e.g.}{=} \frac{1}{Z} e^{-\sum_c E(\underline{x}_c)}$
*$Z = \sum_{\underline{x}} \prod_{c \in C} \psi_c(\underline{x}_c)$: Normalization constant
*$\Pi_{c \in C}$: Product of all maximal cliques
*$\psi_c(\underline{x}_c) \overset{e.g.}{=} e^{-E(\underline{x}_c)}$: Potential function over the clique c (not necessarily a prob.)
*$E(\underline{x}_c)$: Energy function over the clique c

Moralization or Marrying the Parents (BN to MRF) Always possible, but some dependency structure will be lost
1. Add edges b/w all pairs of parents of the same child
2. Remove all directed edges (i.e. make it undirected)

Hidden Markov Model (HMM):
***State-transition Prob.**
$P(z_n \mid z_{n-1}) \equiv P[Z_n = z_n \mid Z_{n-1} = z_{n-1}],\ Z \leq n \leq N$
***Initial State Dist.** $P(z_1) \equiv P[Z_1 = z_1]$
***Emission Prob.**
$P(x_n \mid z_n) \equiv P[X_n = x_n \mid Z_n = z_n],\ 1 \leq n \leq N$
***Note:** Can be continuous (density).

Notes:
(1) Z_n nodes are head-to-tail or tail-to-tail and $Z_1, \ldots, Z_N$ are unobserved
$\implies$ No indep. among X_n's, also $\{X_n\}$ are not a MC.
(2) Latent var. $Z_1, \ldots, Z_N$ are MC $\implies Z_{n+1} \perp Z_{n-1} \mid Z_n$
$\implies \{X_1, X_2\} \perp \{X_3, \ldots, X_N\} \mid \{Z_3\}$

Common Problems
1. Given HMM, find $P(\underline{x})$ for any $\underline{x} = \{x_1, \ldots, x_N\}$
2. Given HMM and $\underline{x}$, find most likely z_n or sequence of states $\underline{z} = \{z_1, \ldots, z_N\}$

Message Passing Algos: Given HMM, find $P(\underline{x})\ \forall \underline{x}$

Forward:
$\alpha(z_n) \equiv P[\underbrace{X_1 = x_1, \ldots, X_n = x_n}_{\text{obs so far}}, \underbrace{Z_n = z_n}_{\text{cur. state}}],\ 1 \leq n \leq N$
$\alpha(z_n) \equiv P[x_1, \ldots, x_n, z_n]$
*$\alpha(z_1) = P(x_1, z_1) = P(z_1)P(x_1 \mid z_1)$

$$\boxed{\alpha(z_n) = P(x_n \mid z_n) \sum_{z_{n-1}} P(z_n \mid z_{n-1})\alpha(z_{n-1})}$$

$$\boxed{\alpha(z_N) = P(\underline{x}, z_N) \implies P(\underline{x}) = \sum_{z_N} \alpha(z_N)}$$

* $\{\alpha(z_n)\}_{z_n = 1, \ldots, K}$: Message from Z_n to Z_{n+1}
***Complexity:** $O(K^2 N)$
*$O(K)$ for each $\alpha(z_n)$, $O(K^2)$ for message at time n

Backward:
$\beta(z_n) \equiv P[\underbrace{X_{n+1} = x_{n+1}, \ldots, X_N = x_N}_{\text{future obs}} \mid \underbrace{Z_n = z_n}_{\text{cur. state}}]$
$\beta(z_n) \equiv P[x_{n+1}, \ldots, x_N \mid z_n]$
*$\beta(z_N) = 1\ \forall z_n$

$$\boxed{\beta(z_n) = \sum_{z_{n+1}} P(x_{n+1} \mid z_{n+1})P(z_{n+1} \mid z_n)\beta(z_{n+1})}$$

$$\boxed{\beta(z_1) = P(x_2, \ldots, x_n \mid z_1) \implies P(\underline{x}) = \sum_{z_1} P(z_1)P(x_1 \mid z_1)\beta(z_1)}$$

*$\{\beta(z_n)\}_{z_n = 1, \ldots, K}$: Message from Z_n to Z_{n-1}
***Complexity:** $O(K^2 N)$
*$O(K)$ for each $\beta(z_n)$, $O(K^2)$ for message at time n

Forward Backward (Same Time): $\alpha(z_n)\beta(z_n) = P(\underline{x}, z_n)$
$P(\underline{x}) = \sum_{z_n} \alpha(z_n)\beta(z_n)\ \forall n$

Approx. Algo: Given HMM and $\underline{x}$, find most likely z_n
$\gamma(z_n) \equiv P(z_n \mid \underline{x}),\ 1 \leq n \leq N$

$$\boxed{\gamma(z_n) = \frac{\alpha(z_n)\beta(z_n)}{\sum_{z'_n} \alpha(z'_n)\beta(z'_n)}}$$

$$\boxed{z^*_n = \arg\max_{z_n} \gamma(z_n)}$$

*Complexity: $O(K^2 N)$

Scaling: $\alpha(z_n), \beta(z_n)$ can be small for large/small n

1. Forward:
$\hat{\alpha}(z_n) \equiv \frac{\alpha(z_n)}{P(x_1, \ldots, x_n)} = P(z_n \mid x_1, \ldots, x_n)$
*Does not shrink as $n \uparrow$
$c_n \equiv P(x_n \mid x_1, \ldots, x_{n-1})$
Then $P(x_1, \ldots, x_n) = \prod_{m=1}^{n} c_m$

$$\boxed{\hat{\alpha}(z_n) = \frac{1}{c_n} P(x_n \mid z_n) \sum_{z_{n-1}} P(z_n \mid z_{n-1})\hat{\alpha}(z_{n-1})}$$

$$\boxed{c_n = \sum_{z_n} P(x_n \mid z_n) \sum_{z_{n-1}} P(z_n \mid z_{n-1})\hat{\alpha}(z_{n-1})}$$

2. Backward:
$\hat{\beta}(z_n) = \frac{\beta(z_n)}{\prod_{m=n+1}^{N} c_m}$

$$\boxed{\hat{\beta}(z_n) = \frac{1}{c_{n+1}} \sum_{z_{n+1}} P(x_{n+1} \mid z_{n+1})P(z_{n+1} \mid z_n)\hat{\beta}(z_{n+1})}$$

$$\boxed{c_{n+1} = \sum_{z_{n+1}} P(x_{n+1} \mid z_{n+1})P(z_{n+1} \mid z_n)\hat{\beta}(z_{n+1})}$$

3. Forward-Backward $\boxed{\gamma(z_n) = \hat{\alpha}(z_n)\hat{\beta}(z_n)}$

Forward-Backward Algo *Have to fwd, then bwd pass.
0. $c_1 = P(x_1) = \sum_{z_1} P(z_1)P(x_1 \mid z_1)$
$\hat{\alpha}(z_1) = \frac{1}{c_1} P(z_1)P(x_1 \mid z_1)$
1. Fwd message passing to compute $\hat{\alpha}(z_n)$ and c_n, $2 \leq n \leq N$
2. $\hat{\beta}(z_N) = \beta(z_N) = 1$
Bwd message passing to compute $\hat{\beta}(z_n)$, $1 \leq n \leq N-1$
3. $\gamma(z_n) = \hat{\alpha}(z_n)\hat{\beta}(z_n)$
4. $z_n^* = \arg\max_{z_n} \gamma(z_n) \forall n$
Viterbi Algo: Given HMM and $\underline{x}$, find most likely $\underline{z}$
$\hat{\underline{z}} = \arg\max_{\underline{z}} P(\underline{z} \mid \underline{x}) = \arg\max_{\underline{z}} P(\underline{x}, \underline{z})$?
Dynamic Programming: Path length $-\log P(\underline{x}, \underline{z})$
$= -\left[\log P(z_1) + \sum_{n=2}^{N} \log P(z_n \mid z_{n-1}) + \sum_{n=1}^{N} \log P(x_n \mid z_n)\right]$
If $(\hat{z}_1, \ldots, \hat{z}_M)$ is the shrotest path to state $\hat{z}_M$, the $(\hat{z}_1, \ldots, \hat{z}_n, z_{M-1})$ is the shortest path to state z_{M-1}
So to find the shortest path to any state z_M^o
1. For each state z_{M-1}, find shortest path to it.
2. Then consider distances b/w all K pairs of (z_{M-1}, z_M^o) and find the shortest path to z_M^o
$\min_{z_1,\ldots,z_{M-1}}$ path length(z_M^o)
$\min_{z_{M-1}} \left[\min_{z_1,\ldots,z_{M-2}} \text{path length}(z_{M-1}) + \text{dist}(z_{M-1}, z_M^o)\right]$
Current Highest Log-Prob to State z_n:
$w_n(z_n) = \max_{z_1,\ldots,z_{n-1}} \log P(x_1, \ldots, x_n, z_1, \ldots, z_n)$
*$w_1(z_1) = \max_{z_1} \log P(z_1) + \log P(x_1 \mid z_1)$
*$w_{n+1}(z_{n+1}) = \max_{z_n} w_n(z_n) + \log P(z_{n+1} \mid z_n) + \log P(x_{n+1} \mid z_{n+1})$

At $n = N$, $\max_{z_N} w_N(z_N)$ is the highest log-prob.
Then backtrack to find the most likely sequence of states.
Trellis Diagram of MC Unfolding of MC over time
Viterbi Algorithm:
0. $w_1(z_1) = \log P(x_1, z_1) = \log P(z_1) + \log P(x_1 \mid z_1)$
1. For $1 \leq n \leq N-1$:
$w_{n+1}(z_{n+1}) = \max_{z_n} w_n(z_n) + \log P(z_{n+1} \mid z_n) + \log P(x_{n+1} \mid z_{n+1})$
$\psi_n(z_{n+1}) = \arg\max_{z_n} w_n(z_n) + \log P(z_{n+1} \mid z_n)$
2. $\hat{z}_N = \arg\max_{z_N} w_N(z_N)$
3. Output: $\hat{\underline{z}} = (\hat{z}_1, \ldots, \hat{z}_N)$ where $\hat{z}_n = \psi_n(\hat{z}_{n+1})$