# ROB311 Quiz 3

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March 30, 2025

## Contents

# One-Shot Multi-Agent Decision Problems

## 1 Multi-Agent Problems

**Summary:** In a **Multi-Agent problem**, we assume that:
- Set of states for environment is $\mathcal{S}$
- $P$ agents within environment.
- For each state $s \in \mathcal{S}$:
  - possible actions for agent $i$ is $\mathcal{A}_i(s)$
  - set of action profiles is $\mathcal{A}(s) = \prod_{i=1}^{P} \mathcal{A}_i(s)$
- possible state-action pairs are $\mathcal{T} = \{(s, a) \text{ s.t. } s \in \mathcal{S}, a \in \mathcal{A}(s)\}$
- environment in some origin state, $s_0$
- environment destroyed after $N$ transitions
- agent $j$ wants to find policy $\pi_j(a_j \mid s)$ so that $\mathbb{E}[r_j(p)]$ is maximized
- agents act independently given the environment's state: $\pi(a \mid s) = \prod_{j \in [P]} \pi_j(a_j \mid s)$

| Name | Function: |
|---|---|
| State transition given state-action pair defined by $\text{tr} : \mathcal{T} \to \mathcal{S}$ | $\text{tr}(s, a) =$ state transition from $s$ under $a$ |
| Reward to each agent, $i$ defined by $r_i$: $\mathcal{Q} \times \mathcal{S} \to \mathbb{R}_+$ | $r_i(s, a, \text{tr}(s, a)) =$ rwd to agent $i$ for $(s, a, tr(s, a))$ |
| State evolution of environment after $N$ transitions<br>• Given sequence of actions: $p.a = \langle a^{(1)}, \dots, a^{(n)} \rangle$<br>• $s_N = \tau(s_{n-1}, a^{(n)})$ | $p = \langle (s_0, a^{(1)}, s_1), \dots, (s_{N-1}, a^{(N)}, s_N) \rangle$ |
| reward to agent $i$ | $r_i(p) = \sum_{n=1}^{N} r_i(s_{n-1}, a^{(n)}, s_n)$ |
| expected-reward (value) of playing $a$ from $s$ for agent $j$<br>• $\mathcal{A}(s) = \emptyset$ if $s \in \mathcal{G}$ | $q_j(s, a) = r_j(s, a, \tau(s, a)) + \sum_{a' \in \mathcal{A}(\tau(s,a))} \pi(a' \mid \tau(s, a)) q_j(\tau(s, a), a')$ |

### 1.1 Policy Equilibria

**Notes**:
- **No Regret:** $\pi$ is no-regret if $\pi_j$ maximizes $q_j$ when $\pi_{-j}$ is fixed.
- If all agents play perfectly, then we expect

$$\pi(a \mid s) = \begin{cases} 1 & \text{if } a = a^*(s) \\ 0 & \text{otherwise} \end{cases}$$

  - $a_j^*(s) = \arg \max_{a_j \in \mathcal{A}_j(s)} q_j(s, a_j, a_{-j}^*)$ is the best action for agent $j$ given the other agents' policies.

## 1.2 Single Action Games

**Summary:** In a **Single Action Game**, we assume that:

- $N = 1$ (one-shot game)
- Initial state is $s_0 \in \mathcal{S}$
- Agent $j$ wants to find policy, $\pi_i(a_i \mid s_0)$ so $\mathbb{E}[r_i(p)]$ is maximized

## 1.3 Actions (Deterministic)

**Summary:** Allow each agent to choose action deterministically.

| **Name** | **Function:** |
|---|---|
| Action $j$ for agent $i$ <br> • One-hot vector of $M_i$ components, $\mathbf{e}_{i,j}$ | $[0 \cdots 0\ 1\ 0 \cdots 0]^T$ |
| Agent $i$'s set of possible actions <br> • Agent $i$'s chosen action with $a_i \in \mathcal{A}_i$ | $\mathcal{A}_i = \{a_i \in \{0,1\}^{M_i} \mid \sum_{j\in[M_i]} a_{i,j} = 1\}$ |
| Action profile is a tuple of actions <br> • **Notational Convenience:** $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_P)$ so that $a = (a_i, a_{-i})$. | $a = (a_1, \ldots, a_P)$ |
| Optimal action profile <br> • An action profile w.r.t. which any other action profile leaves at least one player worse off. | $a^+$ s.t. $\forall a \exists i$ s.t. $r_i(a) < r_i(a^+)$ |
| Set of optimal action profiles | $\text{aOpt} = \{a^+ \mid \forall a \exists i : r_i(a) < r_i(a^+)\}$ |
| Best-action mapping, $\text{ba}_i$: $\mathcal{A}_{-i} \to \mathcal{A}_i$ | $\text{ba}_i(a_{-i}) = \arg\max_{a_i\in\mathcal{A}_i} r_i(a_i, a_{-i})$ <br> $= \{a_i \in \mathcal{A}_i \mid r_i(a_i, a_{-i}) = \max_{a_i'\in\mathcal{A}_i} r_i(a_i', a_{-i})\}$ |
| Agent $i$ will **not regret** playing $a_i^*$ when others play $a_{-i}^*$ if | $r_i(a_i^*, a_{-i}^*) \geq r_i(a_i, a_{-i}^*)\ \forall a_i \in \mathcal{A}_i$ <br> or $a_i^* \in \text{ba}_i(a_{-i}^*)$ |
| Action equilibria is any action, $a^*$ in which no agent regrets | $a_i^* \in \text{ba}_i(a_{-i}^*)\ \forall i \in [P]$ |
| Existence of action equilibria | May not always exist, i.e., it may be that $\text{aEq} = \emptyset$ |

## 1.4 Strategies (Probabilistic)

**Summary:** Allow each agent to choose action based on a distribution/strategy.

| **Name** | **Function:** |
|---|---|
| Stategy for agent $i$ <br> • Vector of $M_i$ components, that are non-negative and sum to 1 | $[0.05 \cdots 0.2\ 0.7\ 0 \cdots 0.05]^T$ |
| Agent $i$'s set of possible strategies <br> • Agent $i$'s chosen strategy with $x_i \in \Delta_i$ | $\Delta_i = \Delta^{M_i} = \left\{ x_i \in [0,1]^{M_i}, \sum_{j \in [M_i]} x_{i,j} = 1 \right\}$ |
| Expected reward | $\bar{r}_i(x_1, \ldots, x_P) = \mathbb{E}[r_i(a)] = \sum_{a_i \in \mathcal{A}_i} \pi(a) r_i(a)$ |
| Stategy profile is a tuple of strategies <br> • **Notational Convenience:** $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_P)$ so that $x = (x_i, x_{-i})$. | $x = (x_1, \ldots, x_P)$ |
| No-regret strategies <br> • Agent $i$ will not regret using $x_i^*$ when others use $x_{-i}^*$. | $\bar{r}_i(x_1, \ldots, x_P) \geq \bar{r}_i(x_i', x_{-i})\ \forall x_i' \in \Delta_i$ <br> $x_i^* \in \text{bs}_i(x_{-i}^*)$ |
| Best strategy mapping $\text{bs}_i$: $\Delta_{-i} \to \Delta_i$ | $\text{bs}_i(x_{-i}) = \arg\max_{x_i \in \Delta_i} \bar{r}_i(x_i, x_{-i})$ <br> $= \{x_i \in \Delta_i \mid \bar{r}_i(x_i, x_{-i}) = \max_{x_i' \in \Delta_i} \bar{r}_i(x_i', x_{-i})\}$ |
| Strategy equilibria is any strategy, $x^*$ in which no agent regrets | $x_i^* \in \text{bs}_i(x_{-i}^*)\ \forall i \in [P]$ |
| Joint best-strategy mapping $\text{bs} : \Delta \to \Delta$ | $\text{bs}(x) = (\text{bs}_1(x_{-1}), \ldots, \text{bs}_P(x_{-P}))$ |
| Existence of strategy equilibria <br> • Fixed pt. always exists. | Any strategy equilibrium, $x^*$ is a fixed point of $x^* = \text{bs}(x^*)$ |

### 1.4.1 Simplifying Games

**Notes**: May be able to reduce $M_i$ by eliminating useless actions/strategies:

- **Equivalent Stategies:** $x_i^{(1)} \equiv s_i^{(2)}$

$$\bar{r}_i(x_i^{(1)}, x_{-i}) = \bar{r}_i(x_i^{(2)}, x_{-i})\ \forall x_{-i}$$

- **Dominated Strategies:** $x_i$

$$\exists x_i' \text{ s.t. } \bar{r}_i(x_i', x_{-i}) \geq \bar{r}_i(x_i, x_{-i})\ \forall x_{-i} \text{ and } \exists x_{-i}' \text{ s.t. } \bar{r}_i(x_i', x_{-i}') > \bar{r}_i(x_i, x_{-i}')$$

Can remove dominated and equilvalent strategies w/o changing the game.

## 1.5 Examples

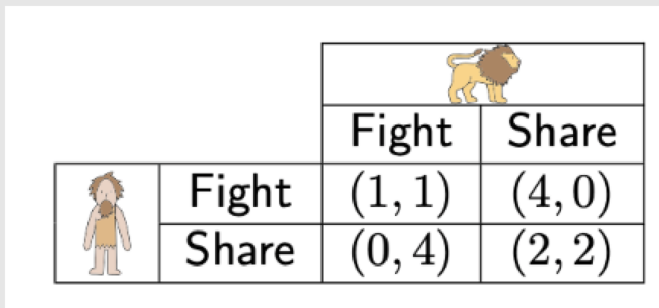
### 1.5.1 Finding Action Equilibria

**Process:** To find action equilibria:

1. For each $i$, compute $\text{ba}_i(a_{-i})$ for all $a_{-i}$
2. Define $\text{bap}_i$ so that $\text{bap}_i = \{(a_i', a_{-i}),\ \forall a_i' \in \text{ba}_i(a_{-i}),\ \forall a_{-i} \in \mathcal{A}_{-i}\}$
3. Action equilibria are then $\text{aEq} = \bigcap_{i\in[P]} \text{bap}_i$.

**Example**:

1. **Given:** Suppose lion and cavemen both want meat. Each must decide whether to fight for the food or share it.



| | | Fight | Share |
|---|---|---|---|
| | Fight | $(1,1)$ | $(4,0)$ |
| | Share | $(0,4)$ | $(2,2)$ |

Figure 1

2. **Problem:** Find the action equilibria
3. **Solution:**
   (a) **Best Action Profiles:**
   - Cavemen: $\text{bap}_{\text{cavemen}} = \{(\text{Fight, Fight}), (\text{Fight, Share})\}$. Cavemen fights no matter what.
     - If lion fights, then cavemen fights to get maximum reward in this scenario of +1.
     - If lion shares, then caveman fights to get maximum reward in this scenario of +4.
   - Lion: $\text{bap}_{\text{lion}} = \{(\text{Fight, Fight}), (\text{Share, Fight})\}$. Lion fights no matter what.
     - If caveman fights, then lion fights to get maximum reward in this scenario of +1.
     - If caveman shares, then lion fights to get maximum reward in this scenario of +4.

   (b) **Best Action Equilibria:** Intersection of the best action profiles.
   - $\text{aEq} = \text{bap}_{\text{cavemen}} \cap \text{bap}_{\text{lion}} = \{(\text{Fight, Fight})\}$

**Example**:

1. **Given:** Suppose lion and cavemen both want meat. Each must decide whether to fight for the food or share it.

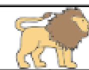
| | |  | |
|---|---|---|---|
| | | Fight | Share |
|  | Fight | $(3, 0)$ | $(0, 3)$ |
| | Share | $(0, 1)$ | $(1, 0)$ |

Figure 2

2. **Problem:** Find the action equilibria
3. **Solution:**
   (a) **Best Action Profiles:**
   - Cavemen: $\text{bap}_{\text{cavemen}} = \{(\text{Fight, Fight}), (\text{Share, Share})\}$. Cavemen fights no matter what.
     - If lion fights, then cavemen fights to get maximum reward in this scenario of +3.
     - If lion shares, then caveman shares to get maximum reward in this scenario of +1.
   - Lion: $\text{bap}_{\text{lion}} = \{(\text{Fight, Share}), (\text{Share, Fight})\}$. Lion fights no matter what.
     - If caveman fights, then lion shares to get maximum reward in this scenario of +3.
     - If caveman shares, then lion fights to get maximum reward in this scenario of +1.

   (b) **Best Action Equilibria:** Intersection of the best action profiles.
   - $\text{aEq} = \text{bap}_{\text{cavemen}} \cap \text{bap}_{\text{lion}} = \emptyset$

### 1.5.2 Optimal Action Profiles

**Example**:

1. **Given:** Suppose lion and cavemen both want meat. Each must decide whether to fight for the food or share it.

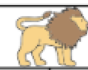
| | |  | |
|---|---|---|---|
| | | Fight | Share |
|  | Fight | $(1, 1)$ | $(4, 0)$ |
| | Share | $(0, 4)$ | $(2, 2)$ |

Figure 3

2. **Problem:** Find the optimal action profiles:
3. **Solution:**
   (a) aOpt $= \{$(Share, Share), (Fight, Share), (Share, Fight)$\}$
   - FINISH

#### 1.5.3 Finding/Convergence Strategy Equilibria

**Process: Finding**

1. For each $i$, compute $\text{bs}_i(x_{-i})$ for all $x_{-i}$
2. Define $\text{bsp}_i$ so that $\text{bsp}_i = \{(x'_i, x_{-i}),\ \forall x'_i \in \text{bs}_i(x_{-i}),\ \forall x_{-i} \in \Delta_{-i}\}$
3. Strategy equilibria are then $\text{bEq} = \bigcap_{i \in [P]} \text{bsp}_i$.

**Process: Convergence**

1. For each $i$, compute $\text{bs}_i(x_{-i})$ for all $x_{-i}$
2. Define $\text{bsp}_i$ so that $\text{bsp}_i = \{(x'_i, x_{-i}),\ \forall x'_i \in \text{bs}_i(x_{-i}),\ \forall x_{-i} \in \Delta_{-i}\}$
3. Strategy equilibria are then $\text{sEq} = \bigcap_{i \in [P]} \text{bsp}_i$.
   - Requires each agent, $j$, to know $\bar{r}_1, \ldots, \bar{r}_P$

**Example**:

1.

**Example**:

1. **Given/Problem:** Find all equilibria of the following one-shot game or state that none exist.

| | **B1 (y)** | **B2 (1-y)** |
|---|---|---|
| **A1 (x)** | (5, 3) | (1, 0) |
| **A2 (1-x)** | (0, 1) | (2, 4) |

   - (#,#) is the payoff to P1 and P2 respectively for a given action profile.
2. **Solution:**
   (a) **Define Probabilities:**
   - Let $y$ be the probability that B1 plays action B1 so $1-y$ is the probability that B1 plays action B2.
   - Let $x$ be the probability that A1 plays action A1 so $1-x$ is the probability that A1 plays action A2.

   (b) **Expected Rewards:**
   - P1:

$$\begin{aligned} E[x] = 5xy + 1x(1-y) + 0(1-x)y + 2(1-x)(1-y) &= 5xy + x - xy + 2 - 2x - 2y + 2xy \\ &= 5xy - xy + 2xy + x - 2x - 2y + 2 \\ &= 6xy - x - 2y + 2 \quad \text{simplify} \\ &= \underbrace{(6y-1)}_{c} x + 2 - 2y \quad \text{linear in } x \end{aligned}$$

   - P2:

$$\begin{aligned} E[y] = 3xy + 0x(1-y) + 1(1-x)y + 4(1-x)(1-y) &= 3xy + 0 + y - xy + 4 - 4x - 4y + 4xy \\ &= 3xy - xy + 4xy + y - 4x - 4y + 4 \\ &= 6xy - 4x - 3y + 4 \quad \text{simplify} \\ &= \underbrace{(6x-3)}_{c} y + 4 - 4x \quad \text{linear in } y \end{aligned}$$

   - **Note:** $E[x]$ is linear in $x$ and $E[y]$ is linear in $y$.

   (c) **Constrained Argmax Expected Rewards w.r.t $x \in [0,1]$ (since P1):** If it was cost, then minimize. Also don't care about constant term in $y$ since we are derivating w.r.t $x$.
   - P1:

$$x = \begin{cases} 1 & \text{if } y > \frac{1}{6} \text{ i.e. } c > 0 \text{ since positive want maximum positive} \\ [0,1] & \text{if } y = \frac{1}{6} \text{ i.e. } c = 0 \text{ doesn't matter since } 0 \\ 0 & \text{if } y < \frac{1}{6} \text{ i.e. } c < 0 \text{ since negative want maximum negative} \end{cases}$$

   - P2:

$$y = \begin{cases} 1 & \text{if } x > \frac{3}{6} \text{ i.e. } c > 0 \text{ since positive want maximum positive} \\ [0,1] & \text{if } x = \frac{3}{6} \text{ i.e. } c = 0 \text{ doesn't matter since } 0 \\ 0 & \text{if } x < \frac{3}{6} \text{ i.e. } c < 0 \text{ since negative want maximum negative} \end{cases}$$

   (d) **Finding all equilibrium:** Lines on the graph represents where your reward is maximized.

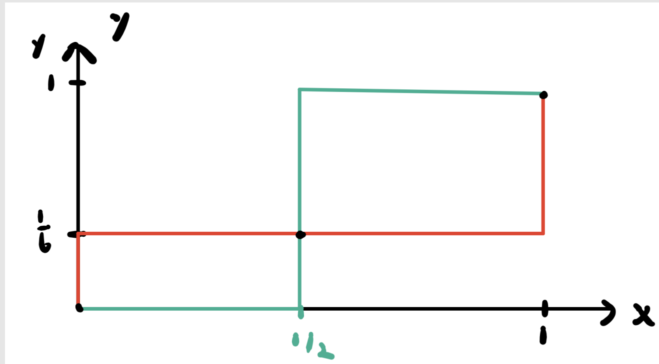



Figure 4

- **Case 1:** $x = 0$ and $y = 0$
  - $P(\text{P1 chooses A1}) = 0$
  - $P(\text{P1 chooses A2}) = 1$
  - $P(\text{P2 chooses B1}) = 0$
  - $P(\text{P2 chooses B2}) = 1$
- **Case 2:** $x = 1/2$ and $y = 1/6$
  - $P(\text{P1 chooses A1}) = 1/2$
  - $P(\text{P1 chooses A2}) = 1/2$
  - $P(\text{P2 chooses B1}) = 1/6$
  - $P(\text{P2 chooses B2}) = 5/6$
- **Case 3:** $x = 1$ and $y = 1$
  - $P(\text{P1 chooses A1}) = 1$
  - $P(\text{P1 chooses A2}) = 0$
  - $P(\text{P2 chooses B1}) = 1$
  - $P(\text{P2 chooses B2}) = 0$

(e) **Unstable Equilibrium:** P1 moves left and right b/c $x$ is associated with $x$-axis. P2 moves up and down b/c $y$ is associated with $y$-axis.

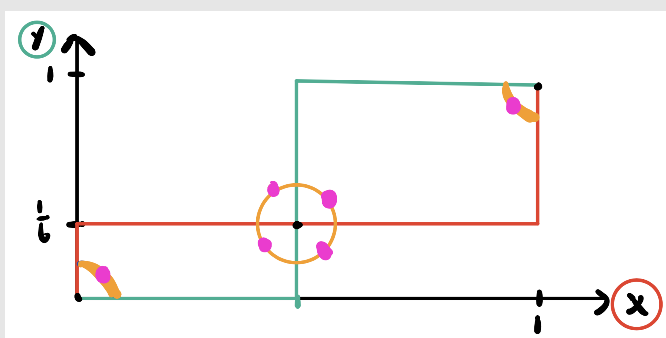



Figure 5

- Stability means that in a radius disc around the equilibrium, if you move a little bit, you will still be in the equilibrium (have to check all relevant quadrants)
  - If one quadrant is unstable, then don't need to check the other quadrants as the equilibrium point is unstable.
  - Simulatenous (both players move at the same time) and sequential (one player moves first and the other player moves second)
- **Case 1:** $x = 0$ and $y = 0$ is stable
  - Q1: Always converges to $(0, 0)$ since P1 moves left to red and P2 moves down to turquoise.
- **Case 2:** $x = 1/2$ and $y = 1/6$ is unstable
  - Q1 (Top Left): P1 moves right to red and P2 moves up to turquoise $\implies (1, 1)$
  - Q2 (Top Right): P1 moves right to red and P2 moves up to turquoise $\implies (1, 1)$
  - Q3 (Bottom Left): P1 moves left to red and P2 moves down to turquoise $\implies (0, 0)$

  - Q4 (Bottom Right): P1 moves left to red and P2 moves down to turquoise $\implies (0, 0)$
- **Case 3:** $x = 1$ and $y = 1$ is stable
  - Q1: Always converges to $(1, 1)$ since P1 moves left to red and P2 moves down to turquoise.

**Example**:

**Example**:

### 1.5.4 Simplifying Games

**Example**: