

ROB311 Quiz 2

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1 Basic Control Problem

2 Feedback + Integral Control

3 Bode Plots

3.1 Bode Plots

Process:

3.1.1 Constant Gain

3.1.2 Pole or Zero at $\omega = 0$

3.1.3 Non-Zero Pole or Zero

3.1.4 Complex Conjugate Poles

3.2 Robustness Margins

Motivation: Approximate the GM and PM from the Bode plot:

- $L(s)$ is a strictly proper rational fn.
- $L(s)$ has no poles in \mathbb{C}^+ (no open loop variable poles)

3.2.1 Gain Margin

Definition:

$$|L(j\omega_{gc})| = 1 \iff |L(j\omega_{gc})|_{dB} = 0$$

3.2.2 Phase Margin

Definition:

$$|L(j\omega_{gc})| = 1 \implies |L(j\omega_{gc})|_{dB} = 0$$

4 Robustness Margins

5 Root Locus, Bode, and Nyquist

6 Control Design in the Frequency Domain

6.1 Goal

Motivation:

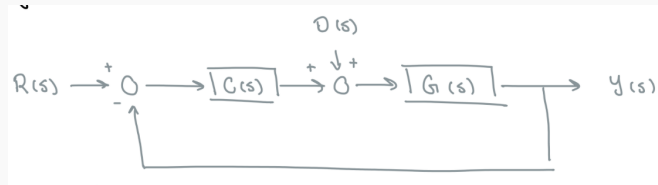


Figure 1

Design $C(s)$ so that the feedback loop:

- BIBO Stable: Verify using Nyquist criterion
 - $\text{roots}(1 + C(s)P(s)) \subseteq \mathbb{C}^-$
 - $C(s)G(s)$ has no pole-zero cancellations in $\overline{\mathbb{C}^+}$
- Satisfies certain performance specifications: Tune using Bode plots

6.2 Proportional Derivative (PD) Controller

Motivation: Increase PM at higher frequencies.

Definition:

$$C(s) = K(T_D s + 1) \quad (1)$$

- $K, T_D > 0$

Since $U(s) = C(s)E(s)$,

$$u(t) = \underbrace{KT_D e(t)}_D + \underbrace{K e(t)}_P \quad (2)$$

6.2.1 Bode Plot

Notes:

$$|C(j\omega)|_{\text{dB}} = 20 \log |K| + 20 \log |j\omega T_D + 1|$$

$$\angle C(j\omega) = \angle K + \angle(j\omega T_D + 1)$$

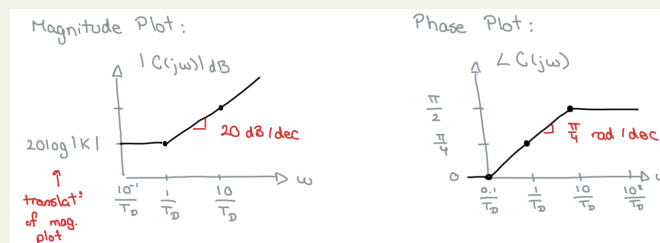


Figure 2

6.3 Proportional Integral (PI) Controller

Motivation: Increase the "system type" for better tracking (IMP) w/o affecting high frequencies.

Definition:

$$C(s) = K \left(1 + \frac{1}{T_I s} \right) = K \frac{T_I s + 1}{T_I s} \quad (3)$$

- $K, T_I > 0$

Since $U(s) = C(s)E(s)$,

$$u(t) = \underbrace{K e(t)}_P + \underbrace{\frac{K}{T_I} \int_0^t e(\tau) d\tau}_I \quad (4)$$

6.3.1 Bode Plot

Notes:

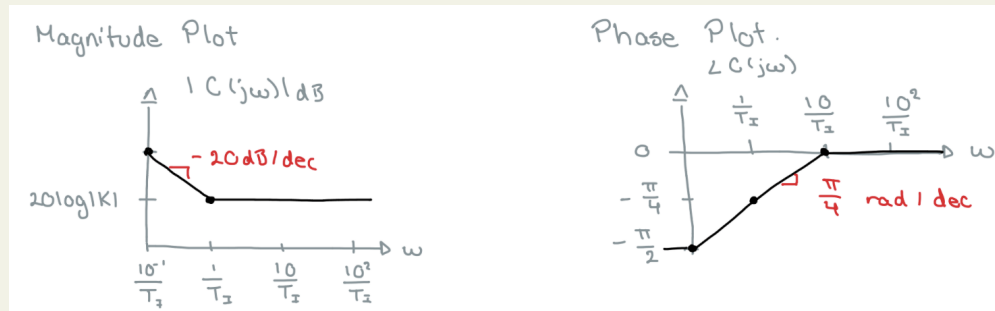


Figure 3

6.3.2 Design Procedure

Process:

1. Choose K to meet asymptotic tracking or bandwidth (loosely increase ω_{gc}) requirements (often set $K = 1$)
2. Find the crossover frequency ω_{gc} of $KG(j\omega)$. Suppose we are happy w/ the PM and ω_{gc} .
3. Set $\frac{1}{T_I} \ll \omega_{gc}$. Typically want $\frac{1}{T_I}$ b/w $0.01\omega_{gc}$ and $0.1\omega_{gc}$

6.4 Proportional Integral Derivative (PID) Controller

Definition:

$$C(s) = K(T_D s + 1) \left(1 + \frac{1}{T_I s} \right) = K_p + \frac{K_I}{s} + K_D s \quad (5)$$

- $K, T_I, T_D > 0$

6.4.1 Design Procedure

Process:

1. Design K, T_D (i.e. the PD controller) to increase the PM.
2. Design T_I (i.e. the PI controller) to increase system type (satisfy IMP) w/o affecting high frequencies.

6.5 Examples

Example:

1. **Given:** $G(s) = \frac{1}{j\omega(j\omega + 1)}$, $C(s) = K(T_D s + 1)$
2. **Problem:** Sketch Bode plots of $C(s)G(s)$ for PD controllers:
 - $K = 1$, $T_D = 10 \rightarrow 20 \log |K| = 0$
 - $K = 10$, $T_D = 10 \rightarrow 20 \log |K| = 20$
 - Corner frequency: $\omega_c = \frac{1}{T_D} = 10^{-1}$
3. **Solution:**

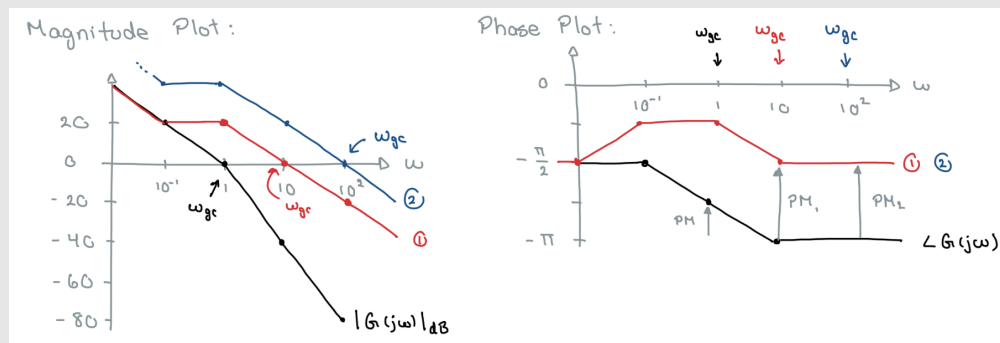


Figure 4

7 Lead Controller

Motivation:

- Approximates a PD controller as $\alpha \rightarrow 0$.
- Used to add phase at a particular frequency ω_{\max} in order to increase the PM (decreases the % OS).
- Increases ω_{gc}

Definition:

$$C(s) = K \frac{T_s + 1}{\alpha T_s + 1} \quad (6)$$

- $K, T > 0$
- $0 < \alpha < 1$

7.1 Bode Plot

Notes:

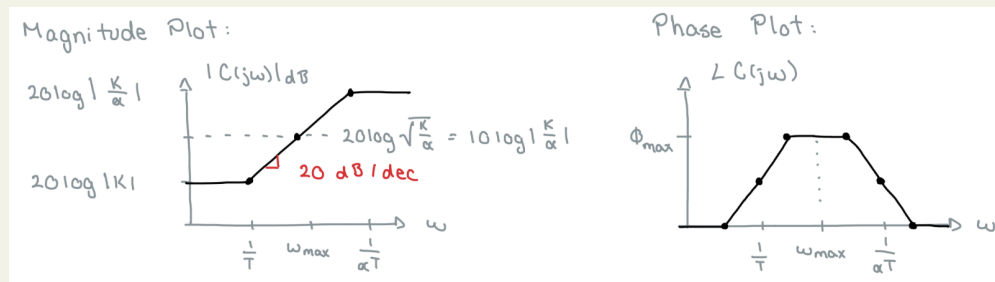


Figure 5

- $\omega_{\max} = \frac{1}{\sqrt{\alpha} T}$
- $\phi_{\max} = \alpha \sin\left(\frac{1 - \alpha}{1 + \alpha}\right)$

7.2 Design Procedure

Process:

1. Choose K to satisfy asymptotic tracking (i.e. steady-state error) specifications or bandwidth (i.e. ω_{gc}) requirements.
2. Draw Bode plot of $KG(j\omega)$ and find PM. Given a desired $\text{PM}_{\text{desired}}$, let $\Delta\text{PM} = \text{PM}_{\text{desired}} - \text{PM}$. Set $\phi_{\max} = \Delta\text{PM} + \text{Extra Margin}$
 - Rule of thumb: Extra Margin = 30°
 - Why? Since lead controller increases ω_{gc} , this ends up decreasing the resulting PM. So to compensate, we add the extra margin.
3. $\alpha = \frac{1 - \sin(\phi_{\max})}{1 + \sin(\phi_{\max})}$
4. Solve for the frequency ω_{gc}^{new} s.t. $20 \log |KG(j\omega_{gc}^{\text{new}})| = -20 \log \left| \frac{1}{\sqrt{\alpha}} \right| = 10 \log |\alpha|$
5. Impose $\omega_{\max} = \omega_{gc}^{\text{new}}$ and solve for $\frac{1}{T} = \sqrt{\alpha} \omega_{gc}^{\text{new}}$
6. Draw Bode plot of $C(j\omega)G(j\omega)$ and verify the PM is adequate.
7. Draw Nyquist plot to verify BIBO stability.

8 Lag Controller