

**Notation:**  $P_{X|Y}(x \mid y) = P[X = x \mid Y = y]$

\*Subscript indicates the RV, and the value indicates the realization.

**Intro:**

**Random Experiment:** An outcome for each run.

**Sample Space  $\Omega$ :** Set of all possible outcomes.

**Event:** Measurable subsets of  $\Omega$ .

**Prob. of Event  $A$ :**  $P(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } \Omega}$

**Axioms:** (1)  $P(A) \geq 0 \forall A \in \Omega$ , (2)  $P(\Omega) = 1$ ,  
(3) If  $A \cap B = \emptyset$ , then  $P(A \cup B) = P(A) + P(B) \forall A, B \in \Omega$

**Additive Rule:**  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

**Cond. Prob.**  $P(A|B) = \frac{P(A \cap B)}{P(B)}$

\*Prob. measured on new sample space  $B$ .  
\* $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

**Independence:**  $P(A|B) = P(A) \Leftrightarrow P(A \cap B) = P(A)P(B)$

**Total Prob. Thm:** If  $H_1, H_2, \dots, H_n$  form a partition of  $\Omega$ , then  $P(A) = \sum_{i=1}^n P(A|H_i)P(H_i)$ .

**Partition:**  $H_1, \dots, H_n$  is a partition if  
(1)  $H_i \cap H_j = \emptyset$  for  $i \neq j$ , (2)  $H_1 \cup H_2 \cup \dots \cup H_n = \Omega$ .

**Bayes' Rule:**  $P(H_k|A) = \frac{P(H_k \cap A)}{P(A)} = \frac{P(A|H_k)P(H_k)}{\sum_{i=1}^n P(A|H_i)P(H_i)}$

\*Posteriori:  $P(H_k|A)$ , Likelihood:  $P(A|H_k)$ , Prior:  $P(H_k)$

**1 RV:**

**Cumulative Distribution Fn (CDF):**  $F_X(x) = P[X \leq x]$

**Prob. Mass Fn (PMF):**  $P_X(x_j) = P[X = x_j] \ j = 1, 2, \dots$

**Prob. Density Fn (PDF):**  $f_X(x) = \frac{d}{dx} F_X(x)$

\* $P[a \leq X \leq b] = \int_a^b f_X(x) \, dx$

**Exp.:**

\* $E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x) \, dx$

\* $E[h(X)] = \sum_{k=-\infty}^{\infty} h(k) P_X(x_i = k)$

\* $E[aX + b] = aE[X] + b$

**Variance:**  $\sigma_X^2 = \text{Var}[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$

\* $\text{Var}[aX + b] = a^2 \text{Var}[X]$

\* $\text{Var}[aX + bY] = a^2 \text{Var}[X] + b^2 \text{Var}[Y] + 2ab \text{Cov}[X, Y]$

**2 RVs:**

**Joint PMF:**  $P_{X,Y}(x, y) = P[X = x, Y = y]$

**Joint PDF:**  $f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$

\* $P[(X, Y) \in A] = \int \int_{(x,y) \in A} f_{X,Y}(x, y) \, dx \, dy$

**Exp.:**  $E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) \, dx \, dy$

**Correlation:**  $E[XY]$

\***Indep:**  $E[XY] = E[X]E[Y]$

**Covar.:**  $\text{Cov}[X, Y] = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - E[X]E[Y]$

**Corr. Coeff.:**  $\rho_{X,Y} = E \left[ \left( \frac{X - \mu_X}{\sigma_X} \right) \left( \frac{Y - \mu_Y}{\sigma_Y} \right) \right] = \frac{\text{Cov}[X, Y]}{\sigma_X \sigma_Y}$

\* $-1 \leq \rho_{X,Y} \leq 1$

**Marginal PMF:**  $P_X(x) = \sum_{j=1}^{\infty} P_{X,Y}(x, y_j) \mid P_Y(y)$

**Marginal PDF:**  $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \mid f_Y(y)$

**Cond. PMF:**  $P_{X|Y}(x|y) = \frac{P_{X,Y}(x, y)}{P_Y(y)} \mid P_{Y|X}(y|x)$

**Cond. PDF:**  $f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} \mid f_{Y|X}(y|x)$

**Bayes' Rule**

\* $f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{f_{X|Y}(x|y) f_Y(y)}{\int_{-\infty}^{\infty} f_{X|Y}(x|y') f_Y(y') \, dy'}$

\* $P_{Y|X}(y|x) = \frac{P_{X,Y}(x, y)}{P_X(x)} = \frac{P_{X|Y}(x|y) P_Y(y)}{\sum_{j=1}^{\infty} P_{X|Y}(x|y_j) P_Y(y_j)}$

$P_{\Theta|\underline{X}}(\theta|\underline{x}) = \begin{cases} \frac{P_{\underline{X}|\Theta}(\underline{x}|\theta) P_{\Theta}(\theta)}{P_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ disc.} \\ \frac{f_{\underline{X}|\Theta}(\underline{x}|\theta) P_{\Theta}(\theta)}{f_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ cont.} \end{cases}$

$f_{\Theta|\underline{X}}(\theta|\underline{x}) = \begin{cases} \frac{P_{\underline{X}|\Theta}(\underline{x}|\theta) f_{\Theta}(\theta)}{P_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ disc.} \\ \frac{f_{\underline{X}|\Theta}(\underline{x}|\theta) f_{\Theta}(\theta)}{f_{\underline{X}}(\underline{x})} & \text{if } \underline{X} \text{ cont.} \end{cases}$

\* $f_{\underline{X}}(\underline{x}) = \int_{-\infty}^{\infty} f_{\underline{X}|\Theta}(\underline{x}|\theta) f_{\Theta}(\theta) \, d\theta$

\* $P_{\underline{X}}(\underline{x}) = \sum_{j=1}^{\infty} P_{\underline{X}|\Theta}(\underline{x}|\theta_j) P_{\Theta}(\theta_j)$

**Ind.:**  $f_{X|Y}(x|y) = f_X(x) \forall y \Leftrightarrow f_{X,Y}(x, y) = f_X(x) f_Y(y)$

**Thm:** If independent, then uncorrelated unless Gaussian.

**Uncorrelated:**  $\text{Cov}[X, Y] = 0 \Leftrightarrow \rho_{X,Y} = 0$

**Orthogonal:**  $E[XY] = 0$

**Cond. Exp.:**  $E[Y] = E[E[Y|X]]$  or  $E[E[h(Y)|X]]$

$E[Y] = \int_{-\infty}^{\infty} E[Y|X] f_X(x) \, dx$

\* $E[Y \mid X] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) \, dy$

\* $E[E[Y|X]]$  w.r.t.  $X \mid E[Y|X]$  w.r.t.  $Y$ .

**Q Fcn:**  $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} \, dt$

**Counting:**

**Permutations:**  $P(n, k) = \frac{n!}{(n-k)!}$

\*Order matters

**Combinations:**  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

\*Order doesn't matter

**Multinomial Coeff.:**  $\binom{n}{n_1, \dots, n_m} = \frac{n!}{n_1! \dots n_m!}$

**PMF/PDF:** Make sure normalized to 1.

**Bernoulli:** Probability of success  $x = 0$  or failure  $x = 1$  in 1 trial, w/ success occurring w/ prob.  $p$

**Binomial:** Probability of obtaining  $k$  successes in  $n$  i.i.d. trials, each w/ success prob.  $p$ .

**Geometric:** Probability that the 1st success occurs on the  $k$ th i.i.d. trials, each w/ success prob.  $p$ .

**Negative Binomial:** Probability that the  $r$ th success occurs on the  $k$ th i.i.d. trials, each w/ success prop.  $p$ .

**Hypergeometric:** Probability of drawing  $k$  successes in  $r$  draws from a population of  $N$ , which contains  $m$  successes, w/o replacement.

**Poisson:** Probability of observing  $k$  events in a fixed interval, given a constant rate  $\lambda$ , assuming events indep.

**Multinomial:** Probability of obtaining counts  $(n_1, \dots, n_r)$  in  $n$  trials, where each outcome belongs to one of  $r$  categories w/ probabilities  $(p_1, \dots, p_r)$ .

**Uniform:** Any value in  $[a, b]$  is equally likely, w/ constant density  $\frac{1}{b-a}$ .

**Exponential:** Probability of waiting time  $x$  until the 1st event in a Poisson process w/ rate  $\lambda$ , modelling time b/w indep. events.

**Gamma:** Probability of waiting time  $x$  until  $a$  events in a Poisson process w/ rate  $\lambda$ , generalizing the exponential distribution.

**Gaussian:** Probability of observing  $x$  in a normal distribution w/ mean  $\mu$  and variance  $\sigma^2$ , modelling continuous data.

**Beta:**  $[0, 1]$  w/  $\alpha, \beta$ , used as a prior in Bayesian inference.

**Estimation:** Estimate unknown parameter  $\theta$  from  $n$  i.i.d. mea-

surements  $X_1, X_2, \dots, X_n, \hat{\Theta}(\underline{X}) = g(X_1, X_2, \dots, X_n)$

**Estimation Error:**  $\hat{\Theta}(\underline{X}) - \theta$ .

**Unbiased:**  $\hat{\Theta}(\underline{X})$  is unbiased if  $E[\hat{\Theta}(\underline{X})] = \theta$ .

**\*Asymptotically unbiased:**  $\lim_{n \rightarrow \infty} E[\hat{\Theta}(\underline{X})] = \theta$ .

**Consistent:**  $\hat{\Theta}(\underline{X})$  is consistent if  $\hat{\Theta}(\underline{X}) \rightarrow \theta$  as  $n \rightarrow \infty$  or  $\forall \epsilon > 0, \lim_{n \rightarrow \infty} P[|\hat{\Theta}(\underline{X}) - \theta| < \epsilon] \rightarrow 1$ .

**Sufficient:** A statistic is sufficient if the expression depends only on the statistic, it should be made up of  $x_1, x_2, \dots, x_n$ .

**Sample Mean:**  $M_n = \frac{1}{n} S_n = \frac{1}{n} \sum_{i=1}^n X_i$ .

\*Given a sequence of i.i.d. RVs,  $X_1, X_2, \dots, X_n$ ,  $M_n$  is unbiased and consistent.

**Sample Variance:**  $S_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - M_n)^2$ .

\*Given a sequence of i.i.d. RVs,  $X_1, X_2, \dots, X_n$ ,  $S_n^2$  is biased and consistent.

\*Use  $S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - M_n)^2$  for unbiased.

**Chebychev's Inequality:**  $P[|X - E[X]| \geq \epsilon] \leq \frac{\text{Var}[X]}{\epsilon^2}$

\* $P[|X - E[X]| < \epsilon] \geq 1 - \frac{\text{Var}[X]}{\epsilon^2}$

**Weak Law of Large #s:**  $\lim_{n \rightarrow \infty} P[|M_n - \mu| < \epsilon] = 1 \quad \forall \epsilon > 0$ .

**ML Estimation:** Choose  $\theta$  that is most likely to generate the obs.  $x_1, x_2, \dots, x_n$ .

\*Disc:  $\hat{\Theta} = \arg \max_{\theta} P_{\underline{X}}(\underline{x}|\theta) \xrightarrow{\log} \hat{\theta} = \arg \max_{\theta} \sum_{i=1}^n \log P_X(x_i|\theta)$

\*Cont:  $\hat{\Theta} = \arg \max_{\theta} f_{\underline{X}}(\underline{x}|\theta) \xrightarrow{\log} \hat{\theta} = \arg \max_{\theta} \sum_{i=1}^n \log f_X(x_i|\theta)$

**Maximum A Posteriori (MAP) Estimation:**

\*Disc:  $\hat{\theta} = \arg \max_{\theta} P_{\Theta|\underline{X}}(\theta|\underline{x}) = \arg \max_{\theta} P_{\underline{X}|\Theta}(\underline{x}|\theta) P_{\Theta}(\theta)$

\*Cont:  $\hat{\theta} = \arg \max_{\theta} f_{\Theta|\underline{X}}(\theta|\underline{x}) = \arg \max_{\theta} f_{\underline{X}|\Theta}(\underline{x}|\theta) f_{\Theta}(\theta)$

\* $f_{\Theta|\underline{X}}(\theta|\underline{x})$ : Posteriori,  $f_{\underline{X}|\Theta}(\underline{x}|\theta)$ : Likelihood,  $f_{\Theta}(\theta)$ : Prior

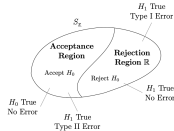
**Least Mean Squares (LMS) Estimation:** Assume prior  $P_{\Theta}(\theta)$  or  $f_{\Theta}(\theta)$  w/ obs.  $\underline{X} = \underline{x}$ .

\* $\hat{\theta} = g(\underline{x}) = E[\Theta|\underline{X} = \underline{x}] \mid \hat{\Theta} = g(\underline{X}) = E[\Theta|\underline{X}]$

**Conditional Exp.**  $E[X|Y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$

**Binary Hyp. Testing:**  $H_0$ : Null Hyp.,  $H_1$ : Alt. Hyp.

$\Omega_{\underline{X}}$ : Set of all possible obs.  $\underline{x}$ .



**TI Err. (False Rejection):** Reject  $H_0$  when  $H_0$  is true.

\* $\alpha(R) = P[\underline{X} \in R \mid H_0]$  (false alarm)

**II Err. (False Accept.):** Accept  $H_0$  when  $H_1$  is true.

\* $\beta(R) = P[\underline{X} \in R^c \mid H_1]$  (missed detection)

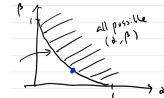
**Prob. of Error:**  $P = \alpha \pi_0 + \beta(1 - \pi_0)$

**Likelihood Ratio Test:**  $\forall \underline{x} \quad L(\underline{x}) = \frac{P_{\underline{X}}(\underline{x}|H_1)}{P_{\underline{X}}(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \text{ or } \xi$

\***Max. Likelihood Test:** 1, **Likelihood Ratio Test:**  $\xi$

**Neyman-Pearson Lemma:** Given a false rejection prob. ( $\alpha$ ), the LRT offers the smallest possible false accept. prob. ( $\beta$ ), and vice versa.

\*LRT produces  $(\alpha, \beta)$  pairs that lie on the efficient frontier.



**Bayesian Hyp. Testing:**

**MAP Rule:**  $L(\underline{x}) = \frac{p_{\underline{X}}(\underline{x}|H_1)}{p_{\underline{X}}(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P[H_0]}{P[H_1]}$

**Min. Cost Bayes' Dec. Rule:**  $C_{ij}$  is cost of choosing  $H_j$  when  $H_i$  is true. Given obs.  $\underline{X} = \underline{x}$ , the expected cost of choosing  $H_j$  is  $A_j(\underline{x}) = \sum_{i=0}^1 C_{ij} P[H_i|\underline{X} = \underline{x}]$ .

**Min. Cost Dec. Rule:**  $L(\underline{x}) = \frac{P_{\underline{X}}(\underline{x}|H_1)}{P_{\underline{X}}(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(C_{01} - C_{00})P[H_0]}{(C_{10} - C_{11})P[H_1]}$ .

\* $C_{01}$ : False accept. cost,  $C_{10}$ : False reject. cost.

**Naive Bayes Assumption:** Assume  $X_1, \dots, X_n$  (features) are ind., then  $p_{\underline{X}|\Theta}(\underline{x}|\theta) = \prod_{i=1}^n p_{X_i|\Theta}(x_i|\theta)$ .

**Notation:**  $P_{\underline{X}|\Theta}(\underline{x}|\theta)$ , only put RVs in subscript, not values.

$P_{\underline{X}}(\underline{x}|H_i)$ , didn't put  $H$  in subscript b/c it's not a RV.

**Beta Prior**  $\Theta$  is a Beta R.V. w/  $\alpha, \beta > 0$

$f_{\Theta}(\theta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} & \text{if } 0 < \theta < 1 \\ 0 & \text{otherwise} \end{cases}$

\* $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$

**Prop.:** 1.  $\Gamma(x+1) = x\Gamma(x)$ . For  $m \in \mathbb{Z}^+$ ,  $\Gamma(m+1) = m!$ .

2.  $\beta(\alpha, \beta) = \frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!} = \beta \binom{\alpha+\beta-1}{\alpha-1}$

3. Expected Value:  $E[\Theta] = \frac{\alpha}{\alpha+\beta}$  for  $\alpha, \beta > 0$

4. Mode (max of PDF):  $\frac{\alpha-1}{\alpha+\beta-2}$  for  $\alpha, \beta > 1$

**Drawing Beta Dist.** 1. Label  $x$ -axis from 0 to 1. 2. Identify mode.

3. Determine shape based on  $\alpha$  and  $\beta$ :  $\alpha = \beta = 1$  (uniform),  $\alpha = \beta > 1$  (bell-shaped, peak at 0.5),  $\alpha = \beta < 1$  (U-shaped w/ high density near 0 and 1),  $\alpha > \beta$  (left-skewed),  $\alpha < \beta$  (right-skewed).

**Uniform PDF**  $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$

\* $E[X] = \frac{a+b}{2}$ ,  $\text{Var}[X] = \frac{(b-a)^2}{12}$

**Random Vector:**  $\underline{X} = (X_1, \dots, X_n) = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = [X_1 \quad \dots \quad X_n]^T$

**Mean Vector:**  $\underline{m}_{\underline{X}} = E[\underline{X}] = [\mu_1, \dots, \mu_n]^T$

**Corr. Mat.:**  $R_{\underline{X}} = \begin{bmatrix} E[X_1^2] & \dots & E[X_1 X_n] \\ E[X_2 X_1] & \dots & E[X_2 X_n] \\ \vdots & \ddots & \vdots \\ E[X_n X_1] & \dots & E[X_n^2] \end{bmatrix}$

\*Real, symmetric ( $R = R^T$ ), and PSD ( $\forall \underline{a}, \underline{a}^T R \underline{a} \geq 0$ ).

**Covar. Mat.:**  $K_{\underline{X}} = \begin{bmatrix} \text{Var}[X_1] & \cdots & \text{Cov}[X_1, X_n] \\ \text{Cov}[X_2, X_1] & \cdots & \text{Cov}[X_2, X_n] \\ \vdots & \ddots & \vdots \\ \text{Cov}[X_n, X_1] & \cdots & \text{Var}[X_n] \end{bmatrix}$

\* $K_{\underline{X}} = R_{\underline{X}} \text{--} \underline{m}_{\underline{X}} = R_{\underline{X}} \text{--} \underline{m} \underline{m}^T$

\*Diagonal  $K_{\underline{X}} \iff X_1, \dots, X_n$  are (mutually) uncorrelated.

**Lin. Trans.**  $\underline{Y} = A \underline{X}$  (A rotates and stretches  $\underline{X}$ )

**Mean:**  $E[\underline{Y}] = A \underline{m}_{\underline{X}}$

**Covar. Mat.:**  $K_{\underline{Y}} = A K_{\underline{X}} A^T$

**Diagonalization of Covar. Mat. (Uncorrelated):**

$\forall \underline{X}$ , set  $P = [\underline{e}_1, \dots, \underline{e}_n]$  of  $K_{\underline{X}}$ , if  $\underline{Y} = P^T \underline{X}$ , then

$K_{\underline{Y}} = P^T K_{\underline{X}} P = \Lambda$

\* $\underline{Y}$ : Uncorrelated RVs,  $K_{\underline{X}} = P \Lambda P^T$

**Find an Uncorrelated  $K_{\underline{Y}}$**

1. Find eigenvalues, normalized eigenvectors of  $K_{\underline{X}}$ .

2. Set  $K_{\underline{Y}} = \Lambda$ , where  $\underline{Y} = P^T \underline{X}$

**PDF of L.T.** If  $\underline{Y} = A \underline{X}$  w/ A not singular, then

$f_{\underline{Y}}(\underline{y}) = \frac{f_{\underline{X}}(\underline{x})}{|\det A|} \Big|_{\underline{x} = A^{-1} \underline{y}}$

**Find  $f_{\underline{Y}}(\underline{y})$**  1. Given  $f_{\underline{X}}(\underline{x})$  and RV relations, define A

2. Determine  $|\det A|$ ,  $A^{-1}$ , then  $f_{\underline{Y}}(\underline{y})$ .

**Gaussian RVs:**  $\underline{X} \sim \mathcal{N}(\underline{\mu}, \Sigma)$

PDF of jointly Gaus.  $X_1, \dots, X_n \equiv$  Guas. vector:

$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2} |\det \Sigma|^{1/2}} e^{-\frac{1}{2}(\underline{x} - \underline{\mu})^T \Sigma^{-1} (\underline{x} - \underline{\mu})}$

\*1D:  $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2}$

\* $\underline{\mu} = \underline{m}_{\underline{X}}$ ,  $\Sigma = K_{\underline{X}}$  ( $\Sigma$  not singular)

\*Indep.:  $f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2} \prod_{i=1}^n \sigma_i} e^{-\frac{1}{2} \sum_{i=1}^n \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2}$

\*IID:  $f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2} \sigma^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2}$

\*Cond. PDF:  $f_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y}) = \mathcal{N}(\underline{\mu}_{\underline{X}|\underline{Y}}, \Sigma_{\underline{X}|\underline{Y}})$

**Properties of Gaussian Vector:**

1. PDF is completely determined by  $\underline{\mu}$ ,  $\Sigma$ .

2.  $\underline{X}$  uncorrelated  $\iff \underline{X}$  independent.

3. Any L.T.  $\underline{Y} = A \underline{X}$  is Gaus. vector w/  $\underline{\mu}_{\underline{Y}} = A \underline{\mu}_{\underline{X}}$ ,  $\Sigma_{\underline{Y}} = A \Sigma_{\underline{X}} A^T$ .

4. Any subset of  $\{X_i\}$  are jointly Gaus.

5. Any cond. PDF of a subset of  $\{X_i\}$  given the other elements is Gaus.

**Diagonalization of Guassian Covar. (Indep.)**

$\forall \underline{X}$ , set  $P = [\underline{e}_1, \dots, \underline{e}_n]$  of  $\Sigma_{\underline{X}}$ , if  $\underline{Y} = P^T \underline{X}$ , then

$\Sigma_{\underline{Y}} = P^T \Sigma_{\underline{X}} P = \Lambda$

\* $\underline{Y}$ : Indep. Gaussian RVs,  $\Sigma_{\underline{X}} = P \Lambda P^T$

**How to go from Y to X?** 1. Given,  $\underline{X} \sim \mathcal{N}(\underline{\mu}, \Sigma)$

2.  $\underline{V} \sim \mathcal{N}(\underline{0}, I)$  3.  $\underline{W} = \sqrt{\Lambda} \underline{V}$  4.  $\underline{Y} = P \underline{W}$  4.  $\underline{X} = \underline{Y} + \underline{\mu}$

**Gaussian Discriminant Analysis:**

Obs:  $\underline{X} = \underline{x} = (x_1, \dots, x_D)$

Hyp:  $\underline{x}$  is generated by  $\mathcal{N}(\underline{\mu}_c, \Sigma_c)$ ,  $c \in C$

Dec: Which "Gaussian bump" generated  $\underline{x}$ ?

Prior:  $P[C = c] = \pi_c$  (Gaussian Mixture Model)

**MAP:**  $\hat{c} = \arg \max_c P_C[c|\underline{X} = \underline{x}] = \arg \max_c f_{\underline{X}|C}(\underline{x} \mid c) \pi_c$

**LGD:** Given  $\Sigma_c = \Sigma \forall c$ , find  $c$  w/ best  $\underline{\mu}_c$

$\hat{c} = \arg \max_c \underline{\beta}_c^T \underline{x} + \gamma_c$

\* $\underline{\beta}_c^T = \underline{\mu}_c^T \Sigma^{-1} \mid \gamma_c = \log \pi_c - \frac{1}{2} \underline{\mu}_c^T \Sigma^{-1} \underline{\mu}_c$

**Bin. Hyp. Decision Boundary**  $\underline{\beta}_0^T \underline{x} + \gamma_0 = \underline{\beta}_1^T \underline{x} + \gamma_1$

\*Linear in space of  $\underline{x}$

**QGD:** Given  $\Sigma_c$  are diff., find  $c$  w/ best  $\underline{\mu}_c$ ,  $\Sigma_c$

$\hat{c} = \arg \max_c -\frac{1}{2} \log |\Sigma_c| - \frac{1}{2} (\underline{x} - \underline{\mu}_c)^T \Sigma_c^{-1} (\underline{x} - \underline{\mu}_c) + \log \pi_c$

**Bin. Hyp. Decision Boundary** Quadratic in space of  $\underline{x}$

**How to find  $\underline{x}_c^1, \underline{\mu}_c^c, \Sigma_c$ :** Given  $n$  points gen. by GMM, then

$n_c$  points  $\{\underline{x}_1^c, \dots, \underline{x}_{n_c}^c\}$  come from  $\mathcal{N}(\underline{\mu}_c, \Sigma_c)$

$\hat{\pi}_c = \frac{n_c}{n}$  (categorical RV)

$\hat{\mu}_c = \frac{1}{n_c} \sum_{i=1}^{n_c} \underline{x}_i^c$ , (sample mean)

$\Sigma_c = \frac{1}{n_c} \sum_{i=1}^{n_c} (x_i^c - \hat{\mu}_c)(x_i^c - \hat{\mu}_c)^T$  (biased sampled var.)

**Gaussian Estimation:**

**MAP Estimator for  $\underline{X}$  Given  $\underline{Y}$  When  $\underline{W} = (\underline{X}, \underline{Y}) \sim \mathcal{N}(\underline{\mu}, \Sigma)$**

Given  $\underline{X} = \{X_1, \dots, X_n\}$ ,  $\underline{Y} = \{Y_1, \dots, Y_m\}$

$\hat{\underline{x}}_{\text{MAP}}(\underline{y}) = \hat{\underline{x}}_{\text{LMS}}(\underline{y}) = \underline{\mu}_{\underline{X}|\underline{Y}} = \underline{\mu}_{\underline{X}} + \Sigma_{\underline{X}\underline{Y}} \Sigma_{\underline{Y}\underline{Y}}^{-1} (\underline{y} - \underline{\mu}_{\underline{Y}})$

\* $\hat{\underline{x}}_{\text{MAP/LMS}}$ : Linear fcn of  $\underline{y}$

**Covar. Matrices:**  $\Sigma = \begin{bmatrix} \Sigma_{\underline{X}\underline{X}} & \Sigma_{\underline{X}\underline{Y}} \\ \Sigma_{\underline{Y}\underline{X}} & \Sigma_{\underline{Y}\underline{Y}} \end{bmatrix}$

\* $\Sigma_{\underline{X}\underline{X}} = \Sigma_{\underline{X}} = E \left[ (\underline{X} - \underline{\mu}_{\underline{X}})(\underline{X} - \underline{\mu}_{\underline{X}})^T \right] \mid \Sigma_{\underline{Y}\underline{Y}} = \Sigma_{\underline{Y}}$

\* $\Sigma_{\underline{X}\underline{Y}} = E \left[ (\underline{X} - \underline{\mu}_{\underline{X}})(\underline{Y} - \underline{\mu}_{\underline{Y}})^T \right] \mid \Sigma_{\underline{Y}\underline{X}} = \Sigma_{\underline{X}\underline{Y}}^T$

**Prec. Matrices:**  $\Lambda = \Sigma^{-1}$

**Mean and Covar. Mat. of  $\underline{X}$  Given  $\underline{Y}$ :**

\* $\underline{\mu}_{\underline{X}|\underline{Y}} = \underline{\mu}_{\underline{X}} + \Sigma_{\underline{X}\underline{Y}} \Sigma_{\underline{Y}\underline{Y}}^{-1} (\underline{y} - \underline{\mu}_{\underline{Y}})$

\* $\Sigma_{\underline{X}|\underline{Y}} = \Sigma_{\underline{X}} - \Sigma_{\underline{X}\underline{Y}} \Sigma_{\underline{Y}\underline{Y}}^{-1} \Sigma_{\underline{Y}\underline{X}}$

\***Reducing Uncertainty:** 2nd term is PSD, so given  $\underline{Y} = \underline{y}$ , always reducing uncertainty in  $\underline{X}$ .

**ML Estimator for  $\theta$  w/ Indep. Guas:**

Given  $\underline{X} = \{X_1, \dots, X_n\}$ :  $\hat{\theta}_{\text{ML}} = \frac{\sum_{i=1}^n \frac{x_i}{\sigma_i^2}}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}$  (weighted avg.  $\underline{x}$ )

\* $X_i = \theta + Z_i$ : Measurement  $\mid Z_i \sim \mathcal{N}(0, \sigma_i^2)$ : Noise (indep.)

\* $\frac{1}{\sigma_i^2}$ : Precision of  $X_i$  (i.e. weight)

\*Larger  $\sigma_i^2 \implies$  less weight on  $X_i$  (less reliable measurement)

\***SC:** If  $\sigma_i^2 = \sigma^2 \forall i$  (iid), then  $\hat{\theta}_{\text{ML}} = \frac{1}{n} \sum_{i=1}^n x_i$ .

**MAP Estimator for  $\theta$  w/ Indep. Gaus., Gaus. Prior:**

Given  $\underline{X} = \{X_1, \dots, X_n\}$ , prior  $\Theta \sim \mathcal{N}(x_0, \sigma_0^2)$

$\hat{\theta}_{\text{MAP}} = \frac{\sum_{i=0}^n \frac{x_i}{\sigma_i^2}}{\frac{1}{\sigma_0^2} + \sum_{i=1}^n \frac{1}{\sigma_i^2}} x_0 + \frac{\sum_{i=1}^n \frac{1}{\sigma_i^2}}{\frac{1}{\sigma_0^2} + \sum_{i=1}^n \frac{1}{\sigma_i^2}} \hat{\theta}_{\text{ML}}$

\* $X_i = \theta + Z_i$ : Measurement  $\mid Z_i \sim \mathcal{N}(0, \sigma_i^2)$ : Noise (indep.)

\* $f_{\Theta}$ : Gaussian prior  $\equiv$  prior meas.  $x_0$  w/  $\sigma_0^2$ .

\***SC**: As  $n \rightarrow \infty$ ,  $\hat{\theta}_{\text{MAP}} \rightarrow \hat{\theta}_{\text{ML}}$ . As  $\sigma_0^2 \rightarrow \infty$ ,  $\hat{\theta}_{\text{MAP}} \rightarrow \hat{\theta}_{\text{ML}}$   
**LMMSE Estimator for  $\underline{X}$  Given  $\underline{Y}$  w/ non-Guas.  $\underline{X}$ ,  $\underline{Y}$** :

$\hat{\underline{\mu}}_{\text{LMMSE}}(\underline{y}) = \underline{\mu}_{\underline{X}} + \Sigma_{\underline{X}\underline{Y}} \Sigma_{\underline{Y}\underline{Y}}^{-1} (\underline{y} - \underline{\mu}_{\underline{Y}})$

\***Linear Gaussian System**: Given  $\underline{Y} = A\underline{X} + \underline{b} + \underline{Z}$   
\* $\underline{X} \sim \mathcal{N}(\underline{\mu}_{\underline{X}}, \Sigma_{\underline{X}})$ ,  $\underline{Z} \sim \mathcal{N}(\underline{0}, \Sigma_{\underline{Z}})$ : Noise (indep. of  $\underline{x}$ )

\* $A\underline{X} + \underline{b}$ : channel distortion,  $\underline{Y}$ : Observed sig.

**MAP/LMS Estimator for  $\underline{X}$  Given  $\underline{Y}$  w/  $\underline{W} = (\underline{X}, \underline{Y})$**

Given  $\underline{W} = \begin{bmatrix} \underline{X} \\ A\underline{X} + \underline{b} + \underline{Z} \end{bmatrix} = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} \underline{X} \\ \underline{Z} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \underline{b} \end{bmatrix}$

$\hat{\underline{\mu}}_{\text{MAP/LMS}} = \underline{\mu}_{\underline{X}} + \Sigma_{\underline{X}} A^T (A \Sigma_{\underline{X}} A^T + \Sigma_{\underline{Z}})^{-1} (\underline{y} - A \underline{\mu}_{\underline{X}} - \underline{b})$

\* $\Sigma_{\underline{X}\underline{Y}} = \Sigma_{\underline{X}} A^T$ ,  $\Sigma_{\underline{Y}\underline{Y}} = A \Sigma_{\underline{X}} A^T + \Sigma_{\underline{Z}}$

$\hat{\underline{\mu}}_{\text{MAP/LMS}} = \left( \Sigma_{\underline{X}}^{-1} + A^T \Sigma_{\underline{Z}}^{-1} A \right)^{-1} \left( A^T \Sigma_{\underline{Z}}^{-1} (\underline{y} - \underline{b}) + \Sigma_{\underline{X}}^{-1} \underline{\mu}_{\underline{X}} \right)$

\***Use**: Good to use when  $\underline{Z}$  is indep.

**Covar. Mat of  $\underline{X}$  Given  $\underline{Y} = \underline{y}$** :  $\Sigma_{\underline{X}|\underline{y}} = \left( \Sigma_{\underline{X}}^{-1} + A^T \Sigma_{\underline{Z}}^{-1} A \right)^{-1}$

**Linear Regression**: Estimate unknown target fn  $Y = g(\underline{X})$  w/ iid obs.  $\{(\underline{x}_1, y_1), \dots, (\underline{x}_n, y_n)\}$  (MLE/MAP)  
\* $\underline{y} = [y_1 \quad \dots \quad y_n]^T$

\* $\underline{X} = \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix} \in \mathbb{R}^{n \times D}$

**ML Estimator**:  $Y = h(\underline{x}) = \underline{w}^T \underline{x} + Z \approx g(\underline{X})$ , then  $\hat{\underline{w}}_{\text{ML}} = (X^T X)^{-1} X^T \underline{y}$

\*Assume  $X^T X$  has full rank (i.e. invertible) since  $n \gg D$

\* $n$ : # of obs.,  $D$ : # of features.

\* $\underline{x} = \{x_1, \dots, x_D\}$ : Input features

\* $\underline{w} = \{w_1, \dots, w_D\}$ : Weights (parameter)

\* $Z \sim \mathcal{N}(0, \sigma^2)$ : Noise (i.i.d.)

\* $\underline{Y}$ : Target/observed output

\* $\underline{X}^\dagger = (X^T X)^{-1} X^T$ : Pseudo-inverse of  $X$  (minimizes  $\|X\underline{w} - \underline{y}\|_2^2 \iff$  maximizes the likelihood of training data)

**Non-Linear Trans**:  $\hat{y} = \underline{w}^T \underline{\phi}(\underline{x}) + Z$  w/ same assumptions,

then  $\hat{\underline{w}}_{\text{ML}} = (X^T X)^{-1} X^T \underline{y}$

\* $\underline{\phi}(\underline{x})$ : Non-linear transformation of  $\underline{x}$

-E.g. of 1 dim  $x$ :  $\underline{\phi}(x) = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^M \end{bmatrix}$ : Polynomial regression

\* $M$ : Degree of polynomial,  $D = 1 + M$ : # of features.

\* $\underline{X} = \begin{bmatrix} \underline{\phi}(\underline{x}_1)^T \\ \vdots \\ \underline{\phi}(\underline{x}_n)^T \end{bmatrix} \in \mathbb{R}^{n \times D}$

**Underfitting vs. Overfitting**:

\*Underfitting: Model too simple, high bias, low variance.

-Results in high train/test error.

\*Overfitting: Model too complex, low bias, high variance.

-Results in low train error, high test error.

**MAP Estimator (Bayesian Linear Regression)**: Assume

prior  $w_i \sim \mathcal{N}(0, \tau^2)$  (i.i.d.) and  $\hat{y} = \underline{w}^T \underline{x} + Z$ , then

$\hat{\underline{w}}_{\text{MAP}} = (X^T X + \lambda I)^{-1} X^T \underline{y}$

\* $\lambda = \frac{\sigma^2}{\tau^2}$ : Regularization parameter

\* $\underline{X}$ : Can be linear or non-linear transformation of  $\underline{x}$

\* $\underline{x} = \{x_1, \dots, x_D\}$ : Input features

\* $\underline{w} = \{w_1, \dots, w_D\}$ : Weights (parameter)

\* $Z \sim \mathcal{N}(0, \sigma^2)$ : Noise (i.i.d.)

\* $\underline{Y}$ : Target/observed output

**Notes**:

1. Useful when training data set size is small i.e.  $n \ll D$ .

2. Regularization: Prevents overfitting by penalizing large weights.

\* $\tau = \infty \implies \lambda = 0$ : No regularization so  $\hat{\underline{w}}_{\text{MAP}} = \hat{\underline{w}}_{\text{ML}}$

\* $\tau = 0 \implies \lambda = \infty$ : Infinite regularization so  $\hat{\underline{w}}_{\text{MAP}} = \underline{0}$

\* $\tau \downarrow \implies \lambda \uparrow$ : More regularization, simpler model.

\* $\tau \uparrow \implies \lambda \downarrow$ : Less regularization, more complex model.

**Gaussian Linear System** Given training data  $\underline{Y} = \underline{X}\underline{w} + \underline{Z}$

$\hat{\underline{w}}_{\text{MAP}} = \underline{\mu}_{\underline{w}|\underline{Y}} = (X^T X + \lambda I)^{-1} X^T \underline{y}$

\* $\underline{w} \sim \mathcal{N}(\underline{0}, \tau^2 I)$ ,  $\underline{Z} \sim \mathcal{N}(\underline{0}, \sigma^2 I)$

\* $E[\underline{w}(\underline{Y})] \rightarrow \underline{w}$  as  $n \rightarrow \infty$

\*Note: Matching it to canonical form.

**Covar. Mat**:  $\Sigma_{\underline{w}|\underline{y}} = \left( \frac{1}{\sigma^2} X^T X + \frac{1}{\tau^2} I \right)^{-1} \preceq \tau^2 I$

-Less uncertainty in  $\underline{w}$  w/ more data. As  $n \uparrow$ ,  $\Sigma_{\underline{w}|\underline{y}} \downarrow$

**Bayesian Prediction** Given some new  $\underline{x}'$  (test data sample), find its label  $y'$

**Plug-In Approx**:  $\hat{Y}' = \underline{x}'^T \hat{\underline{w}}_{\text{MAP}}(\mathcal{D}) + Z'$

\* $\mathcal{D}$ : Training data set,  $Z' \sim \mathcal{N}(0, \sigma^2)$ : Noise

**Bayesian Prediction**: Use  $Y' = \underline{x}'^T \underline{w} + Z'$  and

$f_{\underline{w}|\underline{Y}}(\underline{w} | \underline{y}) = \mathcal{N}(\underline{\mu}_{\underline{w}|\underline{Y}}, \Sigma_{\underline{w}|\underline{Y}})$  to return  $f_{Y'}(y' | \mathcal{D})$  where  $Y'$  is Gaussian given  $\mathcal{D}$  w/

\* $\mu_{Y'|\mathcal{D}} = \underline{x}'^T \mu_{\underline{w}|\underline{Y}}$

\* $\sigma_{Y'|\mathcal{D}}^2 = \underline{x}'^T \Sigma_{\underline{w}|\underline{Y}} \underline{x}' + \sigma^2$

**Linear Classification (Hyp. Test)**:

**Binary Logistic Regression**: Estimate  $\underline{w}$  s.t. it is a soft decision

$P_{Y|\underline{X}}(1 | \underline{x}) = \frac{P_{\underline{X}|\underline{Y}}(\underline{x}|1) P_{Y}(1)}{P_{\underline{X}|\underline{Y}}(\underline{x}|0) P_{Y}(0) + P_{\underline{X}|\underline{Y}}(\underline{x}|1) P_{Y}(1)}$

$P_{Y|\underline{X}}(1 | \underline{x}) = \frac{1}{1 + e^{-\alpha}} = \sigma(\alpha)$

\* $P_{Y|\underline{X}}(0 | \underline{X}) = 1 - \sigma(\alpha) = \frac{1}{1 + e^{\alpha}} = \sigma(-\alpha)$

\* $\alpha = \log \frac{P_{\underline{X}|\underline{Y}}(\underline{x}|1) P_{Y}(1)}{P_{\underline{X}|\underline{Y}}(\underline{x}|0) P_{Y}(0)} = \underline{w}^T \underline{x}$

$-\alpha \rightarrow \infty \implies$  more likely to be in class 1

$-\alpha \rightarrow -\infty \implies$  more likely to be in class 0.

$-\alpha = 0 \implies$  equally likely to be in class 0 or 1.

**Non-Linear Trans**. Use  $\sigma(\underline{w}^T \underline{\phi}(\underline{x}))$

**ML Estimator**: Given  $\mathcal{D} = \{(\underline{x}_i, y_i)\}, i = 1, \dots, n$ , then

$\hat{\underline{w}}_{\text{ML}} = \arg \min_{\underline{w}} - \sum_{i=1}^n \log P_{Y|\underline{X}}(y_i | \underline{x}_i, \underline{w})$

**Cross Entropy** b/w actual  $y_i$  and  $P_{Y|\underline{X}}(\cdot | \underline{x}_i, \underline{w})$  is

$P_{Y|\underline{X}}(y_i | \underline{x}_i, \underline{w}) = \sum_{l=1}^L - (y_i \log P(1 | \underline{x}_i, \underline{w}) + (1 - y_i) \log P(0 | \underline{x}_i, \underline{w}))$

\*Note: Measures the distance between 2 distributions.

\*Dropped the subscripts.

**Gradient Descent:** No closed-form soln. so use GD.

**MAP Estimator:** Given  $\mathcal{D} = \{(\underline{x}_i, y_i)\}, i = 1, \dots, n$ , then  $\underline{w}_{\text{MAP}} = \arg \min_{\underline{w}} - \sum_{i=1}^n \log P_Y(y_i | \underline{x}_i, \underline{w}) + \lambda ||\underline{w}||^2$

$\underline{w} \sim \mathcal{N}(\underline{\mu}, \Sigma)$ : Prior on  $\underline{w}$

\*Necessary: B/c same boundary  $\underline{w}^T \underline{x} = 0$  for any scaling of  $\underline{w}$ .

**Multiclass Logistic Regression:**  $Y \in \{1, 2, \dots, C\}$ , then use softmax fn  $P_Y(k | \underline{x}, \underline{w}_1, \dots, \underline{w}_C) = \frac{e^{\underline{w}_k^T \underline{x}}}{\sum_{c=1}^C e^{\underline{w}_c^T \underline{x}}}$

\* $W = [\underline{w}_1, \dots, \underline{w}_C] \in \mathbb{R}^{D \times C}$ : Weights matrix

**ML Estimator:** Given  $\mathcal{D} = \{(\underline{x}_i, y_i)\}, i = 1, \dots, n$ , then  $\hat{W}_{\text{ML}} = \arg \min_W - \sum_{i=1}^n \log P(y_i | \underline{x}_i, W)$

**MAP Estimator:** Given  $\mathcal{D} = \{(\underline{x}_i, y_i)\}, i = 1, \dots, n$ , then  $\hat{W}_{\text{MAP}} = \arg \min_W - \sum_{i=1}^n \log P(y_i | \underline{x}_i, W) + \sum_{c=1}^C \lambda_c ||\underline{w}_c||^2$

**Markov:**

**Notation:**

\* $P[X_n = x_n, \dots, X_0 = x_0] = P(x_n, \dots, x_0)$

\*Index the possible values of  $X_n$  w/ integers  $0, 1, 2, \dots$

**Markov Chain (Memoryless/Markovian Property):** A sequence of discrete-valued RVs  $X_0, X_1, \dots$  is a (discrete-time) Markov chain if  $P[X_{k+1} = x_{k+1} | X_k = x_k, X_{k-1} = x_{k-1}, \dots, X_0 = x_0] = \underbrace{P[X_{k+1} = x_{k+1}]}_{\text{Future}} \underbrace{P[X_k = x_k]}_{\text{Present}} \underbrace{P[X_{k-1} = x_{k-1}, \dots, X_0 = x_0]}_{\text{Past}}$

$P[X_{k+1} = x_{k+1} | X_k = x_k] \forall k, x_1, \dots, x_{k+1}$

\***Markovian:**  $P(x_n, \dots, x_0) = P(x_n | x_{n-1}) \dots P(x_1 | x_0) P(x_0)$

\***Equiv. Form:**  $k+1 \rightarrow n_{k+1}, k \rightarrow n_k$  and so on for any  $n_{k+1} > n_k > \dots > n_0$  (i.e. farther in future/past)

**State Distribution:** State distribution of the MC at time  $n$  is  $P_j(n) \equiv P[X_n = j], j = 0, 1, \dots \mid \underline{P}(n) \equiv [P_0(n), P_1(n), \dots]$

\*Subscript: Value of  $X_n$ , Argument: Time step

\*Row vector NOT col vector.

**Transition Probabilities:**

$P_{ij}(n, n+1) \equiv P[X_{n+1} = j | X_n = i] \forall i, j, n$

**Homogeneous MC:**  $P_{ij}(n, n+1) = P_{ij} \forall i, j, n$

\*Time invariant,  $P_{ij}$  does not depend on  $n$

**Transition Probability Matrix:**  $P = \begin{bmatrix} P_{00} & P_{01} & \dots \\ P_{10} & P_{11} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$

**Notes:** (1) **Stochastic Matrix:** (1) All entries of  $P$  are non-negative and (2) each row sums to 1:  $\sum_j P_{ij} = 1 \forall i$

(2) State Dist. at time  $n+1$ :  $\underline{P}(n) = \underline{P}(n-1)P$

\* $\underline{P}(n) = \underline{P}(0)P^n$  in terms of initial distribution  $\underline{P}(0)$

(3) State Dist. at time  $n+m$ :  $\underline{P}(n+m) = \underline{P}(m)P^n \forall n, m$

**n-step Transition Probabilities:** Stochastic matrix  $P^n$  s.t.  $P^{(n)}_{ij} \equiv P[X_{k+n} = j | X_k = i]$  for  $n \geq 0$  are the entries of  $P^n$

**Limiting Distribution** A MC has a limiting distribution  $\underline{q}$  if for any initial distribution  $\underline{P}(0)$

$\underline{P}(\infty) \equiv \lim_{n \rightarrow \infty} \underline{P}(n) = \underline{q}$  or  $\underline{P}(0)P^\infty \equiv \underline{P}(0) \lim_{n \rightarrow \infty} P^n = \underline{q}$

**Theorem:** A MC has a limiting distribution  $\underline{q}$  iff  $q_i = \lim_{n \rightarrow \infty} P^{(n)}_{ij} \forall i, j$

\*i.e. every row of  $P^\infty$  equals  $\underline{q}$  (row vector)

**Steady State (Stationary) Distribution  $\underline{\pi}$**  is a steady state distribution of a MC if  $\underline{\pi} = \underline{\pi}P$

\* $1 = \sum_j \pi_j$

**Theorem:** If a limiting dist. exists  $\underline{q} = \underline{P}(\infty)$ , then it is also a steady state dist.

**Ergodic:** For a finite-state, irreducible, and aperiodic MC, then

(1) Limiting dist.  $\underline{q} = \lim_{n \rightarrow \infty} \underline{P}(n)$  exists and  $q_j = \lim_{n \rightarrow \infty} P^{(n)}_{ij} \forall i, j$

(2) Steady state dist.  $\underline{\pi}$  is unique.

(3)  $\underline{\pi} = \underline{q}$

**How Fast Does  $\underline{P}(n)$  Converge to  $\underline{\pi}$ ?** (1)  $\underline{\pi}^T = \underline{\pi}^T P^T$

\* $\underline{\pi}^T$  is an eigenvector of  $P^T$  w/ eigenvalue 1

(2) Suppose  $P^T$  has eigenvectors  $U \equiv [\underline{\pi}^T, \underline{u}_2, \dots, \underline{u}_D]$  and eigenvalues  $\Lambda \equiv \text{diag}[1, \lambda_2, \dots, \lambda_D]$ , then  $P^T U = U \Lambda \implies P^T = U \Lambda U^{-1}$  so  $n$  times  $P^n = (P^T)^n = (U \Lambda U^{-1})^n = U \Lambda^n U^{-1}$

Therefore,  $\Lambda^n = \text{diag}[1, \lambda_2^n, \dots, \lambda_D^n]$

(3) For ergodic MC,  $P^n \rightarrow [\underline{\pi}, \dots, \underline{\pi}]^T$  (i.e. rank 1)

Therefore, # of non-zero eigenvalues is 1, so the rest of the eigenvalues must be  $|\lambda_i| < 1 \forall i \geq 2$  s.t.  $\Lambda^n = \text{diag}[1, 0, \dots, 0]$

**Rate of Convergence:** Depends on the 2nd largest eigenvalue of  $P^T$  i.e.  $(\lambda_2)^n$  is the rate of convergence.

**Bayesian Network (DAG):** Network of RVs  $X_1, \dots, X_n$  w/ directed edges

\***Not State-Transition Diagram:** 1 RV w/ different values w/ different probabilities to each value.

\***Fully Connected Graph (General):** No special dependency structure, so doesn't give additional info as we can write joint dist. from defn. (true for any graph).

\***Non-Fully Connected Graph (Bayes' Net):** Conveys useful info about the dependency structure.

**Factorization of Joint Dist.** Suppose the dependencies among RVs can be represented by a DAG, then  $P(x_1, \dots, x_n) = \prod_{i=1}^N P(x_i | \text{pa}\{X_i\})$

\***General:**  $P(\underline{x}) = \prod_{i=1}^N P(x_i | x_{i-1}, x_{i-2}, \dots)$

$-P(x_1 | x_0) = P(x_1)$

**Topological Ordering:** Often index the RVs s.t. each child has an index greater than those of the parents.

**Fact:** Every DAG has at least one topological ordering.

**Conditional Independence:**  $A \perp B | C$  if

(1)  $P(a, b | c) = P(a | c)P(b | c) \forall a, b, c$  (i.e. A and B are indep. given C)

(2)  $P(a | b, c) = P(a | c) \forall a, b, c$  (i.e. B gives no add. info about A given C)

**Common Cause (T-T):**  $A \perp B | C$ , o.w.  $A \not\perp B$

**Causal Chain (H-T/T-H):**  $A \perp B | C$ , o.w.  $A \not\perp B$

**Common Effect (H-H):**  $A \perp B$ , o.w.  $A \not\perp B | C$  or its descendants

\***Explaining Away:** If  $A \rightarrow B \leftarrow C$ , then if you observe  $B$ , then the other cause  $A$  is less likely to be the cause for the effect  $B$ .

**Directed Separation (D-seperation):** For non-overlapping subsets of RVs  $\mathcal{A}, \mathcal{B}, \mathcal{C}$ , if all undirected paths blocked, then  $\mathcal{A}$  and  $\mathcal{B}$  are **d-separated** by  $\mathcal{C}$ , i.e.  $\mathcal{A} \perp \mathcal{B} | \mathcal{C}$

**Blocked Path:** An undirected path is blocked if it includes a node s.t.

- The node is head-to-tail or tail-to-tail (Cases 1 and 2) and it is in set  $\mathcal{C}$

2. The node is head-to-head, but neither itself nor any of its descendants are in set  $C$  (Case 3)

**Markov Boundary (Blanket):** Minimal set of RVs  $\mathcal{M}$  that isolate  $X_i$  from all the remaining RVs, i.e.

$$X_i \perp \mathcal{N} \setminus (\{X_i\} \cup \mathcal{M}) \mid \mathcal{M}$$

\* $\mathcal{N}$ : Set of all RVs

\* $\mathcal{M}$  = parents  $\cup$  children  $\cup$  co-parents: Blocks all paths b/w  $X_i$  and the remaining nodes.

**Markov Random Field:** Represent RVs as an **undirected** graph s.t. conditional independence  $\mathcal{A} \perp \mathcal{B} \mid \mathcal{C}$  hold iff all paths b/w  $\mathcal{A}$  and  $\mathcal{B}$  so through  $\mathcal{C}$ .

\***Markov blanket** of  $X_i$  := set of neighbours of  $X_i$

\***No Order:** Simplifies, but no way to order the RVs, so lose directivity, lose info.

**Independence:** See if all paths b/w  $\mathcal{A}$  and  $\mathcal{B}$  are blocked by  $C$  (i.e. given  $C$ )

**Clique:** A set of nodes s.t. there is link b/w any pair of them

**Maximal Clique:** A clique s.t. we cannot add another node in the set and maintain a clique.

**Hammersley-Clifford Theorem:** Let  $\underline{x}_c$  denote the values of RVs in set  $C$ . Any strictly postivie dist.  $P(\underline{x})$  that satisfies a Markov random field can be factorized as

$$P(\underline{x}) = \frac{1}{Z} \prod_{c \in C} \psi_c(\underline{x}_c) \stackrel{\text{e.g.}}{=} \frac{1}{Z} e^{-\sum_c E(\underline{x}_c)}$$

\* $Z = \sum_{\underline{x}} \prod_{c \in C} \psi_c(\underline{x}_c)$ : Normalization constant

\* $\Pi_{c \in C}$ : Product of all maximal cliques

\* $\psi_c(\underline{x}_c) \stackrel{\text{e.g.}}{=} e^{-E(\underline{x}_c)}$ : Potential function over the clique  $c$  (not necessarily a prob.)

\* $E(\underline{x}_c)$ : Energy function over the clique  $c$

**Moralization or Marrying the Parents (BN to MRF)** Always possible, but some dependency structure will be lost

1. Add edges b/w all pairs of parents of the same child

2. Remove all directed edges (i.e. make it undirected)

**Hidden Markov Model (HMM):**

\***State-transition Prob.**

$$P(z_n \mid z_{n-1}) \equiv P[Z_n = z_n \mid Z_{n-1} = z_{n-1}], \quad Z \leq n \leq N$$

\***Initial State Dist.**  $P(z_1) \equiv P[Z_1 = z_1]$

\***Emission Prob.**

$$P(x_n \mid z_n) \equiv P[X_n = x_n \mid Z_n = z_n], \quad 1 \leq n \leq N$$

\***Note:** Can be continuous (density).

**Notes:**

(1)  $Z_n$  nodes are H-T or T-T and  $Z_1, \dots, Z_N$  are unobserved  $\implies$  No indep. among  $X_n$ 's, also  $\{X_n\}$  are not a MC.

(2) Latent var.  $Z_1, \dots, Z_N$  are MC  $\implies Z_{n+1} \perp Z_{n-1} \mid Z_n \implies$  e.g.  $\{X_1, X_2\} \perp \{X_3, \dots, X_N\} \mid \{Z_3\}$

**Common Problems**

1. Given HMM, find  $P(\underline{x})$  for any  $\underline{x} = \{x_1, \dots, x_N\}$

2. Given HMM/ $\underline{x}$ , find most likely  $z_n$

3. Given HMM/ $\underline{x}$ , find most likely seq. of states  $\underline{z} = \{z_1, \dots, z_N\}$

**Message Passing Algos: Given HMM, find  $P(\underline{x}) \forall \underline{x}$**

**Forward:**

**Defn of  $\alpha$ :**

$$\alpha(z_n) \equiv P[\underbrace{X_1 = x_1, \dots, X_n = x_n}_{\text{obs so far}}, \underbrace{Z_n = z_n}_{\text{cur. state}}, \quad 1 \leq n \leq N$$

$$\alpha(z_n) \equiv P[x_1, \dots, x_n, z_n]$$

**Recursion:**

$$\alpha(z_1) = P(x_1, z_1) = P(z_1)P(x_1 \mid z_1)$$

$$\alpha(z_n) = P(x_n \mid z_n) \sum_{z_{n-1}} P(z_n \mid z_{n-1})\alpha(z_{n-1})$$

$$P(\underline{x}) = \sum_{z_N} \alpha(z_N)$$

**Algo:**  $\{\alpha(z_n)\}_{z_n=1, \dots, K}$ : Message from  $Z_n$  to  $Z_{n+1}$

\***Complexity:**  $O(K^2N)$

\* $O(K)$  for each  $\alpha(z_n)$ ,  $O(K^2)$  for message at time  $n$

**Backward:**

**Defn of  $\beta$ :**

$$\beta(z_n) \equiv P[\underbrace{X_{n+1} = x_{n+1}, \dots, X_N = x_N}_{\text{future obs}} \mid \underbrace{Z_n = z_n}_{\text{cur. state}}]$$

$$\beta(z_n) \equiv P[x_{n+1}, \dots, x_N \mid z_n]$$

**Recursion:**

$$\beta(z_N) = 1 \quad \forall z_N$$

$$\beta(z_n) = \sum_{z_{n+1}} P(x_{n+1} \mid z_{n+1})P(z_{n+1} \mid z_n)\beta(z_{n+1})$$

$$P(\underline{x}) = \sum_{z_1} P(z_1)P(x_1 \mid z_1)\beta(z_1)$$

**Algo:**  $\{\beta(z_n)\}_{z_n=1, \dots, K}$ : Message from  $Z_{n-1}$  to  $Z_n$

\***Complexity:**  $O(K^2N)$

\* $O(K)$  for each  $\beta(z_n)$ ,  $O(K^2)$  for message at time  $n$

**Forward Backward (Same Time):**

$$\alpha(z_n)\beta(z_n) = P(\underline{x}, z_n) \implies P(\underline{x}) = \sum_{z_n} \alpha(z_n)\beta(z_n) \quad \forall n$$

**Given HMM and  $\underline{x}$ , find most likely  $z_n$**

**Defn of  $\gamma$ :**

$$\gamma(z_n) \equiv P(z_n \mid \underline{x}), \quad 1 \leq n \leq N$$

**Solution:**

$$\gamma(z_n) = \frac{\alpha(z_n)\beta(z_n)}{\sum_{z_n'} \alpha(z_n')\beta(z_n')}$$

$$z_n^* = \arg \max_{z_n} \gamma(z_n)$$

\***Complexity:**  $O(K^2N)$

**Scaling Problem:**  $\alpha/\beta$  can be small for large/small  $n$

**1. Forward:**

**Defn of  $\hat{\alpha}$ :**

$$\hat{\alpha}(z_n) \equiv \frac{\alpha(z_n)}{P(x_1, \dots, x_n)} = P(z_n \mid x_1, \dots, x_n)$$

\*Does not shrink as  $n \uparrow$

**Defn of  $c_n$ :**

$$c_n \equiv P(x_n \mid x_1, \dots, x_{n-1})$$

**Recursion:**

$$\hat{\alpha}(z_n) = \frac{1}{c_n} P(x_n \mid z_n) \sum_{z_{n-1}} P(z_n \mid z_{n-1})\hat{\alpha}(z_{n-1})$$

$$c_n = \sum_{z_n} P(x_n \mid z_n) \sum_{z_{n-1}} P(z_n \mid z_{n-1})\hat{\alpha}(z_{n-1})$$

$$P(x_1, \dots, x_n) = \prod_{m=1}^N c_m$$

**2. Backward:**

**Defn of  $\hat{\beta}$ :**

$$\hat{\beta}(z_n) = \frac{\beta(z_n)}{\prod_{m=n+1}^N c_m}$$

Recursion:

$$\hat{\beta}(z_n) = \frac{1}{c_{n+1}} \sum_{z_{n+1}} P(x_{n+1} \mid z_{n+1}) P(z_{n+1} \mid z_n) \hat{\beta}(z_{n+1})$$

3. Forward-Backward 

$\gamma(z_n) = \hat{\alpha}(z_n) \hat{\beta}(z_n)$

**Forward-Backward Algo** \*Have to fwd, then bwd pass.

0.  $c_1 = P(x_1) = \sum_{z_1} P(z_1) P(x_1 \mid z_1)$

1.  $\hat{\alpha}(z_1) = \frac{1}{c_1} P(z_1) P(x_1 \mid z_1)$

Fwd message passing to compute  $\hat{\alpha}(z_n)$  and  $c_n$ ,  $2 \leq n \leq N$

2.  $\hat{\beta}(z_N) = \beta(z_N) = 1$

Bwd message passing to compute  $\hat{\beta}(z_n)$ ,  $1 \leq n \leq N - 1$

3.  $\gamma(z_n) = \hat{\alpha}(z_n) \hat{\beta}(z_n)$

4.  $z_n^* = \arg \max_{z_n} \gamma(z_n) \forall n$

**Viterbi Algo: Given HMM and  $\underline{x}$ , find most likely  $\underline{z}$**

$\hat{z} = \arg \max_{\underline{z}} P(\underline{z} \mid \underline{x}) = \arg \max_{\underline{z}} P(\underline{x}, \underline{z})?$

**Path Length:**  $-\log P(\underline{x}, \hat{z})$

$= - \left[ \log P(z_1) + \sum_{n=2}^N \log P(z_n \mid z_{n-1}) + \sum_{n=1}^N \log P(x_n \mid z_n) \right]$

**Current Highest Log-Prob to State  $z_n$ :**

$w_n(z_n) = \max_{z_1, \dots, z_{n-1}} \log P(x_1, \dots, x_n, z_1, \dots, z_n)$

**\*Base Case:**  $w_1(z_1) = \log P(z_1) + \log P(x_1 \mid z_1)$

**Trellis Diagram of MC** Unfolding of MC over time

**Viterbi Algorithm:**

0.  $w_1(z_1) = \log P(x_1, z_1) = \log P(z_1) + \log P(x_1 \mid z_1)$

1. For  $1 \leq n \leq N - 1$ :

$w_{n+1}(z_{n+1}) = \max_{z_n} w_n(z_n) + \log P(z_{n+1} \mid z_n) + \log P(x_{n+1} \mid z_{n+1})$

$\psi_n(z_{n+1}) = \arg \max_{z_n} w_n(z_n) + \log P(z_{n+1} \mid z_n)$

2.  $\hat{z}_N = \arg \max_{z_N} w_N(z_N)$

3. Output:  $\hat{z} = (\hat{z}_1, \dots, \hat{z}_N)$  where  $\hat{z}_n = \psi_n(\hat{z}_{n+1})$

**\*Complexity:**  $O(K^2 N)$ ,  $O(K^2)$  per step.