Lecture notes on undergraduate mathematical analysis

Seick Kim

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Chapter 1

Differentiable Mappings

1.1 Basic notation

Euclidean n-space $\mathbb{R}^n$ is defined as the set of all n-tuples $(x_1, \ldots, x_n)$ of real numbers x_i. If x denotes an element of $\mathbb{R}^n$, then x is an n-tuple of numbers, the ith one of which is denoted x_i; thus we can write

$$x = (x_1, \ldots, x_n).$$

The element of $x \in \mathbb{R}^n$ can be interpreted as the the (column) vector

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \ \cdots \ x_n]^\top .$$

The standard basis vectors in $\mathbb{R}^n$ are the vectors $\boldsymbol{e}_j$ with n entries, the jth entry 1 and the others zero. For example in $\mathbb{R}^3$ there are three standard basis vectors

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The dot product $\boldsymbol{x} \cdot \boldsymbol{y}$ of two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + \cdots + x_n y_n.$$

The length $|\boldsymbol{x}|$ of a vector $\boldsymbol{x} \in \mathbb{R}^n$ is

$$|\boldsymbol{x}| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} = \sqrt{x_1^2 + \cdots + x_n^2}.$$

1.2 Linear transformations

Definition 1.2.1. A mapping T of a vector space X into a vector space Y is said to be a *linear transformation* if

$$T(\boldsymbol{x}_1 + \boldsymbol{x}_2) = T\boldsymbol{x}_1 + T\boldsymbol{x}_2, \quad T(c\boldsymbol{x}) = cT\boldsymbol{x}$$

for all $\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{x}_2 \in X$ and all scalars c. Note that one often writes $T\boldsymbol{x}$ instead of $T(\boldsymbol{x})$ if T is linear.

Let $L(X, Y)$ be the set of all linear transformations of the vector space X into the vector space Y. Instead of $L(X, X)$, we shall simply write $L(X)$. If $T_1, T_2 \in L(X, Y)$ and if c_1 and c_2 are scalars, define $c_1T_1 + c_2T_2$ by

$$(c_1T_1 + c_2T_2)\boldsymbol{x} = c_1T_1\boldsymbol{x} + c_2T_2\boldsymbol{x} \quad (\boldsymbol{x} \in X).$$

It is clear that $c_1T_1 + c_2T_2 \in L(X, Y)$.

If X, Y, Z are vector spaces, and if $T \in L(X, Y)$ and $S \in L(Y, Z)$, we define their product ST to be the composition of T and S:

$$(ST)\boldsymbol{x} = S(T\boldsymbol{x}) \quad (\boldsymbol{x} \in X). \tag{1.2.2}$$

Then $ST \in L(X, Z)$.

Definition 1.2.3. For $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, define the *norm* $\|A\|$ of A to be the sup of all numbers $|A\boldsymbol{x}|$, where $\boldsymbol{x}$ ranges over all vectors in $\mathbb{R}^n$ with $|\boldsymbol{x}| \leq 1$.

Proposition 1.2.4. *(a) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\|A\| < \infty$ and A is a uniformly continuous mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$.*

(b) If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and c is a scalar, then

$$\|A + B\| \leq \|A\| + \|B\|, \quad \|cA\| = |c|\,\|A\|.$$

With the distance between A and B defined as $\|A - B\|$, $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space.

(c) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$, then

$$\|BA\| \leq \|B\|\,\|A\|.$$

Proof. Exercise. ■

Proposition 1.2.5. *Let $GL(\mathbb{R}^n) = \{A \in L(\mathbb{R}^n) : A \text{ is invertible}\}$.*

(a) If $A \in GL(\mathbb{R}^n)$, $B \in L(\mathbb{R}^n)$, and $\|B - A\|\,\|A^{-1}\| < 1$, then $B \in GL(\mathbb{R}^n)$.

(b) $GL(\mathbb{R}^n)$ is an open subset of $L(\mathbb{R}^n)$, and the mapping $A \to A^{-1}$ is continuous on $GL(\mathbb{R}^n)$.

Proof. (a) Put $\|A^{-1}\| = 1/\alpha$, put $\|B - A\| = \beta$. Then $\beta < \alpha$. For every $\boldsymbol{x} \in \mathbb{R}^n$,

$$\begin{aligned} \alpha|\boldsymbol{x}| &= \alpha|A^{-1}A\boldsymbol{x}| \leq \alpha\|A^{-1}\|\,|A\boldsymbol{x}| \\ &= |A\boldsymbol{x}| \leq |(A - B)\boldsymbol{x}| + |B\boldsymbol{x}| \leq \beta|\boldsymbol{x}| + |B\boldsymbol{x}|, \end{aligned}$$

so that

$$(\alpha - \beta)|\boldsymbol{x}| \leq |B\boldsymbol{x}| \qquad (\boldsymbol{x} \in \mathbb{R}^n). \tag{1.2.6}$$

Since $\alpha - \beta > 0$, (1.2.6) shows that $B\boldsymbol{x} \neq \boldsymbol{0}$ if $\boldsymbol{x} \neq 0$. Hence B is one-to-one, and thus B is invertible.

(b) Next, replace $\boldsymbol{x}$ by $B^{-1}\boldsymbol{y}$ in (1.2.6). The resulting inequality

$$(\alpha - \beta)|B^{-1}\boldsymbol{y}| \leq |BB^{-1}\boldsymbol{y}| = |\boldsymbol{y}|$$

shows that $\|B^{-1}\| \leq (\alpha - \beta)^{-1}$. The identity

$$B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$$

combined with proposition 1.2.4(c) implies therefore that

$$\|B^{-1} - A^{-1}\| \leq \|B^{-1}\| \|A - B\| \|A^{-1}\| \leq \frac{\beta}{\alpha(\alpha - \beta)}.$$

This establishes the continuity assertion, since $\beta \to 0$ as $B \to A$. ■

1.3 Differentiation

Definition 1.3.1. Suppose U is an open subset of $\mathbb{R}^n$, $\boldsymbol{f}$ maps U into $\mathbb{R}^m$, and $\boldsymbol{x} \in U$. If there is any linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\boldsymbol{h} \to \boldsymbol{0}} \frac{|\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - L\boldsymbol{h}|}{|\boldsymbol{h}|} = \boldsymbol{0}, \tag{1.3.2}$$

then we say that $\boldsymbol{f}$ is differentiable at $\boldsymbol{x}$ and we write $D\boldsymbol{f}(\boldsymbol{x}) = L$. If $\boldsymbol{f}$ is differentiable at every $\boldsymbol{x} \in U$, then we say that $\boldsymbol{f}$ is differentiable in U.

Proposition 1.3.3. *If* (1.3.2) *holds with* $L = L_1$ *and* $L = L_2$, *then* $L_1 = L_2$. *Therefore, if* $\boldsymbol{f}$ *is differentiable at* $\boldsymbol{x}$, *then* $D\boldsymbol{f}(\boldsymbol{x})$ *is uniquely determined.*

Notation 1.3.4. When $n = 1$, then $D\boldsymbol{f}(\boldsymbol{x}) \in L(\mathbb{R}^1, \mathbb{R}^m)$ can be identified with a vector $D\boldsymbol{f}(\boldsymbol{x})\,1 \in \mathbb{R}^m$ and we often use the notation $\boldsymbol{f}'(\boldsymbol{x})$ instead of $D\boldsymbol{f}(\boldsymbol{x})$.

Definition 1.3.5. We call $D\boldsymbol{f}(\boldsymbol{x})$ the derivative of $\boldsymbol{f}$ at $\boldsymbol{x}$; $D\boldsymbol{f}(\boldsymbol{x})$ is often called the differential of $\boldsymbol{f}$ at $\boldsymbol{x}$, or the total derivative of $\boldsymbol{f}$ at $\boldsymbol{x}$.

The relation (1.3.2) can be written in the form

$$\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) = D\boldsymbol{f}(\boldsymbol{x})\,\boldsymbol{h} + \boldsymbol{r}(\boldsymbol{h}),$$

where the remainder $\boldsymbol{r}(\boldsymbol{h})$ satisfies

$$\lim_{\boldsymbol{h} \to \boldsymbol{0}} \frac{|\boldsymbol{r}(\boldsymbol{h})|}{|\boldsymbol{h}|} = \boldsymbol{0}.$$

Also, the relation (1.3.2) may be written by saying that for every $\epsilon > 0$, there is a $\delta > 0$ such that $\boldsymbol{x} \in U$ and $|\boldsymbol{x} - \boldsymbol{x}_0| < \delta$ implies

$$|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0) - D\boldsymbol{f}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)| \leq \epsilon |\boldsymbol{x} - \boldsymbol{x}_0|.$$

Intuitively, $\boldsymbol{x} \mapsto \boldsymbol{f}(\boldsymbol{x}_0) + D\boldsymbol{f}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)$ is supposed to be the *best affine approximation* to $\boldsymbol{f}$ near the point $\boldsymbol{x}_0$.

Example 1.3.6. If $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $\boldsymbol{x} \in \mathbb{R}^m$, then $DT(\boldsymbol{x}) = T$.

Proposition 1.3.7. *Suppose* $U \subset \mathbb{R}^n$ *is open and* $\boldsymbol{f} : U \to \mathbb{R}^m$ *is differentiable at* $\boldsymbol{x}_0 \in U$. *Then* $\boldsymbol{f}$ *is continuous at* $\boldsymbol{x}_0$. *In fact, there are a constant* $M > 0$ *and a* $\delta_0 > 0$ *such that* $|\boldsymbol{x} - \boldsymbol{x}_0| < \delta_0$ *implies* $|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)| \leq M |\boldsymbol{x} - \boldsymbol{x}_0|$.

Proof. By using ϵ-δ formulation of (1.3.2), we see that there is a δ_0 (corresponding to $\epsilon = 1$) so that $|\boldsymbol{x} - \boldsymbol{x}_0| < \delta_0$ implies

$$|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0) - D\boldsymbol{f}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)| \leq |\boldsymbol{x} - \boldsymbol{x}_0|,$$

which implies

$$|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)| \leq |D\boldsymbol{f}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)| + |\boldsymbol{x} - \boldsymbol{x}_0|.$$

Let $M = \|D\boldsymbol{f}(\boldsymbol{x}_0)\| + 1$. ■

Theorem 1.3.8 (The chain rule)**.** *Suppose U is an open set in $\mathbb{R}^n$, $\boldsymbol{f}$ maps U into $\mathbb{R}^m$, $\boldsymbol{f}$ is differentiable at $\boldsymbol{x}_0 \in U$, $\boldsymbol{g}$ maps an open set containing $\boldsymbol{f}(U)$ into $\mathbb{R}^k$, and $\boldsymbol{g}$ is differentiable at $\boldsymbol{f}(\boldsymbol{x}_0)$. Then the mapping $\boldsymbol{F}$ of U into $\mathbb{R}^k$ defined by*

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))$$

is differentiable at $\boldsymbol{x}_0$, and

$$D\boldsymbol{F}(\boldsymbol{x}_0) = D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0))\, D\boldsymbol{f}(\boldsymbol{x}_0). \tag{1.3.9}$$

On the right side of (1.3.9), we have the product of two linear transformations, as defined in (1.2.2).

Proof. By the triangle inequality,

$$\begin{aligned}
&|\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x}_0) - D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0))\, D\boldsymbol{f}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)| \\
&\qquad \leq |\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0)) - D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0))(\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0))| \\
&\qquad\quad + |D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0))\,(\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0) - D\boldsymbol{f}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0))|.
\end{aligned}$$

By proposition 1.3.7, there exist a δ_0 and an $M > 0$ such that

$$|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)| \leq M|\boldsymbol{x} - \boldsymbol{x}_0| \quad \text{whenever} \quad |\boldsymbol{x} - \boldsymbol{x}_0| < \delta_0.$$

By the definition of the derivative of $\boldsymbol{g}$, given $\epsilon > 0$, there is a $\delta_1 > 0$ such that $|\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}_0)| < \delta_1$ implies

$$|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0)) - D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0))(\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}_0))| \leq \frac{\epsilon}{2M}\,|\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}_0)|.$$

Thus $|\boldsymbol{x} - \boldsymbol{x}_0| < \delta_2 = \min(\delta_0, \delta_1/M)$ implies

$$|\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0)) - D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0))(\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0))| \leq \frac{\epsilon}{2}|\boldsymbol{x} - \boldsymbol{x}_0|.$$

On the other hand, by the definition of derivative of $\boldsymbol{f}$, there is a $\delta_3 > 0$ such that $|\boldsymbol{x} - \boldsymbol{x}_0| < \delta_3$ implies

$$|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0) - D\boldsymbol{f}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)| \leq \frac{\epsilon}{(2\|D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0))\| + 1)}\,|\boldsymbol{x} - \boldsymbol{x}_0|,$$

Then $|\boldsymbol{x} - \boldsymbol{x}_0| < \delta_3$ implies

$$|D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0))[\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0) - D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0))(\boldsymbol{x} - \boldsymbol{x}_0)]| \leq \frac{\epsilon}{2}|\boldsymbol{x} - \boldsymbol{x}_0|.$$

Let $\delta = \min(\delta_2, \delta_3)$. Thus, $|\boldsymbol{x} - \boldsymbol{x}_0| < \delta$ implies

$$|\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x}_0) - D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0))\, D\boldsymbol{f}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)| \leq \epsilon|\boldsymbol{x} - \boldsymbol{x}_0|.$$ ■

Definition 1.3.10. We again consider a function $\boldsymbol{f} : U \subset \mathbb{R}^n \to \mathbb{R}^m$. Let

$$\boldsymbol{f}(\boldsymbol{x}) = [f_1(\boldsymbol{x}) \cdots f_m(\boldsymbol{x})]^\top.$$

For $\boldsymbol{x} \in U$, $1 \leq i \leq m$, $1 \leq j \leq n$, we define

$$D_j f_i(\boldsymbol{x}) = \lim_{t \to 0} \frac{f_i(\boldsymbol{x} + t\boldsymbol{e}_j) - f_i(\boldsymbol{x})}{t}$$

provided the limit exists. The notation $\dfrac{\partial f_i}{\partial x_j}$ is often used in place of $D_j f_i$, and $D_j f_i$ is called a *partial derivative.*

Theorem 1.3.11. *Suppose $\boldsymbol{f}$ maps an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and $\boldsymbol{f}$ is differentiable at a point $\boldsymbol{x} \in U$. Then the partial derivatives $D_j f_i(\boldsymbol{x})$ exist, and if $[D\boldsymbol{f}(\boldsymbol{x})]$ is the matrix that represents $D\boldsymbol{f}(\boldsymbol{x})$ with respect to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$, we have*

$$[D\boldsymbol{f}(\boldsymbol{x})] = \begin{bmatrix} D_1 f_1(\boldsymbol{x}) & \cdots & D_n f_1(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ D_1 f_m(\boldsymbol{x}) & \cdots & D_n f_m(\boldsymbol{x}) \end{bmatrix}. \tag{1.3.12}$$

This matrix is called the Jacobian matrix *of $\boldsymbol{f}$.*

Proof. Fix j. Since $\boldsymbol{f}$ is differentiable at $\boldsymbol{x}$,

$$\boldsymbol{f}(\boldsymbol{x} + t\boldsymbol{e}_j) - \boldsymbol{f}(\boldsymbol{x}) = D\boldsymbol{f}(\boldsymbol{x})(t\boldsymbol{e}_j) + \boldsymbol{r}(t\boldsymbol{e}_j),$$

where $|\boldsymbol{r}(t\boldsymbol{e}_j)|/t \to \boldsymbol{0}$ as $t \to 0$. The linearity of $D\boldsymbol{f}(\boldsymbol{x})$ shows therefore that

$$\lim_{t \to 0} \frac{\boldsymbol{f}(\boldsymbol{x} + t\boldsymbol{e}_j) - \boldsymbol{f}(\boldsymbol{x})}{t} = D\boldsymbol{f}(\boldsymbol{x})\boldsymbol{e}_j. \tag{1.3.13}$$

If we now represent $\boldsymbol{f}$ in terms of its components, then (1.3.13) becomes

$$\lim_{t \to 0} \left[\frac{f_1(\boldsymbol{x} + t\boldsymbol{e}_j) - f_1(\boldsymbol{x})}{t} \quad \cdots \quad \frac{f_m(\boldsymbol{x} + t\boldsymbol{e}_j) - f_m(\boldsymbol{x})}{t} \right]^\top = D\boldsymbol{f}(\boldsymbol{x})\boldsymbol{e}_j.$$

It follows that each quotient in this column vector has a limit, as $t \to 0$, so that each $(D_j f_i)(\boldsymbol{x})$ exists, and then (1.3.12) follows from (1.3.13). ■

Proposition 1.3.14. *Suppose $\boldsymbol{f}$ maps a convex open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^m$, $\boldsymbol{f}$ is differentiable in U, and there is a real number M such that*

$$\|D\boldsymbol{f}(\boldsymbol{x})\| \leq M$$

for every $\boldsymbol{x} \in U$. Then

$$|\boldsymbol{f}(\boldsymbol{b}) - \boldsymbol{f}(\boldsymbol{a})| \leq M|\boldsymbol{b} - \boldsymbol{a}|$$

for all $\boldsymbol{a} \in U$, $\boldsymbol{b} \in U$.

Proof. Fix $\boldsymbol{a} \in U$, $\boldsymbol{b} \in U$. Define $\boldsymbol{g}(t) = \boldsymbol{f}(t\boldsymbol{b} + (1-t)\boldsymbol{a})$ for $0 \le t \le 1$. Then

$$\boldsymbol{g}'(t) = D\boldsymbol{g}(t) = D\boldsymbol{f}(t\boldsymbol{b} + (1-t)\boldsymbol{a})(\boldsymbol{b} - \boldsymbol{a}).$$

so that

$$|\boldsymbol{g}'(t)| \le \|D\boldsymbol{f}(t\boldsymbol{b} + (1-t)\boldsymbol{a})\|\, |\boldsymbol{b} - \boldsymbol{a}| \le M|\boldsymbol{b} - \boldsymbol{a}|$$

Then

$$|\boldsymbol{f}(\boldsymbol{b}) - \boldsymbol{f}(\boldsymbol{a})| = |\boldsymbol{g}(1) - \boldsymbol{g}(0)| \le \int_0^1 |\boldsymbol{g}'(t)|\, dt \le M|\boldsymbol{b} - \boldsymbol{a}|. \qquad \blacksquare$$

Definition 1.3.15. A differentiable mapping $\boldsymbol{f}$ of an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is said to be continuously differentiable in U if $D\boldsymbol{f}$ is a continuous mapping of U into $L(\mathbb{R}^n, \mathbb{R}^m)$. More explicitly, it is required that for every $\boldsymbol{x} \in U$ and every $\epsilon > 0$ corresponds a $\delta > 0$ such that

$$\|D\boldsymbol{f}(\boldsymbol{x}) - D\boldsymbol{f}(\boldsymbol{y})\| < \epsilon$$

if $\boldsymbol{y} \in U$ and $|\boldsymbol{x} - \boldsymbol{y}| < \delta$. If this is so, we also say that $\boldsymbol{f}$ is a C^1-mapping, or that $\boldsymbol{f} \in C^1(U)$.

Lemma 1.3.16. *If X is a metric space, if $a_{11}, \dots, a_{mn}$ are real continuous functions on X, and if, for each $x \in X$, A_x is the linear transformation of $\mathbb{R}^n$ into $\mathbb{R}^m$ whose matrix has entries $a_{ij}(x)$, then the mapping $x \mapsto A_x$ is a continuous mapping of X into $L(\mathbb{R}^n, \mathbb{R}^m)$.*

Proof. Exercise. $\blacksquare$

Theorem 1.3.17. *Suppose $\boldsymbol{f}$ maps an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Then $\boldsymbol{f} \in C^1(U)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on U for $1 \le i \le m$, $1 \le j \le n$.*

Proof. Assume first that $\boldsymbol{f} \in C^1(U)$. By (1.3.12), we have

$$|D_j f_i(\boldsymbol{y}) - D_j f_i(\boldsymbol{x})| \le |D_j \boldsymbol{f}(\boldsymbol{x}) - D_j \boldsymbol{f}(\boldsymbol{y})| \le \|D\boldsymbol{f}(\boldsymbol{y}) - D\boldsymbol{f}(\boldsymbol{x})\|.$$

for all i, j, and for all $\boldsymbol{x} \in U$. Hence $D_j f_i$ is continuous. For the converse, it suffices to consider the case $m = 1$. Fix $\boldsymbol{x} \in U$ and $\epsilon > 0$. Since U is open, there is an open ball $B \subset U$, with center at $\boldsymbol{x}$ and radius r, and the continuity of the functions $D_j f$ shows that r can be chosen so that

$$|D_j f(\boldsymbol{y}) - D_j f(\boldsymbol{x})| < \frac{\epsilon}{n} \qquad (\boldsymbol{y} \in B,\ 1 \le j \le n).$$

Suppose $\boldsymbol{h} = [h_1 \ \cdots \ h_n]^\top$, $|\boldsymbol{h}| < r$, put $\boldsymbol{v}_0 = \boldsymbol{0}$, and $\boldsymbol{v}_k = [h_1 \ \cdots \ h_k \ 0 \ \cdots \ 0]^\top$ for $1 \le k < n$. Then

$$f(\boldsymbol{x} + \boldsymbol{h}) - f(\boldsymbol{x}) = \sum_{j=1}^n [f(\boldsymbol{x} + \boldsymbol{v}_j) - f(\boldsymbol{x} + \boldsymbol{v}_{j-1})].$$

Since $|\boldsymbol{v}_k| < r$ for $1 \le k \le n$ and since B is convex, the segments with end points $\boldsymbol{x} + \boldsymbol{v}_{j-1}$ and $\boldsymbol{x} + \boldsymbol{v}_j$ lie in B. Since $\boldsymbol{v}_j = \boldsymbol{v}_{j-1} + h_j \boldsymbol{e}_j$, the mean value theorem shows that

$$f(\boldsymbol{x} + \boldsymbol{v}_j) - f(\boldsymbol{x} + \boldsymbol{v}_{j-1}) = h_j D_j f(\boldsymbol{x} + \boldsymbol{v}_{j-1} + \theta_j h_j \boldsymbol{e}_j)$$

for some $\theta_j \in (0,1)$. Therefore,

$$\left| f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - \sum_{j=1}^{n} h_j D_j f(\boldsymbol{x}) \right| \leq \sum_{j=1}^{n} |h_j| \frac{\epsilon}{n} \leq |\boldsymbol{h}|\epsilon$$

for all $\boldsymbol{h}$ such that $|\boldsymbol{h}| < r$. This says that f is differentiable at $\boldsymbol{x}$ and that $Df(\boldsymbol{x})$ has the matrix representation

$$[Df(\boldsymbol{x})] = [D_1 f(\boldsymbol{x}) \ \cdots \ D_n f(\boldsymbol{x})].$$

Since $D_1 f, \ldots, D_n f$ are continuous functions on U, lemma 1.3.16 shows that $f \in C^1(U)$. ■

Definition 1.3.18. Suppose f is a real function defined on an open set $U \subset \mathbb{R}^n$, with partial derivatives $D_1 f, \ldots, D_n f$. If the functions $D_i f$ are themselves differentiable, then the second-order partial derivatives of f are defined by

$$D_{ij} f = D_i D_j f \quad (i, j = 1, \ldots, n).$$

If all these functions $D_{ij} f$ are continuous in U, we say that f is of class C^2 in U. By induction, $C^k(U)$ is defined as follows, for all integers $k \geq 2$: To say that $f \in C^k(U)$ means that the partial derivatives $D_1 f, \ldots, D_n f$ belong to $C^{k-1}(U)$. A mapping $\boldsymbol{f}$ of U into $\mathbb{R}^m$ is said to be of class C^k if each component of $\boldsymbol{f}$ is of class C^k.

Theorem 1.3.19. *$D_{ij} f = D_{ji} f$ if $f \in C^2(U)$.*

Proof. We may assume without loss of generality that $i = 1$, $j = 2$ and $U \subset \mathbb{R}^2$. Suppose $Q \subset U$ is a closed rectangle with sides parallel to the coordinate axes, having (a, b) and $(a+h, b+k)$ as opposite vertices ($h \neq 0$, $k \neq 0$). Put

$$\Delta(f, Q) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$$

and $u(t) = f(t, b+k) - f(t, b)$ so that

$$\Delta(f, Q) = u(a+h) - u(a).$$

Two applications of the mean value theorem show that there is an x between a and $a+h$, and that there is a y between b and $b+k$, such that

$$\Delta(f, Q) = hu'(x) = h[D_1 f(x, b+k) - D_1 f(x, b)] = hk D_{21} f(x, y).$$

Choose $\epsilon > 0$. If h and k are sufficiently small, we have

$$|D_{21} f(a, b) - D_{21} f(x, y)| < \epsilon$$

for all $(x, y) \in Q$. Thus

$$\left| \frac{\Delta(f, Q)}{hk} - D_{21} f(a, b) \right| < \epsilon.$$

Fix h, and let $k \to 0$. Then the last inequality implies that

$$\left| \frac{D_2 f(a+h, b) - D_2 f(a, b)}{h} - D_{21} f(a, b) \right| \leq \epsilon.$$

Since ϵ was arbitrary, and since the above inequality holds for all sufficiently small $h \neq 0$, it follows that $D_{12} f(a, b) = D_{21} f(a, b)$. ■

1.4 Inverse function theorem

Definition 1.4.1. Let X be a metric space, with metric d. A mapping $T : X \to X$ is a *contraction mapping* or *contraction*, if there exists a constant c, with $0 \le c < 1$, such that

$$d(T(x), T(y)) \le c\, d(x, y)$$

for all $x, y \in X$.

Lemma 1.4.2 (Contraction mapping principle)**.** *Theorem If X is a complete metric space, and if $T : X \to X$ is a contraction, then there exists one and only one $x \in X$ such that $T(x) = x$.*

Proof. Pick $x_0 \in X$ arbitrarily, and define x_n recursively, by setting

$$x_{n+1} = T(x_n) \qquad (n = 0, 1, 2, \ldots).$$

For $n \ge 1$ we then have

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \le d(x_n, x_{n-1}).$$

Hence induction gives

$$d(x_{n+1}, x_n) = c^n d(x_1, x_0) \qquad (n = 0, 1, 2, \ldots).$$

If $n < m$, it follows that

$$\begin{aligned} d(x_n, x_m) &\le \sum_{i=n+1}^{m} d(x_i, x_{i-1}) \le (c^n + c^{n+1} + \cdots + c^{m-1}) d(x_1, x_0) \\ &\le \frac{c^n}{1-c} d(x_1, x_0). \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\lim_{n\to\infty} x_n = x$ for some $x \in X$. Since T is a contraction, T is continuous on X. Hence

$$T(x) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

The uniqueness is a triviality, for if $T(x) = x$ and $T(y) = y$, then

$$d(x, y) = d(T(x), T(y)) \le cd(x, y),$$

which can only happen when $d(x, y) = 0$. ■

The inverse function theorem states, roughly speaking, that a continuously differentiable mapping $\boldsymbol{f}$ is invertible in a neighborhood of any point $\boldsymbol{x}$ at which the linear transformation $D\boldsymbol{f}(\boldsymbol{x})$ is invertible:

Theorem 1.4.3 (Inverse function theorem)**.** *Suppose $\boldsymbol{f}$ is a C^1-mapping of an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $D\boldsymbol{f}(\boldsymbol{a})$ is invertible for some $\boldsymbol{a} \in U$, and $\boldsymbol{b} = \boldsymbol{f}(\boldsymbol{a})$. Then*

(a) there exist open sets V and W in $\mathbb{R}^n$ such that $\boldsymbol{a} \in V$, $\boldsymbol{b} \in W$, $\boldsymbol{f}$ is one-to-one on V, and $\boldsymbol{f}(V) = W$;

(b) if $\boldsymbol{g}$ is the inverse of $\boldsymbol{f}$ defined in W by

$$\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x})) = \boldsymbol{x} \qquad (\boldsymbol{x} \in W),$$

then $\boldsymbol{g} \in C^1(W)$.

Writing the equation $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ in component form, we arrive at the following interpretation of the conclusion of the theorem: The system of n equations

$$\begin{gathered} y_1 = f_1(x_1, \ldots, x_n) \\ \vdots \\ y_n = f_n(x_1, \ldots, x_n) \end{gathered}$$

can be solved for $x_1, \ldots, x_n$ in terms of $y_1, \ldots, y_n$, if we restrict $\boldsymbol{x}$ and $\boldsymbol{y}$ to small enough neighborhoods of $\boldsymbol{a}$ and $\boldsymbol{b}$; the solutions are unique and continuously differentiable.

Proof (a). Put $D\boldsymbol{f}(\boldsymbol{a}) = A$, and choose λ so that $2\lambda\|A^{-1}\| = 1$. Since $D\boldsymbol{f}$ is continuous at $\boldsymbol{a}$, there is an open convex set $V \subset U$ containing $\boldsymbol{a}$, such that

$$\|D\boldsymbol{f}(\boldsymbol{x}) - A\| < \lambda \qquad (\boldsymbol{x} \in V).$$

We associate to each $\boldsymbol{y} \in \mathbb{R}^n$ a function T_y, defined by

$$T_y(\boldsymbol{x}) = \boldsymbol{x} + A^{-1}(\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x})) \qquad (\boldsymbol{x} \in V).$$

Note that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}$ if and only if $\boldsymbol{x}$ is a fixed point of T_y. Since

$$DT_y(\boldsymbol{x}) = I - A^{-1}D\boldsymbol{f}(\boldsymbol{x}) = A^{-1}(A - D\boldsymbol{f}(\boldsymbol{x})),$$

we have $\|DT_y(\boldsymbol{x})\| < \frac{1}{2}$ for all $\boldsymbol{x} \in V$. Hence

$$|T_y(\boldsymbol{x}_1) - T_y(\boldsymbol{x}_2)| \le \frac{1}{2}|\boldsymbol{x}_1 - \boldsymbol{x}_2| \qquad (\boldsymbol{x}_1, \boldsymbol{x}_2 \in V) \tag{1.4.4}$$

by proposition 1.3.14. It follows that T_y has at most one fixed point in V, so that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}$ for at most one $\boldsymbol{x} \in V$. Thus $\boldsymbol{f}$ is one-to-one in V. Next, put $W = \boldsymbol{f}(V)$. Clearly, $\boldsymbol{b} \in W$ and the proof of part (a) of the theorem is complete once we show that W is open. Pick $\boldsymbol{y}_0 \in W$. Then $\boldsymbol{y}_0 = \boldsymbol{f}(\boldsymbol{x}_0)$ for some $\boldsymbol{x}_0 \in V$. Let B be an open ball with center at $\boldsymbol{x}_0$ and radius $r > 0$, so small that its closure $\bar{B}$ lies in V. We will show that $\boldsymbol{y} \in W$ whenever $|\boldsymbol{y} - \boldsymbol{y}_0| < \lambda r$. This proves, of course, that W is open.

For any $\boldsymbol{y}$ satisfying $|\boldsymbol{y} - \boldsymbol{y}_0| < \lambda r$, we have

$$|T_y(\boldsymbol{x}_0) - \boldsymbol{x}_0| = |A^{-1}(\boldsymbol{y} - \boldsymbol{y}_0)| < \|A^{-1}\|\lambda r = \frac{r}{2}.$$

If $\boldsymbol{x} \in \bar{B}$, it therefore follows from (1.4.4) that

$$|T_y(\boldsymbol{x}) - \boldsymbol{x}_0| \le |T_y(\boldsymbol{x}) - T_y(\boldsymbol{x}_0)| + |T_y(\boldsymbol{x}_0) - \boldsymbol{x}_0| < \frac{1}{2}|\boldsymbol{x} - \boldsymbol{x}_0| + \frac{r}{2} \le r;$$

hence $T_y(\boldsymbol{x}) \in B$. Note that (1.4.4) holds if $\boldsymbol{x}_1 \in \bar{B}$, $\boldsymbol{x}_2 \in \bar{B}$. Thus T_y is a contraction of $\bar{B}$ into $\bar{B}$. Therefore, lemma 1.4.2 implies that T_y has a fixed point $\boldsymbol{x} \in \bar{B}$. For this $\boldsymbol{x}$, $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}$. Thus $\boldsymbol{y} \in \boldsymbol{f}(\bar{B}) \subset \boldsymbol{f}(V) = W$.

(b). Pick $\boldsymbol{y} \in W$, $\boldsymbol{y}+\boldsymbol{k} \in W$. Then there exist $\boldsymbol{x} \in V$, $\boldsymbol{x}+\boldsymbol{h} \in V$, so that $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{y}+\boldsymbol{k} = \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})$. Since

$$T_y(\boldsymbol{x}+\boldsymbol{h}) - T_y(\boldsymbol{x}) = \boldsymbol{h} + A^{-1}[\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})] = \boldsymbol{h} - A^{-1}\boldsymbol{k},$$

by (1.4.4), we have $|\boldsymbol{h} - A^{-1}\boldsymbol{k}| \le \frac{1}{2}|\boldsymbol{h}|$. Hence $|A^{-1}\boldsymbol{k}| \ge \frac{1}{2}|\boldsymbol{h}|$, and

$$|\boldsymbol{h}| \le 2\|A^{-1}\|\,|\boldsymbol{k}| = \lambda^{-1}|\boldsymbol{k}|. \tag{1.4.5}$$

By proposition 1.2.5, $D\boldsymbol{f}(\boldsymbol{x})$ has an inverse. Since

$$\begin{aligned}\boldsymbol{g}(\boldsymbol{y}+\boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{y}) - (D\boldsymbol{f}(\boldsymbol{x}))^{-1}\boldsymbol{k} &= \boldsymbol{h} - (D\boldsymbol{f}(\boldsymbol{x}))^{-1}\boldsymbol{k} \\ &= -(D\boldsymbol{f}(\boldsymbol{x}))^{-1}[\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - D\boldsymbol{f}(\boldsymbol{x})\boldsymbol{h}],\end{aligned}$$

we have by (1.4.5)

$$\begin{aligned}&\frac{|\boldsymbol{g}(\boldsymbol{y}+\boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{y}) - (D\boldsymbol{f}(\boldsymbol{x}))^{-1}\boldsymbol{k}|}{|\boldsymbol{k}|} \\ &\qquad\le \frac{\|D\boldsymbol{f}(\boldsymbol{x})^{-1}\|}{\lambda}\,\frac{|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - D\boldsymbol{f}(\boldsymbol{x})\boldsymbol{h}|}{|\boldsymbol{h}|}.\end{aligned}$$

As $\boldsymbol{k} \to \boldsymbol{0}$, (1.4.5) shows that $\boldsymbol{h} \to \boldsymbol{0}$. The right hand side of the last inequality thus tends to 0. Hence the same is true of the left. We have thus proved that

$$D\boldsymbol{g}(\boldsymbol{y}) = (D\boldsymbol{f}(\boldsymbol{x}))^{-1} = \{D\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{y}))\}^{-1} \qquad (\boldsymbol{y} \in W).$$

Finally, note that $\boldsymbol{g}$ is a continuous mapping of W onto V (since $\boldsymbol{g}$ is differentiable), that $D\boldsymbol{f}$ is a continuous mapping of V into the set $GL(\mathbb{R}^n)$ of all invertible elements of $L(\mathbb{R}^n)$, and that inversion is a continuous mapping of $GL(\mathbb{R}^n)$ onto $GL(\mathbb{R}^n)$, by proposition 1.2.5. If we combine these facts, we see that $\boldsymbol{g} \in C^1(W)$. ■

Corollary 1.4.6. *If $\boldsymbol{f}$ is a C^1-mapping of an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and if $D\boldsymbol{f}(\boldsymbol{x})$ is invertible for every $\boldsymbol{x} \in U$, then $\boldsymbol{f}(V)$ is an open subset of $\mathbb{R}^n$ for every open set $V \subset U$. In other words, $\boldsymbol{f}$ is an open mapping of U into $\mathbb{R}^n$.*

1.5 Implicit function theorem

Notation 1.5.1. If $\boldsymbol{x} = [x_1 \ \cdots \ x_n]^\top \in \mathbb{R}^n$ and $\boldsymbol{y} = [y_1 \ \cdots \ y_m]^\top \in \mathbb{R}^m$, let us write $(\boldsymbol{x}, \boldsymbol{y}) = [\boldsymbol{x} \ \ \boldsymbol{y}]^\top$ for the vector $[x_1 \ \cdots \ x_n \ \ y_1 \ \cdots \ y_m]^\top \in \mathbb{R}^{n+m}$. In what follows, the first entry in $[\boldsymbol{x} \ \ \boldsymbol{y}]^\top$ or in a similar symbol will always be a vector in $\mathbb{R}^n$, the second will be a vector in $\mathbb{R}^m$.

Suppose $\boldsymbol{f}$ maps an open set $U \subset \mathbb{R}^{n+m}$ into $\mathbb{R}^m$ and $\boldsymbol{f}$ is differentiable at $(\boldsymbol{a}, \boldsymbol{b}) \in U$. We split $D\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) \in L(\mathbb{R}^{n+m}, \mathbb{R}^m)$ into two two linear transformation $D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})$ and $D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})$, defined by

$$D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})\,\boldsymbol{h} = D\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) \begin{bmatrix} \boldsymbol{h} \\ \boldsymbol{0} \end{bmatrix}, \quad D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})\,\boldsymbol{k} = D\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{k} \end{bmatrix}$$

for $\boldsymbol{h} \in \mathbb{R}^n$, $\boldsymbol{k} \in \mathbb{R}^m$. Then $D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) \in L(\mathbb{R}^m)$. In matrix representation with respect to the standard bases, we have

$$[D\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})] = \left[\ [D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})] \ \middle|\ [D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})]\ \right].$$

We shall denote by I_n the $n \times n$ identity matrix, and by $0_{m\times n}$ the zero $m \times n$ matrix.

Theorem 1.5.2 (Implicit function theorem)**.** *Let $\boldsymbol{f}$ be a C^1-mapping of an open set $U \subset \mathbb{R}^{n+m}$ into $\mathbb{R}^m$, such that $\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{0}$ for some point $(\boldsymbol{a}, \boldsymbol{b}) \in U$. If $D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})$ is invertible, then there exists open sets $V \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^n$, with $(\boldsymbol{a}, \boldsymbol{b}) \in V$ and $\boldsymbol{a} \in W$, having the following property: To every $\boldsymbol{x} \in W$ corresponds a unique $\boldsymbol{y} \in \mathbb{R}^m$ such that*

$$(\boldsymbol{x}, \boldsymbol{y}) \in V \quad \textit{and} \quad \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}.$$

If this $\boldsymbol{y}$ is defined to be $\boldsymbol{g}(\boldsymbol{x})$, then $\boldsymbol{g}$ is a C^1-mapping of W into $\mathbb{R}^m$, $\boldsymbol{g}(\boldsymbol{a}) = \boldsymbol{b}$,

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})) = \boldsymbol{0} \qquad (\boldsymbol{x} \in W), \tag{1.5.3}$$

and

$$D\boldsymbol{g}(\boldsymbol{a}) = -(D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}))^{-1} D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}).$$

The function $\boldsymbol{g}$ is "implicitly" defined by (1.5.3). The equation $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$ can be written as a system of m equations in $n + m$ variables:

$$\begin{aligned} f_1(x_1, \dots, x_n, y_1, \dots, y_m) &= 0 \\ &\vdots \\ f_m(x_1, \dots, x_n, y_1, \dots, y_m) &= 0 \end{aligned} \tag{1.5.4}$$

The assumption that $D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})$ is invertible means that the $m \times m$ matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_m} \end{bmatrix}$$

calculated at $(\boldsymbol{a}, \boldsymbol{b})$ defines an invertible linear operator on $\mathbb{R}^m$. If, furthermore, (1.5.4) holds when $\boldsymbol{x} = \boldsymbol{a}$ and $\boldsymbol{y} = \boldsymbol{b}$, then the conclusion of the theorem is that (1.5.4) can be solved for $y_1, \dots, y_m$ in terms of $x_1, \dots, x_n$ for every $\boldsymbol{x}$ near $\boldsymbol{a}$, and that these solutions are continuously differentiable functions of $\boldsymbol{x}$.

Proof. Define $\boldsymbol{F}$ by

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \end{bmatrix} \quad \text{for } \ (\boldsymbol{x}, \boldsymbol{y}) \in U.$$

Then $\boldsymbol{F}$ is a C^1-mapping of U into $\mathbb{R}^{n+m}$. We claim that $D\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{b})$ is an invertible element of $L(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$: By theorem 1.3.11, we have

$$[D\boldsymbol{F}] = \left[\begin{array}{c|c} I_n & 0_{n\times m} \\ [D_{\boldsymbol{x}}\boldsymbol{f}] & [D_{\boldsymbol{y}}\boldsymbol{f}] \end{array} \right]$$

and thus $D\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{b})$ is invertible if and only if $D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})$ is invertible.

The inverse function theorem can therefore be applied to $\boldsymbol{F}$. It shows that there exist open sets V and V' in $\mathbb{R}^{n+m}$, with $(\boldsymbol{a}, \boldsymbol{b}) \in V$, $(\boldsymbol{a}, \boldsymbol{0}) \in V'$, such that $\boldsymbol{F}$ is a one-to-one mapping of V onto V'. We let W be the set of all $\boldsymbol{x} \in \mathbb{R}^n$

such that $(\boldsymbol{x}, \mathbf{0}) \in V'$. Note that $\boldsymbol{a} \in W$. It is clear that W is open since V' is open. If $\boldsymbol{x} \in W$, then $(\boldsymbol{x}, \mathbf{0}) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y})$ for some $(\boldsymbol{x}, \boldsymbol{y}) \in V$. Then, $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{0}$ for this $\boldsymbol{y}$. Suppose, with the same $\boldsymbol{x}$, that $(\boldsymbol{x}, \boldsymbol{y}') \in V$ and $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}') = \mathbf{0}$. Then $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}') = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y})$. Since $\boldsymbol{F}$ is one-to-one in V, it follows that $\boldsymbol{y}' = \boldsymbol{y}$. This proves the first part of the theorem.

For the second part, define $\boldsymbol{g}(\boldsymbol{x})$, for $\boldsymbol{x} \in W$, so that $(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})) \in V$ and (1.5.3) holds. Then

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})) = \begin{bmatrix} \boldsymbol{x} \\ \mathbf{0} \end{bmatrix} \qquad (\boldsymbol{x} \in W).$$

If $\boldsymbol{G}$ is the mapping of V' onto V that inverts $\boldsymbol{F}$, then $\boldsymbol{G} \in C^1$, by the inverse function theorem, and

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{g}(\boldsymbol{x}) \end{bmatrix} = \boldsymbol{G}(\boldsymbol{x}, \mathbf{0}) \qquad (\boldsymbol{x} \in W).$$

Since $\boldsymbol{G} \in C^1$, this shows that $\boldsymbol{g} \in C^1$.

Finally, to compute $D\boldsymbol{g}(\boldsymbol{a})$, put $\boldsymbol{\Phi}(\boldsymbol{x}) = [\boldsymbol{x} \;\; \boldsymbol{g}(\boldsymbol{x})]^\top$ so that $\boldsymbol{f}(\boldsymbol{\Phi}(\boldsymbol{x})) = \mathbf{0}$ in W. The chain rule shows therefore that

$$\begin{aligned} 0_{m\times n} &= [D\boldsymbol{f}(\boldsymbol{\Phi}(\boldsymbol{x}))]\,[D\boldsymbol{\Phi}(\boldsymbol{x})] = \Big[\; [D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{\Phi}(\boldsymbol{x}))] \;\Big|\; [D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{\Phi}(\boldsymbol{x}))] \;\Big] \begin{bmatrix} I_n \\ [D\boldsymbol{g}] \end{bmatrix} \\ &= [D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{\Phi}(\boldsymbol{x}))] + [D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{\Phi}(\boldsymbol{x}))]\,[D\boldsymbol{g}]. \end{aligned}$$

When $\boldsymbol{x} = \boldsymbol{a}$, then $\boldsymbol{\Phi}(\boldsymbol{x}) = (\boldsymbol{a}, \boldsymbol{b})$, and thus

$$0_{m\times n} = [D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})] + [D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})]\,[D\boldsymbol{g}(a)].$$ ■

Theorem 1.5.5 (Domain-straightening theorem)**.** *Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$ be a C^1 function. Let $\boldsymbol{a} \in U$ and suppose $f(\boldsymbol{a}) = 0$ and $Df(\boldsymbol{a}) \neq 0$. Then there is an open set V, and open set W containing $\boldsymbol{a}$, and a function $\boldsymbol{h} : V \to W$ of class C^1, with inverse $\boldsymbol{h}^{-1} : W \to V$ of class C^1, such that*

$$f(\boldsymbol{h}(x_1, \dots, x_n)) = x_n.$$

Proof. Exercise. ■

Theorem 1.5.6 (Range-straightening theorem)**.** *Let $U \subset \mathbb{R}^m$ be an open set and $\boldsymbol{f} : U \to \mathbb{R}^n$ a function of class C^1, and assume $m \leq n$. Let $\boldsymbol{a} \in U$ and suppose the rank of $D\boldsymbol{f}(\boldsymbol{a})$ is m. Then there are open sets V and V' in $\mathbb{R}^n$ with $\boldsymbol{f}(\boldsymbol{a}) \in V$ and a function $\boldsymbol{g} : V \to V'$ of class C^1 with inverse $\boldsymbol{g}^{-1} : V' \to V$ also of class C^1 and a neighborhood W of $\boldsymbol{a}$ in $\mathbb{R}^m$ such that*

$$\boldsymbol{g} \circ \boldsymbol{f}(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$$

for all $(x_1, \dots, x_m) \in W$.

Proof. Exercise. ■

1.6 Determinants

Definition 1.6.1. A determinant is a function which assigns to each n-tuples of vectors $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ in $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) an element of $\mathbb{R}$ (resp. $\mathbb{C}$), $\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ such that the following conditions are satisfied.

1. (Multilinearity) det is a linear function of each of the vectors $\boldsymbol{v}_j$, if the others are held fixed; i.e.,

$$\det(\ldots, \boldsymbol{v}_j + \boldsymbol{v}'_j, \ldots) = \det(\ldots, \boldsymbol{v}_j, \ldots) + \det(\ldots, \boldsymbol{v}'_j, \ldots),$$
$$\det(\ldots, c\boldsymbol{v}_j, \ldots) = c \det(\ldots, \boldsymbol{v}_j, \ldots).$$

2. (Antisymmetry) det changes sign if two of the vectors $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are interchanged; i.e.

$$\det(\ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_j, \ldots) = -\det(\ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_i, \ldots).$$

3. (Normalization)

$$\det(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) = 1.$$

Note that condition 2 implies that $\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = 0$ if two of vectors $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are equal. In fact, we may replace the condition 2 by an equivalent condition

2'. (Alternativity) Whenever $\boldsymbol{v}_j = \boldsymbol{v}_{j+1}$ for some j, we have

$$\det(\ldots, \boldsymbol{v}_j, \boldsymbol{v}_j, \ldots) = 0.$$

We leave it as an exercise to check property $2'$ implies property 2.

Proposition 1.6.2 (Uniqueness of determinant)**.** *If D and D' are two functions satisfying the above conditions 1, 2, and 3, then $D \equiv D'$.*

Proof. Let $\Delta(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = D(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) - D'(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$. It is clear that Δ satisfies condition 1 and 2. Also, we have $\Delta(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) = 0$ and thus, by anti-symmetry, for any n-tuples $(i_1, \ldots, i_n)$ with $1 \le i_r \le n$, we have

$$\Delta(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_n}) = 0.$$

By multi-linearity, we then have $\Delta(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = 0$. ■

Proposition 1.6.3 (Existence of determinant)**.** *There exists a function D satisfying the above conditions 1, 2, and 3.*

Proof. We use induction on n. For $n = 1$, the function $D(v) = v$, $v \in \mathbb{R}$ (resp. $v \in \mathbb{C}$) satisfies the requirements. Now suppose that D is a function on $\mathbb{R}^{n-1}$ (resp. $\mathbb{C}^{n-1}$) that satisfies the conditions 1, 2, and 3. Fix an index i, $1 \le i \le n$, and the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) be given by

$$\boldsymbol{v}_j = [a_{1,j} \ \cdots \ a_{n,j}]^\top, \quad a_{k,j} \in \mathbb{R} \ \text{(resp. } \mathbb{C}\text{)}, \quad 1 \le j \le n.$$

Then define

$$D(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = (-1)^{i+1} a_{i,1} D_{i,1} + \cdots + (-1)^{i+n} a_{i,n} D_{i,n},$$

where, for $1 \leq i \leq n$, $D_{i,j}$ is the determinant of the vectors $\boldsymbol{v}_1^{(j)}, \ldots \boldsymbol{v}_{n-1}^{(j)}$ in $\mathbb{R}^{n-1}$ (resp. $\mathbb{C}^{n-1}$) obtained from the $n-1$ vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}, \boldsymbol{v}_{j+1}, \ldots, \boldsymbol{v}_n$ by deleting ith component in each case. Note that if we set A to be the matrix whose jth column is the vector $\boldsymbol{v}_j$, then D is the is the *cofactor expansion* along the ith row of the matrix A. We leave it as an exercise to check that D satisfies the required properties. (It is easier to prove condition $2'$ instead of condition 2.) ∎

Definition 1.6.4. For an $n \times n$ matrix A, we denote (by abuse of notation)

$$\det A = \det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n), \quad \text{where } \boldsymbol{v}_j \text{ is the } j\text{th column of } A.$$

In other words,

$$\det A = \det(A\boldsymbol{e}_1, \ldots, A\boldsymbol{e}_n).$$

Definition 1.6.5. Let A be an $m \times n$ matrix with coefficients in $\mathbb{R}$ (resp. $\mathbb{C}$). An elementary column operation applied to A is a rule that produces another $m \times n$ matrix A' from A by one of the following types of operations:

(1) Replacing ith column of $\boldsymbol{v}_i$ of A by $m\boldsymbol{v}_i$, for some $m \neq 0$.

(2) Replacing ith column of $\boldsymbol{v}_i$ of A by $\boldsymbol{v}_i + \lambda\boldsymbol{v}_j$, for some other column $\boldsymbol{v}_j$, $j \neq i$ and $\lambda \in \mathbb{R}$ (resp. $\lambda \in \mathbb{C}$).

(3) Interchanging two columns of A.

Theorem 1.6.6. *Let A be an $n \times n$ matrix with coefficients in $\mathbb{R}$ (resp. $\mathbb{C}$) having $\boldsymbol{v}_j$ as its jth column.*

i. Let A' be a matrix obtained from A by an elementary column operation of type (1). Then
$$\det A' = m \det A.$$

ii. Let A' be a matrix obtained from A by an elementary column operation of type (2). Then
$$\det A' = \det A.$$

iii. Let A' be a matrix obtained from A by an elementary column operation of type (3). Then
$$\det A' = -\det A.$$

Corollary 1.6.7. *A matrix A is invertible if and only if* $\det A \neq 0$.

Theorem 1.6.8. *If A and B are $n \times n$ matrices, then*

$$\det A \det B = \det AB.$$

Proof. If A is invertible, consider the function

$$D(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \frac{\det(A\boldsymbol{v}_1, \ldots, A\boldsymbol{v}_n)}{\det A}.$$

It is easy to see that D is multilinear and antisymmetric. Also, we have $D(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) = 1$. Therefore, by the uniqueness of determinant, we must have $D(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$. By taking $\boldsymbol{v}_j$ to be jth column of B, we get

$$\det A \det B = \det(A\boldsymbol{v}_1, \ldots, A\boldsymbol{v}_n) = \det AB.$$

If $\det A = 0$, then A is not invertible. We leave it as an exercise to check that AB is not invertible if A is not invertible, and thus $\det AB = 0$. ∎

A *permutation* of a set X is a bijective map $f : X \to X$. The set of permutations of the set $X = \{1, 2, \ldots, n\}$ is often denoted S_n.

Proposition 1.6.9. *The set S_n forms a group under the operation $\sigma\tau$, defined by*

$$\sigma\tau(k) = \sigma(\tau(k)), \quad k \in \{1, 2, \ldots, n\}, \quad \sigma, \tau \in S_n.$$

Proof. We leave it as an exercise. ∎

A *transposition* is a permutation that exchanges two symbols and leaves all others fixed.

Proposition 1.6.10. *Every permutation $\sigma \in S_n$ is a product of transpositions.*

Proof. We proceed by induction on n. When $n = 1$, there is nothing to prove. Let $n > 1$ and assume that the assertion is proved for $n - 1$. For $\sigma \in S_n$, let $\sigma(n) = k$. If $k \neq n$, then let τ be the transposition that interchanges k and n. If $k = n$, then let $\tau = id$. Then $\tau\sigma$ is a permutation that leaves n fixed. We may therefore view $\tau\sigma \in S_{n-1}$, and by induction, there exist transpositions $\tau_1, \ldots, \tau_k \in S_{n-1}$ such that

$$\tau\sigma = \tau_1 \cdots \tau_k.$$

Therefore, we can now write $\sigma = \tau\tau_1 \cdots \tau_k$. ∎

Theorem 1.6.11. *There exists a map*

$$\epsilon : S_n \to \{-1, 1\}$$

called the signature (or sign), such that

1. $\epsilon(\sigma_1\sigma_2) = \epsilon(\sigma_1)\epsilon(\sigma_2)$ *for all* $\sigma_1, \sigma_2 \in S_n$.
2. $\epsilon(\tau) = -1$ *for all transpositions* $\tau \in S_n$.

Proof. To see the existence, define for every $\sigma \in S_n$ the *permutation matrix* M_σ by

$$M_\sigma \boldsymbol{e}_i = \boldsymbol{e}_{\sigma(i)}.$$

For every $\sigma \in S_n$ define

$$\epsilon(\sigma) := \det M_\sigma.$$

It is easy to see that this map satisfies the above properties 1 and 2. ∎

Corollary 1.6.12. *If $\sigma \in S_n$ is expressed as a product of transpositions,*

$$\sigma = \tau_1 \cdots \tau_k,$$

then k is even or odd according as $\epsilon(\sigma) = 1$ or -1.

Definition 1.6.13. A permutation $\sigma \in S_n$ is *even* if $\epsilon(\sigma) = 1$ and *odd* if $\epsilon(\sigma) = -1$.

It is clear that any rule satisfying properties 1 and 2 in Theorem 1.6.11 must give the same value on any permutation.

Corollary 1.6.14. *For $n \geq 2$, let $\varphi : S_n \to \{-1, 1\}$ satisfy the property 1 of Theorem 1.6.11; i.e.*

$$\varphi(\sigma\sigma') = \varphi(\sigma)\varphi(\sigma') \quad \text{for all } \sigma, \sigma' \in S_n.$$

Then $\varphi(\sigma) = 1$ for all σ or $\varphi(\sigma) = \epsilon(\sigma)$ for all σ. Thus, if φ is multiplicative and not identically 1*, then $\varphi = \epsilon$.*

The nth alternating group A_n is the group of even permutations in S_n. That is, a permutation is in A_n when it is a product of an even number of transpositions. Such products are clearly closed under multiplication and inversion, so A_n is a subgroup of S_n. Alternatively,

$$A_n = \{\sigma \in S_n : \epsilon(\sigma) = 1\}.$$

It is easy to see that $A_1 = S_1$ and A_n is a proper subgroup of S_n if $n > 1$.

Theorem 1.6.15. *Let A be an $n \times n$ matrix with entries $a(i, j)$ in the ith row and jth column. Then*

$$\det A = \sum_{\sigma \in S_n} \epsilon(\sigma) a(1, \sigma(1)) a(2, \sigma(2)) \cdots a(n, \sigma(n)).$$

Note that $\epsilon(\sigma) = \epsilon(\sigma^{-1})$ and

$$a(1, \sigma(1)) \cdots a(n, \sigma(n)) = a(\sigma^{-1}(1), 1) \cdots a(\sigma^{-1}(n), n).$$

Therefore, we see that

$$\det A = \det A^\top.$$

Definition 1.6.16. If $(j_1, \ldots, j_n)$ is ordered n-tuple of integers, define

$$s(j_1, \ldots, j_n) = \prod_{p<q} \operatorname{sgn}(j_q - j_p),$$

where $\operatorname{sgn} x = 1$ if $x > 0$, $\operatorname{sgn} x = -1$ if $x < 0$, $\operatorname{sgn} x = 0$ if $x = 0$.

Proposition 1.6.17. *Let A be an $n \times n$ matrix with entries $a(i, j)$ in the ith row and jth column. Then*

$$\det A = \sum s(j_1, \ldots, j_n) a(1, j_1) \cdots a(n, j_n),$$

where sum extends over all order n-tuples of integers $(j_1, \ldots, j_n)$ with $1 \leq j_r \leq n$.

Proof. It is easy to see that $s(j_1, \ldots, j_n) \neq 0$ if and only if there is $\sigma \in S_n$ such that $j_k = \sigma(k)$ for $k = 1, \ldots, n$. We shall show that $s(j_1, \ldots, j_n) = \epsilon(\sigma)$. Consider the multivariable polynomial

$$P(X_1, \ldots, X_n) = \prod_{1 \leq i < j \leq n} (X_j - X_i).$$

Note that

$$P(X_{j_1}, \ldots, X_{j_n}) = (-1)^N P(X_1, \ldots, X_n),$$

where N is the number of inversions in $(j_1, \ldots, j_n)$; i.e. the number of pairs (p, q) such that $p < q$ and $j_p > j_q$. Therefore, we have

$$P(X_{j_1}, \ldots, X_{j_n}) = s(j_1, \ldots, j_n) P(X_1, \ldots, X_n).$$

We leave it as an exercise to verify that $\varphi : S_n \to \{-1, 1\}$ defined by

$$\varphi(\sigma) = \frac{P(X_{\sigma(1)}, \ldots, X_{\sigma(n)})}{P(X_1, \ldots, X_n)}$$

satisfies properties 1 and 2 of Theorem 1.6.11. Therefore, we must have $\varphi \equiv \epsilon$. ■

Definition 1.6.18. If $\boldsymbol{f}$ maps an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and $\boldsymbol{f}$ is differentiable at a point $\boldsymbol{x} \in U$, the determinant of the linear operator $D\boldsymbol{f}(\boldsymbol{x})$ is called the *Jacobian* of $\boldsymbol{f}$ at $\boldsymbol{x}$. In symbols,

$$J_{\boldsymbol{f}}(\boldsymbol{x}) = \det D\boldsymbol{f}(\boldsymbol{x}).$$

We shall also use the notation

$$\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}$$

for $J_{\boldsymbol{f}}(\boldsymbol{x})$, if $(y_1, \ldots, y_n) = \boldsymbol{f}(x_1, \ldots, x_n)$.

Chapter 2

Riemann Integral

2.1 The integral over an n-dimensional interval

Definition 2.1.1. A closed interval I in $\mathbb{R}^n$ is given by the product of n one-dimensional closed and bounded intervals

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

where $a_i \leq b_i$ are real numbers, $i = 1, \ldots, n$. In other words, we have

$$I = \{\boldsymbol{x} \in \mathbb{R}^n : a_i \leq x_i \leq b_i,\ i = 1, \ldots, n\}.$$

An open interval in $\mathbb{R}^n$ is the product of n one-dimensional open intervals. Also, a cube is a closed interval for which $b_1 - a_1 = b_2 - a_2 = \cdots = b_n - a_n$.

Definition 2.1.2. The volume $v(I)$ of a closed interval $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and also of an open interval $(a_1, b_1) \times \cdots \times (a_n, b_n)$ is

$$v(I) = \prod_{i=1}^{n} (b_i - a_i).$$

Definition 2.1.3. A partition P of a closed interval $[a, b]$ is a finite set of points $t_0, t_1, \ldots, t_N$, where $a = t_0 \leq t_1 \leq \cdots \leq t_N = b$. The partition P divides the interval $[a, b]$ into N subintervals $[t_{i-1}, t_i]$. A partition of an interval $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is a collection $P = (P_1, \ldots, P_n)$, where each P_i is a partition of the interval $[a_i, b_i]$. If P_i divides $[a_i, b_i]$ into N_i subintervals, then $P = (P_1, \ldots, P_n)$ divides $[a_1, b_1] \times \cdots \times [a_n, b_n]$ into $N = N_1 \cdots N_n$ subintervals. These subintervals will be called subintervals of the partition P.

Definition 2.1.4. An n-dimensional interval is a closed interval $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ such that $a_i < b_i$ for all $i = 1, \ldots, n$.

Here and below, I shall be always an n-dimensional (closed) interval unless otherwise stated.

Definition 2.1.5. Suppose $f : I \to \mathbb{R}$ is a bounded function, and P is a partition of I. For each subinterval S of the partition let

$$M_S(f) = \sup\{f(x) : x \in S\},$$
$$m_S(f) = \inf\{f(x) : x \in S\}.$$

The upper and lower sums of f for P are defined by

$$U(f,P) = \sum_S M_S(f)v(S) \quad \text{and} \quad L(f,P) = \sum_S m_S(f)v(S).$$

Clearly $L(f,P) \leq U(f,P)$, and an even stronger assertion is true.

Proposition 2.1.6. *Suppose the partition P^* is a refinement of P (that is, each subinterval of P^* is contained in a subinterval of P). Then*

$$L(f,P) \leq L(f,P^*) \quad \textit{and} \quad U(f,P^*) \leq U(f,P).$$

Corollary 2.1.7. *If P_1 and P_2 are any two partitions, then*

$$L(f,P_1) \leq U(f,P_2).$$

It follows from corollary 2.1.7 that the least upper bound of all lower sums for f is less than or equal to the greatest lower bound of all upper sums for f.

Definition 2.1.8. If f is a bounded function on I, then the upper integral of f on I is defined by

$$\overline{\int_I} f = \inf\{U(f,P) : P \text{ is a partition of } I\}$$

and the lower integral of f on I by

$$\underline{\int_I} f = \sup\{L(f,P) : P \text{ is a partition of } I\}.$$

respectively.

Definition 2.1.9. A function $f : I \to \mathbb{R}$ is called (Riemann) integrable on I if it is bounded and

$$\overline{\int_I} f = \underline{\int_I} f.$$

This common number is then denoted $\int_I f$, and called the (Riemann) integral of f over I. Often, the notation

$$\int_I f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \quad \text{or} \quad \int_I f(\boldsymbol{x})\, d\boldsymbol{x}$$

is used. If $f : [a,b] \to \mathbb{R}$, the notation $\int_a^b f$ or $\int_a^b f(x)\, dx$ is also used.

A simple but useful criterion for integrability is provided by

Theorem 2.1.10. *A bounded function $f : I \to \mathbb{R}$ is integrable if and only if for every $\epsilon > 0$ there is a partition P of I such that*

$$U(f,P) - L(f,P) < \epsilon.$$

Definition 2.1.11. A subset A of $\mathbb{R}^n$ has (n-dimensional) measure 0 if for every $\epsilon > 0$ there is a cover $\{I_i\}_{i=1}^{\infty}$ of A by open intervals such that

$$\sum_{i=1}^{\infty} v(I_i) < \epsilon.$$

Proposition 2.1.12. *If A has measure 0 and $B \subset A$, then B has measure 0.*

Proposition 2.1.13. *In definition 2.1.11, closed intervals may be used instead of open intervals.*

A set with only finitely many points clearly has measure 0. If A has infinitely many points which can be arranged in a sequence $a_1, a_2, a_3, \ldots$, then A also has measure 0, for if $\epsilon > 0$, we can choose I_i to be an open interval containing a_i with $v(I_i) < \epsilon/2^i$. Then

$$\sum_{i=1}^{\infty} v(I_i) < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon.$$

Proposition 2.1.14. *If $A = \bigcup_{i=1}^{\infty} A_i$ and each A_i has measure 0, then A has measure 0.*

Proof. Let $\epsilon > 0$. Since A_i has measure 0, there is a cover $\{I_{i,j}\}_{j=1}^{\infty}$ of A_i by open intervals such that

$$\sum_{j=1}^{\infty} v(I_{i,j}) < \epsilon/2^i.$$

Then the collection of all $I_{i,j}$ is a cover of A and clearly $\sum_{i,j=1}^{\infty} v(I_{i,j}) < \epsilon$. ■

Definition 2.1.15. Let f be a bounded real function defined on $A \subset \mathbb{R}^n$. The oscillation $\operatorname{osc}(f, \boldsymbol{x})$ of f at $\boldsymbol{x} \in A$ is defined by

$$\operatorname{osc}(f, \boldsymbol{x}) = \lim_{\delta \to 0} \left(\sup\nolimits_{A \cap B(\boldsymbol{x}, \delta)} f - \inf\nolimits_{A \cap B(\boldsymbol{x}, \delta)} f \right).$$

Proposition 2.1.16. *A bounded function f is continuous at $\boldsymbol{x}$ if and only if* $\operatorname{osc}(f, \boldsymbol{x}) = 0$.

Proof. Exercise. ■

Lemma 2.1.17. *Let $f : I \to \mathbb{R}$ be a bounded function such that* $\operatorname{osc}(f, \boldsymbol{x}) < \epsilon$ *for all $\boldsymbol{x} \in I$. Then there is a partition P of I with*

$$U(f, P) - L(f, P) < \epsilon v(I).$$

Proof. For each $\boldsymbol{x} \in I$ there is an interval I_x, containing $\boldsymbol{x}$ in its interior, such that

$$M_{I_x}(f) - m_{I_x}(f) < \epsilon.$$

Since I is compact and the interiors of I_x cover I, a finite number $I_{x_1}, \ldots, I_{x_N}$ of the intervals I_x cover I. Let P be a partition for I such that each subinterval S of P is contained in some I_{x_i}. Then $M_S(f) - m_S(f) < \epsilon$ for each subinterval S of P, so that

$$U(f, P) - L(f, P) = \sum_S \left[M_S(f) - m_S(f) \right] v(S) < \epsilon v(I). \qquad ■$$

Lemma 2.1.18. *Let $A \subset \mathbb{R}^n$ be closed. If $f : A \to \mathbb{R}$ is any bounded function, and $\epsilon > 0$, then the set $E = \{\boldsymbol{x} \in A : \operatorname{osc}(f, \boldsymbol{x}) \geq \epsilon\}$ is closed.*

Proof. We wish to show that E^c is open. If $\boldsymbol{x} \in E^c$, then either $\boldsymbol{x} \notin A$ or else $\boldsymbol{x} \in A$ and $\operatorname{osc}(f, \boldsymbol{x}) < \epsilon$. In the first case, since A is closed, there is an open ball $B(\boldsymbol{x}, r)$ such that $B(\boldsymbol{x}, r) \subset A^c \subset E^c$. In the second case, there is a $\delta > 0$ such that

$$\sup_{A \cap B(\boldsymbol{x},\delta)} f - \inf_{A \cap B(\boldsymbol{x},\delta)} f < \epsilon.$$

For any $\boldsymbol{y} \in B(\boldsymbol{x}, \delta)$, there is an $\delta_1 > 0$ such that $B(\boldsymbol{y}, \delta_1) \subset B(\boldsymbol{x}, \delta)$ so that

$$\operatorname{osc}(f, \boldsymbol{y}) \leq \sup_{A \cap B(\boldsymbol{y},\delta_1)} f - \inf_{A \cap B(\boldsymbol{y},\delta_1)} f < \epsilon.$$

Therefore, $B(\boldsymbol{x}, \delta) \in E^c$. ■

Theorem 2.1.19 (Lebesgue's theorem)**.** *Let I be an n-dimensional interval and let $f : I \to \mathbb{R}$ be a bounded function. Let*

$$A = \{\boldsymbol{x} \in I : f \textit{ is not continuous at } \boldsymbol{x}\}.$$

Then f is integrable if and only if A is a set of measure 0.

Proof. Suppose first that A has measure 0. Let $\epsilon > 0$ and let

$$A_\epsilon = \{\boldsymbol{x} \in I : \operatorname{osc}(f, \boldsymbol{x}) \geq \epsilon\}.$$

Then $A_\epsilon \subset A$, so that A_ϵ has measure 0. By lemma 2.1.18, A_ϵ is compact, and thus there is a finite collection $I_1, \ldots, I_N$ of intervals, whose interiors cover A_ϵ, such that $\sum_{i=1}^N v(I_i) < \epsilon$. Let P any partition of I that contains the end points of the component intervals of $I_1, \ldots, I_N$. Then every subintervals S of P is in one of two groups.

(1) $\mathcal{S}_1$, which consists of subintervals S, such that $S \subset I_i$ for some i.

(2) $\mathcal{S}_2$, which consists of subintervals S with $S \cap A_\epsilon = \emptyset$.

If $S \in \mathcal{S}_2$, then $\operatorname{osc}(f, \boldsymbol{x}) < \epsilon$ for $\boldsymbol{x} \in S$. Lemma 2.1.17 implies that there is a refinement P' of P such that

$$\sum_{S' \subset S} [M_{S'}(f) - m_{S'}(f)]\, v(S') < \epsilon v(S) \quad \text{for any} \quad S \in \mathcal{S}_2.$$

Therefore,

$$\begin{aligned} U(f, P') - L(f, P') &= \sum_{S' \subset S \in \mathcal{S}_1} + \sum_{S' \subset S \in \mathcal{S}_2} [M_{S'}(f) - m_{S'}(f)]\, v(S') \\ &\leq 2 \sup_I |f| \sum_{i=1}^N v(I_i) + \sum_{S \in \mathcal{S}_2} \epsilon v(S) < \left(2 \sup_I |f| + v(I) \right) \epsilon, \end{aligned}$$

which shows that we can find a partition P' with $U(f, P') - L(f, P')$ as small as desired. Thus f is integrable.

Suppose, conversely, that f is integrable. Since $A = \bigcup_{k=1}^\infty A_{1/k}$, it suffices to prove that each $A_{1/k}$ has measure 0. If $\epsilon > 0$, let P be a partition of I such that

$$U(f, P) - L(f, P) < \frac{\epsilon}{k}.$$

Let $\mathcal{S}$ be the collection of subintervals S of P which intersect $A_{1/k}$. Then $\mathcal{S}$ is a cover of $A_{1/k}$. Now if $S \in \mathcal{S}$, then

$$M_S(f) - m_S(f) \geq 1/k.$$

Thus

$$\sum_{S\in\mathcal{S}} v(S) \leq k \sum_{S\in\mathcal{S}} [M_S(f) - m_S(f)]v(S) \leq k \sum_{S} [M_S(f) - m_S(f)]v(S) < \epsilon. \quad \blacksquare$$

Corollary 2.1.20. *If $f : I \to \mathbb{R}$ is a continuous function, then f is integrable on I.*

Corollary 2.1.21. *Assume $f : I \to \mathbb{R}$ is integrable on I.*

(a) If f vanishes except on a set of measure zero, then $\int_I f = 0$.

(b) If f is non-negative and if $\int_I f = 0$, then f vanishes except on a set of measure zero.

Proof. (a) Suppose f vanishes except on a set E of measure zero. Let P be a partition of I. If S is a subinterval determined by P, then S is not contained in E, so that f vanishes at some point of S. Then $m_S(f) \leq 0$ and $M_S(f) \geq 0$. It follows that $L(f, P) \leq 0$ and $U(f, P) \geq 0$. Since these inequalities hold for all P,

$$\underline{\int_I} f \leq 0 \quad \text{and} \quad \overline{\int_I} f \geq 0$$

Since $\int_I f$ exists, it must equal zero.

(b) We show that if f is continuous at $\boldsymbol{a}$, then $f(\boldsymbol{a}) = 0$. It follows that f must vanish except possibly at points where f fails to be continuous; the set of such points has measure zero by the preceding theorem. We suppose that f is continuous at $\boldsymbol{a}$ and that $f(\boldsymbol{a}) > 0$ and derive a contradiction. Set $\epsilon = f(\boldsymbol{a})$. Since f is continuous at $\boldsymbol{a}$, there is a $\delta > 0$ such that

$$f(\boldsymbol{x}) > \epsilon/2 \quad \text{for} \quad |\boldsymbol{x} - \boldsymbol{a}| < \delta \quad \text{and} \quad \boldsymbol{x} \in I.$$

Choose a partition P of I of mesh less than $\delta/\sqrt{n}$. If S_0 is a subinterval determined by P that contains $\boldsymbol{a}$, then $m_{S_0}(f) \geq \epsilon/2$. On the other hand, $m_S(f) \geq 0$ for all subinterval S. It follows that

$$L(f, P) = \sum_S m_S(f)v(S) \geq (\epsilon/2)v(S_0) > 0.$$

But $L(f, P) \leq \int_Q f = 0$. $\blacksquare$

2.2 The integral over a bounded set

We have thus far dealt only with the integrals of functions over intervals. Integrals over other sets are easily reduced to this type.

Definition 2.2.1. If $E \subset \mathbb{R}^n$, the characteristic function χ_E of E is defined by

$$\chi_E(\boldsymbol{x}) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Lemma 2.2.2. *Let I and I' be two intervals in $\mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{R}$ is a bounded function that vanishes outside $I \cap I'$, then*

$$\int_I f = \int_{I'} f;$$

one integral exists if and only if the other does.

Proof. We consider first the case where $I \subset I'$. Let E be the set of points of $\operatorname{Int} I$ at which f fails to be continuous. Then both the maps $f : I \to \mathbb{R}$ and $f : I' \to \mathbb{R}$ are continuous except at points of E and possibly at points of ∂I. Existence of each integral is thus equivalent to the requirement that E have measure zero. Now suppose both integrals exist. Let P be a partition of I', and let P^* be the refinement of P obtained from P by adjoining the end points of the component intervals of I. Then I is a union of subintervals S determined by P^*. If S is a subinterval determined by P^* that is not contained in I, then f vanishes at some point of S, whence $m_S(f) \leq 0$. It follows that

$$L(f, P^*) \leq \sum_{S \subset I} m_S(f) v(S) \leq \int_I f.$$

We conclude that $L(f, P) \leq \int_I f$. An entirely similar argument shows that $U(f, P) \geq \int_I f$. Since P is an arbitrary partition of I', it follows that

$$\int_I f = \int_{I'} f.$$

The proof for an arbitrary pair of intervals I, I' involves choosing an interval I'' containing them both, and noting that

$$\int_I f = \int_{I''} f = \int_{I'} f. \qquad \blacksquare$$

Definition 2.2.3. If f is a bounded function that vanishes outside a bounded set E, let I be any interval which contains E and define

$$\int_{\mathbb{R}^n} f = \int_I f \quad \text{provided the latter integral exists.}$$

Definition 2.2.4. Let E be a bounded set in $\mathbb{R}^n$ and let $f : E \to \mathbb{R}$ be a bounded function. We abuse notation, and define $f\chi_E : \mathbb{R}^n \to \mathbb{R}$ by

$$f\chi_E(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{x}) & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

so that $f\chi_E$ is an extension of f to $\mathbb{R}^n$. We define the integral of f over E by the equation

$$\int_E f = \int_{\mathbb{R}^n} f\chi_E \quad \text{provided the latter integral exists.}$$

Proposition 2.2.5. *Let E be a bounded set in $\mathbb{R}^n$. The function χ_E is integrable if and only if the boundary of E has measure 0.*

Proof. Use theorem 2.1.19; details are left as an exercise. ■

Definition 2.2.6. A bounded set $E \subset R^n$ whose boundary has measure zero is called *Jordan measurable*, and we define (n-dimensional) volume of E by $v(E) = \int_{\mathbb{R}^n} \chi_E$. Note that this definition agrees with our previous definition of volume when E is an interval.

For $k \in \mathbb{Z}$, let $\mathscr{Q}_k$ be the collection of cubes whose side length is 2^{-k} and whose vertices are in the lattice $(2^{-k}\mathbb{Z})^n$. Note that any two cubes in $\mathscr{Q}_k$ have disjoint interiors, and that the cubes in $\mathscr{Q}_{k+1}$ are obtained from the cubes in $\mathscr{Q}_k$ by bisecting the sides. If $E \subset \mathbb{R}^n$, we define the inner and outer approximations to E by the grid of cubes $\mathscr{Q}_k$ to be

$$\underline{A}(E,k) = \bigcup\{Q \in \mathscr{Q}_k,\, Q \subset E\}, \quad \overline{A}(E,k) = \bigcup\{Q \in \mathscr{Q}_k,\, Q \cap E \neq \emptyset\}.$$

Then the volume $v(\underline{A}(E,k))$ is just 2^{-nk} times the number of cubes in $\mathscr{Q}_k$ that lie in $\underline{A}(E,k)$; likewise for $v(\overline{A}(E,k))$. Also, the sets $\underline{A}(E,k)$ increase with k while the sets $\overline{A}(E,k)$ decrease, because each cube in $\mathscr{Q}_k$ is a union of cubes in $\mathscr{Q}_{k+1}$. Hence the limits

$$\underline{v}(E) = \lim_{k\to\infty} v(\underline{A}(E,k)), \quad \overline{v}(E) = \lim_{k\to\infty} v(\overline{A}(E,k))$$

exist (in the extended real number system).

Proposition 2.2.7. *A set $E \subset \mathbb{R}^n$ is Jordan measurable if and only if it is bounded and $\underline{v}(E) = \overline{v}(E) < \infty$. If E is Jordan measurable, then $v(E) = \underline{v}(E) = \overline{v}(E)$.*

Proof. Exercise. ■

Proposition 2.2.8. *If $U \subset \mathbb{R}^n$ is open, then*

$$U = \bigcup_{k=1}^{\infty} \underline{A}(U,k).$$

Moreover, U is a countable union of cubes with disjoint interiors.

Proof. Exercise. ■

2.3 Fubini's theorem

Theorem 2.3.1 (Fubini's theorem)**.** *Let $I_1 \subset \mathbb{R}^n$ and $I_2 \subset \mathbb{R}^m$ be intervals, and let $f : I_1 \times I_2 \to \mathbb{R}$ be integrable. Then the mappings*

$$\boldsymbol{x} \mapsto \underline{\int}_{I_2} f(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y} \quad \text{and} \quad \boldsymbol{x} \mapsto \overline{\int}_{I_2} f(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}$$

are both integrable on I_1 and

$$\int_{I_1\times I_2} f = \int_{I_1}\left(\underline{\int}_{I_2} f(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}\right)d\boldsymbol{x} = \int_{I_1}\left(\overline{\int}_{I_2} f(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}\right)d\boldsymbol{x}.$$

(The integrals on the right side are called iterated integrals for f.)

Proof. Let P_1 be a partition of I_1 and P_2 a partition of I_2. Together they give a partition P of $I_1 \times I_2$ for which any subinterval S is of the form $S_1 \times S_2$, where S_1 is a subinterval of the partition P_1, and S_2 is a subinterval of the partition P_2. Thus

$$\begin{aligned} L(f, P) = \sum_S m_S(f) v(S) = \sum_{S_1, S_2} m_{S_1 \times S_2}(f) v(S_1 \times S_2) \\ = \sum_{S_1} \left(\sum_{S_2} m_{S_1 \times S_2}(f) v(S_2) \right) v(S_1) \end{aligned}$$

Now, clearly

$$m_{S_1 \times S_2}(f) \leq m_{S_2}(f(\boldsymbol{x}, \cdot)) \quad \text{for } \boldsymbol{x} \in S_1.$$

Consequently, for $\boldsymbol{x} \in S_1$ we have

$$\sum_{S_2} m_{S_1 \times S_2}(f) v(S_2) \leq \sum_{S_2} m_{S_2}(f(\boldsymbol{x}, \cdot)) v(S_2) \leq \underline{\int}_{I_2} f(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{y} =: g(\boldsymbol{x}).$$

Therefore, we have

$$L(f, P) = \sum_{S_1} \left(\sum_{S_2} m_{S_1 \times S_2}(f) v(S_2) \right) v(S_1) \leq \sum_{S_1} m_{S_1}(g) = L(g, P_1).$$

By entirely analogous argument, we have

$$U(h, P_1) \leq U(f, P), \quad \text{where } \; h(\boldsymbol{x}) := \overline{\int}_{I_2} f(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{y}.$$

We thus obtain

$$L(f, P) \leq L(g, P_1) \leq U(g, P_1) \leq U(h, P_1) \leq U(f, P).$$

Since f is integrable,

$$\sup L(f, P) = \inf U(f, P) = \int_{I_1 \times I_2} f.$$

Hence, by theorem 2.1.10, we find that g is integrable on I_1 and

$$\int_{I_1 \times I_2} f = \int_{I_1} g = \int_{I_1} \left(\underline{\int}_{I_2} f(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{y} \right) d\boldsymbol{x}.$$

The assertion for the upper integral follows similarly from the inequalities

$$L(f, P) \leq L(g, P_1) \leq L(h, P_1) \leq U(h, P_1) \leq U(f, P).$$ ■

Corollary 2.3.2. *Under the same hypothesis as in theorem 2.3.1, we have*

$$\int_{I_1 \times I_2} f = \int_{I_2} \left(\underline{\int}_{I_1} f(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{x} \right) d\boldsymbol{y} = \int_{I_2} \left(\overline{\int}_{I_1} f(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{x} \right) d\boldsymbol{y}.$$

Moreover, if f is continuous, then

$$\int_{I_1 \times I_2} f = \int_{I_2} \left(\int_{I_1} f(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{x} \right) d\boldsymbol{y} = \int_{I_1} \left(\int_{I_12} f(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{y} \right) d\boldsymbol{x}.$$

Corollary 2.3.3. *If $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $f : I \to \mathbb{R}$ is continuous, we can apply Fubini's theorem repeatedly to obtain*

$$\int_I f = \int_{a_n}^{b_n} \left(\cdots \left(\int_{a_1}^{b_1} f(x_1, \ldots, x_n)\, dx_1 \right) \cdots \right) dx_n.$$

The order of iterated n integrations in the above can be interchanged.

Corollary 2.3.4. *Let $\phi, \psi : [a, b] \to \mathbb{R}$ be continuous functions such that $\phi(x) \leq \psi(x)$ for all $x \in [a, b]$, let*

$$E = \{(x, y) : a \leq x \leq b,\ \phi(x) \leq y \leq \psi(x)\},$$

and let $f : E \to \mathbb{R}$ be continuous. Then

$$\int_E f = \int_a^b \left(\int_{\phi(x)}^{\psi(x)} f(x, y)\, dy \right) dx.$$

Definition 2.3.5. The support of a (real or complex) function f on $\mathbb{R}^n$, denoted by $\operatorname{supp}(f)$, is the smallest closed set outside of which f vanishes; that is

$$\operatorname{supp}(f) = \overline{\{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) \neq 0\}}.$$

Theorem 2.3.6. *Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a nonsingular linear transformation. If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function with a compact support, then*

$$\int_{\mathbb{R}^n} f = |\det T| \int_{\mathbb{R}^n} f \circ T. \tag{2.3.7}$$

Proof. We first note that if (2.3.7) is true for the transformations T and S, it is also true for $T \circ S$, since

$$\begin{aligned}\int_{\mathbb{R}^n} f = |\det T| \int_{\mathbb{R}^n} f \circ T &= |\det T|\, |\det S| \int_{\mathbb{R}^n} (f \circ T) \circ S \\ &= |\det (T \circ S)| \int_{\mathbb{R}^n} f \circ (T \circ S).\end{aligned}$$

Next we recall the fact from elementary linear algebra that T can be written as the product of finitely many transformations of three *elementary* types: (1) switch two coordinates; (2) replace the jth coordinate by itself plus a multiple of another coordinate; (3) multiply the jth coordinate by a nonzero scalar. In symbols:

$$\begin{aligned} T_1(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_n) &= (x_1, \ldots, x_k, \ldots, x_j, \ldots, x_n), \\ T_2(x_1, \ldots, x_j, \ldots, x_n) &= (x_1, \ldots, x_j + c x_k, \ldots, x_n) \qquad (j \neq k), \\ T_3(x_1, \ldots, x_j, \ldots, x_n) &= (x_1, \ldots, c x_j, \ldots, x_n) \qquad (c \neq 0). \end{aligned}$$

Hence is suffices to prove (2.3.7) when T is of the types T_1, T_2, T_3 described above. But this is a simple consequence of the Fubini's theorem. For T_1, we interchange the order of integration in the variables x_j and x_k, and for T_2 and T_3 we integrate first with respect to x_j and use the one-dimensional formulas

$$\int_{\mathbb{R}} g(t)\, dt = |c| \int_{\mathbb{R}} g(ct)\, dt, \qquad \int_{\mathbb{R}} g(t + a)\, dt = \int_{\mathbb{R}} g(t)\, dt$$

for any continuous function $g : \mathbb{R} \to \mathbb{R}$ with a compact support. Since it is easily verified that $\det T_1 = -1$, $\det T_2 = 1$, and $\det T_3 = c$, (2.3.7) is proved. ■

Remark 2.3.8. The same proof of theorem 2.3.6 shows that (2.3.7) is true if $f = \chi_I$ and I is an n-dimensional interval. Therefore, we have $v(T(I)) = |\det T| v(I)$, whenever T is a nonsingular linear transformation and I is an n-dimensional interval.

2.4 Partitions of unity

Lemma 2.4.1. *Let B and B' are open balls centered at $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $B \subset \overline{B} \subset B'$. Then there is a C^∞ function $\phi : \mathbb{R}^n \to [0,1]$ such that $\phi \equiv 1$ on $\overline{B}$ and* $\operatorname{supp}(\phi) \subset B'$.

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be such that

$$h(t) := \begin{cases} e^{-1/t} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then h is a C^∞ function (exercise). Let r and r' be the radii of balls B and B'; note that $0 < r < r'$. Fix any number R such that $r < R < r'$ and let $g : \mathbb{R}^n \to \mathbb{R}$ be such that

$$g(\boldsymbol{x}) = \frac{h(R^2 - |\boldsymbol{x}|^2)}{h(R^2 - |\boldsymbol{x}|^2) + h(|\boldsymbol{x}|^2 - r^2)}$$

Since the denominator is never zero, this is a C^∞ function. When $|\boldsymbol{x}| \geq R$, the numerator is zero, otherwise it is positive; and when $|\boldsymbol{x}| \leq r$, the value of $g(\boldsymbol{x})$ is identically 1. Hence, $\phi(\boldsymbol{x}) := g(\boldsymbol{x} - \boldsymbol{x}_0)$ has the desired properties. ∎

Lemma 2.4.2. *Let F be a closed set in a metric space (X, d). Define* $\operatorname{dist}(x, F)$ *by*

$$\operatorname{dist}(x, F) = \inf_{y \in F} d(x, y).$$

Then $\operatorname{dist}(x, F)$ *is a nonnegative continuous function such that*

$$|\operatorname{dist}(x, F) - \operatorname{dist}(y, F)| \leq d(x, y) \quad \textit{for all}\, x, y \in X,$$

and $\operatorname{dist}(x, F) = 0$ *if and only if* $x \in F$.

Proof. Exercise. ∎

Lemma 2.4.3 (C^∞ Urysohn lemma)**.** *Let E and F be two disjoint closed sets in $\mathbb{R}^n$. Then there exists a C^∞ function $\psi : \mathbb{R}^n \to [0,1]$ such that $\psi \equiv 1$ on E and $\psi \equiv 0$ on F.*

Proof. The proof is elementary if E is compact. Let $B(\boldsymbol{x}_i, r_i)$, $i = 1, \ldots, k$, be a finite collection of open balls in $\mathbb{R}^n \setminus F$ such that $\bigcup_{i=1}^k B(\boldsymbol{x}_i, r_i/2) \supset E$. Let ϕ_i be a C^∞ function that has properties in lemma 2.4.1 with $B = B(\boldsymbol{x}_i, r_i/2)$ and $B' = B(\boldsymbol{x}_i, r_i)$. Define ψ by

$$\psi = 1 - \prod_{i=1}^k (1 - \phi_i).$$

It is clear that ψ is a C^∞ function. On each $\boldsymbol{x} \in E$ at least one ϕ_i have the value 1, $\psi(\boldsymbol{x}) = 1$, so $\psi \equiv 1$ on E. Outside $\bigcup_{i=1}^k B(\boldsymbol{x}_i, r_i)$ each ϕ_i vanishes so $\psi(\boldsymbol{x}) = 0$ and, since F lies outside of this union, $\psi \equiv 0$ on F.

If E is not compact, then proof is more involved. Let E' and F' be two disjoint closed sets containing E and F, respectively, in the interior. Define f by

$$f(\boldsymbol{x}) = \frac{\operatorname{dist}(\boldsymbol{x}, F')}{\operatorname{dist}(\boldsymbol{x}, E') + \operatorname{dist}(\boldsymbol{x}, F')}.$$

Since $E' \cap F' = \emptyset$, the denominator is never zero, this is a continuous function on $\mathbb{R}^n$. Also, note that $f \equiv 1$ on E' and $f \equiv 0$ on F'. However, f is not necessarily a differentiable function. We get around this by introducing the *mollification* of f. Let h be as in the proof of lemma 2.4.1 and for each $\epsilon > 0$ define η_ϵ by

$$\eta_\epsilon(\boldsymbol{x}) = c(n)\, \epsilon^{-n} h(1 - |\boldsymbol{x}/\epsilon|^2),$$

where $c(n)$ is a number such that $c(n) \int_{\mathbb{R}^n} h(1 - |\boldsymbol{x}|^2)\, d\boldsymbol{x} = 1$. Then η_ϵ vanishes outside $B(\mathbf{0}, \epsilon)$ and by theorem 2.3.6 we have

$$\int_{\mathbb{R}^n} \eta_\epsilon = c(n) \int_{\mathbb{R}^n} h(1 - |\boldsymbol{x}|^2)\, d\boldsymbol{x} = 1. \tag{2.4.4}$$

Define f^ϵ by

$$f^\epsilon(\boldsymbol{x}) := \eta_\epsilon * f = \int_{\mathbb{R}^n} \eta_\epsilon(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y}.$$

We leave it as an exercise to verify that $f^\epsilon \in C^\infty(\mathbb{R}^n)$. By (2.4.4), we have

$$\begin{aligned} |f^\epsilon(\boldsymbol{x}) - f(\boldsymbol{x})| &= \left| \int_{|\boldsymbol{y}-\boldsymbol{x}| \le \epsilon} \eta_\epsilon(\boldsymbol{x} - \boldsymbol{y})[f(\boldsymbol{y}) - f(\boldsymbol{x})]\, d\boldsymbol{y} \right| \\ &\le \int_{|\boldsymbol{y}-\boldsymbol{x}| \le \epsilon} \eta_\epsilon(\boldsymbol{x} - \boldsymbol{y})\, |f(\boldsymbol{y}) - f(\boldsymbol{x})|\, d\boldsymbol{y} \le \sup_{|\boldsymbol{y}-\boldsymbol{x}| \le \epsilon} |f(\boldsymbol{y}) - f(\boldsymbol{x})|. \end{aligned}$$

Since $f \equiv 1$ on E' and $f \equiv 0$ on F', this implies that if ϵ is sufficiently small, then $\psi = f^\epsilon$ satisfies $\psi \equiv 1$ on E and $\psi \equiv 0$ on F. ■

Theorem 2.4.5 (C^∞ partitions of unity)**.** *Let $E \subset \mathbb{R}^n$ and let $\mathscr{O}$ be an open cover of E. Then there is a sequence $(\phi_i)_{i \in \mathbb{N}}$ of C^∞ functions $\phi_i : \mathbb{R}^n \to \mathbb{R}$ such that*

(a) $\phi_i(\boldsymbol{x}) \ge 0$ *for all* $\boldsymbol{x}$.

(b) each $\boldsymbol{x} \in \mathbb{R}^n$ has an open set V containing $\boldsymbol{x}$ on which only finitely many ϕ_i's are nonzero.

(c) $\sum_{i=1}^\infty \phi_i(\boldsymbol{x}) = 1$ *for* $\boldsymbol{x} \in E$.

(d) for each i, there exists $U \in \mathscr{O}$ such that $\operatorname{supp}(\phi_i) \subset U$.

(e) $\operatorname{supp}(\phi_i)$ *is compact.*

A collection of continuous functions $\phi_1, \phi_2, \phi_3, \ldots$ satisfying (a) to (c) is called a partition of unity for E. If it satisfies (d), it is said to be subordinate to the cover $\mathscr{O}$. If it satisfies (e), it is said to have compact supports.

Proof. Case 1. E is compact. Then a finite number $U_1, \ldots, U_k$ of open sets in set $\mathscr{O}$ cover E. Associate with each $\boldsymbol{x} \in E$ an open ball B_x centered at $\boldsymbol{x}$, with $B_x \subset \overline{B_x} \subset U_j$ for some j. Since $(B_x)_{\boldsymbol{x}\in E}$ is an open cover of E, there exist $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ such that $E \subset \bigcup_{i=1}^m B_{x_i}$. Let F_j be the closure of the union of those B_{x_i}'s that are subsets of U_j. Then F_j is a compact subset of U_j and by lemma 2.4.3, there exist C^∞ functions $\psi_j : \mathbb{R}^n \to [0,1]$ such that $\psi_j \equiv 1$ on F_j and $\operatorname{supp}(\psi_j) \subset U_j$. Since the F_j's cover E, we have $\sum_{j=1}^k \psi_j \geq 1$ on E, so by lemma 2.4.3 again, there exists a C^∞ function $f : \mathbb{R}^n \to [0,1]$ such that $f \equiv 1$ on E and

$$\operatorname{supp}(f) \subset \{\boldsymbol{x} : \textstyle\sum_{j=1}^k \psi_j(\boldsymbol{x}) > 0\}.$$

Let $\psi_{k+1} = 1 - f$ so that $\lambda := \sum_{j=1}^{k+1} \psi_j > 0$ everywhere. For $j = 1, \ldots, k$, let

$$\phi_j(\boldsymbol{x}) = \psi_j(\boldsymbol{x})/\lambda(\boldsymbol{x}).$$

Then $\operatorname{supp}(\phi_j) = \operatorname{supp}(\psi_j) \subset U_j$ and $\sum_{j=1}^k \phi_j = 1$ on E.

Case 2. E is open. We have $E = \bigcup_{j=1}^\infty E_j$, where

$$E_j = \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| \leq j \text{ and } \operatorname{dist}(\boldsymbol{x}, E^c) \geq 1/j\}.$$

We leave it as an exercise the verify that E_j is compact and $E_j \subset \operatorname{Int} E_{j+1}$. For convenience in notation, let E_j denote the empty set for $j \leq 0$. For each $j = 1, 2, \ldots$ let $\mathscr{O}_j$ consist of all $U \cap (\operatorname{Int} E_{j+1} \setminus E_{j-2})$ for U on $\mathscr{O}$. Then $\mathscr{O}_j$ is an open cover of the compact set $K_j := E_j \setminus \operatorname{Int} E_{j-1}$. By case 1 there is a partition of unity $\mathscr{F}_j = \{\phi_{j,1}, \ldots, \phi_{j,n_j}\}$ for K_j, subordinate to $\mathscr{O}_j$. For each $\boldsymbol{x} \in E$ the sum

$$\sigma(\boldsymbol{x}) = \textstyle\sum_{i=1}^\infty \psi_i(\boldsymbol{x}) := \sum_{j=1}^\infty \sum_{k=1}^{n_j} \phi_{j,k}(\boldsymbol{x}).$$

is a finite sum in some open set containing $\boldsymbol{x}$ since if $\boldsymbol{x} \in E_j$ we have $\psi_i(\boldsymbol{x}) = 0$ for $\psi_i \in \mathscr{F}_k$ with $k \geq j+2$. Because each $\boldsymbol{x} \in E$ has a neighborhood on which $\sigma(\boldsymbol{x})$ equals a finite sum of C^∞ functions, σ is of class C^∞. Since $\sum_k \phi_{j,k} = 1$ on K_j, we have $\sigma \geq 1$ on E. By lemma 2.4.3, there is a C^∞ function $f : \mathbb{R}^n \to [0,1]$ such that

$$f \equiv 0 \text{ on } \{\boldsymbol{x} : \sigma(\boldsymbol{x}) \geq 1\} \quad \text{and} \quad f \equiv 1 \text{ on } \{\boldsymbol{x} : \sigma(\boldsymbol{x}) = 0\}.$$

If we define $\phi_i = \psi_i/(\sigma + f)$, then the collection of all ϕ_i's is a desired partition of unity.

Case 3. E is arbitrary. Let O be the union of all U in $\mathscr{O}$. By case 2 there is a partition of unity for O; this is also a partition of unity for E. ■

Definition 2.4.6. An open cover $\mathscr{O}$ of an open set $E \subset \mathbb{R}^n$ is *admissible* if each $U \in \mathscr{O}$ is contained in E. Let $\phi_1, \phi_2, \ldots$ be a partition of unity having compact supports, subordinate to an admissible cover $\mathscr{O}$ for E; it is not necessary to assume that it is of class C^∞. Suppose that $f : E \to \mathbb{R}$ is bounded in some open set around each point of E and

$$\{\boldsymbol{x} \in E : f \text{ is discontinuous at } \boldsymbol{x}\}$$

has measure zero, then each $\int_E \phi_i|f|$ exists. We define f to be integrable (in the extended sense) if

$$\sum_{i=1}^{\infty} \int_E \phi_i|f| < \infty.$$

This implies absolute convergence of $\sum_{i=1}^{\infty} \int_E \phi_i f$, which we define to be $\int_E f$.

This definition does not depend on $\mathscr{O}$ or a partition of unity $\phi_1, \phi_2, \ldots$.

Theorem 2.4.7. *(1) If $\psi_1, \psi_2, \ldots$ is another partition of unity having compact supports, subordinate to another admissible cover $\mathscr{O}'$ of E, then $\sum_{i=1}^{\infty} \int_E \psi_i f$ also converges, and*

$$\sum_{i=1}^{\infty} \int_E \phi_i f = \sum_{i=1}^{\infty} \int_E \psi_i f.$$

(2) If E and f are bounded, then f is integrable in the extended sense.

(3) If E is Jordan measurable and f is bounded, then this definition of $\int_E f$ agrees with the old one.

Proof. (1) Since $\phi_i = 0$ except on some compact set K, and there are only finitely many ψ_j's which are non-zero on K, we can write

$$\sum_{i=1}^{\infty} \int_E \phi_i f = \sum_{i=1}^{\infty} \int_E \sum_{j=1}^{\infty} \psi_j \phi_i f = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_E \psi_j \phi_i f.$$

This result applied to $|f|$ shows the convergence of $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_E \psi_j \phi_i |f|$, and hence of $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left|\int_E \psi_j \phi_i f\right|$. This absolute convergence justifies interchanging the order of summation in the above equation; the resulting double sum clearly equals $\sum_{j=1}^{\infty} \int_E \psi_j f$. Finally, this result applied to $|f|$ proves convergence of $\sum_{j=1}^{\infty} \int_E \psi_j |f|$.

(2) If E is contained in an interval I and $|f(\boldsymbol{x})| \le M$ for $\boldsymbol{x} \in E$, then for any k, we have

$$\sum_{i=1}^{k} \int_E \phi_i |f| \le M \sum_{i=1}^{k} \int_E \phi_i \le M v(I)$$

since $\sum_{i=1}^{k} \phi_i \le 1$ on E. Therefore $\sum_{i=1}^{\infty} \int_E \phi_i |f|$ converges.

(3) For any $\epsilon > 0$ there exists a compact Jordan measurable set $F \subset E$ such that $v(E \setminus F) < \epsilon/M$; we may take $F = \underline{A}(E, k)$ for some large k (see proposition 2.2.7). Since there are only finitely many ϕ_i's which are nonzero on F and $\sum_{i=1}^{\infty} \phi_i = 1$, there exists m such that $\sum_{i=1}^{m} \phi_i \ge \chi_F$ on E. If $\int_E f$ have its old meaning, then

$$\left| \int_E f - \sum_{i=1}^{m} \int_E \phi_i f \right| \le M \int_E \left(1 - \sum_{i=1}^{m} \phi_i\right) \le M \int_E \chi_{E \setminus F} < \epsilon. \qquad \blacksquare$$

Chapter 3

Change of variables

3.1 Primitive mappings

Definition 3.1.1. If $\boldsymbol{G}$ maps an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$, and if there is an integer m and a real function g with domain U such that

$$\boldsymbol{G}(\boldsymbol{x}) = [x_1 \ \cdots \ x_{m-1} \ \ g(\boldsymbol{x}) \ \ x_{m+1} \ \cdots \ x_n]^\top \qquad (\boldsymbol{x} \in U)$$

then we call $\boldsymbol{G}$ primitive. A primitive mapping is thus one that changes at most one coordinates.

If g is differentiable at some point $\boldsymbol{a} \in U$, so is $\boldsymbol{G}$. The Jacobian of $\boldsymbol{G}$ at $\boldsymbol{a}$ is given by

$$\det D\boldsymbol{G}(\boldsymbol{a}) = D_m g(\boldsymbol{a}). \tag{3.1.2}$$

Theorem 3.1.3. *Suppose $\boldsymbol{F}$ is a C^1 mapping of an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $\mathbf{0} \in U$, $\boldsymbol{F}(\mathbf{0}) = \mathbf{0}$, and $D\boldsymbol{F}(\mathbf{0})$ is invertible. Then there is a neighborhood of $\mathbf{0}$ in $\mathbb{R}^n$ in which a representation*

$$\boldsymbol{F}(\boldsymbol{x}) = T_1 \cdots T_{n-1} \boldsymbol{G}_n \circ \cdots \circ \boldsymbol{G}_1(\boldsymbol{x}) \tag{3.1.4}$$

is valid. In (3.1.4)*, each $\boldsymbol{G}_i$ is a primitive C^1 mapping in some neighborhood of $\mathbf{0}$; $\boldsymbol{G}_i(\mathbf{0}) = \mathbf{0}$, $D\boldsymbol{G}_i(\mathbf{0})$ is invertible, and each T_i is either a linear transformation that interchange two coordinates or the identity operator.*

Proof. In the proof, we shall use the projections $P_0, \ldots, P_n$ in $\mathbb{R}^n$, defined by $P_0 \boldsymbol{x} = \mathbf{0}$ and

$$P_m \boldsymbol{x} = [x_1 \ \cdots \ x_m \ \ 0 \ \cdots \ 0]^\top$$

for $1 \le m \le n$. Thus P_m is the projection whose range and null space are spanned by $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ and $\{\boldsymbol{e}_{m+1}, \ldots, \boldsymbol{e}_n\}$, respectively.

Put $\boldsymbol{F} = \boldsymbol{F}_1$. Assume $1 \le m \le n-1$, and make the following induction hypothesis, which evidently holds for $m = 1$: V_m is a neighborhood of $\mathbf{0}$, $\boldsymbol{F}_m \in C^1(V_m)$, $\boldsymbol{F}_m(\mathbf{0}) = \mathbf{0}$, $D\boldsymbol{F}_m(\mathbf{0})$ is invertible and

$$P_{m-1} \boldsymbol{F}_m(\boldsymbol{x}) = P_{m-1} \boldsymbol{x} \qquad (\boldsymbol{x} \in V_m). \tag{3.1.5}$$

By (3.1.5), we have

$$\boldsymbol{F}_m(\boldsymbol{x}) = [x_1 \ \cdots \ x_{m-1} \ \ \alpha_m(\boldsymbol{x}) \ \cdots \ \alpha_n(\boldsymbol{x})]^\top,$$

where $\alpha_m, \ldots, \alpha_n$ are real C^1 functions in V_m. Hence

$$D_m \boldsymbol{F}_m(\mathbf{0}) = [0 \ \cdots \ 0 \ \ D_m\alpha_m(\mathbf{0}) \ \cdots \ D_m\alpha_n(\mathbf{0})]^\top.$$

Since $D\boldsymbol{F}_m(\mathbf{0})$ is invertible, $D_m\boldsymbol{F}_m(\mathbf{0})$ is not $\mathbf{0}$, and therefore there is k such that $m \leq k \leq n$ and $D_m\alpha_k(\mathbf{0}) \neq 0$.

Let T_m be the linear transformation that interchanges mth and kth coordinates (if $k = m$, T_m is the identity) and define

$$\boldsymbol{G}_m(\boldsymbol{x}) = [x_1 \ \cdots \ x_{m-1} \ \ \alpha_k(\boldsymbol{x}) \ \ x_{m+1} \ \cdots \ x_n]^\top \qquad (\boldsymbol{x} \in V_m).$$

Then $\boldsymbol{G}_m \in C^1(V_m)$, $\boldsymbol{G}_m$ is primitive, and by (3.1.2) $D\boldsymbol{G}_m(\mathbf{0})$ is invertible, since $D_m\alpha_k(\mathbf{0}) \neq 0$. The inverse function theorem shows therefore that there is an open set U_m, with $\mathbf{0} \in U_m \subset V_m$, such that $\boldsymbol{G}_m$ is 1-1 mapping of U_m onto a neighborhood V_{m+1} of $\mathbf{0}$, in which $\boldsymbol{G}_m^{-1}$ is continuously differentiable. Define $\boldsymbol{F}_{m+1}$ by

$$\boldsymbol{F}_{m+1}(\boldsymbol{y}) = T_m\boldsymbol{F}_m \circ \boldsymbol{G}_m^{-1}(\boldsymbol{y}) \qquad (\boldsymbol{y} \in V_{m+1}). \tag{3.1.6}$$

Then $\boldsymbol{F}_{m+1} \in C^1(V_{m+1})$, $\boldsymbol{F}_{m+1}(\mathbf{0}) = \mathbf{0}$, and $D\boldsymbol{F}_{m+1}(\mathbf{0})$ is invertible. Also for $\boldsymbol{x} \in U_m$,

$$\begin{aligned} P_m\boldsymbol{F}_{m+1}(\boldsymbol{G}_m(\boldsymbol{x})) &= P_mT_m\boldsymbol{F}_m(\boldsymbol{x}) \\ &= [x_1 \ \cdots \ x_{m-1} \ \ \alpha_k(\boldsymbol{x}) \ \ 0 \ \cdots \ 0]^\top = P_m\boldsymbol{G}_m(\boldsymbol{x}) \end{aligned}$$

so that

$$P_m\boldsymbol{F}_{m+1}(\boldsymbol{y}) = P_m\boldsymbol{y} \qquad (\boldsymbol{y} \in V_{m+1}).$$

Our induction hypothesis holds therefore with $m + 1$ in place of m. Since $T_mT_m = I$, (3.1.6) with $\boldsymbol{y} = \boldsymbol{G}_m(\boldsymbol{x})$, is equivalent to

$$\boldsymbol{F}_m(\boldsymbol{x}) = T_m\boldsymbol{F}_{m+1}(\boldsymbol{G}_m(\boldsymbol{x})) \qquad (\boldsymbol{x} \in U_m).$$

If we apply this with $m = 1, \ldots, n-1$, we successively obtain

$$\begin{aligned} \boldsymbol{F} = \boldsymbol{F}_1 &= T_1\boldsymbol{F}_2 \circ \boldsymbol{G}_1 = T_1T_2\boldsymbol{F}_3 \circ \boldsymbol{G}_2 \circ \boldsymbol{G}_1 = \cdots \\ &= T_1 \cdots T_{n-1}\boldsymbol{F}_n \circ \boldsymbol{G}_{n-1} \circ \cdots \circ \boldsymbol{G}_1 \end{aligned}$$

in some neighborhood of $\mathbf{0}$. By (3.1.5), $\boldsymbol{F}_n$ is primitive. ■

3.2 Change of variables: continuous integrand

Theorem 3.2.1. *Suppose $\boldsymbol{g}$ is a one-to-one C^1 mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$ such that $\det D\boldsymbol{g}(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x} \in E$. If $f : \boldsymbol{g}(E) \to \mathbb{R}$ is continuous and integrable over $\boldsymbol{g}(E)$, then*

$$\int_{\boldsymbol{g}(E)} f = \int_E f \circ \boldsymbol{g} \, |\det D\boldsymbol{g}|.$$

Proof. Take the case $n = 1$, and suppose g is a one-to-one C^1 mapping of $\mathbb{R}$ onto $\mathbb{R}$. Then $\det Dg(x) = g'(x)$ and we have

$$\int_{\mathbb{R}} f(y)\, dy = \int_{\mathbb{R}} f(g(x))|g'(x)|\, dx$$

for all continuous f with compact support.

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be any continuous function with compact support in $\boldsymbol{g}(E)$. If $\boldsymbol{g}$ is a one-to-one primitive C^1 mapping, then by Fubini's theorem and (3.1.2), we have (cf. definition 2.2.4)

$$\int_{\boldsymbol{g}(E)} \phi f = \int_{\mathbb{R}^n} \phi f = \int_{\mathbb{R}^n} (\phi f) \circ \boldsymbol{g}\, |\det D\boldsymbol{g}| = \int_E (\phi f) \circ \boldsymbol{g}\, |\det D\boldsymbol{g}|. \tag{3.2.2}$$

If the theorem is true for $\boldsymbol{g} : E \to \mathbb{R}^n$ and $\boldsymbol{h} : V \to \mathbb{R}^n$, where $\boldsymbol{g}(E) \subset V$, then by the chain rule, we have

$$\begin{aligned}\int_{\boldsymbol{h}\circ\boldsymbol{g}(E)} f &= \int_{\boldsymbol{h}(\boldsymbol{g}(E))} f = \int_{\boldsymbol{g}(E)} f \circ \boldsymbol{h}\, |\det D\boldsymbol{h}| \\ &= \int_E ((f \circ \boldsymbol{h}) \circ \boldsymbol{g})\, (|\det D\boldsymbol{h}| \circ \boldsymbol{g})\, |\det D\boldsymbol{g}| \\ &= \int_E f \circ (\boldsymbol{h} \circ \boldsymbol{g})\, |\det D(\boldsymbol{h} \circ \boldsymbol{g})|.\end{aligned}$$

Therefore, the theorem is true for $\boldsymbol{h} \circ \boldsymbol{g} : E \to \mathbb{R}^n$.

Each point $\boldsymbol{a} \in E$ has a neighborhood $U \subset E$ in which

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{a}) + T_1 \cdots T_{k-1} \boldsymbol{G}_k \circ \cdots \boldsymbol{G}_1(\boldsymbol{x} - \boldsymbol{a}),$$

where $\boldsymbol{G}_i$ and T_i are as in theorem 3.1.3. Setting $V = \boldsymbol{g}(U)$, it follows that (3.2.2) holds if the support of ϕ lies in V. Thus each point $\boldsymbol{y} \in \boldsymbol{g}(E)$ lies in an open set $V_y \subset \boldsymbol{g}(E)$ such that (3.2.2) holds for ϕ whose support lies in V_y. Note that $\{V_y\}$ is an admissible cover of $\boldsymbol{g}(E)$. Let $\{\phi_i\}_{i=1}^\infty$ be a partition of unity subordinate to this cover. If we set $\psi_i = \phi_i \circ \boldsymbol{g}$, then since $\boldsymbol{g}$ is one-to-one, $\{\psi_i\}_{i=1}^\infty$ becomes a partition of unity subordinate to an admissible cover of E. Hence by (3.2.2), we get

$$\begin{aligned}\int_{\boldsymbol{g}(E)} f &= \sum_{i=1}^\infty \int_{\boldsymbol{g}(E)} \phi_i f = \sum_{i=1}^\infty \int_E (\phi_i f) \circ \boldsymbol{g}\, |\det D\boldsymbol{g}| \\ &= \sum_{i=1}^\infty \int_E \psi_i\, (f \circ \boldsymbol{g})\, |\det D\boldsymbol{g}| = \int_E f \circ \boldsymbol{g}\, |\det D\boldsymbol{g}|. \quad \blacksquare\end{aligned} \tag{3.2.3}$$

3.3 Change of variables: general case

Lemma 3.3.1. *Let U be open in $\mathbb{R}^n$ and let $\boldsymbol{g} : U \to \mathbb{R}^n$ be a C^1 map. If $E \subset U$ has measure zero in $\mathbb{R}^n$, then the set $\boldsymbol{g}(E)$ also has measure zero in $\mathbb{R}^n$.*

Proof. Step 1. Let $\epsilon, \delta > 0$. We first show that if a set S has measure zero in $\mathbb{R}^n$, then S can be covered by countably many closed cubes, each of width less than δ, having total volume less than ϵ. To prove this fact, it suffices to show that if I is an n-dimensional interval

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

then I can be covered by finitely many cubes, each of width less than δ, having total volume less than $2v(I)$. Choose $\lambda > 0$ so that the interval

$$I_\lambda = [a_1 - \lambda, b_1 + \lambda] \times \cdots \times [a_n - \lambda, b_n + \lambda]$$

has volume less than $2v(I)$. Then choose N so that $1/N \leq \min(\delta, \lambda)$. Consider all rational numbers of the form m/N, where m is an arbitrary integer. Let c_i be the largest such number for which $c_i < a_i$, and let d_i be the smallest such number for which $d_i > b_i$. Then $[a_i, b_i] \subset [c_i, d_i] \subset [a_i - \lambda, b_i + \lambda]$. Let I' be the interval

$$I' = [c_1, d_1] \times \cdots \times [c_n, d_n],$$

which contains I and is contained in I_λ. Then $v(I') < 2v(I)$. Each of the component intervals $[c_i, d_i]$ of I' is partitioned by points of the form m/N into subintervals of length $1/N$. Then I' is partitioned into subintervals that are cubes of width $1/N$ (which is less than δ); these subintervals cover I and the total volume of these cubes equals $v(I')$.

Step 2. Let Q be a cube contained in U. Let

$$\|D\boldsymbol{g}(\boldsymbol{x})\| \leq M \quad \text{for} \quad \boldsymbol{x} \in Q.$$

We show that if Q has width l, then $\boldsymbol{g}(Q)$ is contained in a cube in $\mathbb{R}^n$ of width $\sqrt{n}Ml$. Let $\boldsymbol{a}$ be the center of Q; then $|\boldsymbol{x} - \boldsymbol{a}| < \sqrt{n}l/2$ for all $\boldsymbol{x} \in Q$. By proposition 1.3.14, we have

$$|\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{a})| \leq \sqrt{n}Ml/2.$$

It follows from this inequality that if $\boldsymbol{x} \in Q$ then $\boldsymbol{g}(\boldsymbol{x})$ lies in the ball consisting of all $\boldsymbol{y} \in \mathbb{R}^n$ such that

$$|\boldsymbol{y} - \boldsymbol{g}(\boldsymbol{a})| \leq \sqrt{n}Ml/2.$$

This ball is contained in a cube with width $\sqrt{n}Ml$, as desired.

Step 3. Now we prove the lemma. Suppose $E \subset U$ and has measure 0. We show that $\boldsymbol{g}(E)$ also has measure 0. Let C_j be a sequence of compact sets whose union is U, such that $C_j \subset \operatorname{Int} C_{j+1}$ for each j; see proof of theorem 2.4.5. Let $E_k = C_k \cap E$; it suffices to show that $\boldsymbol{g}(E_k)$ has measure 0. Given $\epsilon > 0$, we shall cover $\boldsymbol{g}(E_k)$ by cubes of total volume less than ϵ. Since C_k is compact, we can choose $\delta > 0$ so that the $\sqrt{n}\delta$-neighborhood of C_k lies in $\operatorname{Int} C_{k+1}$. Choose M so that

$$\|D\boldsymbol{g}(\boldsymbol{x})\| \leq M \quad \text{for} \quad \boldsymbol{x} \in C_{k+1}.$$

Using Step 1, cover E_k by countably many cubes, each of width less than δ, having total volume less than $\epsilon' := \epsilon/(\sqrt{n}M)^n$. Let $Q_1, Q_2, \ldots$ denote those cubes that actually intersect E_k. Because Q_i has width less than δ, it is contained in C_{k+1}. Then $\|D\boldsymbol{g}(\boldsymbol{x})\| \leq M$ for $\boldsymbol{x} \in Q_i$, so that $\boldsymbol{g}(Q_i)$ lies in a cube Q_i' whose width is $\sqrt{n}M$ times that of of Q_i, by Step 2. The cube Q_i' has volume

$$v(Q_i') = (\sqrt{n}M)^n v(Q_i).$$

Therefore, the cubes Q_i', which cover $\boldsymbol{g}(E_k)$, have total volume less than ϵ. ■

Corollary 3.3.2. *Let U be open in $\mathbb{R}^n$ and let $\boldsymbol{g} : U \to \mathbb{R}^n$ be a one-to-one C^1 map such that* $\det D\boldsymbol{g} \neq 0$ *on U. If E is a compact subset of U, then we have*

$$\boldsymbol{g}(\operatorname{Int} E) = \operatorname{Int} \boldsymbol{g}(E) \quad \textit{and} \quad \boldsymbol{g}(\partial E) = \partial \boldsymbol{g}(E).$$

If E is Jordan measurable, so is $\boldsymbol{g}(E)$.

Proof. Exercise. ■

Lemma 3.3.3. *Let $\boldsymbol{g} : U \to \mathbb{R}$ be continuously differentiable, where $U \subset \mathbb{R}^n$ is open, and let $E = \{\boldsymbol{x} \in U : \det D\boldsymbol{g}(\boldsymbol{x}) = 0\}$. Then $\boldsymbol{g}(E)$ has measure 0.*

Proof. Let $Q \subset U$ be a cube of width l. If N is sufficiently large and Q is divided into N^n cubes of width l/N, then for each of these cubes S, if $\boldsymbol{x} \in S$ we have

$$|D\boldsymbol{g}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{x})| < \epsilon|\boldsymbol{x} - \boldsymbol{y}| \leq \epsilon\sqrt{n}(l/N)$$

for all $\boldsymbol{y} \in S$. If S intersects E we can choose $\boldsymbol{x} \in S \cap E$; since $\det D\boldsymbol{g}(\boldsymbol{x}) = 0$, the set $\{D\boldsymbol{g}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x}) : \boldsymbol{y} \in S\}$ lies in an $(n-1)$-dimensional subspace V of $\mathbb{R}^n$. Therefore the set $\{\boldsymbol{g}(\boldsymbol{y}) : \boldsymbol{y} \in S\}$ lies within $\epsilon\sqrt{n}(l/N)$ of the $(n-1)$-plane $V + \boldsymbol{g}(\boldsymbol{x})$. On the other hand, by proposition 1.3.14 there is a number M such that

$$|\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y})| \leq M|x - y| \leq M\sqrt{n}(l/N).$$

Thus, if S intersects E, the set $\{\boldsymbol{g}(\boldsymbol{y}) : \boldsymbol{y} \in S\}$ is contained in a cylinder whose height is less than $2\epsilon\sqrt{n}(l/N)$ and whose base is an $(n-1)$-dimensional sphere of radius less than $M\sqrt{n}(l/N)$. This cylinder has volume less than $C(l/N)^n\epsilon$ for some constant C. There are at most N^n such cubes S, so $\boldsymbol{g}(Q \cap E)$ lies in a set of volume less than $C(l/N)^n\epsilon N^n = Cl^n\epsilon$. Since this is true for all $\epsilon > 0$, the set $\boldsymbol{g}(Q \cap E)$ has measure 0. Since we can cover all of U with a sequence of such cubes Q, the desired result follows. ■

Theorem 3.3.4. *Suppose $\boldsymbol{g}$ is a one-to-one C^1 mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$. If $f : \boldsymbol{g}(E) \to \mathbb{R}$ is integrable, then*

$$\int_{\boldsymbol{g}(E)} f = \int_E f \circ \boldsymbol{g}\,|\det D\boldsymbol{g}|.$$

Proof. Let $E' = \{\boldsymbol{x} \in E : \det D\boldsymbol{g}(x) \neq 0\}$. By lemmas 3.3.3 and 3.3.1, the proof is complete if we show

$$\int_{\boldsymbol{g}(E')} f = \int_{E'} f \circ \boldsymbol{g}\,|\det D\boldsymbol{g}|.$$

Note that E' and $\boldsymbol{g}(E')$ are open sets. By theorem 3.2.1, for any open set $U \subset E'$ and a constant function $f \equiv c$, we have

$$\int_{\boldsymbol{g}(U)} c = \int_U c\,|\det D\boldsymbol{g}|. \tag{3.3.5}$$

Let I be an n-dimensional interval in $\boldsymbol{g}(E')$ and P a partition for I. For each subinterval S of P let f_S denote the constant function $m_S(f)$. By (3.3.5)

$$\begin{aligned} L(f, P) = \sum_S m_S(f)v(S) &= \sum_S \int_{\operatorname{Int} S} f_S = \sum_S \int_{\boldsymbol{g}^{-1}(\operatorname{Int} S)} (f_S \circ \boldsymbol{g})\,|\det D\boldsymbol{g}| \\ &\leq \sum_S \int_{\boldsymbol{g}^{-1}(\operatorname{Int} S)} (f \circ \boldsymbol{g})\,|\det D\boldsymbol{g}| \leq \int_{\boldsymbol{g}^{-1}(I)} (f \circ \boldsymbol{g})\,|\det D\boldsymbol{g}|. \end{aligned}$$

Since $\int_I f$ is the least upper bound of all $L(f, P)$, this proves that

$$\int_I f \leq \int_{\boldsymbol{g}^{-1}(I)} (f \circ \boldsymbol{g})\,|\det D\boldsymbol{g}|.$$

By corollary 3.3.2, the equality (3.3.5) is still true if U is a Jordan measurable set in E'. Therefore, a similar argument, letting $f_S = M_S(f)$ and using corollary 2.1.21, shows that

$$\int_I f \geq \int_{\boldsymbol{g}^{-1}(I)} (f \circ \boldsymbol{g}) \, |\det D\boldsymbol{g}|.$$

We have shown that if f is an integrable function on $\boldsymbol{g}(E')$, then for any interval $I \subset \boldsymbol{g}(E')$, we have

$$\int_I f = \int_{\boldsymbol{g}^{-1}(I)} (f \circ \boldsymbol{g}) \, |\det D\boldsymbol{g}|.$$

For each $\boldsymbol{x} \in \boldsymbol{g}(E')$, let $I_x \subset \boldsymbol{g}(E')$ be an n-dimensional interval containing $\boldsymbol{x}$. Let $(\phi_i)_{i \in \mathbb{N}}$ be a partition of unity subordinate to the cover $(\operatorname{Int} I_x)_{\boldsymbol{x} \in E'}$ and let $\psi_i = \phi_i \circ \boldsymbol{g}$. Then similar to (3.2.3), we get

$$\int_{\boldsymbol{g}(E')} f = \sum_{i=1}^{\infty} \int_{\boldsymbol{g}(E')} \phi_i f = \sum_{i=1}^{\infty} \int_{E'} \psi_i \, (f \circ \boldsymbol{g}) \, |\det D\boldsymbol{g}| = \int_{E'} f \circ \boldsymbol{g} \, |\det D\boldsymbol{g}|. \quad \blacksquare$$

Chapter 4

Differential forms

4.1 Determinants and volumes

A volume function V in $\mathbb{R}^n$ is a function which assigns to each n-tuples of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ in $\mathbb{R}^n$ a real number $V(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ such that

1. $V(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) \geq 0$.
2. $V(\ldots, \boldsymbol{v}_i + \boldsymbol{v}_j, \ldots) = V(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n), \quad i \neq j$.
3. $V(\ldots, a\boldsymbol{v}_i, \ldots) = |a| V(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n), \quad a \in \mathbb{R}$.
4. $V(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) = 1$.

Such a function can be interpreted as the volume of the n-dimensional parallelepiped, with edges $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, which consists of all vectors $\boldsymbol{x} = \sum t_i \boldsymbol{v}_i$ for $0 \leq t_i \leq 1$. The connection between volume functions and determinants is given in the following theorem.

Theorem 4.1.1. *There is one and only one volume function $V(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ on $\mathbb{R}^n$, which is given by*

$$V(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = |\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)|.$$

Proof. Clearly, $|\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)|$ is a volume function. Now let V be a volume function, and define

$$D(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \operatorname{sgn}(\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)) V(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n),$$

where $\operatorname{sgn} x = 1$ if $x > 0$, $\operatorname{sgn} x = -1$ if $x < 0$, $\operatorname{sgn} x = 0$ if $x = 0$. Then one verifies easily that D satisfies the axioms for a determinant function and hence that

$$D(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n).$$

The theorem then follows from the definition of D. ∎

4.2 Forms in $\mathbb{R}^n$

Definition 4.2.1. A k-form on $\mathbb{R}^n$ is a function ω that takes k vectors in $\mathbb{R}^n$ and returns a number $\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)$, such that ω is multilinear and antisymmetric (or alternating) as a function of the vectors.

Antisymmetry means that if you change two of the arguments of ω, you change the sign of ω:

$$\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_k) = -\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_k).$$

Multilinearity means that if $\boldsymbol{v}_i = a\boldsymbol{u} + b\boldsymbol{w}$, then

$$\omega(\boldsymbol{v}_1, \ldots, (a\boldsymbol{u} + b\boldsymbol{w}), \ldots, \boldsymbol{v}_k) = a\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{u}, \ldots, \boldsymbol{v}_k) + b\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{w}, \ldots, \boldsymbol{v}_k).$$

Let $i_1, \ldots, i_k$ be any k integers between 1 and n. Then $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is that function of k vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ in $\mathbb{R}^n$ that puts these vectors side by side, make $n \times k$ matrix

$$\begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,k} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n,1} & v_{n,2} & \cdots & v_{n,k} \end{bmatrix}$$

and selects k rows: first row i_1, then row i_2, etc., and finally row i_k, making a square $k \times k$ matrix, and finally takes its determinant. For instance,

$$dx_2 \wedge dx_1 \wedge dx_4 \left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right) = \det \begin{bmatrix} 2 & -2 & 1 \\ 1 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = 7.$$

Evaluating 2-form $dx_1 \wedge dx_2$ on the vectors $\boldsymbol{a} = [a_1\ a_2\ a_3]^\top$ and $\boldsymbol{b} = [b_1\ b_2\ b_3]^\top$, we have

$$dx_1 \wedge dx_2 \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) = \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1. \tag{4.2.2}$$

If we project $\boldsymbol{a}$ and $\boldsymbol{b}$ onto the (x_1, x_2)-plane, we get the vectors

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix};$$

the determinant in (4.2.2) gives the *signed area of the parallelogram that they span*. Thus, $dx_1 \wedge dx_2$ deserves to be called the (x_1, x_2)-component of signed area. Similarly, $dx_2 \wedge dx_3$ and $dx_1 \wedge dx_3$ deserve to be called (x_2, x_3)- and (x_1, x_3)-components of signed area.

Definition 4.2.3. An elementary k-form on $\mathbb{R}^n$ is a expression of the form

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $1 \leq i_1 < \cdots < i_k \leq n$. Evaluated on the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$, it gives the determinant of $k \times k$ matrix obtained by selecting rows $i_1, \ldots, i_k$ of the $n \times k$ matrix whose columns are the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$. The only elementary 0-form is the form, denoted 1, which evaluated on zero vectors and returns 1.

4.3 Multilinear algebra

In this section, we suppose that $\mathcal{V}$ is a finite dimensional vector space over $\mathbb{R}$ and let $\mathcal{V}^*$ denote its dual space. Then $\mathcal{V}^*$ is the space whose elements are linear functions from $\mathcal{V}$ to $\mathbb{R}$. We will denote the k-fold product $\mathcal{V} \times \cdots \times \mathcal{V}$ by $\mathcal{V}^k$. The set of all multilinear functions $T : \mathcal{V}^k \to \mathbb{R}$, denoted $\mathscr{T}^k(\mathcal{V})$, becomes a vector space over $\mathbb{R}$ if for $S, T \in \mathscr{T}^k(\mathcal{V})$ and $a \in \mathbb{R}$ we define

$$\begin{aligned}(S+T)(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k) &= S(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k) + T(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k),\\ (aS)(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k) &= aS(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k).\end{aligned}$$

If $S \in \mathscr{T}^k(\mathcal{V})$ and $T \in \mathscr{T}^l(\mathcal{V})$, we define the *tensor product* $S \otimes T \in \mathscr{T}^{k+l}(\mathcal{V})$ by

$$S \otimes T(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_{k+l}) = S(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k) T(\boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_{k+l}).$$

Note that $S \otimes T \neq T \otimes S$ in general. The following properties of $\otimes$ hold.

1. $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$.
2. $S \otimes (T_1 + T_2) = S \otimes T_1 + S \otimes T_2$.
3. $(aS) \otimes T = S \otimes (aT) = a(S \otimes T)$.
4. $(T_1 \otimes T_2) \otimes T_3 = T_1 \otimes (T_2 \otimes T_3)$.

Both $(T_1 \otimes T_2) \otimes T_3$ and $T_1 \otimes (T_2 \otimes T_3)$ are usually denoted simply $T_1 \otimes T_2 \otimes T_3$; higher-order products $T_1 \otimes \cdots \otimes T_r$ are defined similarly.

Theorem 4.3.1. *Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ be a basis for $\mathcal{V}$, and let $\varphi_1, \dots, \varphi_n$ be the dual basis, $\varphi_i(\boldsymbol{v}_j) = \delta_{ij}$. The the set of all k-fold tensor products*

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}, \quad 1 \leq i_1, \dots, i_k \leq n$$

is a basis for $\mathscr{T}^k(\mathcal{V})$, which therefore has dimension n^k.

Proof. Note that

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}(\boldsymbol{v}_{j_1}, \dots, \boldsymbol{v}_{j_k}) = \delta_{i_1,j_1} \cdots \delta_{i_k,j_k} = \begin{cases} 1 & \text{if } j_1 = i_1, \dots, j_k = i_k, \\ 0 & \text{otherwise.} \end{cases}$$

If $\boldsymbol{w}_1, \dots, \boldsymbol{w}_k$ are k vectors with $\boldsymbol{w}_i = \sum_{j=1}^n a_{i,j} \boldsymbol{v}_j$ and T is in $\mathscr{T}^k(\mathcal{V})$, then

$$\begin{aligned}T(\boldsymbol{w}_1, \dots, \boldsymbol{w}_k) &= \sum_{j_1,\dots,j_k=1}^n a_{1,j_1} \cdots a_{k,j_k} T(\boldsymbol{v}_{j_1}, \dots, \boldsymbol{v}_{j_k})\\ &= \sum_{i_1,\dots,i_k=1}^n T(\boldsymbol{v}_{i_1}, \dots, \boldsymbol{v}_{i_k})\, \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}(\boldsymbol{w}_1, \dots, \boldsymbol{w}_k).\end{aligned}$$

Thus $T = \sum_{i_1,\dots,i_k=1}^n T(\boldsymbol{v}_{i_1}, \dots, \boldsymbol{v}_{i_k})\, \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$. Consequently the $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ span $\mathscr{T}^k(\mathcal{V})$. Suppose now that there are numbers $a_{i_1,\dots,i_k}$ such that

$$\sum_{i_1,\dots,i_k=1}^n a_{i_1,\dots,i_k} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} = 0.$$

Applying both sides of this equation to $(v_{j_1}, \dots, v_{j_k})$ yields $a_{j_1,\dots,j_k} = 0$. Thus the $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ are linearly independent. ∎

Definition 4.3.2. Let $f : \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then the linear transformation $f^* : \mathscr{T}^k(\mathcal{W}) \to \mathscr{T}^k(\mathcal{V})$ is defined by

$$f^*T(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) = T(f(\boldsymbol{v}_1), \ldots, f(\boldsymbol{v}_k)),$$

for $T \in \mathscr{T}^k(\mathcal{W})$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathcal{V}$.

Proposition 4.3.3. *Let $f : \mathcal{V} \to \mathcal{W}$ and $g : \mathcal{W} \to \mathcal{X}$ are linear transformations. Then*

$$(f \circ g)^*(T) = g^*(f^*T).$$
$$f^*(S \otimes T) = f^*S \otimes f^*T.$$

Definition 4.3.4. A k-tensor $\omega \in \mathscr{T}^k(\mathcal{V})$ is called *alternating (or antisymmetric)* if

$$\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_k) = -\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_k)$$

for all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathcal{V}$. The set of all alternating k tensors is denoted $\Lambda^k(\mathcal{V})^*$.

It is clear that $\Lambda^k(\mathcal{V})^*$ is a subspace of $\mathscr{T}^k(\mathcal{V})$.

Since it requires considerable work to produce the determinant, it is not surprising that alternation k-tensors are difficult to write down. There is, however, a uniform way of expressing all of them.

Definition 4.3.5. If $T \in \mathscr{T}^k(\mathcal{V})$, we define $\mathscr{A}(T)$ by

$$\mathscr{A}(T)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon(\sigma) T(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(k)}).$$

Theorem 4.3.6.

(1) *If $T \in \mathscr{T}^k(\mathcal{V})$, then $\mathscr{A}(T) \in \Lambda^k(\mathcal{V})^*$.*

(2) *If $\omega \in \Lambda^k(\mathcal{V})^*$, then $\mathscr{A}(\omega) = \omega$.*

(3) *If $T \in \mathscr{T}^k(\mathcal{V})$, then $\mathscr{A}(\mathscr{A}(T)) = \mathscr{A}(T)$.*

Proof. Exercise. ∎

The above theorem shows that $\mathscr{A} : \mathscr{T}^k(\mathcal{V}) \to \Lambda^k(\mathcal{V})^*$ is a surjective projection. The following is straightforward.

Proposition 4.3.7. *Let $\omega \in \Lambda^k(\mathcal{V})^*$ and $\sigma \in S_k$. Then*

$$\omega(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(k)}) = \epsilon(\sigma)\omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k).$$

To determine the dimensions of $\Lambda^k(\mathcal{V})^*$, we would like a theorem analogous to Theorem 4.3.1. Of course, if $\omega \in \Lambda^k(\mathcal{V})^*$ and $\eta \in \Lambda^l(\mathcal{V})^*$, then $\omega \otimes \eta$ is usually not in $\Lambda^{k+l}(\mathcal{V})^*$.

Definition 4.3.8. For $\omega \in \Lambda^k(\mathcal{V})^*$ and $\eta \in \Lambda^l(\mathcal{V})^*$, we define the wedge product $\omega \wedge \eta \in \Lambda^{k+l}(\mathcal{V})^*$ by

$$\omega \wedge \eta = \frac{(k+l)!}{k!\, l!} \mathscr{A}(\omega \otimes \eta).$$

(The reason for the strange coefficients will appear later.)

Proposition 4.3.9. *The following properties hold.*

1. $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$.
2. $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$.
3. $a\omega \wedge \eta = \omega \wedge a\eta = a(\omega \wedge \eta)$.
4. $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.
5. $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$.

Proof. We only prove property 4 since the rest are straightforward. Let $\tau \in S_{k+l}$ be defined by

$$\tau(i) = \begin{cases} i + k & \text{for } i = 1, \dots, l, \\ i - l & \text{for } i = l + 1, \dots, l + k. \end{cases}$$

Then we have

$$\begin{aligned} &\mathscr{A}(\omega \otimes \eta)(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k+l}) \\ &\qquad = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \epsilon(\sigma) \omega(\boldsymbol{v}_{\sigma(1)}, \dots, \boldsymbol{v}_{\sigma(k)}) \eta(\boldsymbol{v}_{\sigma(k+1)}, \dots, \boldsymbol{v}_{\sigma(k+l)}) \\ &\qquad = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \epsilon(\sigma) \eta(\boldsymbol{v}_{\sigma\tau(1)}, \dots, \boldsymbol{v}_{\sigma\tau(l)}) \omega(\boldsymbol{v}_{\sigma\tau(l+1)}, \dots, \boldsymbol{v}_{\sigma\tau(l+k)}) \\ &\qquad = \epsilon(\tau) \mathscr{A}(\eta \otimes \omega)(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k+l}). \end{aligned}$$

We leave it as an exercise to verify that $\epsilon(\tau) = (-1)^{kl}$. ■

Theorem 4.3.10.

(1) *If $S \in \mathscr{T}^k(\mathcal{V})$ and $T \in \mathscr{T}^l(\mathcal{V})$ and $\mathscr{A}(S) = 0$, then*

$$\mathscr{A}(S \otimes T) = \mathscr{A}(T \otimes S) = 0$$

(2) $\mathscr{A}(\mathscr{A}(\omega \otimes \eta) \otimes \theta) = \mathscr{A}(\omega \otimes \eta \otimes \theta) = \mathscr{A}(\omega \otimes \mathscr{A}(\eta \otimes \theta))$.

(3) *If $\omega \in \Lambda^k(\mathcal{V})^*$, $\eta \in \Lambda^l(\mathcal{V})^*$, and $\theta \in \Lambda^m(\mathcal{V})^*$, then*

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!\, l!\, m!} \mathscr{A}(\omega \otimes \eta \otimes \theta).$$

Proof. (1) If $G \subset S_{k+l}$ consists of all σ which leave $k+1, \dots, k+l$ fixed, then

$$\begin{aligned} &\sum_{\sigma \in G} \epsilon(\sigma) S(\boldsymbol{v}_{\sigma(1)}, \dots, \boldsymbol{v}_{\sigma(k)}) T(\boldsymbol{v}_{\sigma(k+1)}, \dots, \boldsymbol{v}_{\sigma(k+l)}) \\ &\qquad = \left(\sum_{\sigma' \in S_k} \epsilon(\sigma') S(\boldsymbol{v}_{\sigma'(1)}, \dots, \boldsymbol{v}_{\sigma'(k)}) \right) T(\boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_{k+l}) = 0. \end{aligned}$$

Suppose now that $\sigma_0 \notin G$. Let $G\sigma_0 = \{\sigma\sigma_0 : \sigma \in G\}$ and let $w_i = v_{\sigma_0(i)}$ for $i = 1, \dots, k+l$. Then

$$\sum_{\sigma \in G\sigma_0} \epsilon(\sigma) S(\boldsymbol{v}_{\sigma(1)}, \dots, \boldsymbol{v}_{\sigma(k)}) T(\boldsymbol{v}_{\sigma(k+1)}, \dots, \boldsymbol{v}_{\sigma(k+l)})$$
$$= \epsilon(\sigma_0) \left(\sum_{\sigma' \in S_k} \epsilon(\sigma') S(\boldsymbol{w}_{\sigma'(1)}, \dots, \boldsymbol{w}_{\sigma'(k)}) \right) T(\boldsymbol{w}_{k+1}, \dots, \boldsymbol{w}_{k+l}) = 0.$$

Notice that $G \cap G\sigma_0 = \emptyset$. We can then continue in this way, breaking S_{k+l} up into disjoint subsets; the sum over each subset is 0, so that

$$(k+l)! \mathscr{A}(S \otimes T)(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k+l})$$
$$= \sum_{\sigma \in S_{k+l}} \epsilon(\sigma) S(\boldsymbol{v}_{\sigma(1)}, \dots, \boldsymbol{v}_{\sigma(k)}) T(\boldsymbol{v}_{\sigma(k+1)}, \dots, \boldsymbol{v}_{\sigma(k+l)}) = 0.$$

The relation $\mathscr{A}(T \otimes S) = 0$ is proved similarly.

(2) We have

$$\mathscr{A}(\mathscr{A}(\eta \otimes \theta) - \eta \otimes \theta) = \mathscr{A}(\eta \otimes \theta) - \mathscr{A}(\eta \otimes \theta) = 0.$$

Hence by (1) we have

$$0 = \mathscr{A}\left(\omega \otimes (\mathscr{A}(\eta \otimes \theta) - \eta \otimes \theta)\right) = \mathscr{A}\left(\omega \otimes \mathscr{A}(\eta \otimes \theta)\right) - \mathscr{A}(\omega \otimes \eta \otimes \theta).$$

The other equality is proved similarly.

(3) We have

$$(\omega \wedge \eta) \wedge \theta = \frac{(k+l+m)!}{(k+l)!\, m!} \mathscr{A}((\omega \wedge \eta) \otimes \theta)$$
$$= \frac{(k+l+m)!}{(k+l)!\, m!} \frac{(k+l)!}{k!\, l!} \mathscr{A}(\omega \otimes \eta \otimes \theta).$$

The other equality is proved similarly.

■

Naturally $(\omega \wedge \eta) \wedge \theta$ and $\omega \wedge (\eta \wedge \theta)$ are both denoted simply $\omega \wedge \eta \wedge \theta$, and higher-order products $\omega_1 \wedge \dots \wedge \omega_r$ are defined similarly.

Theorem 4.3.11. *If $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ is a basis for $\mathcal{V}$ and $\varphi_1, \dots, \varphi_n$ is the dual basis, then the set of all*

$$\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k} \quad 1 \le i_1 < i_2 < \dots < i_k \le n$$

is a basis for $\Lambda^k(\mathcal{V})^*$, which has therefore the dimension

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Proof. If $\omega \in \Lambda^k(\mathcal{V})^* \subset \mathscr{T}^k(\mathcal{V})$, then we can write

$$\omega = \sum_{j_1,\ldots,j_k} a_{j_1,\ldots,j_k} \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k}.$$

Thus

$$\omega = \mathscr{A}(\omega) = \sum_{j_1,\ldots,j_k} a_{j_1,\ldots,j_k} \mathscr{A}(\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k}).$$

Note that each $\mathscr{A}(\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k})$ is a constant times one of the $\varphi_{j_1} \wedge \cdots \wedge \varphi_{j_k}$ and $\varphi_{j_1} \wedge \cdots \wedge \varphi_{j_k} = s(j_1, \ldots, j_k)\, \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$, where $i_1, \ldots, i_k$ is non-decreasing rearrangement of $j_1, \ldots, j_k$ and $s(j_1, \ldots, j_k)$ is as defined in definition 1.6.16. Therefore, these elements span $\Lambda^k(\mathcal{V})^*$. Linear independence is proved as in Theorem 4.3.1. ■

Theorem 4.3.12. *The wedge product $\omega \wedge \eta$ of the forms $\omega \in \Lambda^k(\mathcal{V})^*$ and $\eta \in \Lambda^l(\mathcal{V})^*$ is given by*

$$(\omega \wedge \eta)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+l}) = \sum_{\sigma \in S(k,l)} \epsilon(\sigma)\omega(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(k)})\eta(\boldsymbol{v}_{\sigma(k+1)}, \ldots, \boldsymbol{v}_{\sigma(k+l)}),$$

where $\sigma \in S(k,l)$ means that $\sigma \in S_{k+l}$ and is such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$.

Proof. We note that any $\sigma \in S_{k+l}$ can be written as $\sigma = \tau_1\tau_2\sigma'$, where

1. τ_1 is a permutation of $\{\sigma(1), \ldots, \sigma(k)\}$ and leaves others fixed.
2. τ_2 is a permutation of $\{\sigma(k+1), \ldots, \sigma(k+l)\}$ and leaves others fixed.
3. $\sigma' \in S(k,l)$.

By proposition 4.3.7, we have

$$\begin{aligned}\epsilon(\tau_1)\omega(\boldsymbol{v}_{\sigma'(1)}, \ldots, \boldsymbol{v}_{\sigma'(k)})\, \epsilon(\tau_2)\eta(\boldsymbol{v}_{\sigma'(k+1)}, \ldots, \boldsymbol{v}_{\sigma'(k+l)}) \\ = \omega(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(k)})\eta(\boldsymbol{v}_{\sigma(k+1)}, \ldots, \boldsymbol{v}_{\sigma(k+l)}).\end{aligned}$$

Therefore,

$$\begin{aligned}\epsilon(\sigma)\omega(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(k)})\eta(\boldsymbol{v}_{\sigma(k+1)}, \ldots, \boldsymbol{v}_{\sigma(k+l)}) \\ = \epsilon(\sigma')\omega(\boldsymbol{v}_{\sigma'(1)}, \ldots, \boldsymbol{v}_{\sigma'(k)})\eta(\boldsymbol{v}_{\sigma'(k+1)}, \ldots, \boldsymbol{v}_{\sigma'(k+l)}).\end{aligned}$$

Note that for each $\sigma' \in S(k,l)$ corresponds to $k!\,l!$ members of $\sigma \in S_{k+l}$. Therefore, we have

$$\begin{aligned}&(\omega \wedge \eta)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+l}) \\ &\qquad = \frac{(k+l)!}{k!\,l!}\mathscr{A}(\omega \otimes \eta)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+l}) \\ &\qquad = \frac{1}{k!\,l!}\sum_{\sigma \in S_{k+l}} \epsilon(\sigma)\omega(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(k)})\eta(\boldsymbol{v}_{\sigma(k+1)}, \ldots, \boldsymbol{v}_{\sigma(k+l)}) \\ &\qquad = \sum_{\sigma' \in S(k,l)} \epsilon(\sigma')\omega(\boldsymbol{v}_{\sigma'(1)}, \ldots, \boldsymbol{v}_{\sigma'(k)})\eta(\boldsymbol{v}_{\sigma'(k+1)}, \ldots, \boldsymbol{v}_{\sigma'(k+l)}).\end{aligned}$$ ■

Theorem 4.3.13. *For $\varphi_1, \ldots, \varphi_k \in \Lambda^1(\mathcal{V})^*$, we have*

$$\varphi_1 \wedge \cdots \wedge \varphi_k(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) = \det\left[\varphi_i(\boldsymbol{v}_j)\right]. \tag{4.3.14}$$

Here $[\varphi_i(\boldsymbol{v}_j)]$ denotes the $k \times k$ matrix with $\varphi_i(\boldsymbol{v}_j)$ as its (i,j)-entry.

Proof. We will prove the formula (4.3.14) by induction on k. For $k = 1$, (4.3.14) is obvious. Suppose (4.3.14) is true for some $k \geq 1$. Then by theorem 4.3.12, we have

$$\begin{aligned}(\varphi_1 \wedge \cdots \wedge \varphi_k) &\wedge \varphi_{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}) \\ &= \sum \epsilon(\sigma)(\varphi_1 \wedge \cdots \wedge \varphi_k)(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(k)})\varphi_{k+1}(\boldsymbol{v}_{\sigma(k+1)}),\end{aligned} \tag{4.3.15}$$

where the sum is over all $\sigma \in S_{k+1}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$. For such a σ, if $\sigma(k+1) = j$, then we must have

$$(\sigma(1), \ldots, \sigma(k)) = (1, \ldots, j-1, j+1, \ldots, k+1)$$

and thus $\epsilon(\sigma) = (-1)^{k+1-j} = (-1)^{k+1+j}$. Therefore, the RHS of (4.3.15) becomes

$$\sum_{j=1}^{k+1} (-1)^{k+1+j} \varphi_{k+1}(\boldsymbol{v}_j)\,(\varphi_1 \wedge \cdots \wedge \varphi_k)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}, \boldsymbol{v}_{j+1}, \ldots, \boldsymbol{v}_{k+1}). \tag{4.3.16}$$

By the induction hypothesis

$$(\varphi_1 \wedge \cdots \wedge \varphi_k)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}, \boldsymbol{v}_{j+1}, \ldots, \boldsymbol{v}_{k+1})$$

is determinant of the $k \times k$ submatrix that remains after $(k+1)$th row and jth column are deleted from $A = [\varphi_i(\boldsymbol{v}_j)]$. Therefore, (4.3.16) is simply the cofactor expansion along $(k+1)$th row of $A = [\varphi_i(\boldsymbol{v}_j)]$. ∎

4.4 Forms on $\mathbb{R}^n$: continued

If we take $\varphi_1 = dx_{i_1}$, …, $\varphi_k = dx_{i_k}$ in theorem 4.3.13, then we have

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) = \det \begin{bmatrix} v_{i_1,1} & v_{i_1,2} & \cdots & v_{i_1,k} \\ v_{i_2,1} & v_{i_2,2} & \cdots & v_{i_2,k} \\ \vdots & \vdots & \ddots & \vdots \\ v_{i_k,1} & v_{i_k,2} & \cdots & v_{i_k,k} \end{bmatrix},$$

which agrees with definition 4.2.3. In $\mathbb{R}^2$, for example, we see that the 2-form

$$dx_1 \wedge dx_2(\boldsymbol{a}, \boldsymbol{b}) = \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1,$$

is indeed equal to the wedge product of the 1-forms dx_1 and dx_2, which evaluated on the same two vectors, gives (by theorem 4.3.12)

$$dx_1 \wedge dx_2(\boldsymbol{a}, \boldsymbol{b}) = dx_1(\boldsymbol{a})dx_2(\boldsymbol{b}) - dx_1(\boldsymbol{b})dx_2(\boldsymbol{a}) = a_1 b_2 - a_2 b_1.$$

Similarly, set $\omega = dx_1 \wedge dx_2$ and $\varphi = dx_3$ and use definition 4.2.3 to get

$$\begin{aligned}
dx_1 \wedge dx_2 \wedge dx_3(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \\
&= u_1v_2w_3 - u_1v_3w_2 - u_2v_1w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1.
\end{aligned}$$

If instead we use theorem 4.3.12 for the wedge product, we get

$$\begin{aligned}
(dx_1 \wedge dx_2) \wedge dx_3(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= dx_1 \wedge dx_2(\boldsymbol{u}, \boldsymbol{v}) dx_3(\boldsymbol{w}) - dx_1 \wedge dx_2(\boldsymbol{u}, \boldsymbol{w}) dx_3(\boldsymbol{v}) \\
&\quad + dx_1 \wedge dx_2(\boldsymbol{v}, \boldsymbol{w}) dx_3(\boldsymbol{u}) \\
&= \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} w_3 - \det \begin{bmatrix} u_1 & w_1 \\ u_2 & w_2 \end{bmatrix} v_3 + \det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} u_3 \\
&= u_1v_2w_3 - u_2v_1w_3 - u_1v_3w_2 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1.
\end{aligned}$$

The following is an immediate corollary of theorem 4.3.11.

Corollary 4.4.1. *The elementary k-forms form a basis for $\Lambda^k(\mathbb{R}^n)^*$: every k-form can be uniquely written*

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

The coefficients $a_{i_1 \ldots i_k}$ are given by

$$a_{i_1 \ldots i_k} = \omega(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_k}).$$

The space $\Lambda^k(\mathbb{R}^n)^*$ has dimension equal to the binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

The following is a corollary of proposition 4.3.7

Corollary 4.4.2. *Let $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ be an elementary k-form with $1 \le i_1 < \cdots < i_k \le n$. Then*

$$dx_{i_{\sigma(1)}} \wedge \cdots \wedge dx_{i_{\sigma(k)}} = \epsilon(\sigma)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

4.5 Form fields

Definition 4.5.1. If $p \in \mathbb{R}^n$, the set of all pairs $(p, \boldsymbol{v})$, $\boldsymbol{v} \in \mathbb{R}^n$, is denoted $\mathbb{R}^n_p$ and called the tangent space of $\mathbb{R}^n$ at p. This set is made into a vector space, by defining

$$\begin{aligned}
(p, \boldsymbol{v}) + (p, \boldsymbol{w}) &= (p, \boldsymbol{v} + \boldsymbol{w}) \\
a\,(p, \boldsymbol{v}) &= (p, a\boldsymbol{v}).
\end{aligned}$$

A vector $\boldsymbol{v} \in \mathbb{R}^n$ is often pictured as an arrow from 0 to $\boldsymbol{v}$; the vector $(p, \boldsymbol{v}) \in \mathbb{R}^n_p$ may be pictured as an arrow with the same direction and length, but with initial point p.

We will usually write $(p, \boldsymbol{v})$ as $\boldsymbol{v}_p$, the vector $\boldsymbol{v}$ at p. The standard basis of $\mathbb{R}^n_p$ is $(\boldsymbol{e}_1)_p, \ldots, (\boldsymbol{e}_n)_p$.

Definition 4.5.2. A *a differential k-form* (or simply a k-form) on an open subset $U \subset \mathbb{R}^n$ is a map ω that maps $p \in U$ to

$$\omega(p) \in \Lambda^k(\mathbb{R}^n_p)^*.$$

If $\varphi_1(p), \ldots, \varphi_n(p)$ is the dual basis to $(\boldsymbol{e}_1)_p, \ldots, (\boldsymbol{e}_n)_p$, then

$$\omega(p) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \ldots i_k}(p)\, \varphi_{i_1}(p) \wedge \cdots \wedge \varphi_{i_k}(p) \tag{4.5.3}$$

for certain functions $a_{i_1 \ldots i_k}$.

Definition 4.5.4. A k-form ω on U is called continuous, differentiable, C^r, etc., if the functions $a_{i_1 \ldots i_k}$ in (4.5.3) are. The sum $\omega + \eta$, product $f\omega$, and the wedge product $\omega \wedge \eta$ are defined in the obvious way. A function f is considered as a 0-form and $f\omega$ is also written $f \wedge \omega$.

Definition 4.5.5. Let f be a 0-form on $U \subset \mathbb{R}^n$ of class C^1; i.e., it is just real function $f \in C^1(U)$. Then its (exterior) differential df is a 1-form given by

$$df(p)(\boldsymbol{v}_p) = Df(p)\boldsymbol{v}.$$

It is customary to let x_i denote (by abuse of notation) the ith projection function $\pi_i : \mathbb{R}^n \to \mathbb{R}$

$$x_i(p) = \pi_i(p) = p_i \quad \text{for} \ \ p = (p_1, \ldots p_n).$$

Since

$$dx_i(p)(\boldsymbol{v}_p) = D\pi_i(p)\boldsymbol{v} = v_i,$$

we see that $dx_1(p), \ldots, dx_n(p)$ is just the dual basis to $(\boldsymbol{e}_1)_p, \ldots, (\boldsymbol{e}_n)_p$. Thus every k-form ω can be written

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{4.5.6}$$

Definition 4.5.7. If $i_1, \ldots, i_k$ are integers such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, and if I is the ordered k-tuple $\{i_1, \ldots, i_k\}$, then we call I an increasing k-index, and we use the brief notation

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

These forms dx_I are the so-called basic k-forms in $\mathbb{R}^n$. Then the k-form ω in (4.5.6) is briefly written as

$$\omega = \sum_I a_I dx_I.$$

Proposition 4.5.8. *Let $f : U \to \mathbb{R}$ be a differential 0-form. Then*

$$df = \sum_{i=1}^n D_i f dx_i. \tag{4.5.9}$$

Proof.

$$
\begin{aligned}
df(p)(\boldsymbol{v}_p) = Df(p)\boldsymbol{v} &= \begin{bmatrix} D_1 f(p) & \cdots & D_n f(p) \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\
&= \sum_{i=1}^{n} D_i f(p) dx_i(p)(\boldsymbol{v}_p).
\end{aligned}
$$

■

An important construction associated with forms is a generalization of the (exterior differential) operator d which changes 0-forms into 1-forms.

Definition 4.5.10. Let $\omega = \sum a_I dx_I$ be a differential k-form of class C^1 in $U \subset \mathbb{R}^n$. The *(exterior) differential* $d\omega$ of ω is a $(k+1)$-form defined by

$$
d\omega = \sum_I da_I \wedge dx_I = \sum_I \sum_{i=1}^{n} (D_i a_I)\, dx_i \wedge dx_I.
$$

Proposition 4.5.11. *We present some properties of exterior differentiation.*

(a) If ω_1 and ω_2 are of class C^1 in U, then

$$
d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2.
$$

(b) If ω is a k-form and η is an l-form of class C^1 in U, then

$$
d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.
$$

(c) If ω is of class C^2 in U, then $d(d\omega) = d^2\omega = 0$.

Proof. (a) straightforward.

(b) Because of (a), it is enough to consider the special case

$$
\omega = f dx_I, \quad \eta = g dx_J,
$$

where $f, g \in C^1(U)$, dx_I is basic k-form, and dx_J is a basic l-form. If k or l or both are 0, simply omit dx_I or dx_J in the above. We may assume that I and J have no element in common. Then

$$
\omega \wedge \eta = fg dx_I \wedge dx_J = \epsilon fg dx_{[I,J]}, \quad (\epsilon = \pm 1)
$$

where $[I, J]$ is the increasing $(k+l)$-index which is obtained by arranging the members of I and J in increasing order. By (4.5.9) and Leibniz rule, we have $d(fg) = gdf + fdg$, and thus

$$
\begin{aligned}
d(\omega \wedge \eta) &= \epsilon d(fg) \wedge dx_{[I,J]} = d(fg) \wedge dx_I \wedge dx_J \\
&= gdf \wedge dx_I \wedge dx_J + fdg \wedge dx_I \wedge dx_J \\
&= d\omega \wedge \eta + (-1)^k f dx_I \wedge dg \wedge dx_J \\
&= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.
\end{aligned}
$$

(c) Let us first assume ω is a 0-form; i.e., $\omega = f$. Then

$$d(df) = d\left(\sum_{j=1}^{n} D_j f dx_j\right) = \sum_{j=1}^{n} d(D_j f) \wedge dx_j = \sum_{j=1}^{n}\sum_{i=1}^{n} D_{ij} f dx_i \wedge dx_j.$$

Since $D_{ij}f = D_{ji}f$ and $dx_i \wedge dx_j = dx_j \wedge dx_i$, $i \neq j$, we obtain that

$$d(df) = \sum_{i<j} (D_{ij}f - D_{ji}f)\, dx_i \wedge dx_j = 0.$$

Now let $\omega = \sum_I a_I dx_I$. By (a), we can restrict ourselves to the case $\omega = f dx_I$. By (b), we have that

$$d(d\omega) = d(df \wedge dx_I) = d(df) \wedge dx_I - df \wedge d(dx_I) = 0,$$

since $d(df) = 0$ and $d(dx_I) = 0$. ∎

Definition 4.5.12. Suppose U is an open set in $\mathbb{R}^n$, $\boldsymbol{f}$ is a C^1-mapping of U into an open set $V \subset \mathbb{R}^m$. Then $\boldsymbol{f}$ produces a linear transformation $\boldsymbol{f}_* : \mathbb{R}^n_p \to \mathbb{R}^m_{\boldsymbol{f}(p)}$ defined by

$$\boldsymbol{f}_*(\boldsymbol{v}_p) = (\boldsymbol{D}f(p)\boldsymbol{v})_{\boldsymbol{f}(p)}.$$

This linear transformation $\boldsymbol{f}_*$ induces a linear transformation

$$\boldsymbol{f}^* : \Lambda^k(\mathbb{R}^m_{\boldsymbol{f}(p)}) \to \Lambda^k(\mathbb{R}^n_p).$$

If ω is a k-form in $V \subset \mathbb{R}^m$, then $\boldsymbol{f}^*\omega$ is the k-form in $U \subset \mathbb{R}^n$ defined as follows. For $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n_p$, we have (cf. definition 4.3.2)

$$(\boldsymbol{f}^*\omega)(p)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) = \omega(\boldsymbol{f}(p))(\boldsymbol{f}_*(\boldsymbol{v}_1), \ldots, \boldsymbol{f}_*(\boldsymbol{v}_k)).$$

We set the convention if g is a 0-form on U, then

$$\boldsymbol{f}^*(g) = g \circ \boldsymbol{f}.$$

Proposition 4.5.13.

(a) $\boldsymbol{f}^*(\omega_1 + \omega_2) = \boldsymbol{f}^*\omega_1 + \boldsymbol{f}^*\omega_2$.

(b) $\boldsymbol{f}^*(g\omega) = (g \circ \boldsymbol{f})\, \boldsymbol{f}^*\omega$.

(c) $\boldsymbol{f}^*(\omega \wedge \eta) = \boldsymbol{f}^*\omega \wedge \boldsymbol{f}^*\eta$.

Proof. Exercise. ∎

Proposition 4.5.14.

$$(\boldsymbol{g} \circ \boldsymbol{f})^*\omega = \boldsymbol{f}^*(\boldsymbol{g}^*\omega).$$

Proof. Exercise. ∎

Let $x = (x_1, \ldots, x_n)$ be coordinates in $\mathbb{R}^n$, $y = (y_1, \ldots, y_m)$ be coordinates in $\mathbb{R}^m$ and let $\boldsymbol{f} : U \to V$ be written as

$$y_i = f_i(x) = f_i(x_1, \ldots, x_n).$$

Let $\omega = \sum_I a_I dy_I$ be a k-form in $V \subset \mathbb{R}^m$. By using the above properties for $\boldsymbol{f}^*$, we obtain

$$\boldsymbol{f}^*\omega = \sum_I (a_I \circ \boldsymbol{f})\,(\boldsymbol{f}^* dy_{i_1}) \wedge \cdots \wedge (\boldsymbol{f}^* dy_{i_k}).$$

Since

$$\begin{aligned}
\boldsymbol{f}^*(dy_i)(p)(\boldsymbol{v}_p) &= dy_i(\boldsymbol{f}(p))(\boldsymbol{f}_*(\boldsymbol{v}_p)) \\
&= dy_i(\boldsymbol{f}(p))\left(\textstyle\sum_{j=1}^n D_j f_1(p) v_j, \ldots, \sum_{j=1}^n D_j f_n(p) v_j\right)_{\boldsymbol{f}(p)} \\
&= \textstyle\sum_{j=1}^n D_j f_i(p) v_j = \sum_{j=1}^n D_j f_i(p)\, dx_j(p)(\boldsymbol{v}_p) \\
&= df_i(p)(\boldsymbol{v}_p),
\end{aligned}$$

we have

$$\boldsymbol{f}^*(dy_i) = df_i \tag{4.5.15}$$

and thus we get

$$\boldsymbol{f}^*\omega = \sum_I (a_I \circ \boldsymbol{f})\, df_{i_1} \wedge \cdots \wedge df_{i_k}.$$

Proposition 4.5.16. *Let $U \subset \mathbb{R}^n$ be open and $\boldsymbol{f} : U \to \mathbb{R}^n$ be a C^1-map. Then*

$$\boldsymbol{f}^*(h\, dx_1 \wedge \cdots \wedge dx_n) = (h \circ \boldsymbol{f})(\det D\boldsymbol{f})\, dx_1 \wedge \cdots \wedge dx_n. \tag{4.5.17}$$

Proposition 4.5.18.

$$d(\boldsymbol{f}^*\omega) = \boldsymbol{f}^*(d\omega).$$

Proof. Let $x = (x_1, \ldots, x_n)$ be coordinates in $\mathbb{R}^n$ and $y = (y_1, \ldots, y_m)$ be coordinates in $\mathbb{R}^m$. First consider the case when ω is a 0-form; i.e., $\omega = g$. By using the chain rule and (4.5.15), we get

$$\begin{aligned}
d(\boldsymbol{f}^*(g)) = d(g \circ \boldsymbol{f}) &= \textstyle\sum_i D_i(g \circ \boldsymbol{f}) dx_i = \sum_i \sum_j (D_j g \circ \boldsymbol{f})\, D_i f_j\, dx_i \\
&= \textstyle\sum_j (D_j g \circ \boldsymbol{f})\, df_j = \boldsymbol{f}^*(\sum_j D_j g\, dy_j) = \boldsymbol{f}^*(dg).
\end{aligned}$$

Next, note that by Proposition 4.5.11, we have

$$d(\boldsymbol{f}^*(dy_I)) = d(df_{i_1} \wedge \cdots \wedge df_{i_k}) = 0.$$

Now, for $\omega = \sum_I a_I dy_I$,

$$\begin{aligned}
d(\boldsymbol{f}^*(\omega)) &= d\left(\sum_I (a_I \circ \boldsymbol{f})\, df_{i_1} \wedge \cdots \wedge df_{i_k}\right) \\
&= \sum_I \big(d(\boldsymbol{f}^*(a_I)) \wedge df_{i_1} \wedge \cdots \wedge df_{i_k} + \boldsymbol{f}^*(a_I) \wedge d(df_{i_1} \wedge \cdots \wedge df_{i_k})\big) \\
&= \sum_I \boldsymbol{f}^*(da_I) \wedge \boldsymbol{f}^*(dy_I) = \sum_I \boldsymbol{f}^*(da_I \wedge dy_I) = \boldsymbol{f}^*(\omega).
\end{aligned}$$

■

4.6 Integration of differential forms

Definition 4.6.1. The k-simplex Q^k consists of all points $x = (x_1, \dots, x_k)$ in $\mathbb{R}^k$ for which

$$x_1 + \cdots + x_k \le 1 \quad \text{and} \quad x_i \ge 0 \quad \text{for} \quad i = 1, \dots, k.$$

We denote by I^k a k-dimensional interval; see definition 2.1.1.

Definition 4.6.2. We say that $\boldsymbol{f}$ is a C^r-mapping of a compact set $D \subset \mathbb{R}^k$ into $\mathbb{R}^n$ if there is a C^r-mapping $\boldsymbol{g}$ of an open set $W \subset \mathbb{R}^k$ into $\mathbb{R}^n$ such that $D \subset W$ and that $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})$ for all $\boldsymbol{x} \in D$.

A k-surface in $U \subset \mathbb{R}^n$ is a C^1-mapping $\boldsymbol{\Phi}$ from a compact set $D \subset \mathbb{R}^k$ into U. D is called the *parameter domain* of $\boldsymbol{\Phi}$; most of the time, we shall confine ourselves to the simple situation in which D is either a k-dimensional interval I^k or the k-simplex Q^k.

Let ω be a k-form in E. The integral of ω over $\boldsymbol{\Phi}$ is defined by

$$\int_{\boldsymbol{\Phi}} \omega = \int_D \boldsymbol{\Phi}^*(\omega)(\boldsymbol{e}_1, \dots, \boldsymbol{e}_k).$$

Thus, if $\omega = \sum a_{i_1 \dots i_k}(\boldsymbol{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, then (cf. (4.5.17))

$$\begin{aligned}\int_{\boldsymbol{\Phi}} \omega = \int_D \boldsymbol{\Phi}^*(\omega)(\boldsymbol{e}_1, \dots, \boldsymbol{e}_k) &= \int_D \omega(\boldsymbol{\Phi}(\boldsymbol{u}))(D\boldsymbol{\Phi}(\boldsymbol{u})\boldsymbol{e}_1, \dots, D\boldsymbol{\Phi}(\boldsymbol{u})\boldsymbol{e}_k)\, d\boldsymbol{u} \\ &= \int_D \sum a_{i_1 \dots i_k}(\boldsymbol{\Phi}(\boldsymbol{u})) \frac{\partial(\phi_{i_1}, \dots, \phi_{i_k})}{\partial(u_1, \dots, u_k)}\, d\boldsymbol{u}, \qquad (4.6.3)\end{aligned}$$

where $\phi_1, \dots, \phi_n$ are components of $\boldsymbol{\Phi}$ and the Jacobian in (4.6.3) is determined by the mapping

$$[u_1 \ \cdots \ u_k]^\top \to [\phi_{i_1}(\boldsymbol{u}) \ \cdots \ \phi_{i_k}(\boldsymbol{u})]^\top.$$

If we use the convention in definition 4.5.7 and denote the above mapping by $\boldsymbol{\phi}_I$, then (4.6.3) becomes

$$\int_{\boldsymbol{\Phi}} \omega = \int_D \sum a_I(\boldsymbol{\Phi}(\boldsymbol{u})) \det D\boldsymbol{\phi}_I(\boldsymbol{u})\, d\boldsymbol{u}.$$

Note that (4.6.3) also implies that

$$\int_{\boldsymbol{\Phi}} \omega = \int_{\boldsymbol{\Delta}} \boldsymbol{\Phi}^*(\omega), \qquad (4.6.4)$$

where $\boldsymbol{\Delta} : D \to \mathbb{R}^k$ is the k-surface defined by $\boldsymbol{\Delta}(\boldsymbol{u}) = \boldsymbol{u}$; i.e., $\boldsymbol{\Delta}$ is the inclusion map of D into $\mathbb{R}^k$. If $\omega = g\, dx_1 \wedge \cdots \wedge dx_k$, then

$$\int_{\boldsymbol{\Delta}} g\, dx_1 \wedge \cdots \wedge dx_k = \int_D g(\boldsymbol{x})\, d\boldsymbol{x}.$$

Let $\boldsymbol{\Phi}$ be a k-surface in $U \subset \mathbb{R}^n$ with the parameter domain D. If $\boldsymbol{f} : D \to \mathbb{R}^k$ is a one-to-one C^1 map such that $\det D\boldsymbol{f}(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \in D$. If we set $D' = \boldsymbol{f}(D)$, then $\boldsymbol{\Psi} = \boldsymbol{\Phi} \circ \boldsymbol{f}^{-1}$ is a k-surface in U with parameter domain D' and

$$\begin{aligned}\int_{\boldsymbol{\Psi}} \omega = \int_{D'} \sum (a_I \circ \boldsymbol{\Psi}) \det D\boldsymbol{\psi}_I &= \int_D \sum (a_I \circ \boldsymbol{\Psi} \circ \boldsymbol{f})[(\det D\boldsymbol{\psi}_I) \circ \boldsymbol{f}] \det(D\boldsymbol{f}) \\ &= \int_D \sum (a_I \circ \boldsymbol{\Phi}) \det D\boldsymbol{\phi}_I = \int_{\boldsymbol{\Phi}} \omega.\end{aligned}$$

Example 4.6.5. (a) Let γ be a 1-surface (a curve of class C^1) in $\mathbb{R}^3$, with parameter domain $[0,1]$. Write (x,y,z) in place of (x_1,x_2,x_3), and put

$$\omega = x\,dy + y\,dx$$

Then

$$\int_\gamma \omega = \int_0^1 \gamma_1(t)\gamma_2'(t) + \gamma_2(t)\gamma_1'(t)\,dt = \gamma_1(1)\gamma_2(1) - \gamma_1(0)\gamma_2(0).$$

Note that in this example, $\int_\gamma \omega$ depends only on the initial point $\gamma(0)$ and on the end point $\gamma(1)$ of γ. In particular, $\int_\gamma \omega = 0$ for every closed curve γ.

(b) Fix $a > 0$, $b > 0$, and define

$$\gamma(t) = (a\cos t, b\sin t) \qquad (0 \leq t \leq 2\pi),$$

so that γ is a closed curve in $\mathbb{R}^2$ (Its range is an ellipse.) Then

$$\int_\gamma x\,dy = \int_0^{2\pi} ab\cos^2 t\,dt = \pi ab,$$

whereas

$$\int_\gamma y\,dx = -\int_0^{2\pi} ab\sin^2 t\,dt = -\pi ab.$$

Note that $\int_\gamma x\,dy$ is the area of the region bounded by γ. This is a special case of Green's theorem.

(c) Let D be the 3-dimensional interval defined by

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi.$$

Define $\mathbf{\Phi}(r,\theta,\phi) = [x\ \ y\ \ z]^\top$, where

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

Then $\det D\mathbf{\Phi} = r^2\sin\theta$. Hence

$$\int_{\mathbf{\Phi}} dx \wedge dy \wedge dz = \int_D \det D\mathbf{\Phi} = \frac{4\pi}{3}.$$

Note that $\mathbf{\Phi}$ maps D onto the closed unit ball in $\mathbb{R}^3$, that the mapping is 1-1 in the interior of D (but certain boundary points are identified by $\mathbf{\Phi}$), and that the integral is equal to the volume of $\mathbf{\Phi}(D)$.

Proposition 4.6.6. *Suppose $\boldsymbol{f}$ is a C^1-mapping of an open set $U \subset \mathbb{R}^n$ into an open set $V \subset \mathbb{R}^m$, $\mathbf{\Phi}$ is a k-surface in U, and ω is is k-form in V. Then*

$$\int_{\boldsymbol{f}\circ\mathbf{\Phi}} \omega = \int_{\mathbf{\Phi}} \boldsymbol{f}^*\omega.$$

Proof. Let D be the parameter domain of $\mathbf{\Phi}$ (hence also of $\boldsymbol{f}\circ\mathbf{\Phi}$) and define $\mathbf{\Delta}$ be as in (4.6.4). Then by proposition 4.5.14

$$\int_{\boldsymbol{f}\circ\mathbf{\Phi}} \omega = \int_{\mathbf{\Delta}} (\boldsymbol{f}\circ\mathbf{\Phi})^*(\omega) = \int_{\mathbf{\Delta}} \mathbf{\Phi}^*(\boldsymbol{f}^*\omega) = \int_{\mathbf{\Phi}} \boldsymbol{f}^*\omega. \qquad \blacksquare$$

4.7 Simplexes and chains

Definition 4.7.1. Assume that $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_k$ are vectors in $\mathbb{R}^n$. The *oriented affine k-simplex*

$$\boldsymbol{\sigma} = [\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_k]$$

is defined to be the k-surface in $\mathbb{R}^n$ with parameter domain $D = Q^k$ which is given by the affine mapping

$$\boldsymbol{\sigma}(u_1, \ldots, u_k) = \boldsymbol{\sigma}(u_1\boldsymbol{e}_1 + \cdots + u_k\boldsymbol{e}_k) = \boldsymbol{p}_0 + \sum_{i=1}^{k} u_i(\boldsymbol{p}_i - \boldsymbol{p}_0).$$

We call $\boldsymbol{\sigma}$ *oriented* to emphasize the ordering of vertices $\boldsymbol{p}_0, \ldots, \boldsymbol{p}_k$ is taken into account. An *oriented 0-simplex* is defined to be a point with a sign attached. We write $\boldsymbol{\sigma} = +\boldsymbol{p}_0$ or $\boldsymbol{\sigma} = -\boldsymbol{p}_0$. If $\boldsymbol{\sigma} = \epsilon\boldsymbol{p}_0$ ($\epsilon = \pm 1$) and f is a 0-form, we define

$$\int_{\boldsymbol{\sigma}} f = \epsilon f(\boldsymbol{p}_0).$$

Note that $\boldsymbol{\sigma} = [\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_k]$ is an affine map characterized by

$$\boldsymbol{\sigma}(\boldsymbol{0}) = \boldsymbol{p}_0 \quad \text{and} \quad \boldsymbol{\sigma}(\boldsymbol{e}_i) = \boldsymbol{p}_i \ \text{ for } \ 1 \le i \le k.$$

Proposition 4.7.2. *If $\boldsymbol{\sigma} = [\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_k]$ is an oriented affine k-simplex in an open set $E \subset \mathbb{R}^n$ and if $\bar{\boldsymbol{\sigma}} = [\boldsymbol{p}_{\tau(0)}, \boldsymbol{p}_{\tau(1)}, \ldots, \boldsymbol{p}_{\tau(k)}]$, where τ is a permutation of $\{0, 1, \ldots, k\}$, then*

$$\int_{\bar{\boldsymbol{\sigma}}} \omega = \epsilon(\tau) \int_{\boldsymbol{\sigma}} \omega \tag{4.7.3}$$

for every k-form ω in U. Therefore, we adopt the (abused) notation

$$\bar{\boldsymbol{\sigma}} = \epsilon(\tau)\boldsymbol{\sigma} \quad (\bar{\boldsymbol{\sigma}} = \pm\boldsymbol{\sigma}).$$

Proof. Note that $\boldsymbol{\sigma}$ is characterized by

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{p}_0 + A\boldsymbol{u} \ \text{ for } \ \boldsymbol{u} \in Q^k, \tag{4.7.4}$$

where A is the $n \times k$ matrix whose ith column $A\boldsymbol{e}_i$ is $\boldsymbol{x}_i = \boldsymbol{p}_i - \boldsymbol{p}_0$. Suppose $1 \le j \le k$, and suppose $\bar{\boldsymbol{\sigma}}$ is obtained from $\boldsymbol{\sigma}$ by interchanging $\boldsymbol{p}_0$ and $\boldsymbol{p}_j$. Then $\epsilon = -1$, and

$$\bar{\boldsymbol{\sigma}}(\boldsymbol{u}) = \boldsymbol{p}_j + B\boldsymbol{u} \quad (\boldsymbol{u} \in Q^k),$$

where B is $n \times k$ matrix whose columns $B\boldsymbol{e}_1, \ldots, B\boldsymbol{e}_k$ are

$$B\boldsymbol{e}_j = \boldsymbol{p}_0 - \boldsymbol{p}_j = -\boldsymbol{x}_j, \quad B\boldsymbol{e}_i = \boldsymbol{p}_i - \boldsymbol{p}_j = \boldsymbol{x}_i - \boldsymbol{x}_j \ \text{ for } \ 1 \le i \le k, \ \ i \ne j.$$

Therefore,

$$\begin{aligned}\bar{\boldsymbol{\sigma}}(\boldsymbol{u}) &= \boldsymbol{p}_0 + \boldsymbol{x}_j + u_1(\boldsymbol{x}_1 - \boldsymbol{x}_j) + \cdots + u_j(-\boldsymbol{x}_j) + \cdots + u_k(\boldsymbol{x}_k - \boldsymbol{x}_j) \\ &= \boldsymbol{p}_0 + u_1\boldsymbol{x}_1 + \cdots + (1 - (u_1 + \cdots + u_k))\boldsymbol{x}_j + \cdots + u_k\boldsymbol{x}_k.\end{aligned}$$

We may assume that $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ are linearly independent vectors in $\mathbb{R}^n$; otherwise both sides of (4.7.3) become zero. By equating $\bar{\boldsymbol{\sigma}}(\boldsymbol{u}) = \boldsymbol{\sigma}(\boldsymbol{v})$, we find

$$v_i = \begin{cases} u_i & (i \ne j), \\ 1 - (u_1 + \cdots + u_k) & (i = j) \end{cases}$$

For $1 \leq i_1 < \cdots < i_k \leq n$, denote by $A(i_1, \ldots, i_k)$ and $B(i_1, \ldots, i_k)$ the $k \times k$ matrices obtained by selecting rows $i_1, \ldots, i_k$ of A and B respectively. Note that if we subtract the jth column of B from each of the others, the determinant of $B(i_1, \ldots, i_k)$ is not affected, and we obtain columns

$$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{j-1}, -\boldsymbol{x}_j, \boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_k.$$

These differ from those of A only in the sign of the jth column. Therefore,

$$\begin{aligned}
\int_{\bar{\boldsymbol{\sigma}}} \sum a_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} &= \int_{Q^k} \sum a_{i_1 \ldots i_k}(\bar{\boldsymbol{\sigma}}(\boldsymbol{u})) \det B(i_1, \ldots, i_k)\, d\boldsymbol{u} \\
&= -\int_{Q^k} \sum a_{i_1 \ldots i_k}(\bar{\boldsymbol{\sigma}}(\boldsymbol{u})) \det A(i_1, \ldots, i_k)\, d\boldsymbol{u} \\
&= -\int_{Q^k} \sum a_{i_1 \ldots i_k}(\boldsymbol{\sigma}(\boldsymbol{v})) \det A(i_1, \ldots, i_k)\, d\boldsymbol{v} \\
&= -\int_{\boldsymbol{\sigma}} \sum a_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.
\end{aligned}$$

Hence (4.7.3) holds for this case. Suppose next that $0 < i < j \leq k$ and that $\bar{\boldsymbol{\sigma}}$ is obtained from $\boldsymbol{\sigma}$ by interchanging $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$. Then

$$\bar{\boldsymbol{\sigma}}(\boldsymbol{u}) = \boldsymbol{p}_0 + C\boldsymbol{u},$$

where C has the same columns as A, except that the ith and jth columns have been interchanged. This again implies that (4.7.3) holds, since $\epsilon = -1$. The general case follows, since every permutation of $\{0, 1, \ldots, k\}$ is a composition of the special cases we have just dealt with. ∎

Definition 4.7.5. If $\bar{\boldsymbol{\sigma}} = \epsilon \boldsymbol{\sigma}$ (using the above convention) and if $\epsilon = 1$, we say that $\bar{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}$ have the *same orientation*; if $\epsilon = -1$, $\bar{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}$ are said to have opposite orientations. When $n = k$ and when the vectors $\boldsymbol{p}_i - \boldsymbol{p}_0$ are independent. In that case, the matrix A that appears in (4.7.4) is invertible, and its determinant (which is the same as the Jacobian of $\boldsymbol{\sigma}$) is not 0. Then $\boldsymbol{\sigma}$ is said to be positively (or negatively) oriented if $\det A$ is positive (or negative).

In particular, the simplex $[\boldsymbol{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_k]$ in $\mathbb{R}^k$, given by the identity mapping, has positive orientation.

Definition 4.7.6. An *affine k-chain* $\boldsymbol{\Gamma}$ in an open set $U \subset \mathbb{R}^n$ is a collection of finitely many oriented affine k-simplexes $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_r$ in U. These need not be distinct. If $\boldsymbol{\Gamma}$ is as above, and if ω is a k-form in U, we define

$$\int_{\boldsymbol{\Gamma}} \omega = \sum_{i=1}^{r} \int_{\boldsymbol{\sigma}_i} \omega.$$

This suggests the use of (abused) notation $\boldsymbol{\Gamma} = \sum_{i=1}^{r} \boldsymbol{\sigma}_i = \boldsymbol{\sigma}_1 + \cdots + \boldsymbol{\sigma}_r$.

Definition 4.7.7. For $k \geq 1$, the *boundary* of the oriented affine k-simplex $\boldsymbol{\sigma} = [\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_k]$ is defined to be the affine $(k-1)$-chain

$$\partial \boldsymbol{\sigma} = \sum_{j=0}^{k} (-1)^j [\boldsymbol{p}_0, \ldots, \widehat{\boldsymbol{p}_j}, \ldots, \boldsymbol{p}_k] \qquad (\widehat{\boldsymbol{p}_j} \text{ means } \boldsymbol{p}_j \text{ is omitted}).$$

For example, if $\boldsymbol{\sigma} = [\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2]$, then

$$\partial\boldsymbol{\sigma} = [\boldsymbol{p}_1, \boldsymbol{p}_2] - [\boldsymbol{p}_0, \boldsymbol{p}_2] + [\boldsymbol{p}_0, \boldsymbol{p}_1] = [\boldsymbol{p}_0, \boldsymbol{p}_1] + [\boldsymbol{p}_1, \boldsymbol{p}_2] + [\boldsymbol{p}_2, \boldsymbol{p}_0].$$

Definition 4.7.8. Let $\boldsymbol{f}$ be a C^r-mapping of an open set $U \subset \mathbb{R}^n$ into an open set $V \in \mathbb{R}^m$; $\boldsymbol{f}$ need not be one-to-one. If $\boldsymbol{\sigma}$ is an oriented affine k-simplex in D, then the corresponding mapping $\boldsymbol{\Phi} = \boldsymbol{f} \circ \boldsymbol{\sigma}$ (which we shall sometimes write in the simpler form $\boldsymbol{f\sigma}$) is a k-surface in V, with parameter domain Q^k. We call $\boldsymbol{\Phi}$ an oriented k-simplex of class C^r.

Definition 4.7.9. A finite collection $\boldsymbol{\Psi}$ of oriented k-simplexes $\boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_r$ of class C^r in V is called a k-chain of class C^r in V. If ω is a k-form in V, we define

$$\int_{\boldsymbol{\Psi}} \omega = \sum_{i=1}^{r} \int_{\boldsymbol{\Phi}_i} \omega$$

and use the corresponding notation $\boldsymbol{\Psi} = \sum \boldsymbol{\Phi}_i$. If $\boldsymbol{\Gamma} = \sum \boldsymbol{\sigma}_i$ is an affine chain and $\boldsymbol{\Phi}_i = \boldsymbol{f} \circ \boldsymbol{\sigma}_i$, we also write $\boldsymbol{\Psi} = \boldsymbol{f} \circ \boldsymbol{\Gamma}$ or (by abuse of notation)

$$\boldsymbol{f}\left(\sum \boldsymbol{\sigma}_i\right) = \sum \boldsymbol{f\sigma}_i.$$

The boundary of $\partial\boldsymbol{\Phi}$ is the oriented k-simplex $\boldsymbol{\Phi} = \boldsymbol{f} \circ \boldsymbol{\sigma}$ is defined to be the $(k-1)$-chain

$$\partial\boldsymbol{\Phi} = \boldsymbol{f}(\partial\boldsymbol{\sigma}).$$

Definition 4.7.10 (Positively oriented boundary)**.** Let Q^n be the standard simplex in $\mathbb{R}^n$, let $\boldsymbol{\sigma}_0$ be the identity mapping with domain Q^n. Then, $\boldsymbol{\sigma}_0$ may be regarded as a positively oriented n-simplex in $\mathbb{R}^n$. Its boundary $\partial\boldsymbol{\sigma}_0$ is an affine $(n-1)$-chain. This chain is called the positively oriented boundary of the set Q^n. For example, the positively oriented boundary of Q^3 is

$$[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] - [\boldsymbol{0}, \boldsymbol{e}_2, \boldsymbol{e}_3] + [\boldsymbol{0}, \boldsymbol{e}_1, \boldsymbol{e}_3] - [\boldsymbol{0}, \boldsymbol{e}_1, \boldsymbol{e}_2].$$

Now let $\boldsymbol{f}$ be a one-to-one mapping of Q^n into $\mathbb{R}^n$, of class C^1, whose Jacobian is positive (at least in the interior of Q^n). Let $E = \boldsymbol{f}(Q^n)$. By the inverse function theorem, E is the closure of an open subset of $\mathbb{R}^n$. We define the positively oriented boundary of the set E to be the $(n-1)$-chain

$$\partial\boldsymbol{f} = \boldsymbol{f}(\partial\boldsymbol{\sigma}_0)$$

and we may denote this $(n-1)$-chain by ∂E. Let

$$\Omega = E_1 \cup \cdots \cup E_r,$$

where $E_i = \boldsymbol{f}_i(Q^n)$, each $\boldsymbol{f}_i$ has the properties that $\boldsymbol{f}$ had above, and the interiors of the sets E_i are pairwise disjoint. Then the $(n-1)$-chain

$$\partial\boldsymbol{f}_1 + \cdots + \partial\boldsymbol{f}_r = \partial\Omega.$$

is called the positively oriented boundary of Ω.

Example 4.7.11. The unit square I^2 in $\mathbb{R}^2$ is the union of $\boldsymbol{\sigma}_1(Q^2)$ and $\boldsymbol{\sigma}_2(Q^2)$, where

$$\boldsymbol{\sigma}_1 = [\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2], \quad \boldsymbol{\sigma}_2 = [\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_2, \boldsymbol{e}_1].$$

Both $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ have Jacobian $1 > 0$. Then

$$\begin{aligned}\partial I^2 &= \partial\boldsymbol{\sigma}_1 + \partial\boldsymbol{\sigma}_2\\ &= [\boldsymbol{e}_1, \boldsymbol{e}_2] - [\mathbf{0}, \boldsymbol{e}_2] + [\mathbf{0}, \boldsymbol{e}_1] + [\boldsymbol{e}_2, \boldsymbol{e}_1] - [\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_1] + [\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_2]\\ &= [\mathbf{0}, \boldsymbol{e}_1] + [\boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_2] + [\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_2] + [\boldsymbol{e}_2, \mathbf{0}].\end{aligned}$$

If $\boldsymbol{\Phi}$ is a 2-surface in $\mathbb{R}^m$, with parameter domain I^2, then $\boldsymbol{\Phi}$ is the same as the 2-chain (regarded as a function on 2-forms)

$$\boldsymbol{\Phi} \circ \boldsymbol{\Gamma} = \boldsymbol{\Phi} \circ \boldsymbol{\sigma}_1 + \boldsymbol{\Phi} \circ \boldsymbol{\sigma}_2.$$

Thus

$$\partial\boldsymbol{\Phi} = \partial(\boldsymbol{\Phi} \circ \boldsymbol{\sigma}_1) + \partial(\boldsymbol{\Phi} \circ \boldsymbol{\sigma}_2) = \boldsymbol{\Phi}(\partial\boldsymbol{\sigma}_1) + \boldsymbol{\Phi}(\partial\boldsymbol{\sigma}_2) = \boldsymbol{\Phi}(\partial I^2).$$

In other words, if the parameter domain of $\boldsymbol{\Phi}$ is the square I^2, we need not refer back to the simplex Q^2, but can obtain $\partial\boldsymbol{\Phi}$ directly from ∂I^2.

4.8 Stokes' theorem

Theorem 4.8.1. *If $\boldsymbol{\Psi}$ is a k-chain of class C^2 in an open set $V \subset \mathbb{R}^m$ and if ω is a $(k-1)$-form of class C^1 in V, then*

$$\int_{\boldsymbol{\Psi}} d\omega = \int_{\partial\boldsymbol{\Psi}} \omega.$$

Proof. By definition 4.7.9, it is enough to prove that

$$\int_{\boldsymbol{\Phi}} d\omega = \int_{\partial\boldsymbol{\Phi}} \omega \tag{4.8.2}$$

for every oriented k-simplex $\boldsymbol{\Phi}$ of class C^2 in V. Fix such a $\boldsymbol{\Phi}$ and put

$$\boldsymbol{\sigma} = [\mathbf{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_k]$$

Thus $\boldsymbol{\sigma}$ is the oriented affine k-simplex with parameter domain Q^k which is defined by the identity mapping. Since $\boldsymbol{\Phi}$ is also defined on Q^k and $\boldsymbol{\Phi}$ is C^2, there is an open set $U \subset \mathbb{R}^k$ which contains Q^k, and there is a C^2-mapping $\boldsymbol{f}$ of U into V such that $\boldsymbol{\Phi} = \boldsymbol{f} \circ \boldsymbol{\sigma}$. By propositions 4.6.6 and 4.5.18, the left side of (4.8.2) is equal to

$$\int_{\boldsymbol{f}\circ\boldsymbol{\sigma}} d\omega = \int_{\boldsymbol{\sigma}} \boldsymbol{f}^*(d\omega) = \int_{\boldsymbol{\sigma}} d(\boldsymbol{f}^*\omega).$$

Another application of proposition 4.6.6 shows, by definition 4.7.9, that the right side (4.8.2) is

$$\int_{\partial(\boldsymbol{f}\circ\boldsymbol{\sigma})} \omega = \int_{\boldsymbol{f}(\partial\boldsymbol{\sigma})} \omega = \int_{\partial\boldsymbol{\sigma}} \boldsymbol{f}^*\omega.$$

Since $\boldsymbol{f}^*\omega$ is a $(k-1)$-form in U, we see that in order to prove (4.8.2) we merely have to show that

$$\int_{\boldsymbol{\sigma}} d\lambda = \int_{\partial\boldsymbol{\sigma}} \lambda \tag{4.8.3}$$

for the special simplex $\boldsymbol{\sigma}$ and for every $(k-1)$-form λ of class C^1 in U. If $k = 1$, the definition of an oriented 0-simplex shows that (4.8.3) merely asserts that

$$\int_0^1 g'(u)\, du = g(1) - g(0)$$

for every continuously differentiable function g on $[0, 1]$, which is true by the fundamental theorem of calculus. From now on we assume that $k > 1$, fix an integer r $(1 \leq r \leq k)$ and choose $g \in C^1(U)$. It is then enough to prove (4.8.3) for the case

$$\lambda = g(\boldsymbol{x})\, dx_1 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k$$

since every $(k-1)$-form is a sum of these special ones, for $r = 1, \ldots, k$. The boundary of the simplex $\boldsymbol{\sigma}$ is

$$\partial\boldsymbol{\sigma} = [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k] + \sum_{i=1}^{k} (-1)^i \boldsymbol{\tau}_i$$

where

$$\boldsymbol{\tau}_i = [\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{i-1}, \boldsymbol{e}_{i+1}, \ldots, \boldsymbol{e}_k]$$

for $i = 1, \ldots, k$. Put

$$\boldsymbol{\tau}_0 = [\boldsymbol{e}_r, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{r-1}, \boldsymbol{e}_{r+1}, \ldots, \boldsymbol{e}_k]$$

Note that $\boldsymbol{\tau}_0$ is obtained from $[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k]$ by $r-1$ successive interchanges of $\boldsymbol{e}_r$ and its left neighbors. Thus

$$\partial\boldsymbol{\sigma} = (-1)^{r-1}\boldsymbol{\tau}_0 + \sum_{i=1}^{k} (-1)^i \boldsymbol{\tau}_i.$$

Each τ_i has Q^{k-1} as parameter domain. If $\boldsymbol{x} = \boldsymbol{\tau}_0(\boldsymbol{u})$ and $\boldsymbol{u} \in Q^{k-1}$, then

$$x_j = \begin{cases} u_j & (1 \leq j < r), \\ 1 - (u_1 + \cdots + u_{k-1}) & (j = r), \\ u_{j-1} & (r < j \leq k). \end{cases}$$

If $1 \leq i \leq k$, $\boldsymbol{u} \in Q^{k-1}$, and $\boldsymbol{x} = \boldsymbol{\tau}_i(\boldsymbol{u})$, then

$$x_j = \begin{cases} u_j & (1 \leq j < i), \\ 0 & (j = i), \\ u_{j-1} & (i < j \leq k). \end{cases}$$

For $0 \leq i \leq k$, let J_i be the Jacobian of the mapping

$$(u_1, \ldots, u_{k-1}) \to (x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_k)$$

induced by $\boldsymbol{\tau}_i$. When $i = 0$ and when $i = r$, the above is the identity mapping. Thus $J_0 = 1$, $J_r = 1$. For other i, the fact that $x_i = 0$ shows that $J_i = 0$. Thus by (4.6.3)

$$\int_{\boldsymbol{\tau}_i} \lambda = 0 \qquad (i \neq 0,\ i \neq r).$$

Consequently,

$$\begin{aligned}\int_{\partial\boldsymbol{\sigma}} \lambda &= (-1)^{r-1} \int_{\boldsymbol{\tau}_0} \lambda + (-1)^r \int_{\boldsymbol{\tau}_r} \lambda \\ &= (-1)^{r-1} \int_{Q^{k-1}} [g(\boldsymbol{\tau}_0(\boldsymbol{u})) - g(\boldsymbol{\tau}_r(\boldsymbol{u}))]\, d\boldsymbol{u}. \end{aligned} \tag{4.8.4}$$

On the other hand,

$$\begin{aligned} d\lambda &= (D_r g)(\boldsymbol{x})\, dx_r \wedge dx_1 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k \\ &= (-1)^{r-1} (D_r g)\, dx_1 \wedge \cdots \wedge dx_k \end{aligned}$$

so that

$$\int_{\boldsymbol{\sigma}} d\lambda = (-1)^{r-1} \int_{Q^k} (D_r g)(\boldsymbol{x})\, d\boldsymbol{x}. \tag{4.8.5}$$

We evaluate (4.8.5) by first integrating with respect to x_r, over the interval

$$[0, 1 - (x_1 + \cdots + x_{r-1} + x_{r+1} + \cdots + x_k)],$$

put $(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_k) = (u_1, \ldots, u_{k-1})$, and see that the integral over Q^k in (4.8.5) is equal to the integral over Q^{k-1} in (4.8.4). Thus (4.8.3) holds. ■

4.9 Closed forms and exact forms

Definition 4.9.1. Let ω be a k-form in an open set $U \subset \mathbb{R}^n$. If there is a $(k-1)$-form λ in U such that $\omega = d\lambda$, then ω is said to be exact in U. If ω is of class C^1 and $d\omega = 0$, then ω is said to be closed.

By proposition 4.5.11(c), every exact form of class C^1 is closed.

Let ω be an exact k-form in U. Then there is a $(k-1)$-form λ in U with $d\lambda = \omega$, and Stokes' theorem asserts that

$$\int_{\boldsymbol{\Psi}} \omega = \int_{\boldsymbol{\Psi}} d\lambda = \int_{\partial\boldsymbol{\Psi}} \lambda$$

for every k-chain $\boldsymbol{\Psi}$ of class C^2 in U. If $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ are such chains, and if they have the same boundaries, it follows that

$$\int_{\boldsymbol{\Psi}_1} \omega = \int_{\boldsymbol{\Psi}_2} \omega.$$

In particular, the integral of an exact k-form in U is 0 over every k-chain in U whose boundary is 0. As an important special case of this, note that integrals of exact 1-forms in U are 0 over closed (differentiable) curves in U.

Let ω be a closed k-form in U. Then $d\omega = 0$, and Stokes' theorem asserts that

$$\int_{\partial \boldsymbol{\Psi}} \omega = \int_{\boldsymbol{\Psi}} d\omega = 0$$

for every $(k+1)$-chain $\boldsymbol{\Psi}$ of class C^2 in U. In other words, integrals of closed k-forms in U are 0 over k-chains that are boundaries of $(k+1)$-chains in U.

Let $\boldsymbol{\Psi}$ be a $(k+1)$-chain in U and let λ be a $(k-1)$-form in U, both of class C^2. Since $d^2\lambda = 0$, two applications of Stokes' theorem show that

$$\int_{\partial\partial \boldsymbol{\Psi}} \lambda = \int_{\partial \boldsymbol{\Psi}} d\lambda = \int_{\boldsymbol{\Psi}} d^2\lambda = 0$$

We conclude that $\partial^2 \boldsymbol{\Psi} = 0$. In other words, the boundary of a boundary is 0.

Example 4.9.2. Let $U = \mathbb{R}^2 \setminus \{\mathbf{0}\}$, the plane with the origin removed. The 1-form

$$\eta = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

is closed in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$. This is easily verified by differentiation. Fix $r > 0$, and define

$$\boldsymbol{\gamma}(t) = \begin{bmatrix} r\cos t \\ r\sin t \end{bmatrix} \qquad (0 \le t \le 2\pi).$$

Then $\boldsymbol{\gamma}$ is a curve (an "oriented 1-simplex") in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$. Since $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(2\pi)$, we have $\partial\boldsymbol{\gamma} = 0$. A direct computation shows that

$$\int_{\boldsymbol{\gamma}} \eta = 2\pi \neq 0.$$

Therefore, η is not exact in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ and $\boldsymbol{\gamma}$ is not the boundary of any 2-chain in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ (of class C^2).

Definition 4.9.3. A set $E \subset \mathbb{R}^n$ is called star-shaped if there exists $\boldsymbol{a} \in E$ such that for all $\boldsymbol{x} \in E$ the line segment from $\boldsymbol{a}$ to $\boldsymbol{x}$ is in E.

Theorem 4.9.4 (Poincaré lemma). *If $U \subset \mathbb{R}^n$ is an open star-shaped set, then every closed form on U is exact.*

Proof. We may assume that U is star-shaped with respect to $\mathbf{0}$. We will define a function I from k-forms to $(k-1)$ forms, such that

$$I(0) = 0 \quad \text{and} \quad \omega = I(d\omega) + d(I\omega) \quad \text{for any form } \omega.$$

It follows that $\omega = d(I\omega)$ if $d\omega = 0$. Let

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Since U is star-shaped with respect to $\mathbf{0}$ we can define

$$I\omega(\boldsymbol{x}) = \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{k} (-1)^{\alpha-1} \left(\int_0^1 t^{k-1} a_{i_1 \ldots i_k}(t\boldsymbol{x})\, dt \right) x_{i_\alpha}$$
$$dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_\alpha}} \wedge \cdots \wedge dx_{i_k} \quad (\widehat{dx_{i_\alpha}} \text{ means } dx_{i_\alpha} \text{ is omitted}).$$

Then we have

$$d(I\omega) = k \sum_{i_1<\cdots<i_k} \left(\int_0^1 t^{k-1} a_{i_1\ldots i_k}(t\boldsymbol{x})\, dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$
$$+ \sum_{i_1<\cdots<i_k} \sum_{\alpha=1}^{k} \sum_{j=1}^{n} (-1)^{\alpha-1} \left(\int_0^1 t^k D_j a_{i_1\ldots i_k}(t\boldsymbol{x})\, dt \right) x_{i_\alpha}$$
$$dx_j \wedge dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_\alpha}} \wedge \cdots \wedge dx_{i_k}$$

We also have

$$d\omega = \sum_{i_1<\cdots<i_k} \sum_{j=1}^{n} D_j a_{i_1\ldots i_k}\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Applying I to the $(k+1)$-form $d\omega$, we obtain

$$I(d\omega) = \sum_{i_1<\cdots<i_k} \sum_{j=1}^{n} \left(\int_0^1 t^k D_j a_{i_1\ldots i_k}(t\boldsymbol{x})\, dt \right) x_j dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$
$$- \sum_{i_1<\cdots<i_k} \sum_{j=1}^{n} \sum_{\alpha=1}^{k} (-1)^{\alpha-1} \left(\int_0^1 t^k D_j a_{i_1\ldots i_k}(t\boldsymbol{x})\, dt \right) x_{i_\alpha}$$
$$dx_j \wedge dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_\alpha}} \wedge \cdots \wedge dx_{i_k}$$

Adding, the triple sums cancel, and we obtain

$$\begin{aligned} d(I\omega) + I(d\omega) &= \sum_{i_1<\cdots<i_k} k \left(\int_0^1 t^{k-1} a_{i_1\ldots i_k}(t\boldsymbol{x})\, dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &\quad + \sum_{i_1<\cdots<i_k} \sum_{j=1}^{n} \left(\int_0^1 t^k x_j D_j a_{i_1\ldots i_k}(t\boldsymbol{x})\, dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i_1<\cdots<i_k} \left(\int_0^1 \frac{d}{dt}[t^k a_{i_1\ldots i_k}(t\boldsymbol{x})]\, dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i_1<\cdots<i_k} a_{i_1\ldots i_k}(t\boldsymbol{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \omega. \end{aligned}$$ ■

Definition 4.9.5. If U and V are open sets in $\mathbb{R}^n$ and $\boldsymbol{f} : U \to V$ is a one-to-one function carrying U onto V such that both $\boldsymbol{f}$ and $\boldsymbol{f}^{-1}$ are of class C^r, then $\boldsymbol{f}$ is called a diffeomorphism (of class C^r); U and V are said to be C^r-diffeomorphic.

Proposition 4.9.6. *Fix k, $1 \le k \le n$. Let $U \subset \mathbb{R}^n$ be an open set in which every closed k-form is exact. If $V \subset \mathbb{R}^n$ is C^2-diffeomorphic to U, then every closed k-form in V is exact in V.*

Proof. Let $\boldsymbol{f} : U \to V$ be a diffeomorphism of class C^2 and let $\boldsymbol{g} = \boldsymbol{f}^{-1}$. Let ω be a k-form in V, with $d\omega = 0$. Then $\boldsymbol{f}^*\omega$ is a k-form in U for which $d(\boldsymbol{f}^*\omega) = 0$. Hence $\boldsymbol{f}^*\omega = d\lambda$ for some $(k-1)$-form λ in U. By proposition 4.5.14

$$\omega = \boldsymbol{g}^*(\boldsymbol{f}^*\omega) = \boldsymbol{g}^*(d\lambda) = d(\boldsymbol{g}^*\lambda).$$

Since $\boldsymbol{g}^*\lambda$ is a $(k-1)$-form in V, ω is exact in V. ■

Corollary 4.9.7. *Every closed form is exact in a set which is C^2-diffeomorphic to a star-shaped open set.*

4.10 Vector analysis

Let $\boldsymbol{F} = [F_1 \ \ F_2 \ \ F_3]^\top$ be a continuous mapping of an open set $U \subset \mathbb{R}^3$ into $\mathbb{R}^3$. Since $\boldsymbol{F}$ associates a vector to each point of U, $\boldsymbol{F}$ is sometimes called a vector field. With every such $\boldsymbol{F}$ is associated a 1-form

$$\lambda_{\boldsymbol{F}} = F_1\,dx + F_2\,dy + F_3\,dz \tag{4.10.1}$$

and a 2-form

$$\omega_{\boldsymbol{F}} = F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy. \tag{4.10.2}$$

Here, we use the customary notation (x, y, z) in place of (x_1, x_2, x_3). It is clear, conversely, that every 1-form λ in U is $\lambda_{\boldsymbol{F}}$ for some vector field F in U, and that every 2-form ω is $\omega_{\boldsymbol{F}}$ for some $\boldsymbol{F}$.

If $u \in C^1(U)$ is a real function, then its *gradient*

$$\nabla u = (D_1 u)\boldsymbol{e}_1 + (D_2 u)\boldsymbol{e}_2 + (D_3 u)\boldsymbol{e}_3$$

is an example of a vector field in U. Suppose now that $\boldsymbol{F}$ is a vector field in U, of class C^1. Its *curl* $\nabla \times \boldsymbol{F}$ is the vector field defined in U by

$$\nabla \times \boldsymbol{F} = (D_2F_3 - D_3F_2)\boldsymbol{e}_1 + (D_3F_1 - D_1F_3)\boldsymbol{e}_2 + (D_1F_2 - D_2F_1)\boldsymbol{e}3$$

and its *divergence* is the real function $\nabla \cdot \boldsymbol{F}$ defined in U by

$$\nabla \cdot \boldsymbol{F} = D_1F_1 + D_2F_2 + D_3F_3.$$

Theorem 4.10.3. *Suppose U is an open set in $\mathbb{R}^3$, $u \in C^2(U)$, and $\boldsymbol{G}$ is a vector field in U, of class C^2.*

(a) $\boldsymbol{F} = \nabla u$, then $\nabla \times \boldsymbol{F} = \boldsymbol{0}$.

(b) If $\boldsymbol{F} = \nabla \times \boldsymbol{G}$, then $\nabla \cdot \boldsymbol{F} = 0$.

Furthermore, if U is C^2-diffeomorphic to a star-shaped set, then (a) and (b) have converses, in which we assume that $\boldsymbol{F}$ is a vector field in U, of class C^2:

(a′) If $\nabla \times \boldsymbol{F} = \boldsymbol{0}$, then $\boldsymbol{F} = \nabla u$ for some $u \in C^2(U)$.

(b′) If $\nabla \cdot \boldsymbol{F} = 0$, then $\boldsymbol{F} = \nabla \times \boldsymbol{G}$ for some vector field $\boldsymbol{G}$ in U, of class C^2.

Proof. If we compare the definitions of ∇u, $\nabla \times \boldsymbol{F}$, and $\nabla \cdot \boldsymbol{F}$ with the differential forms $\lambda_{\boldsymbol{F}}$ and $\omega_{\boldsymbol{F}}$, we obtain the following four statements:

$$\begin{array}{rcl}
\boldsymbol{F} = \nabla u & \text{if and only if} & \lambda_{\boldsymbol{F}} = du, \\
\nabla \times \boldsymbol{F} = \boldsymbol{0} & \text{if and only if} & d\lambda_{\boldsymbol{F}} = 0, \\
\boldsymbol{F} = \nabla \times \boldsymbol{G} & \text{if and only if} & \omega_{\boldsymbol{F}} = d\lambda_{\boldsymbol{G}}, \\
\nabla \cdot \boldsymbol{F} = 0 & \text{if and only if} & d\omega_{\boldsymbol{F}} = 0.
\end{array}$$

■

Definition 4.10.4. The k-form $dx_1 \wedge \cdots \wedge dx_k$ is called the volume element in $\mathbb{R}^k$. It is often denoted by dV and the notation

$$\int_{\boldsymbol{\Phi}} f(\boldsymbol{x})\,dx_1 \wedge \cdots \wedge dx_k = \int_{\boldsymbol{\Phi}} f\,dV$$

is used when $\boldsymbol{\Phi}$ is a positively oriented k-surface in R^k and f is a continuous function on the range of $\boldsymbol{\Phi}$.

The reason for using this terminology is very simple: If D is a parameter domain in $\mathbb{R}^k$, and if $\mathbf{\Phi}$ is a one-to-one C^1-mapping of D into $\mathbb{R}^k$, with positive Jacobian $\det D\mathbf{\Phi}$, then the left side of the above equation is

$$\int_D f(\mathbf{\Phi}(\boldsymbol{u})) \det D\mathbf{\Phi}(\boldsymbol{u})\, d\boldsymbol{u} = \int_{\mathbf{\Phi}(D)} f(\boldsymbol{x})\, d\boldsymbol{x}.$$

Theorem 4.10.5 (Green's theorem)**.** *Let U be an open set in $\mathbb{R}^2$. Suppose $P : U \to \mathbb{R}$ and $Q : U \to \mathbb{R}$ are of class C^1, and D is a closed subset of U, with positively oriented boundary ∂D. Then*

$$\int_{\partial D} P\, dx + Q\, dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dV. \tag{4.10.6}$$

Proof. Put $\lambda = P\, dx + Q\, dy$. Then

$$d\lambda = (D_2 P)\, dy \wedge dx + (D_1 Q)\, dx \wedge dy = (D_1 Q - D_2 P) dV,$$

and (4.10.6) is the same as

$$\int_{\partial D} \lambda = \int_D d\lambda.$$

■

Definition 4.10.7. Let $\mathbf{\Phi}$ be a 2-surface in $\mathbb{R}^3$, of class C^1, with parameter domain $D \subset \mathbb{R}^2$. Associate with each point $(u, v) \in D$ the vector

$$\boldsymbol{N}(u, v) = \frac{\partial(y, z)}{\partial(u, v)}\, \boldsymbol{e}_1 + \frac{\partial(z, x)}{\partial(u, v)}\, \boldsymbol{e}_2 + \frac{\partial(x, y)}{\partial(u, v)}\, \boldsymbol{e}_3,$$

where Jacobians correspond to the equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{\Phi}(u, v).$$

We note that

$$\boldsymbol{N}(u, v) = D_u \mathbf{\Phi} \times D_v \mathbf{\Phi}.$$

If f is a continuous function on $\mathbf{\Phi}(D)$, the area integral of f over $\mathbf{\Phi}$ is defined to be

$$\int_{\mathbf{\Phi}} f\, dA = \int_D f(\mathbf{\Phi}(u, v))\, |\boldsymbol{N}(u, v)|\, du dv.$$

Definition 4.10.8. Let $\boldsymbol{\gamma}$ be a C^1-curve in an open set $U \subset \mathbb{R}^3$, with parameter interval $[a, b]$, let $\boldsymbol{F}$ be a vector field in U, and define $\lambda_{\boldsymbol{F}}$ by (4.10.1). We define

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s},$$

the *line integral* of $\boldsymbol{F}$ along $\boldsymbol{\gamma}$ by the formula

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}} \lambda_{\boldsymbol{F}} = \sum_{i=1}^{3} \int_a^b F_i(\boldsymbol{\gamma}(u))\, \gamma_i'(u)\, du = \int_a^b \boldsymbol{F}(\boldsymbol{\gamma}(u)) \cdot \boldsymbol{\gamma}'(u)\, du.$$

Definition 4.10.9. Let $\mathbf{\Phi}$ be a 2-surface in an open set $U \subset \mathbb{R}^3$, of class C^1, with parameter domain $D \subset \mathbb{R}^2$. Let $\boldsymbol{F}$ be a vector field in U, and define $\omega_{\boldsymbol{F}}$ by (4.10.2). The surface integral of $\boldsymbol{F}$ over $\mathbf{\Phi}$, denoted by

$$\begin{aligned}\int_{\mathbf{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_{\mathbf{\Phi}} \omega_{\boldsymbol{F}} &= \int_{\mathbf{\Phi}} F_1 \, dx \wedge dy + F_2 \, dz \wedge dx + F_3 \, dx \wedge dy \\ &= \int_D \left\{ (F_1 \circ \mathbf{\Phi}) \frac{\partial(y,z)}{\partial(u,v)} + (F_2 \circ \mathbf{\Phi}) \frac{\partial(z,x)}{\partial(u,v)} + (F_3 \circ \mathbf{\Phi}) \frac{\partial(x,y)}{\partial(u,v)} \right\} dudv \\ &= \int_D \boldsymbol{F}(\mathbf{\Phi}(u,v)) \cdot \boldsymbol{N}(u,v) \, dudv.\end{aligned}$$

Let $\boldsymbol{n} = \boldsymbol{n}(u,v)$ be the unit vector in the direction of $\boldsymbol{N}(u,v)$. [If $\boldsymbol{N}(u,v) = \mathbf{0}$ for some $(u,v) \in D$, take $\boldsymbol{n}(u,v) = \boldsymbol{e}_1$.] Then the last integral becomes

$$\int_D \boldsymbol{F}(\mathbf{\Phi}(u,v)) \cdot \boldsymbol{n}(u,v) |\boldsymbol{N}(u,v)| \, dudv = \int_{\mathbf{\Phi}} \boldsymbol{F} \cdot \boldsymbol{n} \, dA.$$

and we can write this in the form

$$\int_{\mathbf{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_{\mathbf{\Phi}} \omega_{\boldsymbol{F}} = \int_{\mathbf{\Phi}} \boldsymbol{F} \cdot \boldsymbol{n} \, dA.$$

We can now state the original form of Stokes' theorem.

Theorem 4.10.10 (Stokes' formula). *If $\boldsymbol{F}$ is a vector field of class C^1 in an open set $U \subset \mathbb{R}^3$, and if $\mathbf{\Phi}$ is a 2-surface of class C^2 in U, then*

$$\int_{\mathbf{\Phi}} (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial \mathbf{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

Proof. By theorem 4.10.3, we have $\omega_{\nabla \times \boldsymbol{F}} = d\lambda_{\boldsymbol{F}}$. Hence

$$\int_{\mathbf{\Phi}} (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\mathbf{\Phi}} \omega_{\nabla \times \boldsymbol{F}} = \int_{\mathbf{\Phi}} d\lambda_{\boldsymbol{F}} = \int_{\partial \mathbf{\Phi}} \lambda_{\boldsymbol{F}} = \int_{\partial \mathbf{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{s}. \qquad \blacksquare$$

Theorem 4.10.11 (Gauss' divergence theorem). *If $\boldsymbol{F}$ is a vector field of class C^1 in an open set $U \subset \mathbb{R}^3$, and if D is a closed subset of U with positively oriented boundary ∂D, then*

$$\int_D (\nabla \cdot \boldsymbol{F}) \, dV = \int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{S}.$$

Proof. Since

$$d\omega_{\boldsymbol{F}} = (\nabla \cdot \boldsymbol{F}) \, dx \wedge dy \wedge dz = (\nabla \cdot \boldsymbol{F}) \, dV,$$

we have

$$\int_D (\nabla \cdot \boldsymbol{F}) \, dV = \int_D d\omega_{\boldsymbol{F}} = \int_{\partial D} \omega_{\boldsymbol{F}} = \int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{S}. \qquad \blacksquare$$

Chapter 5

The Lebesgue theory

5.1 Set functions

Definition 5.1.1. Let X be a nonempty set. A family $\mathscr{R}$ of subsets of X is called a ring if $A \in \mathscr{R}$ and $B \in \mathscr{R}$ implies

$$A \cup B \in \mathscr{R}, \quad A \setminus B \in \mathscr{R}.$$

Since $A \cap B = A \setminus (A \setminus B)$, we also have $A \cap B \in \mathscr{R}$ if $\mathscr{R}$ is a ring.

A ring $\mathscr{R}$ is called a σ-ring if it is closed under countable unions; i.e.

$$\bigcup_{n=1}^{\infty} A_n \in \mathscr{R} \quad \text{whenever} \quad A_n \in \mathscr{R} \ \ (n = 1, 2, \ldots).$$

Since $\bigcap_{n=1}^{\infty} A_n = A_1 \setminus \bigcup_{n=1}^{\infty}(A_1 \setminus A_n)$, we also have $\bigcap_{n=1}^{\infty} A_n \in \mathscr{R}$.

Definition 5.1.2. An algebra of sets on X is a nonempty collection $\mathscr{A}$ of subsets of X that is closed under finite unions and complements; in other words, if $E_1, \ldots, E_n \in \mathscr{A}$, then $\bigcup_{i=1}^{n} E_i \in \mathscr{A}$ and if $E \in \mathscr{A}$, then $E^c \in \mathscr{A}$. A σ-algebra is an algebra that is closed under countable unions.

Proposition 5.1.3. *A σ-algebra is a σ-ring. A σ-ring is a σ-algebra if it contains the universal set X.*

Definition 5.1.4. We say that ϕ is a set function defined on $\mathscr{R}$ if ϕ assigns to every $A \in \mathscr{R}$ a number $\phi(A)$ of the extended real number system. ϕ is additive if $A \cap B = \emptyset$ implies

$$\phi(A \cup B) = \phi(A) + \phi(B),$$

and ϕ is countably additive if $A_i \cap A_j = \emptyset$ $(i \neq j)$ implies

$$\phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \phi(A_n).$$

We shall always assume that the range of ϕ does not contain both $+\infty$ and $-\infty$. Also, we exclude set functions whose only value is $+\infty$ or $-\infty$.

Proposition 5.1.5. *If ϕ is additive, the following properties hold.*

1. $\phi(\emptyset) = 0$.

2. $\phi(A_1 \cup \cdots \cup A_n) = \phi(A_1) + \cdots + \phi(A_n)$ *if* $A_i \cap A_j = \emptyset$ *whenever* $i \neq j$.

3. $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$.

4. If $\phi(A) \geq 0$ *for all* A*, and* $A_1 \subset A_2$*, then* $\phi(A_1) \leq \phi(A_2)$.

5. $\phi(A \setminus B) = \phi(A) - \phi(B)$ *if* $B \subset A$ *and* $|\phi(B)| < +\infty$.

Proof. Exercise. ∎

Proposition 5.1.6. *Suppose* ϕ *is countably additive on a ring* $\mathscr{R}$*. Suppose* $A_n \in \mathscr{R}$ $(n = 1, 2, \ldots)$*,* $A_1 \subset A_2 \subset A_2 \subset \cdots$*,* $A \in \mathscr{R}$*, and* $A = \bigcup_{n=1}^{\infty} A_n$*. Then,*

$$\phi(A_n) \to \phi(A) \quad \text{as } n \to \infty.$$

Proof. Put $B_1 = A_1$, and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then $B_i \cap B_j = \emptyset$ for $i \neq j$, $A_n = B_1 \cup \cdots \cup B_n$, and $A = \bigcup_{n=1}^{\infty} B_n$. Hence

$$\phi(A_n) = \sum_{i=1}^{n} \phi(B_i) \quad \text{and} \quad \phi(A) = \sum_{i=1}^{\infty} \phi(B_i).$$ ∎

5.2 Construction of the Lebesgue measure

Let $\mathbb{R}^d$ denote d-dimensional euclidean space. By an interval in $\mathbb{R}^d$ we mean the set of points $\boldsymbol{x} = (x_1, \ldots, x_d)$ such that

$$a_i \leq x_i \leq b_i \quad (i = 1, \ldots, d) \tag{5.2.1}$$

or the set of points which is characterized by (5.2.1) with any or all of the $\leq$ signs replaced by $<$. Here, we assume $a_i \leq b_i$ are real numbers for $i = 1, \ldots, d$. The possibility that $a_i = b_i$ for any value of i is not ruled out in (5.2.1); in particular, the empty set is included among the intervals.

Definition 5.2.2. If I is an interval, we define

$$m(I) = \prod_{i=1}^{d} (b_i - a_i)$$

no matter whether equality is included or excluded in any of the inequalities (5.2.1). If $A = I_1 \cup \cdots \cup I_n$ and if these intervals are pairwise disjoint, we set

$$m(A) = m(I_1) + \cdots + m(I_n).$$

We note that intervals in $\mathbb{R}^d$ include closed and open intervals as introduced in definition 2.1.1 and also that $m(I)$ agrees with $v(I)$ as defined in definition 2.1.2.

Definition 5.2.3. If A is the union of a finite number of intervals, A is said to be an elementary set. We let $\mathscr{E}$ denote the family of all elementary subsets of $\mathbb{R}^d$.

Proposition 5.2.4. *1.* $\mathscr{E}$ *is a ring.*

2. If $A \in \mathscr{E}$, then A is the union of a finite number of disjoint intervals.

3. If $A \in \mathscr{E}$, $m(A)$ is well defined; that is, if two different decompositions of A into disjoint intervals are used, each gives rise to the same value of $m(A)$.

4. m is additive on $\mathscr{E}$.

Proof. Exercise. ■

Definition 5.2.5. A nonnegative additive set function ϕ defined on $\mathscr{E}$ is said to be regular if the following is true: To every $A \in \mathscr{E}$ and to every $\epsilon > 0$ there exist sets $F \in \mathscr{E}$, $G \in \mathscr{E}$ such that F is closed, G is open, $F \subset A \subset G$, and

$$\phi(G) - \epsilon \leq \phi(A) \leq \phi(F) + \epsilon.$$

Proposition 5.2.6. *The set function m is regular.*

Proof. Exercise. ■

Definition 5.2.7. Let μ be additive, regular, nonnegative, and finite on $\mathscr{E}$. Consider countable coverings of any set $E \in \mathbb{R}^d$ by open elementary sets A_n: $E \subset \bigcup_{n=1}^{\infty} A_n$. Define

$$\mu^*(E) = \inf \sum_{n=1}^{\infty} \mu(A_n)$$

the infimum being taken over all countable coverings of E by open elementary sets. $\mu^*(E)$ is called the outer measure of E, corresponding to μ.

Theorem 5.2.8. *(a) $\mu^*(E) \geq 0$ for all E.*

(b) $\mu^*(E_1) \leq \mu^*(E_2)$ if $E_1 \subset E_2$.

(c) For every $A \in \mathscr{E}$, $\mu^*(A) = \mu(A)$.

(d) If $E = \bigcup_{n=1}^{\infty} E_n$, then

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n). \tag{5.2.9}$$

Note that (c) asserts that μ^* is an extension of μ from $\mathscr{E}$ to the family of all subsets of $\mathbb{R}^d$. The property (5.2.9) is called countable subadditivity.

Proof. Properties (a) and (b) are clear. To prove (c), choose $\epsilon > 0$. The regularity of μ shows that A is contained in an open elementary set G such that $\mu(G) \leq \mu(A) + \epsilon$. Since $\mu^*(A) \leq \mu(G)$ and since ϵ was arbitrary, we have

$$\mu^*(A) \leq \mu(A).$$

The definition of μ^* shows that there is a sequence $(A_n)_{n \in \mathbb{N}}$ of open elementary sets whose union contains A, such that $\sum_{n=1}^{\infty} \mu(A_n) \leq \mu^*(A) + \epsilon$. The regularity of μ shows that A contains a closed elementary set F such that $\mu(F) \geq \mu(A) - \epsilon$; and since F is compact, we have $F \subset A_1 \cup \cdots \cup A_N$ for some N. Hence

$$\mu(A) \leq \mu(F) + \epsilon \leq \sum_{i=1}^{N} \mu(A_n) + \epsilon \leq \mu^*(A) + 2\epsilon.$$

Since ϵ was arbitrary, in conjunction with $\mu^*(A) \le \mu(A)$, this proves (c).

Next, suppose $E = \bigcup_{n=1}^{\infty} E_n$, and assume that $\mu^*(E_n) < +\infty$ for all n. Given $\epsilon > 0$, there are coverings $(A_{n,k})_{k\in\mathbb{N}}$ of E_n by open elementary sets such that

$$\sum_{k=1}^{\infty} \mu(A_{n,k}) \le \mu^*(E_n) + 2^{-n}\epsilon.$$

Then

$$\sum_{k=1}^{\infty} \mu(A_{n,k}) \le \mu^*(E_n) + 2^{-n}\epsilon.$$

and (5.2.9) follows. In the excluded case, i.e., if $\mu^*(E_n) = +\infty$ for some n, (5.2.9) is of course trivial. ∎

We note that $m^*(E) = 0$ if and only if E is a measure zero set in $\mathbb{R}^d$; see definition 2.1.11 and proposition 2.1.13.

Proposition 5.2.10. *$\mathscr{N} = \{E : \mu^*(E) = 0\}$ is a σ-ring on $\mathbb{R}^d$.*

Proof. Exercise. ∎

Definition 5.2.11. For any $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^d$, we define

$$A \triangle B = (A \setminus B) \cup (B \setminus A),$$
$$d(A, B) = \mu^*(A \triangle B).$$

We write $A_n \to A$ if

$$\lim_{n\to\infty} d(A_n, A) = 0.$$

If there is a sequence $(A_n)_{n\in\mathbb{N}}$ of elementary sets such that $A_n \to A$, we say that A is *finitely μ-measurable* and write $A \in \mathfrak{M}_F(\mu)$. If A is the union of a countable collection of finitely μ-measurable sets, we say that A is *μ-measurable* and write $A \in \mathfrak{M}(\mu)$.

Proposition 5.2.12. *1. $d(A, B) = d(B, A)$ and $d(A, A) = 0$.*

2. $d(A, B) \le d(A, C) + d(C, B)$.

3. $d(A, B) = d(A^c, B^c)$, where E^c is the complement of E.

4. $\left.\begin{matrix} d(A_1 \cup A_2, B_1 \cup B_2) \\ d(A_1 \cap A_2, B_1 \cap B_2) \\ d(A_1 \setminus A_2, B_1 \setminus B_2) \end{matrix}\right\} \le d(A_1, B_1) + d(A_2, B_2).$

5. $|\mu^*(A) - \mu^*(B)| \le d(A, B)$ if at least one of $\mu^*(A)$ and $\mu^*(B)$ is finite.

Proof. 1. It is clear that

$$A \triangle B = B \triangle A \quad \text{and} \quad A \triangle A = \emptyset.$$

2. We have $A \triangle B \subset (A \triangle C) \cup (C \triangle B)$ since

$$A \setminus B \subset (A \setminus C) \cup (C \setminus B) \quad \text{and} \quad B \setminus A \subset (C \setminus A) \cup (B \setminus C).$$

3. We have
$$A^c \triangle B^c = A \triangle B. \tag{5.2.13}$$

4. The first formula of 4 is obtained from
$$(A_1 \cup A_2) \triangle (B_1 \cup B_2) \subset (A_1 \triangle B_1) \cup (A_2 \triangle B_2), \tag{5.2.14}$$
which in turn is consequence of
$$(A_1 \cup A_2) \setminus (B_1 \cup B_2) \subset (A_1 \setminus B_1) \cup (A_2 \setminus B_2).$$
Next, by (5.2.13) and (5.2.14) we have
$$\begin{aligned}(A_1 \cap A_2) \triangle (B_1 \cap B_2) &= (A_1^c \cup A_2^c) \triangle (B_1^c \cup B_2^c) \\ &\subset (A_1^c \triangle B_1^c) \cup (A_2^c \triangle B_2^c) = (A_1 \triangle B_1) \cup (A_2 \triangle B_2),\end{aligned}$$
from which the second formula of 4 follows; and the last formula of 4 is obtained if we note that $A_1 \setminus A_2 = A_1 \cap A_2^c$ and use 3.

5. Suppose $0 \le \mu^*(B) \le \mu^*(A)$. Then 2 shows that
$$d(A, \emptyset) \le d(A, B) + d(B, \emptyset),$$
that is $\mu^*(A) \le d(A, B) + \mu^*(B)$. Since $\mu^*(B)$ is finite, it follows that
$$\mu^*(A) - \mu^*(B) \le d(A, B). \qquad \blacksquare$$

Theorem 5.2.15. *$\mathfrak{M}(\mu)$ is a σ-algebra, and μ^* is countably additive on $\mathfrak{M}(\mu)$.*

Proof. Step 1. $\mathfrak{M}_F(\mu)$ is a ring and μ^* is additive on $\mathfrak{M}_F(\mu)$.

Suppose $A \in \mathfrak{M}_F(\mu)$, $B \in \mathfrak{M}_F(\mu)$. Choose $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ such that $A_n \in \mathscr{E}$, $B_n \in \mathscr{E}$, $A_n \to A$, $B_n \to B$. Then proposition 5.2.12 shows that
$$\begin{aligned} A_n \cup B_n &\to A \cup B, \\ A_n \cap B_n &\to A \cap B, \\ A_n \setminus B_n &\to A \setminus B, \\ \mu^*(A_n) &\to \mu^*(A), \end{aligned}$$
and $\mu^*(A) < +\infty$ since $d(A_n, A) \to 0$. Therefore, $\mathfrak{M}_F(\mu)$ is a ring and by proposition 5.1.5,
$$\mu(A_n) + \mu(B_n) = \mu(A_n \cup B_n) + \mu(A_n \cap B_n).$$
Letting $n \to \infty$, we obtain by theorem 5.2.8,
$$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu^*(A \cap B).$$
It follows that μ^* is additive on $\mathfrak{M}_F(\mu)$.

Step 2. If $A \in \mathfrak{M}(\mu)$ and $\mu^*(A) < +\infty$, then $A \in \mathfrak{M}_F(\mu)$.

Let $A \in \mathfrak{M}(\mu)$. Then A can be represented as the union of a countable collection of *disjoint* sets of $\mathfrak{M}_F(\mu)$. For if $A = \bigcup_{n=1}^{\infty} A_n'$ with $A_n' \in \mathfrak{M}_F(\mu)$, write $A_1 = A_1'$ and
$$A_n = A_n' \setminus (A_1' \cup \cdots \cup A_{n-1}') \quad (n = 2, 3, 4, \ldots).$$

Then $A = \bigcup_{n=1}^\infty A_n$ is the required representation. By countable subadditivity of μ^*, we have

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

On the other hand, $A \supset A_1 \cup \cdots \cup A_n$; and by the additivity of μ^* on $\mathfrak{M}_F(\mu)$ we obtain

$$\mu^*(A) \geq \mu^*(A_1 \cup \cdots \cup A_n) = \mu^*(A_1) + \cdots + \mu^*(A_n).$$

The above two inequalities imply

$$\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n). \tag{5.2.16}$$

Suppose $\mu^*(A)$ is finite. Put $B_n = A_1 \cup \cdots \cup A_n$. Then (5.2.16) shows that

$$d(A, B_n) = \mu^* \left(\bigcup_{i=n+1}^{\infty} A_i \right) \leq \sum_{i=n+1}^{\infty} \mu^*(A_i) \to 0 \quad \text{as} \;\; n \to \infty.$$

Hence $B_n \to A$; and since $B_n \in \mathfrak{M}_F(\mu)$, it is easily seen that $A \in \mathfrak{M}_F(\mu)$.

Step 3. μ^* is countably additive on $\mathfrak{M}(\mu)$.

If $A = \bigcup_{n=1}^\infty A_n$, where $\{A_n\}$ is a sequence of disjoint sets of $\mathfrak{M}(\mu)$, we have shown that (5.2.16) holds if $\mu^*(A_n) < +\infty$ for every n, and in the other case (5.2.16) is trivial.

Step 4. $\mathfrak{M}(\mu)$ is a σ-algebra.

If $A_n \in \mathfrak{M}(\mu)$, $n = 1, 2, 3, \ldots$, it is clear that $\bigcup_{n=1}^\infty A_n \in \mathfrak{M}(\mu)$. Suppose $A \in \mathfrak{M}(\mu)$, $B \in \mathfrak{M}(\mu)$, and

$$A = \bigcup_{n=1}^{\infty} A_n, \qquad B = \bigcup_{n=1}^{\infty} B_n,$$

where $A, B \in \mathfrak{M}_F(\mu)$. Then the identity

$$A_n \cap B = \bigcup_{i=1}^{\infty} (A_n \cap B_i)$$

shows that $A_n \cap B \in \mathfrak{M}(\mu)$; and since

$$\mu^*(A_n \cap B) \leq \mu^*(A_n) < +\infty,$$

$A_n \cap B \in \mathfrak{M}_F(\mu)$. Hence $A_n \setminus B \in \mathfrak{M}_F(\mu)$, and $A \setminus B \in \mathfrak{M}(\mu)$ since

$$A \setminus B = \bigcup_{n=1}^{\infty} (A_n \setminus B).$$

Therefore, $\mathfrak{M}(\mu)$ is a σ-ring. Finally, $\mathbb{R}^d \in \mathfrak{M}(\mu)$ since $\mathbb{R}^d = \bigcup_{n=1}^\infty [-n, n]^d$. ∎

Definition 5.2.17. We now replace $\mu^*(A)$ by $\mu(A)$ if $A \in \mathfrak{M}(\mu)$. Thus μ, originally only defined on $\mathscr{E}$, is extended to a countable additive set function on the σ-algebra $\mathfrak{M}(\mu)$. This extended set function is called a *measure*. The special case $\mu = m$ is called the *Lebesgue measure* on $\mathbb{R}^d$.

Definition 5.2.18. Let X be a topological space. We say that $E \subset X$ is a Borel set if E can be obtained by a countable number of operations, starting from open sets, each operation consisting in taking unions, intersections, or complements.

Proposition 5.2.19. *The collection $\mathscr{B}$ of all Borel sets in $\mathbb{R}^d$ is a σ-algebra; in fact, it is the smallest σ-algebra which contains all open sets.*

Proof. Exercise. ∎

Proposition 5.2.20. *(a) $\mathscr{B} \subset \mathfrak{M}(\mu)$; i.e., $\mathfrak{M}(\mu)$ contains all the Borel sets.*

(b) If $E \in \mathfrak{M}(\mu)$ and $\epsilon > 0$, there exist sets F and G such that $F \subset E \subset G$, F closed, G is open, and $\mu(G \setminus E) < \epsilon$, $\mu(E \setminus F) < \epsilon$.

(c) If $E \in \mathfrak{M}(\mu)$, there exist Borel sets F and G such that $F \subset E \subset G$ and $\mu(G \setminus E) = \mu(E \setminus F) = 0$.

Proof. (a) Every open set in $\mathbb{R}^d$ is the union of a countable collection of open intervals, and thus belongs to $\mathfrak{M}(\mu)$.

(b) First consider the case that $\mu(E) = \mu^*(E) < \infty$. For any $\epsilon > 0$, the definition of μ^* shows that there exist open elementary sets A_n such that $E \subset \bigcup_{n=1}^\infty A_n$ and $\sum_{n=1}^\infty \mu(A_n) < \mu(E) + \epsilon$. Define $G = \bigcup_{n=1}^\infty A_m$. Then G is an open set containing A and by (a) and (5.2.9),

$$\mu(G) = \mu^*(G) \le \sum_{n=1}^\infty \mu(A_n) < \mu(E) + \epsilon.$$

Since $G \setminus E \in \mathfrak{M}(\mu)$, the additivity of μ thus shows that

$$\mu(G \setminus E) = \mu(G) - \mu(E) < \epsilon.$$

If $\mu^*(E) = \infty$, let $E_n = E \cap [-n, n]^d \in \mathfrak{M}(\mu)$. By the preceding argument, there is an open set G_n containing E_n for which $\mu(G_n \setminus E_n) < \epsilon/2^n$. The set $G = \bigcup_{n=1}^\infty G_n$ is open, $E \subset G$, and $G \setminus E \subset \bigcup_{n=1}^\infty (G_n \setminus E_n)$. Therefore,

$$\mu(G \setminus E) \le \sum_{n=1}^\infty \mu(G_n \setminus E_n) < \sum_{n=2}^\infty \epsilon/2^n = \epsilon.$$

The second inequality follows by taking the complement.

(c) This follows from (b) if we take $\epsilon = 1/n$ and let $n \to \infty$. ∎

5.3 Measurable functions

Definition 5.3.1. A set X is said to be a measurable space if there exists a σ-algebra $\mathfrak{M}$ of subsets of X (which are called measurable sets). If, in addition, there exists a nonnegative countably additive set function μ (which is called a measure), defined on $\mathfrak{M}$, then X is said to be a measure space.

Example 5.3.2. We can take $X = \mathbb{R}^d$, $\mathfrak{M}$ the collection of all Lebesgue-measurable subsets of $\mathbb{R}^d$, and μ Lebesgue measure. Or, let $X = \mathbb{N}$, the set of all positive integers, $\mathfrak{M}$ the collection of all subsets of $\mathbb{N}$, and $\mu(E)$ the number of elements of E. Another example is provided by probability theory, where events may be considered as sets, and the probability of the occurrence of events is an additive (or countably additive) set function.

Definition 5.3.3. Let f be a function defined on the measurable space X, with values in the extended real number system. The function f is said to be measurable if the set

$$\{x : f(x) > a\}$$

is measurable for every real a.

Example 5.3.4. If $X = \mathbb{R}^d$ and $\mathfrak{M} = \mathfrak{M}(\mu)$ as defined in definition 5.2.11, every continuous f is measurable, since then $\{x : f(x) > a\}$ an open set.

Proposition 5.3.5. *Each of the following four conditions implies the other three:*

1. $\{x : f(x) > a\}$ *is measurable for every real* a.
2. $\{x : f(x) \geq a\}$ *is measurable for every real* a.
3. $\{x : f(x) < a\}$ *is measurable for every real* a.
4. $\{x : f(x) \leq a\}$ *is measurable for every real* a.

Hence any of these conditions may be used instead to define measurability.

Proof. The relations

$$\begin{aligned}
\{x : f(x) \geq a\} &= \bigcap_{n=1}^{\infty} \{x : f(x) > a - 1/n\}, \\
\{x : f(x) < a\} &= X \setminus \{x : f(x) \geq a\}, \\
\{x : f(x) \leq a\} &= \bigcap_{n=1}^{\infty} \{x : f(x) < a + 1/n\}, \\
\{x : f(x) > a\} &= X \setminus \{x : f(x) \leq a\}
\end{aligned}$$

shows successively that 1 implies 2, 2 implies 3, 3 implies 4, and 4 implies 1. ■

Proposition 5.3.6. *If f is measurable, then $|f|$ is measurable.*

Proof.

$$\{x : |f(x)| < a\} = \{x : f(x) < a\} \cap \{x : f(x) > -a\}.$$ ■

Proposition 5.3.7. *Let $\{f_n\}$ be a sequence of measurable functions. For $x \in X$, put*

$$g(x) = \sup_n f_n(x) \quad \textit{and} \quad h(x) = \limsup_{n\to\infty} f_n(x).$$

Then g and h are measurable. The same is of course true of the inf *and* lim inf*.*

Proof.

$$\{x : g(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > a\},$$
$$h(x) = \inf_m g_m(x), \quad \text{where } g_m(x) = \sup_{n\geq m} f_n(x). \qquad \blacksquare$$

Corollary 5.3.8. *1. If f and g are measurable, then* $\max(f, g)$ *and* $\min(f, g)$ *are measurable. If*

$$f^+ = \max(f, 0), \qquad f^- = -\min(f, 0), \tag{5.3.9}$$

it follows, in particular, that f^+ and f^- are measurable.

2. The limit of a convergent sequence of measurable functions is measurable.

Proposition 5.3.10. *Let f and g be measurable real-valued functions defined on X, let F be real and continuous on $\mathbb{R}^2$, and put*

$$h(x) = F(f(x), g(x)) \qquad (x \in X).$$

Then h is measurable. In particular, $f + g$ and fg are measurable.

Proof. Let $G_a = \{(u, v) : F(u, v) > a\}$. Then G_a is an open subset of $\mathbb{R}^2$, and we can write $G_a = \bigcup_{n=1}^{\infty} I_n$, where $(I_n)_{n\in\mathbb{N}}$ is a sequence of open intervals:

$$I_n = \{(u, v) : a_n < u < b_n,\ \ c_n < v < d_n\}.$$

Since

$$\{x : a_n < f(x) < b_n\} = \{x : f(x) > a_n\} \cap \{x : f(x) < b_n\}$$

is measurable, it follows that the set

$$\{x : (f(x), g(x)) \in I_n\} = \{x : a_n < f(x) < b_n\} \cap \{x : c_n < g(x) < d_n\}$$

is measurable. Hence the same is true of

$$\{x : h(x) > a\} = \{x : (f(x), g(x)) \in G_a\} = \bigcup_{n=1}^{\infty} \{x : (f(x), g(x)) \in I_n\}. \qquad \blacksquare$$

Summing up, we may say that all ordinary operations of analysis, including limit operations, when applied to measurable functions, lead to measurable functions; in other words, all functions that are ordinarily met with are measurable.

5.4 Integration

Definition 5.4.1. Let s be a real-valued function defined on X. If the range of s is finite, we say that s is a simple function. Suppose the range of s consists of the distinct numbers $c_1, \dots c_n$. Then

$$s = \sum_{i=1}^{n} c_i \chi_{E_i}, \quad \text{where } \; E_i = \{x : s(x) = c_i\} \quad (i = 1, \dots, n),$$

that is, every simple function is a finite linear combination of characteristic functions. It is clear that s is measurable if and only if the sets $E_1, \dots, E_n$ are measurable.

Theorem 5.4.2. *Let f be a real function on X. There exists a sequence $(s_n)_{n\in\mathbb{N}}$ of simple functions such that $s_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$. If f is measurable, $(s_n)_{n\in\mathbb{N}}$ may be chosen to be a sequence of measurable functions. If $f \geq 0$, $(s_n)_{n\in\mathbb{N}}$ may be chosen to be a monotonically increasing sequence.*

Proof. If $f \geq 0$, define

$$E_{n,i} = \left\{x : \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}\right\}, \qquad F_n = \{x : f(x) \geq n\}$$

for $n = 1, 2, 3, \dots$, $i = 1, 2, \dots, n2^n$. Put

$$s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n\chi_{F_n}.$$

In the general case, let $f = f^+ - f^-$, and apply the preceding construction to f^+ and to f^-. It may be noted that the sequence $\{s_n\}$ given above converges uniformly to f if f is bounded. ■

Definition 5.4.3. Suppose

$$s(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x) \qquad (c_i > 0)$$

is measurable, and suppose $E \in \mathfrak{M}$. We define

$$I_E(s) = \sum_{i=1}^{n} c_i \mu(E \cap E_i).$$

If f is measurable and nonnegative, we define

$$\int_E f \, d\mu = \sup I_E(s), \tag{5.4.4}$$

where the supremum is taken over all measurable simple functions s such that $0 \leq s \leq f$. The left member of (5.4.4) is called the *Lebesgue integral* of f, with respect to the measure μ, over the set E. It should be noted that the integral may have the value $+\infty$.

Proposition 5.4.5. *$\int_E s\,d\mu = I_E(s)$ for every nonnegative simple measurable function s.*

Definition 5.4.6. Let f be measurable, and consider the two integrals

$$\int_E f^+\,d\mu, \qquad \int_E f^-\,d\mu, \tag{5.4.7}$$

where f^+ and f^- are defined as in (5.3.9). If at least one of the integrals in (5.4.7) is finite, we define

$$\int_E f\,d\mu = \int_E f^+\,d\mu - \int_E f^-\,d\mu.$$

If both integrals in (5.4.7) are finite, then $\int_E f\,d\mu$ is finite, and we say that f is integrable (or summable) on E in the Lebesgue sense, with respect to μ; we write $f \in L^1(\mu)$ on E. If $\mu = m$, the usual notation is: $f \in L^1$ on E.

Proposition 5.4.8. *Suppose $E \in \mathfrak{M}$. Then the following properties hold.*

(a) If f is measurable and bounded on E, and if $\mu(E) < +\infty$, then $f \in L^1(\mu)$ on E.

(b) If $a \le f(x) \le b$ for $x \in E$, and $\mu(E) < +\infty$, then

$$a\mu(E) \le \int_E f\,d\mu \le b\mu(E).$$

(c) If f and $g \in L^1(\mu)$ on E, and if $f(x) \le g(x)$ for $x \in E$, then

$$\int_E f\,d\mu \le \int_E g\,d\mu.$$

(d) If $f \in L^1(\mu)$ on E, then $cf \in L^1(\mu)$ on E, for every finite constant c, and

$$\int_E cf\,d\mu = c\int_E f\,d\mu.$$

(e) If $\mu(E) = 0$, and f is measurable, then

$$\int_E f\,d\mu = 0.$$

(f) If $f \in L^1(\mu)$ on E, $A \in \mathfrak{M}$, and $A \subset E$, then $f \in L^1(\mu)$ on A.

Proof. Exercise. ∎

Theorem 5.4.9. *(a) Suppose f is measurable and nonnegative on X. For $A \in \mathfrak{M}$, define*

$$\phi(A) = \int_A f\,d\mu.$$

Then ϕ is countably additive on $\mathfrak{M}$.

(b) The same conclusion holds if $f \in L^1(\mu)$ on X.

Proof. It is clear that (b) holds from (a) if we write $f = f^+ - f^-$ and apply (a) to f^+ and to f^-. To prove (a), we have to show that

$$\phi(A) = \sum_{n=1}^{\infty} \phi(A_n) \tag{5.4.10}$$

if $A_n \in \mathfrak{M}$ $(n = 1, 2, 3, \ldots)$, $A_i \cap A_j = \emptyset$ for $i \neq j$, and $A = \bigcup_{n=1}^{\infty} A_n$.

If f is a characteristic function, then the countable additivity of ϕ is precisely the same as the countable additivity of μ, since

$$\int_A \chi_E \, d\mu = \mu(A \cap E).$$

If f is simple, then the conclusion again holds. In the general case, we have, for every measurable simple function s such that $0 \leq s \leq f$,

$$\int_A s \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} s \, d\mu \leq \sum_{n=1}^{\infty} \phi(A_n).$$

Therefore, by (5.4.4)

$$\phi(A) \leq \sum_{n=1}^{\infty} \phi(A_n).$$

Now if $\phi(A_n) = +\infty$ for some n, (5.4.10) is trivial, since $\phi(A) \geq \phi(A_n)$. Suppose $\phi(A_n) < +\infty$ for every n. Given $\epsilon > 0$, we can choose a measurable function s such that $0 \leq s \leq f$, and such that

$$\int_{A_1} s \, d\mu \geq \int_{A_1} f \, d\mu - \epsilon, \quad \int_{A_2} s \, d\mu \geq \int_{A_2} f \, d\mu - \epsilon.$$

Hence

$$\phi(A_1 \cup A_2) \geq \int_{A_1 \cup A_2} s \, d\mu = \int_{A_1} s \, d\mu + \int_{A_2} s \, d\mu \geq \phi(A_1) + \phi(A_2) - 2\epsilon,$$

so that

$$\phi(A_1 \cup A_2) \geq \phi(A_1) + \phi(A_2).$$

It follows that we have, for every n,

$$\phi(A_1 \cup \cdots \cup A_n) \geq \phi(A_1) + \cdots + \phi(A_n).$$

Since $A \supset A_1 \cup \cdots \cup A_n$, the above implies

$$\phi(A) \geq \sum_{n=1}^{\infty} \phi(A_n).$$

and (5.4.10) follows. ■

Corollary 5.4.11. *If $A \in \mathfrak{M}$, $B \subset A$, and $\mu(A \setminus B) = 0$, then*

$$\int_A f \, d\mu = \int_B f \, d\mu.$$

The preceding corollary shows that sets of measure zero are negligible in integration. Let us write $f \sim g$ on E if the set

$$\{x : f(x) \neq g(x)\} \cap E$$

has measure zero. Then $f \sim f$; $f \sim g$ implies $g \sim f$ and $f \sim g$, $g \sim h$ implies $f \sim h$. That is, the relation $\sim$ is an equivalence relation. If $f \sim g$ on E, we clearly have

$$\int_A f \, d\mu = \int_A g \, d\mu,$$

provided the integrals exist, for every measurable subset A of E.

Definition 5.4.12. If a property P holds for every $x \in E \setminus A$, and if $\mu(A) = 0$, we say that P holds for almost all $x \in E$, or that P holds almost everywhere on E. (This concept of "almost everywhere" depends of course on the particular measure under consideration. In the literature, unless something is said to the contrary, it usually refers to Lebesgue measure.)

Proposition 5.4.13. *If $f \in L^1(\mu)$ on E, then $|f| \in L^1(\mu)$ on E and*

$$\left| \int_E f \, d\mu \right| \leq \int_E |f| \, d\mu.$$

Proof. Write $E = A \cup B$, where $f(x) \geq 0$ on A and $f(x) < 0$ on B. By theorem 5.4.9,

$$\int_E |f| \, d\mu = \int_A |f| \, d\mu + \int_B |f| \, d\mu = \int_A f^+ \, d\mu + \int_B f^- \, d\mu =< +\infty$$

so that $|f| \in L^1(\mu)$. Since $f \leq |f|$ and $-f \leq |f|$, we see that

$$\int_E f \, d\mu \leq \int_E |f| \, d\mu, \quad -\int_E f \, d\mu \leq \int_E |f| \, d\mu. \qquad \blacksquare$$

Proposition 5.4.14. *Suppose f is measurable on E, $|f| \leq g$, and $g \in L^1(\mu)$ on E. Then $f \in L^1(\mu)$ on E.*

Proof. Exercise. $\blacksquare$

Theorem 5.4.15 (Lebesgue's monotone convergence theorem)**.** *Suppose $E \in \mathfrak{M}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions such that*

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \qquad (x \in E).$$

Let f be defined by

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (x \in E).$$

Then

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

Proof. Since $\int_E f_n$ is an increasing sequence of numbers, so its limit exists (possibly equal to $+\infty$). Moreover, $\int_E f_n \leq \int_E f$ for all n, so we have

$$\lim_{n\to\infty} \int_E f_n \, d\mu \leq \int_E f \, d\mu.$$

To establish the reverse inequality, fix c such that $0 < c < 1$, and let s be a simple measurable function such that $0 \leq s \leq f$. Put

$$E_n = \{x : f_n(x) \geq cs(x)\} \quad (n = 1, 2, 3, \ldots).$$

Then $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of measurable sets whose union is E, and we have

$$\int_E f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq c \int_{E_n} s \, d\mu.$$

By theorem 5.4.9 and proposition 5.1.6,

$$\lim_{n\to\infty} \int_{E_n} s \, d\mu = \int_E s \, d\mu$$

and hence

$$\lim_{n\to\infty} \int_E f_n \, d\mu \geq c \int_E s \, d\mu.$$

Since this is true for all $c < 1$, it remains true for $c = 1$, and taking the supremum over all simple $0 \leq s \leq f$, we obtain

$$\lim_{n\to\infty} \int_E f_n \, d\mu \geq \int_E f \, d\mu. \qquad \blacksquare$$

Proposition 5.4.16. *Suppose $f = f_1 + f_2$, where $f_1, f_2 \in L^1(\mu)$ on E. Then $f \in L^1(\mu)$ on E, and*

$$\int_E f \, d\mu = \int_E f_1 \, d\mu + \int_E f_2 \, d\mu. \tag{5.4.17}$$

Proof. First, suppose $f_1 \geq 0$, $f_2 \geq 0$. If f_1 and f_2 are simple, (5.4.17) follows trivially. Otherwise, choose monotonically increasing sequences $(s'_n)_{n\in\mathbb{N}}$, $(s''_n)_{n\in\mathbb{N}}$ of nonnegative measurable simple functions which converge to f_1, f_2. Theorem 5.4.2 shows that this is possible. Put $s_n = s'_n + s''_n$. Then

$$\int_E s_n \, d\mu = \int_E s'_n \, d\mu + \int_E s''_n \, d\mu,$$

and (5.4.17) follows if we let $n \to \infty$ and appeal to the monotone convergence theorem. Next, suppose $f_1 \geq 0$, $f_2 \leq 0$. Put

$$A = \{x : f(x) \geq 0\}, \quad B = \{x : f(x) < 0\}.$$

Then, f, f_1 and $-f_2$ are nonnegative on A. Hence

$$\int_A f_1 \, d\mu = \int_A f \, d\mu + \int_A (-f_2) \, d\mu = \int_A f \, d\mu - \int_A f_2 \, d\mu.$$

Similarly, $-f$, f_1, and $-f_2$ are nonnegative on B, so that

$$\int_B (-f_2)\, d\mu = \int_B f_1\, d\mu + \int_B (-f)\, d\mu,$$

or

$$\int_B f_1\, d\mu = \int_B f\, d\mu - \int_B f_2\, d\mu,$$

and (5.4.17) follows. In the general case, E can be decomposed into four sets E_i on each of which $f_1(x)$ and $f_2(x)$ are of constant sign. The two cases we have proved so far imply

$$\int_{E_i} f\, d\mu = \int_{E_i} f_1\, d\mu + \int_{E_i} f_2\, d\mu \quad (i = 1, 2, 3, 4),$$

and (5.4.17) follows by adding these four equations. ∎

Corollary 5.4.18. *Suppose $E \in \mathfrak{M}$. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative measurable functions and*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in E),$$

then

$$\int_E f\, d\mu = \sum_{n=1}^{\infty} \int_E f_n\, d\mu.$$

Theorem 5.4.19 (Fatou's lemma)**.** *Suppose $E \in \mathfrak{M}$. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative measurable functions and*

$$f(x) = \liminf_{n\to\infty} f_n(x) \qquad (x \in E),$$

then

$$\int_E f\, d\mu \le \liminf_{n\to\infty} \int_E f_n\, d\mu.$$

Proof. For $n = 1, 2, 3, \ldots$ and $x \in E$ put

$$g_n(x) = \inf_{k\ge n} f_k(x).$$

Then g_n is measurable on E, $g_n(x) \le f_n(x)$, $\lim_{n\to\infty} g_n(x) = f(x)$, and

$$0 \le g_1(x) \le g_2(x) \le \cdots .$$

By the monotone convergence theorem

$$\int_E f\, d\mu = \lim_{n\to\infty} \int_E g_n\, d\mu \le \liminf_{n\to\infty} \int_E f_n\, d\mu.$$ ∎

Theorem 5.4.20 (Lebesgue's dominated convergence theorem)**.** *Suppose $E \in \mathfrak{M}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions such that*

$$\lim_{n\to\infty} f_n(x) = f(x).$$

If there exists a function $g \in L^1(\mu)$ on E, such that

$$|f_n(x)| \le g(x) \quad \textit{for all} \ \ n = 1, 2, 3, \ldots,$$

then

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

Proof. First, note that $f_n \in L^1(\mu)$ and $f \in L^1(\mu)$ on E. Since $f_n + g \ge 0$, Fatou's lemma shows that

$$\int_E (f+g) \, d\mu \le \liminf_{n\to\infty} \int_E (f_n + g) \, d\mu,$$

or

$$\int_E f \, d\mu \le \liminf_{n\to\infty} \int_E f_n \, d\mu.$$

Since $g - f_n \ge 0$, we see similarly that

$$\int_E (g-f) \, d\mu \le \liminf_{n\to\infty} \int_E (g - f_n) \, d\mu,$$

so that

$$\int_E f \, d\mu \ge \limsup_{n\to\infty} \int_E f_n \, d\mu.$$

Therefore,

$$\liminf_{n\to\infty} \int_E f_n \, d\mu \ge \int_E f \, d\mu \ge \limsup_{n\to\infty} \int_E f_n \, d\mu,$$

and the result follows. ■

Corollary 5.4.21. *If $\mu(E) < \infty$, $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded on E, and $f_n(x) \to f(x)$ on E, then*

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

Theorem 5.4.22. *Let f be a bounded real-valued function on $[a, b]$.*

a. If f is Riemann integrable, then f is Lebesgue measurable (and hence integrable on $[a, b]$ since it is bounded), and

$$\int_a^b f(x) \, dx = \int_{[a,b]} f \, dm.$$

b. f is Riemann integrable if and only if

$$\{x \in [a, b] : f \textit{ is discontinuous at } x\}$$

has Lebesgue measure zero.

Proof. There is a sequence $(P_k)_{k\in\mathbb{N}}$ of partitions of $[a,b]$, such that P_{k+1} is a refinement of P_k, such that the distance between adjacent points of P_k is less than $1/k$, and such that

$$\lim_{k\to\infty} L(f,P_k) = \underline{\int_a^b} f(x)\,dx, \quad \lim_{k\to\infty} U(f,P_k) = \overline{\int_a^b} f(x)\,dx$$

If $P_k = \{t_0, t_1, \ldots, t_n\}$, with $t_0 = a$, $t_n = b$, define $U_k(a) = L_k(a) = f(a)$; put $U_k(x) = \sup_{[t_{i-1},t_i]} f$ and $L_k(x) = \inf_{[t_{i-1},t_i]} f$ for $t_{i-1} < x \leq t_i$, $1 \leq i \leq n$. Then

$$L(f,P_k) = \int_{[a,b]} L_k\,dm, \quad U(f,P_k) = \int_{[a,b]} U_k\,dm,$$

and

$$L_1(x) \leq L_2(x) \leq \cdots \leq f(x) \leq \cdots \leq U_2(x) \leq U_1(x)$$

for all $x \in [a,b]$, since P_{k+1} refines P_k. Therefore, there exist

$$L(x) = \lim_{k\to\infty} L_k(x), \quad U(x) = \lim_{k\to\infty} U_k(x).$$

Observe that L and U are bounded measurable functions on $[a,b]$, that

$$L(x) \leq f(x) \leq U(x) \qquad (a \leq x \leq b),$$

and that

$$\int_{[a,b]} L\,dm = \underline{\int_a^b} f(x)\,dx, \quad \int_{[a,b]} U\,dm = \overline{\int_a^b} f(x)\,dx$$

by the dominated convergence theorem.

To complete the proof, note that f is Riemann integrable if and only if its upper and lower Riemann integrals are equal, hence if and only if

$$\int_{[a,b]} L\,dm = \int_{[a,b]} U\,dm = \int_a^b f(x)\,dx.$$

Since $L \leq U$, the above equation implies that $L(x) = U(x)$ almost everywhere on $[a,b]$, and thus

$$L(x) = f(x) = U(x)$$

almost everywhere on $[a,b]$, so that f is measurable and

$$\int_{[a,b]} f\,dm = \int_{[a,b]} U\,dm = \int_a^b f(x)\,dx.$$

This proves (a) and the proof of (b) is given in theorem 2.1.19. ∎

Definition 5.4.23. Suppose f is a complex-valued function defined on a measurable space X, and $f = u + iv$, where u and v are real. We say that f is measurable if and only if both u and v are measurable.

It is easy to verify that sums and products of complex measurable functions are again measurable. Since

$$|f| = \sqrt{u^2 + v^2},$$

proposition 5.3.10 shows that $|f|$ is measurable for every complex measurable function f.

Definition 5.4.24. Suppose μ is a measure on X, E is a measurable subset of X, and f is a complex function on X. We say that $f \in L^1(\mu)$ on E provided that f is measurable and

$$\int_E |f|\, d\mu < +\infty$$

and we define

$$\int_E f\, d\mu = \int_E u\, d\mu + i \int_E v\, d\mu.$$

Since $|u| \le |f|$, $|v| \le |f|$, and $|f| \le |u| + |v|$, it is clear that $f \in L^1(\mu)$ on E if and only if $u \in L^1(\mu)$ and $v \in L^1(\mu)$ on E.

Proposition 5.4.8(a), (d), (e), (f), theorem 5.4.9(b), propositions 5.4.13, 5.4.14, 5.4.16, and Lebesgue's dominated convergence theorem can now be extended to Lebesgue integrals of complex functions. The proofs are quite straightforward. That of proposition 5.4.13 is the only one that offers anything of interest:

If $f \in L^1(\mu)$ on E, there is a complex number c, $|c| = 1$, such that

$$c \int_E f\, d\mu \ge 0.$$

Put $g = cf = u + iv$, u and v real, then

$$\left| \int_E f\, d\mu \right| = c \int_E f\, d\mu = \int_E g\, d\mu = \int_E u\, d\mu \le \int_E |f|\, d\mu.$$

The third of the above equalities holds since the preceding ones show that $\int_E g\, d\mu$ is real.

5.5 Functions of class L^2

Definition 5.5.1. Let X be a measurable space. We say that a complex function $f \in L^2(\mu)$ on X if f is measurable and if

$$\int_X |f|^2\, d\mu < +\infty.$$

If μ is the Lebesgue measure, we say $f \in L^2$. For $f \in L^2(\mu)$ (we shall omit the phrase "on X" from now on) we define

$$\|f\| = \left\{ \int_X |f|^2\, d\mu \right\}^{1/2}$$

and call $\|f\|$ the $L^2(\mu)$ norm of f.

Proposition 5.5.2 (Schwarz inequality)**.** *Suppose $f \in L^2(\mu)$ and $g \in L^2(\mu)$. Then $fg \in L^1(\mu)$, and*

$$\int_X |fg|\, d\mu \le \|f\|\, \|g\|.$$

Proof. It follows from the inequality

$$0 \le \int_X (|f| + \lambda|g|)^2\, d\mu = \|f\|^2 + 2\lambda \int_X |fg|\, d\mu + \lambda^2 \|g\|^2$$

which holds for every real λ. ∎

Proposition 5.5.3. *If $f \in L^2(\mu)$ and $g \in L^2(\mu)$, then $f + g \in L^2(\mu)$, and*

$$\|f + g\| \le \|f\| + \|g\|.$$

Proof. The Schwarz inequality shows that

$$\begin{aligned}\|f+g\|^2 &= \int |f|^2 + \int f\bar{g} + \int \bar{f}g + \int |g|^2 \\ &= \int |f|^2 + 2\Re \int f\bar{g} + \int |g|^2 \\ &\le \|f\|^2 + 2\|f\|\,\|g\| + \|g\|^2 = (\|f\| + \|g\|)^2.\end{aligned}$$ ∎

If we define the distance between two functions f and g in $L^2(\mu)$ to be $\|f - g\|$, we see that the conditions for a metric are satisfied, except for the fact that $\|f - g\| = 0$ does not imply that $f(x) = g(x)$ for all x, but only for almost all x. Thus, if we identify functions which differ only on a set of measure zero, $L^2(\mu)$ is a metric space.

Theorem 5.5.4. *The continuous functions form a dense subset of L^2 on $[a, b]$. More explicitly, this means that for any $f \in L^2$ on $[a, b]$, and any $\epsilon > 0$, there is a function g, continuous on $[a, b]$, such that*

$$\|f - g\| = \left\{ \int_a^b (f - g)^2 \, dx \right\}^{1/2} < \epsilon.$$

Proof. We shall say that f is approximated in L^2 by a sequence $\{g_n\}$ if

$$\lim_{n\to\infty} \|f - g_n\| = 0.$$

Let A be a closed subset of $[a, b]$, and χ_A its characteristic function. Put

$$g_n(x) = \frac{1}{1 + n \operatorname{dist}(x, A)} \qquad (n = 1, 2, 3, \ldots).$$

Then g_n is continuous on $[a, b]$, $g_n(x) = 1$ on A, and $g_n(x) \to 0$ on $[a, b] \setminus A$. Hence

$$\lim_{n\to\infty} \|g_n - \chi_A\|^2 = \lim_{n\to\infty} \int_{[a,b]\setminus A} g_n^2 \, dx = 0$$

by the dominated convergence theorem. Therefore, characteristic functions of closed sets can be approximated in L^2 by continuous functions. By proposition 5.2.20(b) the same is true for the characteristic function of any measurable set, and hence also for simple measurable functions. If $f \ge 0$ and $f \in L^2$, let $(s_n)_{n\in\mathbb{N}}$ be a monotonically increasing sequence of simple nonnegative measurable functions such that $s_n(x) \to f(x)$. Since $|f - s_n|^2 \le f^2$, the dominated convergence theorem shows that $\|f - s_n\| \to 0$. The general case follows. ∎

Definition 5.5.5. We say that a sequence of complex functions $(\phi_n)_{n\in\mathbb{N}}$ is an orthonormal set of functions on a measurable space X if

$$\int_X \phi_n \bar{\phi}_m \, d\mu = \begin{cases} 0 & (n \ne m) \\ 1 & (n = m) \end{cases}.$$

In particular, we must have $\phi_n \in L^2(\mu)$. If $f \in L^2(\mu)$ and if

$$c_n = \int_X f\bar{\phi}_n \, d\mu \qquad (n = 1, 2, 3, \ldots)$$

we write

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n.$$

Definition 5.5.6. Let f and $f_n \in L^2(\mu)$ $(n = 1, 2, 3, \ldots)$. We say that f_n converges to f in $L^2(\mu)$ if $\|f_n - f\| \to 0$. We say that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mu)$ if for every $\epsilon > 0$ there is an integer N such that $n \geq N$, $m \geq N$ implies $\|f_n - f_m\| \leq \epsilon$.

Theorem 5.5.7. *If $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mu)$, then there exists a function $f \in L^2(\mu)$ such that f_n converges to f in $L^2(\mu)$. This says, in other words, that $L^2(\mu)$ is a complete metric space.*

Proof. Since $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, we can find a sequence $n_1, n_2, n_3, \ldots$ such that

$$\|f_{n_k} - f_{n_{k+1}}\| < \frac{1}{2^k} \qquad (k = 1, 2, 3, \ldots).$$

Choose a function g in $L^2(\mu)$. By the Schwarz inequality,

$$\int_X \sum_{k=1}^{\infty} |g(f_{n_k} - f_{n_{k+1}})| \, d\mu = \sum_{k=1}^{\infty} \int_X |g(f_{n_k} - f_{n_{k+1}})| \, d\mu \leq \|g\|.$$

It follows that

$$|g(x)| \sum_{k=1}^{\infty} |f_{n_k}(x) - f_{n_{k+1}}(x)| < +\infty$$

almost everywhere on X. Therefore

$$\sum_{k=1}^{\infty} |f_{n_k}(x) - f_{n_{k+1}}(x)| < +\infty$$

almost everywhere on X. For if the series were divergent on a set E of positive measure, we could take $g(x)$ to be nonzero on a subset of E of positive measure, thus obtaining a contradiction. Since the kth partial sum of the series

$$\sum_{k=1}^{\infty} (f_{n_k}(x) - f_{n_{k+1}}(x)),$$

which converges almost everywhere on X, is $f_{n_{k+1}}(x) - f_{n_1}(x)$, we see that the equation

$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

defines $f(x)$ for almost all $x \in X$, and it does not matter how we define $f(x)$ at the remaining points of X. We shall now show that this function f has the desired properties. Let $\epsilon > 0$ be given, and choose N as indicated in definition 5.5.6. If $n_k > N$, Fatou's lemma shows that

$$\|f - f_{n_k}\| \leq \liminf_{i\to\infty} \|f_{n_i} - f_{n_k}\| \leq \epsilon.$$

Thus $f - f_{n_k} \in L^2(\mu)$, and since $f = (f - f_{n_k}) + f_{n_k}$, we see that $f \in L^2(\mu)$. Also, since ϵ is arbitrary,

$$\lim_{k\to\infty} \|f - f_{n_k}\| = 0.$$

Finally, the inequality

$$\|f - f_n\| = \|f - f_{n_k}\| + \|f_{n_k} - f_n\|$$

shows that f_n converges to f in $L^2(\mu)$; for if we take n and n_k large enough, each of the two terms on the right of the above equation can be made arbitrarily small. ■

Theorem 5.5.8 (Riesz-Fischer theorem)**.** *Let $(\phi_n)_{n\in\mathbb{N}}$ be orthonormal on X. Suppose $\sum_{n=1}^{\infty}|c_n|^2$ converges, and put $s_n = c_1\phi_1 + \cdots + c_n\phi_n$. Then there exists a function $f \in L^2(\mu)$ such that s_n converges to f in $L^2(\mu)$, and such that*

$$f \sim \sum_{n=1}^{\infty} c_n\phi_n.$$

Proof. For $n > m$,

$$\|s_n - s_m\|^2 = |c_{m+1}|^2 + \cdots + |c_n|^2,$$

so that $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mu)$. By theorem 5.5.7, there is a function $f \in L^2(\mu)$ such that $\lim_{n\to\infty}\|f - s_n\| = 0$. Now, for $n > k$,

$$\int_X f\bar{\phi}_k \, d\mu - c_k = \int_X f\bar{\phi}_k \, d\mu - \int_X s_n\bar{\phi}_k \, d\mu$$

so that

$$\left|\int_X f\bar{\phi}_k \, d\mu - c_k\right| = \|f - s_n\| \, \|\phi_k\| \leq \|f - s_n\|.$$

Letting $n \to \infty$, we see that

$$c_k = \int_X f\bar{\phi}_k \, d\mu \qquad (k = 1, 2, 3, \ldots).$$ ■

Definition 5.5.9. An orthonormal set $(\phi_n)_{n\in\mathbb{N}}$ is said to be complete if, for $f \in L^2(\mu)$, the equations

$$\int_X f\bar{\phi}_n \, d\mu = 0 \qquad (n = 1, 2, 3, \ldots)$$

imply that $\|f\| = 0$.

Theorem 5.5.10. *Let $(\phi_n)_{n\in\mathbb{N}}$ be a complete orthonormal set. If $f \in L^2(\mu)$ and if*

$$f \sim \sum_{n=1}^{\infty} c_n\phi_n,$$

then

$$\int_X |f|^2 \, d\mu = \sum_{n=1}^{\infty} |c_n|^2.$$

The above equation is called Parseval's identity.

Proof. By the Bessel inequality, $\sum |c_n|^2$ converges. Putting

$$s_n = c_1\phi_1 + \cdots + c_n\phi_n,$$

the Riesz-Fischer theorem shows that there is a function $g \in L^2(\mu)$ such that $g \sim \sum c_n\phi_n$, and such that $\|g - s_n\| \to 0$. Hence $\|s_n\| \to \|g\|$. Since

$$\|s_n\|^2 = |c_1|^2 + \cdots + |c_n|^2,$$

we have

$$\int_X |g|^2 \, d\mu = \sum_{n=1}^{\infty} |c_n|^2.$$

Now, the completeness of $(\phi_n)_{n\in\mathbb{N}}$ shows that $\|f - g\| = 0$. ■

Combining Riesz-Fischer theorem and theorem 5.5.10, we arrive at the very interesting conclusion that every complete orthonormal set induces a one-to-one correspondence between the functions $f \in L^2(\mu)$ (identifying those which are equal almost everywhere) on the one hand and the sequences $(c_n)_{n\in\mathbb{N}}$ for which $\sum |c_n|^2$ converges, on the other. The representation

$$f \sim \sum_{n=1}^{\infty} c_n\phi_n$$

together with the Parseval identity, shows that $L^2(\mu)$ may be regarded as an infinite-dimensional euclidean space (the so-called "Hilbert space"), in which the point f has coordinates c_n, and the functions ϕ_n are the coordinate vectors.

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