Probability Distribution

1 **Binomial** (n,p), $\quad (x = 0,1,\ldots,n)$ R:*x*binom

$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$
$M(t) = (pe^t + 1 - p)^n$
$E(X) = np$, $\mathrm{Var}(X) = np(1-p)$

NOTE: $n \to \infty$ with $\mu = np \Rightarrow \text{Poisson}(\mu)$
$F_{\text{Bin}(n,p)}(x) = F_{\text{Beta}(n-x,x+1)}(1-p)$
$F_{\text{Bin}(n,p)}(r-1) = 1 - F_{\text{NegBin}(r,p)}(n-r)$
$X_i \sim \text{Bin}(n_i,p) \Rightarrow \sum X_i \sim \text{Bin}(\sum n_i, p)$

2 **Geometric** (p) R:*x*geom

$f(x) = p(1-p)^x$, $\quad (x = 0,1,\ldots)$
$F(t) = 1-(1-p)^{\lfloor t \rfloor + 1}$, where $\lfloor t \rfloor = \max\{m \in \mathbb{Z} : m \le t\}$
$S(t) = P[X \ge t] = (1-p)^{\lceil t \rceil}$ with $\lceil t \rceil = \min\{m \in \mathbb{Z} : m \ge t\}$
$R(t) = P[X > t] = (1-p)^{\lfloor t \rfloor + 1}$
$M_X(t) = \dfrac{p}{1-(1-p)e^t}$, $\quad (t < -\log(1-p))$
$E(X) = \dfrac{1-p}{p}$, $\mathrm{Var}(X) = \dfrac{1-p}{p^2}$

$f(y) = p(1-p)^{y-1}$, $\quad (y = 1,2,\ldots)$
$F(t) = 1-(1-p)^{\lfloor t \rfloor}$
$S(t) = P[Y \ge t] = (1-p)^{\lceil t \rceil - 1}$
$R(t) = P[Y > t] = (1-p)^{\lfloor t \rfloor}$
$M_Y(t) = \dfrac{pe^t}{1-(1-p)e^t}$, $(t < -\log(1-p))$
$E(Y) = \dfrac{1}{p}$, $\mathrm{Var}(Y) = \dfrac{1-p}{p^2}$

NOTE: $\min\limits_{1 \le i \le n}(X_i) \sim \text{Geo}\left(1-(1-p)^n\right)$: self-reproducing

3 **Hypergeometric** (N,M,n) R:*x*hyper

$$f(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}, \quad (\max(0, M-(N-n)) \le x \le \min(n,M))$$

$E(X) = n\left(\dfrac{M}{N}\right)$, $\mathrm{Var}(X) = n\dfrac{M}{N}\dfrac{(N-M)(N-n)}{N(N-1)}$

4 **Negative Binomial** (r,p) R:*x*nbinom

$f(x) = \binom{r+x-1}{x} p^r (1-p)^x$, $(x = 0,1,\ldots)$
$M_X(t) = \left(\dfrac{p}{1-(1-p)e^t}\right)^r$
$E(X) = r\frac{1-p}{p}$, $\mathrm{Var}(X) = r\frac{1-p}{p^2}$

$f(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$, $(y = r, r+1, \ldots)$
$M_Y(t) = \left(\dfrac{pe^t}{1-(1-p)e^t}\right)^r$
$E(Y) = r\frac{1}{p}$, $\mathrm{Var}(Y) = r\frac{1-p}{p^2}$

NOTE: $X = V_1 + \cdots + V_r$, $(V_i \sim \text{Geometric}(p))$
$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \quad (|x| < 1)$
$\mu = r(1-p), r \to \infty \Rightarrow \text{Poisson}(\mu)$
$F_{\text{Bin}(n,p)}(r-1) = 1 - F_{\text{NegBin}(r,p)}(n-r)$
$X_i \sim \text{NB}(r_i,p) \Rightarrow \sum X_i \sim \text{NB}(\sum r_i, p)$

5 **Poisson** (μ) R:*x*pois

$f(x) = \dfrac{e^{-\mu}\mu^x}{x!}$, $\quad (x = 0,1,\ldots)$
$M(t) = \exp\left(\mu(e^t - 1)\right)$
$E(X) = \mu$, $\mathrm{Var}(X) = \mu$

NOTE: $X_i \sim \text{Poi}(\mu_i) \Rightarrow \sum X_i \sim \text{Poi}(\sum \mu_i)$

$$P_{\text{Poi}(\mu)}[X \ge n] = F_{\Gamma(n,1)}(\mu) = F_{\Gamma(n,\mu)}(1) = F_{\chi^2_{2n}}(2\mu)$$

1 **Beta** (α,β), $\quad (\alpha > 0, \beta > 0)$ R:*x*beta

$f(x) = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $\ (0 < x < 1)$
$M(t) = 1 + \sum_{k=1}^{\infty}\left(\prod_{r=1}^{k-1} \frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$
$E(X) = \dfrac{\alpha}{\alpha+\beta}$, $\ \mathrm{Var}(X) = \dfrac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

NOTE: $F_{\text{Beta}(n-x,x+1)}(1-p) = F_{\text{Bin}(n,p)}(x)$

2 **BS** (α,β), $\ (\alpha > 0, \beta > 0)$: Birnbaum-Saunders

$f(t) = \frac{1}{2\alpha\beta}\sqrt{\frac{\beta}{t}}\left(1+\frac{\beta}{t}\right)\phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right]$, $\ (t > 0)$
$F(t) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right]$

$$\begin{aligned} F^{-1}(p) &= \frac{1}{4}\left[\alpha\sqrt{\beta}\Phi^{-1}(p) + \sqrt{\alpha^2\beta\{\Phi^{-1}(p)\}^2 + 4\beta}\right]^2 \\ &= \beta\left\{1 + \gamma(p)^2 + \gamma(p)\sqrt{\gamma(p)^2 + 2}\right\}, \\ &\quad \text{where } \gamma(p) = \alpha\Phi^{-1}(p)/\sqrt{2} \end{aligned}$$

$E(T) = \beta(1 + \frac{1}{2}\alpha^2)$, $\quad \mathrm{Var}(T) = (\alpha\beta)^2(1 + \frac{5}{4}\alpha^2)$

NOTE: $\text{median}(T) = \beta$, $cT \sim \text{BS}(\alpha, c\beta)$, $T^{-1} \sim \text{BS}(\alpha, \beta^{-1})$
$X = \left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \sim N(0, \alpha^2)$
$\log T \sim \text{sinh-Normal}(\log\beta, \alpha)$

3 **Cauchy** (α,β) R:*x*cauchy

$f(x) = \dfrac{\beta}{\pi}\dfrac{1}{\beta^2 + (x-\alpha)^2}$
$F(x) = \dfrac{1}{\pi}\left[\arctan\left(\dfrac{x-\alpha}{\beta}\right) + \dfrac{\pi}{2}\right]$
$\phi(t) = \exp(it\alpha - \beta|t|)$

NOTE: $\text{Cauchy}(0,1) = t(1)$
$X, Y \sim N(0,1) \Rightarrow X/Y \sim \text{Cauchy}(0,1)$
$cX \sim \text{Cauchy}(c\alpha, c\beta)$
$X_i \sim \text{Cauchy}(\alpha,\beta) \Rightarrow \frac{1}{n}\sum X_i \sim \text{Cauchy}(\alpha,\beta)$

4 **Chi Squared** (n) R:*x*chisq

$f(x) = \dfrac{1}{\Gamma(n/2)2^{n/2}} x^{n/2-1} e^{-x/2}$, $(x \ge 0)$
$E(X) = n$, $\mathrm{Var}(X) = 2n$
$M(t) = \left(\frac{1}{1-2t}\right)^{n/2}$, $\quad (t < \frac{1}{2})$
$E(X^m) = 2^m\Gamma(m + n/2)/\Gamma(n/2)$

NOTE: $\chi^2(n) = \text{Gamma}(n/2, 2)$
$\chi^2(2) = \text{Exponential}(\beta = 2)$
$X_i \sim \chi^2(n_i) \Rightarrow \sum X_i \sim \chi^2(\sum n_i)$
$X_i \sim N(0,1) \Rightarrow \sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$
$F_{\chi^2(p+2)}(x) = F_{\chi^2(p)}(x) - 2xf_{\chi^2(p)}(x)/p$
$F_{\chi^2(1)}(x) = 2\Phi(\sqrt{x}) - 1$
$F_{\chi^2(2)}(x) = 1 - \sqrt{2\pi}\phi(\sqrt{x})$
$F_{\chi^2(3)}(x) = 2\Phi(\sqrt{x}) - 1 - 2\sqrt{x}\phi(\sqrt{x})$

5 **Exponential** (β) R:*x*exp

$f(x) = \dfrac{1}{\beta}e^{-x/\beta}$ $\ (x \ge 0, \ \beta > 0)$
$F(x) = 1 - e^{-x/\beta}$
$M(t) = \dfrac{1}{1-\beta t}$, $(t < \frac{1}{\beta})$
$E(X) = \beta, \mathrm{Var}(X) = \beta^2$

NOTE: Memoryless property
$cX \sim \text{Exponential}(c\beta)$
$\sum X_i \sim \text{Gamma}(n, \theta)$
$\min\limits_{1 \le i \le n}(X_i) \sim \text{Exponential}(\beta/n)$: self-reproducing
$Y = X^{1/\alpha} \sim \text{Weibull}(\alpha, \beta)$
$Y = \sqrt{2X/\beta} \sim \text{Rayleigh}(1)$
$Y = \alpha - \gamma\log(X/\beta) \sim \text{Gumbel}(\alpha, \gamma)$

6 $F(m,n)$ R:xf

$F(m,n) = \dfrac{\chi_m^2/m}{\chi_n^2/n}$

$E(X) = \dfrac{n}{n-2} \quad (n>2),$

$\mathrm{Var}(X) = 2\left(\dfrac{n}{n-2}\right)^2 \dfrac{m+n-2}{m(n-4)} \quad (n>4)$

NOTE: $[F(m,n)]^{-1} = F(n,m)$
$F_{1-\alpha}(m,n) = [F_\alpha(n,m)]^{-1}$
$F(1,k) = t^2(k)$
If $X \sim F(m,n)$, $mX/(mX+n) \sim \mathrm{Beta}(\frac{m}{2},\frac{n}{2})$.

7 **Gamma** (α,θ) R:$x$gamma

$f(x) = \dfrac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1}\exp(-x/\theta), \ (0<x<\infty)$

$M(t) = (1-\theta t)^{-\alpha} \ (t<1/\theta)$

$E(X) = \alpha\theta$, $\mathrm{Var}(X) = \alpha\theta^2$

NOTE: $X \sim \mathrm{Gamma}(n,\theta) = \mathrm{Erlang}(n,\theta)$
$X = V_1 + \cdots + V_n$, $(V_i \sim \mathrm{Exponential}(\theta))$
$2X/\theta \sim \chi^2(2n)$

$$F_X(x) = 1 - \sum_{k=0}^{n-1} \left(\frac{x}{\theta}\right)^k \frac{e^{-x/\theta}}{k!} = \sum_{k=n}^{\infty} \left(\frac{x}{\theta}\right)^k \frac{e^{-x/\theta}}{k!}$$

$\mathrm{Gamma}(n/2,2) = \chi^2(n)$, $\Gamma(1/2) = \sqrt{\pi}$
$X_i \sim \mathrm{Gamma}(\alpha_i,\theta) \Rightarrow \sum_{i=1}^r X_i \sim \mathrm{Gamma}(\sum_{i=1}^r \alpha_i, \theta)$
$X/d \sim \mathrm{Gamma}(\alpha, \theta/d)$
$E[X^c] = \Gamma(\alpha+c)\theta^c/\Gamma(\alpha) \ (c>-\alpha)$

8 **Laplace** (μ,σ): Double Exponential

$f(x) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}$

$F(x) = \begin{cases} \frac{1}{2} e^{-|x-\mu|/\sigma} & (x<\mu) \\ 1 - \frac{1}{2} e^{-|x-\mu|/\sigma} & (x \ge \mu) \end{cases}$

$M(t) = \dfrac{e^{\mu t}}{1-(\sigma t)^2}, \ (|t| < \frac{1}{\sigma})$

$E(X) = \mathrm{median}(X) = \mu$, $\mathrm{Var}(X) = 2\sigma^2$

9 **Logistic**,(μ,β) R:$x$logis

$f(x) = \dfrac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2} = \dfrac{1}{2\beta\left[1+\cosh\left(\frac{x-\alpha}{\beta}\right)\right]}$

$F(x) = \dfrac{1}{1+e^{-(x-\mu)/\beta}}$

$M(t) = e^{\mu t}\Gamma(1-\beta t)\Gamma(1+\beta t)$, $|t| < \frac{1}{\beta}$

$E(X) = \mathrm{median}(X) = \mu$, $\mathrm{Var}(X) = \frac{\pi^2\beta^2}{3}$

10 **Lognormal** (μ,σ^2) R:$x$lnorm

$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}x}\exp(-\frac{1}{2}(\frac{\log(x)-\mu}{\sigma})^2)$

$E(X^k) = e^{k\mu + k^2\sigma^2/2}$, $\mathrm{Var}(X) = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$

NOTE: $F_{logN(\mu,\sigma^2)}(x) = F_{N(\mu,\sigma^2)}(\log x) = \Phi(\frac{\log x - \mu}{\sigma})$
Self-reproducing under multiplication and division

11 **Normal** (μ,σ^2) R:$x$norm

$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)$

$M(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$

NOTE: If $X \sim N(\mu,\sigma^2)$, $Y = e^X \sim logN(\mu,\sigma^2)$

$$F_{N(0,1)}(x) = \frac{1}{2} + \frac{1}{2}\mathrm{sign}(x) F_{\Gamma(1/2,1)}(\frac{1}{2}x^2) = \frac{1}{2} + \frac{1}{2}\mathrm{sign}(x) F_{\chi^2(1)}(x)$$

$\phi'(z) = -z\phi(z)$, $\phi''(z) = (z^2-1)\phi(z)$
$E(X^3) = \mu^3 + 3\mu\sigma^2$, $E(X^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$

12 **Rayleigh** (β)

$f(x) = \dfrac{x}{\beta^2}\exp\left(-\dfrac{x^2}{2\beta^2}\right)$

$F(x) = 1 - \exp\left(-\dfrac{x^2}{2\beta^2}\right)$

$E(X) = \beta\sqrt{\pi/2}$, $E(X^k) = (\sqrt{2}\beta)^k \frac{k}{2}\Gamma(\frac{k}{2})$

$\mathrm{median}(X) = \beta\sqrt{2\ln 2}$

$\mathrm{Var}(X) = (2-\pi/2)\beta^2$

NOTE: $\mathrm{Rayleigh}(\beta) = \mathrm{Weibull}(2, 2\beta^2)$
$cX \sim \mathrm{Rayleigh}(c\beta)$
$(X/\beta)^2 \sim \chi^2(2)$
$X^2 \sim \mathrm{Exponential}(2\beta^2) = \mathrm{Gamma}(1, 2\beta^2)$
$\min_{1\le i\le n}(X_i) \sim \mathrm{Rayleigh}(\beta/\sqrt{n})$: self-reproducing

13 **Slash** (α,β)

$f(x) = \frac{1}{\sqrt{2\pi}(x-\alpha)^2/\beta^2}\left(1-\exp(-\frac{1}{2}(\frac{x-\alpha}{\beta})^2)\right)$

$F(x) = \Phi(\frac{x-\alpha}{\beta}) - (\frac{x-\alpha}{\beta}) f_{\mathrm{Slash}(0,1)}(\frac{x-\alpha}{\beta})$

NOTE: $X = \alpha + \beta\dfrac{Z}{U}$,
where $Z \sim N(0,1)$ and $U \sim \mathrm{Uniform}(0,1)$.

14 **Student** $t(k)$ R:xt

$f(x) = \dfrac{\Gamma((k+1)/2)}{\Gamma(k/2)}\dfrac{1}{\sqrt{k\pi}}\left(1+\dfrac{x^2}{k}\right)^{-(k+1)/2}, \ (k \ge 1)$

$E(X) = 0 \ (k>1)$, $\mathrm{Var}(X) = k/(k-2) \ (k>2)$

NOTE: $X \sim \dfrac{N(0,1)}{\sqrt{\chi_k^2/k}}$, $\quad F_\alpha(1,k) = t_{\alpha/2}(k)^2$

$$F_{t(k)}(x) = \begin{cases} 1 - \frac{1}{2} F_{\mathrm{Beta}(k/2,1/2)}(\frac{k}{k+x^2}) & (x \ge 0) \\ \frac{1}{2} F_{\mathrm{Beta}(k/2,1/2)}(\frac{k}{k+x^2}) & (x<0) \end{cases}$$

15 **Uniform** (a,b) R:$x$unif

$f(x) = \dfrac{1}{b-a}$

$M(t) = \dfrac{e^{tb}-e^{ta}}{t(b-a)}, \ (t \ne 0)$

$E(X) = \mathrm{median}(X) = \dfrac{a+b}{2}$, $\mathrm{Var}(X) = \dfrac{(b-a)^2}{12}$

NOTE: $X \sim \mathrm{Uniform}(0,1)$, $-\log X \sim \mathrm{Exponential}(1)$

16 **Wald** (μ,λ): Inverse Gaussian (IG) R:$x$invgauss{statmod}

$f(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi x^3}}\exp\left[-\lambda\frac{(x-\mu)^2}{2\mu^2 x}\right] = \sqrt{\frac{\lambda}{x^3}}\phi\left(\sqrt{\frac{\lambda}{x}}\frac{x-\mu}{\mu}\right)$

$E(X) = \mu$, $\mathrm{Var}(X) = \mu^3/\lambda$

$F(x) = \Phi\left(\sqrt{\frac{\lambda}{x}}\frac{x-\mu}{\mu}\right) + \exp\left(\frac{2\lambda}{\mu}\right)\Phi\left(-\sqrt{\frac{\lambda}{x}}\frac{x+\mu}{\mu}\right)$

$M(t) = \exp\left[\frac{\lambda}{\mu}\left(1-\sqrt{1-2\mu^2 t/\lambda}\right)\right]$

$\hat\mu_{\mathrm{mle}} = \bar X$, $\hat\lambda_{\mathrm{mle}} = \left[\frac{1}{n}\sum\{X_i^{-1} - \bar X^{-1}\}\right]^{-1}$.

NOTE: $\lambda(X-\mu)^2/(\mu^2 X) \sim \chi^2(1)$
$X_i \sim \mathrm{IG}(\mu,\lambda) \Rightarrow kX_i \sim \mathrm{IG}(k\mu, k\lambda)$,
$\sum X_i \sim \mathrm{IG}(n\mu, n^2\lambda)$, $n\lambda/\hat\lambda_{\mathrm{mle}} \sim \chi^2(n-1)$

17 **Weibull** (α,θ) R:$x$weibull

$f(x) = \dfrac{\alpha}{\theta}\left(\dfrac{x}{\theta}\right)^{\alpha-1}\exp\left(-(x/\theta)^\alpha\right), \ (x \ge 0, \alpha>0, \theta>0)$

$F(x) = 1 - \exp\left(-(x/\theta)^\alpha\right)$

$E(X^k) = \theta^k\Gamma(1+\frac{k}{\alpha})$, $\quad \mathrm{median}(X) = \theta\,(\ln 2)^{1/\alpha}$

$\mathrm{Var}(X) = \theta^2\left[\Gamma(1+\frac{2}{\alpha}) - \Gamma^2(1+\frac{1}{\alpha})\right]$

NOTE: $\theta = 1 - e^{-1} \approx 63.2\%$ percentile
$Y = X^\alpha \sim \mathrm{Exponential}(\theta^\alpha)$
$\mathrm{Weibull}(1,\theta) = \mathrm{Exponential}(\theta)$
$\mathrm{Weibull}(2,\sqrt{2}\theta) = \mathrm{Rayleigh}(\theta)$
$\min_{1\le i\le n}(X_i) \sim \mathrm{Weibull}(\alpha, \theta/n^{1/\alpha})$: self-reproducing