QF 5212: Introduction to Quantitative Finance

Lecture 10

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Recap

In the last lecture:

- We have discussed optimal stopping approaches.
- We have discussed the difference between American Puts and American Calls.
- We have seen the free boundary problem and the linear Complementarity problem.

Table of Contents I

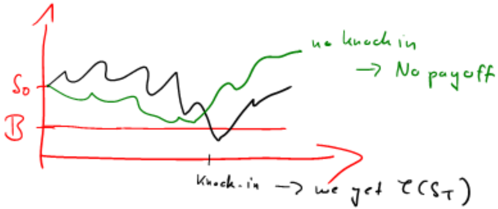
Barrier Option

A **barrier option** consists of

1) a payoff $\varphi(S_T)$, e.g. $\varphi(S_T) = (S_T - K)^+$
2) and a barrier B.

The payoff $\varphi(S_T)$ is paid at maturity T *conditional* on the stock price crossing (or not crossing) the barrier B.

Barrier Options



Two types of barrier condition:

- **Knock-in** barrier option:
  - The option is activated only if the price of the underlying asset hits the pre-specified barrier before maturity.
  - Once a barrier is knocked in, the option remains in existence until maturity.

- **Knock-out** barrier option:
  - The option stops to exist if the price of the underlying asset hits the pre-specified barrier before maturity.

Two types of barrier:

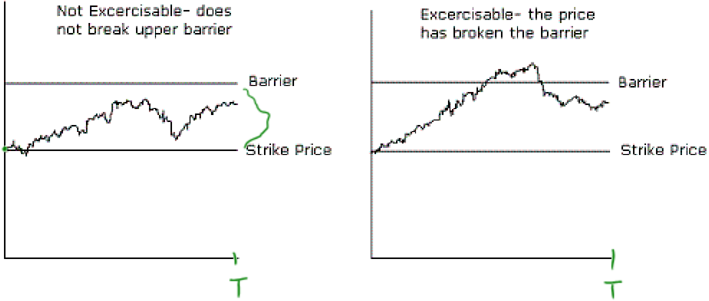
- Lower barrier: $B < S_0$.
- Upper barrier: $B > S_0$.

Barrier Options

There are four main types of barrier options:

- **Up-and-out**: asset price starts below the barrier level and has to move up for the option to be knocked out.
- **Down-and-out**: asset price starts above the barrier level and has to move down for the option be knocked out.
- **Up-and-in**: asset price starts below the barrier level and has to move up for the option to become activated.
- **Down-and-in**: asset price starts above the barrier level and has to move down for the option to become activated.

Up-and-in Call Option



Example of **up-and-in** call option.

- Today, I buy an at-the-money option with strike 100$, maturity 1 year, and a knock-in barrier at 120$.
- If the price never hits 120$, the option is worthless.
- If the price touches the barrier between today and maturity, the option behaves as a regular call.

Barrier Options

Why do investors buy barrier options?

- Because they are cheaper than vanilla options!
- They are customizable.
- But... they are less liquid.

Example:

- Stock is worth 100$ today. I believe that the price will go up, but not by too much.
- The call with strike 105$ and maturity 3 months has price 3$.
- The same call with a knock-out barrier at 110$ is sold at 2$.
- If I believe that it's unlikely that the stock will go up by 10% over the next 3 months, I might be willing to buy the barrier option, which is 33% cheaper.

Down-and-Out

Example

A **down-and-out call option** with strike K and lower barrier B has payoff at maturity T

$$\begin{cases} \underbrace{(S_T - K)^+}_{} \overset{\varphi(S_T)}{} & \text{if } S_t > B \text{ for all } t \leq T \Leftrightarrow \text{if } \min_{0 \leq t \leq T} S_t > B \\ 0 & \text{otherwise} \end{cases}$$

The barrier is hit if $\min_{0 \leq t \leq T} S_t \leq B$.
Then the option becomes void and worthless.

Down-and-Out

Down-and-Out

The payoff of a **down-and-out call option** can be rewritten as

$$(S_T - K)1_F, \quad = \begin{cases} S_T - K & \text{if } S_T > K \text{ and } \min\limits_{0\le t\le T} S_t > B \\ 0 & \text{otherwise} \end{cases}$$

where $F = \{\underbrace{S_T \geq K}_{\text{payoff of call option}}, \underbrace{\min_{0\leq t\leq T} S_t > B}_{\text{Barrier never gets hit}}\}$.

Denote the price of a down-and-out call option with maturity T at time 0 by

$$c_{\text{do}}(S, B, K, T),$$

where S is the price of the stock at time 0, B is the lower barrier, K is the strike.

Down-and-In

Down-and-In

The payoff of a **down-and-in call option** is

$$(S_T - K)1_F,$$

where $F = \{\underbrace{S_T \geq K}_{\text{payoff of call option}}, \underbrace{\min_{0\leq t\leq T} S_t \leq B}_{\text{hit the barrier before maturity}}\}$.

Denote the price of a down-and-in call option with maturity T at time 0 by

$$c_{\text{di}}(S, B, K, T),$$

where S is the price of the stock at time 0, B is the lower barrier, K is the strike.

In-out Parity

Consider a portfolio:

- long one down-and-out call;
- long one down-and-in call.

The payoff of this portfolio is:

	down-and-out	down-and-in
Barrier gets hit	0	$(S_T - K)^+$
Barrier never gets hit	$(S_T - K)^+$	0

In-out Parity

The payoff of the portfolio is

$$(S_T - K)^+.$$

Therefore

$$(S_T - K)\mathbb{1}_F + (S_T - K)\mathbb{1}_G = (S_T - K)^+$$

$$F = \{ S_T \geq K, \min_{0 \leq t \leq T} S_t > B \}, \quad G = \{ S_T \geq K, \min_{0 \leq t \leq T} S_t \leq B \}$$

down-and-out call + down-and-in call = standard call.

$$\Rightarrow \quad \mathbb{E}_Q[e^{-rT}(S_T - K)\mathbb{1}_F] + \mathbb{E}_Q[e^{-rT}(S_T - K)\mathbb{1}_G]$$

$$= \mathbb{E}_Q[e^{-rT}(S_T - K)^+]$$

In-out Parity

More in general,

knock-out option $+$ knock-in option $=$ standard option.

It is enough to study the price of the knock-in option.
The price of the knock-out option follows from in-out parity.

Pricing of Barrier Options

In the Black–Scholes model, find the price of a **down-and-in call option**:

$$c_{\mathrm{di}}(S, B, K, T),$$

where

- S is the current stock price (time 0),
- B is the lower barrier (so $B < S$),
- K is the strike price,
- T is the time to maturity.

Pricing of Barrier Options

By risk-neutral valuation,

$(S_T - K) \mathbb{1}_F = S_T \mathbb{1}_F - K \mathbb{1}_F$

$$c_{\text{di}}(S, B, K, T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T - K) \cdot 1_F],$$

where

$$F = \left\{ S_T \geq K, \underbrace{\min_{0 \leq t \leq T} S_t \leq B}_{\text{hitting the barrier before maturity.}} \right\}.$$

Equivalently,

$$c_{\text{di}}(S, B, K, T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T \cdot 1_F] - e^{-rT}\mathbb{E}^{\mathbb{Q}}[K \cdot 1_F].$$

Pricing of Barrier Options

Recall that $B_T = B_0 e^{rT}$. Compute separately,

$e^{-rT} = \frac{B_0}{B_T}$

- first term = $B_0 \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T \cdot 1_F}{B_T}\right]$,
- and second term = $B_0 \mathbb{E}^{\mathbb{Q}}\left[\frac{K \cdot 1_F}{B_T}\right]$.

Compute the first term using the change of numeraire technique!

Change of Numeraire Formula

Given a numeraire N_t, the value V_0 at time 0 of the payoff X paid at time T satisfies

$$\frac{V_0}{N_0} = \mathbb{E}^{\mathbb{Q}^N}\left[\frac{X}{N_T}\right].$$

$$V_0 = N_0 \cdot \mathbb{E}^{\mathbb{Q}^N}\left[\frac{X}{N_T}\right]$$

Pricing of Barrier Options

The value V_0 at time 0 of the payoff $S_T \cdot 1_F$ paid at time T is

$$\underline{V_0 = B_0 \mathbb{E}^{\mathbb{Q}} \left[\frac{S_T \cdot 1_F}{B_T} \right]} = S_0 \mathbb{E}^{\mathbb{Q}^S} \left[\frac{\cancel{S_T} \cdot 1_F}{\cancel{S_T}} \right],$$

Change of numeraire

where

- in the first equality we used B_t as numeraire,
- in the second equality we used S_t as numeraire.

Pricing of Barrier Options

Then,

$$B_0 \mathbb{E}^{\mathbb{Q}} \left[\frac{S_T \cdot 1_F}{B_T} \right] = S_0 \mathbb{E}^{\mathbb{Q}^s} \left[\frac{S_T \cdot 1_F}{S_T} \right] = S_0 \mathbb{E}^{\mathbb{Q}^s} [1_F] = S_0 \mathbb{Q}^S(F).$$

Also for the second term:

$$B_0 \mathbb{E}^{\mathbb{Q}} \left[\frac{K \cdot 1_F}{B_T} \right] = Ke^{-rT} \mathbb{Q}(F).$$

Summarizing,

$$c_{\text{di}}(S, B, K, T) = S_0 \mathbb{Q}^S(F) - Ke^{-rT} \mathbb{Q}(F).$$

Pricing of Barrier Options

Next, we need to compute $\mathbb{Q}^S(F)$ and $\mathbb{Q}(F)$.

Under $\mathbb{Q}^S$,

$$\frac{dS_t}{S_t} = (r + \sigma^2)dt + \sigma dW_t^S.$$

Under $\mathbb{Q}$,

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^B.$$

$S_t = S_0 e^{\eta t + \sigma W_t}$

Pricing of Barrier Options

$\log(S_t) = \log(S_0) + \eta t + \sigma W_t = \underline{\log(S_0)} + \int_0^t \eta \, ds + \int_0^t \sigma \, dW_s$

It's more convenient to consider **log-prices**:

Under $\underline{\mathbb{Q}}^S$,

$$d(\log S_t) = \overbrace{\left(r + \frac{\sigma^2}{2}\right)}^{\eta} dt + \sigma dW_t^S.$$

Under $\mathbb{Q}$,

$$d(\log S_t) = \overbrace{\left(r - \frac{\sigma^2}{2}\right)}^{\eta} dt + \sigma dW_t^B.$$

Itô's lemma: (under $\mathbb{Q}$)

$$d\log(S_t) = \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2$$

$$= \frac{1}{\cancel{S_t}}\cancel{S_t} r\,dt + \frac{1}{\cancel{S_t}}\cancel{S_t}\sigma\, dW_t^B - \frac{1}{2}\frac{\cancel{S_t^2}\cdot\sigma^2}{\cancel{S_t^2}}dt = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma\, dW_t^B$$

Pricing of Barrier Options

$\min_{0 \le t \le T} S_t \le B$

$\Leftrightarrow$ there ex $0 \le t \le T$ s.t. $S_t \le B$

$\Leftrightarrow$ there ex. $0 \le t \le T$ s.t. $\log(S_t) \le \log(B)$

$\Leftrightarrow$ $\min_{0 \le t \le T} \log(S_t) \le \log(B)$

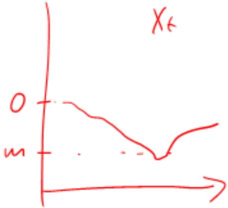
In terms of log-prices, F becomes:

$$
\begin{aligned}
F &= \left\{ S_T \geq K, \min_{0 \leq t \leq T} S_t \leq B \right\} \\
&= \left\{ \log(S_T) \geq \log(K), \min_{0 \leq t \leq T} \log(S_t) \leq \log(B) \right\} \\
&= \left\{ \underbrace{\log(S_T) - \log(S_0)}_{X_t} \geq \underbrace{\log(K) - \log(S_0)}_{x}, \right. \\
&\qquad \left. \underbrace{\min_{0 \leq t \leq T} (\underbrace{\log(S_t) - \log(S_0)}_{X_t})}_{m_T} \leq \underbrace{\log(B) - \log(S_0)}_{m} \right\}
\end{aligned}
$$

Pricing of Barrier Options

Define

$$
\begin{aligned}
X_t &:= \log(S_t) - \log(S_0), \\
m_T &:= \min_{0 \le t \le T} X_t = \min_{0 \le t \le T} (\log(S_t) - \log(S_0)), \\
x &:= \log(K) - \log(S_0), \\
m &:= \log(B) - \log(S_0). \quad < 0
\end{aligned}
$$



Then

$$F = \{X_T \ge x, m_T \le m\}$$

and

$$
X_t = \begin{cases} \overbrace{\left(r + \frac{\sigma^2}{2}\right)}^{u} t + \sigma W_t^S & \text{under } \mathbb{Q}^S \\ \underbrace{\left(r - \frac{\sigma^2}{2}\right)}_{\eta} t + \sigma W_t^B & \text{under } \mathbb{Q}. \end{cases}
$$

We're considering a lower barrier: $B < S_0$, so $m < 0$!

Pricing of Barrier Options

Goal: Compute $\mathbb{Q}^*(\overbrace{X_T \geq x, m_T \leq m}^{F})$, where $X_t = \eta t + \sigma W_t^*$ and W_t^* is a Wiener process under $\mathbb{Q}^*$.

Then,

- for $\eta = r + \frac{\sigma^2}{2}$, we get $\mathbb{Q}^S(F)$,
- for $\eta = r - \frac{\sigma^2}{2}$, we get $\mathbb{Q}(F)$.

Then, we get the price of the down-and-in call

$$c_{\mathrm{di}}(S, B, K, T) = S_0 \mathbb{Q}^S(F) - Ke^{-rT} \mathbb{Q}(F).$$

Pricing of Barrier Options

Compute $\mathbb{P}(X_T \geq x, m_T \leq m)$, where $X_t = \eta t + \sigma W_t$ and W_t is a Wiener process under $\mathbb{P}$.

- Step 1: we study the case $\eta = 0$, $\sigma = 1$. That means $X_t = W_t$.
- Step 2: we extend it to general η and σ using Girsanov Theorem.

Reflection Principle

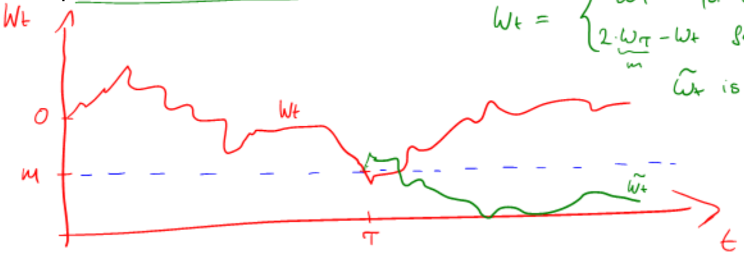
Let W_t be a Wiener process under $\mathbb{P}$, and $m_T = \min_{0 \le t \le T} W_t$.

Assume $m \le x$ and let $\tau = \inf\{t \ge 0 : W_t \le m\}$.

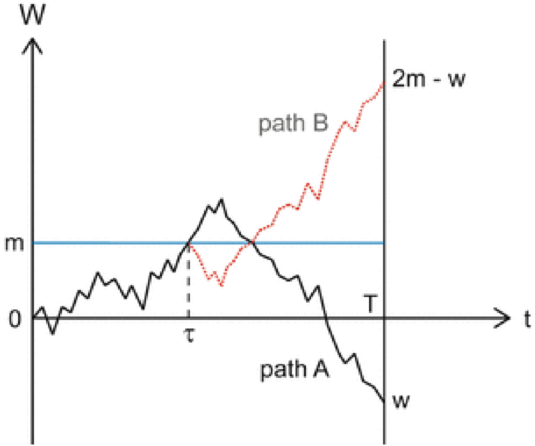
- τ is the first time the Wiener process W_t hits m.
- The path after time τ has the same distribution as the reflection of the path about the value m.

$$\tilde{W}_t = \begin{cases} W_t & \text{for } 0 \le t \le \tau \\ 2 \cdot \underbrace{W_\tau}_{m} - W_t & \text{for } t \ge \tau \end{cases}$$

$\tilde{W}_t$ is a Brownian motion.



Reflection Principle

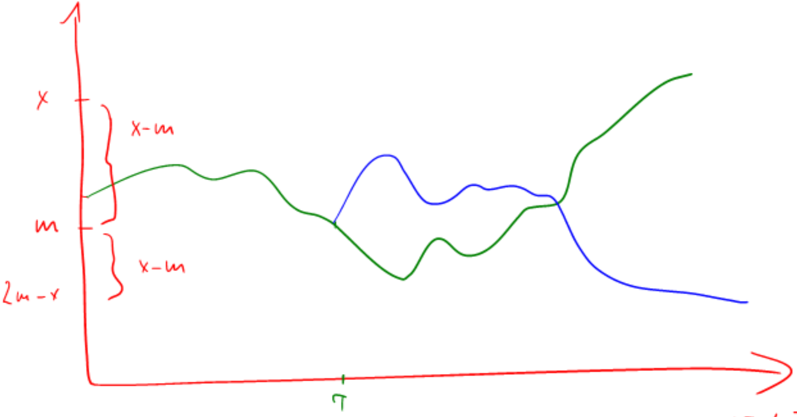


The path after time τ has the same distribution as the reflection of the path about the value *m*.

Reflection Principle

$\tau = \inf\{t \geq 0 \mid W_t \leq m\}$

- Assume $\tau \leq T$.
  Every path of the Wiener process starting at time τ that ends at time T above x has a “reflected” path that ends at time T below $2m - x$.



$\mathbb{P}(W_T \geq x \mid \tau \leq T) = \mathbb{P}(\tilde{W}_T \geq x \mid \tau \leq T) = \mathbb{P}(2m - W_t \geq x \mid \tau \leq T)$
$= \mathbb{P}(W_t \leq 2m - x \mid \tau \leq T)$

Reflection Principle

Exercise

Prove that, for every $a \geq 0$, $\mathbb{P}(\sup_{0 \leq u \leq t} W_u \geq a) = 2\mathbb{P}(W_t \geq a)$.

$$\mathbb{P}\Big(\sup_{0\le u\le t} W_u \ge a\Big) = \mathbb{P}\Big(\sup_{0\le u\le t} W_u \ge a,\ W_t \ge a\Big) + \mathbb{P}\Big(\sup_{0\le u\le t} W_u \ge a,\ W_t < a\Big)$$

$\tau := \inf\{t \ge 0 \mid W_t \ge a\}$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A, B)}{\mathbb{P}(B)}$$

$$\cdot\ \mathbb{P}\Big(\sup_{0\le u\le t} W_u \ge a,\ W_t < a\Big) = \mathbb{P}(\tau \le t,\ W_t < a)$$

$$= \mathbb{P}(W_t < a \mid \tau \le t)\cdot\mathbb{P}(\tau \le t)$$

Reflection principle

$$\rightarrow = \mathbb{P}(\underbrace{2a - W_t}_{\tilde{W}_t} < a \mid \tau \le t)\cdot\mathbb{P}(\tau \le t)$$

$$= \mathbb{P}(W_t \geqslant a \mid \tau \le t)\ \mathbb{P}(\tau \le t)$$

$$= \mathbb{P}(W_t \ge a,\ \tau \le t) = \mathbb{P}\Big(W_t \ge a,\ \sup_{0\le u\le t} W_t \ge a\Big)$$

Reflection Principle

$A \cap B = B$

$$\Rightarrow \mathbb{P}\left(\sup_{0 \le u \le t} W_u \ge a\right) = 2 \cdot \mathbb{P}\Big(\overbrace{\sup_{0 \le u \le t} W_u \ge a}^{A},\ \overbrace{W_t \ge a}^{B}\Big)$$

$$= 2 \cdot \mathbb{P}(W_t \ge a)$$

Reflection Principle

Define

$$A := \{W_T \geq x, m_T \leq m\}, \qquad B := \{W_T \leq 2m - x\}.$$

We assume $m \leq x$. And we let $\tau = \inf\{t \geq 0 : W_t \leq m\}$. Then,

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(W_T \geq x, \underbrace{m_T \leq m}_{\text{hitting the barrier before } T}) \\
&= \mathbb{P}(W_T \geq x, \underbrace{\tau \leq T}_{\text{hitting the barrier before } T}) \\
\text{Reflection principle} \rightarrow &= \mathbb{P}(W_T \leq 2m - x, \tau \leq T) \\
&= \mathbb{P}(W_T \leq 2m - x) = \mathbb{P}(B) = N\left(\frac{2m - x}{\sqrt{T}}\right).
\end{aligned}$$

if $W_T \leq 2m - x$

$\Rightarrow \tau \leq T$

$\tau \leq T$

More in general, if W_t hits the value m before time T, then W_T and $2m - W_T$ have the same law!

$= \underbrace{2 \cdot W_\tau - W_T}_{\tilde{W}_T}$

for $\tau = \inf\{t \mid W_t \leq m\}$

So,

$$\mathbb{E}\left[1_A g(W_T)\right] = \mathbb{E}\left[1_B g(2m - W_T)\right].$$

- We solved the problem for $\eta = 0$ and $\sigma = 1$.
- Consider the general case

$$dX_t = \eta dt + \sigma dW_t, \qquad m_t = \min_{0 \leq s \leq t} X_s.$$

Pricing of Barrier Options

- Write
$$X_T = \eta T + \sigma W_T = \sigma \underbrace{\left(\frac{\eta}{\sigma} T + W_T\right)}_{\tilde{W}_T} = \sigma \tilde{W}_T,$$
where $\tilde{W}_t = W_t + \frac{\eta}{\sigma} t.$ $= W_t + \int_0^t \frac{\eta}{\sigma} \, ds$
- We need to find $\mathbb{P}(X_T \geq x, m_T \leq m)$!
- Strategy:
  1) Find a measure $\tilde{\mathbb{P}}$ such that $(\tilde{W}_t)_{t \geq 0}$ is a Wiener process under $\tilde{\mathbb{P}}$.
  2) Compute $\tilde{\mathbb{P}}(X_T \geq x, m_T \leq m)$.
  3) Change back to measure $\mathbb{P}$ and find $\mathbb{P}(X_T \geq x, m_T \leq m)$.

Pricing of Barrier Options

- First step:
  Find a measure $\tilde{\mathbb{P}}$ such that $(\tilde{W}_t)_{t\geq 0}$ is a Wiener process under $\tilde{\mathbb{P}}$.

$L_T = \exp\left\{ -\int_0^T \frac{\eta}{\sigma} ds - \frac{1}{2}\int_0^T \frac{\eta^2}{\sigma^2} dW_s \right\}$

- Use **Girsanov Theorem**!
  The change of measure $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = L_T$, where $dL_t = -\frac{\eta}{\sigma} L_t dW_t$, works.

- The change of measure is

$$L_T = \exp\left\{ -\frac{\eta^2}{2\sigma^2} T - \frac{\eta}{\sigma} W_T \right\}.$$

Pricing of Barrier Options

- Second step:
  Compute $\tilde{\mathbb{P}}(X_T \geq x, m_T \leq m)$.

- Define $\tilde{x} = \frac{x}{\sigma}$ and $\tilde{m} = \frac{m}{\sigma}$.

- We have already found that

$$\tilde{\mathbb{P}}(X_T \geq x, m_T \leq m) = \overbrace{\tilde{\mathbb{P}}(\underbrace{\tilde{W}_T \geq \tilde{x}, \tilde{m}_T \leq \tilde{m}}_{\hat{A}})}^{\text{as in step 1.}}$$

$$\tilde{\mathbb{P}}\left(\tilde{W}_T \sigma \geq x, \min_{0 \le t \le T}\{\sigma \hat{W}_t\} \le m\right)$$

$$= \tilde{\mathbb{P}}(\underbrace{\tilde{W}_T \leq 2\tilde{m} - \tilde{x}}_{\hat{B}}) = N\left(\frac{2m - x}{\sigma\sqrt{T}}\right).$$

- Let $\tilde{A} = \{\tilde{W}_T \geq \tilde{x}, \tilde{m}_T \leq \tilde{m}\}$ and $\tilde{B} = \{\tilde{W}_T \leq 2\tilde{m} - \tilde{x}\}$.
  Given a function g, we also know that $\mathbb{E}^{\tilde{\mathbb{P}}}[1_{\tilde{A}} g(\tilde{W}_T)] = \mathbb{E}^{\tilde{\mathbb{P}}}[1_{\tilde{B}} g(2\tilde{m} - \tilde{W}_T)]$.

Pricing of Barrier Options

- Third step:
  Change back to measure $\mathbb{P}$ and find $\mathbb{P}(X_T \geq x, m_T \leq m)$.

- The change of measure is

$\tilde{W}_T = W_T + \frac{\eta}{\sigma} \cdot T$

$W_T = \tilde{W}_T - \frac{\eta}{\sigma} \cdot T$

$$
\begin{aligned}
L_T^{-1} = \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} &= \left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right)^{-1} \\
&= \exp\left\{\frac{\eta^2}{2\sigma^2} T + \frac{\eta}{\sigma} W_T\right\} \\
&= \exp\left\{-\frac{\eta^2}{2\sigma^2} T + \frac{\eta}{\sigma} \tilde{W}_T\right\} \\
&=: g(\tilde{W}_T).
\end{aligned}
$$

Pricing of Barrier Options

Therefore,

$$
\begin{aligned}
\mathbb{P}(X_T \geq x, m_T \leq m) &= \mathbb{E}^{\mathbb{P}}[1_{\{X_T \geq x, m_T \leq m\}}] \\
&= \mathbb{E}^{\mathbb{P}}\left[1_{\tilde{A}}\right] = \mathbb{E}^{\tilde{\mathbb{P}}}\left[1_{\tilde{A}} \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right] \\
&= \mathbb{E}^{\tilde{\mathbb{P}}}[1_{\tilde{A}} g(\tilde{W}_T)] \\
&= \mathbb{E}^{\tilde{\mathbb{P}}}[1_{\tilde{B}} g(2\tilde{m} - \tilde{W}_T)] \\
&= \mathbb{E}^{\tilde{\mathbb{P}}}\left[1_{\{\underbrace{\tilde{W}_T \leq 2\tilde{m} - \tilde{x}}_{\tilde{B}}\}} \exp\left\{-\frac{\eta^2}{2\sigma^2} T + \frac{\eta}{\sigma}\left(2\tilde{m} - \tilde{W}_T\right)\right\}\right]
\end{aligned}
$$

Pricing of Barrier Options

pdf: $f(x) = \frac{1}{\sqrt{2\pi \cdot T}} \cdot e^{-\frac{x^2}{2T}}$

$$= \mathbb{E}^{\tilde{\mathbb{P}}}\left[1_{\{\tilde{W}_T \le 2\tilde{m} - \tilde{x}\}} \exp\left\{-\frac{\eta^2}{2\sigma^2}T + \frac{\eta}{\sigma}\left(2\tilde{m} - \tilde{W}_T\right)\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{2\tilde{m}-\tilde{x}} \exp\left\{-\frac{\eta^2}{2\sigma^2}T + \frac{\eta}{\sigma}(2\tilde{m} - w) - \frac{w^2}{2T}\right\} dw$$

$$= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\frac{2m-x}{\sigma}} \exp\left\{2\frac{\eta m}{\sigma^2} - \frac{1}{2T}\left(\frac{\eta}{\sigma}T + w\right)^2\right\} dw$$

binomial formula.

$$= e^{\frac{2\eta m}{\sigma^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{2m-x+\eta T}{\sigma\sqrt{T}}} \exp\left\{-\frac{1}{2}z^2\right\} dz$$

$$z = \frac{\frac{\eta}{\sigma}T + w}{\sqrt{T}}$$

$$= e^{\frac{2\eta m}{\sigma^2}} N\left(\frac{2m - x + \eta T}{\sigma\sqrt{T}}\right)$$

$dz = \frac{1}{\sqrt{T}} dw$

$dw = \sqrt{T}\, dz$

$w \le \frac{2m-x}{\sigma}$

$\Leftrightarrow z = \frac{\frac{\eta}{\sigma}T + w}{\sqrt{T}} \le \frac{\frac{\eta}{\sigma}T + \frac{2m-x}{\sigma}}{\sqrt{T}}$

Pricing of Barrier Options

If $m \leq x$ and $m < 0$, we have shown that

$$\mathbb{P}(X_T \geq x, m_T \leq m) = e^{\frac{2\eta m}{\sigma^2}} N\left(\frac{2m - x + \eta T}{\sigma\sqrt{T}}\right)$$

Let's put all the pieces together!

Recall that

$$
\begin{aligned}
c_{\text{di}}(S,B,K,T) &= S_0\mathbb{Q}^S(F) - Ke^{-rT}\mathbb{Q}(F),\\
F &= \{X_T \geq x, m_T \leq m\},\\
x &= \log(K) - \log(S_0),\\
m &= \log(B) - \log(S_0),\\
\eta &= \begin{cases} r + \frac{\sigma^2}{2} & \text{for } \mathbb{Q}^S(F) \\ r - \frac{\sigma^2}{2} & \text{for } \mathbb{Q}(F). \end{cases}
\end{aligned}
$$

$2m - x$

$= 2\log(B) - 2\log(S_0) - \log(K) + \log(S_0)$

$= 2\log(B) - (\log(S_0) + \log(K))$

$= \log(B^2) - \log(S_0 \cdot K)$

$= \log \frac{B^2}{S_0 K}$

Therefore,

$$
\mathbb{Q}^S(F) = \overbrace{e^{\frac{2\overbrace{(r+\frac{\sigma^2}{2})}^{\eta}\overbrace{\log\frac{B}{S_0}}^{m}}{\sigma^2}}}^{\left(\frac{B}{S_0}\right)^{2\frac{r}{\sigma^2}+1}} N\left(\frac{\overbrace{\log\frac{B^2}{S_0K}}^{2m-x} + \overbrace{(r+\frac{\sigma^2}{2})}^{\eta}T}{\sigma\sqrt{T}}\right),
$$

$$
\mathbb{Q}(F) = \underbrace{e^{\frac{2\overbrace{(r-\frac{\sigma^2}{2})}^{\eta}\overbrace{\log\frac{B}{S_0}}^{m}}{\sigma^2}}}_{\left(\frac{B}{S_0}\right)^{2\frac{r}{\sigma^2}-1}} N\left(\frac{\overbrace{\log\frac{B^2}{S_0K}}^{2m-x} + \overbrace{(r-\frac{\sigma^2}{2})}^{\eta}T}{\sigma\sqrt{T}}\right),
$$

Pricing of Barrier Options

Define

$$d_{\mathrm{b}} = \frac{\log \frac{B^2}{S_0 K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$
$$\alpha = \frac{2r}{\sigma^2} + 1.$$

Then

Price of a Down-and-in Barrier Option

$$c_{\mathrm{di}}(S, B, K, T) = S_0 \left(\frac{B}{S_0}\right)^{\alpha} N(d_b) - Ke^{-rT}\left(\frac{B}{S_0}\right)^{\alpha-2} N\left(d_b - \sigma\sqrt{T}\right).$$

PDE Approach

Consider a knock-out option.

Prior to knock-out, the option is alive and its price $V(t, S_t)$ at time t must satisfy the Black–Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

The barrier condition enters through

- the *boundary conditions*
- and the *solution domain*.

PDE Pricing

- When the barrier is hit, the option becomes worthless:

$$V(t, B) = 0, \qquad \text{for all } t \in [0, T].$$

- The solution domain (for a lower barrier) is

$$[0, T] \times (B, +\infty).$$

- The final condition is the payoff of the corresponding vanilla option. If the payoff at T is $\varphi(S_T)$, then

$$V(T, S) = \varphi(S).$$

Corollary

Let $V_{do}(t, S_t; B, \varphi)$ be the price at time t of a down-and-out option with final payoff $\varphi(S_T)$ and lower knock-out barrier B. Then

$$V_{do}(t, S_t; B, \alpha\varphi + \beta\psi) = \alpha V_{do}(t, S_t; B, \varphi) + \beta V_{do}(t, S_t; B, \psi).$$

$$\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(\alpha\varphi + \beta\psi)\mathbb{1}_{\mathcal{I}} \mid \mathcal{F}_t]$$

$$= \alpha\, \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}\varphi \mathbb{1}_{\mathcal{I}} \mid \mathcal{F}_t] + \beta\, \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}\psi \mathbb{1}_{\mathcal{I}} \mid \mathcal{F}_t]$$

Put-call Parity

$(K - S_T)^+ = \begin{cases} K - S_T & \text{if } K \geq S_T \\ 0 & \text{otherwise} \end{cases} \overset{\circledast}{=} K - S_T + (S_T - K)^+$

From the Corollary, we get the put-call parity for barrier options.

Put-call Parity

$$p_{\mathrm{do}}(S, B, K, T) = K \cdot b_{\mathrm{do}}(S, B, K, T) - s_{\mathrm{do}}(S, B, K, T) + c_{\mathrm{do}}(S, B, K, T),$$

where p_{do} is the price of the down-and-out put, b_{do} is the price of the down-and-out contract with payoff 1 and s_{do} is the price of the down-and-out contract with payoff S_T.

$F = \{ \min_{0 \le t \le T} S_t > B \}$

$\mathbb{E}_Q [e^{-rT} (K - S_T)^+ \mathbb{1}_F]$

$\overset{\circledast}{=} \mathbb{E}_Q [e^{-rT} K \mathbb{1}_F] - \mathbb{E}_Q [e^{-rT} S_T \mathbb{1}_F] + \mathbb{E}_Q [e^{-rT} (S_T - K)^+ \mathbb{1}_F]$

Summary of Lecture 10

- We have seen different types of Barrier options.
- We have presented an approach to the valuation of Barrier options.
- We have seen the reflection principle.
- We have derived a put-call parity for Barrier options.