Pricing of out-of-the-money Barrier Options Using Importance Sampling

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Abstract

Barrier options are path-dependent exotic derivatives similar to vanilla options but are only activated or deactivated once the price of the underlying asset reaches a certain level, which is called the barrier. For this thesis, we first derive the formula for the theoretical price of a barrier option exercised in the European style. Next, we shall try to price the barrier option using Monte Carlo simulations and compare the simulated price to the theoretical price. However, when it comes to out-of-the-money barrier options, the likelihood of hitting the barrier is low, and thus most simulations do not contribute meaningful information. This is computationally inefficient. Therefore, importance sampling is applied to alter the probability distribution of the underlying asset's price path to increase the probability of the option's barrier being hit, which in turn reduces the variance and improves the accuracy of the simulation.

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Chapter 1

Introduction

Options have been used for risk management, speculation and hedging strategies. Among the various types of options available, there is a category of exotic options which are path-dependent, called barrier options. Instead of solely depending on the underlying asset's price at expiration, barrier options could be activated or deactivated when the price of the underlying asset hits the barrier level during the life of the option.

In this thesis, we begin with the Black-Scholes model and derive the theoretical price of a European-style barrier option, which means the option can only be exercised at expiration. We shall use [Ses24] as a reference. Knowledge from stochastic calculus and probability theory will be applied to aid us. The theoretical price will serve as a benchmark for comparing simulated pricing methods.

Next, we shall also attempt to price barrier options using Monte Carlo simulation. For this method, we first generate a large number of random price paths for the underlying asset and then take the average of resulting payoffs to approximate the theoretical payoff. However, when applied to out-of-the-money barrier options, since hitting the barrier is a low-probability event, most generated price paths do not reach the barrier and the simulated paths do not provide useful information.

To address this, the importance sampling technique is applied, which first appeared in [BBG97]. It is a variance reduction method that changes the probability distribution of the underlying asset's price paths and increases the chances of the price reaching the barrier.

Lastly, we also conduct a series of numerical experiments to show the challenges of barrier option pricing and provide evidence for the effectiveness of importance sampling. Ultimately, this work aims to deepen the understanding of pricing complex derivatives and provide pragmatic solutions for improving the computational efficiency of financial models.

Chapter 2

Barrier Option Pricing in the Black-Scholes Model

2.1 The Black-Scholes Model

We first introduce the idea of a Brownian motion.

Definition 2.1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. Then $(W(t)_{t\geq 0})$ is a* ***Brownian motion*** *if for all $0 = t_0 < t_1 < \cdots < t_m$ the increments*

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \cdots, W(t_m) - W(t_{m-1})$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}[W(t_{i+1} - W(t_i)] = 0,$$

$$\mathrm{Var}[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i.$$

Now, we shall discuss the Black-Scholes model. Let the risk-free rate be r, the rate of return of the stock be μ, the volatility of the stock be σ, the stock price at time t be $S(t)$, and $(W(t)_{t\geq 0})$ be a Brownian motion. Then under the Black-Scholes model, the stock price follows a geometric Brownian motion, i.e.

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

However, when it comes to pricing derivatives, we shall take the expectation of the payoff under the risk-neutral measure $\mathbb{Q}$, in which the stock price is governed by the following equation

$$dS(t) = rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t),$$

where $(W^{\mathbb{Q}}(t)_{t\geq 0})$ is a Brownian motion under the measure $\mathbb{Q}$.

Now, we are interested in deriving a formula for the theoretical price of a down-and-in call barrier option on this stock, with barrier level B, and strike price K, which matures at time T. Initially, the stock price is above the barrier B and below the strike price K. At time T, the stock price is $S(T)$ and the payoff $(S(T)-K)^+ := \max\{S(T)-K, 0\}$ is paid only if the stock price goes below the barrier B before maturity.

2.2 Formula for Theoretical Price

From here on, the term barrier option shall refer to down-and-in barrier options. The following derivation is from [Ses24].

The payoff of the barrier option is non-zero when the final stock price is greater than the strike price and the barrier is hit before maturity, i.e.

$$S(T) \geq K, \min_{0\leq t\leq T} S(T) \leq B.$$

Now let v_0 be the price of the barrier option, by risk-neutral valuation,

$$v_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S(T)-K)\cdot \mathbb{1}_F],$$

where $\mathbb{Q}$ is the risk-neutral measure and $F = \{S(T) \geq K, \min_{0\leq t\leq T} S(t) \leq B\}$.

We see that

$$v_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[S(T)\cdot \mathbb{1}_F] - Ke^{-rT}\mathbb{Q}(F),$$

where $\mathbb{Q}(F)$ is the probability of F under the measure $\mathbb{Q}$. To evaluate the first term, we use the stock as a numeraire, with $\mathbb{Q}^S$ being the risk-neutral measure under the stock being a numeraire, and we get

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)\cdot \mathbb{1}_F}{1}\right] = S(0)\mathbb{E}^{\mathbb{Q}^S}\left[\frac{S(T)\cdot \mathbb{1}_F}{S(T)}\right].$$

Thus

$$v_0 = S(0)\mathbb{Q}^S(F) - Ke^{-rT}\mathbb{Q}(F).$$

Now we shall focus on computing $\mathbb{Q}^S(F), \mathbb{Q}(F)$ and it turns out that it would be more convenient to consider *log-prices*. In this case, we have

$$F = \{X(T) \geq x, m(T) \leq m\},$$

where $X(t) := \log(S(t)/S(0)), m(T) := \min_{0\leq t\leq T} X(t), x := \log(K/S(0)), m := \log(B/S(0))$ and under $\mathbb{Q}^S$,

$$d(\log S(t)) = (r + \sigma^2/2)dt + \sigma dW^S(t),$$

and under $\mathbb{Q}$

$$d(\log S(t)) = (r - \sigma^2/2)dt + \sigma dW^B(t).$$

Thus,

$$X(t) = \begin{cases} (r + \sigma^2/2)t + \sigma W^S(t) & \text{under } \mathbb{Q}^S \\ (r - \sigma^2/2)t + \sigma W^B(t) & \text{under } \mathbb{Q}. \end{cases}$$

To proceed, we start with the following theorem, which is often called the reflection principle and appears as Theorem 2.19 of [MP10].

Theorem 2.2.1. *If τ is a stopping time and $(W(t))_{0\le t\le T}$ is a standard Brownian motion, then the process $(W^*(t))_{0\le t\le T}$ called **Brownian motion reflected at** τ and defined by*

$$W^*(t) = W(t)\mathbb{1}_{\{t\le\tau\}} + (2W(\tau) - W(t))\mathbb{1}_{\{t>\tau\}}$$

is also a standard Brownian motion.

This gives us the following lemma

Lemma 2.2.1.1. *Let $W(T)$ be a Brownian motion, $m(T) = \min_{0\le t\le T} W(T)$ then*

$$\mathbb{P}(W(T) \ge x, m(T) \le m) = \mathbb{P}(W(T) \le 2m - x) \text{ for } m < 0, m \le x.$$

Proof. Let $\tau = \inf\{t \ge 0 : W(t) = m\}$ and let $\{W^*(t) : t \ge 0\}$ be Brownian motion reflected at the stopping time τ. Then

$$\begin{aligned}
&\mathbb{P}(W(T) \ge x, m(T) \le m) \\
&= \mathbb{P}(W(T) \ge x, \tau \le T) \\
&= \mathbb{P}(W(T) \ge x \mid \tau \le T)\mathbb{P}(\tau \le T) \\
&= \mathbb{P}(W^*(T) \ge x \mid \tau \le T)\mathbb{P}(\tau \le T) && \text{(Theorem 2.2.1)} \\
&= \mathbb{P}(2m - W(T) \ge x \mid \tau \le T)\mathbb{P}(\tau \le T) \\
&= \mathbb{P}(W(T) \le 2m - x \mid \tau \le T)\mathbb{P}(\tau \le T) \\
&= \mathbb{P}(W(T) \le 2m - x, \tau \le T) \\
&= \mathbb{P}(W(T) \le 2m - x) && (2m - x = m + (m - x) \le m)
\end{aligned}$$

□

Now consider the following stochastic process $(X(t))_{t\ge0}$ where

$$dX(t) = \eta dt + \sigma dW(t).$$

Then

$$X(t) = \eta t + \sigma W(t) = \sigma \left(\frac{\eta}{\sigma} t + W(t)\right) := \sigma\tilde{W}(t).$$

Our plan is as follows:

Step 1: Find a measure $\tilde{\mathbb{P}}$ such that $(\tilde{W}(t))_{t\geq 0}$ is a Brownian motion under $\tilde{\mathbb{P}}$.

Step 2: Compute $\tilde{\mathbb{P}}(X(T) \geq x, m(T) \leq m)$.

Step 3: Change back to measure $\mathbb{P}$ and find $\mathbb{P}(X(T) \geq x, m(T) \leq m)$.

For step 1, we shall use Girsanov theorem, which is Theorem 5.2.3 in [Shr04].

Theorem 2.2.2. *Let $(W(t))_{0\leq t\leq T}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $(\mathcal{F}(t))_{0\leq t\leq T}$ be a filtration for this Brownian motion. Let $(\theta(t))_{0\leq t\leq T}$ be an adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du\right\},$$

$$\tilde{W}(t) = W(t) + \int_0^t \theta(u)du,$$

and assume that

$$\mathbb{E}\left[\int_0^T \theta^2(u)Z^2(u)du\right] < +\infty.$$

Set $Z = Z(T)$. Then $\mathbb{E}[Z] = 1$ and under the probability measure $\tilde{\mathbb{P}}$ given by

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)dP(\omega) \text{ for all } A \in \mathcal{F},$$

the process $(\tilde{W}(t))_{0\leq t\leq T}$, is a Brownian motion.

We see that to let $(\tilde{W}(t))_{0\leq t\leq T}$ be a Brownian motion, the measure $\tilde{\mathbb{P}}$ is given by the adapted process $(\theta(t))_{0\leq t\leq T}$ where $\theta(u) = \eta/\sigma$. Thus we have that the change of measure is

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = Z = \exp\left\{-\frac{\eta}{\sigma}W(T) - \frac{\eta^2}{2\sigma^2}T\right\}.$$

Next, for step 2, let $\tilde{m}(T) = \min_{0\leq t\leq T} \tilde{W}(t), \tilde{x} = x/\sigma, \tilde{m} = m/\sigma$.

$$\begin{aligned}
&\tilde{\mathbb{P}}(X(T) \geq x, m(T) \leq m) \\
&= \tilde{\mathbb{P}}(\tilde{W}(T) \geq \tilde{x}, \tilde{m}(T) \leq \tilde{m}) \\
&= \mathbb{P}(\tilde{W}(T) \leq 2\tilde{m} - \tilde{x}) && \text{(Lemma 2.2.1.1)} \\
&= N\left(\frac{2\tilde{m} - \tilde{x}}{\sqrt{T}}\right) \\
&= N\left(\frac{2m - x}{\sigma\sqrt{T}}\right).
\end{aligned}$$

where $N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$ since by Definition 2.1.1, $\tilde{W}(T)$ follows a normal distribution with mean 0 and variance T, denoted by $\tilde{W}(T) \sim \mathcal{N}(0, T)$.

Lastly, for step 3, first note that given a function g, again by the reflection principle (Theorem 2.2.1), we have that

$$\mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\tilde{A}} g(\tilde{W}(T))] = \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\tilde{B}} g(2\tilde{m} - \tilde{W}(T))], \tag{2.1}$$

where $\tilde{A} = \{\tilde{W}(T) \geq \tilde{x}, \tilde{m}(T) \leq \tilde{m}\}$ and $\tilde{B} = \{\tilde{W}(T) \leq 2\tilde{m} - \tilde{x}\}$. Now, let g be the change of measure from $\tilde{\mathbb{P}}$ back to $\mathbb{P}$, i.e.

$$\begin{aligned}
&g(\tilde{W}(T)) \\
&= \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} \\
&= \left(\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right)^{-1} \\
&= \exp\left\{\frac{\eta}{\sigma} W(T) + \frac{\eta^2}{2\sigma^2} T\right\} \\
&= \exp\left\{\frac{\eta}{\sigma} \tilde{W}(T) - \frac{\eta^2}{2\sigma^2} T\right\}. && (W(T) = \tilde{W}(T) - \eta/\sigma T)
\end{aligned}$$

And finally, we have

$$\begin{aligned}
&\mathbb{P}(X(T) \geq x, m(T) \leq m) \\
&= \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\tilde{A}}] \\
&= \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\tilde{A}} g(\tilde{W}(T))] \\
&= \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\tilde{B}} g(2\tilde{m} - \tilde{W}(T))] && \text{(Equation 2.1)} \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{2\tilde{m}-\tilde{x}} g(2\tilde{m} - w) e^{-\frac{w^2}{2T}} \, dw && (\tilde{W}(T) \sim \mathcal{N}(0, T).) \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{2\tilde{m}-\tilde{x}} \exp\left\{-\frac{w^2}{2T} + \frac{\eta}{\sigma}(2\tilde{m} - w) - \frac{\eta^2}{2\sigma^2} T\right\} dw \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\frac{2m-x}{\sigma}} \exp\left\{\frac{2\eta m}{\sigma^2} - \frac{1}{2T}\left(\frac{\eta T}{\sigma} + w\right)^2\right\} dw \\
&= e^{\frac{2\eta m}{\sigma^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{2m-x+\eta T}{\sigma\sqrt{T}}} \exp\left\{-\frac{1}{2} z^2\right\} dz && (z = (\eta T/\sigma + w)/\sqrt{T}) \\
&= e^{\frac{2\eta m}{\sigma^2}} N\left(\frac{2m - x + \eta T}{\sigma\sqrt{T}}\right).
\end{aligned}$$

Recall that

$$\begin{aligned} v_0 &= S(0)\mathbb{Q}^S(F) - Ke^{-rT}\mathbb{Q}(F), \\ F &= \{X(T) \geq x, m(T) \leq m\}, \\ x &= \log(K/S(0)), \\ m &= \log(B/S(0)), \\ \eta &= \begin{cases} r + \sigma^2/2 & \text{for } \mathbb{Q}^S(F) \\ r - \sigma^2/2 & \text{for } \mathbb{Q}(F) \end{cases} \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{Q}^S(F) &= e^{\frac{2(r+\sigma^2/2)\log(B/S(0))}{\sigma^2}} N\left(\frac{\log \frac{B^2}{S(0)K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right), \\ \mathbb{Q}(F) &= e^{\frac{2(r-\sigma^2/2)\log(B/S(0))}{\sigma^2}} N\left(\frac{\log \frac{B^2}{S(0)K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Thus, we have obtained the formula for the theoretical price, which is stated as the following theorem

Theorem 2.2.3. *Suppose the risk-free rate is r and the stock follows a geometric Brownian motion, with volatility σ. Let v_0 be the price of a down-and-in barrier call option on the stock with barrier B, strike price K, and matures at time T. We define*

$$d_b = \frac{\log \frac{B^2}{S(0)K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \qquad \alpha = \frac{2r}{\sigma^2} + 1.$$

Then

$$v_0 = S(0)\left(\frac{B}{S(0)}\right)^{\alpha} N(d_b) - Ke^{-rT}\left(\frac{B}{S(0)}\right)^{\alpha-2} N(d_b - \sigma\sqrt{T}).$$

Chapter 3

Monte Carlo Simulations

3.1 Direct Method

Now, instead of using Theorem 2.2.3, we shall attempt to calculate the price by using Monte Carlo simulations. Now, we start by simulating the final stock price. Note that for a stock that follows the geometric Brownian motion, with its volatility being σ and the risk-free rate being r, we have the following analytical solution for the stock price at time t, i.e.

Proposition 3.1.1. *Let $(W(t)_{t\geq 0})$ be a Brownian motion. Suppose the stochastic process $(S(t)_{t\geq 0})$ satisfies*

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \tag{3.1}$$

then we have

$$S(t) = S(0)e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W(t)}. \tag{3.2}$$

Proof. We shall verify that Equation 3.2 satisfies Equation 3.1. Let

$$f(t,x) = S(0)e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma x},$$

then $S(t) = f(t, W(t))$. Then by Itô's lemma, we have

$$\begin{aligned} df(t,W(t)) &= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dW(t) \\ &= \left[\left(r - \frac{\sigma^2}{2}\right)f(t,W(t)) + \frac{\sigma^2}{2}f(t,W(t))\right]dt + \sigma f(t,W(t))dW(t) \\ &= rf(t,W(t))dt + \sigma f(t,W(t))dW(t). \end{aligned}$$

$\square$

For the i-th simulation of the final stock price at time $t = T$, denoted by $\hat{S}^{(i)}(T)$ we have

$$\hat{S}^{(i)}(T) = S(0)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\epsilon_i},$$

where ϵ_i's follows the standard normal distribution. Since the mean of $\hat{S}^{(i)}(T)$ is $\mathbb{E}^{\mathbb{Q}}[S(T)]$, and by the law of large numbers, the average of $\hat{S}^{(i)}(T)$'s should be close to $\mathbb{E}^{\mathbb{Q}}[S(T)]$ after doing a large number of simulations. Thus after n simulations, our estimate of the expected final stock price would be

$$\mathbb{E}^{\mathbb{Q}}[S(T)] \approx \frac{1}{n}\sum_{i=1}^{n} \hat{S}^{(i)}(T).$$

However, the barrier options are path-dependent, so we also need to know the prices between $t = 0$ and $t = T$. For the rest of the paper, we shall not emphasize whether it is the i-th simulation when unnecessary, i.e. we will omit the notation (i). We proceed by dividing the duration T into m intervals and we define $\Delta t = T/m$ and we shall monitor the stock price at times $\Delta t, 2\Delta t, \cdots, T$. Here we let $t_j := j\Delta t$. By Equation 3.2, we see that there is a relationship between the stock price at time t_j and t_{j+1}, i.e.

$$S(t_{j+1}) = S(t_j)e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t+\sigma[W(t_{j+1})-W(t_j)]}.$$

Since $W(t_{j+1}) - W(t_j)$ follows a normal distribution with mean 0 and variance Δt, we let the simulated stock price at time t_{j+1} be

$$\hat{S}(t_{j+1}) = \hat{S}(t_j)e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}\epsilon_{j+1}},$$

in which ϵ_{j+1} follows a standard normal distribution and is independent of the other $\epsilon_{j'}$'s where $j + 1 \neq j'$. We can further simplify the expression above by letting X_{j+1} be a random variable which follows the normal distribution with mean $\left(r - \frac{\sigma^2}{2}\right)\Delta t$ and variance $\sigma^2\Delta t$, and this gives us

$$\hat{S}(t_{j+1}) = \hat{S}(t_j)e^{X_{j+1}} = S(0)e^{\sum_{k=1}^{j+1} X_k}.$$

Therefore if we let $U_j = \sum_{k=1}^{j} X_k$ we have that

$$\hat{S}(t_j) = S(0)e^{U_j}.$$

We wish to calculate the price of a down-and-in barrier call option on the stock mentioned above with barrier B, strike price K and expiration at time T. Since we are already in a risk-neutral world, we simply have to compute the discounted expected payoff. Thus, the current price of the barrier option v_0 is

$$v_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{S(T)\geq K, \min_{0\leq t\leq T} S(t)\leq B\}} \cdot (S(T) - K)].$$

To approximate v_0 using Monte Carlo simulations, we first take m samples from the normal distribution with mean $\left(r - \sigma^2/2\right)\Delta t$ and variance $\sigma^2 \Delta t$, which would be our $X_1, \cdots, X_m$. Then we calculate the cumulative sums $U_1, \cdots, U_m$. Now let $b = \log(S(0)/B), c = \log(K/S(0))$ and $P^{(i)}$ be the payoff for the i-th simulation. Then $P^{(i)}$ is equal to $\hat{S}^{(i)}(T) - K$ if there exists some $U_j, j \leq m$ such that $U_j \leq -b$, which means the barrier is hit, and $U_m \geq c$, which means the final stock price is greater the strike price, otherwise it is 0. Thus we have

$$v_0 \approx e^{-rT} \cdot \frac{1}{n} \sum_{i=1}^{n} P^{(i)}.$$

To make the mean of our sampling close to the theoretical value, we can make m large. In other words, monitor the stock price more closely, so that our simulation becomes close to continuous. We can see this from Table 3.1, where we have $S(0) = 95, r = 0.05, \sigma = 0.15, B = 93, K = 97, T = 1, n = 10,000$.

Table 3.1: Simulated Prices Obtained through Monte Carlo Simulations and Absolute Error from the Theoretical Price for various m's (Monitoring Frequency)

m	Simulated Price	Absolute Error
25	3.0260	1.5164
50	3.4152	1.1272
100	3.6621	0.8803
500	4.2046	0.3378
5000	4.4998	0.0426

Theoretical Price = 4.5424.

On the other hand, we also need a large enough n so that when we take the average, it is close to the mean of our sampling. However note that when the option is deep out of the money, our approximation converges slower, which can be seen from Figure 3.1, Figure 3.2, Figure 3.3, Figure 3.4. This is because for most of the simulations, either the barrier is not hit or it is hit but the strike price is not reached, which gives us a zero payoff and does not contribute much to understanding the behaviour of the barrier option's price.

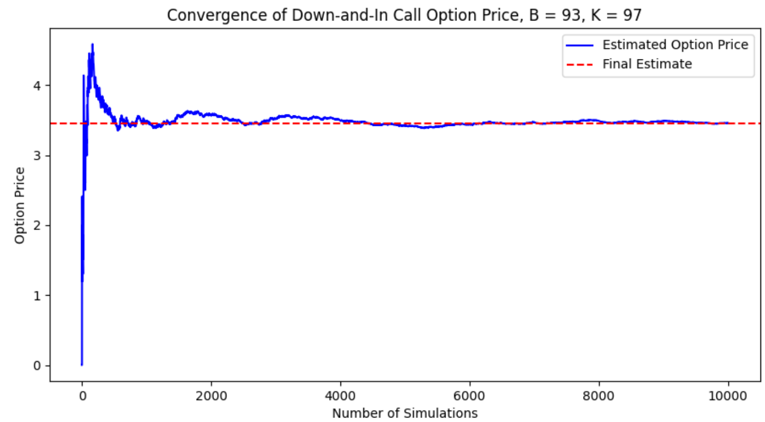

Figure 3.1: Simulated option price as the number of simulations increase when $B = 93, K = 97, S(0) = 95$. The price stabilizes after the 5000th simulation.

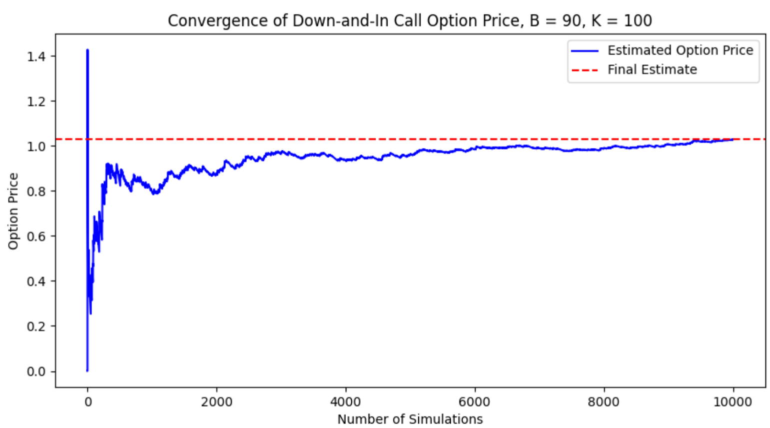

Figure 3.2: Simulated option price as the number of simulations increase when $B = 90, K = 100, S(0) = 95$. More simulations are required for convergence.

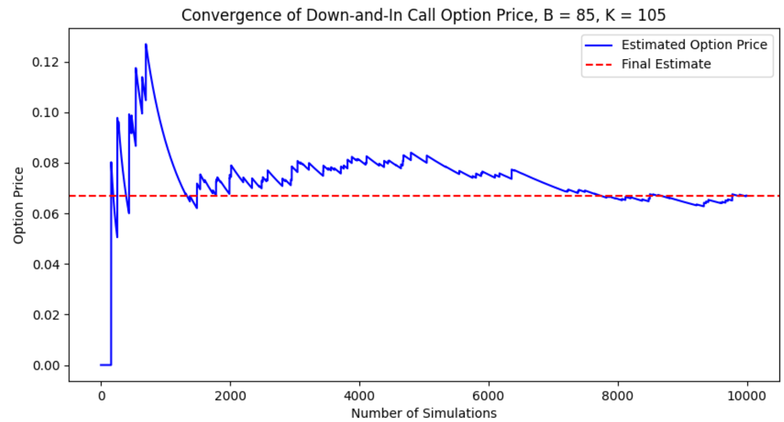

Figure 3.3: Simulated option price as the number of simulations increase when $B = 90, K = 100, S(0) = 95$. More fluctuation is observed.

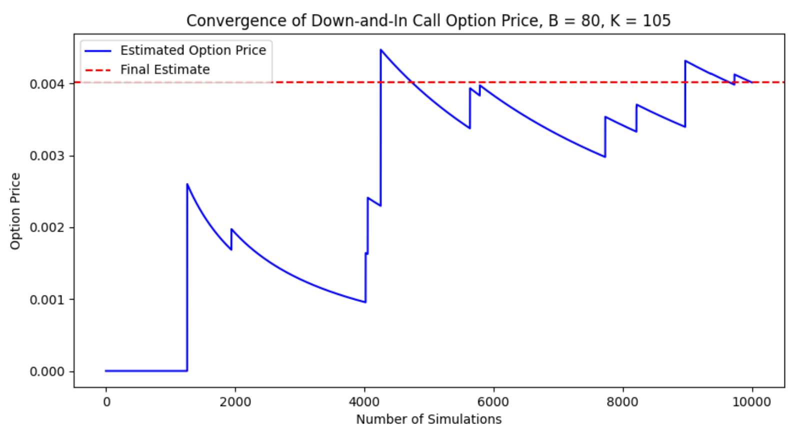

Figure 3.4: Simulated option price as the number of simulations increase when $B = 80, K = 105, S(0) = 95$. Much greater fluctuation is observed

As the barrier option becomes deeper out of the money, notice that there is greater fluctuation in the graph, which indicates a greater variance in the simulated price. Thus, more simulations are required to stabilize the mean, which leads to slower convergence.

Thus, for an option that is out of the money, if we want our simulated price to be close to the theoretical value, we need a large m and a large n. This requires significant computational resources. Therefore, we shall apply a variance reduction technique, importance sampling, so that our simulated price converges faster and we do not require such a large n. Furthermore, we can used the saved computational resources to simulate with a larger m, yielding better approximations.

3.2 Importance Sampling

[BBG97] pioneered the use of importance sampling in pricing down-and-in options. Importance sampling is used to alter the probability distribution of the stock price, which leads to the barrier being hit and the stock price going above the strike price more often. Thus we have more price paths that give nonzero payoffs, which in turn increases the efficiency of our simulation.

To increase the probability of the stock price hitting the barrier, which in turn leads to more simulations giving non-zero payoffs, we shall let its drift be some negative number, and after hitting the barrier we want the stock price to have a positive drift so that it moves upwards and goes above the strike price. Now let t_τ be the time when the stock price first hits the barrier. When doing the simulation, instead of letting the X_j's all follow a normal distribution with mean $(r - \sigma^2/2)\Delta t$, variance $\sigma^2\Delta t$, which we call it $f(x)$, we let $X_1, \cdots, X_\tau$ follow a normal distribution with mean $\mu_1\Delta t$, variance $\sigma^2\Delta t$, which we denote by $f_{\mu_1}(x)$ and for $X_{\tau+1}, \cdots, X_m$ they follow a normal distribution with mean $\mu_2\Delta t$, variance $\sigma^2\Delta t$ which we denote by $f_{\mu_2}(x)$. Now our task here is to figure out μ_1 and μ_2 so that $P(\tau < m, U_m > c)$ is increased. We see from [OSG08] that

$$\begin{aligned}
&P(\tau < m, U_m > c) \\
&= \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{\tau<m,U_m>c\}}] \\
&= \mathbb{E}_{\mu_1,\mu_2}\left[\mathbb{1}_{\{\tau<m,U_m>c\}} \prod_{j=1}^{\tau} \frac{f(X_j)}{f_{\mu_1}(X_j)} \prod_{j=\tau+1}^{m} \frac{f(X_j)}{f_{\mu_2}(X_j)}\right] \\
&:= \mathbb{E}_{\mu_1,\mu_2}\left[\mathbb{1}_{\{\tau<m,U_m>c\}} L_{\mu_1,\mu_2}\right],
\end{aligned}$$

where L_{μ_1,μ_2} is the likelihood ratio and $\mathbb{E}_{\mu_1,\mu_2}[\cdot]$ is the expectation taken under the probability measure that lets the stock drift be μ_1 before hitting the barrier and μ_2 after hitting the barrier.

We now do the detailed calculations to determine the likelihood ratio

$$
\begin{aligned}
\frac{f(x)}{f_{\mu_i}(x)} &= \frac{(2\pi\sigma\sqrt{\Delta t})^{-1} \exp - \frac{[x-(r-\sigma^2/2)\Delta t]^2}{2\sigma^2 \Delta t}}{(2\pi\sigma\sqrt{\Delta t})^{-1} \exp - \frac{[x-\mu_i \Delta t]^2}{2\sigma^2 \Delta t}} \\
&= \exp \frac{[x-\mu_i\Delta t]^2 - [x-(r-\sigma^2/2)\Delta t]^2}{2\sigma^2\Delta t} \\
&= \exp \frac{[x-\sigma^2\theta_i\Delta t-(r-\sigma^2/2)\Delta t]^2 - [x-(r-\sigma^2/2)\Delta t]^2}{2\sigma^2\Delta t} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\theta_i = \tfrac{\mu_i - r + \sigma^2/2}{\sigma^2}) \\
&= \exp \frac{-2x\sigma^2\theta_i\Delta t + 2\sigma^2\theta_i(r-\sigma^2/2)\Delta t^2 + \sigma^4\theta_i^2\Delta t^2}{2\sigma^2\Delta t} \\
&= \exp(-\theta_i x + \psi(\theta_i)). \qquad (\psi(\theta_i) = \theta_i(r-\sigma^2/2)\Delta t + \sigma^2\theta_i^2\Delta t/2)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
L_{\mu_1,\mu_2} &= \prod_{j=1}^{\tau} \exp\left(-\theta_1 X_j + \psi(\theta_1)\right) \prod_{j=\tau+1}^{m} \exp\left(-\theta_2 X_j + \psi(\theta_2)\right) \\
&= \exp\left\{ \sum_{j=1}^{\tau} [-\theta_1 X_j + \psi(\theta_1)] + \sum_{j=\tau+1}^{m} [-\theta_2 X_j + \psi(\theta_1)] \right\} \\
&= \exp(-\theta_1 U_\tau + \psi(\theta_1)\tau - \theta_2(U_m - U_\tau) + \psi(\theta_2)(m-\tau))
\end{aligned}
$$

The result above can be found at [BBG97]. Here we briefly repeat the heuristics used by [BBG97] to determine μ_1, μ_2. The observation here is that most of the variability comes from τ since when b, c are large we have $U_\tau \approx b, U_m \approx c$, and to remove the term from the expression, we let $\psi(\theta_1) = \psi(\theta_2)$, which simplifies the expression to be

$$
L_{\mu_1,\mu_2} = \exp[-(\theta_1 - \theta_2)U_\tau - \theta_2 U_m + m\psi(\theta_2)].
$$

The condition $\psi(\theta_1) = \psi(\theta_2)$ gives us

$$
\begin{aligned}
\theta_1(r-\sigma^2/2)\Delta t + \sigma^2\theta_1^2\Delta t/2 &= \theta_2(r-\sigma^2/2)\Delta t + \sigma^2\theta_2^2\Delta t/2 \\
(\theta_1 - \theta_2)(r - \sigma^2/2) &= -\sigma^2(\theta_1^2 - \theta_2^2)/2 \\
r - \sigma^2/2 &= -\sigma^2(\theta_1 + \theta_2)/2 \\
r - \sigma^2/2 &= -(\mu_1 + \mu_2 - 2r + \sigma^2)/2 \\
\mu_1 &= -\mu_2.
\end{aligned}
$$

Now we let $\mu_1 = -\mu_2 = -\mu$ and [BBG97] has chosen μ so that the time taken to go from 0 to $-b$ and then back to c is equal to m, which means

$$
\frac{b}{\mu\Delta t} + \frac{b+c}{\mu\Delta t} = m.
$$

This gives us $\mu = (2b + c)/T$, note that $m\Delta t = T$. We can then further simplify the likelihood ratio to be

$$L_{\mu_1,\mu_2} = \exp\left[2\mu U_\tau - \theta_2 U_m + m\psi(\theta_2)\right].$$

Now let us look at Table 3.2. These simulations have common parameters $S(0) = 95, r = 0.05, \sigma = 0.15, m = 250, n = 10,000$. We first show that the variance is indeed reduced.

Table 3.2: Variance (Var) and Rate of Non-Zero Payoffs ($R_{\neq 0}$)

B	K	Var	$R_{\neq 0}$	Var^{IS}	$R_{\neq 0}^{\text{IS}}$
85	105	4×10^{-4}	2%	2×10^{-5}	47%
80	105	2×10^{-5}	0.1%	4×10^{-8}	47%
75	96	3×10^{-7}	0.06%	3×10^{-9}	47%

IS means with Importance Sampling, otherwise, it means without.

Note that when the option becomes deeper out of the money, the simulation has a smaller variance because the theoretical price of the option is very small, but not because of reduced variability. As we can see, when importance sampling is applied, we consistently see smaller variances. Figure 3.5 would provide further support for the effectiveness of importance sampling.

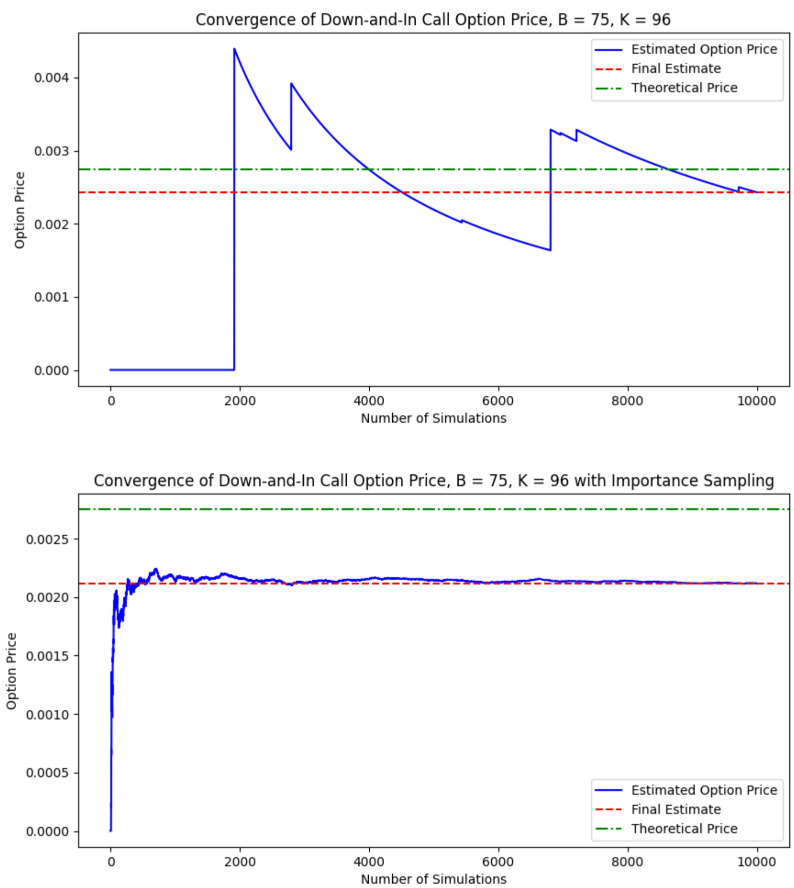

Figure 3.5: Faster Convergence when Importance Sampling is applied.

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