PRICING OF OUT-OF-THE-MONEY BARRIER OPTIONS USING IMPORTANCE SAMPLING AND NEURAL NETWORKS

1. Steps

(1) We want to price up-and-in Barrier call options[1], i.e., options with a payoff of the form

$$\varphi(S) = \mathbb{1}_{\{\max_{0\leq t\leq T} S_t \geq B\}} \cdot (S_T - K)^+$$

where $S = (S_t)_{0\leq t\leq T}$ denotes the price process of an underlying asset, K is the strike of the associated call option, $T > 0$ the maturity of the option, and B the barrier with $B > S_0$, i.e., the call option becomes *activated* as soon as we hit B before T, otherwise the option becomes worthless.

(2) According to risk-neutral valuation theory, an arbitrage free price of the option can be computed as

$$e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\varphi(S)\right]$$

where $\mathbb{Q}$ is a risk-neutral probability, and where $r > 0$ refers to the interest rate.

(3) The price can be computed via Monte-Carlo simulation, i.e, we sample the asset N times under the measure $\mathbb{Q}$, obtain the samples of the asset paths $S^{(1)}, \cdots, S^{(N)}$, and compute

$$e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\varphi(S)\right] \approx e^{-rT}\frac{1}{N}\sum_{i=1}^{N}\varphi(S^{(i)}) \tag{1.1}$$

where the approximation is due to the law of large numbers.

(4) However, if we are far out of the money, i.e., S_0 is much smaller than B, then only very few samples will lead to positive payoffs, i.e., $\mathbb{1}_{\{\max_{0\leq t\leq T} S_t \geq B\}}$ is 0 in almost all samples, which in turn implies a high variance of the approximation in (1.1), as the average is taken only w.r.t. a few samples.

(5) We apply *importance sampling*, i.e., we sample under an equivalent measure $\widetilde{\mathbb{Q}}$ under which the event $\{\max_{0\leq t\leq T} S_t \geq B\}$ is more likely to happen, this leads via the Radon-Nikodym theorem to

$$e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\varphi(S)\right] = e^{-rT}\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\widetilde{\mathbb{Q}}}\varphi(S)\right] \approx e^{-rT}\frac{1}{N}\sum_{i=1}^{N}\varphi(\widetilde{S}^{(i)})\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\widetilde{\mathbb{Q}}}(\tilde{S}^{(i)}) \tag{1.2}$$

where $\widetilde{S}^{(1)}, \ldots, \widetilde{S}^{(N)}$ denote the samples of the stock price evolution on $[0, T]$ under $\widetilde{\mathbb{Q}}$.

(6) Learn the relation between the inputs to the options contract i.e., $x = (K, S_0, r, T, B)$ and the corresponding price y (computed via importance sampling). To do this, we sample different inputs $(x_i)_{i=1,\ldots,n}$, compute the associated prices $(y_i)_{i=1,\ldots,n}$, and use this data set to train a neural network $\mathcal{NN}$ which upon training allows to predict prices, given an input x.

(7) We can use the trained neural network $\mathcal{NN}$ to study sensitivities of the option w.r.t. input parameters. This allows in particular to compute the corresponding "greeks", which then could be used for hedging.

2. Tasks

(1) Get familiar with the concepts that are unclear. I am happy to provide literature on any of the concepts. Just let me know.

(2) It is our (or your) choice with which model we want to work to simulate the price paths of S. I would propose to start with the Black–Scholes model, as we have a closed form pricing formula for Barrier options in that model which can serve as a benchmark. The formula can be found, e.g., here [2].

[1] This can be replaced by any other type of Barrier options if we want

(3) The Black–Scholes model assumes that the stock price evolves according to

$$dS_t = S_t r dt + S_t \sigma dt$$

which is a stochastic differential equation. If you are not familiar with this concept I am happy to explain the fundamental idea and provide literature. For the project more important is however to understand how to use such a model to sample paths. One method is the so called Euler-Maruyama method, explained for example here: [1].

References

[1] Stéphane Crépey. Financial modeling. *Springer Finance, DOI*, 10:978–3, 2013.
[2] Steven E Shreve et al. *Stochastic calculus for finance II: Continuous-time models*, volume 11. Springer, 2004.