Pricing of out-of-the-money Barrier Options Using Importance Sampling

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April 16, 2025

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A barrier option is an option whose payoff is conditional upon the underlying asset's price breaching a barrier level during the option's lifetime.

The four main types of barrier options are:

- Up-and-out: spot price starts below the barrier level and has to move up for the option to be knocked out.
- Down-and-out: spot price starts above the barrier level and has to move down for the option to become null and void.
- Up-and-in: spot price starts below the barrier level and has to move up for the option to become activated.
- Down-and-in: spot price starts above the barrier level and has to move down for the option to become activated.

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Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. Then $(W(t)_{t \geq 0})$ is a **Brownian motion** if for all $0 = t_0 < t_1 < \cdots < t_m$ the increments

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \cdots, W(t_m) - W(t_{m-1})$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}[W(t_{i+1} - W(t_i)] = 0,$$

$$\mathrm{Var}[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i.$$

The Black-Scholes Model

Price Dynamics of the Stock under the Black-Scholes Framework

Let the risk-free rate be r, the rate of return of the stock be μ, the volatility of the stock be σ, the stock price at time t be $S(t)$, and $(W(t)_{t\geq 0})$ be a Brownian motion. Then under the Black-Scholes model, the stock price follows a geometric Brownian motion, i.e.

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

However, when it comes to pricing derivatives, we shall take the expectation of the payoff under the risk-neutral measure $\mathbb{Q}$, in which the following equation governs the stock price

$$dS(t) = rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t),$$

where $(W^{\mathbb{Q}}(t)_{t\geq 0})$ is a Brownian motion under the measure $\mathbb{Q}$.

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Formula for Theoretical Price

Transforming the Problem into Calculating Probabilities

The following derivation of the theoretical price is obtained from the lecture notes of Prof. Julian [4]. Let B be the barrier, K be the strike price, and suppose the option matures at time T. By risk-neutral valuation, the price of the barrier option v_0 is

$$\begin{aligned} v_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S(T)-K)\cdot \mathbb{1}_F] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[S(T)\cdot \mathbb{1}_F] - Ke^{-rT}\mathbb{Q}(F) \end{aligned}$$

where $F = \{S(T) \geq K, \min_{0\leq t\leq T} S(t) \leq B\}$ and $\mathbb{Q}$ is the risk-neutral measure. Then, by a change of numeraire, we get

$$v_0 = S(0)\mathbb{Q}^S(F) - Ke^{-rT}\mathbb{Q}(F),$$

where $\mathbb{Q}^S$ is the risk-neutral measure with the stock as numeraire.

Formula for Theoretical Price

The Reflection Principle

Theorem

(Reflection Principle from [2]) *If τ is a stopping time and $(W(t))_{0\leq t\leq T}$ is a standard Brownian motion, then the process $(W^*(t))_{0\leq t\leq T}$ called **Brownian motion reflected at** τ and defined by*

$$W^*(t) = W(t)\mathbb{1}_{\{t\leq\tau\}} + (2W(\tau) - W(t))\mathbb{1}_{\{t>\tau\}}$$

is also a standard Brownian motion.

Lemma

Let $W(T)$ be a Brownian motion, $m(T) = \min_{0\leq t\leq T} W(T)$ then for $m < 0, m \leq x$ we have

$$\mathbb{P}(W(T) \geq x, m(T) \leq m) = \mathbb{P}(W(T) \leq 2m - x).$$

Formula for Theoretical Price

A slightly General Case

Let stochastic process $(X(t)_{t\geq 0})$ follow $dX(t) = \eta dt + \sigma dW(t)$. We shall calculate $\mathbb{P}(X(t) \geq x, m(T) \leq m)$, where $m(T)$ is the minimum of $X(t)$ for $0 \leq t \leq T$. We use the following theorem:

Theorem

(Girsanov Theorem) *Let $(W(t))_{0\leq t\leq T}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $(\mathcal{F}(t))_{0\leq t\leq T}$ be a filtration for this Brownian motion. Let $(\theta(t))_{0\leq t\leq T}$ be an adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du\right\},$$

$$\tilde{W}(t) = W(t) + \int_0^t \theta(u)du,$$

and assume that

$$\mathbb{E}\left[\int_0^T \theta^2(u)Z^2(u)du\right] < +\infty.$$

Set $Z = Z(T)$. *Then* $\mathbb{E}[Z] = 1$ *and under the probability measure* $\tilde{\mathbb{P}}$ *given by*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)dP(\omega) \text{ for all } A \in \mathcal{F},$$

the process $(\tilde{W}(t))_{0\leq t\leq T}$, *is a Brownian motion.*

Formula for Theoretical Price

Calculations under the measure $\tilde{\mathbb{P}}$

Since $X(t) = \eta t + \sigma W(t) = \sigma\left(\frac{\eta}{\sigma} t + W(t)\right) := \sigma \tilde{W}(t)$, by Girsanov, let $\theta = \frac{\eta}{\sigma}$, then under the measure $\tilde{\mathbb{P}}$ given by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = Z = \exp\left\{-\frac{\eta}{\sigma} W(T) - \frac{\eta^2}{2\sigma^2} T\right\},$$

$(\tilde{W}(t)_{t\geq 0})$ is a Brownian motion. Let $\tilde{x} = x/\sigma, \tilde{m} = m/\sigma$ then

$$\begin{aligned}
&\tilde{\mathbb{P}}(X(T) \geq x, m(T) \leq m) \\
&= \tilde{\mathbb{P}}(\tilde{W}(T) \geq \tilde{x}, \tilde{m}(T) \leq \tilde{m}) \qquad (\tilde{m}(T) := \min_{0\leq t\leq T} \tilde{W}(T)) \\
&= \mathbb{P}(\tilde{W}(T) \leq 2\tilde{m} - \tilde{x}) \qquad \text{(Reflection Principle)} \\
&= N\left(\frac{2m - x}{\sigma\sqrt{T}}\right).
\end{aligned}$$

Formula for Theoretical Price

Switching back to $\mathbb{P}$

Now, note that

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = \left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right)^{-1} = \exp\left\{\frac{\eta}{\sigma}\tilde{W}(T) - \frac{\eta^2}{2\sigma^2}T\right\}$$

and by the reflection principle, for any function g, we have

$$\mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\tilde{A}}g(\tilde{W}(T))] = \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\tilde{B}}g(2m/\sigma - \tilde{W}(T))],$$

where $\tilde{A} = \{\tilde{W}(T) \geq \tilde{x}, \tilde{m}(T) \leq \tilde{m}\}$ and $\tilde{B} = \{\tilde{W}(T) \leq 2\tilde{m} - \tilde{x}\}$. Thus, we now let

$$g(\tilde{W}(T)) = \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}$$

and so $\mathbb{P}(X(T) \geq x, m(T) \leq m)$ is equal to

$$\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\tilde{A}}] = \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\tilde{A}}g(\tilde{W}(T))] = \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\tilde{B}}g(2\tilde{m} - \tilde{W}(T)).$$

Formula for Theoretical Price

The Key Calculation

We proceed as follows:

$$
\begin{aligned}
&\mathbb{P}(X(T) \geq x, m(T) \leq m) \\
&= \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\tilde{B}} g(2\tilde{m} - \tilde{W}(T))] \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{2\tilde{m}-\tilde{x}} g(2\tilde{m} - w) e^{-\frac{w^2}{2T}} \, dw \qquad (\tilde{W}(T) \sim \mathcal{N}(0, T).) \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{2\tilde{m}-\tilde{x}} \exp\left\{ -\frac{w^2}{2T} + \frac{\eta}{\sigma}(2\tilde{m} - w) - \frac{\eta^2}{2\sigma^2} T \right\} dw \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\frac{2m-x}{\sigma}} \exp\left\{ \frac{2\eta m}{\sigma^2} - \frac{1}{2T} \left(\frac{\eta T}{\sigma} + w \right)^2 \right\} dw \\
&= e^{\frac{2\eta m}{\sigma^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{2m-x+\eta T}{\sigma\sqrt{T}}} \exp\left\{ -\frac{1}{2} z^2 \right\} dz \quad (z = (\eta T/\sigma + w)/\sqrt{T}) \\
&= e^{\frac{2\eta m}{\sigma^2}} N\left(\frac{2m - x + \eta T}{\sigma\sqrt{T}} \right).
\end{aligned}
$$

Formula for Theoretical Price

Back to the Original Problem

To make use of the result obtained, we consider *log-prices*. Let $X(t) := \log(S(t)/S(0)), m(T) := \min_{0\le t\le T} X(t), x := \log(K/S(0)), m := \log(B/S(0))$. Then under $\mathbb{Q}^S$,

$$d(\log S(t)) = (r + \sigma^2/2)dt + \sigma dW^S(t),$$

and under $\mathbb{Q}$

$$d(\log S(t)) = (r - \sigma^2/2)dt + \sigma dW^B(t).$$

Thus,

$$X(t) = \begin{cases} (r + \sigma^2/2)t + \sigma W^S(t) & \text{under } \mathbb{Q}^S \\ (r - \sigma^2/2)t + \sigma W^B(t) & \text{under } \mathbb{Q}. \end{cases}$$

Formula for Theoretical Price

The Final Result

So we have the following result:

Theorem

Suppose the risk-free rate is r and the stock follows a geometric Brownian motion, with volatility σ. Let v_0 be the price of a down-and-in barrier call option on the stock with barrier B, strike price K, and matures at time T. We define

$$d_b = \frac{\log \frac{B^2}{S(0)K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \qquad \alpha = \frac{2r}{\sigma^2} + 1.$$

Then

$$v_0 = S(0)\left(\frac{B}{S(0)}\right)^{\alpha} N(d_b) - Ke^{-rT}\left(\frac{B}{S(0)}\right)^{\alpha-2} N(d_b - \sigma\sqrt{T}).$$

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Direct Method

Analytical Solutions for Geometric Brownian Motions.

For the stochastic equation $dS(t) = rS(t)dt + \sigma S(t)dW(t)$, we have the analytical solution

$$S(t) = S(0)e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W(t)}.$$

We shall monitor the stock m times over the period T, and let $t_j = j\Delta t, j = 0, \cdots, m$, where $\Delta t = T/m$. Then

$$S(t_{j+1}) = S(t_j)e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t+\sigma[W(t_{j+1})-W(t_j)]}.$$

Moreover, since $W(t_{j+1}) - W(t_j) \sim \mathcal{N}(0, \Delta t)$, we shall simulate the price paths in the following way:

Direct Method

Simulating Price Paths

Let $\hat{S}(t)$ be the simulated price at time t, then

$$\hat{S}(t_{j+1}) = \hat{S}(t_j) e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\epsilon_{j+1}},$$

in which ϵ_{j+1} follows a standard normal distribution. We can further simplify the expression above by letting X_{j+1} be a random variable which follows the normal distribution with mean $\left(r - \frac{\sigma^2}{2}\right)\Delta t$ and variance $\sigma^2\Delta t$, and this gives us

$$\hat{S}(t_{j+1}) = \hat{S}(t_j) e^{X_{j+1}} = S(0) e^{\sum_{k=1}^{j+1} X_k}.$$

Therefore if we let $U_j = \sum_{k=1}^{j} X_k$ we have that

$$\hat{S}(t_j) = S(0) e^{U_j}.$$

Direct Method

Approximating Price of the Barrier Option, Examples

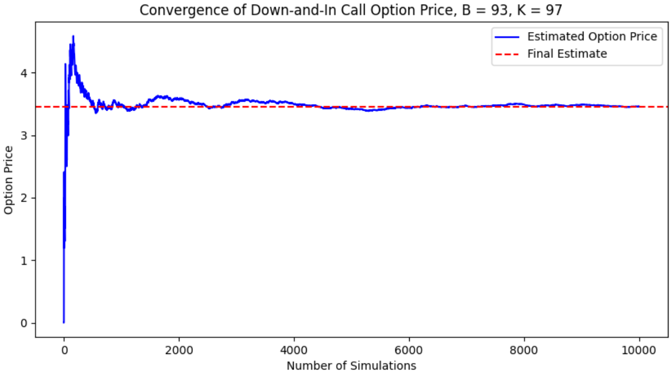

Figure: Simulated option price as the number of simulations increase when $B = 93, K = 97, S(0) = 95$. The price stabilizes after the 5000th simulation.

Direct Method

Approximating Price of the Barrier Option, Examples

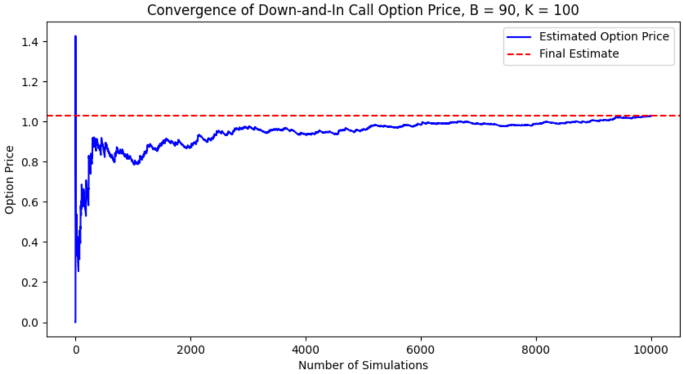

Figure: Simulated option price as the number of simulations increase when $B = 90, K = 100, S(0) = 95$. More simulations are required for convergence.

Direct Method

Approximating Price of the Barrier Option, Examples

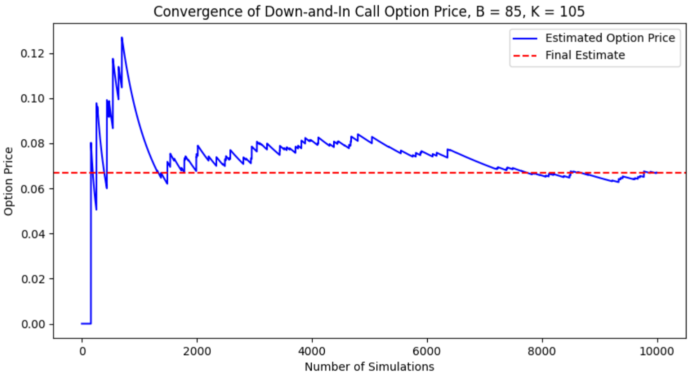

Figure: Simulated option price as the number of simulations increase when $B = 90, K = 100, S(0) = 95$. More fluctuation is observed.

Direct Method

Approximating Price of the Barrier Option, Examples

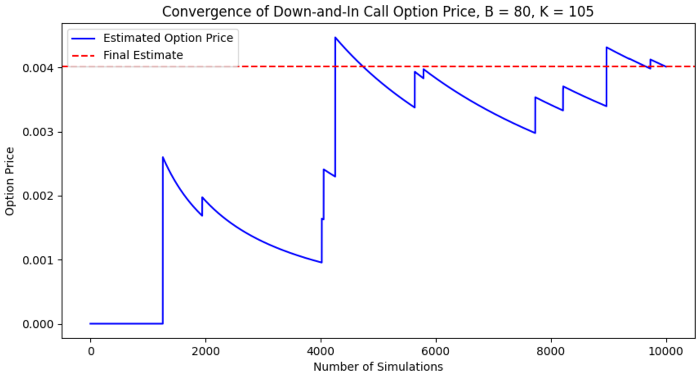

Figure: Simulated option price as the number of simulations increase when $B = 80, K = 105, S(0) = 95$. Much greater fluctuation is observed

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Importance Sampling

Boyle et al. [1] pioneered importance sampling in pricing down-and-in options. Importance sampling is used to alter the probability distribution of the stock price, which leads to the barrier being hit and the stock price going above the strike price more often. Thus, we have more price paths that give nonzero payoffs, increasing our simulation's efficiency.

Changing the Drift

To increase the probability of the stock price hitting the barrier, we shall let the drift of the stock price be some negative number, and after hitting the barrier at time τ we want the stock price to have a positive drift so that it moves upwards and goes above the strike price.

Importance Sampling

The Likelihood Function

Before time τ, we shall sample from $\mathcal{N}(\mu_1 \Delta t, \sigma^2 \Delta t)$, which has probability density function f_{μ_1} and then from $\mathcal{N}(\mu_2 \Delta t, \sigma^2 \Delta t)$, which has probability density function f_{μ_2}. Let f be the probability density function of $\mathcal{N}((r - \sigma^2/2)\Delta t, \sigma^2 \Delta t)$, then from [3] we have

$$
\begin{aligned}
& P(\tau < m, U_m > c) \\
&= \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{\tau < m, U_m > c\}}] \\
&= \mathbb{E}_{\mu_1,\mu_2}\left[\mathbb{1}_{\{\tau < m, U_m > c\}} \prod_{j=1}^{\tau} \frac{f(X_j)}{f_{\mu_1}(X_j)} \prod_{j=\tau+1}^{m} \frac{f(X_j)}{f_{\mu_2}(X_j)}\right] \\
&:= \mathbb{E}_{\mu_1,\mu_2}\left[\mathbb{1}_{\{\tau < m, U_m > c\}} L_{\mu_1,\mu_2}\right],
\end{aligned}
$$

where L_{μ_1,μ_2} is the likelihood ratio and $\mathbb{E}_{\mu_1,\mu_2}[\cdot]$ is the expectation taken under the probability measure that lets the stock drift be μ_1 before hitting the barrier and μ_2 after hitting the barrier.

Importance Sampling

Calculations

Detailed calculations show that

$$\frac{f(x)}{f_{\mu_i}(x)} = \exp(-\theta_i x + \psi(\theta_i)),$$

where $\theta_i = \frac{\mu_i - r + \sigma^2/2}{\sigma^2}$ and $\psi(\theta_i) = \theta_i(r - \sigma^2/2)\Delta t + \sigma^2\theta_i^2\Delta t/2$. Thus, we have

$$L_{\mu_1,\mu_2} = \exp(-\theta_1 U_\tau + \psi(\theta_1)\tau - \theta_2(U_m - U_\tau) + \psi(\theta_2)(m - \tau)).$$

Empirical evidence suggests that we should remove the τ term from the likelihood function which can be done by letting $\psi(\theta_1) = \psi(\theta_2)$ and leads to $\mu_1 = -\mu_2$. And we choose $\mu_1 = (2b + c)/T$ where $b = \log(S(0)/B), c = \log(K/S(0))$.

Importance Sampling

The final result

Now let us look at Table 31. These simulations have common parameters $S(0) = 95, r = 0.05, \sigma = 0.15, m = 250, n = 10,000$. We first show that the variance is indeed reduced.

Table: Variance (Var) and Rate of Non-Zero Payoffs ($R_{\neq 0}$)

B	K	Var	$R_{\neq 0}$	$\mathrm{Var}^{\mathrm{IS}}$	$R^{\mathrm{IS}}_{\neq 0}$
85	105	4×10^{-4}	2%	2×10^{-5}	47%
80	105	2×10^{-5}	0.1%	4×10^{-8}	47%
75	96	3×10^{-7}	0.06%	3×10^{-9}	47%

IS means with Importance Sampling, otherwise, it means without.

Importance Sampling

The final result

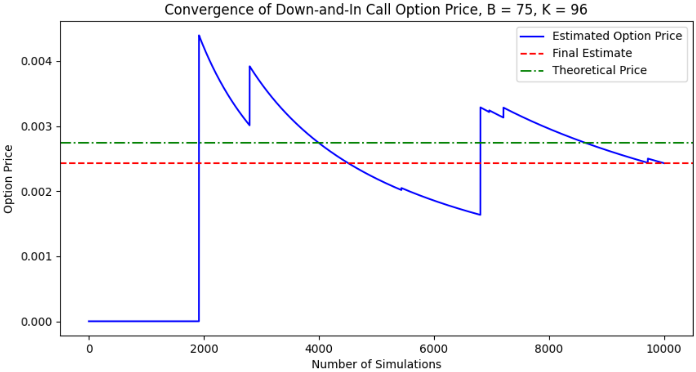

Figure: Convergence without Importance Sampling.

Importance Sampling

The final result

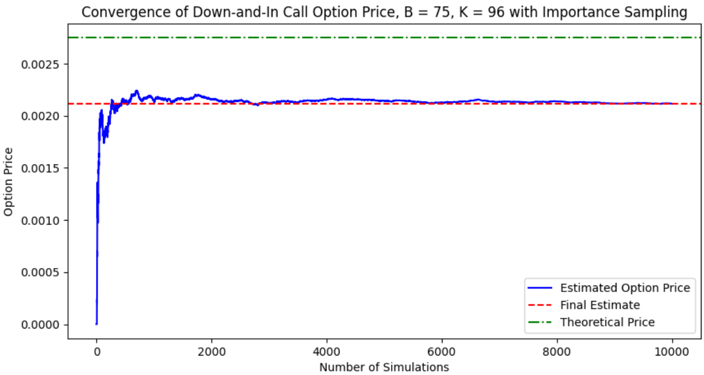

Figure: Convergence with Importance Sampling.

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