HW 2, M362M, Fall 2024

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Problem 1

1.1

$$
\begin{aligned}
&\Pr[X_1 \neq X_2] \\
=&\sum_{i=1}^{k} \Pr[X_2 \neq X_1 \mid X_1 = i] \Pr[X_1 = i] \\
=&\sum_{i=1}^{k} \Pr[X_2 \neq i] \Pr[X_1 = i] \\
=&\sum_{i=1}^{k} \frac{1}{k} \cdot \frac{k-1}{k} \\
=&\frac{k-1}{k}
\end{aligned}
$$

1.2

$$
\begin{aligned}
&\Pr[X_1 \neq X_2, X_1 \neq X_3, X_2 \neq X_3] \\
=&\sum_{i=1}^{k} \Pr[X_2 \neq i, X_3 \neq i, X_2 \neq X_3] \Pr[X_1 = i] \\
=&\sum_{i=1}^{k} \sum_{j \neq i}^{k} \Pr[X_3 \neq i, X_3 \neq j] \Pr[X_2 = j] \cdot \frac{1}{k} \\
=&\sum_{i=1}^{k} \sum_{j \neq i}^{k} \frac{k-2}{k} \cdot \frac{1}{k} \cdot \frac{1}{k} \\
=&\frac{(k-1)(k-2)}{k^2}
\end{aligned}
$$

1.3

Statement:

$$P_n := \Pr[X_i \neq X_j, \forall\, i,j = 1,\cdots,n, i \neq j] = \frac{(k-1)!}{(k-n)!} \cdot \frac{1}{k^{n-1}}, n \leq k.$$

We have covered the cases $n = 2, 3$, now suppose the statement holds for $n = N$, we show that it holds when $n = N + 1 \leq k$.

$$\begin{aligned}
P_{N+1} &= \sum_{A \subset \{1,\cdots,k\}, |A|=N} \Pr[X_{N+1} \notin A \mid \{X_i\}_{i=1}^N = A] \cdot \Pr[\{X_i\}_{i=1}^N = A] \\
&= \sum_{A \subset \{1,\cdots,k\}, |A|=N} \frac{k-N}{k} \cdot \Pr[\{X_i\}_{i=1}^N = A] \\
&= \frac{k-N}{k} \cdot \sum_{A \subset \{1,\cdots,k\}, |A|=N} \Pr[\{X_i\}_{i=1}^N = A] \\
&= \frac{k-N}{k} \cdot P_N \\
&= \frac{k-N}{k} \cdot \frac{(k-1)!}{(k-N)!} \cdot \frac{1}{k^{N-1}} && \text{(Induction Hypothesis, } N+1 \leq k\text{)} \\
&= \frac{(k-1)!}{(k-(N+1))!} \cdot \frac{1}{k^{(N+1)-1}}.
\end{aligned}$$

And we are done.

1.4.1

```r
k = 365
n = 23
unique_birthdays_count = 0
n_sim = 10000

simulate_birthday = function(k,n) {
  birthdays = sample(1:k, n, replace = TRUE)

  # Returns TRUE if all birthdays are unique
  return (anyDuplicated(birthdays) == 0)
  # Alternative: return(length(unique(birthdays)) == n)
}

for (i in 1:n_sim) {
  if (simulate_birthday(k,n)) {
    unique_birthdays_count = unique_birthdays_count + 1
  }
}
# Alternative: replicate(n_simulations, simulate_birthday_problem(k, n))
#            -> unique_birthdays_count = mean(results)

prob_unique_birthdays = unique_birthdays_count/n_sim

cat("Probability of everyone having a unique birthday in a group of",
            n, "people :", prob_unique_birthdays)
## Probability of everyone having a unique birthday in a group of 23 people : 0.5077
```

1.4.2

Note that $P_1 = 1$ and we can simply P_n to get

$$P_n = \prod_{i=1}^{n-1} \left(\frac{k-i}{k} \right), n \geq 2.$$

```
#Load library
library(glue)

k = 365
n = 23

prob_theoretical = function(k,n) {
  ans = 1
  if (n != 1) {
    for (i in 1:(n-1)) {
      ans = ans * (k-i)/k
    }
  }
  return (round(ans,4))
}

glue("Probability (Theoretical): {prob_theoretical(k,n)} \n
      Probability (Simulation) : {prob_unique_birthdays} \n
      The results are quite close.")
## Probability (Theoretical): 0.4927
##
## Probability (Simulation) : 0.5077
##
## The results are quite close.
```

Problem 2

2.5

Given that $X \sim \text{ Bin}(n = 1000, p = 0.7)$, we have

$$\Pr[X > 750] = \sum_{k=751}^{1000} \binom{1000}{k} 0.7^k \, 0.3^{1000-k}.$$

2.6

Consider $Y \sim \mathcal{N}(\mu = np, \sigma^2 = np(1-p))$, where Y is a normally distributed random variable with mean $\mu = 700$ and variance $\sigma^2 = 210$.

$$\begin{aligned} \Pr[X > 750] \approx & \Pr[Y \geq 750.5] \\ = & \Pr[Z \geq 3.4503] && (Z \sim \mathcal{N}(1,0), Z = (Y-\mu)/\sigma) \\ = & 0.0003 && \text{(Calculator)} \end{aligned}$$

2.7

```r
n = 1000
p = 0.7
threshold = 750
n_sim = 10000

simulate_binomial = function(n,p) {
  # 0: Tail (1-p), 1 : Head (p)
  toss = sample(c(0,1), n, replace = TRUE, prob=c(1-p, p))

  # Returns the number of heads
  return (sum(toss))
}
# Alternative: sum(rbinom(n, 1, p))

results = replicate(n_sim, simulate_binomial(n,p))

prob_over_threshold = mean(results > threshold)

cat(paste(" Probability to have more than", threshold,
            "success(es) in"),"\n",
    paste(n, "trials with probability to succeed being",p, "is",
          prob_over_threshold))
##  Probability to have more than 750 success(es) in
##  1000 trials with probability to succeed being 0.7 is 2e-04
```

Problem 3

3.8

Let $N \sim \text{Pois}(\lambda)$, and 1_H be a discrete variable that depends on N and only takes on the values -1 and 1. Also, it has the following properties:

$$\Pr[1_H = -1 \mid N] = \frac{N}{N+1}, \qquad \Pr[1_H = 1 \mid N] = \frac{1}{N+1}.$$

$$
\begin{aligned}
\Pr[1_H = 1] =& \sum_{k=0}^{\infty} \Pr[1_H = 1 \mid N = k] \cdot \Pr[N = k] \\
=& \sum_{k=0}^{\infty} \Pr[1_H = 1 \mid N = k] \cdot \frac{\lambda^k e^{-\lambda}}{k!} \\
=& \sum_{k=0}^{\infty} \frac{1}{k+1} \cdot \frac{\lambda^k e^{-\lambda}}{k!} \\
=& \frac{1}{\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^{k+1} e^{-\lambda}}{(k+1)!} \\
=& \frac{1}{\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k} e^{-\lambda}}{k!} \\
=& \frac{1}{\lambda} \cdot (1 - \Pr[N = 0]) \\
=& \frac{1}{\lambda} \cdot \left(1 - \frac{\lambda^0 e^{-\lambda}}{0!}\right) \\
=& \frac{1 - e^{-\lambda}}{\lambda},
\end{aligned}
$$

$$
\Pr[1_H = -1] = 1 - \Pr[1_H = 1] = \frac{\lambda - 1 + e^{-\lambda}}{\lambda},
$$

$$
\mathbb{E}[1_H] = (-1) \cdot \Pr[1_H = -1] + 1 \cdot \Pr[1_H = 1] = \frac{2(1 - e^{-\lambda}) - \lambda}{\lambda}
$$

3.9

Note that

$$
\Pr[N = k \mid 1_H = i] = \frac{\Pr[1_H = i \mid N = k] \cdot \Pr[N = k]}{\Pr[1_H = i]}.
$$

Therefore

$$
\Pr[N = k \mid 1_H = -1] = \frac{k\lambda^{k+1} e^{-\lambda}}{(k+1)!(\lambda - 1 + e^{-\lambda})}
$$

$$
\Pr[N = k \mid 1_H = 1] = \frac{\lambda^{k+1} e^{-\lambda}}{(k+1)!(1 - e^{-\lambda})}.
$$

(Please Turn Over)

3.10

```
lambda = 5

# Generates a random number which follows
# the Poisson distribution with parameter lambda
N = rpois(1, lambda)

winnings = sample(c(-1,1), 1, replace = TRUE, prob=c(N/(N+1), 1/(N+1)))

if (winnings == 1) {
  toss = "Head"
} else {
  toss = "Tail"
}

cat(" N = ", N, "\n",
    "Probability of getting Head: ", 1/(N+1), "\n",
    "Probability of getting Tail: ", N/(N+1), "\n",
    "Result of Coin Toss: ", toss, "\n",
    "Winnings: ", winnings)
##  N =  1
##  Probability of getting Head:  0.5
##  Probability of getting Tail:  0.5
##  Result of Coin Toss:  Head
##  Winnings:  1
```

End of Homework