HW 5 M362M, Fall 2024

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Problem 1

We always have $Y \geq 0$ and it only takes integral values. Thus $\mathbb{P}[Y<0]=0$. For $i=0,1,2,\cdots$, we have

$$\begin{aligned}\mathbb{P}[Y=i] &= \sum_{n=1}^{\infty} \mathbb{P}[\xi_1+\cdots+\xi_N=i \mid N=n]\cdot\mathbb{P}[N=n] \\ &= \sum_{n=1}^{\infty} \mathbb{P}[\xi_1+\cdots+\xi_n=i]\cdot\mathbb{P}[N=n]\end{aligned}$$

Note that $\xi_1+\cdots+\xi_n \sim \text{Binom}(n,p_B)$. If $i=0$, then

$$\begin{aligned}\mathbb{P}[Y=0] &= \sum_{n=1}^{\infty}\binom{n}{0}p_B^0(1-p_B)^n\cdot p_g(1-p_g)^{n-1} \\ &= (1-p_B)p_g\cdot\sum_{n=0}^{\infty}\left[(1-p_B)(1-p_g)\right]^n \\ &= \frac{(1-p_B)p_g}{1-(1-p_B)(1-p_g)}\end{aligned}$$

If $i=1,2,\cdots$, $\mathbb{P}[\xi_1+\cdots+\xi_n=i]=0$ for $n<i$, thus

$$\begin{aligned}\mathbb{P}[Y=i] &= \sum_{n=i}^{\infty}\binom{n}{i}p_B^i(1-p_B)^{n-i}\cdot p_g(1-p_g)^{n-1} \\ &= p_g(1-p_g)^{i-1}p_B^i\cdot\sum_{n=i}^{\infty}\binom{n}{i}\left[(1-p_B)(1-p_g)\right]^{n-i}\end{aligned}$$

Now we prove the identity, for all $i\in\mathbb{N}_0, |x|<1$, we have

$$S_i := \sum_{n=i}^{\infty}\binom{n}{i}x^{n-i} = \frac{1}{(1-x)^{i+1}}.$$

We proceed with induction on i. When $i=0$, it is a geometric series, and the result holds. Suppose it holds when $i=k-1$, then for $i=k$,

$$\begin{aligned}
S_k - xS_k &= 1 + \sum_{n=k+1}^{\infty} \left[\binom{n}{k} - \binom{n-1}{k} \right] x^{n-k} \\
&= 1 + \sum_{n=k+1}^{\infty} \binom{n-1}{k-1} x^{n-k} \\
&= S_{k-1} \\
&= \frac{1}{(1-x)^k}
\end{aligned}$$

Therefore

$$S_k = \frac{1}{(1-x)^{k+1}}.$$

Thus the statement holds for all $i \in \mathbb{N}_0$. Back to the original problem, we have

$$\mathbb{P}[Y=i] = \begin{cases} \frac{p_g(1-p_g)^{i-1}p_B^i}{[1-(1-p_B)(1-p_g)]^{i+1}}, & i=1,2,\cdots \\ \\ \frac{(1-p_B)p_g}{1-(1-p_B)(1-p_g)} \quad, & i=0, \\ \\ \quad 0 \quad, & \text{else.} \end{cases}$$

Problem 2

2.1

$$\begin{aligned}
M_{X_n}(t) &= M_{\sum_{i=1}^n \delta_i}(t) \\
&= \prod_{i=1}^n M_{\delta_i}(t) && (\{\delta_i\}_{i=1}^n \text{ independent}) \\
&= [M_{\delta_1}(t)]^n && (\delta_1, \cdots, \delta_n \text{ identically distributed}) \\
&= \left[\frac{1}{2}(e^t + e^{-t})\right]^n
\end{aligned}$$

2.2

We first calculate the first two derivatives of $M_{X_n}(t)$.

$$M'_{X_n}(t) = n\left[\frac{1}{2}(e^t + e^{-t})\right]^{n-1} \cdot \frac{1}{2}(e^t - e^{-t}) = \frac{n}{2^n}(e^t + e^{-t})^{n-1}(e^t - e^{-t}).$$

$$M''_{X_n}(t) = \frac{n}{2^n}\left[(n-1)(e^t + e^{-t})^{n-2}(e^t - e^{-t})^2 + (e^t + e^{-t})^n\right].$$

We are now in a position to calculate $\mathbb{E}[X_n]$ and $\operatorname{Var}(X_n)$.

$$\begin{aligned}
\mathbb{E}[X_n] &= M'_{X_n}(0) \\
&= 0, && (e^0 - e^{-0} = 0)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(X_n) &= \mathbb{E}[X_n^2] - (\mathbb{E}[X_n])^2 \\
&= \mathbb{E}[X_n^2] && (\mathbb{E}[X_n] = 0) \\
&= M''_{X_n}(0) + M'_{X_n}(0) \\
&= n
\end{aligned}$$

Therefore

$$Z_n = \frac{X_n}{\sqrt{n}}$$

2.3

$$\begin{aligned}
M_{Z_n}(t) &= \left[M_{\frac{\delta_1}{\sqrt{n}}}(t)\right]^n \\
&= \left[\frac{1}{2}\left(e^{\frac{t}{\sqrt{n}}} + e^{\frac{-t}{\sqrt{n}}}\right)\right]^n
\end{aligned}$$

2.4

$$\begin{aligned}
M_{Z_\infty}(t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot e^{tx} dx \\
&= e^{\frac{1}{2}t^2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx \\
&= e^{\frac{1}{2}t^2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy && (y = x - t) \\
&= e^{\frac{1}{2}t^2}
\end{aligned}$$

2.5

$$\begin{aligned}
\ln\left(\lim_{n\to\infty} M_{Z_n}(t)\right) &= \lim_{n\to\infty} \ln M_{Z_n}(t) && (\ln(\cdot) \text{ is continuous}) \\
&= \lim_{u\to 0^+} \frac{\ln\left(\frac{e^{ut}+e^{-ut}}{2}\right)}{u^2} && (u = \tfrac{1}{\sqrt{n}}) \\
&= \lim_{u\to 0^+} \frac{te^{ut} - te^{-ut}}{2u \cdot \underbrace{(e^{ut} + e^{-ut})}_{\text{Converges to 2}}} && (\text{L'Hospital's Rule}) \\
&= \lim_{u\to 0^+} \frac{e^{ut} - e^{-ut} + 2t^2 e^{-ut}}{4} && (\text{L'Hospital's Rule}) \\
&= \frac{1}{2}t^2.
\end{aligned}$$

Thus $\lim_{n\to\infty} M_{Z_n}(t) = e^{\frac{1}{2}t^2} = M_{Z_\infty}(t)$

Problem 3

3.1

$$
\begin{aligned}
\mathbb{P}[Z_2 = 4] &= \mathbb{P}[Z_{1,1} = 2, Z_{1,2} = 2] \cdot \mathbb{P}[Z_1 = 2] \\
&= 0.3^2 \times 0.3 \\
&= 0.027
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}[Z_2 = 3] &= \mathbb{P}[Z_{1,1} = 1, Z_{1,2} = 2] \cdot \mathbb{P}[Z_1 = 2] \\
&\quad + \mathbb{P}[Z_{1,1} = 2, Z_{1,2} = 1] \cdot \mathbb{P}[Z_1 = 2] \\
&= 0.6 \times 0.3 \times 0.3 + 0.3 \times 0.6 \times 0.3 \\
&= 0.108
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}[Z_2 = 2] &= \mathbb{P}[Z_{1,1} = 1, Z_{1,2} = 1] \cdot \mathbb{P}[Z_1 = 2] \\
&\quad + \mathbb{P}[Z_{1,1} = 0, Z_{1,2} = 2] \cdot \mathbb{P}[Z_1 = 2] \\
&\quad + \mathbb{P}[Z_{1,1} = 2, Z_{1,2} = 0] \cdot \mathbb{P}[Z_1 = 2] \\
&\quad + \mathbb{P}[Z_{1,1} = 2] \cdot \mathbb{P}[Z_1 = 1] \\
&= 0.6^2 \times 0.3 + 0.1 \times 0.3 \times 0.3 \\
&\quad + 0.3 \times 0.1 \times 0.3 + 0.3 \times 0.6 \\
&= 0.306
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}[Z_2 = 1] &= \mathbb{P}[Z_{1,1} = 1, Z_{1,2} = 0] \cdot \mathbb{P}[Z_1 = 2] \\
&\quad + \mathbb{P}[Z_{1,1} = 0, Z_{1,2} = 1] \cdot \mathbb{P}[Z_1 = 2] \\
&\quad + \mathbb{P}[Z_{1,1} = 1] \cdot \mathbb{P}[Z_1 = 1] \\
&= 0.6 \times 0.1 \times 0.3 + 0.1 \times 0.6 \times 0.3 \\
&\quad + 0.6 \times 0.6 \\
&= 0.396
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}[Z_2 = 0] &= \mathbb{P}[Z_{1,1} = 0, Z_{1,2} = 0] \cdot \mathbb{P}[Z_1 = 2] \\
&\quad + \mathbb{P}[Z_{1,1} = 0] \cdot \mathbb{P}[Z_1 = 1] \\
&\quad + \mathbb{P}[Z_1 = 0] \\
&= 0.1^2 \times 0.3 + 0.1 \times 0.6 + 0.1 \\
&= 0.163
\end{aligned}
$$

$$
\therefore \quad \mathbb{P}[Z_2 = i] = \begin{cases} 0.027, & i = 4 \\ 0.108, & i = 3 \\ 0.306, & i = 2 \\ 0.396, & i = 1 \\ 0.163, & i = 0 \\ \ \ 0 \ \ , & \text{else.} \end{cases}
$$

3.2

Let Z be a random variable with the following probability mass function

$$P(Z=i)=\begin{cases}0.3, & i=2\\ 0.6, & i=1\\ 0.1, & i=0\\ \ 0\ , & \text{else.}\end{cases}$$

Define $G_Z(s)=\mathbb{E}[s^Z], G_{Z_n}(s)=\mathbb{E}[s^{Z_n}]$ for $0\leq s\leq 1$. We claim that

$$G_{Z_n}(s)=\underbrace{G_Z\circ\cdots\circ G_z}_{n\text{ times}}(s).$$

We proceed by induction on n. When $n=1$, the results holds as $Z_1\ Z$. Suppose the statement is true for some $n=k_0$. Then

$$\begin{aligned}
G_{Z_{k_0+1}}(s)&=\mathbb{E}\left[\ \mathbb{E}\left[s^{Z_{k_0+1}}\mid Z_{k_0}\right]\ \right] && \text{(Tower Property)}\\
&=\mathbb{E}\left[\ \mathbb{E}\left[s^{\sum_{j=1}^{Z_{k_0}} Z_{k_0,j}}\middle| Z_{k_0}\right]\ \right]\\
&=\mathbb{E}\left[\left(\mathbb{E}\left[s^Z\right]\right)^{Z_{k_0}}\right] && (\{Z_{k_0,j}\}_{j=1}^{Z_{k_0}}\text{ i.i.d.})\\
&=G_{Z_{k_0}}\left(G_Z(s)\right)\\
&=\underbrace{G_Z\circ\cdots\circ G_z}_{k_0\text{ times}}(G_Z(s)) && \text{(Induction Hypothesis)}\\
&=\underbrace{G_Z\circ\cdots\circ G_z}_{k_0+1\text{ times}}(s)
\end{aligned}$$

Thus the statement is true for all $n\in\mathbb{N}$. Note that

$$G_{Z_n}(s)=\sum_{k=0}^{\infty}s^k\cdot\mathbb{P}[Z_n=k],$$

therefore $G_{Z_n}(0)=\mathbb{P}[Z_n=0]$, and we have

$$\mathbb{P}[Z_n=0]=\underbrace{G_Z\circ\cdots\circ G_Z}_{n\text{ times}}(0).$$

We know that $G_Z(s)=0.1+0.6s+0.3s^2$, and using a computer, $\mathbb{P}[Z_{10}=0]=\underbrace{G_Z\circ\cdots\circ G_Z}_{10\text{ times}}(0)=0.3109$.

(Please Turn Over)

3.3

```r
n_sim = 100000
p_0 = 0.1
p_1 = 0.6
p_2 = 0.3
# This function returns Z_n (n = 1,2,3, ...)
branch = function(n,p_0,p_1,p_2){
  if (n==1){
    # The "1" here represents Z_0 = 1
    return (sample(c(0,1,2), 1, replace=TRUE, prob=c(p_0,p_1,p_2)))
  }
  # This is Z_{n-1}
  prev = branch(n-1,p_0,p_1,p_2)
  if (prev == 0){
    return (0)
  }
  # This is (Z_{n-1,1}, ...,Z_{n-1,Z_{n-1}})
  offsprings = sample(c(0,1,2), prev, replace=TRUE, prob=c(p_0,p_1,p_2))
  # Return Z_n
  return (sum(offsprings))
}
# These are the results of Z_10 from the 100000 simulations
Z_10 = replicate(n_sim, branch(10,p_0,p_1,p_2))

# Probability of extinction (Theoretical)
the_prob = 0.3109

# Probability of extinction (Simulation)
sim_prob = mean(Z_10==0)

cat(" The probability of extinction by the 10th generation", "\n", "\n",
      "Probability (Theoretical) :", round(the_prob, 4), "\n",
      "Probatility (Simulation)  :", round(sim_prob, 4))
##  The probability of extinction by the 10th generation
##
##  Probability (Theoretical) : 0.3109
##  Probatility (Simulation)  : 0.3101
```

End of Homework