Complexity Project

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Abstract

This aim of this project is to study the Oslo model, one of the simplest models to display the properties of self organised criticality. We investigate this property by looking into scaling and data collapse of the model and through that we will estimate the value of some important parameters.

1 Introduction

This aim of this project is to study the Oslo model, one of the simplest models to display the properties of self organised criticality.

2 Task 1

We implemented the system as suggested above. To check whether the program is working, we devise a test in which we set z_{th} can only be one and we expect all z to be one under this condition. The programme performs as expected which means it is relaxing properly. Apart from that we also use the test suggested in the project notes which is to measure average height of system with size L=16,32 and we measure values of 26.5 and 53.9 as suggested by the notes. To conclude, the simulation is implemented correctly.

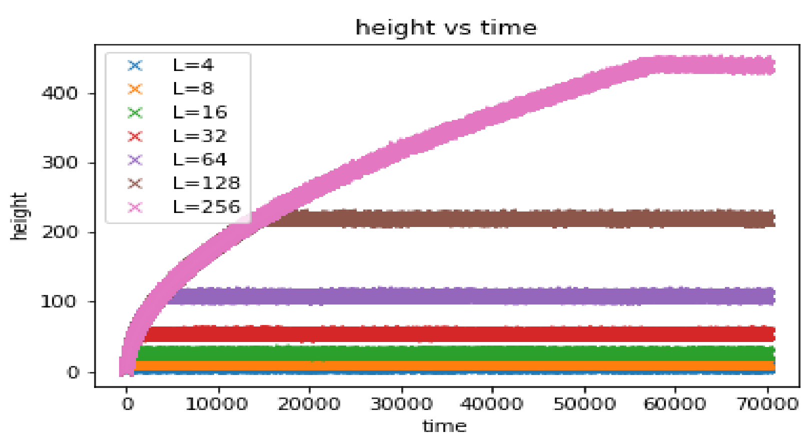
3 Task 2

3.1 Part a

height of the pile, $h(t; L)$ is given as

$$h(t; L) = \sum_{i=1}^{L} z_i(t)$$

To investigate the behaviour of the pile, we measure height for various system sizes and plot it versus time which is defined as the number of grains added as below

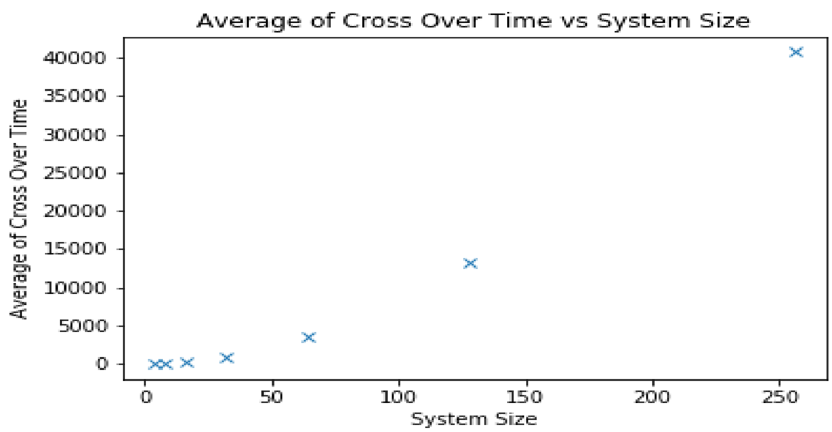


When the system reaches recurrent state, height fluctuates but stay in a fairly fixed range. Similarly, height fluctuates in transient state but it has a strictly increasing trend until it reaches recurrent state. Height increases with system size and it takes longer time for system with larger size to reach recurrent state.

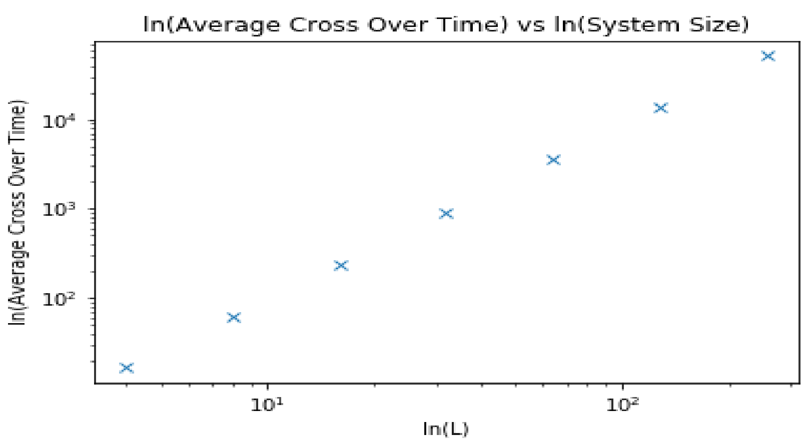
3.2 Part b

We will now investigate the relationship of cross over time, $t_c(L)$ with system size L. Cross over time is the number of grains added before an added grain causes an grain to leave the system for the first time. The graph below is average of cross over time for 10000 iterations plotted against system size

$$t_c(L) = \sum_{i=1}^{L} z_i(t)i$$



Qualitatively, we could vaguely see that $< t_c(L) >$ follows a power law as L increases. To further investigate this scaling, we plot $ln(< t_c(L) >)$ against $ln(L)$.



3.3 Part c

Individual z_i are independent of each other as the only requirement for z_i is that z_i is smaller or equal to z_{th} and z_{th} is chosen to be either 1 or 2 randomly. Thus average of z_i, $< z_i >$ is fixed and independent of each other.

$$< h(t; L) >=< \sum_{i=1}^{L} z_i(t) >$$

$$< h(t; L) >= \sum_{i=1}^{L} < z_i(t) >$$

$$< h(t;L) >=< z_i(t) > L$$

Therefor we conclude that $< h(t;L) >$ scales linearly with L for $L >> 1$. As boundary effects become significantly large if the system size is small.

Now we try to find out how $< t_c(L) >$ scale with L.

$$< t_c(L) >= \sum_{i=1}^{L} < z_i(t) > i$$

$$< t_c(L) >=< z_i(t) > \frac{L(L+1)}{2}$$

We can see that $< t_c(L) >$ does not scale with L^2 strictly but when system size get large, $(L+1) \approx L$ so $< t_c(L) >$ is almost directly proportional to L^2 when $L >> 1$

3.4 Part d

Define the processed height $h^{-}(t;L)$

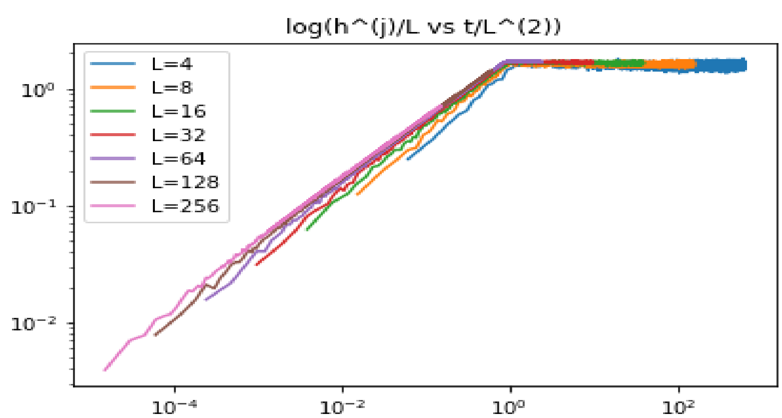
$$h^{-}(t;L) = \frac{1}{M} \sum_{j=1}^{M} h^{j}(t;L)$$

where M is the number of realisations and h^{-} is the height of that realisation at time t.

we assume that $h^{-}(t;L)$ can be written as a scaling function such that

$$h^{-}(t;L) = something \times F(argument)$$

As we know from above that average height is proportional to L.The multiplying factor must be L. For the argument, it must be t/t_c but t_c is proportional to L^2 so we could use t/L^2. To investigate how F behaves we divide the whole equation by L and take logarithm on both sides and plot them against each other as shown below.



We can see that $F(x)$ follows a power law when $x < 1$ as the height is increasing so that it can reach the attractor state later but when $x > 1$ $F(x)$ is constant which stops height from exceeding certain limits for each systems. Let's assume the intercept of the graph is zero and the gradient is m. We could deduce the following by massaging the function.

$$logF = mlog(\frac{t}{L^2})$$

$$F = (\frac{t}{L^2})^m$$

and $F = h^-(t;L)/L$

$$h^-(t;L) = L \times (\frac{t}{L^2})^m$$

We can see that $h^-(t;L)$ is directly proportional to t^m during the transient. By looking at the graph m should be in the scale of 10^0.

3.5 Part e

Firstly we measure the average height when the system has reached the attractor of the dynamics using the equations below.

$$< h(t; L) >_t = \lim_{T\to\infty} \frac{1}{T} \sum_{t=t_0+1}^{t_0+T} h(t; L)$$

where $t_0 > t_c(L)$ and $t_c(L)$ is times when the system reach attractor state.

Secondly we assume $< h(t; L) >_t$ has a scaling corrections form as shown below

$$< h(t; L) >_t = a_0 L(1 - a_1 L^{-\omega_1} + ...)$$

where we ignore higher order terms.

To estimate a_0, we rearrange the equation as shown below

$$L^{-1} < h(t; L) >= a_0(1 - a_1 L^{-\omega_1})$$

We can see that if $L >> 1$, then $a_1 L^{-\omega_1} \approx 0$. Thus

$$L^{-1} < h(t; L) >\approx a_0$$

I have measured the average height by collecting 10000 sample after the system has reached the attractor state to reduce the effect of noise.The value of $L^{-1} < h(t; L) >$ for $L = 256$ is used as an estimate for a_0 as it has the largest system size and I estimate the uncertainty by using the standard deviation of a_0 measured over 5 times.

$$a_0 = 1.7206 \pm 0.0016$$

Then we move onto to estimate the value of ω_1. We could see how we could estimate the value of ω_1 by rearranging the equations as below

$$L^{-1} < h(t; L) >= a_0(1 - a_1 L^{-\omega_1})$$

$$1 - a_0^{-1} L^{-1} < h(t; L) >= a_1 L^{-\omega_1}$$

then we take logarithm on both side

$$log(1 - a_0^{-1} L^{-1} < h(t; L) >) = log(a_1 L^{-\omega_1})$$

set $y = (1 - a_0^{-1} L^{-1} < h(t; L) >)$

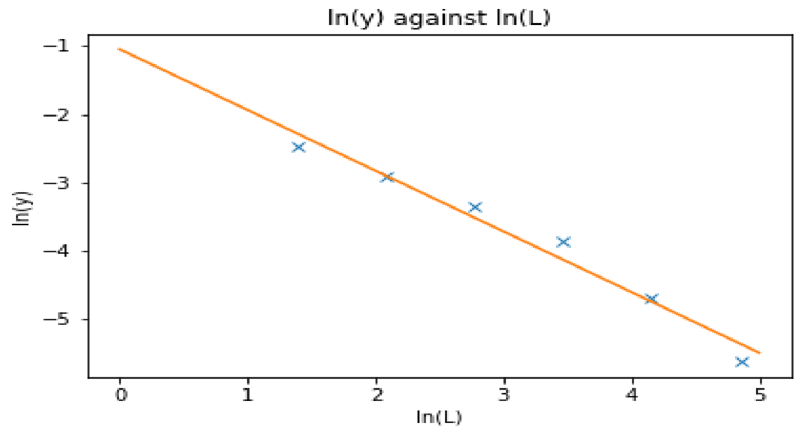
$$log(y) = log(a_1) - log(L)\omega_1$$

This implies that we could find the value of ω_1 by plotting $log(1 - a_0^{-1} L^{-1} < h(t; L) >)$ against $log(L)$ and the absolute value of the slope will be an accurate estimate for ω_1. To do this, we use linear regression to find the best fit line and its slope.

Then we arrive at an estimate for ω_1

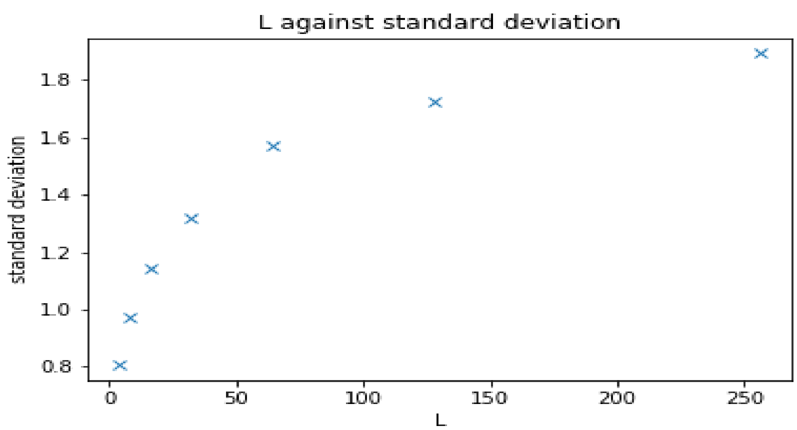
$$\omega_1 = 0.8353 \pm 0.0726$$

the estimate for ω_1 is going to be not very accurate as it depends on the value of a_0



3.6 Part f

Then we try to investigate how standard deviation of $< h(t;L) >_t$ scale with L. We use the same dataset from task 2e and we plot σ_h against L. It seems that σ_h follow a power law.



To further investigate the scaling we plot $log(\sigma_h)$ against $log(L)$

We then use linear regression to find the best fit line and its slope to be around 0.2075 which implies that σ_h is diminishing as L increases. From task 2e, we know that

$$< h(t;L) >_t = a_0 L(1 - a_1 L^{-\omega_1} + ...)$$

and as $L >> 1$

$$< h(t;L) >_t \approx a_0 L$$

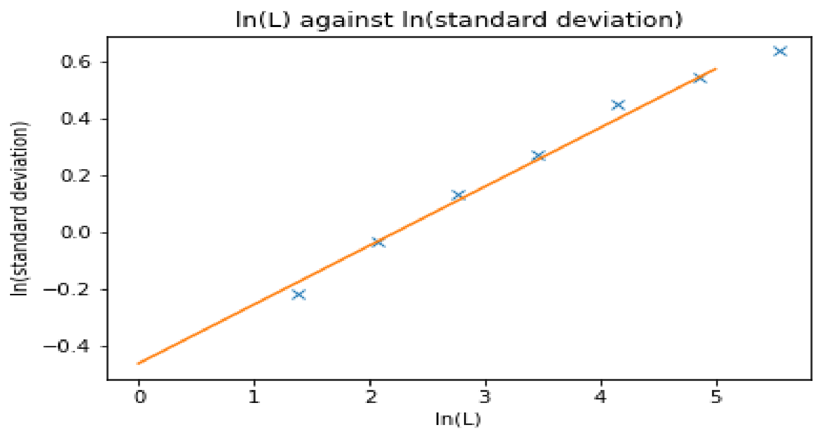
This means that average height scales linearly with L which implies average slope will be 1 and its standard deviation will be smaller as σ_h is diminishing as L increases.

3.7 Part g

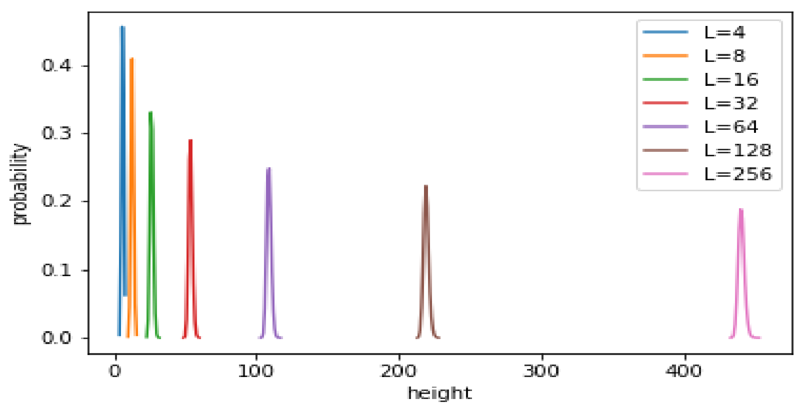
The height probability is defined as

$$P(h;L) = \frac{n_h}{N}$$

where N is the total no. of observed configurations and n_h is no. of observed configurations with height h



As $L >> 1$, we expect $P(h; L)$ to have a gaussian shape distribution as it is directly proportional to the sum of L independent z according to central limit theorem.

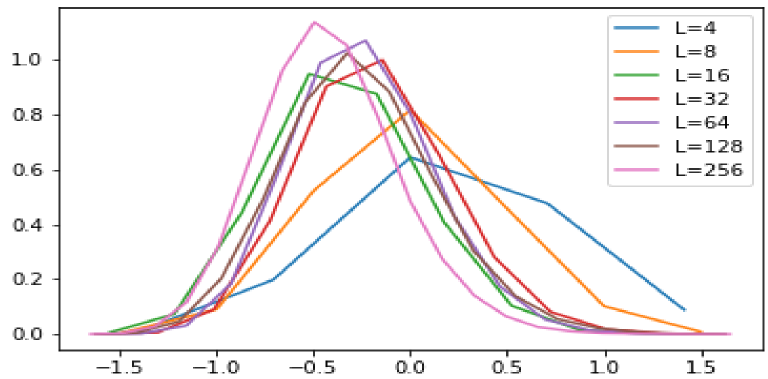


To produce a data collapse, we must make use of the form of gaussian distribution to make each of the graph fall on top of each other.

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

To perform data collapse we could plot $\mu \times P(x)$ against $\frac{x-\mu}{\sigma}$

We can see that the graph with small system size do not collapse as the number of independent z is not enough for central limit theorem to be valid. Thus they do not have a gaussian distribution. For σ_h, the average of L independent variables should have a standard deviation which is reduced by a factor of $\sqrt{L}$ and the graph of task2f shows that.



4 Task 3

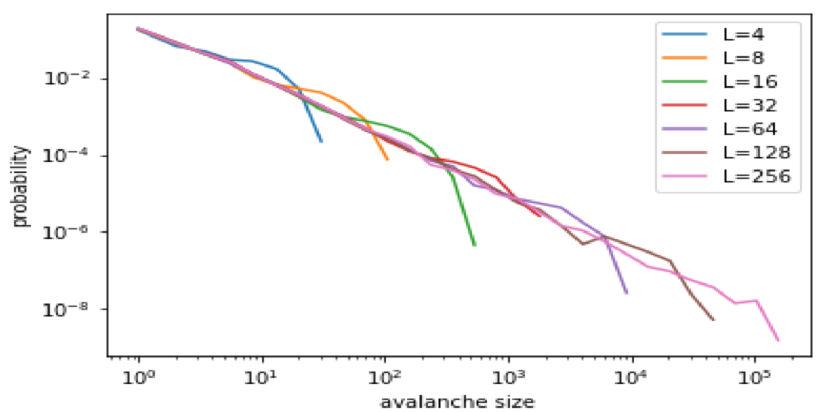
Firstly, we need to define avalanche size probability $P_N(s;L)$

$$P_N(s;L) = \frac{n_s}{N}$$

where n_s is no. of avalanches with size s in a system of size L and N is the total no. of avalanches.

4.1 Part a

We set all the system to their attractor states and collect 10000 samples for avalanche size the we use a logbin function which bins these frequencies in logarithmically increasing bin sizes controlled by the scale parameter which I set to be 1.5.



We can see that probability drops with avalanche size and avalanche with smallest s has the highest probability. Apart from that, avalanche with large s are only restricted for large systems and small systems are kept to have some s

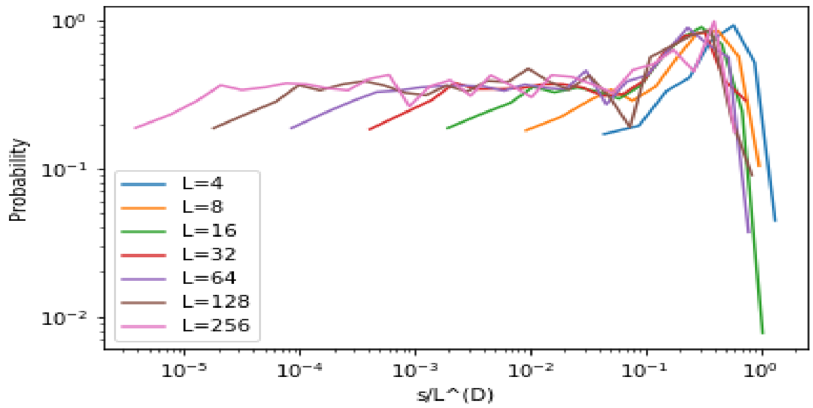
4.2 Part b

We are testing whether probability follows the finite size scaling ansatz.

$$P_N(s;L) = C \times s^{-\tau_s} G(s/L^D)$$

$$P_N(s;L)/(C \times s^{-\tau_s}) = G(s/L^D)$$

We plot the left hand side of the equation against $\frac{s}{L^D}$ using the logbin data generated in part a. We expect a data collapse for the right value of D and τ_s as it is essentially showing us G. We know that τ_s controls the vertical part



and D controls the horizontal part and we found τ_s to be 1.55 and D to be 2.55

4.3 Part c

Firstly we need to define $< s_k >$

$$< s^k >= \lim_{T\to\infty} \frac{1}{T} \sum_{t=t_0+1}^{t_0+T} s_t^k$$

where $s_t$is the measured avalanche size at time t and $t_0 > t_c(L)$

Theoretically $< s^k >$ can be expressed in the following form.

$$< s^k >= \sum_{s=0}^{\infty} s^k P_N(s;L)$$

Assuming the $P_N(s;L)$ follows the finite size scaling form given in task 3b. We could simplify the equation further by using an integral to approximate a sum.

$$< s^k >= \sum_{s=0}^{\infty} s^k C \times s^{-\tau_s} G(s/L^D)$$

$$< s^k >= \sum_{s=0}^{\infty} s^k C \times s^{-\tau_s} G(s/L^D)$$

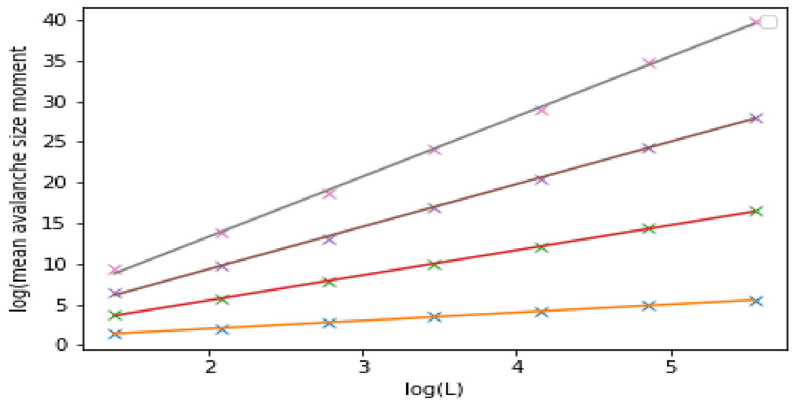
$$< s^k >= C \int_{s=0}^{\infty} s^{k-\tau_s} G(s/L^D) ds$$

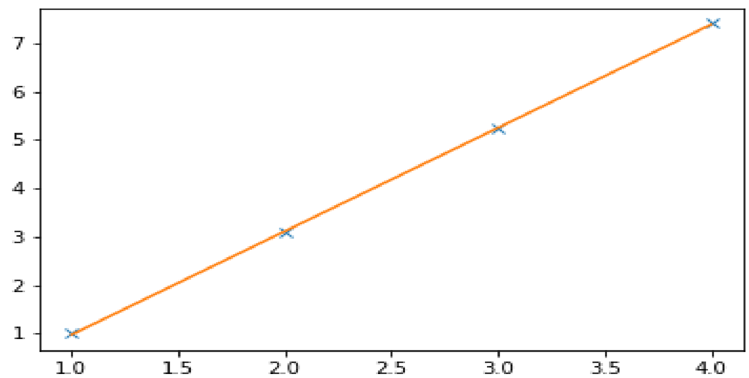
By changing variable from s to u, where

$$u = \frac{s}{L^D}$$

We arrive at the following form

$$< s^k >= CL^{D(1+k-\tau_s)} \int_{s=0}^{\infty} u^{k-\tau_s} G(u) du$$





Now the integral will be a constant if $1+k-\tau_s > 0$, if this is true then $< s^k >$ is directly proportional to $L^{D(1+k-\tau_s)}$. Thus we could estimate D and τ_s by plotting $log(< s^k >)$ against $log(L)$ and measure their slope.

The slope of these 4 lines are the estimates of $1 + k - \tau_s > 0$, we could then plot these slopes against k. This will yield a straight line with value of slope,D and a x-intercept of $\tau_s - 1$

We find that $D = 2.135 \pm 0.014$ and $\tau_s = 1.543$. I think the results from moment analysis is more accurate than those I collected using data collapse as moment analysis is based on averages which reduces random error and it has well defined procedure unlike data collapse that we could not quantify how good the collapse is.