Exercise2

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See in the last 4 pages

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(a^*) Cauchy

$F_X(x) = \int f_X(x)dx = \int \frac{1}{\pi(1+x^2)}$dx

Let x = tanθ , then:

F(θ) = $\int \frac{1}{\pi(1+\frac{sin\theta^2}{cos\theta^2})} dtan\theta$

= $\frac{1}{\pi}\int \frac{1}{1+\frac{sin\theta^2}{cos\theta^2}} \frac{1}{cos\theta^2}d\theta$

=$\frac{1}{\pi}\int 1d\theta$

= $\frac{1}{\pi}$(θ+C)

As θ = arctan(x), then:

$F_X(x)$ = $\frac{1}{\pi}$(arctan(x)+C)

According to the property of CDF, $\lim_{x\to\infty}$F(x) = 1.

As $\lim_{x\to\infty}$arctan(x) = $\frac{\pi}{2}$, then C = $\frac{\pi}{2}$.

$F_X(x) = \frac{arctan(x)}{\pi} + \frac{1}{2}$

(b^*) Logistic

$F_X(x) = \int f_X(x)dx = \int \frac{e^{-x}}{(1+e^{-x})^2}$dx

$=\frac{1}{1+e^{-x}}$+C

As $\lim_{x\to\infty} \frac{1}{1+e^{-x}} = 1$, then C = 0.

$F_X(x) = \frac{1}{1+e^{-x}}$

(c^*) Pareto

$F_X(x) = \int f_X(x)dx = \int \frac{a-1}{(1+x)^a}$dx

$= -(1+x)^{1-a}$ + C

As $\lim_{x\to\infty} -(1+x)^{1-a} = 0$, then C = 1.

$F_X(x) = -(1+x)^{1-a} + 1 \;\; (x > 0)$

$F_X(x) = 0 \;\; (x <= 0)$

(d^*) Weibull

$F_X(x) = \int f_X(x)dx = \int c^{\tau-1}e^{-cx^\tau}dx$

$= -e^{-cx^\tau}$ + C
As $\lim_{x\to\infty} -e^{-cx^\tau} = 0$, then C = 1.

$F_X(x) = -e^{-cx^\tau} + 1 \;\; (x = 1, 2, ...)$

$F_X(x) = 0 \;\; (x <= 0)$

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(a^*) Gamma Distribution

Xs from Gamma Distribution are continuous. Therefore, $E(X) = \int xf_X(x)dx$.

$\int_{-\infty}^{\infty} xf_X(x)dx = \int_0^\infty x\frac{e^{-kx}x^{r-1}k^r}{(r-1)!}dx + \int_{-\infty}^0 0dx = \int_0^\infty \frac{e^{-kx}x^{r-1}k^r}{(r-1)!}dx$

$= \int_0^\infty x\frac{e^{-kx}x^{r-1}k^r}{(r-1)!}dx = \int_0^\infty \frac{e^{-kx}x^rk^r}{(r-1)!}dx = \frac{r}{k}\int_0^\infty \frac{e^{-kx}x^rk^{r+1}}{(r)!}dx = \frac{r}{k}$
E(X)=$\frac{r}{k}$

Var(X)=$E(x^2)-E(x)^2$

$E(X^2)=\int_{-\infty}^{\infty}x^2 f_X(x)dx$

Likewise, $E(X^2)=\frac{r^2+r}{k^2}$. Var(X)=$\frac{r}{k^2}$

E(X)=$\frac{r}{k}$, Var(X)=$\frac{r}{k^2}$

($b*$) Poisson Distribution

E(X)=$\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{(x-1)!}$=$\lambda\sum_{x=1}^{\infty}\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}+0$=$\lambda$

$E(X^2)=\sum_{x=0}^{\infty}\frac{(x-1)e^{-\lambda}\lambda^x+e^{-\lambda}\lambda^x}{(x-1)!}$=$\lambda^2$+$\lambda$

Var(X)=λ

(c*)Pareto Distribution

E(X)=$\int_0^{\infty}\frac{x(a-1)}{(1+x)^a}dx$=$\int_0^{\infty}\frac{x(a-1)+(a-1)-(a-1)}{(1+x)^a}dx$=

=$\int_0^{\infty}\frac{(a-1)}{(1+x)^{a-1}}-\frac{a-1}{(1+x)^a}dx$

=$\frac{a-1}{2-a}(1+x)^{2-a}|_0^{\infty}$+$(1+x)^{1-a}|_0^{\infty}$=$\frac{1}{a-2}$

$E(X^2)$=$\int_0^{\infty}\frac{a-1}{(1+x)^{a-2}}-\frac{2(a-1)}{(1+x)^{a-1}}+\frac{a-1}{(1+x)^a}dx$

=$[\frac{a-1}{3-a}(1+x)^{3-a}-\frac{2(a-1)}{2-a}(1+x)^{2-a}-(1+x)^{1-a}]|_0^{\infty}$=$\frac{2}{(3-a)(2-a)}$

Var(X)=$\frac{1-a}{(3-a)(2-a)^2}$

(d)Negative Binomial

$P(X=x)=\frac{(a+x-1)!}{x!(a-1)!}[\frac{b}{1+b}]^a[\frac{1}{1+B}]^x$

For convenience, I substitute $\frac{b}{1+b}$ with p, then the equation above can be expressed as:

$P(X=x)=C_{a+x-1}^{x}p^a(1-p)^x$

$E(X)=\sum_0^{\infty}xP(x)=\sum_1^{\infty}\frac{(a+x-1)!}{(x-1)!(a-1)!}p^a(1-p)^x$

(As $0*P(0)=0$, summation starting from is equivalent to the original one)

$\sum_{x=1}^{\infty}\frac{(a+x-1)!}{(x-1)!(a-1)!}p^a(1-p)^x$=$\sum_{x=1}^{\infty}aC_{a+x-1}^{x-1}p^a(1-p)^x$

Let t=x-1, then

$\sum_1^\infty aC_{a+x-1}^{x-1}p^a(1-p)^x$=$\sum_{t=0}^\infty aC_{a+t}^t p^a(1-p)^{t+1}$

=$\frac{a(1-p)}{p}\sum_{t=0}^\infty C_{a+t}^t p^{a+1}(1-p)^t$

$C_{a+t}^t p^{a+1}(1-p)^t$ is a density function of negative binomial. Hence, $\sum_{t=0}^\infty C_{a+t}^t p^{a+1}(1-p)^t$=1.

Finally, $E(X)=\frac{a(1-p)}{p}=\frac{a}{b}$

$Var(X)=E(X^2)-E(X)^2$

$E(X^2)=\sum_{x=0}^\infty x^2\frac{(a+x-1)!}{x!(a-1)!}p^a(1-p)^x$

=$\sum_{x=1}^\infty \frac{x(a+x-1)!}{(x-1)!(a-1)!}p^a(1-p)^x$

=$\sum_{x=1}^\infty axC_{a+x-1}^{x-1}p^a(1-p)^x$

Let t=x-1, then

$\sum_{x=1}^\infty axC_{a+x-1}^{x-1}p^a(1-p)^x$=$\sum_{t=0}^\infty a(t+1)C_{a+t}^t p^a(1-p)^{t+1}$

=$\sum_{t=0}^\infty \frac{a(t+1)(1-p)}{p}C_{a+t}^t p^{a+1}(1-p)^t$

=$\frac{a(1-p)}{p}\sum_{t=0}^\infty (t+1)C_{a+t}^t p^{a+1}(1-p)^t$

$\sum_{t=0}^\infty (t+1)C_{a+t}^t p^{a+1}(1-p)^t$=$\sum_{t=0}^\infty C_{a+t}^t p^{a+1}(1-p)^t$+$\sum_{t=0}^\infty tC_{a+t}^t p^{a+1}(1-p)^t$

$\sum_{t=0}^\infty C_{a+t}^t p^{a+1}(1-p)^t$=1

$\sum_{t=0}^\infty tC_{a+t}^t p^{a+1}(1-p)^t$=$\frac{(1+p)(a+1)}{p}$

(The calculating process is slightly omitted here, as the procedure is almost the same as that in calculating E(X))

Just combine the two parts of $\sum_{t=0}^\infty (t+1)C_{a+t}^t p^{a+1}(1-p)^t$,

$E(X^2)=\frac{a(1-p)}{p}\sum_{t=0}^\infty (t+1)C_{a+t}^t p^{a+1}(1-p)^t$=$\frac{a(1-p)}{p}(\frac{(1-p)(a+1)}{p}+1)$

$Var(X)=E(x^2)-E(X)^2=\frac{a(1-p)}{p}(\frac{(1-p)(a+1)}{p}+1)-[\frac{a(1-p)}{p}]^2=\frac{a(1-p)}{p^2}$

Replace p with the original expression $\frac{b}{1+b}$,

$Var(X) = \frac{a(b+1)}{b^2}$

$E(X) = \frac{a}{b}$, $Var(X) = \frac{a(b+1)}{b^2}$.

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Exponential Distribution

As both of x and y are greater than 0, then $P(x + y > X, X > x) = P(x + y > X)$.

$P(x + y > X|x > X) = \frac{P(x+y>X)}{P(x>X)}$.

$P(x > X) = 1 - \int_0^x \frac{1}{\lambda} e^{\frac{-x}{\lambda}} dx {=} \int_x^\infty \frac{1}{\lambda} e^{\frac{-x}{\lambda}} dx$

$= - e^{\frac{-x}{\lambda}} |_x^\infty {=} e^{\frac{-x}{\lambda}}$

Likewise, $P(y > X) = e^{\frac{-y}{\lambda}}$, and $P(x + y > X) = e^{\frac{-(x+y)}{\lambda}}$

$P(x + y > X|x > X) = \frac{P(x+y>X)}{P(x>X)} = P(y > X) = e^{\frac{-y)}{\lambda}}$

Geometric Distribution

$P(x > X) = 1 - \sum_1^x q^{x-1} p$

$\sum_1^x q^{x-1} p$ is a geometric sequence whose sum equals to $\frac{(q^x-1)p}{q-1}$.

$P(x > X) = 1 - \frac{(q^x-1)p}{q-1}$

Likewise, $P(y > X) = 1 - \frac{(q^y-1)p}{q-1}$, and $P(x + y > X) = 1 - \frac{(q^{x+y}-1)p}{q-1}$

Substitute p with 1-q, then we can get:

$P(x > X) = q^x, P(y > X) = q^y, andP(x + y > X) = q^{x+y}$.

It is obvious that $P(x + y > X|x > X) = \frac{P(x+y>X)}{P(x>X)} = P(y > X) = q^y$

So far, I have shown that if X has either the exponential distribution, or a geometric distribution, then X has no memory.

"No memory" means what happens in the process will not affect the probability of the next event. To draw an economics analogy, suck cost should not affect the future decision, as it does not matter after happening.

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(a)

As Y $= e^X$, then $X = lnY$.

$F_X(x) = F_X(lny)$

$f(y) = \frac{\partial F(lny)}{\partial y} = \frac{f_X(lny)}{y} = \frac{1}{y\sqrt{2\pi}\sigma}\exp(-\frac{(lny-\mu)^2}{2\sigma^2})$

$E(Y^n) = E(e^{nx}) = \int e^{xn} f(x)dx$

$= \int e^{nx}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}}dx$

$= \int \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{2\sigma^2 nx - x^2 - \mu^2 + 2\mu x}{2\sigma^2}}dx$

$= \int \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{(\sigma^2 n+\mu)^2 - (x-\sigma^2 n-\mu)^2 - \mu^2}{2\sigma^2}}dx$

$= \frac{(\sigma^2 n+\mu)^2-\mu^2}{e^{2\sigma^2}} \int \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\sigma^2 n-\mu)^2}{2\sigma^2}}dx$

$\int \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\sigma^2 n-\mu)^2}{2\sigma^2}}dx = 1$

Therefore, $E(Y^n) = e^{\frac{\sigma^4 n^2 + 2\sigma^2 n\mu}{2\sigma^2}}$

$E(Y) = e^{\frac{\sigma^2}{2}+\mu}$, $E(Y^2) = e^{2\sigma^2+2\mu}$

$Var(Y) = E(X^2) - E(Y)^2 = e^{\sigma^2+2\mu}(e^{\sigma^2} - 1)$.

Finally, $E(Y) = e^{\frac{\sigma^2}{2}+\mu}$, $Var(Y) = e^{\sigma^2+2\mu}(e^{\sigma^2} - 1)$.

(b) See in the last 4 pages