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PHY-234 Lab

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Mechanics Capstone: Modeling a Double Pendulum

Throughout the semester, Physics Mechanics labs have focused on modeling systems by numerically solving differential equations. One particularly interesting lab was the Damped Driven Pendulum (DDP), as it exhibited chaotic behavior. Drawing on inspiration from the DDP, we challenged ourselves to analyze another chaotic system: the Double Pendulum.

This setup, as illustrated on the right by Figure 11.9 from John Taylor's *Classical Mechanics,* consists of a pendulum with a mass $\boldsymbol{m}_1$ to which a second pendulum of mass $\boldsymbol{m}_2$ hangs from. The lengths and angular displacements with respect to the vertical of pendulums 1 and 2 are denoted by $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, $\boldsymbol{\varnothing}_1$, and $\boldsymbol{\varnothing}_2$ respectively. Our goals were to numerically model the Double Pendulum, make graphs of the angular displacement over time, the state space, and the Poincare sections, examine special limiting cases of the system, and animate the trajectory.

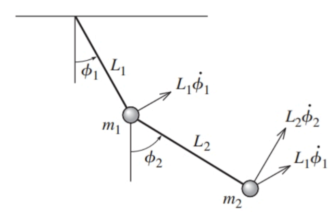


Figure 11.9 A double pendulum. The velocity of m_2 is the vector sum of the two velocities shown, separated by an angle $\phi_2 - \phi_1$.

We modeled the Double Pendulum through Lagrangian mechanics. We chose to neglect air resistance. By separating the velocity into x and y components and setting the top as U = 0, we derived the Lagrangian below (images of equations from our Python notebook):

We can express the Lagrangian as the following, where $\boldsymbol{\Delta\phi} = \boldsymbol{\phi}_1 - \boldsymbol{\phi}_2$:

$$\mathbf{L} = \tfrac{1}{2}(\mathbf{m}_1 + \mathbf{m}_2)\left(\mathbf{l}_1\,\dot{\boldsymbol{\phi}}_1\right)^2 + \tfrac{1}{2}\,\mathbf{m}_2\left(\mathbf{l}_2\,\dot{\boldsymbol{\phi}}_2\right)^2 + \mathbf{m}_2\,\mathbf{l}_1\,\mathbf{l}_2\,\dot{\boldsymbol{\phi}}_1\,\dot{\boldsymbol{\phi}}_2\cos(\boldsymbol{\Delta\phi}) + (\mathbf{m}_1 + \mathbf{m}_2)\,g\,\mathbf{l}_1\cos\boldsymbol{\phi}_1 + \mathbf{m}_2\,g\,\mathbf{l}_2\cos\boldsymbol{\phi}_2$$

These were our corresponding Euler-Lagrange equations:

$$(\mathbf{m}_1 + \mathbf{m}_2)\,\mathbf{l}_1\,\ddot{\boldsymbol{\phi}}_1 + \mathbf{m}_2\,\mathbf{l}_2\,\ddot{\boldsymbol{\phi}}_2\cos(\boldsymbol{\Delta\phi}) + \mathbf{m}_2\,\mathbf{l}_2\,\dot{\boldsymbol{\phi}}_2^2\sin(\boldsymbol{\Delta\phi}) + (\mathbf{m}_1 + \mathbf{m}_2)\,g\,\sin\boldsymbol{\phi}_1 = 0$$

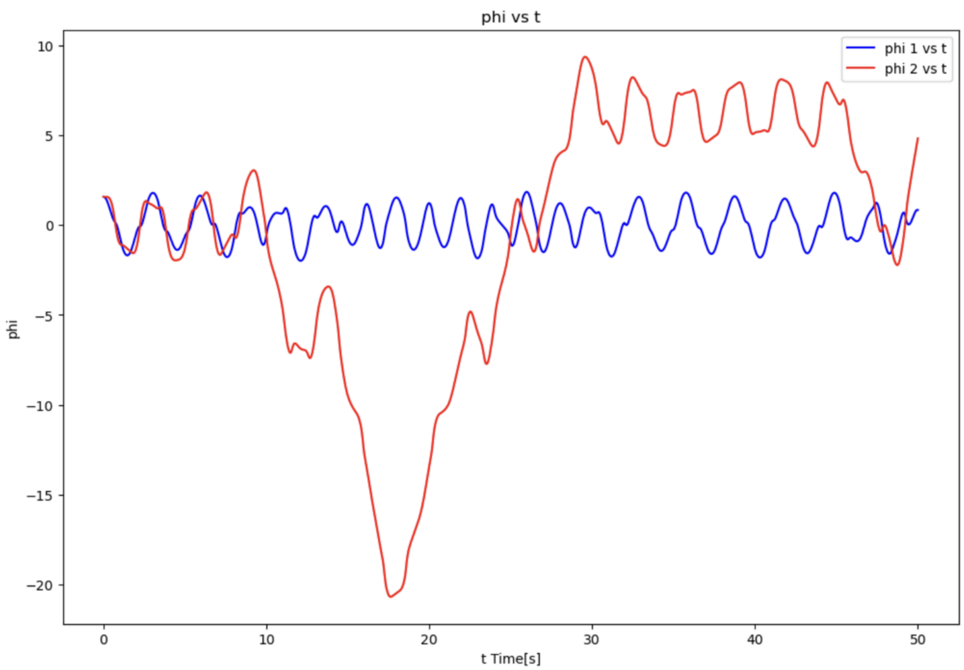
$$\mathbf{l}_1\,\ddot{\boldsymbol{\phi}}_1\cos(\boldsymbol{\Delta\phi}) + \mathbf{l}_2\,\ddot{\boldsymbol{\phi}}_2 - \mathbf{l}_1\,\dot{\boldsymbol{\phi}}_1^2\sin(\boldsymbol{\Delta\phi}) + g\,\sin\boldsymbol{\phi}_2 = 0$$

To decouple them, we used algebra and substitution to derive the following second order differential equations for the angular accelerations of both pendulums:

$$\ddot{\boldsymbol{\phi}}_1 = \frac{-\,\mathbf{m}_2\,\mathbf{l}_2\,\dot{\boldsymbol{\phi}}_2^2\,\sin(\boldsymbol{\Delta\phi}) - \mathbf{m}_2\,\mathbf{l}_1\,\dot{\boldsymbol{\phi}}_1^2\,\sin(\boldsymbol{\Delta\phi})\cos(\boldsymbol{\Delta\phi}) - (\mathbf{m}_1+\mathbf{m}_2)\,g\,\sin\boldsymbol{\phi}_1 + \mathbf{m}_2\,g\,\sin\boldsymbol{\phi}_2\cos(\boldsymbol{\Delta\phi})}{\mathbf{l}_1\left(\mathbf{m}_1+\mathbf{m}_2\sin^2(\boldsymbol{\Delta\phi})\right)}$$

$$\ddot{\boldsymbol{\phi}}_2 = \frac{\mathbf{m}_2\,\mathbf{l}_2\,\dot{\boldsymbol{\phi}}_2^2\,\sin(\boldsymbol{\Delta\phi})\cos(\boldsymbol{\Delta\phi}) + (\mathbf{m}_1+\mathbf{m}_2)\,\mathbf{l}_1\,\dot{\boldsymbol{\phi}}_1^2\,\sin(\boldsymbol{\Delta\phi}) + (\mathbf{m}_1+\mathbf{m}_2)\,g\,\sin\boldsymbol{\phi}_1\cos(\boldsymbol{\Delta\phi}) - (\mathbf{m}_1+\mathbf{m}_2)\,g\,\sin\boldsymbol{\phi}_2}{\mathbf{l}_2\left(\mathbf{m}_1+\mathbf{m}_2\sin^2(\boldsymbol{\Delta\phi})\right)}$$

By creating a ‘deriv’ function like in previous labs, we numerically solved for the equations of $Ø_1$ and $Ø_2$ as functions of time through ‘solve_ivp’. Our setup was two 5 m long pendulums with 2 kg masses on each. We started both pendulums from rest with initial angles of $Ø_1 = \pi/2$ and $Ø_2 = 0$. Here were our graphs of angular displacement from 0 to 50 seconds.



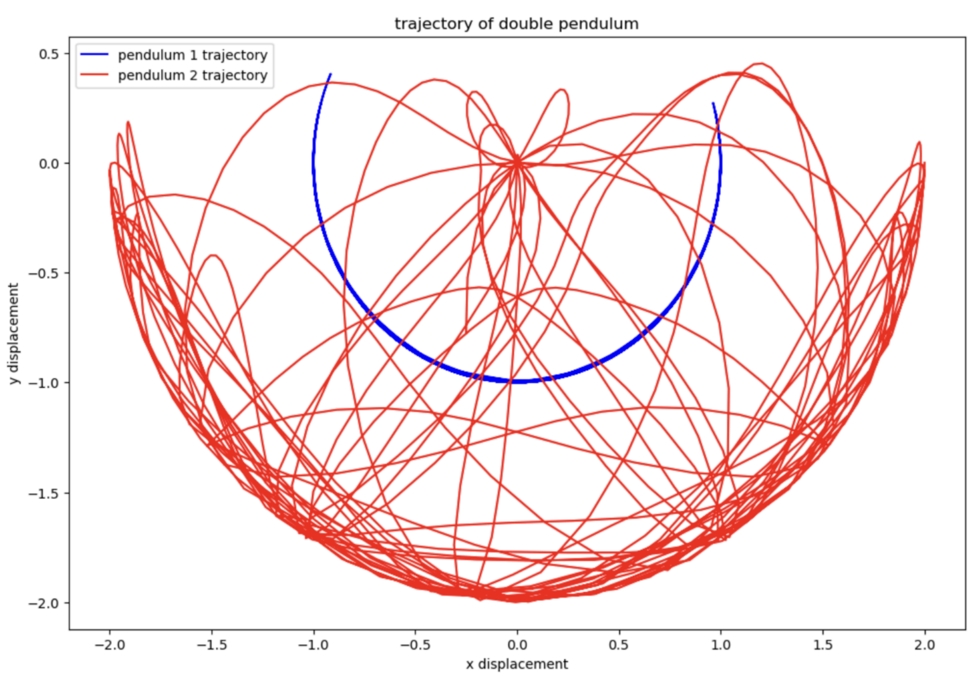
As expected, we observe nonperiodic, or chaotic behavior. $Ø_2$ reached values outside of $-\pi$ and π, which indicates the bottom pendulum made full rotations. $Ø_1$ stayed between the $-\pi$ to π range. This means the top pendulum never made a full rotation.

Below are the positions of mass 1 and mass 2, respectively, in cartesian coordinates.

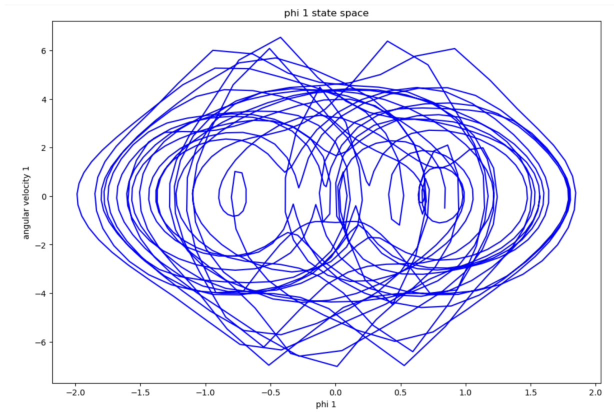
Mass 1: $(\mathbf{l}_1 \sin\phi_1,\ -\mathbf{l}_1 \cos\phi_1)$

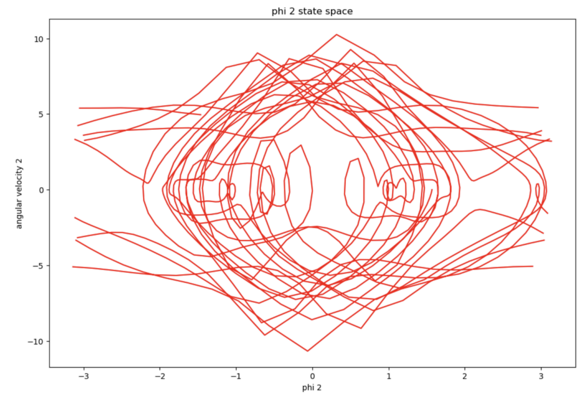
Mass 2: $(\mathbf{l}_1 \sin\phi_1 + \mathbf{l}_2 \sin\phi_2,\ -\mathbf{l}_1 \cos\phi_1 - \mathbf{l}_2 \cos\phi_2)$

Using the positions of the masses above, we graphed the trajectories of both pendulum bobs. This shows how unpredictable the paths of our mass bobs were. Through this graph, it is clear this double pendulum system was indeed chaotic.

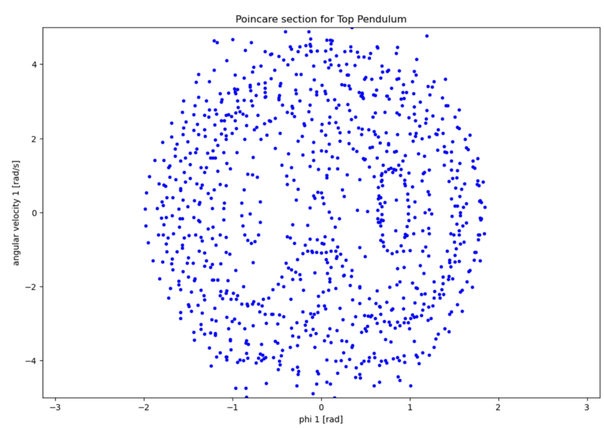


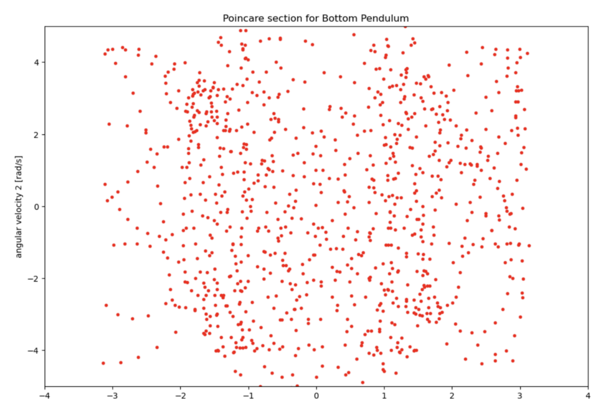
Additionally, we plotted the state space of both pendulums. Because there are no clearly defined, thicker state space orbits, there are no repeated orbits, meaning the system is chaotic.





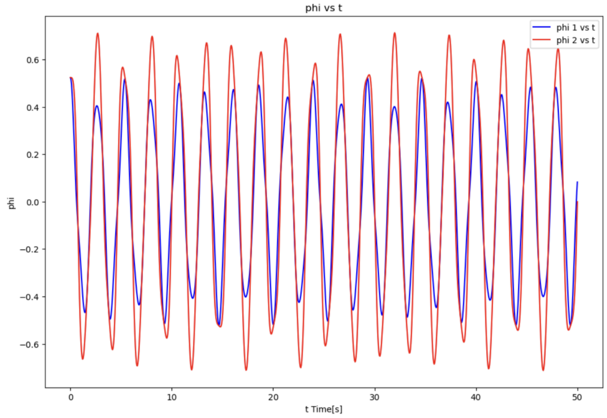
The Poincare sections further support this, as out of the 1000 points plotted over 50 seconds (every 0.05 seconds), no two points overlapped perfectly. This implies no two 0.05 second intervals had both the same angular displacement and angular velocity, a condition that can only be observed in chaotic systems.

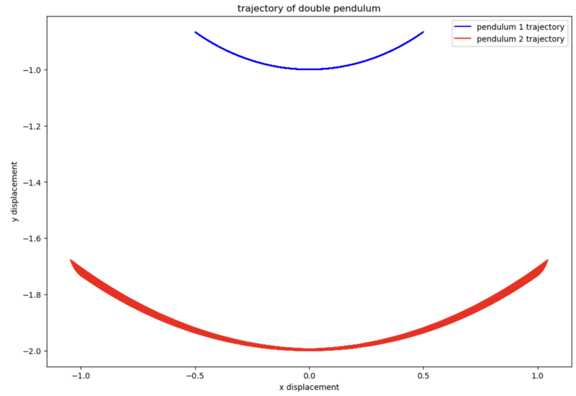




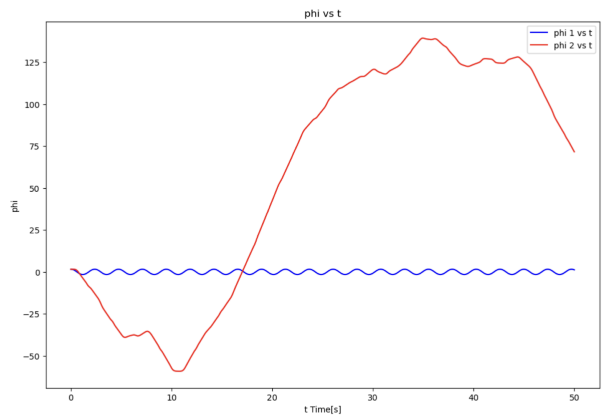
To validate our code and results, we examined two limiting cases of the system. The first case was where our initial angles were both π/6. We can estimate the equations of motion using small angle approximations, which transforms the nonlinear equations of motion into linear ODEs. Because the system does not have enough energy, this results in a double pendulum with small, nonchaotic (periodic) oscillations akin to two simple pendulums.

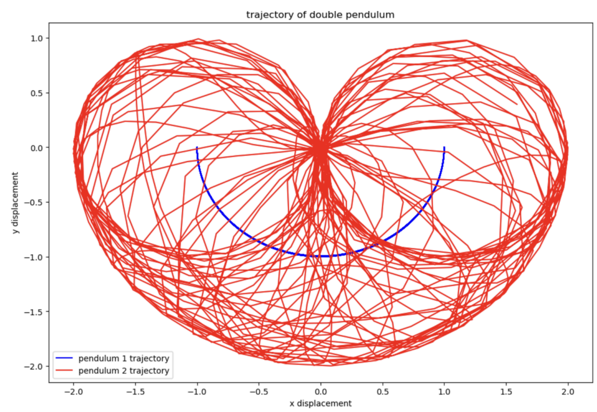
We modeled the first case using the same process as before with the ‘deriv’ and ‘solve_ivp’ functions. As predicted by the small angle approximations, the graphs of angular displacement and the trajectories showed nonchaotic, almost periodic motion (next page).





The second case we analyzed was where pendulum 1 had mass $\boldsymbol{m_1} = 2$ kg and length $\boldsymbol{l_1} =$ 5 m and pendulum 2 had mass $\boldsymbol{m_2} = 0$ kg and length $\boldsymbol{l_2} = 1$ m. Both pendulums started from rest with initial angles of $\pi/2$. Because the bottom pendulum was just a massless string, the top pendulum should behave like a simple pendulum.





Our graphs above were as predicted. The top pendulum (in blue) oscillates in a periodic manner while the bottom pendulum was in a frenzy, as it had no mass.

Through these limiting cases, we see that our code can model a double pendulum system of any pair of masses/lengths and initial conditions accurately. Our work proved this system exhibits chaotic behavior if the initial angular displacement is large enough. However, when the initial angles are relatively small, like in our first limiting case, the system behaves like two simple pendulums. Finally, when there is no mass on the bottom pendulum, the top behaves like a simple pendulum.

To summarize, our research determined what initial angle conditions led to chaotic and nonchaotic behavior. We also analyzed different systems graphically. Angular displacements beyond the -π to π range imply that the pendulum bob has made a complete revolution. No thick curves in state space or overlapping points in Poincare sections means no repetition of motion, or chaos. Under certain initial conditions and pendulum setups, a double pendulum can behave like one or two simple pendulums, as evident in our limiting cases. Through our numerical solutions to our analytically determined equations of motion, we gained a holistic, in-depth understanding of the double pendulum and its typically chaotic behavior.