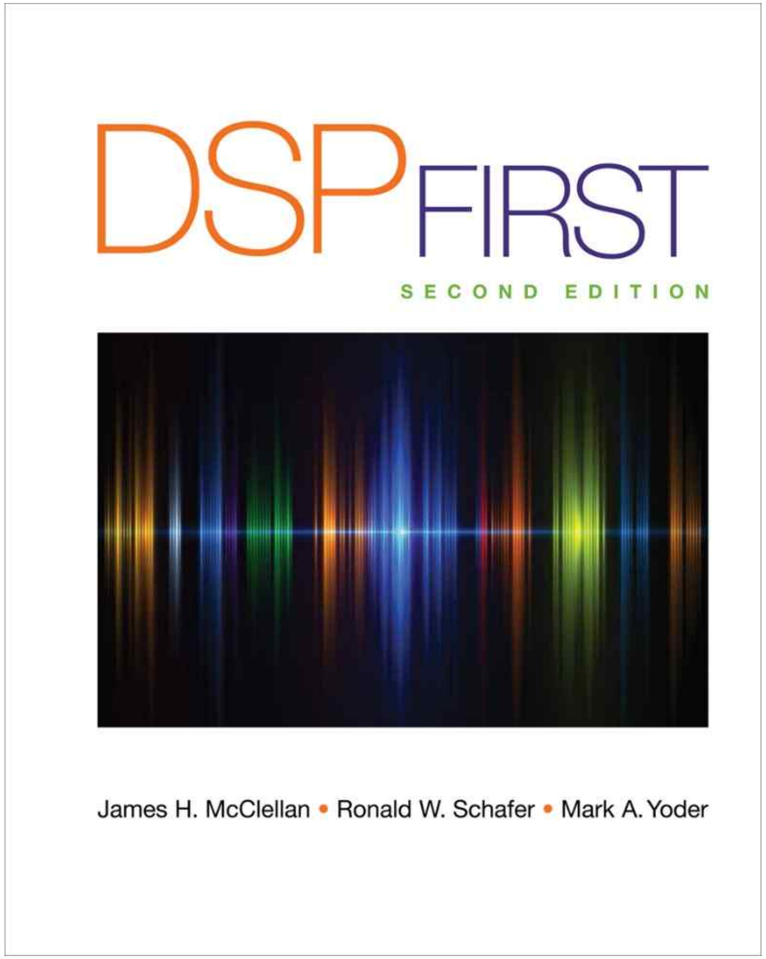
DSP First

Second Edition



CHAPTER 10

IIR Filters

Contents

- FIR vs IIR
- 1st order IIR: the simplest case
  - System function of IIR: $h[n] \Leftrightarrow H[z]$
  - Freq. response $H(e^{j\hat{\omega}})$ of IIR: poles & zeros
- 2nd order IIR
- Stability condition of IIR

FIR Review: Delay by n_d

$$y[n] = x[n - n_d]$$

IMPULSE RESPONSE $h[n] = \delta[n - n_d]$

SYSTEM FUNCTION $H(z) = z^{-n_d}$

FREQUENCY RESPONSE $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}n_d}$

FIR Review: L-pt Averager

$$y[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} x[n-k]$$

IMPULSE RESPONSE $$h[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} \delta[n-k]$$

SYSTEM FUNCTION $$H(z) = \sum_{n=0}^{L-1} \frac{1}{L} z^{-n}$$

What Is Next ?

1. FIND the IMPULSE RESPONSE, $h[n]$ which is INFINITELY LONG

   $\rightarrow$ **Infinite** Impulse Response(IIR) Filters

2. EXPLOIT THREE DOMAINS to Show Relationship for IIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

First Order IIR : ONE FEEDBACK TERM

- ADD PREVIOUS OUTPUTS

$$y[n] = a_1 y[n-1] + \underbrace{b_0 x[n] + b_1 x[n-1]}_{\substack{\text{FIR PART of the FILTER} \\ \rightarrow \text{feed-forward term}}}$$

previous output
→ feedback term

- STILL CAUSAL
  - NOT USING FUTURE OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

$$y[n] - 0.8y[n-1] = 3x[n] - 2x[n-1]$$

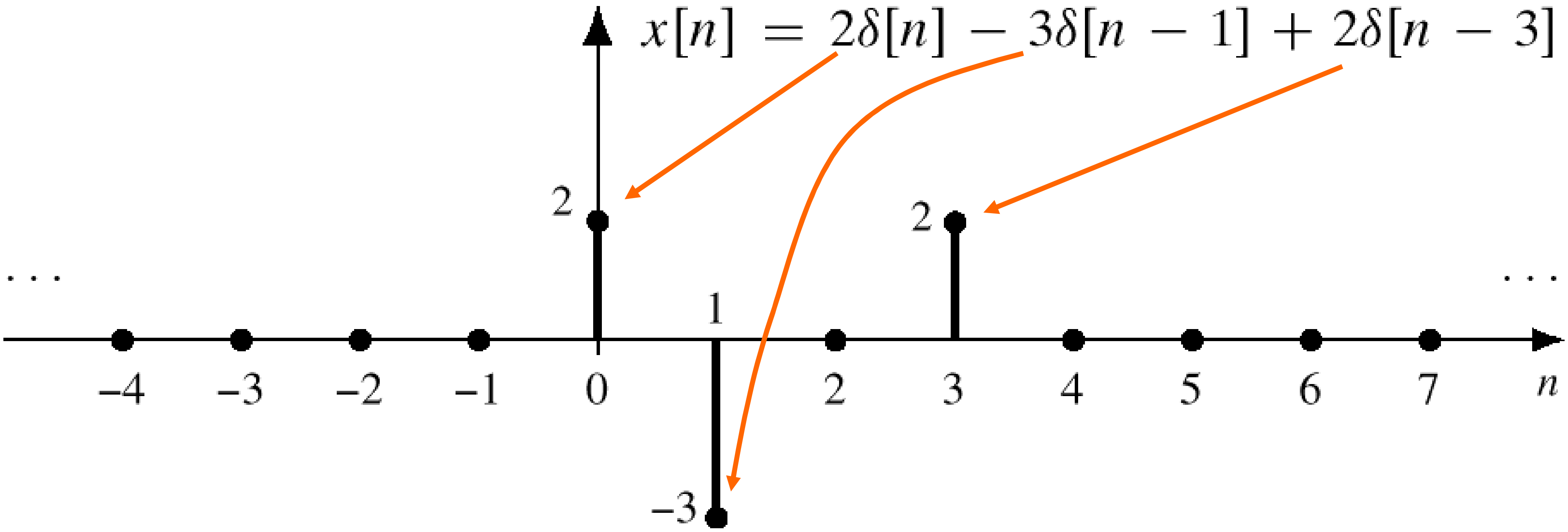
FEEDBACK
COEFFICIENT

- MATLAB
  - `yy = filter([3,-2],[1,-0.8],xx)`

HOW TO COMPUTE y[n] ?

$$y[n] = 0.8y[n-1] + 5x[n]$$

$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]$$



HOW TO COMPUTE y[n] ?

- FEEDBACK DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 5x[n]$$

- NEED y[-1] to get started

$$y[0] = 0.8y[-1] + 5x[0]$$

INITIAL REST CONDITION

- Needs assumptions to get started:

  $x[n] = 0$, for $n < 0 \rightarrow y[n] = 0$, for $n < 0$

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied*.
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE y[0]

$x[n] = \{2, -3, 0, 2\}$

$y[n] = 0.8y[n-1] + 5x[n]$

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5x[0] = 0.8(0) + 5(2) = 10$$

- SAME with MORE FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + \underbrace{a_2 y[n-2]} + \sum_{k=0}^{2} b_k x[n-k]$$

COMPUTE MORE y[n]

$x[n] = \{2, -3, 0, 2\}$

$y[n] = 0.8y[n-1] + 5x[n]$

- CONTINUE THE RECURSION:

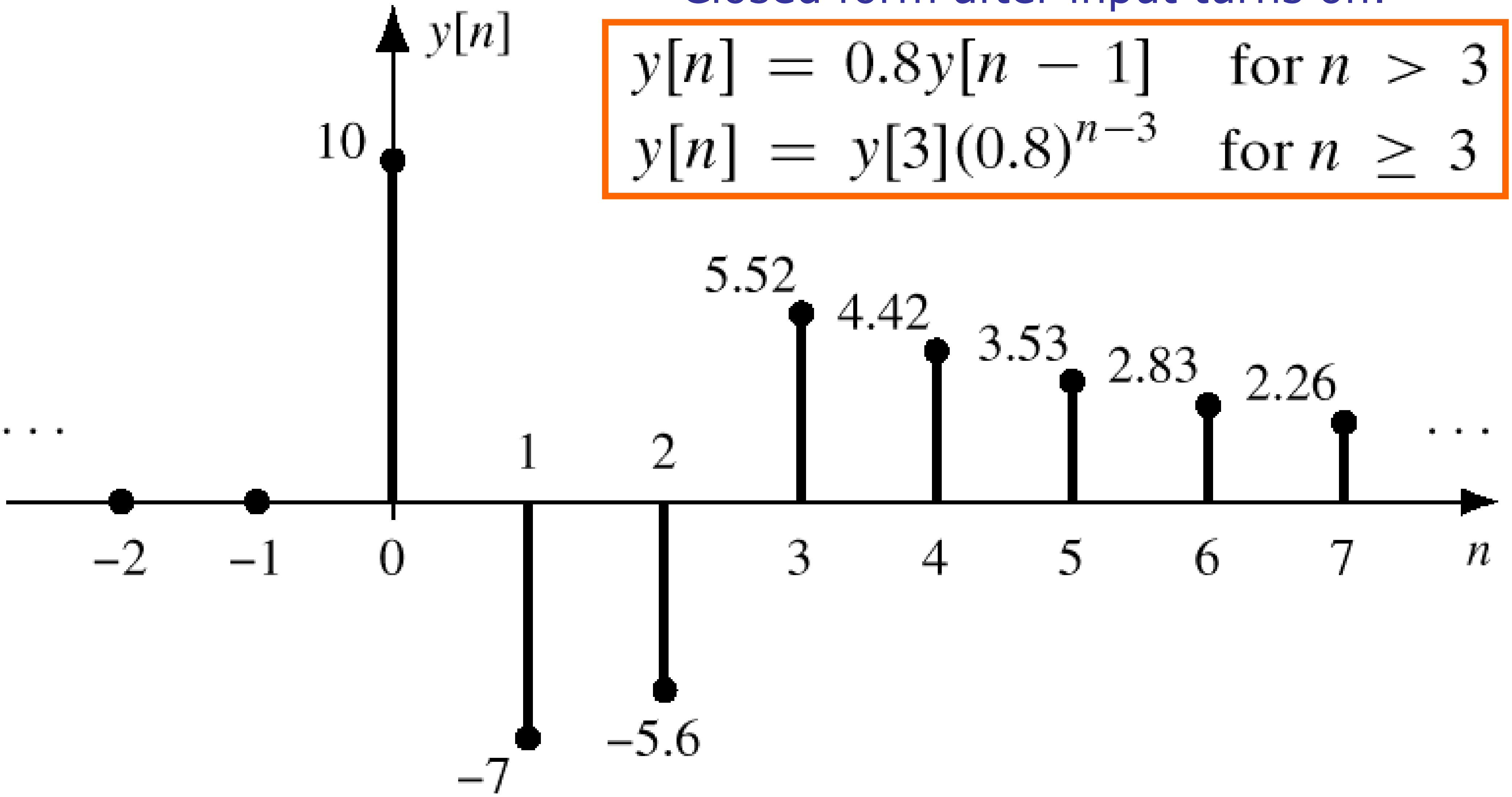
$$
\begin{aligned}
y[1] &= 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7 \\
y[2] &= 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6 \\
y[3] &= 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52 \\
y[4] &= 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416 \\
y[5] &= 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328 \\
y[6] &= 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262
\end{aligned}
$$

0 input at n > 3

PLOT y[n]

Closed form after input turns off.

$$y[n] = 0.8y[n-1] \quad \text{for } n > 3$$
$$y[n] = y[3](0.8)^{n-3} \quad \text{for } n \geq 3$$



IMPULSE RESPONSE of First-Order IIR System With One B_k

$$y[n] = a_1 y[n-1] + b_0 x[n] \Rightarrow h[n] = a_1 h[n-1] + b_0 \delta[n]$$

n	$n < 0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$h[n] = b_0(a_1)^n u[n]$$

$$u[n] = 1, \quad \text{for } n \geq 0$$

IMPULSE RESPONSE of First-Order IIR System With One B_k

- DIFFERENCE EQUATION:

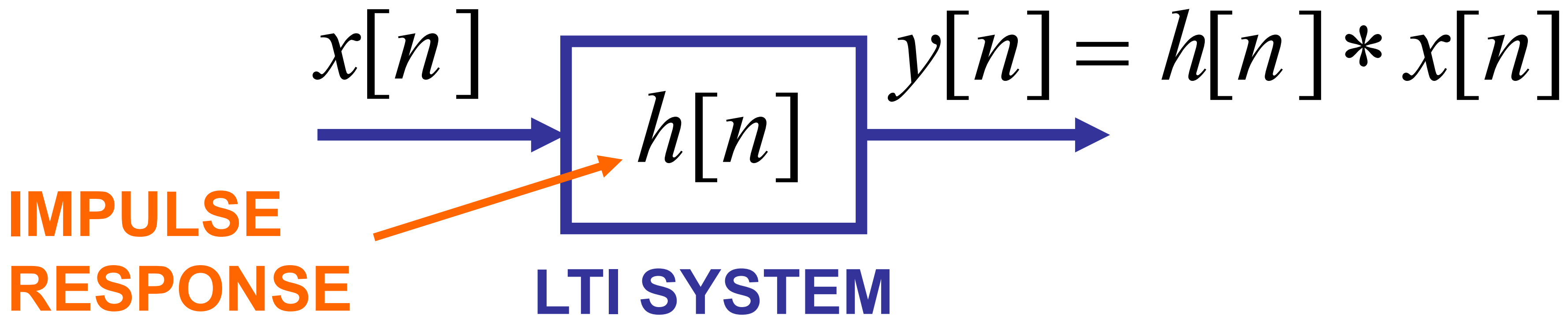
$$y[n] = 0.8y[n-1] + 3x[n]$$

a_0 a_1 b_0

- Find h[n]

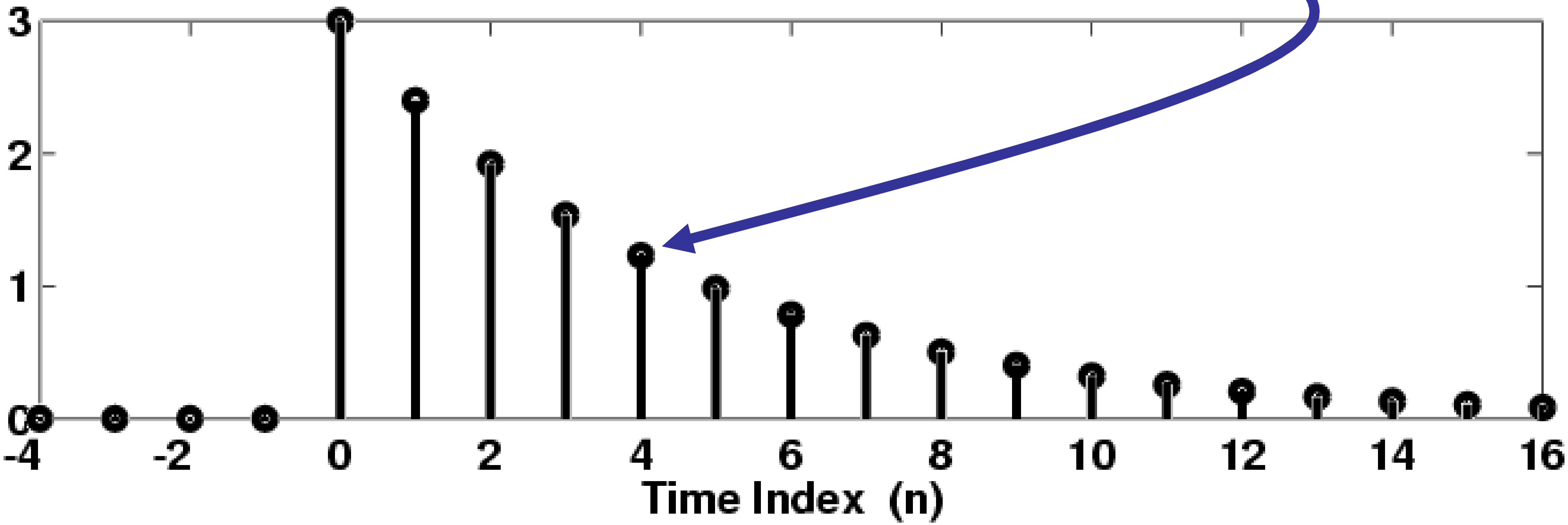
$$h[n] = 3(0.8)^n u[n]$$

- CONVOLUTION in TIME-DOMAIN



PLOT IMPULSE RESPONSE

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



Infinite length !

H(z) from Infinite Length of h[n] : 1st-Order IIR With One B_k

- Z Transform: POLYNOMIAL Representation

$$H(z)=\sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

APPLIES to Any SIGNAL

- SIMPLIFY the SUMMATION

$$H(z)=\sum_{n=-\infty}^{\infty} b_0(a_1)^n u[n]\, z^{-n} = b_0\sum_{n=0}^{\infty} a_1^n z^{-n}$$

H(z) from Infinite Length of h[n] : 1st-Order IIR With One B_k

- Recall Sum of Geometric Sequence(=series = progression) (등비수열=기하수열):

$$\sum_{n=0}^{\infty} ar^n = \frac{a(1-r^n)}{1-r} \quad \Rightarrow \frac{a}{1-r}, |\mathrm{r}| < 1$$

- Yields a COMPACT FORM: $r = a_1 z^{-1}$

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

Summary: H(z) of 1st-Order IIR With One B_k

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$\boxed{h[n] = b_0 (a_1)^n u[n]}$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

FIR part of the filter
→ feed-forward term

previous output
→ feedback term

$h[n]$ of 1st-Order IIR With two B_k's

- ANOTHER FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

When $x[n] = \delta[n]$, $y[0] = h[0] = b_0\delta[0] + b_1\delta[-1] = b_0$

$$y[1] = h[1] = a_1 y[0] + b_0\delta[1] + b_1\delta[0] = a_1 b_0 + b_1$$

$$y[2] = h[2] = a_1(a_1 b_0 + b_1) = a_1{}^2 b_0 + a_1 b_1$$

$$y[3] = h[3] = a_1(a_1{}^2 b_0 + a_1 b_1) = a_1{}^3 b_0 + a_1{}^2 b_1$$

$$y[n] = h[n] = a_1{}^n b_0 + a_1{}^{n-1} b_1, \qquad n \geq 0$$

$$h[n] = b_0(a_1)^n u[n] + b_1(a_1)^{n-1} u[n-1]$$

$H(z)$ of 1st-Order IIR With two B_k's

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$h[n] = b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=-\infty}^{\infty} (b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]) z^{-n}$$

$$= \sum_{n=0}^{\infty} b_0 (a_1)^n z^{-n} + \sum_{n-1=0}^{\infty} b_1 (a_1)^{n-1} z^{-(n-1)} z^{-1}$$

$$\Leftarrow k = n-1$$

$$= \frac{b_0}{1-a_1 z^{-1}} + z^{-1} \sum_{k=0}^{\infty} b_1 (a_1)^k z^{-k} = \frac{b_0}{1-a_1 z^{-1}} + z^{-1} \frac{b_1}{1-a_1 z^{-1}}$$

$$H(z) = \frac{b_0}{1-a_1 z^{-1}} + z^{-1} \frac{b_1}{1-a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1-a_1 z^{-1}}$$

POP QUIZ: Inverse Z Transform

- Given:

$$H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}=\frac{\overset{b_0}{2}}{1-\underset{a_1}{0.8}z^{-1}}+\frac{\overset{b_1}{2}z^{-1}}{1-\underset{a_1}{0.8}z^{-1}}$$

- Find the Impulse Response, h[n]
  - Use the DELAY PROPERTY

$$h[n]=2(0.8)^{n}u[n]+2(0.8)^{n-1}u[n-1]$$

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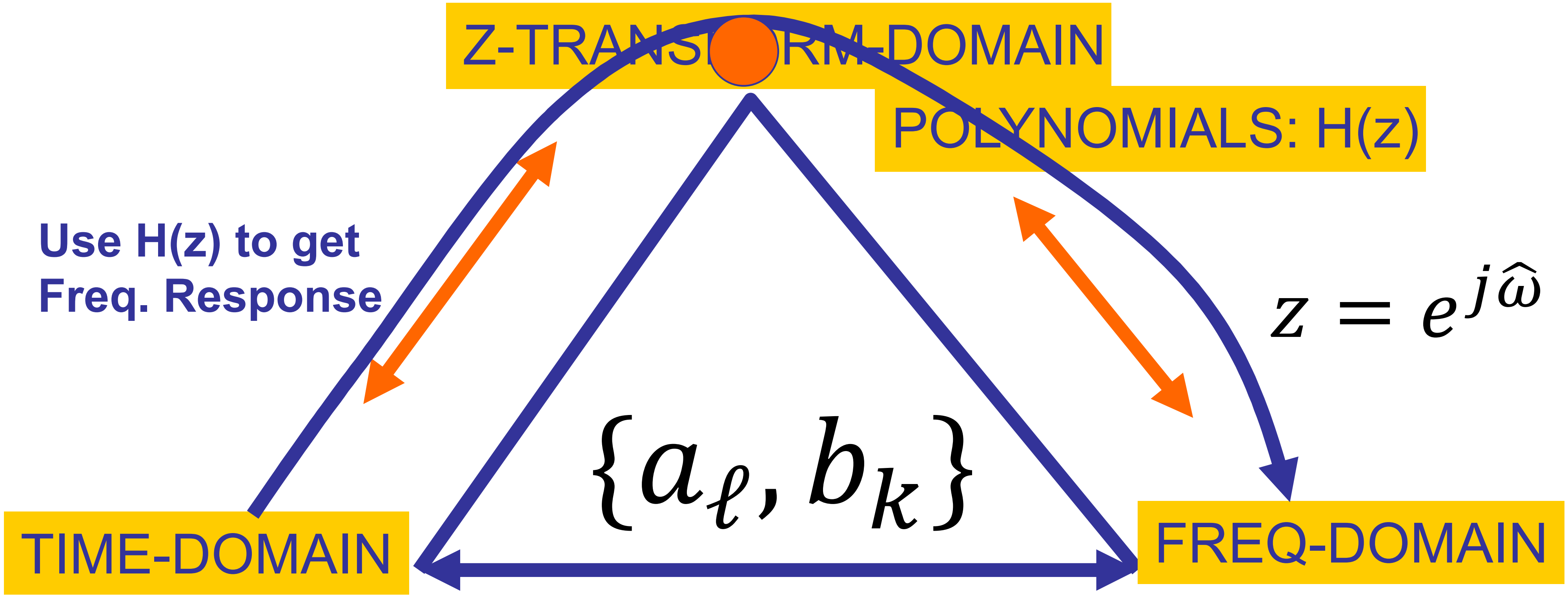
THREE DOMAINS



$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \frac{\sum_{k=0}^{M} b_k\, e^{-j\hat{\omega}k}}{1 - \sum_{\ell=1}^{N} a_\ell\, e^{-j\hat{\omega}\ell}}$$

Recall:
H(z) of 1st-Order IIR With One B_k

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

FIR part of the filter
→ feed-forward term

previous output
→ feedback term

Recall:
$H(z)$ of 1st-Order IIR With two B_k's

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$h[n] = b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left(b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]\right) z^{-n}$$

$$= \sum_{n=0}^{\infty} b_0 (a_1)^n z^{-n} + \sum_{n-1=0}^{\infty} b_1 (a_1)^{n-1} z^{-(n-1)} z^{-1}$$

$$\Leftarrow k = n-1$$

$$= \frac{b_0}{1 - a_1 z^{-1}} + z^{-1} \sum_{k=0}^{\infty} b_1 (a_1)^k z^{-k} = \frac{b_0}{1 - a_1 z^{-1}} + z^{-1} \frac{b_1}{1 - a_1 z^{-1}}$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}} + z^{-1} \frac{b_1}{1 - a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

Review: CONVOLUTION PROPERTY

- MULTIPLICATION of z-TRANSFORMS

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)X(z)$$

- CONVOLUTION in TIME-DOMAIN

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

IMPULSE RESPONSE

Z-Transform of IIR Filter

- DERIVE the SYSTEM FUNCTION H(z)
  - Use DELAY PROPERTY

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

EASIER with DELAY PROPERTY

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n - n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$

How to Get the SYSTEM FUNCTION of IIR more easily ?

- Take Z transform first !

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

- Then use the convolution property !

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{B(z)}{A(z)}$$

Example: SYSTEM FUNCTION by Convolution Property

- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

- READ the FILTER COEFFS:

$$Y(z) = \underbrace{\left(\frac{3-2z^{-1}}{1-0.8z^{-1}}\right)}_{H(z)} X(z)$$

POLES & ZEROS of H(z)

- Zeros of H(z), i.e., where is H(z)=0?
  - Look for Roots of Numerator Polynomial

$$H(z) = \frac{B(z)}{A(z)}, \text{ if } B(z_0) = 0 \Rightarrow H(z_0) = 0$$

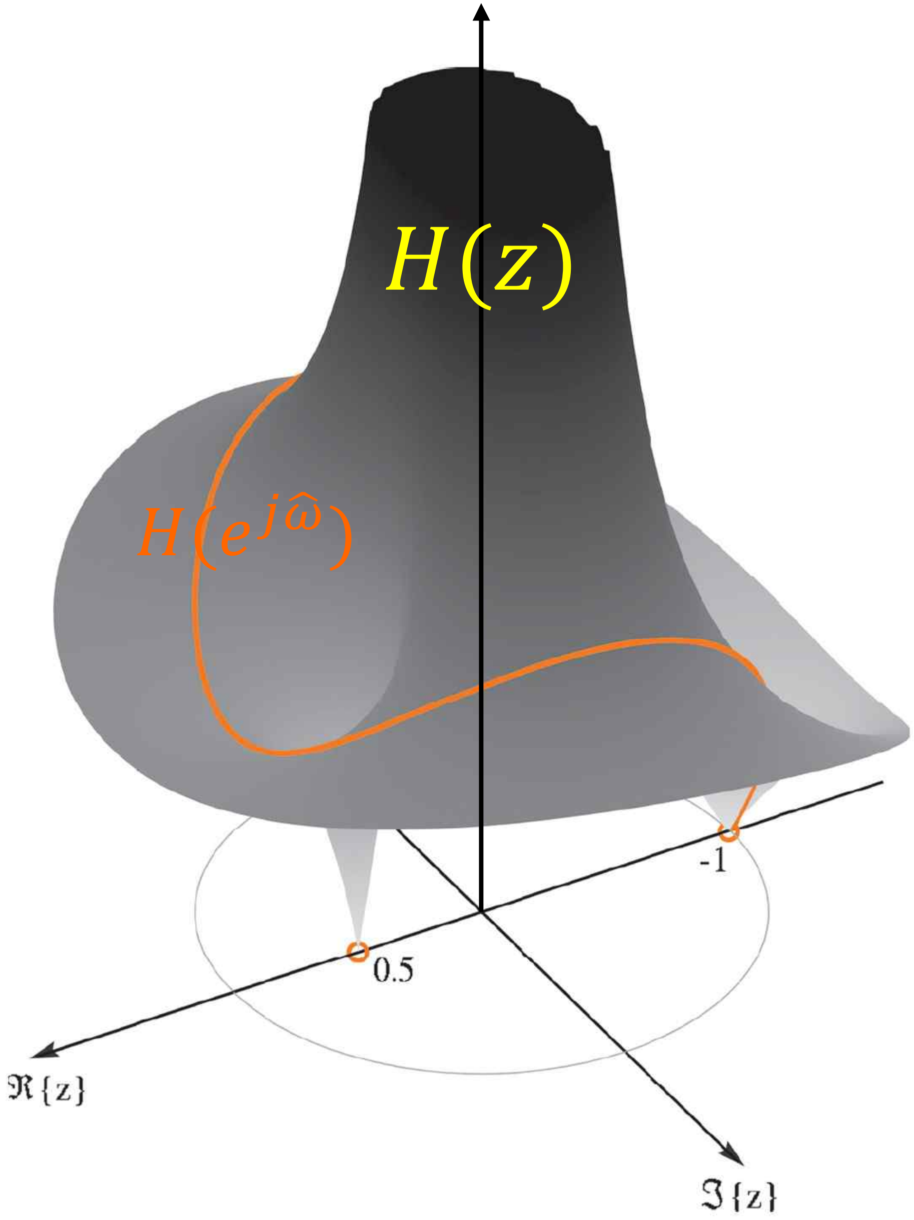
$$\text{if } A(z_0) \neq 0$$

- Poles of H(z), i.e., where is H(z)=infinity?
  - Look for Roots of Denominator Polynomial

$$H(z) = \frac{B(z)}{A(z)}, \text{if } A(z_0) = 0 \Rightarrow H(z_0) \to \infty$$

$$\text{if } B(z_0) \neq 0$$

Recall: $H(e^{j\hat{\omega}})$ is H(z) on Unit Circle



$$H(z) = 1 + 0.5z^{-1} - 0.5z^{-2}$$
$$= (1 + z^{-1})(1 - 0.5z^{-1})$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

zeros at $z = -1$ and $z = 0.5$

[Q] What are DFT $H[k]$ in the plot?

POLES & ZEROS

- ROOTS of Numerator & Denominator

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow \boxed{H(z) = \frac{b_0 z + b_1}{z - a_1}}$$

$$b_0 z + b_1 = 0 \quad \Rightarrow z = -\frac{b_1}{b_0}$$ **ZERO: H(z)=0**

$$z - a_1 = 0 \quad \Rightarrow z = a_1$$ **POLE: H(z) → inf**

EXAMPLE: Poles & Zeros

- VALUE of H(z) at POLES is INFINITE.

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

$$H(z) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

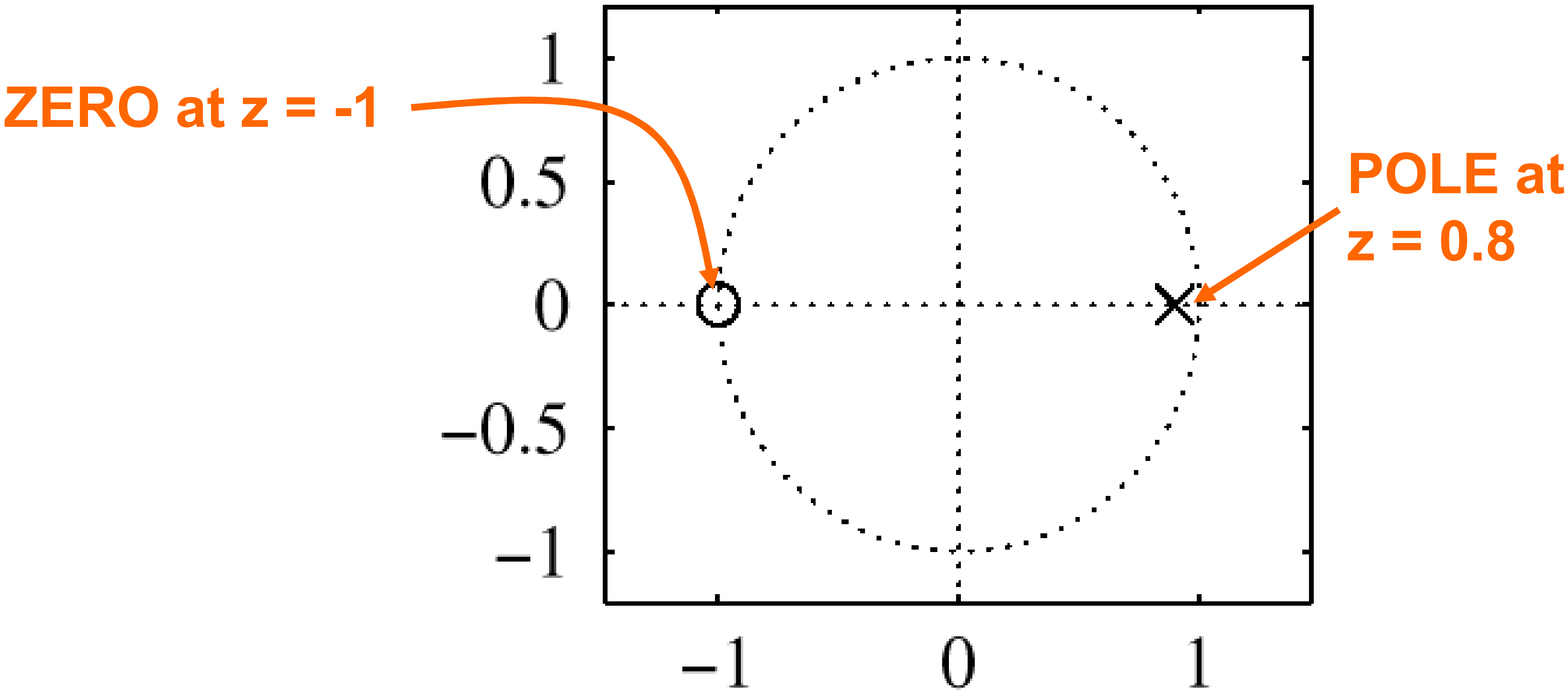
ZERO at z= -1

$$H(z) = \frac{2 + 2(0.8)^{-1}}{1 - 0.8(0.8)^{-1}} = \frac{\frac{9}{2}}{0} \to \infty$$

POLE at z=0.8

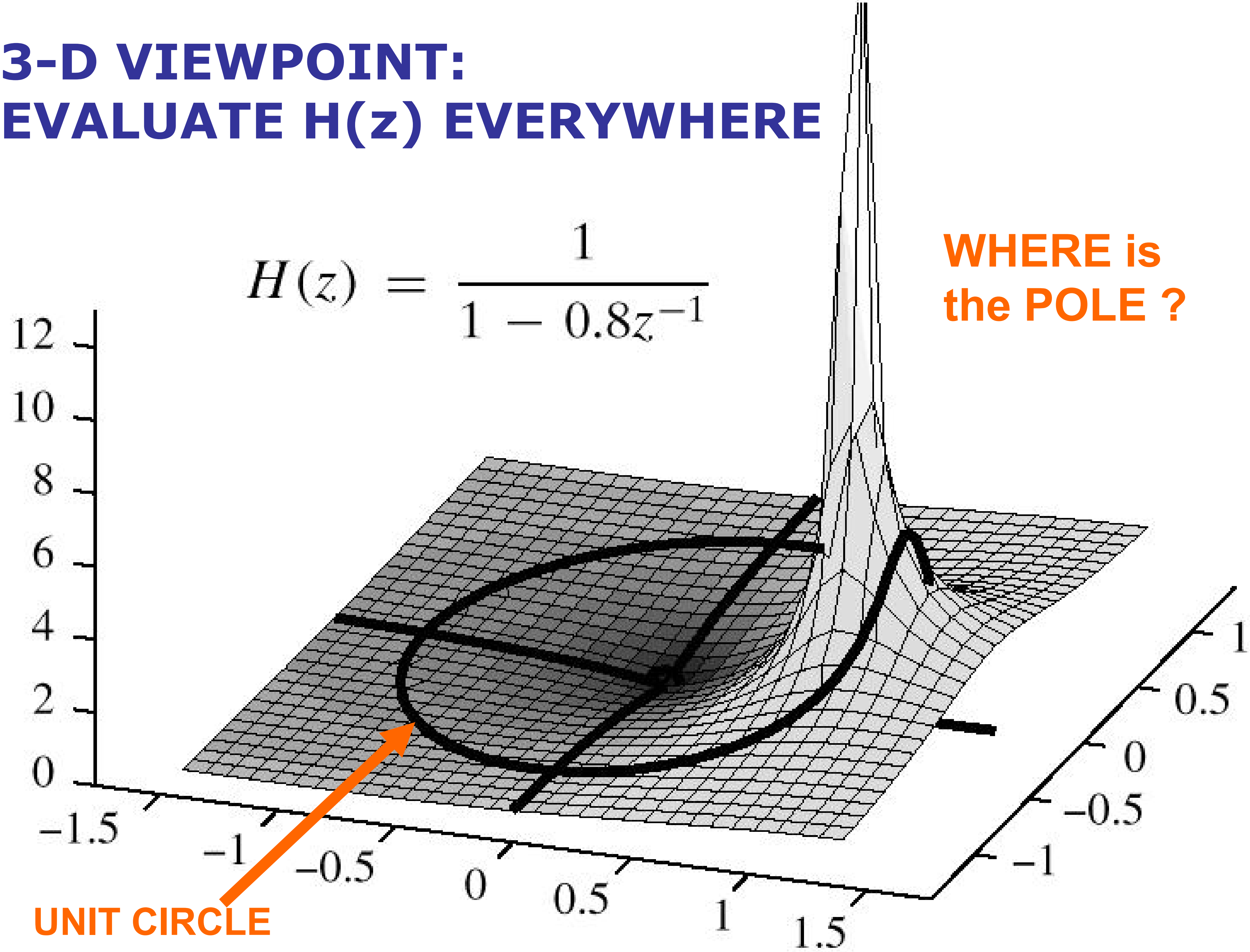
POLE-ZERO PLOT

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$



3-D VIEWPOINT: EVALUATE H(z) EVERYWHERE

$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$



FREQUENCY RESPONSE $H(e^{j\hat{\omega}})$

- EVALUATE on the UNIT CIRCLE

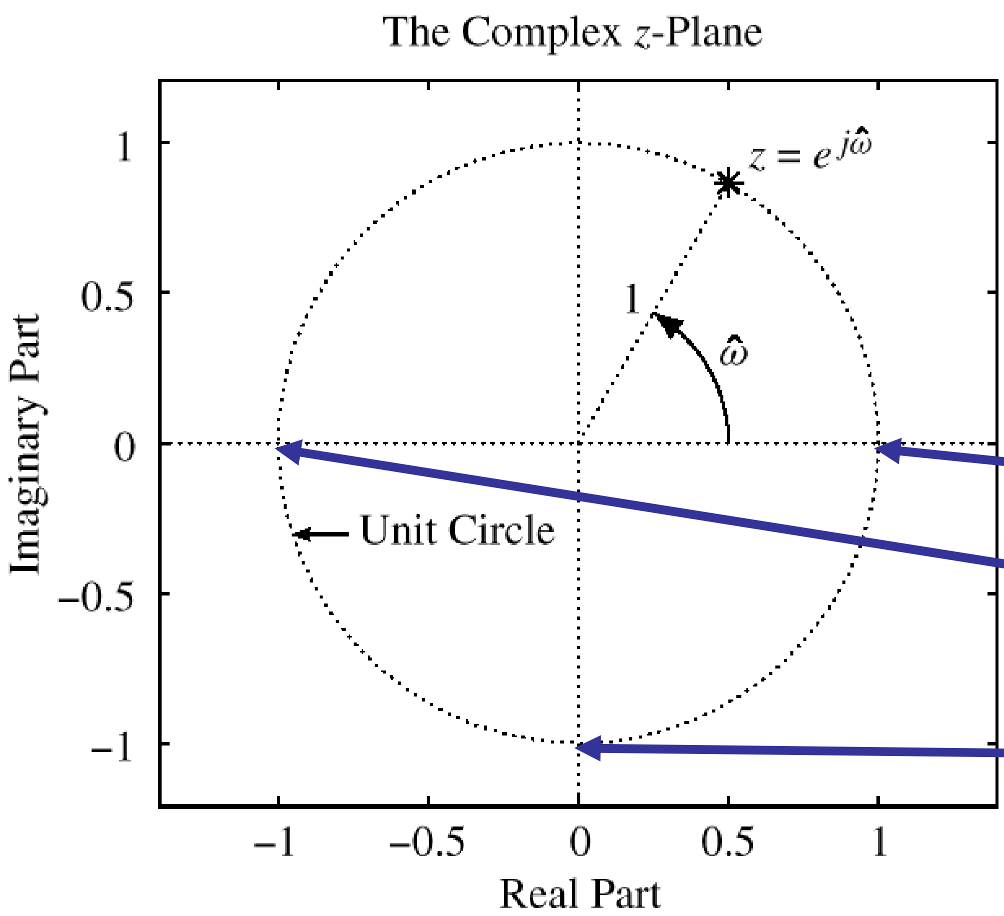
$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$

|H(z)| Along the UNIT CIRCLE

- MAPPING BETWEEN z and $\hat{\omega}$



$$z = e^{j\hat{\omega}}$$

$$z = 1 \quad \leftrightarrow \quad \hat{\omega} = 0$$

$$z = -1 \quad \leftrightarrow \quad \hat{\omega} = \pm\pi$$

$$z = \pm j \quad \leftrightarrow \quad \hat{\omega} = \pm\frac{1}{2}\pi$$

FREQ. RESPONSE FORMULA

$$H(z) = \frac{2+2z^{-1}}{1-0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$

A trick to evaluate $H(e^{j\hat{\omega}})$ more easily

$$\left|H(e^{j\hat{\omega}})\right|^2 = \left|\frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}\right|^2 = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}} \cdot \frac{2+2e^{j\hat{\omega}}}{1-0.8e^{j\hat{\omega}}}$$

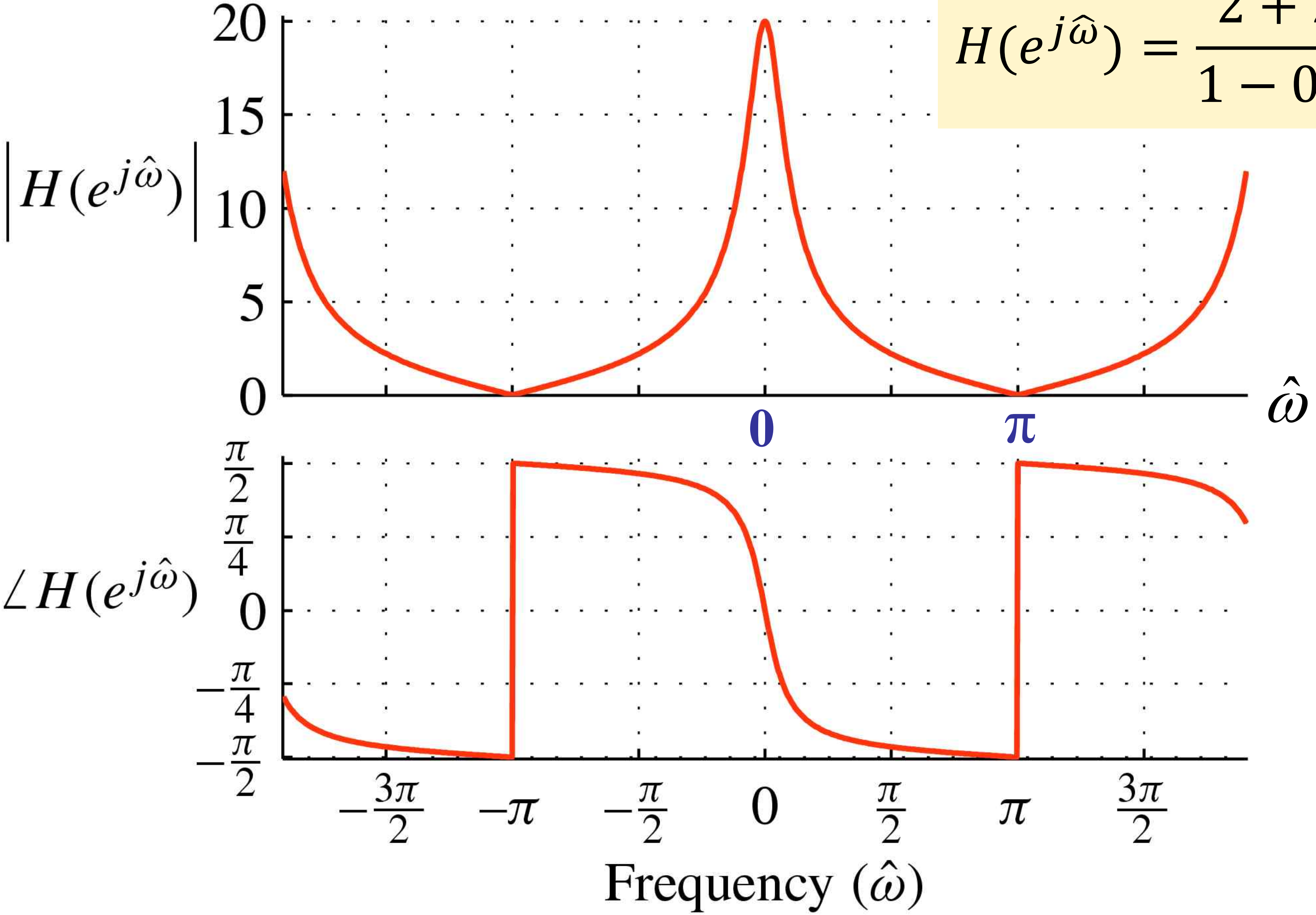
$$= \frac{4+4+4e^{-j\hat{\omega}}+4e^{j\hat{\omega}}}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}} = \frac{8+8\cos\hat{\omega}}{1.64-1.6\cos\hat{\omega}}$$

$@\hat{\omega} = 0, \quad \left|H(e^{j\hat{\omega}})\right|^2 = \frac{8+8}{0.04} = 400 \quad \rightarrow \left|H(e^{j\hat{\omega}})\right| = 20$

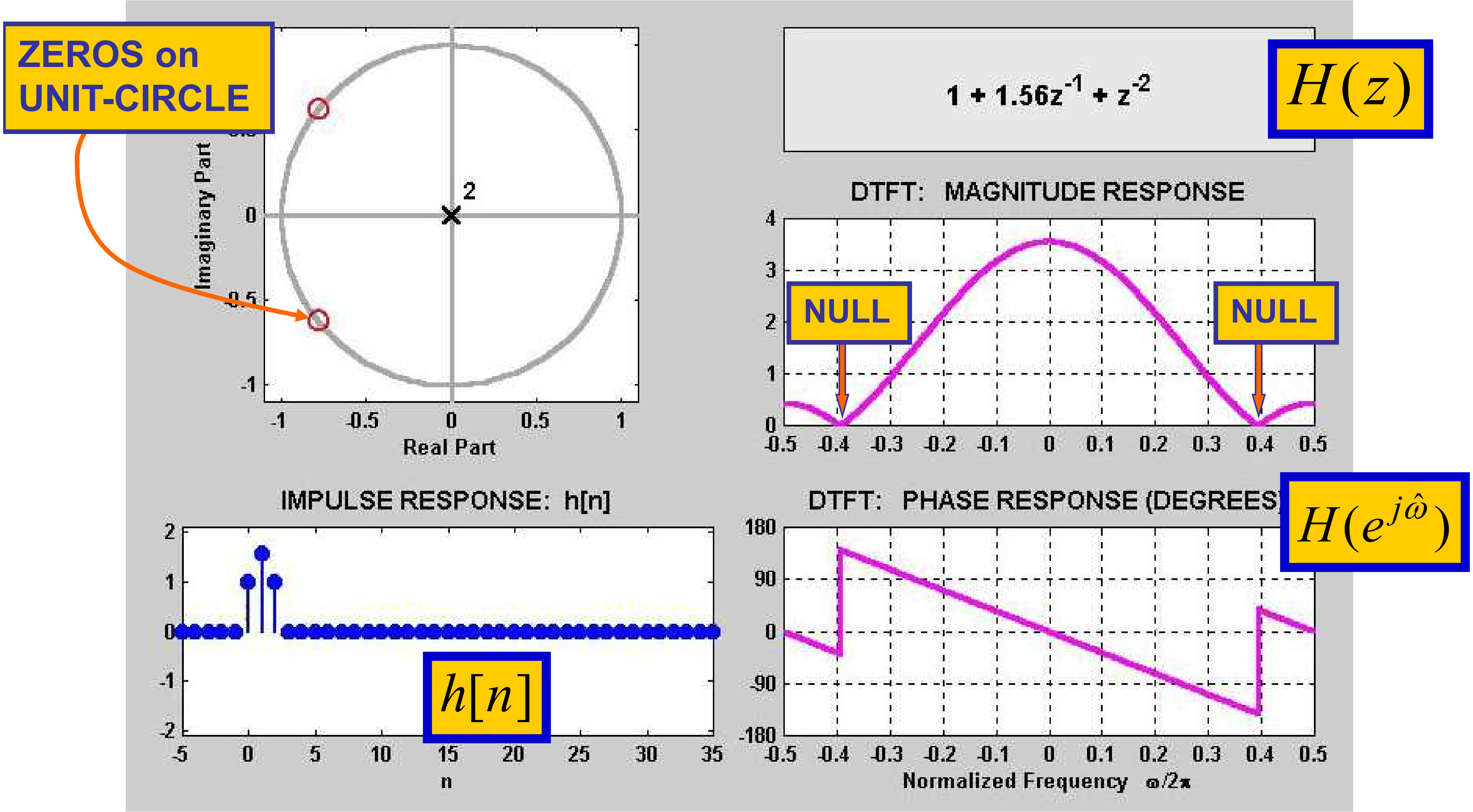
$@\hat{\omega} = \pi, \quad \left|H(e^{j\hat{\omega}})\right|^2 = 0$

Frequency Response Plot

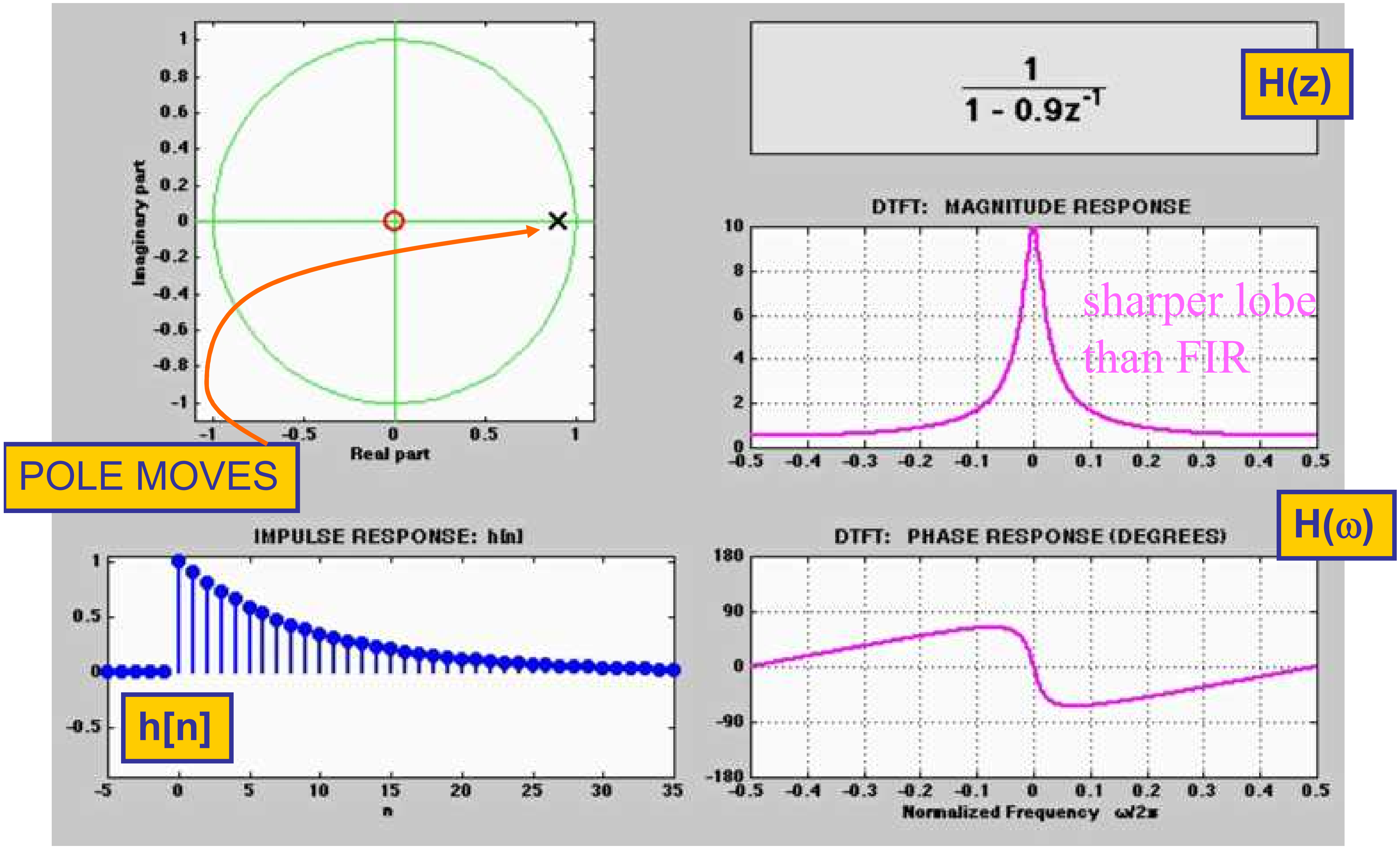
$$H(e^{j\hat{\omega}}) = \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}}$$



Review: 3 DOMAINS MOVIE: FIR



3 DOMAINS MOVIE: IIR



7 IIR MOVIES @ WEBSITE

- `http://dspfirst.gatech.edu/chapters/08feedbac/demos/3_domain/index.html`
- 3 DOMAINS MOVIES: IIR Filters
  - One pole moving and a zero at the origin
  - One pole and one zero; both moving
  - Two complex-conjugate poles moving radially
  - Two complex-conjugate poles moving in angle
  - Movement of a zero in a two-pole Filter
  - Radial Movement of Two out of Four Poles
  - Angular Movement of Two out of Four Poles

SINUSOIDAL RESPONSE $y[n]$

- x[n] = SINUSOID → y[n] is SINUSOID
- Find $y[n]$ by getting MAGNITUDE & PHASE from H(z)

$$\text{if } \; x[n] = e^{j\hat{\omega}n}$$

$$\text{then } \; y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$$

$$\text{where } \; H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

POP QUIZ

- Given: $H(z) = \dfrac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

$$Y(z) - 0.8z^{-1}Y(z) = 2X(z) + 2z^{-1}X(z)$$

$$y[n] = 0.8y[n-1] + 2x[n] + 2x[n-1], \qquad y[n] = 0 \text{ when } n < 0$$

$$h[n] = 0.8h[n-1] + 2\delta[n] + 2\delta[n-1], \qquad h[n] = 0 \text{ when } n < 0$$

1. Find the Impulse Response, h[n]

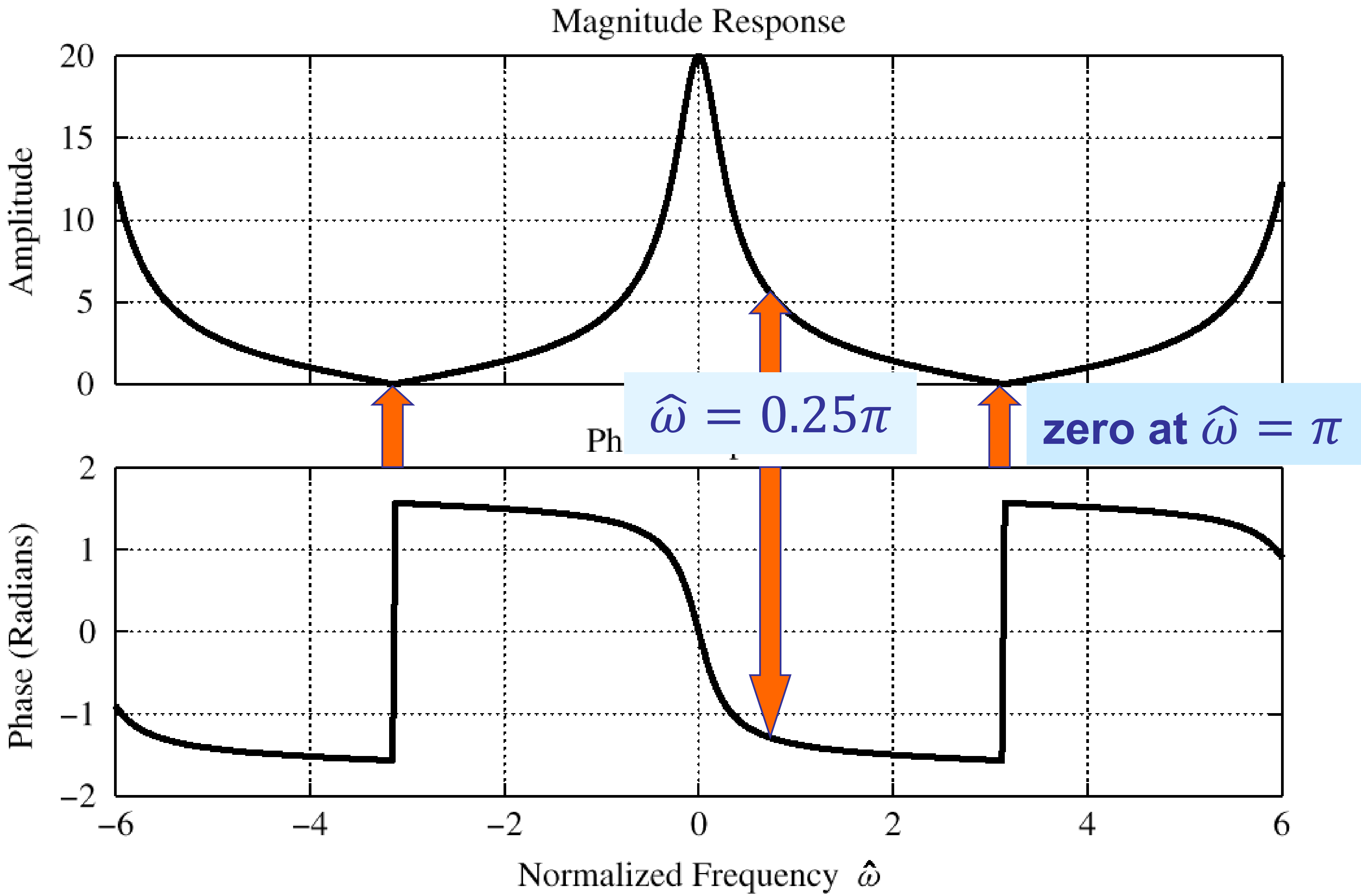
$$h[n] = b_0(a_1)^n u[n] + b_1(a_1)^{n-1} u[n-1]$$

$$h[n] = 2(0.8)^n u[n] + 2(0.8)^{n-1} u[n-1]$$

2. Find the output, y[n]

   when $x[n] = \cos(0.25\pi n)$

POP QUIZ:
Evaluate FREQ. RESPONSE to get the complex amplitude where $\hat{\omega} = 0.25\pi$



POP QUIZ: Eval Freq. Resp.

- Given: $H(z) = \dfrac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$

- Find output, y[n], when $x[n] = \cos(0.25\pi n)$
  - Evaluate H(z) at $z = e^{j0.25\pi}$

$$H(z) = \frac{2 + 2e^{-j0.25\pi}}{1 - 0.8e^{-j0.25\pi}} = 5.182e^{-j1.309}$$

$$y[n] = 5.182\cos(0.25\pi n - 0.417\pi)$$

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SECOND-ORDER IIR FILTERS

- Two FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$\boxed{H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}}$$

2nd order IIR: Two POLES

- poles by quadratic formula (근의 공식)
  - Two poles are either REAL
  - or COMPLEX CONJUGATES.

$$\frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}$$

PROPERTY OF REAL POLYNOMIALS

A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.

2nd ORDER EXAMPLE

$$h[n] = (0.9)^n \cos\left(\frac{\pi}{3}n\right)u[n] = (0.9)^n 0.5\left(e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}\right)u[n]$$

$$= (0.9)^n e^{j\frac{\pi}{3}n} 0.5 u[n] + (0.9)^n e^{-j\frac{\pi}{3}n} 0.5 u[n]$$

$$\boxed{h[n] = b a^n u[n] \quad \leftrightarrow \quad H(z) = \frac{b}{1 - a z^{-1}}}$$

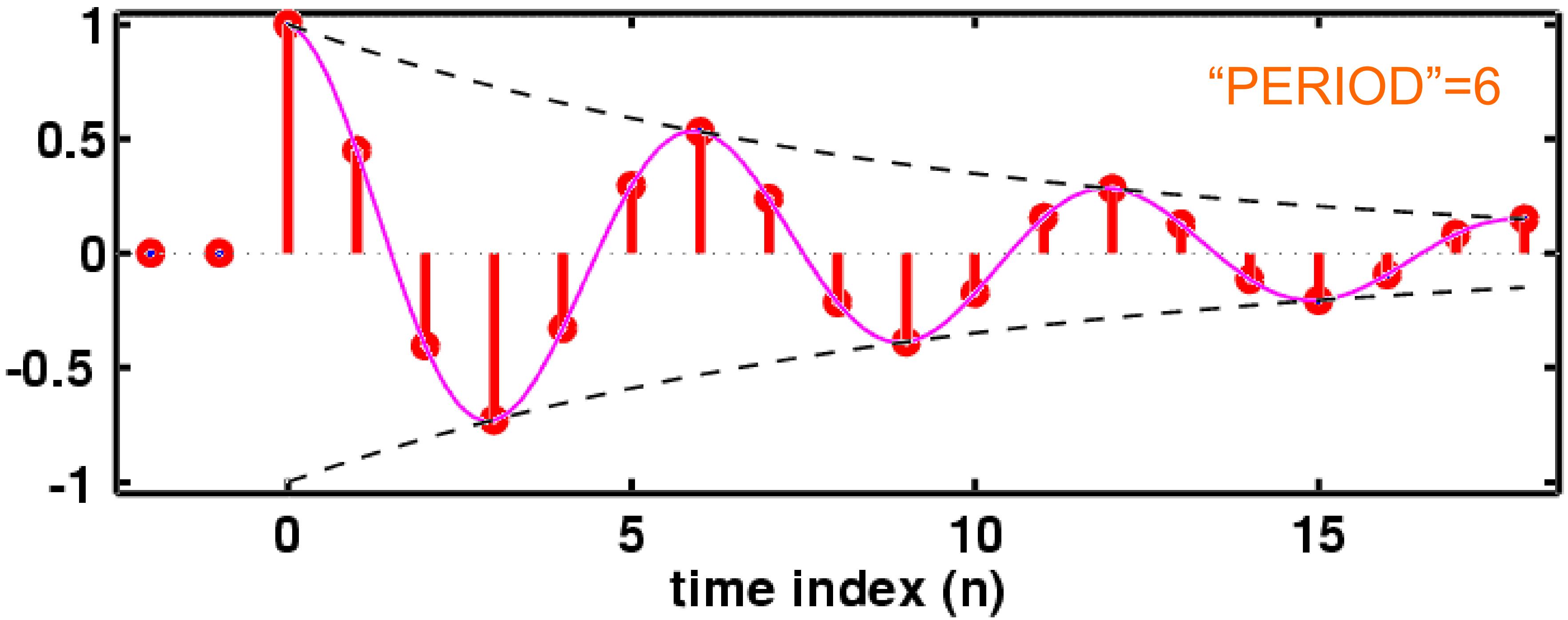
$$a = 0.9 e^{\pm j\frac{\pi}{3}}, \; b = 0.5$$

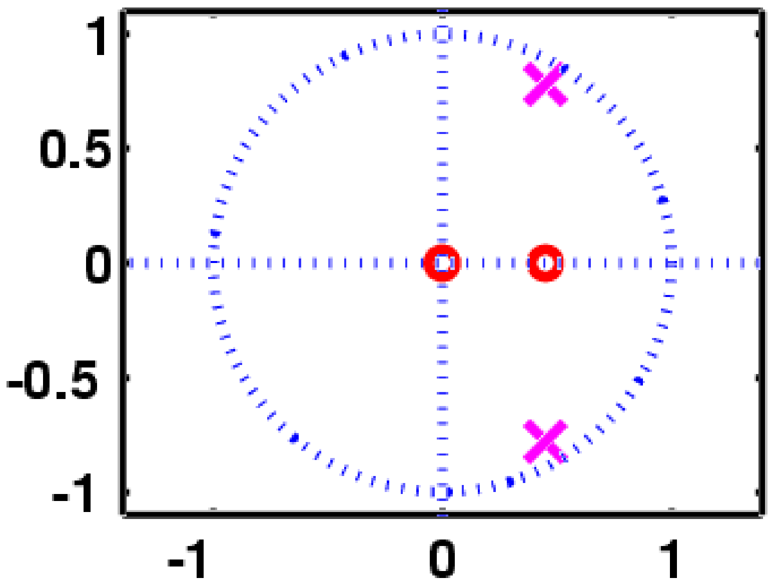
$$H(z) = \frac{0.5}{1 - 0.9 e^{j\pi/3} z^{-1}} + \frac{0.5}{1 - 0.9 e^{-j\pi/3} z^{-1}}$$

$$H(z) = \frac{1 - 0.9\cos(\frac{\pi}{3}) z^{-1}}{(1 - 0.9 e^{j\pi/3} z^{-1})(1 - 0.9 e^{-j\pi/3} z^{-1})}$$

$$H(z) = \frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

h[n]: Decays & Oscillates





$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n]$$

$$H(z) = \frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}} = \frac{z^2 - 0.45z}{z^2 - 0.9z + 0.81}$$

two zeros & two poles

2nd ORDER EX: n-Domain

$$H(z) = \frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

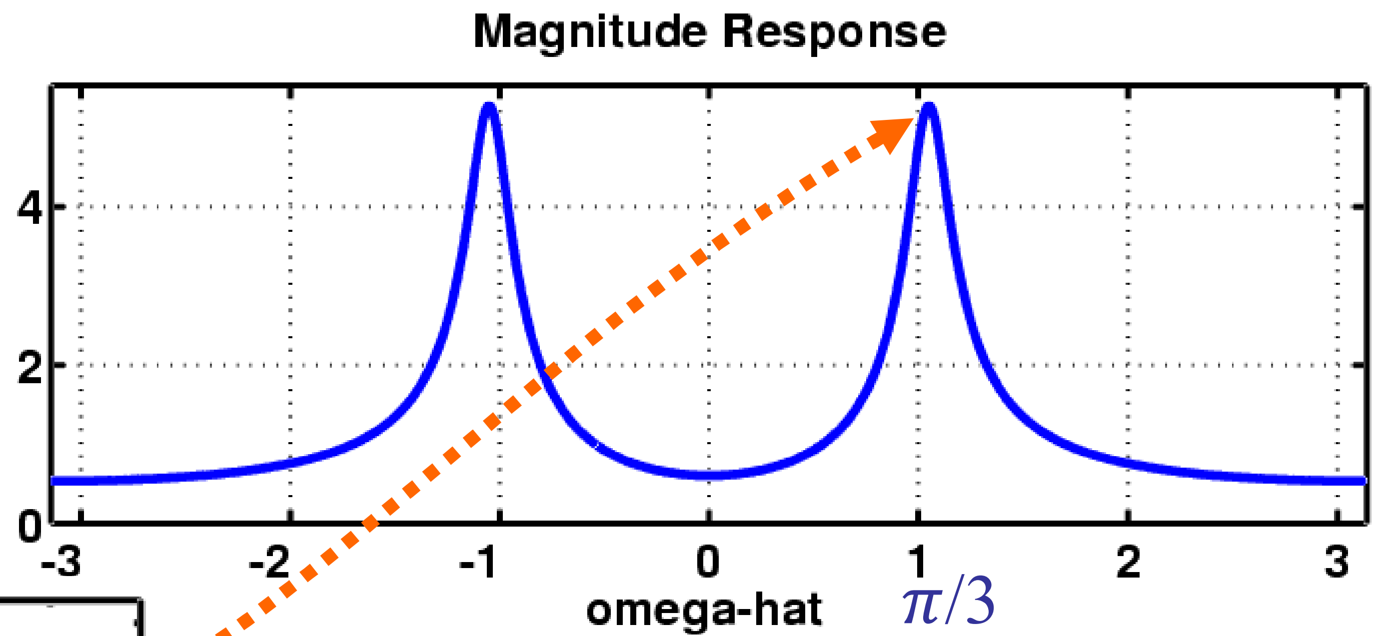
$$y[n] = 0.9y[n-1] - 0.81y[n-2] + x[n] - 0.45x[n-1]$$

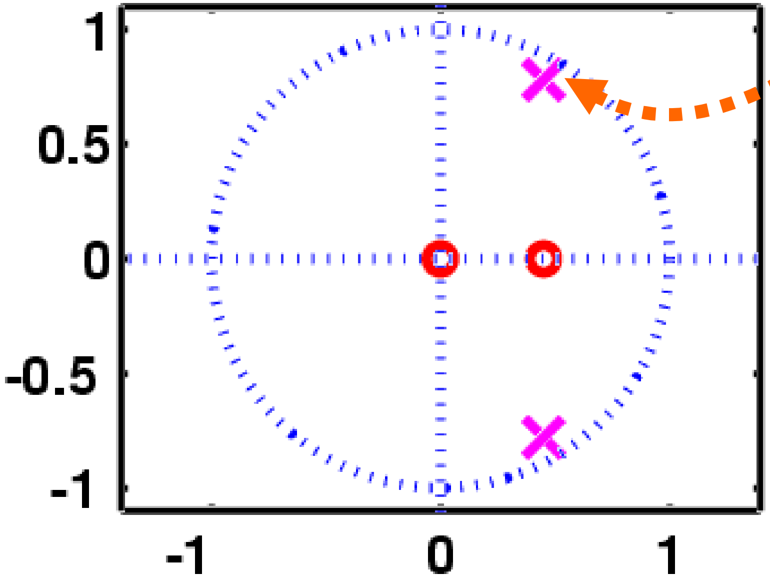
```
aa = [ 1, -0.9, 0.81 ];
bb = [ 1, -0.45 ];
nn = -2:19;
hh = filter( bb, aa, (nn==0) );
HH = freqz( bb, aa, [-pi,pi/100:pi] );
```

$\delta[n]$

always "1" in FIR

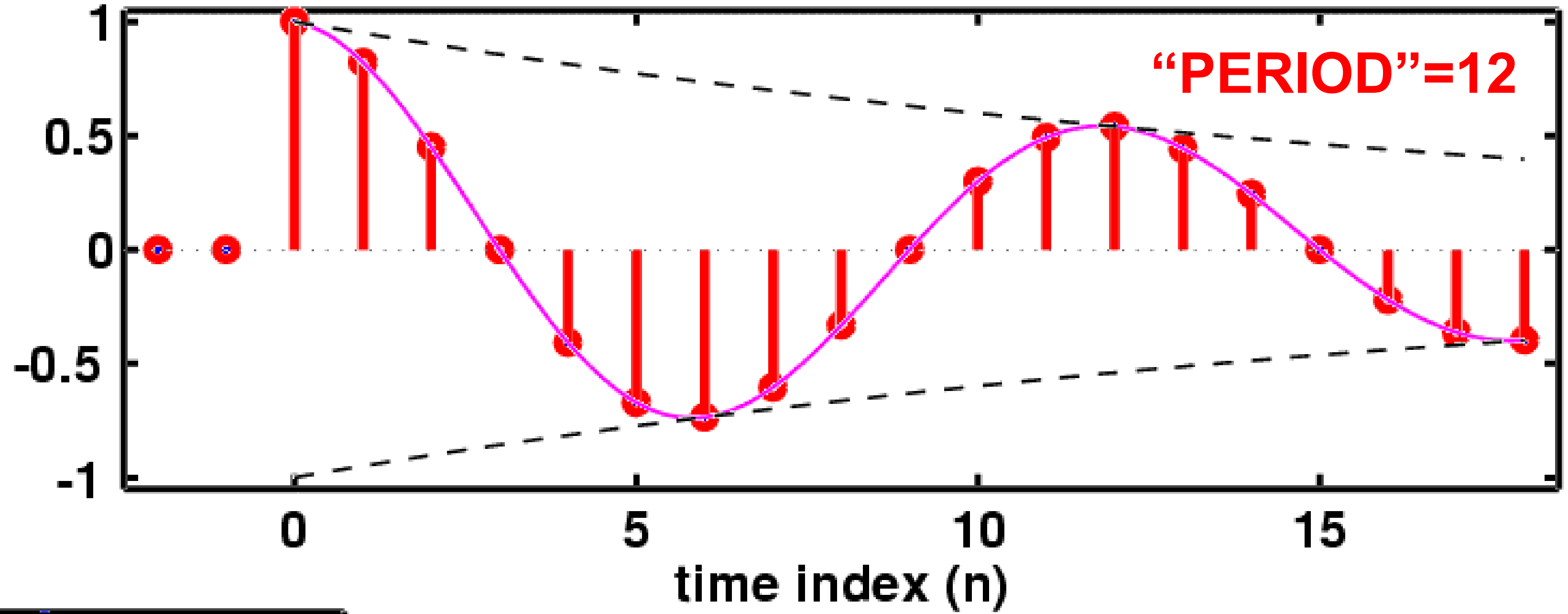
Complex POLE-ZERO PLOT

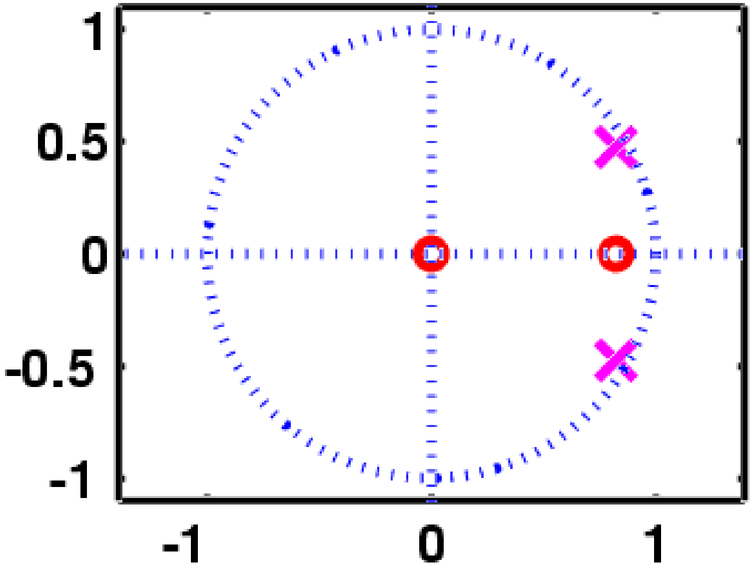




$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

h[n]: Decays & Oscillates

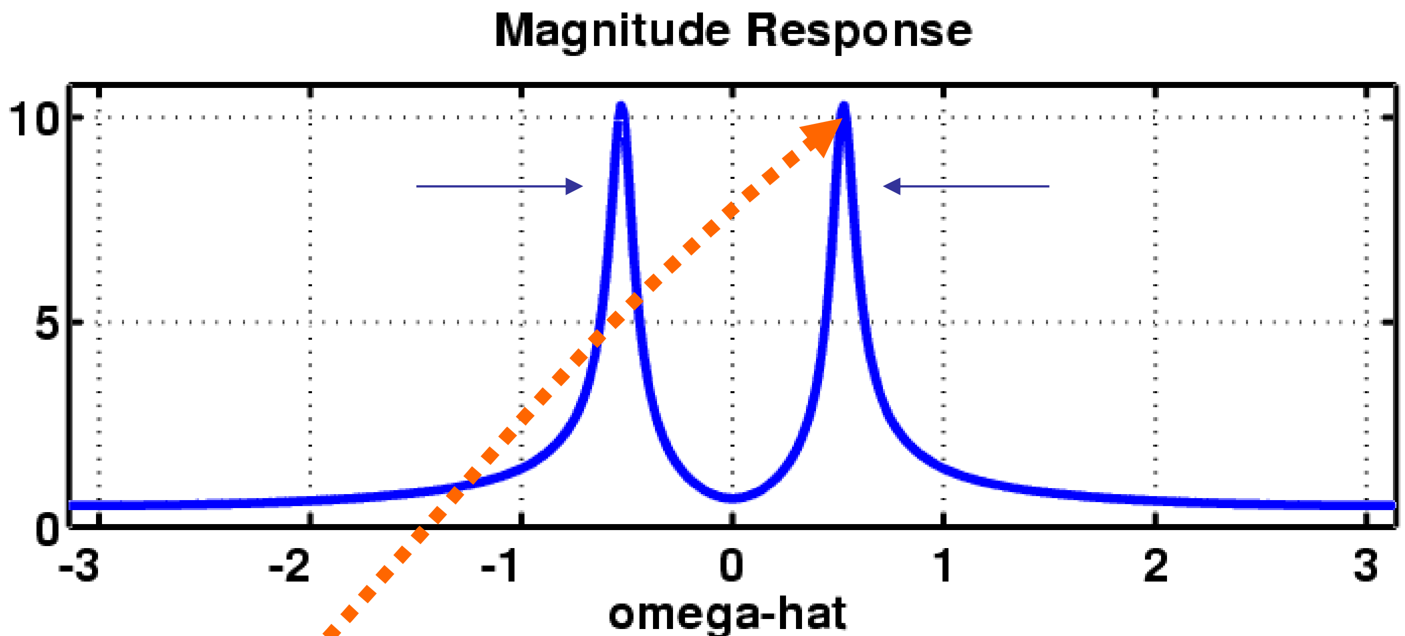


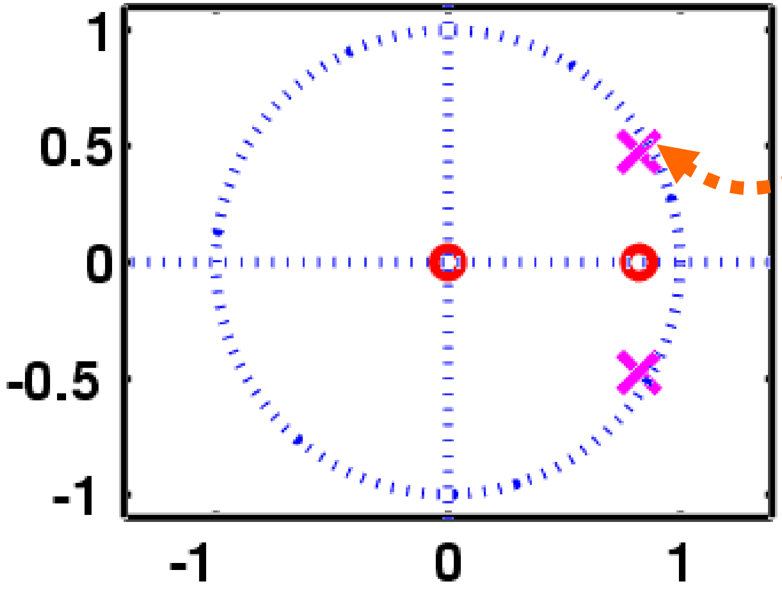


$$h[n] = (0.95)^n \cos(\tfrac{\pi}{6} n) u[n]$$

$$\frac{1 - 0.8227 z^{-1}}{1 - 1.6454 z^{-1} + 0.9025 z^{-2}}$$

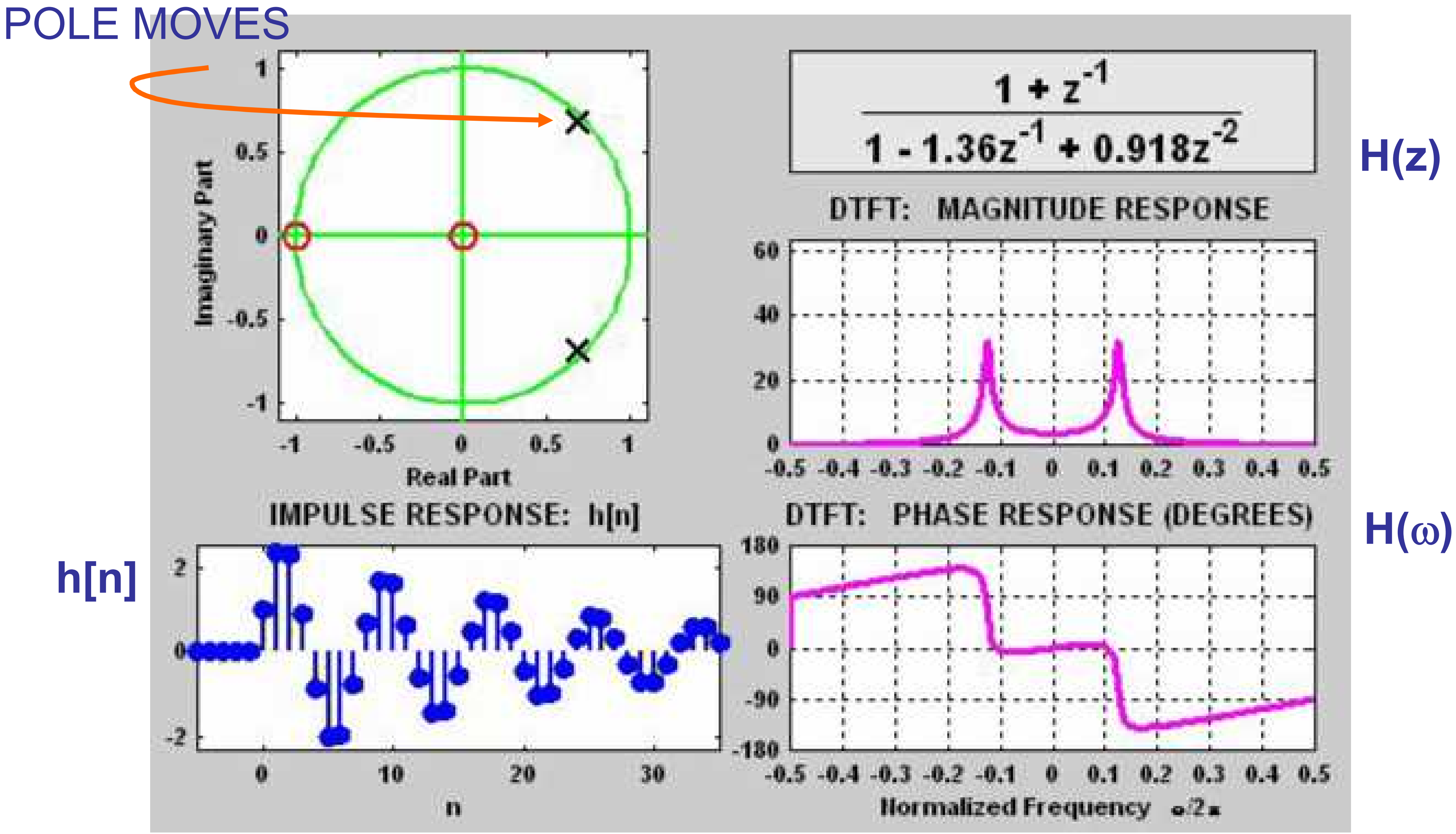
Complex POLE-ZERO PLOT





$$\frac{1-0.8227z^{-1}}{1-1.6454z^{-1}+0.9025z^{-2}}$$

3 DOMAINS MOVIE: IIR



7 IIR MOVIES @ WEBSITE

- **http://dspfirst.gatech.edu/chapters/08feedbac/demos/3_domain/index.html**
- 3 DOMAINS MOVIES: <u>IIR</u> Filters
  - One pole moving and a zero at the origin
  - One pole and one zero; both moving
  - Two complex-conjugate poles moving radially
  - Two complex-conjugate poles moving in angle
  - Movement of a zero in a two-pole Filter
  - Radial Movement of Two out of Four Poles
  - Angular Movement of Two out of Four Poles

A Useful IIR Application: Remove Interference → Hands-on

- Design a **NOTCH filter** (Find a_k and b_k)
  1. To **Reject** completely 0.7π
     - NULLING $\hat{\omega} = 0.7\pi$ → Zeros on unit circle

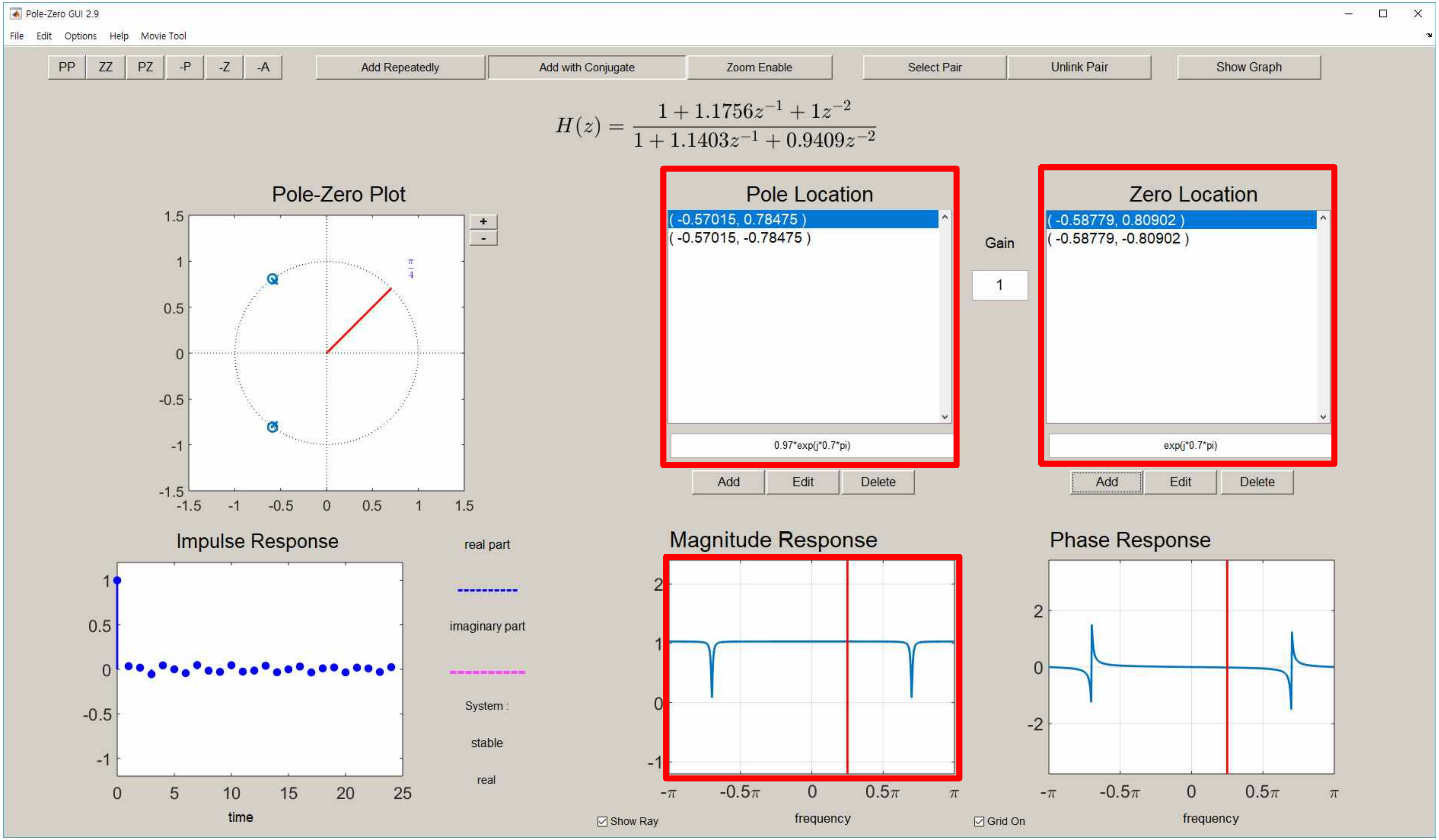
       $$\boxed{\text{two zeros: } z = e^{\pm j0.7\pi}}$$

  2. Make the frequency response magnitude FLAT away from the notch.
     - Use poles at the **same angle(=freq.)**

       $$\boxed{\text{two poles: } z = 0.97e^{\pm j0.7\pi}}$$

- Z-POLYNOMIALS provide the TOOLS
  - PEZDEMO GUI

PeZ Demo: IIR Notch Filter



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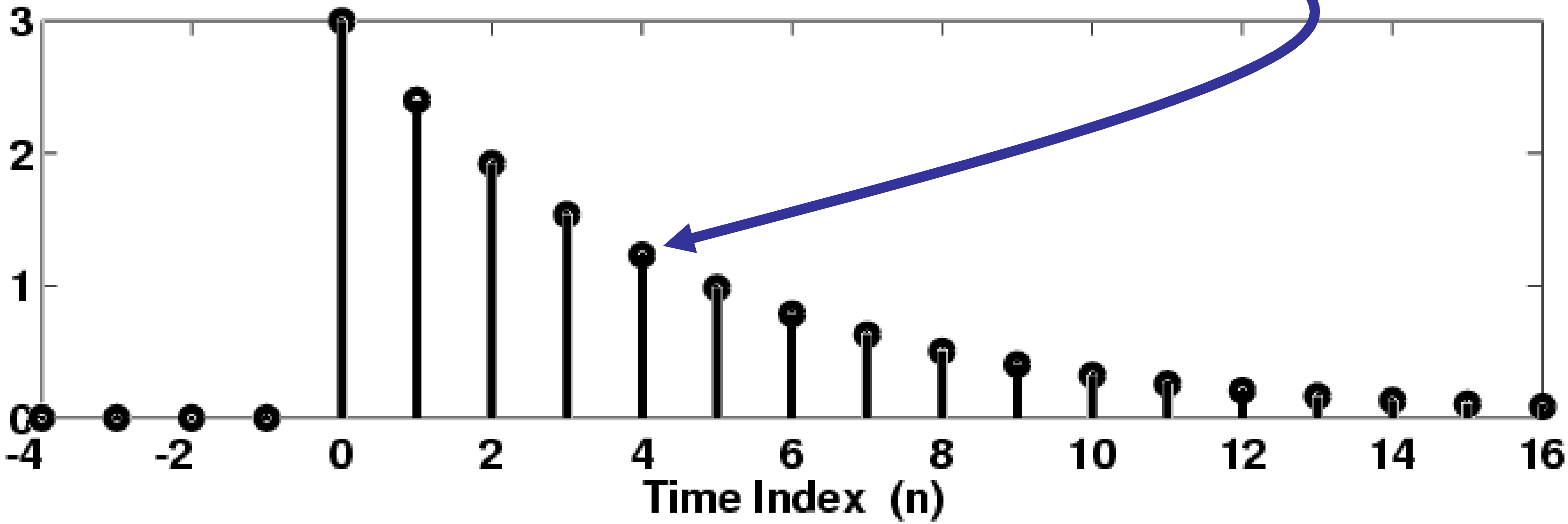
Remind IMPULSE RESPONSE of IIR Filter.

value of a_1

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



Infinite length !

Stability in IIR

- Nec. & suff. condition for IIR:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

$$h[n] = b(a)^n u[n] \Leftrightarrow H(z) = \frac{b}{1-az^{-1}}$$

$$\sum_{n=0}^{\infty} |b||a|^n < \infty \text{ if } \boxed{|a| < 1} \Rightarrow$$

Pole must be Inside unit circle

Region of convergence (ROC)

STABILITY CONDITION

- ALL POLES INSIDE the UNIT CIRCLE
- UNSTABLE EXAMPLE: $|a| \geq 1$

$$y[n] = \mathbf{1.1}y[n-1] + 5x[n]$$

$$\boxed{x[n] = \cos(0.2\pi n)u[n]}$$

POLE @ z=1.1

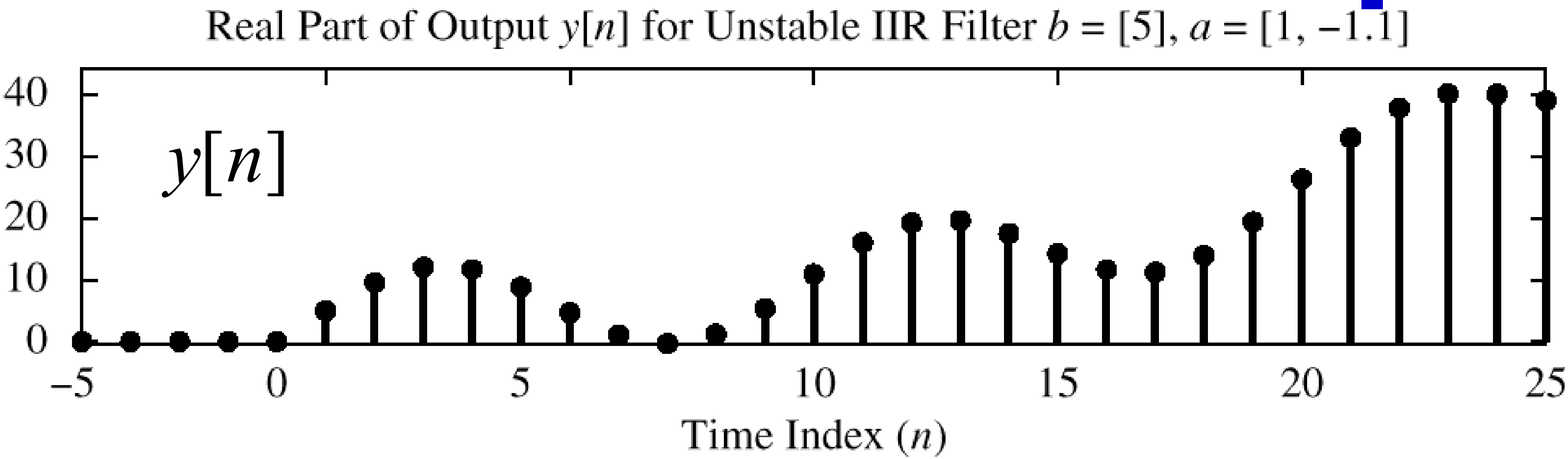


Figure 8.15 Illustration of an unstable IIR system. Pole is at $z = 1.1$.

Summary

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