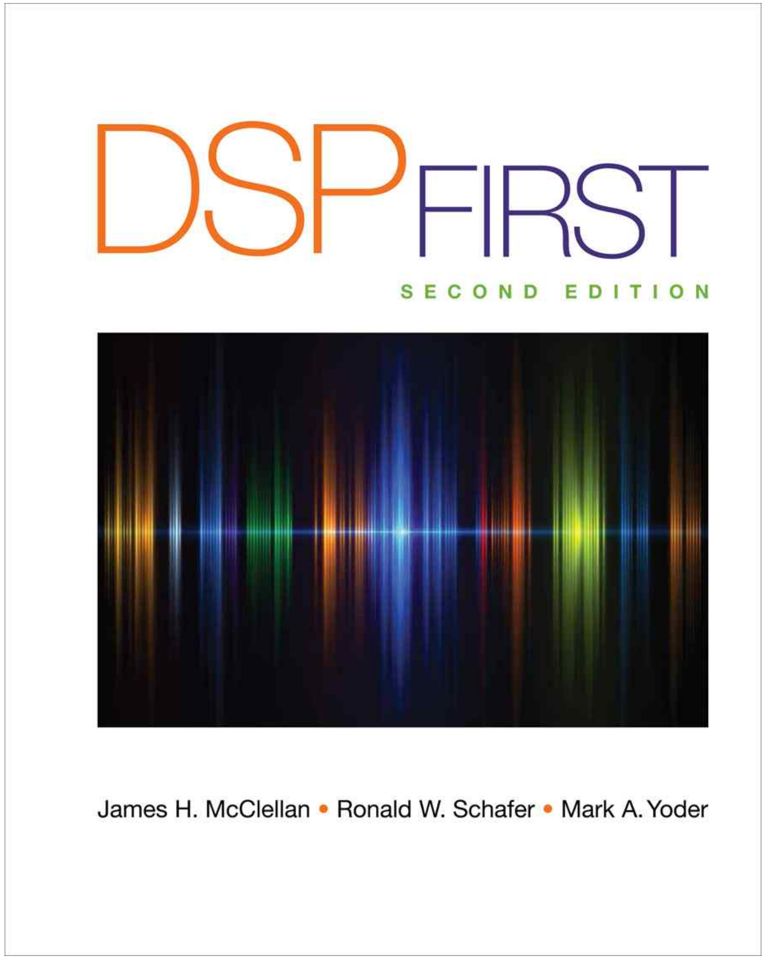
DSP First

Second Edition



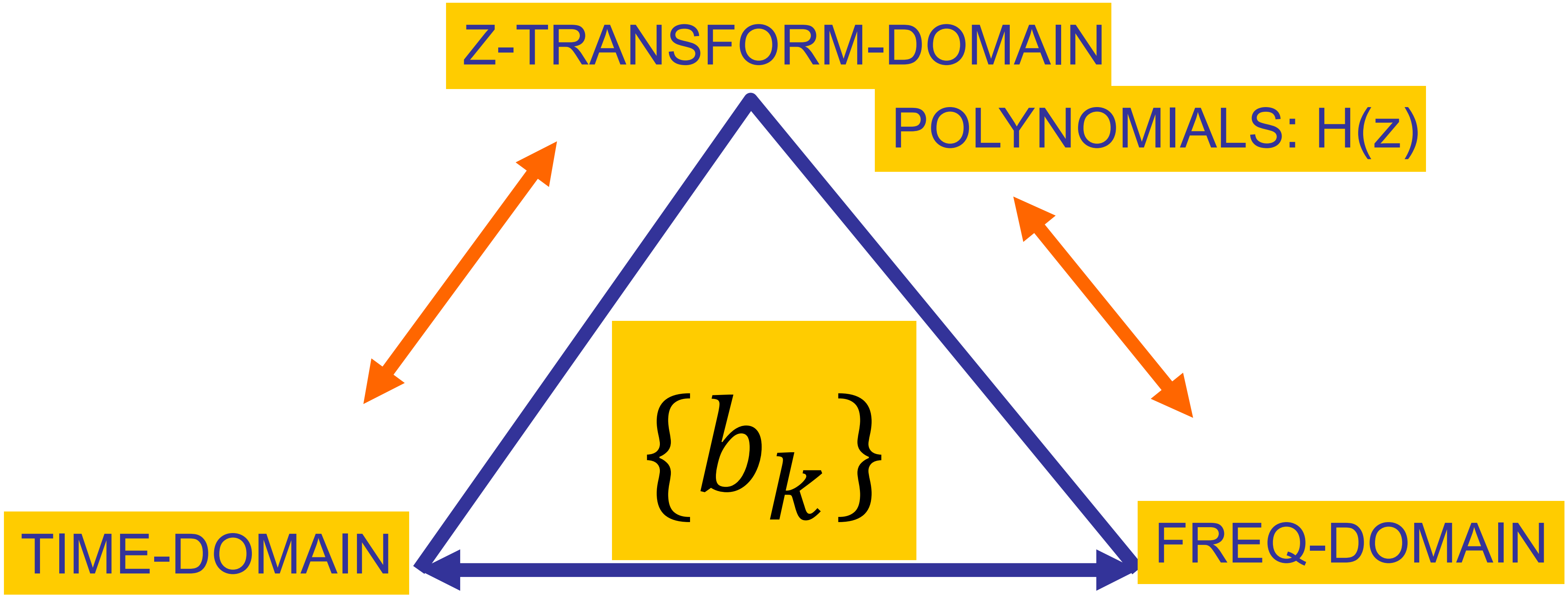
CHAPTER 9

z-Transforms

Contents

- Definition of the z-Transform
- z-Transform Properties and Convolution
- z-Domain vs. frequency domain
- Filter Design in z-Domain: Zeros and Poles of H(z)

TWO (no, THREE) DOMAINS



$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k\, e^{-j\hat{\omega}k}$$

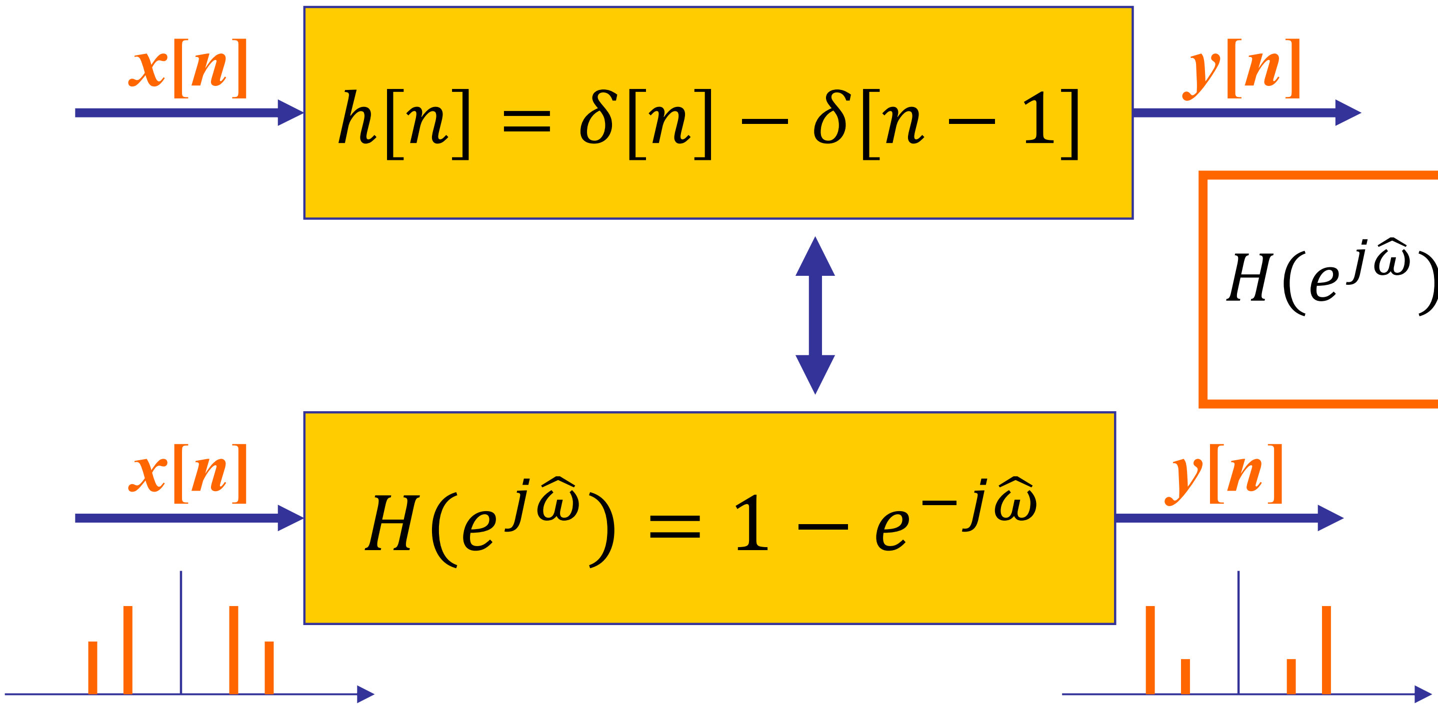
z-TRANSFORM CONCEPT

- Move to a new domain where
  - OPERATIONS are SIMPLER & FAMILIAR
  - Use simple POLYNOMIALS

- TRANSFORM both ways
  - $x[n] \rightarrow X(z)$ (into the z domain)
  - $X(z) \rightarrow x[n]$ (back to the time domain)

"TRANSFORM" EXAMPLE: DTFT

- Equivalent Representations



DTFT

$$H(e^{j\hat{\omega}}) = \sum_{n} h[n] e^{-j\hat{\omega} n}$$

Z-TRANSFORM IDEA

- For any complex number z, $H(z)$ is represented with polynomials.

DTFT

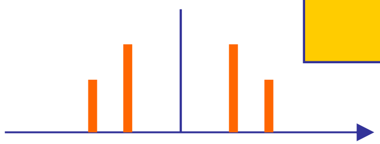
$$H(e^{j\hat{\omega}}) = \sum_{n=0}^{M} h[n]\, e^{-j\hat{\omega} n}$$

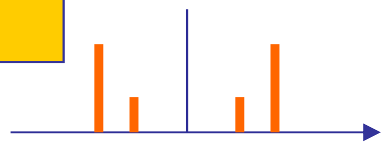
$$\boxed{H(z) = \sum_{n} h[n] z^{-n}}$$

$x[n]$ → $h[n] = \delta[n] - \delta[n-1]$ → $y[n]$

↕

$x[n]$ → $H(z) = 1 - z^{-1}$ → $y[n]$





9-1 Z-Transform DEFINITION

- Finite length signal x[n]

$$x[n] = \sum_{k=0}^{L-1} b_k \delta[n-k]$$

- Z-transform of x[n]

$$X(z) = \sum_{k=0}^{L-1} x[k] z^{-k}$$

9-2 z-Transform Property: Linearity

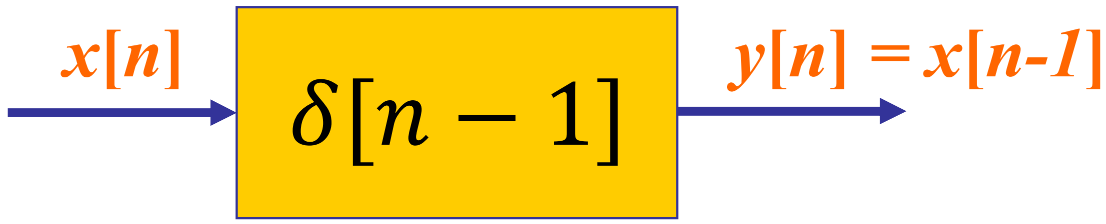
- Can be proven that z-transform satisfies the superposition(addition and scaling) property.

$$x[n] = ax_1[n] + bx_2[n]$$

$$X(z) = \sum_{n=0}^{L-1} x[n]z^{-n} = \sum_{n=0}^{L-1} (ax_1[n] + bx_2[n])z^{-n}$$

$$= a\sum_{n=0}^{L-1} x_1[n]z^{-n} + b\sum_{n=0}^{L-1} x_2[n]z^{-n} = aX_1[z] + bX_2[z]$$

Review: Unit Delay



- y[n]: x[n] is delayed by 1

- Example
  - x[n] = {3,1,4,1,5,9} $\rightarrow$ y[n] = {0,3,1,4,1,5,9}

z-Transform Property: Time-Delay

- UNIT DELAY: find y[n] via polynomials
  - x[n] = {3,1,4,1,5,9,0,0,0,...}

n	$n<0$	0	1	2	3	4	5	6	$n>6$
$y[n]$	0	0	3	1	4	1	5	9	0

Relationship between X(z) and Y(z) in Z-domain ?

$$Y(z) = 0z^0 + 3z^{-1} + z^{-2} + 4z^{-3} + z^{-4} + 5z^{-5} + 9z^{-6}$$

$$Y(z) = z^{-1}(3 + z^{-1} + 4z^{-2} + z^{-3} + 5z^{-4} + 9z^{-5})$$

$$Y(z) = z^{-1}X(z)$$

Multiply by z^{-1} so that Y(Z) causes the delay by 1 from X(z)

DELAY PROPERTY

A delay of one sample multiplies the z-transform by z^{-1}.

$$x[n-1] \iff z^{-1}X(z)$$

Time delay of n_0 *samples multiplies the z-transform by* z^{-n_0}

$$x[n-n_0] \iff z^{-n_0}X(z)$$

Z-Transform EXAMPLE: forward

- ANY SIGNAL has a z-Transform:

$$X(z) = \sum_{n} x[n]z^{-n}$$

n	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

$$X(z) = ?$$

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

Z-Transform EXAMPLE: backward

$$X(z) = 1 - 2z^{-1} + 3z^{-3} - z^{-5}$$

$$x[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ -2 & n = 1 \\ 0 & n = 2 \\ 3 & n = 3 \\ 0 & n = 4 \\ -1 & n = 5 \\ 0 & n > 5 \end{cases}$$

EXPONENT GIVES TIME LOCATION

$$x[n] = ?$$

$$x[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-3] - \delta[n-5]$$

9-3,4 z-Transform of an FIR Filter

- Impulse response of FIR filter

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \quad \longrightarrow \quad h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

- z-Transform of h[n] : SYSTEM FUNCTION

$$H(z) = \sum_{k=0}^{M} h[k] z^{-k}$$

Convolution & z-Transform

- FIR difference equation

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

- z-Transform of FIR filter

$$Y(z) = \underbrace{\sum_{k=0}^{M} b_k}_{linearity} \underbrace{\left(z^{-k} X(z)\right)}_{delay\ property} = \underbrace{\left(\sum_{k=0}^{M} b_k z^{-k}\right)}_{H(z)} X(z)$$

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] \Leftrightarrow Y(z) = H(z)X(z)$$

CONVOLUTION EXAMPLE

- **<u>MULTIPLY</u>** the z-TRANSFORMS:

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$

and $$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

and $$H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

MULTIPLY *H*(*z*)*X*(*z*)

CONVOLUTION EXAMPLE

- Finite-Length input x[n]
- FIR Filter (L=4)

MULTIPLY Z-TRANSFORMS

$$Y(z) = H(z)X(z)$$

$$= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(z^{-1} - z^{-2} + z^{-3} - z^{-4})$$

$$= z^{-1} + (-1 + 2)z^{-2} + (1 - 2 + 3)z^{-3} + (-1 + 2 - 3 + 4)z^{-4}$$

$$+ \quad (-2 + 3 - 4)z^{-5} + (-3 + 4)z^{-6} + (-4)z^{-7}$$

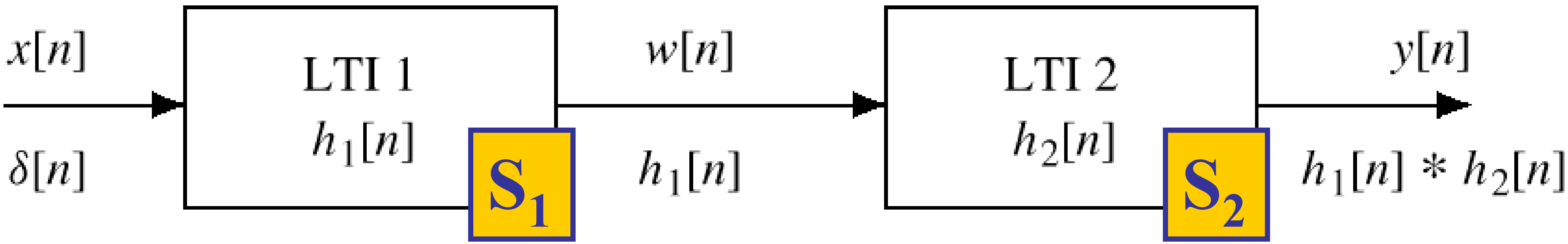
$$= z^{-1} + z^{-2} + 2z^{-3} + 2z^{-4} - 3z^{-5} + z^{-6} - 4z^{-7}$$

$y[n] = ?$

y[n], Y(z)	0	+1	+1	+2	+2	-3	+1	-4

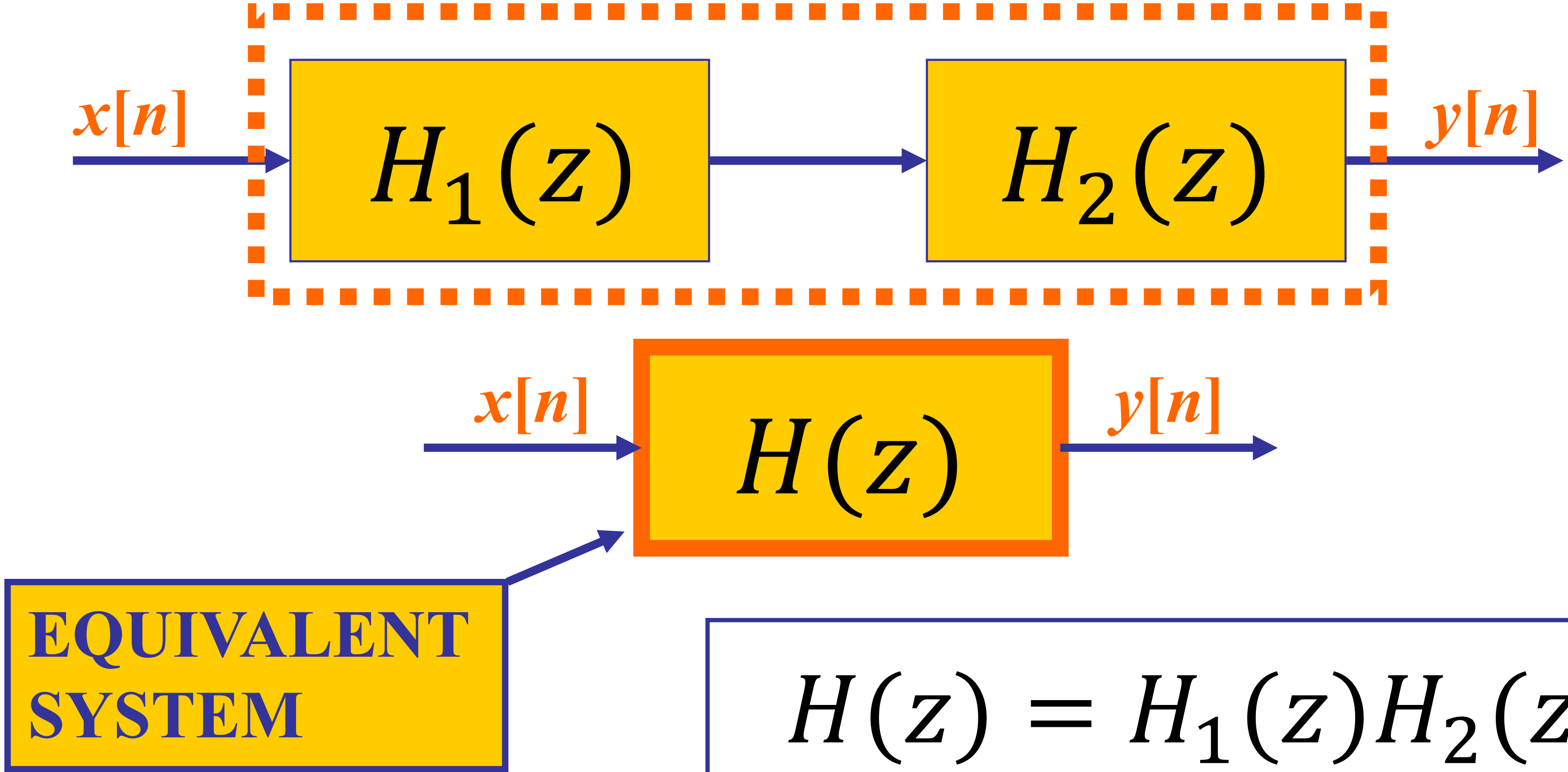
CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
  - NO, LTI SYSTEMS can be rearranged !!!
  - Remember: $h_1[n] * h_2[n]$
  - How to combine $H_1(z)$ and $H_2(z)$?



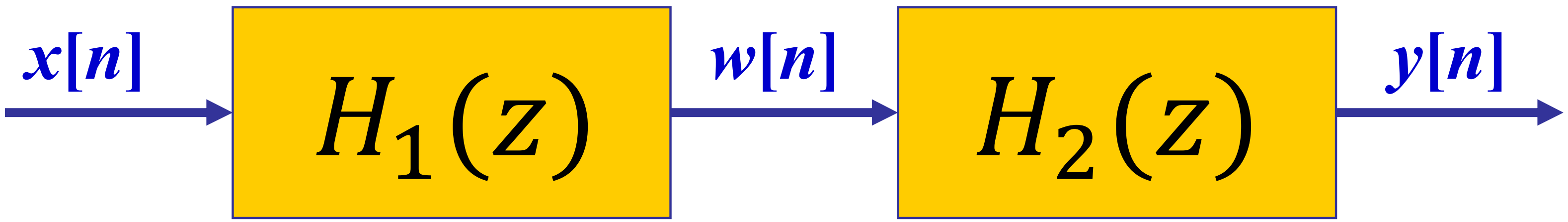
CASCADE EQUIVALENT

- Multiply the System Functions



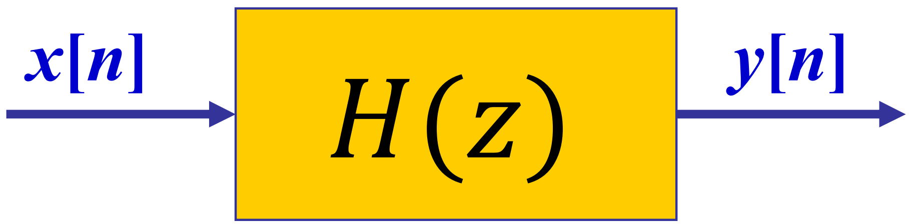
$$H(z) = H_1(z)H_2(z)$$

CASCADE EXAMPLE



$$w[n] = x[n] - x[n-1] \qquad y[n] = w[n] + w[n-1]$$

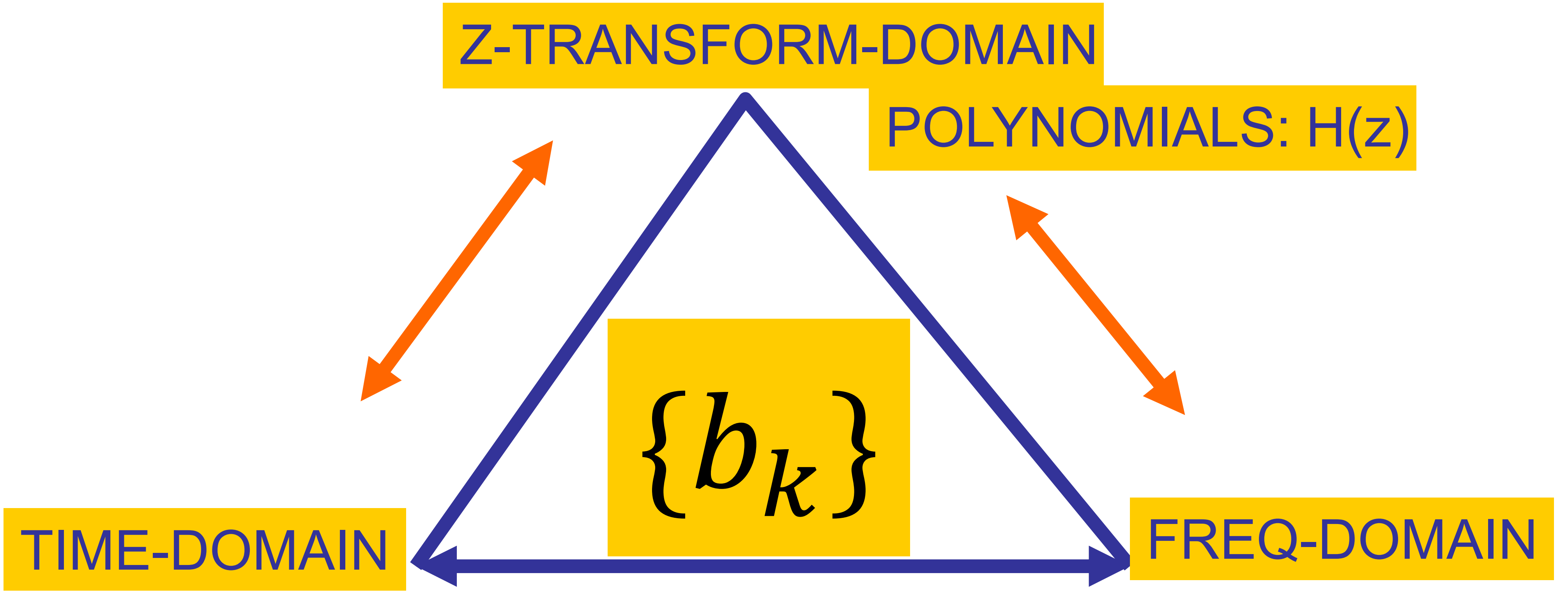
$$H_1(z) = 1 - z^{-1} \qquad H_2(z) = 1 + z^{-1}$$



$$H(z) = (1 - z^{-1})(1 + z^{-1}) = 1 - z^{-2}$$

$$y[n] = x[n] - x[n-2]$$

9-5~7 Relationship between z-Domain and Freq. Domain



$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k\, e^{-j\hat{\omega}k}$$

Review: FREQUENCY RESPONSE ?

- Same Form:

$\hat{\omega}$ − Domain

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k \, e^{-j\hat{\omega}k}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k \, (e^{j\hat{\omega}})^{-k}$$

$$z = e^{j\hat{\omega}}$$

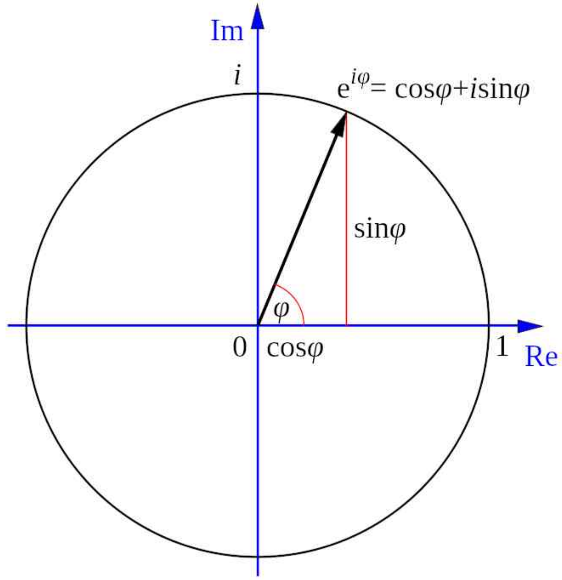
z − Domain

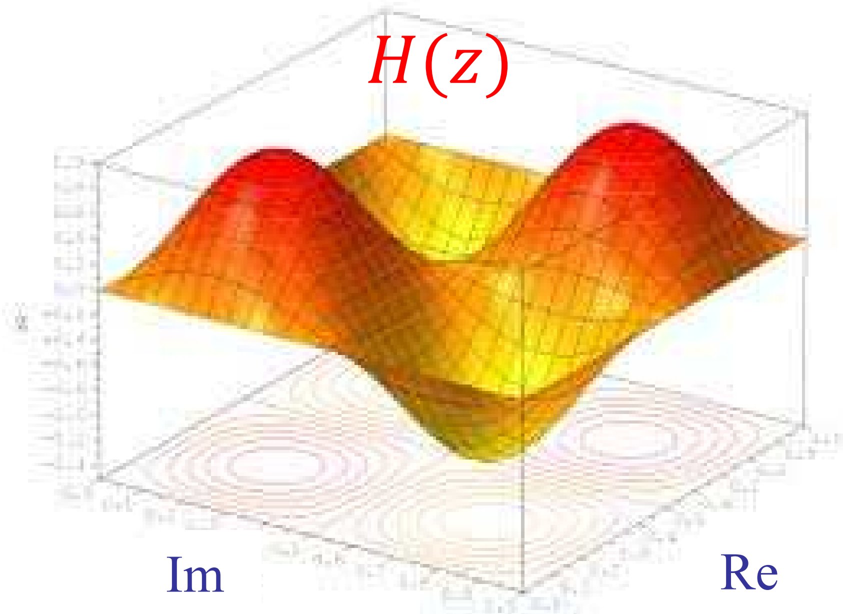
$$H(z) = \sum_{k=0}^{M} b_k \, z^{-k}$$

SAME COEFFICIENTS

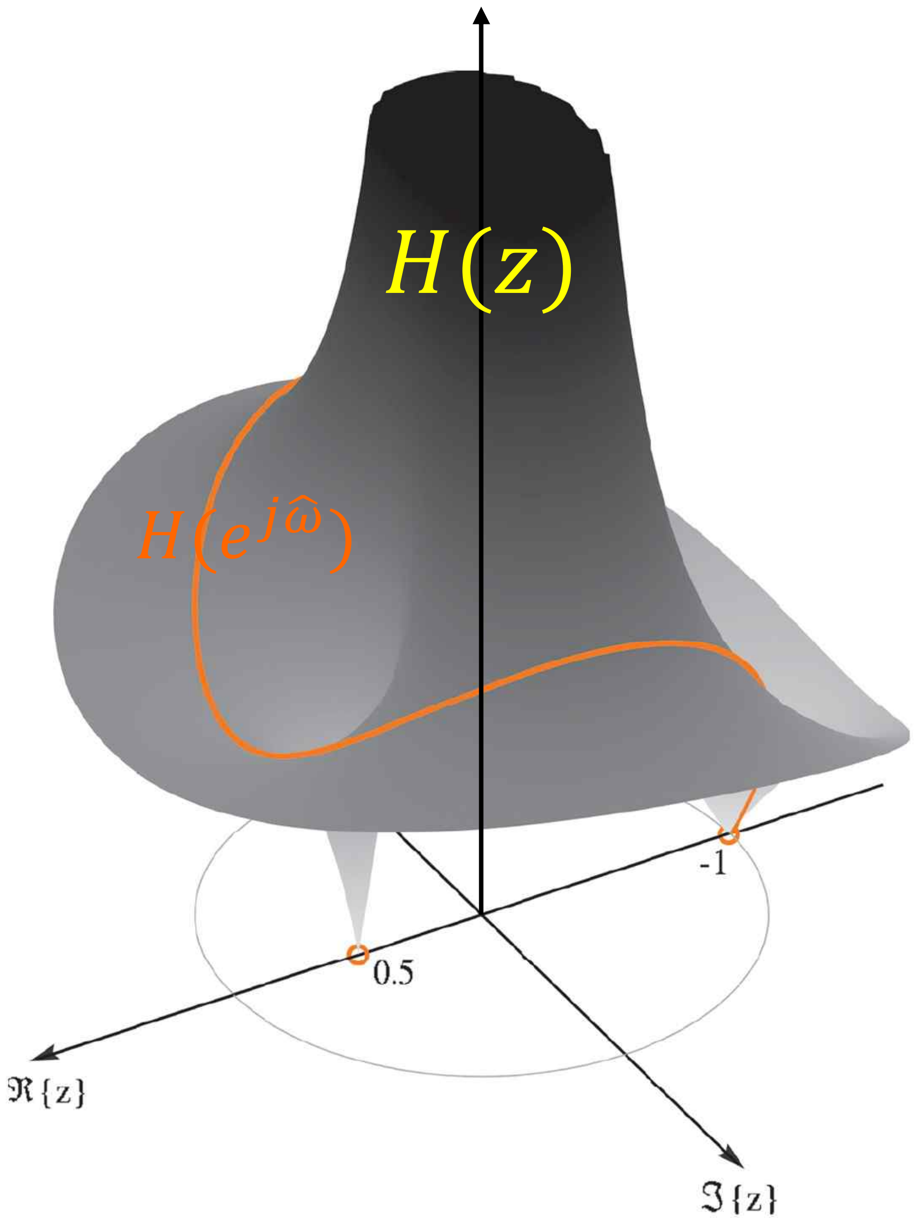
$H(z)$ & $H(e^{j\hat{\omega}})$ in a Complex Plane

- What is $e^{j\hat{\omega}}$ in a complex plane?
- What is z in a complex plane?
- What is $H(z)$ in a complex plane ?
- What is $H(e^{j\hat{\omega}})$ in a complex plane ?





Evaluate H(z) on Unit Circle



$$H(z) = 1 + 0.5z^{-1} - 0.5z^{-2}$$

$$= (1 + z^{-1})(1 - 0.5z^{-1})$$

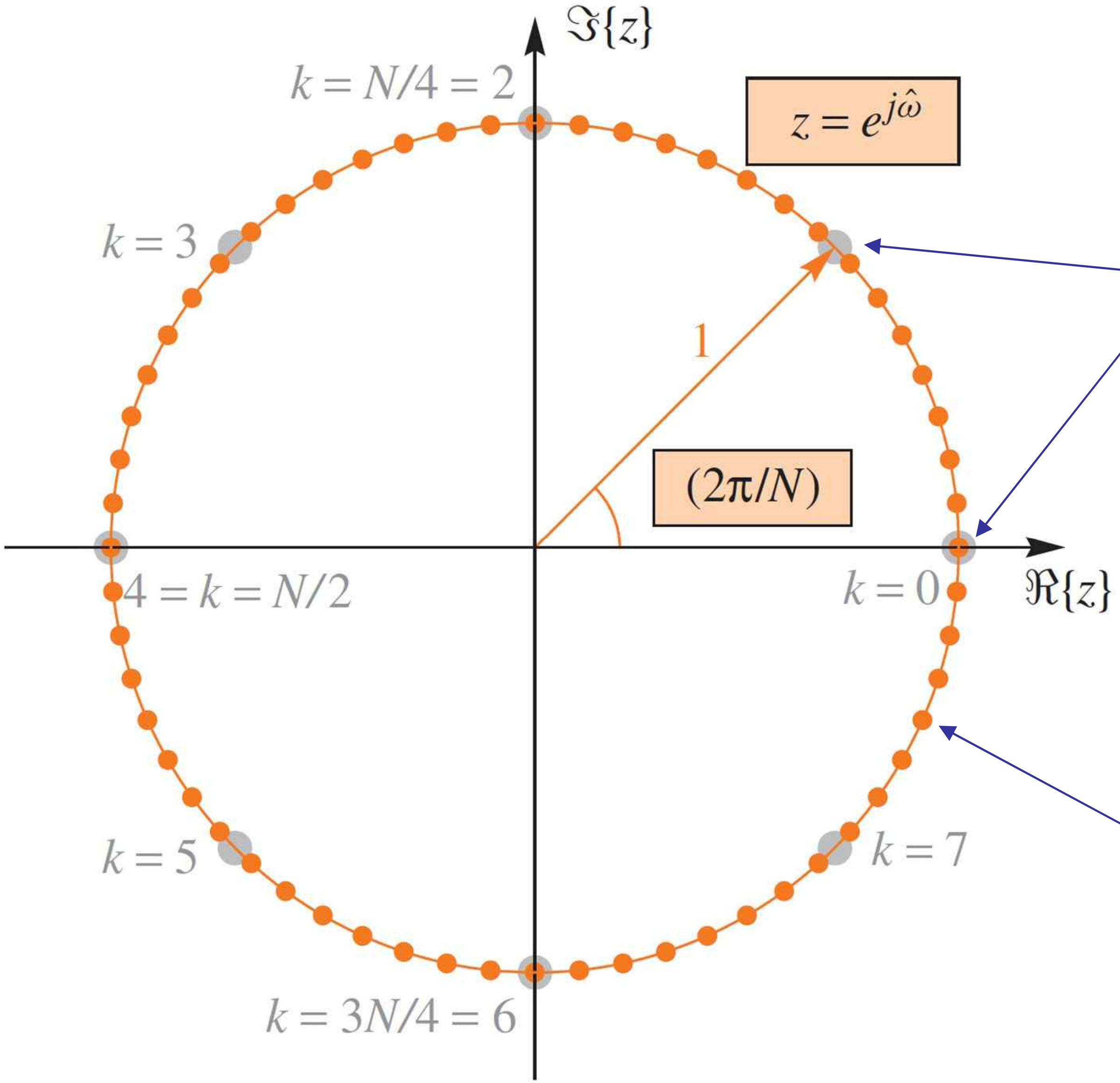
DTFT

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

zeros at $z = -1$ and $z = 0.5$

[Q] What are DFT $H[k]$ in the plot?

DFT By Sampling DTFT



Gray dots:
N = 8-point DFT, i.e., at the points $z_k = e^{j\left(\frac{2\pi}{8}\right)k}$ with $k = 0, 1, \ldots, 7$

Orange dots:
are locations where values of $X(z)$ are evaluated by a 60-point DFT.

Z-Transform:
ANOTHER ANALYSIS TOOL

- z-Transform POLYNOMIALS are EASY !
    - ROOTS, FACTORS, etc.
- Can easily choose ZEROS and POLES so we design whatever filter we want to make ➔

  Location of zeros and poles is important to characterize the filter response.
- The z-domain is COMPLEX.

➔ H(z) is a COMPLEX-VALUED function of a COMPLEX VARIABLE z.

ZEROS & Poles of H(z)

$$H(z) = 1 - 0.5z^{-1} = 1 - \frac{0.5}{z} = \frac{z - 0.5}{z}$$

- Find z, where $H(z) = 0$ → zeros
  - H(z) becomes zero at z = 0.5.

  ➔ H(z) has a zero at z = 0.5.

- Find z, where $H(z) = \infty$ → poles
  - H(z) becomes infinite at z = 0.

  ➔ H(z) has a pole at z = 0.

EX) ZEROS of H(z)

- Find z, where H(z)=0
  - Interesting when z is ON the unit circle.

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = \frac{z^3(1 - 2z^{-1} + 2z^{-2} - z^{-3})}{z^3}$$

$$= \frac{z^3 - 2z^2 + 2z - 1}{z^3} = \frac{(z-1)(z^2 - z + 1)}{z^3}$$

$$= (1 - z^{-1})(1 - z^{-1} + z^{-2})$$

$$\text{Roots: } z = 1, \ \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$$e^{\pm j\pi/3}$$

EX) POLES of H(z)

- Find z, where $H(z) \to \infty$
  - Not very interesting for the FIR case

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

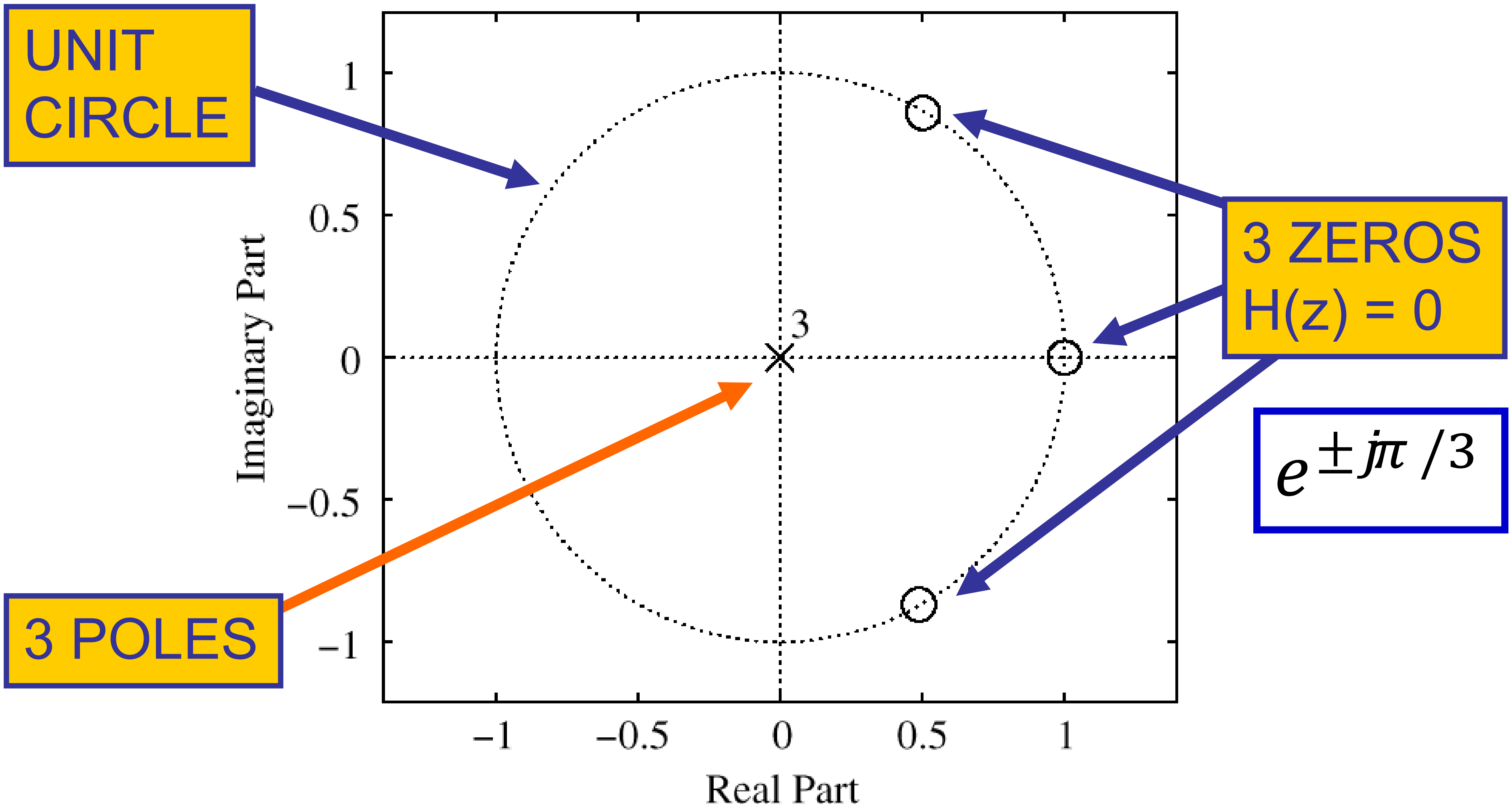
$$H(z) = \frac{z^3 - 2z^2 + 2z - 1}{z^3}$$

Three Poles at: $z = 0$

EX) PLOT ZEROS in z-DOMAIN:

3 Nulling points of the frequency response

$$H(z) = (1 - z^{-1})(1 - z^{-1} + z^{-2}) \qquad z = 1,\ \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$



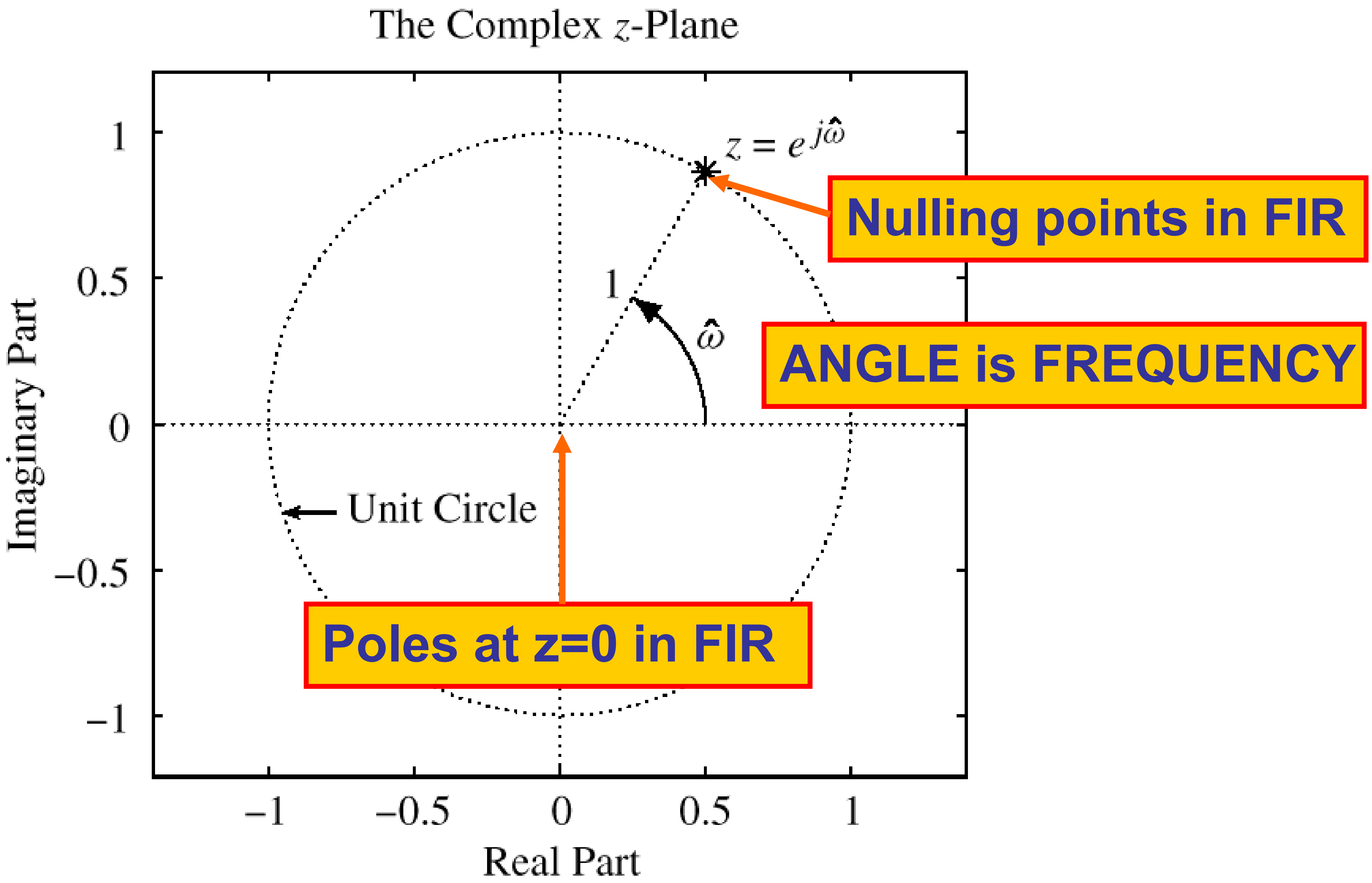
EX) FREQ. RESPONSE from ZEROS

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

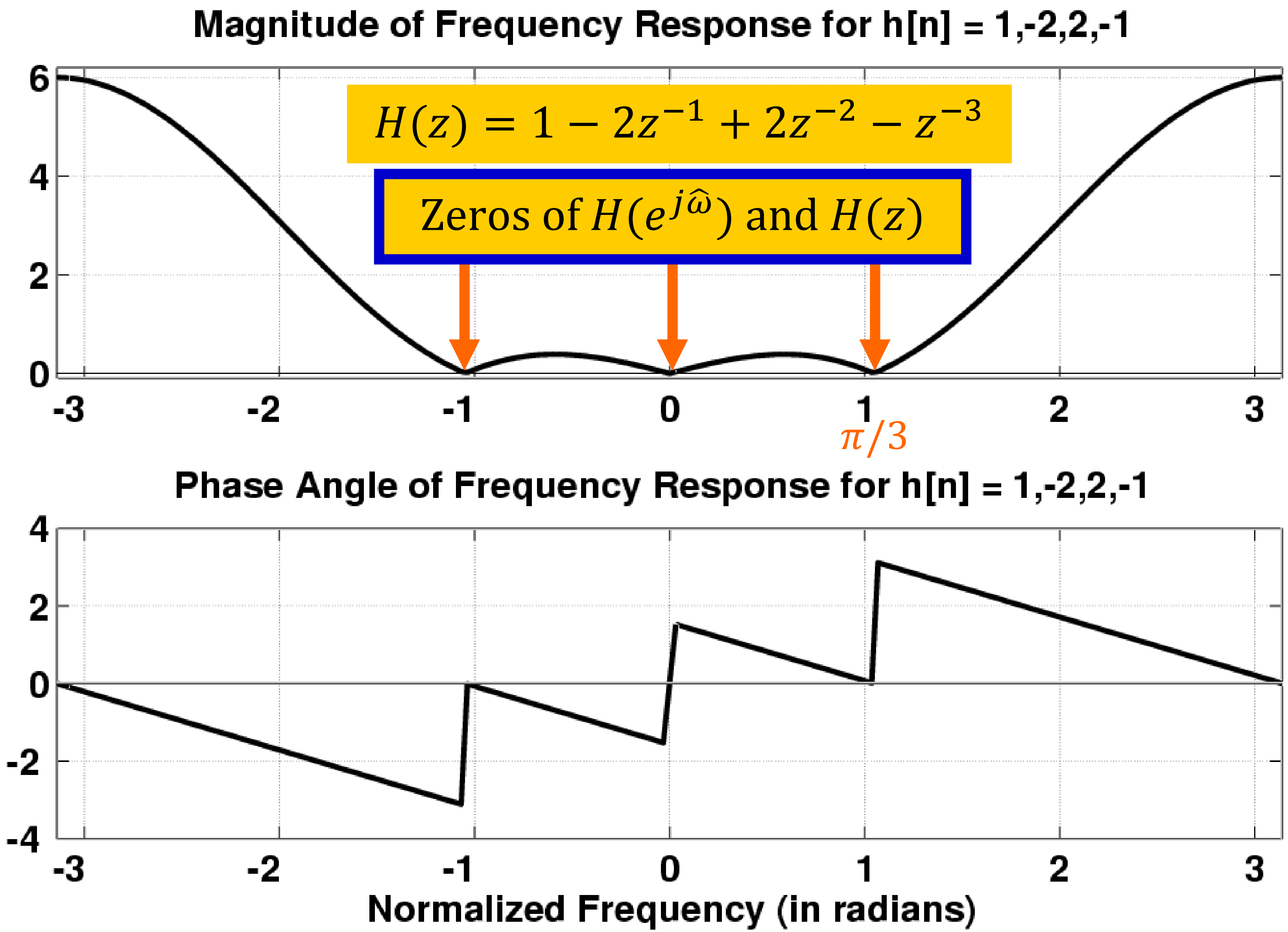
- Relate H(z) to FREQUENCY RESPONSE
- EVALUATE H(z) on the UNIT CIRCLE
  - ANGLE $\hat{\omega}$ is same as FREQUENCY

$z = e^{j\hat{\omega}}$ (as $\hat{\omega}$ varies) defines a CIRCLE, radius $= 1$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$



EX) FIR Frequency Response



EX) NULLING FILTER Design

- PLACE ZEROS to make y[n] = 0

3 ZEROS
H(z) = 0

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

the output resulting from each of the following three signals will be zero:

$H(z_1) = 0$	$x_1[n] = (z_1)^n = 1$	$y_1[n] = 0$
$H(z_2) = 0$	$x_2[n] = (z_2)^n = e^{j\pi n/3}$	$y_2[n] = 0$
$H(z_3) = 0$	$x_3[n] = (z_3)^n = e^{-j\pi n/3}$	$y_3[n] = 0$

Another example:
L-pt RUNNING Average H(z)

sum of geometric series (등비수열) $\sum_{n=0}^{\infty} ar^n = \frac{a(1-r^n)}{1-r} \Rightarrow \frac{a}{1-r}$,|r| < 1

$$H(z) = \sum_{k=0}^{L-1} \frac{1}{L} z^{-k} = \frac{1-z^{-L}}{L(1-z^{-1})} = \frac{z^L - 1}{Lz^{L-1}(z-1)}$$

L-1 POLES at $z = 0$
one POLE at $z = 1$

$$z^L - 1 = 0 \quad \Rightarrow \quad z^L = 1 = e^{j2\pi k}$$

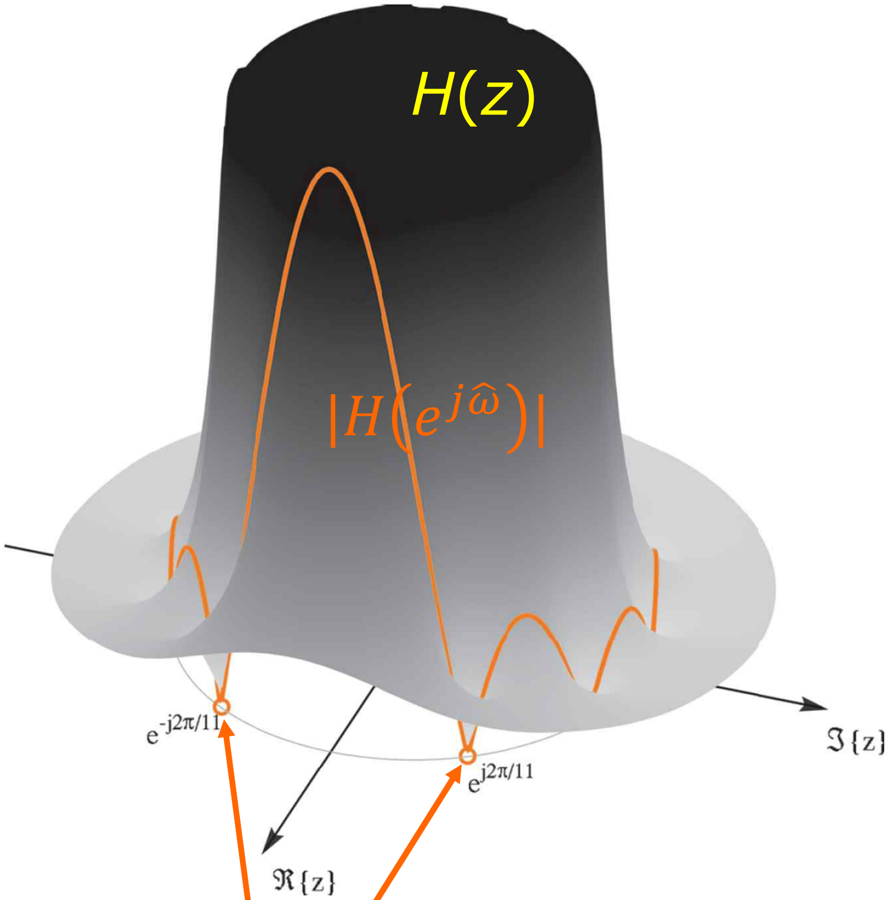
$$z = e^{j(2\pi/L)k} \quad \text{for } k = 1,2,\ldots L-1$$

ZEROS at $\hat{\omega} = \frac{2\pi k}{L}$
on UNIT CIRCLE

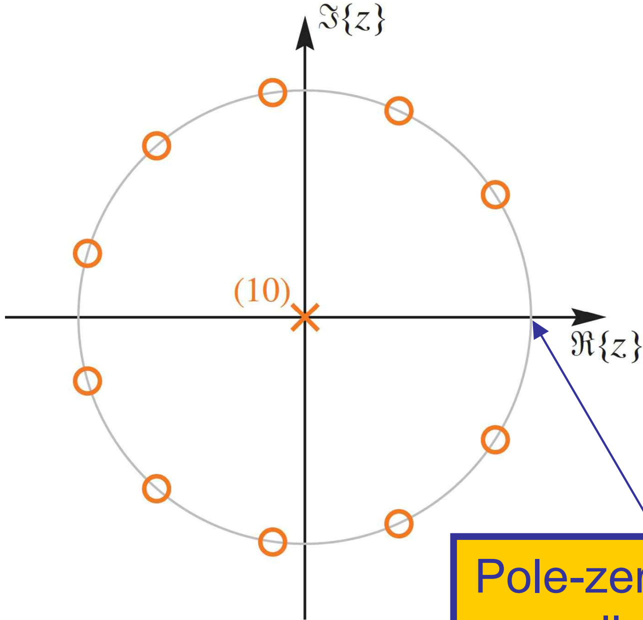
Pole and zero are canceled when $z = 1$ $(k = 0)$.

11-pt RUNNING Average H(z)

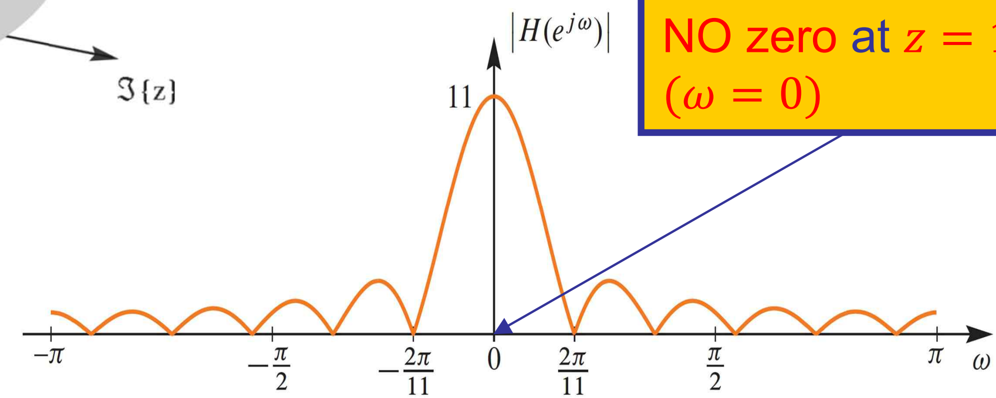
: displayed with the region $|z| \le 1.5$



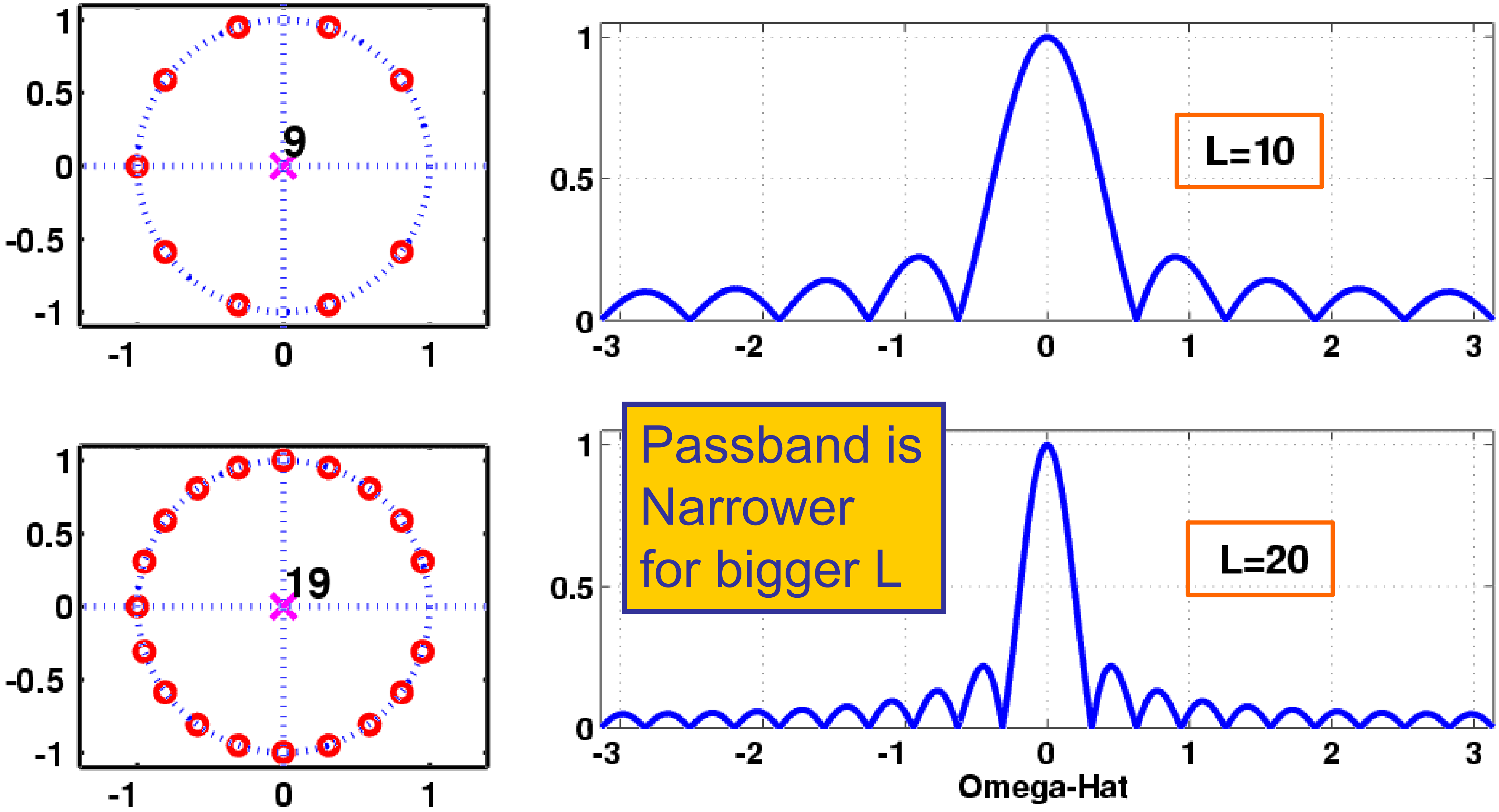
Zeros on the unit circle at $z = e^{\pm j\frac{2\pi}{11}}$



Pole-zero cancellation
NO zero at $z = 1$
$(\omega = 0)$



FILTER DESIGN: CHANGE L



Filter Design Example

→ Z transform needed for easier design

- Design a Lowpass FIR filter → i.e. Find b_k's
- Reject completely 0.7π, 0.8π, and 0.9π
  - NULLING at $\pm 0.7\pi, \pm 0.8\pi, \pm 0.9\pi$
- Estimate the filter length needed to accomplish this task. How many b_k ?

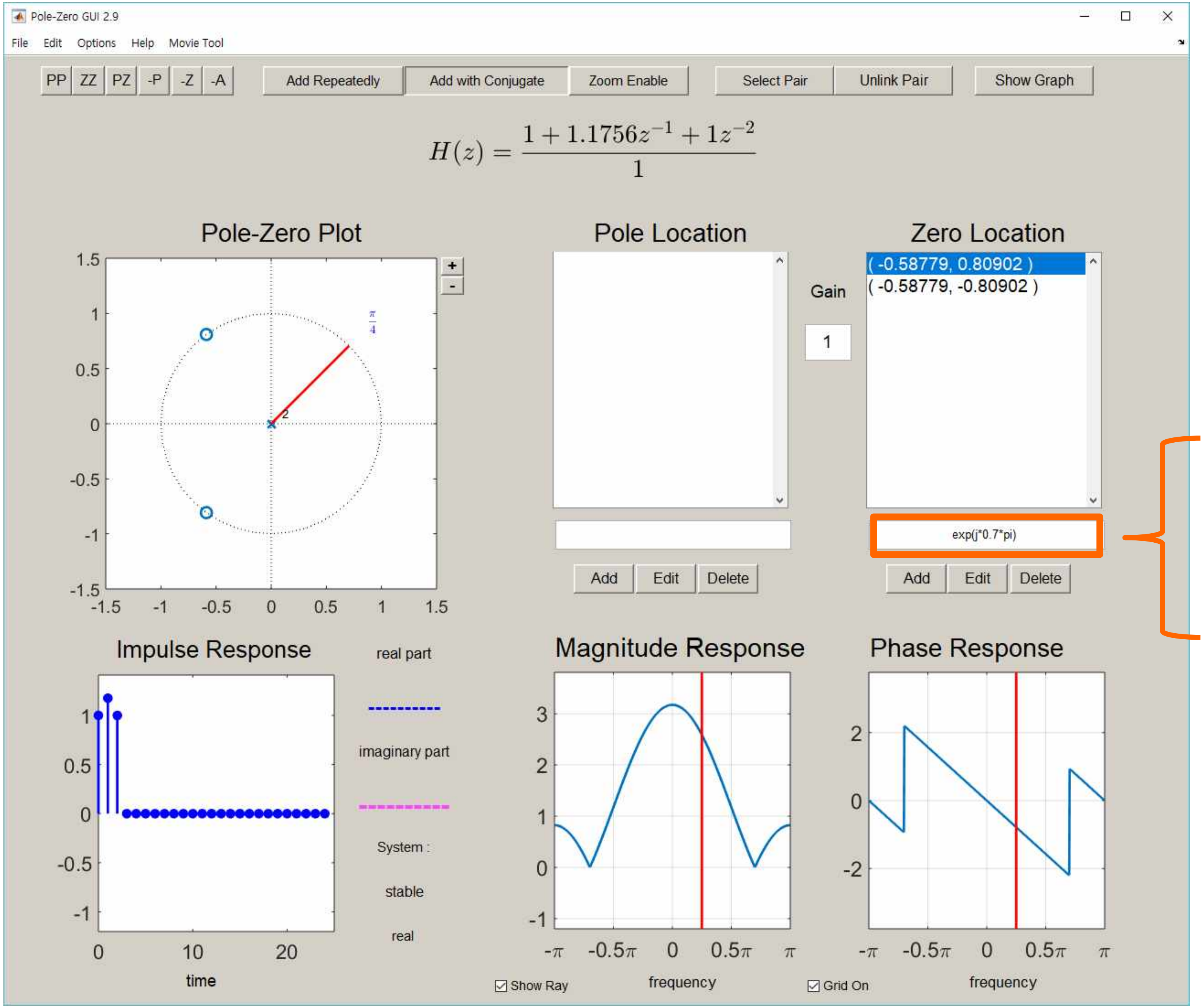
→ Z POLYNOMIALS provide the TOOLS

$$H(z) = \frac{(z - e^{j0.7\pi})(z - e^{-j0.7\pi})(z - e^{j0.8\pi})(z - e^{-j0.8\pi})(z - e^{j0.9\pi})(z - e^{-j0.9\pi})}{z^6}$$

$$= z^{-6}(z - e^{j0.7\pi})(z - e^{-j0.7\pi})(z - e^{j0.8\pi})(z - e^{-j0.8\pi})(z - e^{j0.9\pi})(z - e^{-j0.9\pi})$$

At least 7 b_k's are required!

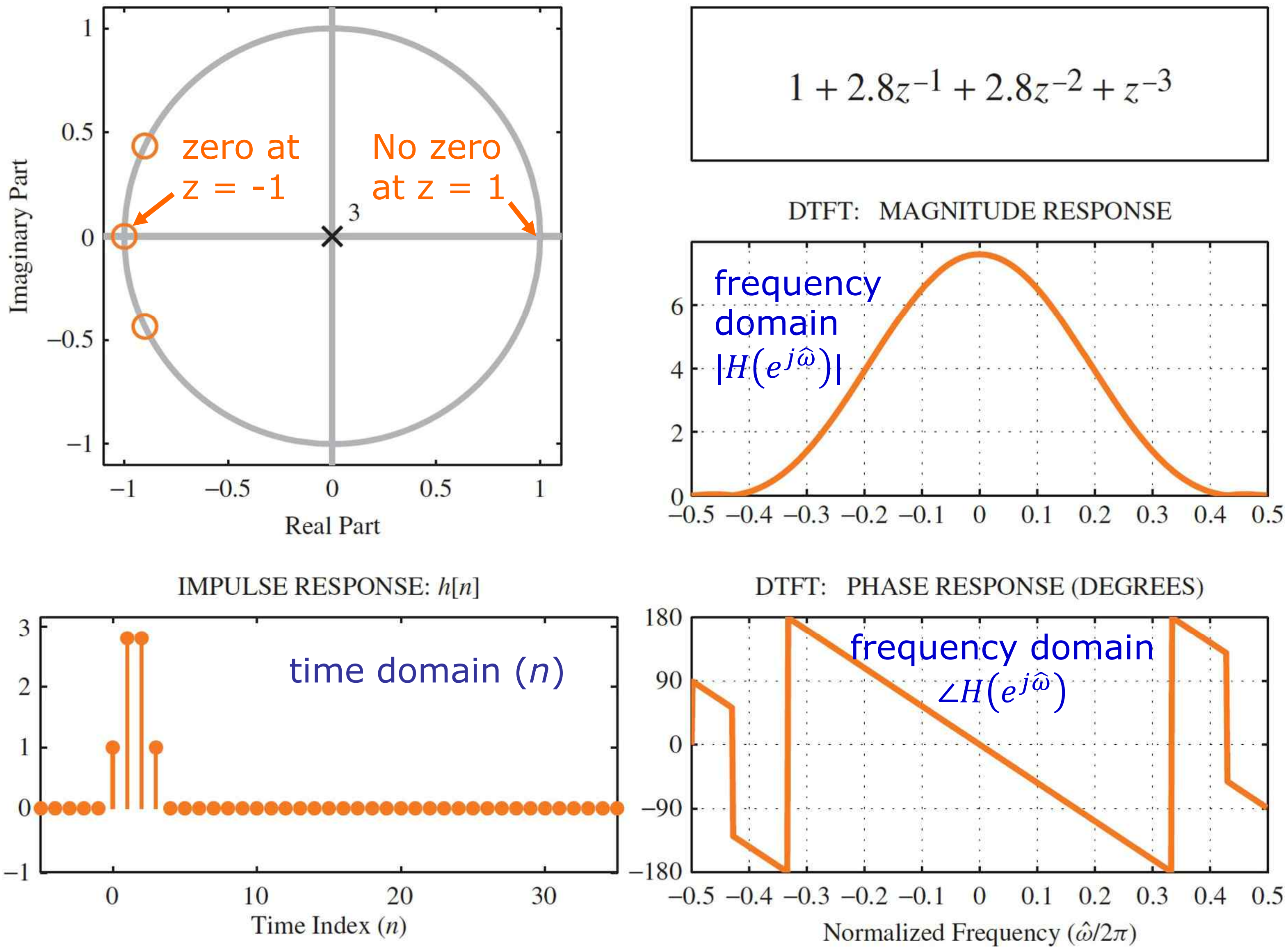
PeZDemo: Zero Placing



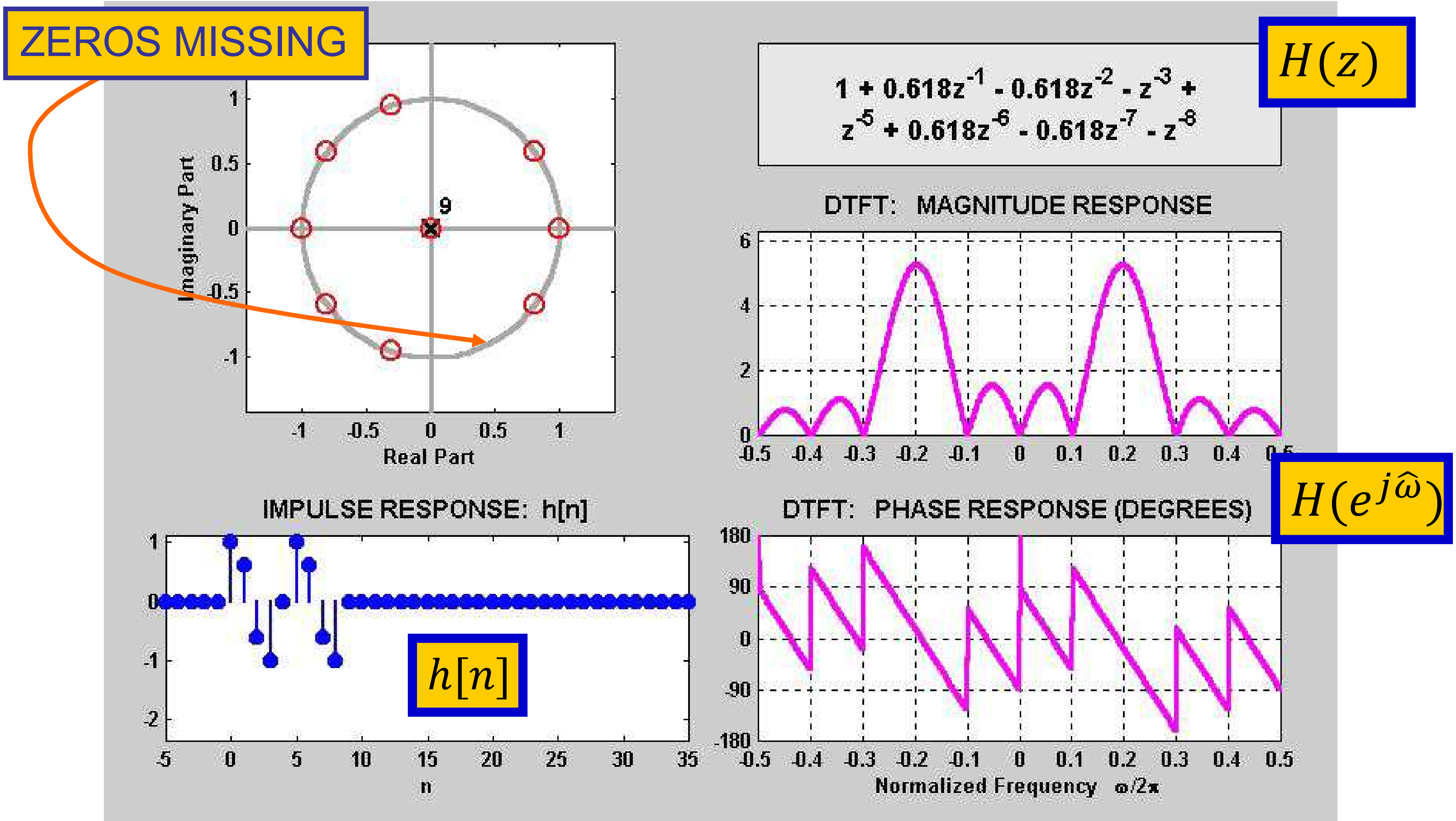
exp(j*0.7*pi)
exp(j*0.8*pi)
exp(j*0.9*pi)

3 Domain Movie: FIR LPF

z transform domain $H(z)$ and Pole Zero Plot



3 DOMAINS MOVIE: FIR BPF



4 MOVIES @ Blackboard

- 3 DOMAINS MOVIES: FIR Filters
  - fir2.mp4: Two zeros moving around UC and inside
  - fir3.mp4: Three zeros; one held fixed at $z=-1$
  - fir10_1.mp4:

    Ten zeros; 9 equally spaced around unit circle; one moving
  - fir10_r.mp4

    Ten zeros; 8 equally spaced around unit circle; two moving

Summary

- Definition of the z-Transform
- z-Transform Properties and Convolution
- z-Domain vs. frequency domain
- Filter Design in z-Domain: Zeros and Poles of H(z)