Project 3 Car Tracking

Problem 1: Bayesian network basics

a. Compute $\mathbb{P}(C_2 = 1 | D_2 = 0)$

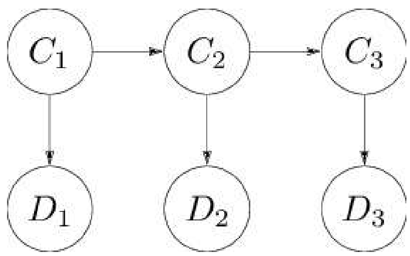

Figure 1: Hidden Markov Model

$$\mathbb{P}(C_2 = 1 | D_2 = 0) = \frac{\mathbb{P}(C_2 = 1, D_2 = 0)}{\mathbb{P}(D_2 = 0)} = \frac{\mathbb{P}(D_2 = 0 | C_2 = 1)\mathbb{P}(C_2 = 1)}{\mathbb{P}(D_2 = 0)}$$

Since

$$\mathbb{P}(C_2 = 1) = \sum_{c_1} \mathbb{P}((C_2 = 1 | c_1)\mathbb{P}(c_1) = \frac{1}{2}\epsilon + \frac{1}{2}(1 - \epsilon) = \frac{1}{2}$$

$$\mathbb{P}(C_2 = 0) = \sum_{c_1} \mathbb{P}(C_2 = 0 | c_1)\mathbb{P}(c_1) = \frac{1}{2}(1 - \epsilon) + \frac{1}{2}\epsilon = \frac{1}{2}$$

$$\mathbb{P}(D_2 = 0) = \mathbb{P}(D_2 = 0 | C_2 = 0)\mathbb{P}(C_2 = 0) + \mathbb{P}(D_2 = 0 | C_2 = 1)\mathbb{P}(C_2 = 1) = \frac{1}{2}$$

$$\mathbb{P}(D_2 = 0 | C_2 = 1) = \eta$$

Therefore we have $\mathbb{P}(C_2 = 1 | D_2 = 0) = \eta$.

b. Compute $\mathbb{P}(C_2 = 1 | D_2 = 0, D_3 = 1)$

$$\mathbb{P}(C_2 = 1 | D_2 = 0, D_3 = 1) = \frac{\mathbb{P}(C_2 = 1, D_2 = 0, D_3 = 1)}{\mathbb{P}(D_2 = 0, D_3 = 1)}$$

$$
\begin{aligned}
\mathbb{P}(C_2 = 1, D_2 = 0, D_3 = 1) &= \mathbb{P}(D_3 = 1|C_2 = 1, D_2 = 0)\mathbb{P}(D_2 = 0|C_2 = 1)\mathbb{P}(C_2 = 1) \\
&= \frac{1}{2}\eta \sum_{c_3} \mathbb{P}(D_3 = 1|c_3)\mathbb{P}(c_3|C_2 = 1) \\
&= \frac{1}{2}\eta[(1-\eta)(1-\epsilon) + \eta\epsilon]
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(C_2 = 0, D_2 = 0, D_3 = 1) &= \mathbb{P}(D_3 = 1|C_2 = 0, D_2 = 0)\mathbb{P}(D_2 = 0|C_2 = 0)\mathbb{P}(C_2 = 0) \\
&= \frac{1}{2}(1-\eta) \sum_{c_3} \mathbb{P}(D_3 = 1|c_3)\mathbb{P}(c_3|C_2 = 0) \\
&= \frac{1}{2}(1-\eta)[(1-\eta)\epsilon + \eta(1-\epsilon)]
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(D_2 = 0, D_3 = 1) &= \sum_{c_3} \mathbb{P}(c_3, D_2 = 0, D_3 = 1) \\
&= \frac{1}{2}\eta[(1-\eta)(1-\epsilon) + \eta\epsilon] + \frac{1}{2}(1-\eta)[(1-\eta)\epsilon + \eta(1-\epsilon)] \\
&= \frac{1}{2}[\epsilon + 2\eta - 4\epsilon\eta + 4\epsilon\eta^2 - 2\eta^2]
\end{aligned}
$$

Therefore we have

$$
\mathbb{P}(C_2 = 1|D_2 = 0, D_3 = 1) = \frac{\mathbb{P}(C_2 = 1, D_2 = 0, D_3 = 1)}{\mathbb{P}(D_2 = 0, D_3 = 1)} = \frac{\eta - \eta^2 - \epsilon\eta + 2\epsilon\eta^2}{\epsilon + 2\eta - 4\epsilon\eta + 4\epsilon\eta^2 - 2\eta^2}
$$

c. Suppose $\eta = 0.2$ and $\epsilon = 0.1$

(i) Compute $\mathbb{P}(C_2 = 1|D_2 = 0)$ and $\mathbb{P}(C_2 = 1|D_2 = 0, D_3 = 1)$

$\mathbb{P}(C_2 = 1|D_2 = 0) = \eta = 0.2000$

$\mathbb{P}(C_2 = 1|D_2 = 0, D_3 = 1) = \frac{\eta-\eta^2-\epsilon\eta+2\epsilon\eta^2}{\epsilon+2\eta-4\epsilon\eta+4\epsilon\eta^2-2\eta^2}\frac{37}{89} \approx 0.4157$

(ii) How did adding the second sensor reading change the result?

After adding the second sensor reading $D_3 = 1$, the posterior probability of $C_2 = 1$ increases, since $\mathbb{P}(C_2 = 1|D_2 = 0) < \mathbb{P}(C_2 = 1|D_2 = 0, D_3 = 1)$ according to our results.

This is because the position of the car at timestep $t + 1$ can be influenced by (or is not independent of) its position at timestep t. Since the probability given indicates that both $\mathbb{P}(C_3 = 1|C_2 = 1)$ and $\mathbb{P}(D_3 = 1|C_3 = 1)$ are large, there's larger probability that $C_3 = 1$ will happen.

(iii) Set ϵ so that $\mathbb{P}(C_2 = 1|D_2 = 0) = \mathbb{P}(C_2 = 1|D_2 = 0, D_3 = 1)$

$$
\frac{\eta - \eta^2 - \epsilon\eta + 2\epsilon\eta^2}{\epsilon + 2\eta - 4\epsilon\eta + 4\epsilon\eta^2 - 2\eta^2} = \eta
$$

From calculation we know $\epsilon = 0.5$, which means $\mathbb{P}(C_t|C_{t-1}) \equiv \frac{1}{2}$, doesn't depend on specific value of C_t

and C_{t-1}, i.e. C_t is independent of C_{t-1}. Consider

$$\mathbb{P}(C_3 = 0) = \sum_{c_2} \mathbb{P}(C_3 = 0|c_2)\mathbb{P}(c_2) = \frac{1}{2} = \mathbb{P}(C_3 = 1)$$

This verifies $\mathbb{P}(C_3 = i) = \mathbb{P}(C_3 = i|C_{2\text{J}}),\ i, j \in \{0, 1\}$, which is the property of conditional probability independence.

Problem 2∼4:

Please check the scripts in $submission.py$.