# On the Instability of Fractional Reserve Banking*

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**Abstract**

This paper develops a dynamic monetary model to study the (in)stability of the fractional reserve banking system. The model shows that the fractional reserve banking system can endanger stability in that equilibrium is more prone to exhibit endogenous cyclic, chaotic, and stochastic dynamics under lower reserve requirements, although it can increase consumption in the steady-state. Introducing endogenous unsecured credit to the baseline model does not change the main results. This paper also provides empirical evidence that is consistent with the prediction of the model. The calibrated exercise suggests that this channel could be another source of economic fluctuations.

**JEL Classification Codes:** E42, E51, G21
**Keywords:** Money, Banking, Instability, Volatility

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> Motivated partly by a desire to avoid such [excessive] price-level fluctuations and possible Wicksellian price-level indeterminacy, quantity theorists have advocated legal restrictions on private intermediation. ... Thus, for example, Friedman (1959, p. 21) ... has advocated 100 percent reserves against bank liabilities called demand deposit. **Sargent and Wallace (1982)**

# 1 Introduction

There have been claims that fractional reserve banking is an important cause of boom-bust cycles, based on the notion that banks create excess credit under fractional reserve banking (e.g., Fisher, 1935; Von Mises, 1953; Minsky, 1957; Minsky, 1970). For instance, Fisher (1935) views fractional reserve banking as one of several important factors in explaining economic fluctuations. Others believe that this is a primary cause of boom-bust cycles. According to Von Mises (1953), the overexpansion of bank credit as a result of fractional reserve banking is the root cause of business cycles. Minsky (1970) claims that economic booms and structural characteristics of the financial system, such as fractional reserve banking, can result in an economic collapse even when fundamentals remain unchanged.

This idea leads to policy debates on fractional reserve banking. Earlier examples include Peel's Banking Act of 1844 and the Chicago plan of banking reform with a 100% reserve requirement proposed by Irving Fisher, Paul Douglas, and others in 1939. Later, Friedman (1959) supported this banking reform, whereas Becker (1956) took the opposite position of supporting free banking with 0% reserve requirement.[1] Recently in 2018, Switzerland had a referendum of 100% reserve banking, which was rejected by 75.72% of the voters. The referendum aimed at making money safe from crisis by constructing full-reserve banking.[2] Whereas the debate on whether a fractional reserve banking system is inherently unstable has been an important policy discussion since a long time ago, the debate has never stopped.

This paper examines the instability of fractional banking by answering the following

[1] Sargent (2011) provides a novel review of the historical debates between narrow banking and free banking as tensions between stability versus efficiency.

[2] The official title of the referendum was *the Swiss federal popular initiative "for crisis-safe money: money creation by the National Bank only! (Sovereign Money Initiative)"* and also titled as "*debt-free money.*"

questions: (i) Can fractional reserve banking be inherently volatile even if we shut down the stochastic component of the economy? (ii) If so, under what condition can fractional reserve banking generate endogenous cycles without the presence of exogenous shocks and changes in fundamentals? To assess the claim that fractional reserve banking causes business cycles, this paper constructs a model of money and banking that captures the role of fractional reserve banking.

In the model, each agent faces an idiosyncratic liquidity shock. Banks can create deposits by extending loans to provide risk-sharing among the depositors whereas the bank's deposit issuance is constrained by the reserve requirement. The real balance of money is determined by two factors: storage value and liquidity premium. The storage value is increasing in the future value of money. However, the liquidity premium, the marginal value of its liquidity function, is decreasing if the money becomes more abundant. When the liquidity premium dominates the storage value, the economy can exhibit endogenous fluctuations. Fractional reserve banking amplifies the liquidity premium because it allows the bank to create inside money through lending. Due to this amplified liquidity premium, the fractional reserve banking system is more prone to endogenous cycles.

In the baseline model, lowering the reserve requirement increases consumption in the steady state. However, lowering the reserve requirements can induce two-period cycles as well as three-period cycles which implies the existence of periodic cycles of all order and chaotic dynamics. This also implies it can induce sunspot cycles. This result holds in the extended model with unsecured credit. It is worth noting that the full reserve requirement does not necessarily exclude the possibility of endogenous cycles. However, the economy will be more susceptible to cycles with lower reserve requirement.[3]

This paper departs from previous works in two ways. First, in contrast to the previous works on banking instability, which mostly focus on bank runs following the seminal model by Diamond and Dybvig (1983), this paper focuses on the volatility of real balances of inside money. It is another important focal point of banking instability because recurring boom-bust cycles associated with banking are probably more prevalent than bank runs. Second, the approach here differs from a traditional approach to economic fluctuations with financial frictions. There are two major points of view to understand economic fluctuations. The first one is that economic fluctua-

[3] Gu, Monnet, Nosal and Wright (2019) show that introducing banks to the economy could induce instability in various settings which is in line with this result.

tions are driven by exogenous shocks disturbing the dynamic system, and the effects of exogenous shocks shrink over time as the system goes back to its balanced path or steady-state. The second one is that they instead reflect an endogenous mechanism that produces boom-bust cycles. While there has been a lot of work on the role of financial friction in the business cycles including Kiyotaki and Moore (1997), Bernanke, Gertler and Gilchrist (1999), and Gertler and Karadi (2011), most of them focused on the first approach, in which all economic fluctuations are caused by exogenous shocks and the financial sectors only serve as an amplifier. This paper, however, takes the second approach and focuses on whether the endogenous cycles arise in the absence of the stochastic components of the economy.

To evaluate the main prediction from the theory that fractional reserve banking induces excess volatility, I test the relationship between the required reserves ratio and the volatility in real balance using cointegrating regression. A significant negative relationship between the two variables are found, and the results are robust to different measures of inflation and different frequency of time series. Both theoretical and empirical evidence indicate a link between the reserve requirement and the (in)stability. The calibrated exercise suggests that this channel could be another source of economic fluctuations.

**Related Literature** This paper builds on Berentsen, Camera and Waller (2007), who introduce financial intermediaries with enforcement technology to Lagos and Wright (2005) framework. The approach to introduce unsecured credit to the monetary economy is related to Lotz and Zhang (2016) and Gu, Mattesini and Wright (2016) which are based on the earlier work by Kehoe and Levine (1993).

This paper is related to the large literature on fractional reserve banking. Freeman and Huffman (1991) and Freeman and Kydland (2000) develop general equilibrium models that explicitly capture the role of fractional reserve banking. Using those models, they explain the observed relationships between key macroeconomic variables over business cycles. Chari and Phelan (2014) study the condition under which fractional reserve banking can be socially useful by preventing bank runs in the cash-in-advance framework. For recent work, Wipf (2020) studies the welfare implications of fractional reserve banking in a New Monetarist economy with imperfect competition and identifies the conditions under which fractional reserve banking can be welfare-improving compared to narrow banking. Andolfatto, Berentsen and Martin (2020) integrate Diamond (1997) into Lagos and Wright (2005) to provide a model in which fractional reserve banking emerges endogenously and a central bank can prevent bank panic as a lender of last resort. Whereas many previous work on instability focuses on bank runs, this paper studies a different type of instability in the sense that fractional reserve banking induces endogenous monetary cycles.

This paper is also related to the large literature on endogenous fluctuations, chaotic dynamics, and indeterminacy that have been surveyed by Brock (1988), Baumol and Benhabib (1989), Boldrin and Woodford (1990), Scheinkman and Woodford (1994) and Benhabib and Farmer (1999). For a model of bilateral trade, Gu, Mattesini, Monnet and Wright (2013) show that credit markets can be susceptible to endogenous fluctuations due to limited commitment. Using a continuous-time New Monetarist economy, Rocheteau and Wang (2022) show that asset liquidity can be a source of price volatility when assets have a non-positive intrinsic value. Altermatt, Iwasaki and Wright (2021) study economies with multiple liquid assets and show that liquidity considerations could imply endogenous fluctuations as self-fulfilling prophecies. Gu et al. (2019) show that introducing financial intermediaries to an economy can engender instability in four distinct setups that capture various functions of banking. The model in this paper is closely related to Gu et al. (2019), whereas the model here is extended to incorporate fractional reserve banking.

The rest of the paper is organized as follows. Section 2 constructs the baseline search-theoretic monetary model. Section 3 provides main results. Section 4 introduces unsecured credit. Section 5 discusses the empirical evaluation of the model. Section 6 calibrates the model to quantify the theory. Section 7 concludes.

# 2 Model

The model is based on Lagos and Wright (2005) with a financial intermediary as in Berentsen et al. (2007). Time is discrete and infinite. In each period, three markets convene sequentially. First, a centralized financial market (FM), followed by a decentralized goods market (DM), and finally a centralized goods market (CM). The FM and CM are frictionless. The DM is subject to search frictions, anonymity, and limited commitment. Therefore, a medium of exchange is needed to execute trades.

There is a continuum of agents who produce and consume perishable goods. At the beginning of the FM, a preference shock is realized: With probability $\sigma$, an agent will

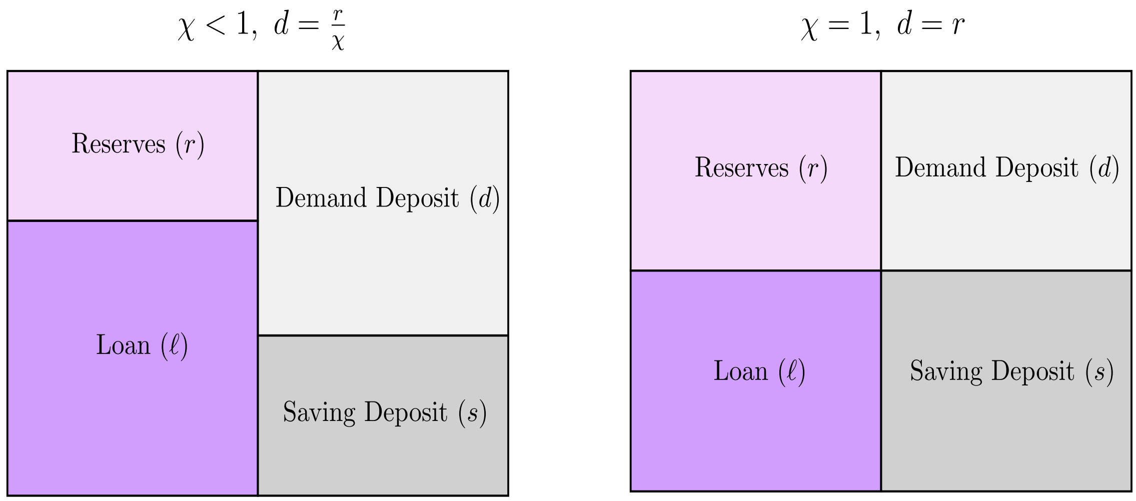


**Figure 1:** Bank's Balance Sheet: fractional reserve banking vs. full reserve banking

be a buyer in the following DM and with probability $1 - \sigma$, she will be a seller. The buyers and the sellers randomly meet and trade bilaterally in the DM. Agents discount their utility each period by $\beta$. Within-period utility is represented by

$$\mathcal{U} = U(X) - H + u(q) - c(q),$$

where $X$ is the CM consumption, $H$ is the CM disutility from production, and $q$ is the DM consumption. As standard $U'$, $u'$, $c' > 0$, $U''$, $u'' < 0$, $c'' \geq 0$, and $u(0) = c(0) = 0$. The CM consumption good $X$ is produced one-for-one with $H$, implying the real wage is 1. The efficient consumption in CM and DM is $X^*$ and $q^*$ that solves $U'(X^*) = 1$ and $u'(q^*) = c'(q^*)$, respectively.

There is a representative bank who accepts deposits and lends loans in the FM. In the FM, the agent can borrow money from the bank for a promise to repay money in the subsequent CM at nominal lending rate $i_l$. The agent can also deposit money to the bank and receive money in the subsequent CM at nominal deposit rate. There are two kinds of deposit: demand deposit and saving deposit. The interest rate on demand deposit and saving deposit is $i_d$ and $i_s$, respectively. The demand deposit can be used as a means of payment in DM trade whereas the savings deposit cannot, and the bank's issue of demand deposits is subject to a reserve requirement, $\chi$. Figure 1 illustrates the bank's balance sheet identity given the reserve requirement. The banking market is perfectly competitive. The bank can enforce the repayment of loans at no cost. Last, there is a central bank that controls the fiat money supply $M_t$. Let $\gamma$ be the growth rate of the fiat money stock. Changes in fiat money supply are accomplished by lump-sum transfer if $\gamma > 0$ and by lump-sum tax if $\gamma < 0$.

## 2.1 Agent's Problem

Let $W_t$, $G_t$, and $V_t$ denotes the agent's value function in the CM, FM, and DM, respectively, in period $t$. There are two payment instruments for the DM transaction: fiat money (outside money) and demand deposit issued by the bank (inside money). I will allow the agents to use unsecured credit as a means of payment in the next section. An agent entering the CM with fiat money nominal balance $m_t$, demand deposit $d_t$, saving deposit $s_t$, and loan $\ell_t$, solves the following problem:

$$\begin{aligned} W_t(m_t, d_t, s_t, \ell_t) &= \max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta G_{t+1}(\hat{m}_{t+1}) \\ \text{s.t. } \phi_t \hat{m}_{t+1} + X_t &= H_t + T_t + \phi_t m_t + (1 + i_{d,t})\phi_t d_t + (1 + i_{s,t})\phi_t s_t - (1 + i_{l,t})\phi_t \ell_t, \end{aligned} \tag{1}$$

where $T_t$ is the lump-sum transfer (or tax if it is negative), $i_{d,t}$ is the demand deposit interest rate, $i_{s,t}$ is the saving deposit interest rate, $i_{l,t}$ is the loan interest rate, $\phi_t$ is the price of money in terms of the CM goods, and $\hat{m}_{t+1}$ is the money balance carried to the FM where banks take deposits and makes loans. The first-order conditions (FOCs) result in $X_t = X^*$ and

$$\phi_t = \beta G'_{t+1}(\hat{m}_{t+1}), \tag{2}$$

where $G'_{t+1}(\hat{m}_{t+1})$ is the marginal value of an additional unit of money taken into the FM of period $t + 1$. The envelope conditions are

$$\frac{\partial W_t}{\partial m_t} = \phi_t, \quad \frac{\partial W_t}{\partial d_t} = \phi_t(1 + i_{d,t}), \quad \frac{\partial W_t}{\partial s_t} = \phi_t(1 + i_{s,t}), \quad \frac{\partial W_t}{\partial \ell_t} = -\phi_t(1 + i_{l,t}),$$

implying $W_t$ is linear in $m_t$, $d_t$, $s_t$, and $\ell_t$.

The value function of an agent at the beginning of FM is

$$G_t(m) = \sigma G_{b,t}(m) + (1 - \sigma)G_{s,t}(m), \tag{3}$$

where $G_{j,t}$ is the value function of type $j$ agent in the FM. Agents choose their money holding $m_j$, demand deposit balance $d_j$, saving deposit balance $s_j$, and loan $\ell_j$ based on the realization of their types in the following DM. The value function $G_{j,t}$ can be written as

$$G_{j,t}(m_t) = \max_{m_{j,t},d_{j,t},s_{j,t},\ell_{j,t}} V_{j,t}(m_{j,t},d_{j,t},s_{j,t},\ell_{j,t})$$
$$\text{subject to} \quad m_{j,t}+d_{j,t}+s_{j,t} \leq m_t + \ell_{j,t} \tag{4}$$

where $V_{j,t}$ is the value function of type $j$ agent in the DM. The FOCs are

$$\frac{\partial V_{j,t}}{\partial m_{j,t}} - \lambda_{j,m} \leq 0 \tag{5}$$
$$\frac{\partial V_{j,t}}{\partial d_{j,t}} - \lambda_{j,m} \leq 0 \tag{6}$$
$$\frac{\partial V_{j,t}}{\partial s_{j,t}} - \lambda_{j,m} \leq 0 \tag{7}$$
$$\frac{\partial V_{j,t}}{\partial \ell_{j,t}} + \lambda_{j,m} \leq 0 \tag{8}$$

where $\lambda_{j,m}$ is the Lagrange multiplier for $m_{j,t}+d_{j,t}+s_{j,t} \leq m + \ell_{j,t}$.

The terms of trade in the DM are determined by an abstract mechanism that is studied in Gu and Wright (2016). The buyer must pay $p = v(q)$ to the seller to get $q$ where $v(q)$ is some payment function satisfying $v'(q) > 0$ and $v(0) = 0$. As shown in Gu and Wright (2016), if the trading protocol satisfies four common axioms, then the terms of trade can be written in the following form:

$$p_t = \begin{cases} \bar{p}_t & \text{if } \bar{p}_t < p^* \\ p^* & \text{if } \bar{p}_t \geq p^* \end{cases} \qquad q_t = \begin{cases} v^{-1}(\bar{p}_t) & \text{if } \bar{p}_t < p^* \\ q^* & \text{if } \bar{p}_t \geq p^*, \end{cases} \tag{9}$$

where $\bar{p}_t = \phi_t \{m_{b,t} + (1+i_{d,t})d_{b,t}\}$ is the is the liquidity position of the buyer, and $p^*$ is the payment required to get efficient consumption $q^*$. Many standard mechanisms, such as Kalai and generalized Nash bargaining, are consistent with this specification.

With probability $\alpha$, a buyer meets a seller in the DM while a seller meets a buyer with probability $\alpha_s$. Since the CM value function is linear, the DM value function for the seller can be written as

$$V_{s,t}(m_{s,t},d_{s,t},s_{s,t},\ell_{s,t}) = \alpha_s[p_t - c(q_t)] + W_t(m_{s,t},d_{s,t},s_{s,t},\ell_{s,t}). \tag{10}$$

Differentiating $V_{s,t}$ yields

$$\frac{\partial V_{s,t}}{\partial m_{s,t}} = \phi_t, \quad \frac{\partial V_{s,t}}{\partial d_{s,t}} = \phi_t(1+i_{d,t}), \quad \frac{\partial V_{s,t}}{\partial s_{s,t}} = \phi_t(1+i_{s,t}), \quad \frac{\partial V_{s,t}}{\partial \ell_t} = -\phi_t(1+i_{l,t}).$$

Combining the seller's FOCs in the FM and the derivatives of $V_{s,t}$ yields

$$\phi_t - \lambda_{m,s} \leq 0, \text{``} = \text{''} 0 \text{ iff } m_{s,t} > 0 \tag{11}$$

$$\phi_t(1+i_{d,t}) - \lambda_{m,s} \leq 0, \text{``} = \text{''} 0 \text{ iff } d_{s,t} > 0. \tag{12}$$

$$\phi_t(1+i_{s,t}) - \lambda_{m,s} \leq 0, \text{``} = \text{''} 0 \text{ iff } s_{s,t} > 0 \tag{13}$$

$$-\phi_t(1+i_{l,t}) + \lambda_{m,s} \leq 0, \text{``} = \text{''} 0 \text{ iff } \ell_{s,t} > 0. \tag{14}$$

Since the seller does not need to carry liquidity to the DM, the DM terms of trade $(p_t, q_t)$ is independent of $m_{s,t}$, $d_{s,t}$ and $s_{s,t}$. Then the seller's choices on $m_t$, $d_t$ and $s_t$ only depend on their returns. For $i_d$, $i_s > 0$, the budget constraint is binding, $\lambda_{s,m} > 0$, and the seller does not hold $m_{s,t}$. If $i_s > i_d$, the seller deposits all the money balance to saving deposit $s_{s,t} = m_t$, and if $i_s = i_d$ the seller is indifferent between depositing to demand deposit and saving deposit. Holding saving deposit is weakly preferred to holding demand deposit in this case. The seller does not have a strict incentive to borrow loans from the bank.

A buyer's DM value function is

$$V_{b,t}(m_{b,t}, d_{b,t}, s_{b,t}, \ell_{b,t}) = \alpha[u(q_t) - p_t] + W(m_{b,t}, d_{b,t}, s_{b,t}, \ell_{b,t}), \tag{15}$$

where $p_t \leq \bar{p}_t = \phi_t \{m_{b,t} + (1+i_{d,t})d_{b,t}\}$. Assuming interior solutions, differentiating $V_{b,t}$ yields

$$\frac{\partial V_{b,t}}{\partial m} = \phi_t[\alpha\lambda(q_t) + 1], \qquad \frac{\partial V_{b,t}}{\partial d} = \phi_t(1+i_{d,t})[\alpha\lambda(q_t) + 1],$$
$$\frac{\partial V_{b,t}}{\partial s} = \phi_t(1+i_{s,t}), \qquad \frac{\partial V_{b,t}}{\partial \ell} = -\phi_t(1+i_{l,t}),$$

where $\lambda(q) = u'(q)/v'(q) - 1$ if $p^* > \bar{p}$ and $\lambda(q) = 0$ if $\bar{p} \geq p^*$. Similar to the seller's case, combining the buyer's FOCs in the FM and the first-order derivatives of $V_{b,t}$ yields

$$\phi_t[\alpha\lambda(q_t)+1]-\lambda_{m,b}\leq 0, \text{``}=\text{''}0 \text{ iff } m_{b,t}>0 \tag{16}$$

$$\phi_t(1+i_{d,t})[\alpha\lambda(q_t)+1]-\lambda_{m,b}\leq 0, \text{``}=\text{''}0 \text{ iff } d_{b,t}>0 \tag{17}$$

$$\phi_t(1+i_{s,t})-\lambda_{m,b}\leq 0, \text{``}=\text{''}0 \text{ iff } s_{b,t}>0 \tag{18}$$

$$-\phi_t(1+i_{l,t})+\lambda_{m,b}\leq 0, \text{``}=\text{''}0 \text{ iff } \ell_{b,t}>0. \tag{19}$$

When $i_d>0$, the buyer does not hold $m_{b,t}=0$ and deposits all the fiat money balance to her demand deposit account, $d_{b,t}=m_t$. When $i_d=0$, the buyer is indifferent between holding $m_{b,t}$ and $d_{b,t}$. Holding the demand deposit is weakly preferred to holding the cash. The buyer does not have a strict incentive to deposit her balance to the saving deposit. With the binding constraint $\lambda_{m,b}>0$, we have

$$1+i_{l,t}=(1+i_{d,t})[\alpha\lambda(q_t)+1]$$

which implies that the choice of $\ell_{b,t}$ equates the marginal cost and the marginal gain from borrowing loans from the bank. Since $i_{s,t}=i_{l,t}$, we have $p_t=v(q_t)=\phi_t(1+i_{d,t})d_{b,t}=\phi_t(1+i_{d,t})(m_t+\ell_{b,t})$,

Using the above results, we can rewrite the value functions in the FM as follows:

$$G_{b,t}(m_t)=\alpha[u(q_t)-p_t]+W(0,m_t+\ell_{b,t},0,\ell_{b,t}) \tag{20}$$

$$G_{s,t}(m_t)=\alpha_s[p_t-c(q_t)]+W(0,0,m_t,0) \tag{21}$$

where $q_t=v^{-1}(p_t)$ and $p_t=\min\{p^*,\phi_t(1+i_{d,t})(m_t+\ell_{b,t})\}$. Take derivative of $G_{j,t}(m_t)$ with respect to $m_t$ to get

$$G'_{b,t}(m_t)=\phi_t(1+i_{d,t})[1+\alpha\lambda(q_t)] \tag{22}$$

$$G'_{s,t}(m_t)=\phi_t(1+i_{s,t}). \tag{23}$$

Since $G'_t(m_t)=\sigma G'_{b,t}(m_t)+(1-\sigma)G'_{s,t}(m_t)$, we have the following:

$$G'_t(m_t)=\phi_t\sigma(1+i_{d,t})[1+\alpha\lambda(q_t)]+\phi_t(1-\sigma)\left(1+i_{s,t}\right) \tag{24}$$

Combine (2) and (24) to get the Euler equation

$$\phi_t = \phi_{t+1}\beta\left[\sigma(1+i_{d,t+1})\left\{1+\alpha\lambda(q_{t+1})\right\}+(1-\sigma)(1+i_{s,t+1})\right] \tag{25}$$

where $v(q_{t+1}) = \phi_{t+1}(1+i_{d,t+1})(m_{t+1}+\ell_{b,t+1})$.

## 2.2 Bank's Problem

A representative bank makes loans $\ell_t$ and issues demand deposits $d_t$ and saving deposits $s_t$. The depositors are paid at the nominal interest rates, $i_{d,t}$ and $i_{s,t}$, by the bank, and the borrowers need to repay their borrowing with a nominal interest rate $i_{l,t}$. The bank also decides the reserve balance $r_t$ and its fiat money holdings $m_{k,t}$. The central bank sets the reserve requirement $\chi$. The representative bank solves the following profit maximization problem.

$$\begin{aligned} \max_{d_t,s_t,\ell_t,m_{k,t},r_t} \quad & (1+i_{l,t})\ell_t - (1+i_{d,t})d_t - (1+i_{s,t})s_t + m_{k,t} \\ \text{subject to} \quad & m_{k,t}+\ell_t = d_t + s_t, \quad \chi d \leq r, \quad r_t \leq m_{k,t} \end{aligned} \tag{26}$$

The first constraint is a balance sheet identity and the second constraint is the reserve requirement constraint. The last constraint says the bank's reserve cannot exceed its fiat money holdings. The FOCs for the bank's problem are

$$(1+i_{l,t}) - \lambda^b \leq 0, \text{“} = \text{”}0 \text{ iff } \ell_t > 0 \tag{27}$$

$$-(1+i_{s,t}) + \lambda^b \leq 0, \text{“} = \text{”}0 \text{ iff } s_t > 0. \tag{28}$$

$$-(1+i_{d,t}) + \lambda^b - \chi\lambda^r \leq 0, \text{“} = \text{”}0 \text{ iff } d_t > 0 \tag{29}$$

$$-\lambda^b + \lambda^r - \lambda^m \leq 0, \text{“} = \text{”}0 \text{ iff } r_t > 0 \tag{30}$$

$$1 + \lambda^m \leq 0, \text{“} = \text{”}0 \text{ iff } m_{k,t} > 0 \tag{31}$$

where $\lambda^b$, $\lambda^r$, and $\lambda^m$ denotes the Lagrange multiplier of the first constraint, the second constraint, and the last constraint in (26), respectively. Assuming interiority, we have

$$i_{l,t} = i_{s,t} \tag{32}$$

$$i_{d,t} = i_{s,t}(1-\chi) \tag{33}$$

Given the bank's problem and the agent's problem, we can define an equilibrium as follows:

**Definition 1.** *Given $(\gamma, \chi)$ and an initial money stock $M_0$, an equilibrium consists of sequences of prices $\{\phi_t, i_{d,t}, i_{s,t}, i_{l,t}\}_{t=0}^{\infty}$, real balances $\{m_t, d_{b,t}, d_{s,t}, s_{b,t}, s_{s,t}, \ell_{b,t}, \ell_{s,t}\}_{t=0}^{\infty}$, and allocations $\{q_t, X_t\}_{t=0}^{\infty}$ satisfying the following:*

- *Agents solve CM, FM and DM problems: (1) and (4)*
- *The terms of trade in the DM satisfy (9), (15) and (10)*
- *A representative bank solves its profit maximization problem: (26)*
- *Markets clear in every period:*
  1. *Money Market: $m_t = M_t$*
  2. *Loan Market: $\sigma\ell_{b,t} + (1-\sigma)\ell_{s,t} = \ell_t$*
  3. *Deposit Market: $\sigma d_{b,t} + (1-\sigma)d_{s,t} = d_t$ and $\sigma s_{b,t} + (1-\sigma)s_{s,t} = s_t$*
- *It is a monetary equilibrium if $\phi_t M_t > 0$*

The next step is to characterize the equilibrium. Assuming interior solutions, combining buyers' FOCs (16)-(19), sellers' FOCs (11)-(14) and bank's FOCs (32)-(33) gives

$$\phi_t(1+i_{s,t}) = \phi_t(1+i_{d,t})\{1+\alpha\lambda(q_t)\} \tag{34}$$

$$\phi_t(1+i_{d,t}) = \phi_t\frac{\chi}{\chi+(\chi-1)\alpha\lambda(q_t)} \tag{35}$$

and when $i_{d,t} = i_{s,t} = 0$, $q_t = q^*$. Combine equations (25), (34), and (35), and use equilibrium condition $m_{t+1} = M_{t+1}$ to get

$$\phi_t = \phi_{t+1}\beta\frac{\chi\{1+\alpha\lambda(q_{t+1})\}}{\chi+(\chi-1)\alpha\lambda(q_{t+1})}. \tag{36}$$

Given $i_d \geq 0$, any equilibrium $\{\phi_t\}_{t=1}^{+\infty}$ satisfying (36) must satisfy either (1) $\lambda(q) < \chi/\{\alpha(1-\chi)\}$ and $\phi_t M_t > 0$ or (2) $q_t = \phi_t = 0$. This leads to the following lemma.

**Lemma 1.** *In a monetary equilibrium, $q_t \in (\hat{q}, q^*]$ and a monetary equilibrium satisfying $q_t \in [0, \hat{q}]$ does not exist where $\hat{q}$ solves*

$$\lambda(\hat{q}) = \min\left\{\frac{\chi}{\alpha(1-\chi)}, \lambda(0)\right\}.$$

**Proof.** See Appendix A. ■

The equation (36) also can be rewritten in terms of the buyer's real balance of liquidity, $z_{t+1} \equiv \phi_{t+1}(m_{b,t+1} + d_{b,t+1})$ by defining $i_{d,t}$ as a function of $z_t$. Given equation (35), if $z_t \geq p^*$ then $i_{d,t} = 0$ where $p^* = v(q^*)$. Define $\hat{p} = v(\hat{q})$. When $\hat{p} < z_t < p^*$, $i_{d,t}$ is a positive fixed point that satisfying

$$i_{d,t} = \frac{\chi}{\chi + (\chi - 1)\alpha L(z_t + z_t i_{d,t})} - 1 \tag{37}$$

where $L(\cdot) \equiv \lambda \circ v^{-1}(\cdot)$. The following lemma establishes the existence and uniqueness of this fixed point.

**Lemma 2.** *Given $z_t \in (\hat{p}, p^*)$, there exists a positive unique fixed point $\hat{i}_{d,t}$ satisfying (37).*

**Proof.** See Appendix A. ■

Given above results, now we can define $i_d(z)$ as function of $z$ where $i_d(z) = \hat{i}_{d,t}(z)$ if $p^* > z > \hat{p}$ and $i_d(z) = 0$ if $z \geq p^*$. Using $i_d(z)$ allows us to rewrite the right-hand side of (36) in terms of real balance of liquidity, $z_{t+1}$:

$$\phi_t = \phi_{t+1}\beta \times \frac{\chi\left\{1 + \alpha L\big(z_{t+1} + i_d(z_{t+1})z_{t+1}\big)\right\}}{\chi + (\chi - 1)\alpha L\big(z_{t+1} + i_d(z_{t+1})z_{t+1}\big)} \tag{38}$$

Then multiplying both sides of (38) by $M_t/\chi$ allows us to reduce the equilibrium condition to one difference equation of real balances $z$:

$$z_t = \frac{z_{t+1}}{1+i} \times \frac{\chi\left\{1 + \alpha L\big(z_{t+1} + i_d(z_{t+1})z_{t+1}\big)\right\}}{\chi + (\chi - 1)\alpha L\big(z_{t+1} + i_d(z_{t+1})z_{t+1}\big)} \tag{39}$$

where $i \equiv \gamma/\beta - 1$.[4] Since $i_d(z_{t+1}) = \chi/\big[\chi + (\chi - 1)\alpha L\big(z_{t+1} + i_d(z_{t+1})z_{t+1}\big)\big] - 1$,

[4]In the stationary equilibrium, $i = \gamma/\beta - 1$ is the nominal interest rate.

equation (39) can also be expressed as below:

$$z_t = f(z_{t+1}) \equiv \frac{z_{t+1}}{1+i}\{1 + i_d(z_{t+1})\}\Big\{1 + \alpha L\big(z_{t+1} + i_d(z_{t+1})z_{t+1}\big)\Big\} \tag{40}$$

# 3 Results

This section establishes key results on the instability of fractional reserve banking. Consider a stationary equilibrium, which is a fixed point that satisfies $z = f(z)$. There always exists an non-monetary equilibrium with $z = 0$. Given $i \in [0, \bar{\iota})$ and $\chi \in (0, 1)$, where $\bar{\iota} = \alpha\lambda(\hat{q})/\{\alpha\lambda(\hat{q})(\chi - 1) + \chi\}$, an unique stationary monetary equilibrium exists and satisfies

$$i\chi = \{1 + i(1 - \chi)\}\, \alpha L(\bar{p}_s). \tag{41}$$

Nash and Kalai bargaining provides simple examples for $\bar{\iota}$. Under the Inada condition $u'(0) = \infty$, with Nash bargaining, $\bar{\iota} = \infty$ while with Kalai, $\bar{\iota} = \infty$ when $\frac{\chi}{\alpha(1-\chi)} \leq \frac{\theta}{1-\theta}$ and $\bar{\iota} = \frac{\alpha\theta}{\alpha\theta(\chi-1)+\chi(1-\theta)}$ otherwise.

Since $L'(\cdot) < 0$ (see Gu and Wright, 2016), the following result holds:

**Proposition 1.** *In the stationary equilibrium, lowering $i$ or lowering $\chi$ increases $q$.*

**Proof.** See Appendix A. ■

The dynamics of monetary equilibrium is characterized by equation (40). We can derive the condition that the economy exhibits a two-period cycle that satisfy $z_1 < z_s \leq p^* < z_2$.

**Proposition 2 (Two-period Monetary Cycle).** *There exists a two-period cycle with $z_1 < z_s \leq p^* < z_2$ if $\chi \in (0, \chi_m)$, where*

$$\chi_m \equiv \frac{(1+i)^2 \alpha L\left(\frac{p^*}{1+i}\right)}{\{(1+i)^2 - 1\}\left\{1 + \alpha L\left(\frac{p^*}{1+i}\right)\right\}}. \tag{42}$$

**Proof.** See Appendix A. ■

Proposition 2 shows that lowering the reserve requirement can induce a two-period cycle under the general trading mechanism. However, in general, a two period cycle with $z_1 < z_s < z_2$, could be either $z_2 > p^*$ or $z_2 < p^*$. Following the standard textbook method (see Azariadis, 1993), we can show that if $f'(z_s) < -1$, there exists a two-period cycle in the neighborhood of $z_s$ which includes $z_2 < p^*$ case. Consider a special case where $-qu''(q)/u'(q) = \eta$, $c(q) = q$ and the buyer makes take-it-or-leave-it (TIOLI) offer. The following proposition provides a closed-form expression of $f'(z_s)$.

**Proposition 3.** *Assume $-qu''(q)/u'(q) = \eta$ and $c(q) = q$. Then, we have*

$$f'(z_s) = \frac{\chi\left[1-\alpha+\alpha(1-\eta)\left(1+\frac{i\chi}{\alpha[1+i(1-\chi)]}\right)\right]}{(1+i)\left\{1+(\chi-1)\left[1-\alpha+\alpha(1-\eta)\left(1+\frac{i\chi}{\alpha[1+i(1-\chi)]}\right)\right]\right\}}$$

**Proof.** See Appendix A. ■

There exists a two-period cycle when either $f'(z_s) < -1$ or $\chi < \chi_m$. To interpret the results, consider equation (39). The first term, $z_{t+1}/(1+i)$ on the right-hand side, reflects the store of value, which is monotonically increasing in $z_{t+1}$. The second term, $\chi\{1+\alpha L(\cdot)\}/[\chi+(\chi-1)\alpha L(\cdot)]$, reflecting the liquidity premium, is decreasing in $q_{t+1}$ but non-monotone in $z_{t+1}$. Because $f'(z_{t+1})$ depends on both terms, $f(z_{t+1})$ is non-monotone in general. If the liquidity premium dominates the storage value at $z = z_s$, we can have $f'(z_s) < -1$, which is a standard condition for the existence of cyclic equilibria. Similarly, when $\chi < \chi_m$, the liquidity premium dominates the storage value at $z = z_1$ where $z_1 < z_s$. Lowering the reserve requirement amplifies the liquidity premium because it allows the bank to create more liquidity through lending. This amplification of liquidity generates endogenous cycles.

It is worth noting that the 100% reserve requirement does not necessarily rule out endogenous cycles. Even if $\chi = 1$, the condition of endogenous cycles around $z_s$, $f'(z_s) < -1$, still can hold when

$$\eta > \frac{2(1+i)}{\alpha+i}.$$

This implies that the 100% reserve requirement can not rule out endogenous cycles when the agents are highly risk averse. Similarly, if $\alpha L\left(\frac{p^*}{1+i}\right) \geq (1+i)^2 - 1$, the 100% reserve requirement can not rule out endogenous cycles as well because $\chi_m \geq 1$ in this case.

Whereas the condition in Proposition 2 is written in terms of $\chi$, this condition is not independent of $i$. Taking derivative with respect of $i$ gives the following

$$\frac{\partial \chi_m}{\partial i} = -\frac{\alpha \left\{ 2\alpha(1+i)\left[L\left(\frac{p^*}{1+i}\right)\right]^2 + p^* i(i+2)L'\left(\frac{p^*}{1+i}\right) + 2(1+i)L\left(\frac{p^*}{1+i}\right)\right\}}{i^2(i+2)^2\left[1+\alpha L\left(\frac{p^*}{1+i}\right)\right]^2}$$

the effect of $i$ on $\chi_m$ is, however, ambiguous in general.

In addition to the condition for two-period cycles, the next result provides the condition for three-period cycles under the general trading mechanism. The existence of three period-cycles implies cycles of all orders as well as chaotic dynamics (see Sharkovskii, 1964 and Li and Yorke, 1975).

**Proposition 4** (**Three-period Monetary Cycle and Chaos**). *A three-period cycle with $z_1 < z_2 < p^* < z_3$ does not exist. There exists a three-period cycle with $z_1 < p^* < z_2 < z_3$ if $\chi \in (0, \hat{\chi}_m)$, where*

$$\hat{\chi}_m \equiv \frac{(1+i)^3 \alpha L\left(\frac{p^*}{1+i}\right)}{\{(1+i)^3 - 1\}\left\{1+\alpha L\left(\frac{p^*}{1+i}\right)\right\}}. \tag{43}$$

**Proof.** See Appendix A. ■

The following corollary is a direct result from Proposition 4.

**Corollary 1** (**Binding Liquidity Constraint**). *In any n-period cycle, the liquidity constraint binds, $z_t < p^*$, at least one periodic point over the cycle.*

**Proof.** See Appendix A. ■

The model can also generate sunspot cycles. Consider a Markov sunspot variable $S_t \in \{1, 2\}$. This sunspot variable is not related to fundamentals but may affect equilibrium. Let $\Pr(S_{t+1} = 1 | S_t = 1) = \zeta_1$ and $\Pr(S_{t+1} = 2 | S_t = 2) = \zeta_2$. The sunspot is realized in the FM. Let $W_t^S$ be the CM value function in state $S$ in period $t$, then $W_t^S$ can be expressed as

$$W_t^S(m_t, d_t, s_t, \ell_t) = \max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta\left[\zeta_s G_{t+1}^S(\hat{m}_{t+1}) + (1-\zeta_s) G_{t+1}^{-S}(\hat{m}_{t+1})\right]$$
$$\text{s.t. } \phi_t^S \hat{m}_{t+1} + X_t = H_t + T_t + \phi_t^S m_t + (1+i_{d,t})\phi_t^S d_t + (1+i_{s,t})\phi_t^S s_t - (1+i_{l,t})\phi_t^S \ell_t.$$

The FOC can be written as

$$-\phi_t^S + \beta\zeta_s G_{t+1}'^S(\hat{m}_{t+1}) + \beta(1-\zeta_s) G_{t+1}'^{-S}(\hat{m}_{t+1}) = 0. \tag{44}$$

Solving the FM problem results in

$$G'^S_{t+1}(m^S_{t+1}) = \phi^S_{t+1}\left[\sigma(1+i^S_{d,t+1})\{1+\alpha\lambda(q^S_{t+1})\} + (1-\sigma)(1+i^S_{s,t+1})\right]. \tag{45}$$

We substitute (45) into (44) and use the money market clearing condition $m_{t+1} = M_{t+1}$ to get the Euler equation.

$$\begin{aligned}\phi^S_t =& \beta\zeta_s\phi^S_{t+1}\left\{1+i_d(z^S_{t+1})\right\}\left\{1+\alpha L\big(z^S_{t+1}+i_d(z^S_{t+1})z^S_{t+1}\big)\right\} \\ &+ \beta(1-\zeta_s)\phi^{-S}_{t+1}\left\{1+i_d(z^{-S}_{t+1})\right\}\left\{1+\alpha L\big(z^{-S}_{t+1}+i_d(z^{-S}_{t+1})z^{-S}_{t+1}\big)\right\}\end{aligned}$$

where $z^S_{t+1} = \phi^S_{t+1}\{m^S_{b,t}+d^S_{b,t}\}$. Then multiply both sides of the Euler equation by $M_t/\chi$ to reduce the equilibrium condition into one difference equation of real balances $z^S_{t+1}$:

$$\begin{aligned}z^S_t =& \frac{\zeta_s z^S_{t+1}}{1+i}\left\{1+i_d(z^S_{t+1})\right\}\left\{1+\alpha L\big(z^S_{t+1}+i_d(z^S_{t+1})z^S_{t+1}\big)\right\} \\ &+ \frac{(1-\zeta_s)z^{-S}_{t+1}}{1+i}\left\{1+i_d(z^{-S}_{t+1})\right\}\left\{1+\alpha L\big(z^{-S}_{t+1}+i_d(z^{-S}_{t+1})z^{-S}_{t+1}\big)\right\} \\ =& \zeta_s f(z^S_{t+1}) + (1-\zeta_s)f(z^{-S}_{t+1}).\end{aligned} \tag{46}$$

Define a sunspot equilibrium as follows:

**Definition 2** (**Proper Sunspot Equilibrium**). *A proper sunspot equilibrium consists of the sequences of real balances $\{z^S_t\}^{\infty}_{t=0,S=1,2}$ and probabilities $(\zeta_1,\zeta_2)$, solving (46) for all $t$.*

Consider stationary sunspot equilibria with $z^1 < z^2$ that only depend on the state, not the time. The liquidity constraint is binding in state $S = 1$. By the standard approach (see again Azariadis, 1993 for the textbook treatment), the conditions for two-period cycles also suffices for two-state sunspot equilibrium. If $f'(z_s) < -1$, there exists $(\zeta_1,\zeta_2) \in (0,1)^2$, $\zeta_1+\zeta_2 < 1$, such that the economy has a proper sunspot equilibrium in the neighborhood of $z_s$. In addition to that, if $\chi < \chi_m$, there exists $(\zeta_1,\zeta_2) \in (0,1)^2$, $\zeta_1+\zeta_2 < 1$, such that the economy has a proper sunspot equilibrium satisfying $z^1 < p^* < z^2$.

**Proposition 5** (**Stationary Sunspot Equilibria**). *The stationary sunspot equilibrium exists if either $\chi < \chi_m$ or $f'(z_s) < -1$.*

**Proof.** See Appendix A. ■

# 4 Money and Unsecured Credit

Consider an alternative payment instrument in the DM - unsecured credit. The buyer can pay for DM goods using unsecured credit that will be redeemed to the seller in the following CM and she can borrow up to her debt limit, $\bar{b}_t$. For simplicity, I assume that the buyer makes a TIOLI offer to the seller in the DM, which means the buyer maximizes her surplus subject to the seller's participation constraint. The DM cost function is $c(q) = q$. Suppose the buyer has issued $b_t$ units of unsecured debt in the previous DM. The CM value function is

$$W_t(m_t, d_t, s_t, \ell_t, -b_t) = \max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta G_{t+1}(\hat{m}_{t+1})$$

$$\text{s.t. } \phi_t \hat{m}_{t+1} + X_t = H_t + T_t + \phi_t m_t + (1 + i_{d,t})\phi_t d_t + (1 + i_{s,t})\phi_t s_t - (1 + i_{l,t})\phi_t \ell_t - b_t,$$

which is the same as before except that the agent needs to pay or collect the debt. The agent's FM problem is identical to the previous section. Then, the seller's DM value function is

$$V_t^s(0, 0, s_{s,t}, 0) = W_t(0, 0, s_{s,t}, 0, 0)$$

The buyer's DM value function is

$$V_t^b(0, d_{b,t}, 0, \ell_{b,t}) = \alpha[u(q_t) - q_t] + W_t(0, d_{b,t}, 0, \ell_{b,t}, 0),$$

where $q_t = \min\{q^*, \bar{b}_t + \phi_t(1 + i_{d,t})d_{b,t}\}$.

Similar to Lemma 2, there exist a positive unique fixed point $\hat{\imath}_{d,t}$ satisfying

$$\hat{\imath}_{d,t} = \frac{\chi}{\chi + (\chi - 1)\alpha[u'(z_t(1 + \hat{\imath}_{d,t}) + \bar{b}_t) - 1]} - 1$$

given $\bar{b}_t$ and $z_t \in (\hat{p} - \bar{b}_t, p^* - \bar{b}_t)$. Define $\iota_d(z, \bar{b})$ as function of $z$ and $\bar{b}$ where $\iota_d(z, \bar{b}) = \hat{\imath}_{d,t}(z)$ if $p^* > z + \bar{b} > \hat{p}$ and $i_d(z) = 0$ if $z + \bar{b} \geq p^*$.

Given $\bar{b}_t$, solving equilibrium yields

$$z_t = \frac{z_{t+1}}{1 + i} \left\{1 + \iota_d(z_{t+1}, \bar{b}_{t+1})\right\} \left\{1 + \alpha[u'\big(z_{t+1} + \iota_d(z_{t+1}, \bar{b}_{t+1})z_{t+1} + \bar{b}_{t+1}\big) - 1]\right\} \quad (47)$$

Next, I am going to endogenize the debt limit. The buyer cannot commit to pay back the debt. If the buyer reneges she is captured with probability $\mu$. The punishment for a defaulter is permanent exclusion from the DM trade but she can still produce for herself in the CM. The value of autarky is $\underline{\mathrm{W}}(0,0,0,0) = [U(X^*) - X^* + T]/(1-\beta)$. The incentive condition for voluntary repayment is

$$\underbrace{-b_t + W_t(m_t, d_t, s_t, \ell_t, 0)}_{\text{value of honoring debts}} \geq \underbrace{(1-\mu)W_t(m_t, d_t, s_t, \ell_t, 0) + \mu \underline{\mathrm{W}}(m_t, d_t, s_t, \ell_t, 0)}_{\text{value of not honoring debts}}.$$

One can write the debt limit $\bar{b}_t$ as $b_t \leq \bar{b}_t \equiv \mu W_t(0,0,0,0) - \mu \underline{\mathrm{W}}(0,0,0,0)$. Recall the CM value function. Using the solution of FM, we can rewrite the buyer's CM value function as

$$\begin{aligned} W_t(0,0,0,0,0) = & \, U(X^*) - X^* + T_t + \beta W_{t+1}(0,0,0,0,0) \\ & + \max_{\hat{m}_{t+1}} \Big\{ -\phi_t \hat{m}_{t+1} + \beta\alpha\sigma[u(q_{t+1}) - q_{t+1}] + \beta\phi_{t+1}\hat{m}_{t+1} \Big\} \end{aligned}$$

where $q_{t+1} = \min\{q^*, z_{t+1}[1 + \iota_d(z_{t+1}, \bar{b}_{t+1})] + \bar{b}_{t+1}\}$. Substituting $W_t(0,0,0,0,0) = \bar{b}_t/\mu + \underline{\mathrm{W}}(0,0,0,0,0)$ yields

$$\begin{aligned} \frac{\bar{b}_t}{\mu} = & \frac{\beta \bar{b}_{t+1}}{\mu} - \phi_t M_{t+1} + \beta\phi_{t+1}M_{t+1} \\ & + \beta\alpha\sigma\Big[u\big(\bar{b}_{t+1} + z_{t+1}\{1 + i_d(z_{t+1})\}\big) - \bar{b}_{t+1} - z_{t+1}\{1 + i_d(z_{t+1})\}\Big] \end{aligned}$$

where $M_{t+1} = \hat{m}_{t+1}$ and $\phi_{t+1}M_{t+1} = \chi z_{t+1}$ and $z_{t+1}$ solves (47). Rearranging terms yields

$$\bar{b}_t = \begin{cases} \beta\bar{b}_{t+1} + \chi\mu[-\gamma z_t + \beta z_{t+1}] + \beta\alpha\mu\sigma S(q_{t+1}) & \text{if } z_{t+1} + \bar{b}_{t+1} < q^* \\ \beta\bar{b}_{t+1} + \chi\mu[-\gamma z_t + \beta z_{t+1}] + \beta\alpha\mu\sigma S(q^*) & \text{if } z_{t+1} + \bar{b}_{t+1} > q^* \end{cases} \tag{48}$$

where $S(q) \equiv u(q) - q$ is the buyer's trade surplus and $q_{t+1} = \min\{q^*, z_{t+1}[1 + \iota_d(z_{t+1}) + \bar{b}_{t+1}]\}$. The equilibrium can be collapsed into a dynamic system satisfying (47)-(48).

In the stationary equilibrium, (47) becomes

$$-1 - \frac{i\chi}{\alpha[1 + (1-\chi)i]} + u'(q) \leq 0, = \text{ if } z > 0 \tag{49}$$

and (48) becomes

$$(1-\beta)\bar{b} = \begin{cases} \chi\mu[\beta - \gamma]z + \beta\alpha\mu\sigma S(z + \bar{b}) & \text{if } z + \bar{b} < q^* \\ \chi\mu[\beta - \gamma]z + \beta\alpha\mu\sigma S(q^*) & \text{if } z + \bar{b} \geq q^*, \end{cases} \tag{50}$$

where $q = \min\{q^*, z + \bar{b}\}$. The stationary equilibrium solves the above two equations, and it falls into one of the three cases: the pure money equilibrium, the pure credit equilibrium, and the money-credit equilibrium. First, if no one can capture the buyer after she reneges, $\mu = 0$, the unsecured credit is not feasible, $\bar{b} = 0$. In this case, the equilibrium will be the pure money equilibrium. Second, when $\bar{b}$ solving (50) satisfies $u'(\bar{b}) < i\chi/\{\alpha[1 + (1-\chi)i]\}$ then money is not valued, $z = 0$. We have the pure credit equilibrium in this case. Third, if the solutions of (49)-(50), $(z, \bar{b})$ are strictly positive then money and credit coexist, which is the money-credit equilibrium.

The debt limit at the stationary equilibrium, $\bar{b}$, is a fixed point satisfying $\bar{b} = \Omega(\bar{b})$ where

$$\Omega(\bar{b}) = \begin{cases} \dfrac{\mu\sigma\alpha}{\rho}[u(\tilde{q}) - \tilde{q}] + \dfrac{i\mu\chi}{\rho}(\tilde{q} - \bar{b}) & \text{if } \tilde{q} > \bar{b} \geq 0 \\ \dfrac{\mu\sigma\alpha}{\rho}[u(\bar{b}) - \bar{b}] & \text{if } q^* > \bar{b} \geq \tilde{q} \\ \dfrac{\mu\sigma\alpha}{\rho}[u(q^*) - q^*] & \text{if } \bar{b} \geq q^* \end{cases} \tag{51}$$

and $\tilde{q}$ solves $u'(\tilde{q}) = 1 + i\chi/[\alpha\{1 + (1-\chi)i\}]$ and $\rho \equiv 1/\beta - 1$. The DM consumption $q_s$ is determined by $q_s = \min\{q^*, \max\{\tilde{q}, \bar{b}\}\}$. Money and credit coexist if and only if $0 < \bar{b} < \tilde{q}$, which holds when $0 < \mu < \min\{1, \tilde{\mu}\}$, where

$$\tilde{\mu} \equiv \frac{\rho\tilde{q}}{\alpha\sigma[u(\tilde{q}) - \tilde{q}]}.$$

Consider the dynamics of equilibria where money and credit coexist. I claim the main results from Section 3 - lowering the reserve requirement can induce endogenous cycles - still hold even after unsecured credit is introduced. For compact notation, let $\iota \equiv \max\{i, r\}$ and $a_j \equiv z_j + \bar{b}_j$. The following proposition establishes the conditions for two-period cycles, three-period cycles, and chaotic dynamics.

**Proposition 6 (Monetary Cycles with Unsecured Credit).** *There exists a two-period cycle of money and credit with $a_1 < q^* < a_2$ if $\chi \in (0, \chi_c)$, where*

$$\chi_c \equiv \frac{(1+\iota)^2\alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]}{\{(1+\iota)^2 - 1\}\left\{1 + \alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]\right\}}.$$

*There exists a three-period cycle of money and credit with $a_1 < q^* < a_2 < a_3$, if*

$\chi \in (0, \hat{\chi}_c)$, *where*

$$\hat{\chi}_c \equiv \frac{(1+\iota)^3 \alpha [u'\left(\frac{q^*}{1+\iota}\right) - 1]}{\{(1+\iota)^3 - 1\}\left\{1 + \alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]\right\}}.$$

**Proof.** See Appendix A. ■

# 5 Empirical Evaluation: Inside Money Volatility

In the previous sections, the theoretical results show that lowering the required reserve ratio can induce instability. To evaluate the model prediction, I examine whether the required reserve ratio is associated with the cyclical volatility of the real balance of the inside money.

Following Jaimovich and Siu (2009) and Carvalho and Gabaix (2013), I measure the cyclical volatility in quarter $t$ as the standard deviation of a filtered log real total checkable deposit during a 41-quarter (10-year) window centered around quarter $t$. Total checkable deposits are from the H.6 Money Stock Measures published by the Federal Reserve Board and converted to real value using the Consumer Price Index (CPI). Seasonally adjusted series are used to smooth the seasonal fluctuation. I adopt the Hodrick-Prescott (HP) filter with a 1600 smoothing parameter as standard. To construct an annual series, quarterly observations are averaged for each year. The sample period is from 1960:I to 2018:IV so that there are annual series from 1965 to 2013. To check whether the results are sensitive to different measures of the price level, I also use the core CPI, the Personal Consumption Expenditures (PCE), and the core PCE to transform the total checkable deposit into real value.

The legal reserve requirement for the demand deposits has been 10% since April 2, 1992. However, the Federal Reserve imposes different reserve requirements depending on the size of a bank's liability. These criteria have changed over time. For example, during 1992:Q1-2019:Q4, this changed 27 times. To consider these changes, I divide the required reserves by total checkable deposits to compute the required reserve ratio.

Figure 2 presents a scatter plot of the cyclical volatility of the real inside money balance and the required reserve ratio. Column (1) of Table 1 reports its regression estimates with Newey-West standard errors. The plot and estimates show a negative relationship between the cyclical volatility of the real inside money balance and the required reserve ratio with statistically significant regression coefficients. However, this

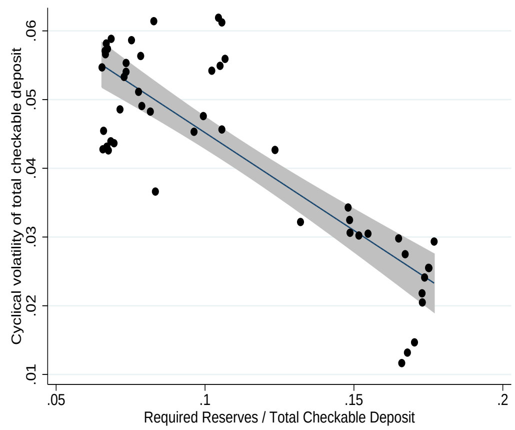


**Figure 2:** Scatter Plot for Inside Money Volatility and Required Reserve Ratio

**Table 1:** Effect of Required Reserve Ratio

| Price level | CPI | | Core CPI | | PCE | | Core PCE | |
|---|---|---|---|---|---|---|---|---|
| Dependent variable: $\sigma_t^{Roll}$ | OLS (1) | CCR (2) | OLS (3) | CCR (4) | OLS (5) | CCR (6) | OLS (7) | CCR (8) |
| $\chi$ | $-0.283^{***}$ | $-0.245^{***}$ | $-0.267^{***}$ | $-0.221^{***}$ | $-0.306^{***}$ | $-0.227^{***}$ | $-0.307^{***}$ | $-0.220^{***}$ |
| | (0.027) | (0.002) | (0.027) | (0.003) | (0.029) | (0.004) | (0.027) | (0.005) |
| `ffr` | | $-0.109^{***}$ | | $-0.125^{***}$ | | $-0.187^{***}$ | | $-0.207^{***}$ |
| | | (0.002) | | (0.003) | | (0.004) | | (0.004) |
| Constant | $0.074^{***}$ | $0.074^{***}$ | $0.070^{***}$ | $0.071^{***}$ | $0.074^{***}$ | $0.075^{***}$ | $0.073^{***}$ | $0.073^{***}$ |
| | (0.003) | (0.000) | (0.004) | (0.000) | (0.004) | (0.000) | (0.004) | (0.000) |
| Obs. | 49 | 49 | 49 | 49 | 49 | 49 | 49 | 49 |
| $adjR^2$ | 0.700 | 0.621 | 0.728 | 0.648 | 0.740 | 0.650 | 0.764 | 0.665 |
| $\lambda_{trace}(r=0)$ | **9.807** | 35.688 | **9.120** | 35.145 | **9.109** | 35.367 | **8.593** | 35.028 |
| 5% CV | 15.41 | 29.68 | 15.41 | 29.68 | 15.41 | 29.68 | 15.41 | 29.68 |
| $\lambda_{trace}(r=1)$ | 3.324 | **10.682** | 2.839 | **10.065** | 2.723 | **9.894** | 2.417 | **9.345** |
| 5% CV | 3.76 | 15.41 | 3.76 | 15.41 | 3.76 | 15.41 | 3.76 | 15.41 |

Note: For (1), (3), (5) and (7), OLS estimates are reported, and Newey-West standard errors with lag 1 are reported in parentheses. For (2), (4), (6), and (8), first-stage long-run variance estimations for CCR are based on the quadratic spectral kernel and Bayesian information criterion. The bandwidth selection is based on Newey-West fixed lag, $4 \times (T/100)^{2/9}$; $\chi$ denotes the required reserve ratio, `ffr` denotes federal funds rates and $\sigma_t^{Roll}$ denotes the cyclical volatility of real inside money balances. ***, **, and * denotes significance at the 1, 5, and 10 percent levels, respectively.

**Table 2:** Unit Root Tests

| | | Phillips-Perron test | | ADF test |
|---|---|---|---|---|
| | | $Z(\rho)$ | $Z(t)$ | $Z(t)$ w/ lag 1 |
| `ffr` | | −6.766 | −1.704 | −2.362 |
| $\chi$ | | −1.492 | −1.173 | −1.341 |
| $\sigma_t^{Roll}$ | (CPI) | −4.708 | −2.191 | −2.090 |
| $\sigma_t^{Roll}$ | (Core CPI) | −4.681 | −2.189 | −1.978 |
| $\sigma_t^{Roll}$ | (PCE) | −4.329 | −2.038 | −2.047 |
| $\sigma_t^{Roll}$ | (Core PCE) | −4.076 | −1.954 | −1.930 |
| $\Delta$`ffr` | | −28.373*** | −5.061*** | −6.357*** |
| $\Delta\chi$ | | −31.818*** | −4.802*** | −3.693*** |
| $\Delta\sigma_t^{Roll}$ | (CPI) | −24.905*** | −3.416** | −2.942** |
| $\Delta\sigma_t^{Roll}$ | (Core CPI) | −24.758*** | −3.509** | −2.942** |
| $\Delta\sigma_t^{Roll}$ | (PCE) | −23.691*** | −3.330** | −2.842* |
| $\Delta\sigma_t^{Roll}$ | (Core PCE) | −22.826*** | −3.296** | −2.768* |

Note: `ffr` denotes federal funds rates, $\chi$ denotes required reserve ratio, and $\sigma_t^{Roll}$ denotes cyclical volatility of real inside money balances. ***, **, and * denotes significance at the 1, 5, and 10 percent levels, respectively.

result can be driven by a spurious regression. Table 2 provides unit root test results for the federal funds rate, the required reserve ratio, and the cyclical volatility of inside money. Both augmented Dickey-Fuller tests and Phillips-Perron tests fail to reject the null hypotheses of unit roots for these series, whereas they reject the null hypotheses of unit roots at their first differences. In addition to that, the Johansen cointegration test in Column (1), suggests that there is no cointegration relationship between two variables. So it is hard to rule out that Column (1)'s results are driven by a spurious regression.

To overcome this issue, I adopt the cointegrating regression with an additional variable, the federal funds rate. Column (2) of Table 1 provides Johansen cointegration test results for the federal funds rate, the required reserves, and the cyclical volatility of inside money. The trace test suggests a cointegration relationship among these three variables, which is consistent with the theoretical result: The instability depends on the reserve requirement and the interest rate. With the cointegration relationship, we may not have to worry about a spurious relationship. Column (2) of Table 1 reports the estimates for the cointegrating relationship. Because of the potential bias from long-run variance, I estimate a canonical cointegrating regression (CCR). The estimates are statistically significant with a sizeable level.

To check the sensitivity of the results, I redo all the analyses using the core CPI, the Personal Consumption Expenditures (PCE), and the core PCE to transform the total checkable deposit into real value. Columns (3), (5), and (7) of Table 1 regress required reserve ratio on the inside money volatility and report its Newey-West standard errors. They also report the trace test statistics of Johansen cointegration test between these two variables. The results are consistent with the benchmark case in Column (1). Columns (4), (6), and (8) of Table 1 report CCR estimates regressing the required reserve ratio and federal funds rate on the inside money volatility and the trace test statistics of Johansen cointegration test between these three variables. All the results are consistent with the benchmark case in Column (2).

Appendix B includes more sensitivity analyses: (1) Using quarterly series instead of annual series; (2) Using time series before 2008; (3) Using alternative data proposed by Lucas and Nicolini (2015). All the results are not sensitive with respect to different frequencies, time periods, and alternative data.

# 6 Calibrated Examples

## 6.1 Parameters and Targets

In this section, I calibrate the model using U.S. data. The model calibration is based on the data from 1983 to 2008 and uses M1J, a new measure of M1 proposed by Lucas and Nicolini (2015). Following is the rationale for focusing since 1983. From 1933, the Glass-Steagall Act and Regulation Q prevented commercial banks from paying interest on demand deposits. Regulation Q was initially loosened in 1980 by allowing the introduction of negotiable withdrawal order accounts (NOW accounts), which are personal checking accounts with limited interest rates. As of December 14, 1982, banks were permitted to issue money market deposit accounts (MMDA) which pay interest. Lucas and Nicolini (2015) argue that these regulatory changes loosening Regulation Q could be responsible for the instability of money demand and show that adding MMDA to the standard measure of M1 restores the stability of money demand. To be consistent with the model environment in which the bank pays interest on demand deposits, the calibration focuses on the data from 1983 when banks were permitted to pay interest on MMDA.

Both models with and without unsecured credit are calibrated. First, assuming there is no unsecured credit, I calibrate the model without unsecured credit. This

**Table 3:** Model Parametrization

| | Data | Model 1<br>No Credit, $\mu = 0$ | Model 2<br>With Credit, $\mu = 1$ |
|---|---|---|---|
| **Parameters** | | | |
| DM utility level, $C$ | | 0.9956 | 1.0110 |
| DM utility curvature, $\eta$ | | 0.0568 | 0.0282 |
| Monitoring probability, $\mu$ | | - | 1.0000 |
| **Targets** | | | |
| avg. $z/y$ | 0.2578 | 0.2578 | 0.2578 |
| elasticity of $z/y$ wrt $i$ | -0.1012 | -0.1012 | -0.1002 |

model will be referred as Model 1 throughout this section. In addition to Model 1, I also calibrate the model with unsecured credit. This model will be referred as Model 2. I set $\mu = 1$ in Model 2.

I set the discount rate $\beta = 0.9709$ to match the real annual interest rate of 3%. Using the average required reserve to deposit ratio for 1983-2008, the benchmark reserve requirement is set to 3.25%. The benchmark value for $i$ is set to 0.0536 as the average annualized nominal interest rate is 5.36%. In Model 1, since there is no unsecured credit, the monitoring probability is set to $\mu = 0$ but in Model 2, the agents can use unsecured credit in the DM trade and the monitoring probability is set to $\mu > 0$. The matching function in the DM is $\mathcal{M}(\mathcal{B}, \mathcal{S}) = \frac{\mathcal{B}\mathcal{S}}{\mathcal{B}+\mathcal{S}}$, where $\mathcal{B}$ and $\mathcal{S}$ denotes the measure of buyers and sellers, respectively. This implies $\alpha = \mathcal{M}(\sigma, 1-\sigma)/\sigma$ and $\alpha_s = \mathcal{M}(\sigma, 1-\sigma)/(1-\sigma)$. The utility functions for the parameterization are

$$U(X) = B\log(X), \qquad u(q) = \frac{Cq^{1-\eta}}{1-\eta}$$

implying $X^* = B$ and the DM cost function is given as $c(q) = q$. Assume the buyer makes a take-it-or-leave-it offer to the seller in the DM trade, implying $\lambda(q) = Cq^{-\eta} - 1$.

This section focuses on the equilibrium where $X > 0$ which requires $U(X) - X > 0$. To guarantee $U(X) - X > 0$, we need to have $\log(B) > 1$ since $B = X^*$. Otherwise, the CM consumption is zero, $X = 0$. For normalization, I set $B = 3$. The parameters $(C, \eta)$ are set to match the money demand relationship. In the model, the money demand relationship is given by real balances of money as a fraction of output

$$Z \equiv \frac{z}{y} = \frac{z}{\sigma\alpha q + B}$$

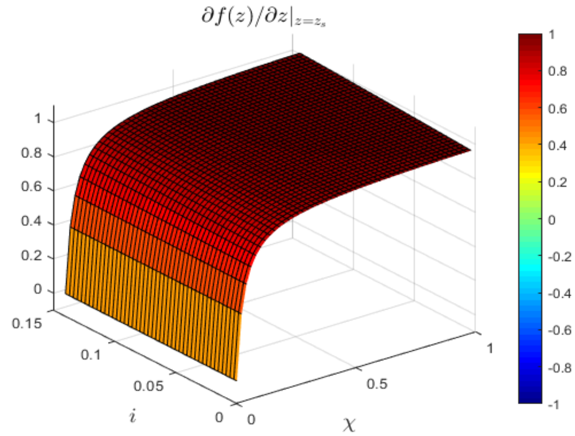


**Figure 3:** Calibrated Examples (Model 1): $f'(z_s)$

and elasticity of $z/y$ with respect to $i$

$$\frac{\partial \log(Z)}{\partial \log(i)} = \frac{\partial Z}{\partial i}\frac{i}{Z}$$

where $y$ is the real output of the economy.

Specifically, the parameter $C$ is set to match the average money stock to GDP ratio, 0.2578, and the parameter $\eta$ is set to match the elasticity of $z/y$ with respect to $i$. The target semi-elasticity is estimated by the following regression

$$\log(Z_t) = \beta_0 + \beta_1 \log(i_t) + \varepsilon_t$$

for 1983-2008 and the estimated elasticity is $\beta_1 = -0.1012$. The closed-form solutions of targets from the models can be found in Appendix C. Table 3 shows the calibrated parameters and the target moments.

## 6.2 Model Implied Thresholds

Given the parameterization from the previous section, this section examines whether the conditions of endogenous cycles hold or not. Figure 3 shows $f'(z_s)$ for given parameters and policy variables in Model 1. Given calibrated parameters and policy variable space, $f'(z_s)$ is always larger than 0. Since $f'(z_s) < -1$ does not hold, the endogenous cycle satisfying $z_1 < z_s < z_2 < p^*$ in the neighborhood of $z_s$ does not exist in Model 1 under calibrated parameters.

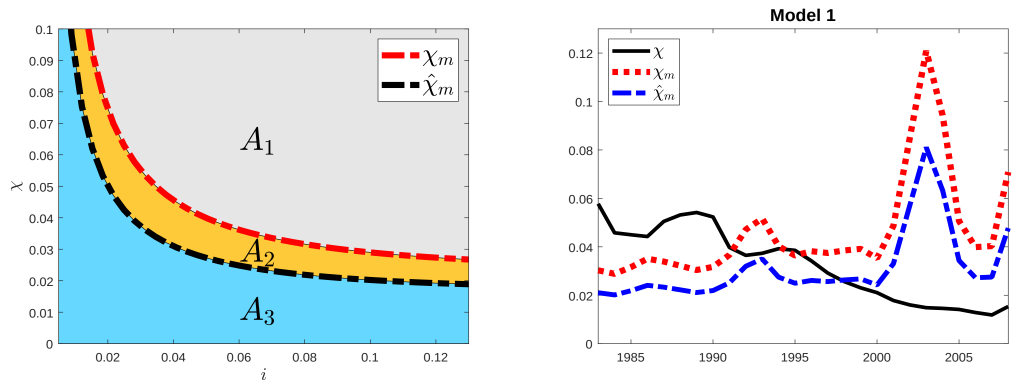


**Figure 4:** Calibrated Examples (Model 1): $\chi_m$, $\hat{\chi}_m$

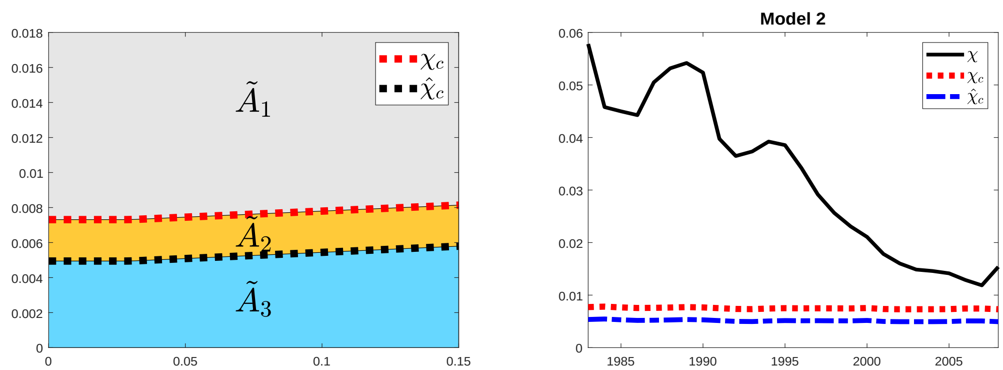


**Figure 5:** Calibrated Examples (Model 2): $\chi_c$ and $\hat{\chi}_c$

However, this does not indicate that this economy does not have endogenous cycles. The left panel of Figure 4 shows the thresholds for two-period cycles $\chi_m$, and three-period cycles and chaotic dynamics $\hat{\chi}_m$ under different $i$ in Model 1. The region $A_1$ denotes the area satisfying $\chi > \chi_m$ which means that there is no cycle. The region $A_2$ denotes the area satisfying $\chi_m > \chi > \hat{\chi}_m$. In $A_2$ region, there are two-period cycles but higher order cycles do not exist. The region $A_3$ denotes the area satisfying $\chi < \hat{\chi}_m$ which implies there exist higher-order cycles and chaotic dynamics. In Model 1, both of the model-implied thresholds are decreasing in $i$. Lowering $i$ and $\chi$ both can result in cyclic equilibria. The right panel of Figure 4 plots the model-implied thresholds over time. It shows that after 2000, both of $\chi_m$ and $\hat{\chi}_m$ are higher than $\chi$. This says the economy could exhibit endogenous fluctuations and chaotic dynamics. It implies that the economy may have endogenous cycles as well as sunspot cycles due to fractional reserve banking, which is independent of the presence of exogenous shocks and changes in fundamentals.

**Table 4:** Welfare cost of 10% inflation : model comparisons

| | $\chi = 1\%$ | $\chi = 3.25\%$ | $\chi = 5\%$ | $\chi = 10\%$ | $\chi = 50\%$ | $\chi = 100\%$ |
|---|---|---|---|---|---|---|
| | | | **Model 1** | | | |
| $1 - \Delta$ | 0.0003 | 0.0010 | 0.0014 | 0.0025 | 0.0048 | 0.0045 |
| | | | **Model 2** | | | |
| $1 - \Delta$ | 0.0003 | 0.0009 | 0.0012 | 0.0017 | 0.0028 | 0.0032 |

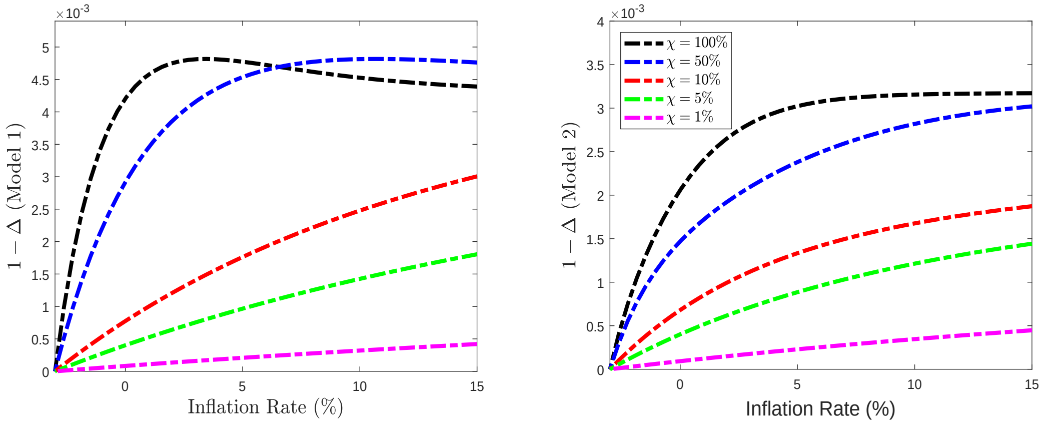


**Figure 6:** Welfare cost of inflation : model comparisons

The left panel of Figure 5 shows the thresholds for the two-period cycles $\chi_c$ and three-period and chaotic dynamics $\hat{\chi}_c$ in Model 2. The thresholds for cycles in Model 2 are much lower than the thresholds for cycles in Model 1 and increasing in $i$. The region $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ denotes the area satisfying $\chi > \chi_c$, $\chi_c > \chi > \hat{\chi}_c$, and $\chi < \hat{\chi}_c$, respectively. The right panel of Figure 5 plots the model-implied thresholds over time. While both of $\chi_c$ and $\hat{\chi}_c$ are higher than $\chi$, they are very close to $\chi$ after 2000.

In both models, the endogenous cycle thresholds are not low enough to be ignored, showing that this channel of volatility needs to be considered in addition to economic fluctuations induced by exogenous shocks that disrupt the dynamic system.

## 6.3 Welfare

In this section, I focus on the welfare implication of fractional reserve banking in the stationary equilibrium. As standard, given the inflation rate $\tilde{\pi}$, the welfare cost of inflation, $1 - \Delta$, is defined as the percentage reduction in total consumption that the agents would accept compared to the inflation rate at the Friedman rule, $\pi^0$. To be consistent with $i = 0$ and $\beta = 0.9709$, I set $\pi^0 = -0.03$. To compute the welfare cost

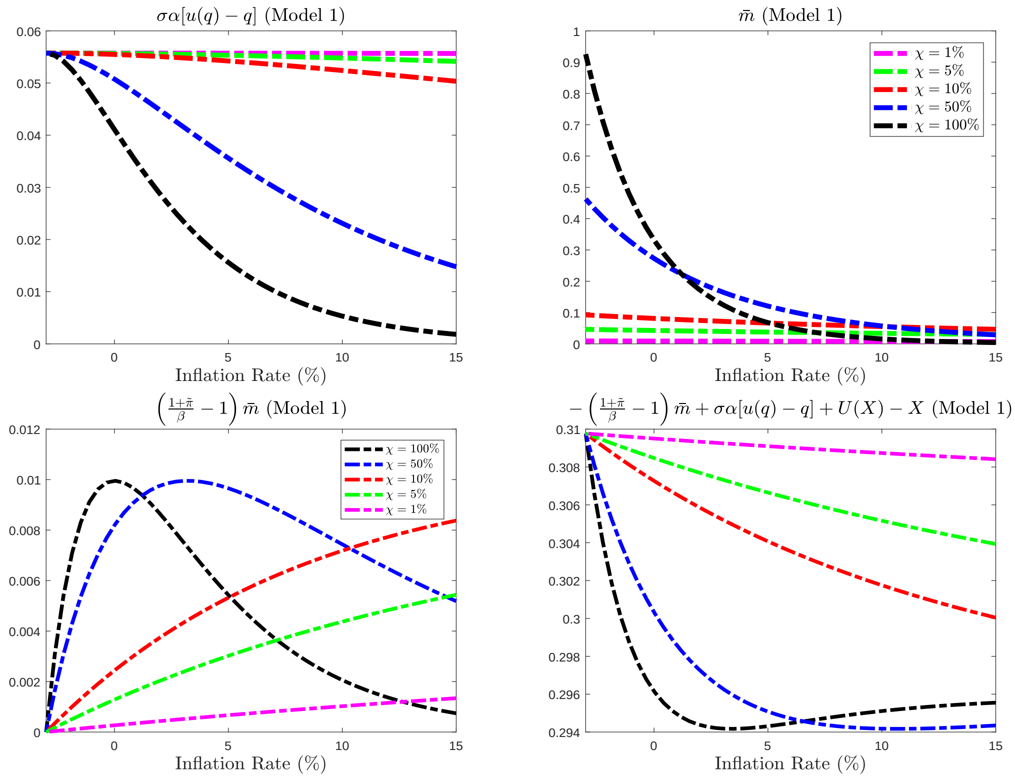


**Figure 7:** DM surplus, money real balances, cost of holding money, and total surplus

of inflation, $\Delta$ solves the following

$$(1-\beta)W(\mathbf{0};\tilde{\pi},\chi) = \sigma\alpha\{u(q^*\Delta)-q^*\} + U(X^*\Delta) - X^*$$

where $\tilde{\pi} = \beta(1+i)-1$ and $\mathbf{0}$ is a vector of state variables.

The welfare cost of 10% inflation is summarized in Table 4. The result of the benchmark reserve requirement (3.25%) is 0.1% of consumption in Model 1 and 0.09% in Model 2. When the reserve requirement is set to 100%, the cost rises to 0.45% in Model 1 and 0.32% in Model 2. Under the 1% reserve requirement, the cost drops to 0.03% both in Model 1 and Model 2. The values are much lower than Lagos and Wright (2005), for several reasons. First and foremost, this model has inside money creation under fraction reserve banking. The agents need to hold less fiat money from CM to DM which lowers the cost of holding money. While the agents hold less fiat money, DM trade is increased since the bank creates inside money. Second, in the DM, agents trade via take-it-or-leave-it-offer from the buyer to the seller rather than Nash bargaining, excluding inefficiency from the holdup problem.

Figure 6 shows the reduction in total consumption, $1-\Delta$, for $\tilde{\pi}$ varying from $\pi^0$ to 15%. While the welfare cost is increasing in $\tilde{\pi}$ and $\chi$ in Model 2, there are two distinct observations in Model 1. The first is that the welfare cost of inflation is not monotone in inflation rate when the reserve requirement is 100% and 50%. The second one is that when inflation rate is high, welfare cost of inflation under the 100% reserve requirement is lower than welfare cost of inflation under the 50% reserve requirement.

This might be a surprising result because Proposition 1 shows that the DM consumption is strictly decreasing in $i$ and $\chi$. It is worth to discuss the channel through which this occurs. To inspect the mechanism, the welfare can be decomposed into three components as follows

$$(1-\beta)W(\mathbf{0}) = \underbrace{-\left(\frac{1+\tilde{\pi}}{\beta}-1\right)\bar{m}}_{\text{opp. cost of holding money}} + \underbrace{\sigma\alpha[u(q)-q]}_{\text{DM surplus}} + \underbrace{U(X^*)-X^*}_{\text{CM surplus}} \tag{52}$$

where $\bar{m}$ is the real balance of fiat money carried from the CM to the FM (e.g., $\bar{m}_t = \phi_{t+1}m_t$). In equation (52), the first term captures the opportunity cost of holding money, the second term is the DM surplus, and the last term is the CM surplus. The CM surplus is constant. As the top-left panel of Figure 7 shows, the DM surplus is strictly decreasing in $i$ and $\chi$. This is a direct result of Proposition 1.

The real balance of fiat money is decreasing in inflation rate, as the top-right panel of Figure 7 shows. However, The effect of $\chi$ is not monotone. At lower inflation, the $\bar{m}$ is higher when $\chi$ is higher. In contrast to this, at high inflation, the $\bar{m}$ is smaller when $\chi$ is higher. This is because it decreases in steeper slopes under higher reserves requirement. The top-right panel of Figure 7 shows that $\bar{m}$ decreases from 0.9255 to 0.0162 as inflation increases from $\pi^0$ to $\pi = 0.10$ under 100% reserve requirement while it decreases from 0.4627 to 0.0579 under the 50% reserve requirement.

There are two forces that make this slope steeper under the higher reserve requirement. First, the agents need to carry more fiat money to the DM. Since holding fiat money bears inflation costs, it increases inflation costs. Second, the deposit rate is decreasing in the reserve requirement. Since the deposit rate is given as $i_d = (1-\chi)(\frac{1+\tilde{\pi}}{\beta}-1)$, an increase in inflation rises the deposit rate which partly offsets the increased opportunity cost of holding money. However, with higher $\chi$, we have a lower deposit rate. Therefore, the higher reserve requirement lowers the incentive to hold money when the inflation rate is high.

Consider the cost of holding money, $(\frac{1+\tilde{\pi}}{\beta}-1)\bar{m}$. While $(\frac{1+\tilde{\pi}}{\beta}-1)$ is linearly increasing in $\tilde{\pi}$, $\bar{m}$ is non-linearly decreasing in $\tilde{\pi}$. As a result, the slope is steeper when $\chi$ is higher and it generates a non-monotonicity of the cost of holding money. Hence, the cost of holding fiat money is not monotone in the inflation rate when the reserve requirement is high. The bottom-left panel of Figure 7 plots the cost of holding money under different $\chi$ and $\tilde{\pi}$ and shows that the first term of (52) is not monotone with respect to $\tilde{\pi}$ as well as $\chi$ and could be decreasing in $\tilde{\pi}$ when inflation and reserve requirement are high. When this effect outweighs the increased surplus from the higher DM consumption, the total surplus in the economy could be increasing in $\tilde{\pi}$ when the inflation and the reserve requirement are high as the bottom-right panel of Figure 7 shows.

## 6.4 Monetary News Shocks

This section studies discrete-time dynamics resulting from news on future changes in monetary policy, following Burdett, Trejos and Wright (2017) and Gu, Han and Wright (2020). Consider a case that a central bank changes $i$ permanently, from $i_0 = 0.02$ to $i_T = 0.01$.[5] Suppose the change in the monetary policy at $T$ is announced at time 0.

First, consider the dynamics without unsecured credit, i.e., $\mu = 0$. The permanent change in $i$ from $i_0$ to $i_T$ implies a shift of equation (40) from $z_t = f_0(z_{t+1}; i_0, \chi)$ to $z_t = f_T(z_{t+1}; i_T, \chi)$. Let $z_s(i, \chi)$ be a steady state for a given monetary policy, $(i, \chi)$. Then, the transition starts in steady state with $z_s(i_0, \chi)$ and ends with $z_s(i_T, \chi)$. This can be solved by backward induction, as follows.

$$z_T = f_T(z_T), \quad z_{T-1} = f_0(z_T), \quad z_{T-2} = f_0(z_{T-1}), \quad \ldots \quad z_0 = f_0(z_1)$$

Figure 8 shows that the transition dynamics can be very different depending on the reserve requirements. In the top-left panel of Figure 8, under the 100% percent reserve requirement, lowering $i$ increases $z_t$ from $z_0 = 0.4639$ to $z_T = 0.6529$ at the end. The most of changes (76% of the total change) are reflected in the period $t = 1$ as soon as the announcement is released. However, under the 50% percent reserve requirement, 68% of changes are reflected in the period $t = 1$ and $z_t$ gradually increases to $z_T$ after $t = 1$.

Under much lower reserve requirements, the transition dynamics become more com-

[5]The permanent change in $i$ implies a permanent change in the rate of monetary expansion, $\gamma$, from $\gamma_0 = \beta(1 + i_0)$ to $\gamma_T = \beta(1 + i_T)$.

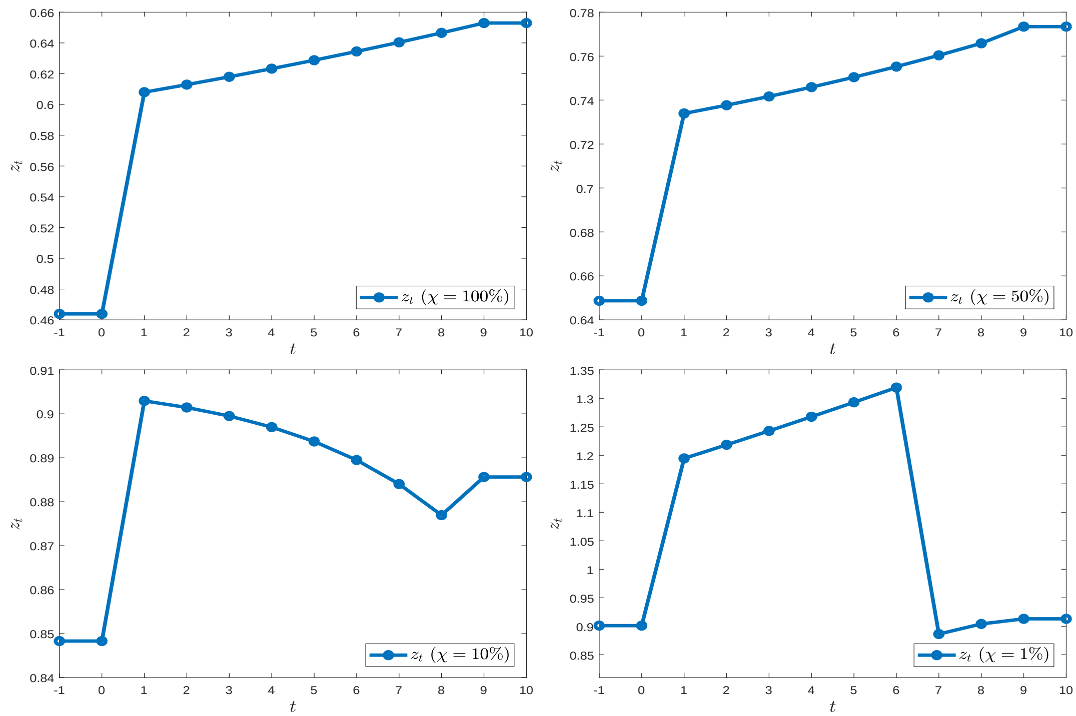


**Figure 8:** Announcement effect (Model 1): from $i_0 = 0.02$ to $i_T = 0.01$

plicated. Under the 10% reserve requirement, lowering $i$ eventually increases $z_t$ from $z_0 = 0.8483$ to $z_T = 0.8856$. However, as soon as the announcement is released, $z_1$ increases to 0.9029 which is higher than $z_T$ and it gradually decreases to $z_T$. Under the 1% reserve requirement, lowering $i$ eventually increases $z_t$ from $z_0 = 0.9011$ to $z_T = 0.9130$. Similar to the 10% reserve requirement case, as soon as the announcement is released, it increases to $z_1 = 1.1946$ which is a lot higher than $z_T$ and gradually increases to $z_6 = 1.3189$. However, it converges to $z_T = 0.9130$ by accompanying a big crash. As the central bank sets lower reserve requirement, the monetary policy announcement likely induces more fluctuations which can result in boom and bust.

Next, I introduce unsecured credit. Similar to the previous experiment, the permanent change from $i_0$ to $i_T$ implies a shift of equations (47) and (48), from $z_t = \Phi_0(z_{t+1}, \bar{b}_{t+1}; i_0, \chi)$ and $\bar{b}_t = \Gamma_0(z_{t+1}, \bar{b}_{t+1}; i_0, \chi)$ to $z_t = \Phi_T(z_{t+1}, \bar{b}_{t+1}; i_T, \chi)$ and $\bar{b}_t = \Gamma_T(z_{t+1}, \bar{b}_{t+1}; i_T, \chi)$, respectively. Then, starting in steady state with $(z_0, \bar{b}_0)$ and ending in steady state with $(z_T, \bar{b}_T)$, the transitional dynamics of the equilibrium with

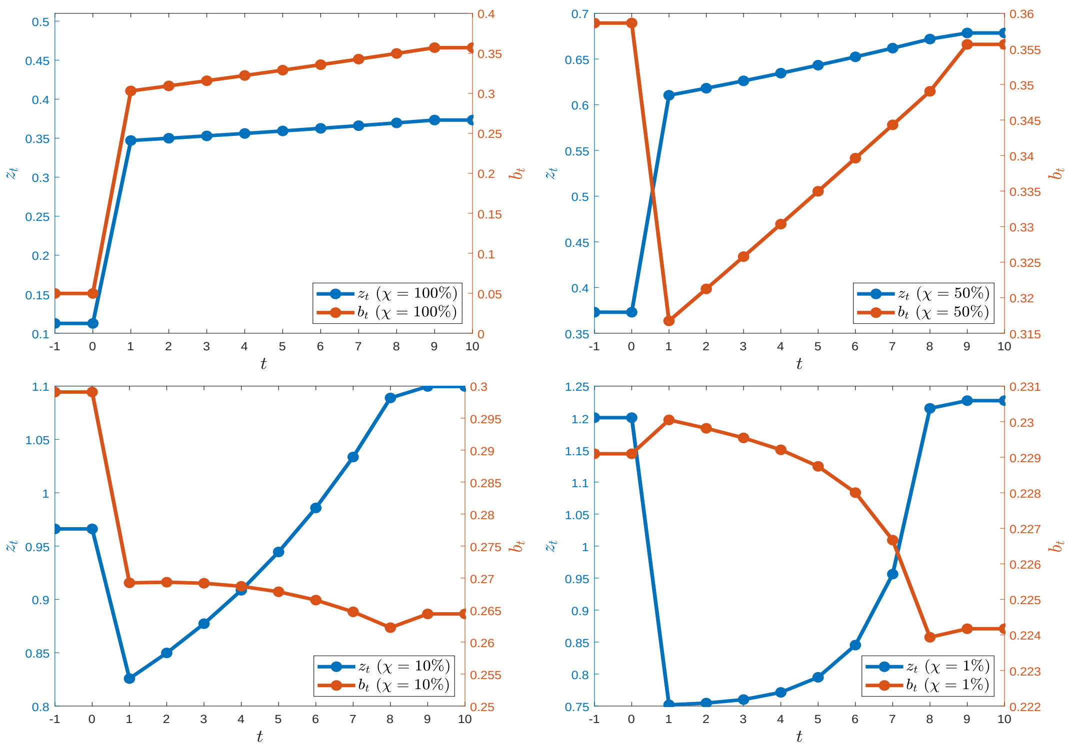


**Figure 9:** Announcement effect (Model 2): from $i_0 = 0.02$ to $i_T = 0.01$

unsecured credit also can be solved by backward induction:

$$
\begin{aligned}
&z_T = \Phi_T(z_T, \bar{b}_T), \quad z_{T-1} = \Phi_0(z_T, \bar{b}_T), \quad z_{T-2} = \Phi_0(z_{T-1}, \bar{b}_{T-1}), \quad \ldots \quad z_0 = \Phi_0(z_1, \bar{b}_1) \\
&\bar{b}_T = \Gamma_T(z_T, \bar{b}_T), \quad \bar{b}_{T-1} = \Gamma_0(z_T, \bar{b}_T), \quad \bar{b}_{T-2} = \Gamma_0(z_{T-1}, \bar{b}_{T-1}), \quad \ldots \quad \bar{b}_0 = \Gamma_0(z_1, \bar{b}_1)
\end{aligned}
$$

Figure 9 illustrates that the transition dynamics could be more complicated in Model 2 than in Model 1. The top-left panel of Figure 9 shows transition dynamics under the 100% percent reserve requirement. This case is similar to the case from Model 1. Lowering $i$ from $i_0 = 0.02$ to $i_T = 0.01$ eventually increases $z_t$ and $b_t$. Most of the changes are reflected in period $t = 1$ as soon as the announcement is released. The top-right panel of Figure 9 shows transition dynamics under the 50% reserve requirement. First, the real balance ultimately increases to $z_T > z_0$ but the unsecured credit decreases to $b_T < b_0$. However, the transition dynamics are not monotone. The money real balance $z_t$ adjusts gradually toward $z_T$ after most of the changes in $z_t$ are reflected at $t = 1$. In contrast to the money real balance, the secured credit $b_t$ drops drastically right after the monetary policy announcement and gradually increases to the $b_T$.

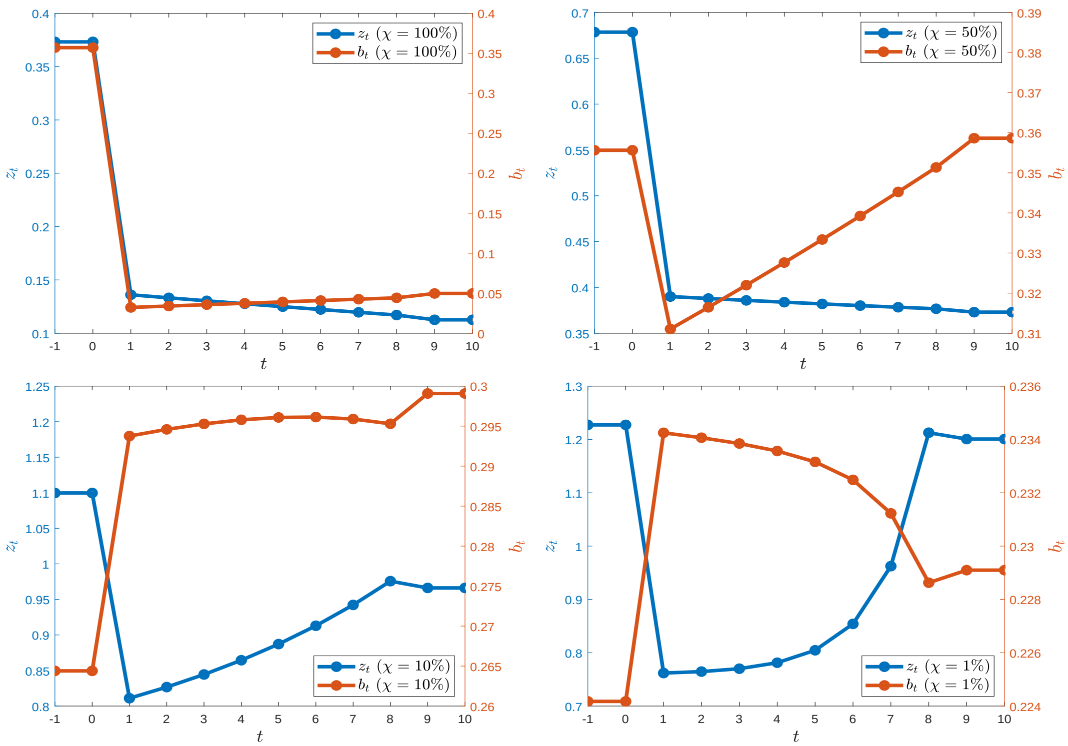


**Figure 10:** Announcement effect (Model 2): from $i_0 = 0.01$ to $i_T = 0.02$

Similar to the experiment in Model 1, the transition dynamics become more complicated when reserve requirements are small. The bottom-left panel of Figure 9 shows the case under the 10% reserve requirement. In this case, the transition dynamics of $z_t$ and $b_t$ both are non-monotone. Right after the announcement, both drop first, then $z_t$ increases to $z_T$ but $b_t$ gradually decreases further until it converges to $b_T$. Under the 1% reserve requirement, the initial responses for the monetary announcement are a huge decrease in real balance $z_t$ and an increase in $b_t$. Their ultimate responses, however, are a small increase in real balance $z_t$ and a decrease in unsecured credit $b_t$. Since the ultimate responses are the opposite of the initial responses, the announcement generates fluctuations in money and credit. Same as in Model 1, in Model 2, as the central bank sets the lower reserve requirement, the monetary policy announcement is more likely to induce more fluctuations which can result in boom and bust.

Figure 10 presents the transition dynamics in Model 2 when the central bank increases $i$ from $i_0 = 0.01$ to $i_T = 0.02$. The top-left panel of Figure 10 shows the transition dynamics under the 100% percent reserve requirement. This case is just the opposite of lowering $i$. Increasing $i$ from $i_0 = 0.01$ to $i_T = 0.02$ eventually increases

$z_t$ and $b_t$ and the most of changes are reflected in period $t = 1$ as soon as the announcement is released. Under the 50% reserve requirement, an increase in $i$ increases $b_T$ but lowers $z_T$. But the initial responses of the announcement are decreases in both $z_t$ and $b_t$ while $b_t$ decreases and $z_t$ increases after the initial response. Under the 10% reserve requirement, in the end, $z_T$ decreases while $b_T$ increases. However, in the initial responses, $b_t$ increases more than $b_T$, and $z_t$ decreases much lower than $z_T$, as soon as the announcement is released. The case under the 1% reserve requirement is similar to 10% reserve requirement. While the ultimate responses are a decrease in $z_T$ and an increase in $b_T$, $b_t$ increases more than $b_T$, and $z_t$ decreases far more than $z_T$. The monetary policy announcement on the increase in $i$ is also likely to induce more fluctuations which can result in boom and bust.

Some of the transition dynamics in Figure 9 and 10 are asymmetric. For example, in the bottom-right panels of Figure 9 and 10, the initial responses are a huge increase in $b_t$ and a huge decrease in $z_t$ in both cases, whereas the one is the effect of the announcement on an increase in $i$ and the other is the effect of the announcement on a decrease in $i$. In the bottom-left panels of Figure 9 and 10, initial responses of increasing interest rate are a huge drop in $z_t$ and a huge increase in $b_t$ while both decrease as an initial response when the central bank lowers $i$. These dynamics are not symmetric and the economy can exhibit many different transition dynamics depending on different reserve requirements.

To summarize the findings, fractional reserve banking can cause fluctuations and asymmetrical movements in money real balances and credit by amplifying the transition dynamics arising from the monetary policy announcements.

# 7 Conclusion

The goal of this paper is to examine the (in)stability of fractional reserve banking. To that end, this paper builds a simple monetary model of fractional reserve banking that can capture the conditions for (in)stability under different specifications. Lowering the reserve requirement increases the consumption at the steady state. However, it can induce instability. The baseline model and its extension establish the conditions for endogenous cycles and chaotic dynamics. The model also features stochastic cycles under explicit conditions. The model shows that fractional reserve banking can endanger stability in the sense that equilibrium is more prone to exhibit cyclic, chaotic, and stochastic dynamics under lower reserve requirements. This is due to the amplified liquidity premium. This result holds in the extended model with unsecured credit.

This paper also provides some empirical evidence that is consistent with the prediction of the model. I test the association between the required reserves ratio and the real inside money volatility using cointegrating regression. I find a significant negative relationship between the two variables. Both theoretical and empirical evidence find a link between the reserve requirement policy and (in)stability. The calibrated exercise suggests that this channel could be another source of economic fluctuations.

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# Appendix

## Appendix A Proofs

**Proof of Lemma 1.** First we check the possibility that the bank is not active, $d_t = s_t = 0$. Assuming this is the case, the buyer's problem gives

$$1 + i_{s,t} < \lambda_{m,b} < 1 + i_{l,t}, \qquad (1 + i_{d,t})[\alpha\lambda(q_t) + 1] < \lambda_{m,b} < 1 + i_{l,t} \tag{53}$$

where $\lambda_{m,b} > 0$. However, from the bank's problem $d_t = s_t = 0$ gives

$$1 + i_l < \lambda^b < 1 + i_s, \qquad 1 + i_l < \lambda^b < 1 + i_d + \chi\lambda^r \tag{54}$$

where $\lambda^b, \lambda^r > 0$. These conditions contradict to each other because the first condition (53) implies $i_{l,t} > i_{d,t}$ and $i_{l,t} > i_{s,t}$ while the second condition (54) implies $i_{l,t} < i_{d,t}$ and $i_{l,t} < i_{s,t}$. Therefore, we always have $d_{b,t}, d_t > 0$ and $i_{d,t} \geq 0$ in the monetary equilibrium. Since the bank is active and $i_{d,t} \geq 0$ satisfies (35), the monetary equilibrium always satisfies $\lambda(q) < \chi/\{\alpha(1-\chi)\}$. Therefore, any equilibrium $\{\phi_t\}_{t=1}^{+\infty}$ satisfying (36) must satisfy either (1) $\lambda(q) < \chi/\{\alpha(1-\chi)\}$ and $\phi_t M_t > 0$ or (2) $q_t = \phi_t = 0$. Then any monetary equilibrium satisfies $\lambda(q) < \chi/\{\alpha(1-\chi)\}$ which means in a monetary equilibrium, $q_t \in (\hat{q}, q^*]$ and a monetary equilibrium satisfying $q_t \in [0, \hat{q}]$ does not exist where $\hat{q}$ solves

$$\lambda(\hat{q}) = \min\left\{\frac{\chi}{\alpha(1-\chi)}, \lambda(0)\right\}.$$

■

**Proof of Lemma 2.** Let $z_t \in (\hat{p}, p^*)$. Recall equation (35):

$$i_{d,t} = \frac{\chi}{\chi + (\chi - 1)\alpha L(z_t + z_t i_{d,t})} - 1$$

The left-hand side (LHS) of (35) is increasing in $i_{d,t}$

$$\frac{\partial i_{d,t}}{\partial i_{d,t}} = 1 > 0.$$

and the LHS of (35) equal to 0 when $i_{d,t} = 0$. The right-hand side (RHS) of (35) is decreasing in $i_{d,t}$

$$\frac{\partial\left[\frac{\chi}{\chi+(\chi-1)\alpha L(z_t+z_t i_{d,t})}\right]}{\partial i_{d,t}} = \frac{-z_t\chi(\chi-1)\alpha L'(z_t+z_t i_{d,t})}{[\chi+(\chi-1)\alpha L(z_t+z_t i_{d,t})]^2} < 0.$$

and the RHS of (35) is bigger than 0 when $i_{d,t}=0$ since

$$\frac{\chi}{\chi+(\chi-1)\alpha L(z_t)} - 1 > 0.$$

Because the LHS is decreasing in $i_{d,t}$ and the RHS is increasing in $i_{d,t}$, there is a unique point that satisfies (35). ■

**Proof of Proposition 1.** Recall equation (41):

$$i\chi = \{1+i(1-\chi)\}\,\alpha L(\bar{p}_s)$$

Using the implicit function theorem, it is straightforward to show the following comparative statics:

$$\frac{\partial \bar{p}_s}{\partial i} = \frac{\chi+(\chi-1)\alpha L(\bar{p}_s)}{\{1+i(1-\chi)\}\alpha L'(\bar{p}_s)} < 0$$

$$\frac{\partial \bar{p}_s}{\partial \chi} = \frac{i\{1+\alpha L(\bar{p}_s)\}}{\{1+i(1-\chi)\}\alpha L'(\bar{p}_s)} < 0$$

Since $v(q_s)=\bar{p}_s$ and $v'(q)>0$, one can show $\partial q/\partial i<0$ and $\partial q/\partial\chi<0$. ■

**Proof of Proposition 2.** Let there exists a two-period cycle satisfying $z_1 < z_s < p^* < z_2$. Since $z_2 > p^*$, we have $z_2 = (1+i)z_1$. Using (40) with $z_1 < \bar{p}_1 < p^*$ gives

$$\chi = \frac{(1+i)^2\alpha L\left(\bar{p}_1\right)}{\{(1+i)^2-1\}\left\{1+\alpha L\left(\bar{p}_1\right)\right\}} \tag{55}$$

This two-period cycle should satisfy $\bar{p}_1 < \bar{p}_s < p^*$ and $z_2 = \bar{p}_2 = (1+i)z_1 \geq p^*$. First one can be easily shown using

$$0 = L(p^*) < L(\bar{p}_s) = \frac{i}{\alpha[1+i(1-\chi)]}\chi < \frac{(1+i)^2-1}{\alpha[1+\{(1+i)^2-1\}(1-\chi)]}\chi = L(\bar{p}_1)$$

since we have $L'(\cdot) < 0$. Because $\partial\bar{p}_1/\partial\chi < 0$ and $\partial\bar{p}_1/\partial z_1 > 0$, the latter one,

$z_1 \geq p^*/(1+i)$, is held when

$$0 < \chi < \frac{(1+i)^2 \alpha L\left(\frac{p^*}{1+i}\right)}{\left\{(1+i)^2 - 1\right\}\left\{1 + \alpha L\left(\frac{p^*}{1+i}\right)\right\}}.$$

■

**Proof of Proposition 3.** Recall (40)

$$\begin{aligned} f(z) =& \frac{z}{1+i}\{1 + i_d(z)\}\Big\{1 + \alpha L\big(z + i_d(z)z\big)\Big\} \\ =& \frac{q}{1+i}\left\{1 + \alpha\lambda\big(q\big)\right\} \end{aligned}$$

and take a derivative with respect to $z$

$$f'(z) = \frac{\partial q}{\partial z}\left[1 + \alpha\lambda(q) + q\alpha\lambda'(q)\right]\frac{1}{1+i}.$$

Since $-qu''(q)/u'(q) = \eta$ and $c(q) = q$, we have

$$f'(z) = \left[\frac{1 - \alpha + \alpha u'(q)(1-\eta)}{1+i}\right]\frac{\partial q}{\partial z}.$$

We want to get $f'(z)|_{z=z_s}$ which is

$$f'(z_s) = \left[\frac{1 - \alpha + \alpha u'(q_s)(1-\eta)}{1+i}\right] \times \frac{\partial q}{\partial z}\Big|_{z=z_s}.$$

Using $-qu''(q)/u'(q) = \eta$ and $c(q) = q$, rewrite equation (41) as

$$u'(q_s) = 1 + \frac{i\chi}{\alpha\left\{1 + i(1-\chi)\right\}}. \tag{56}$$

Then we have

$$f'(z_s) = \left[\frac{1 - \alpha + \alpha(1-\eta)\left\{1 + \frac{i\chi}{\alpha\{1+i(1-\chi)\}}\right\}}{1+i}\right] \times \frac{\partial q}{\partial z}\Big|_{z=z_s}. \tag{57}$$

Recall

$$q = (1+i_d)z = \frac{z\chi}{\chi + (\chi - 1)\alpha[u'(q) - 1]}$$

and applying the implicit function theorem gives

$$\frac{\partial q}{\partial z}\Big|_{z=z_s} = \frac{\chi}{\chi + (1-\chi)\alpha + (\chi-1)\alpha(1-\eta)u'(q_s)}. \tag{58}$$

Combining (57) and (58) gives

$$f'(z_s) = \left[\frac{1-\alpha+\alpha(1-\eta)u'(q_s)}{1+i}\right] \frac{\chi}{1+(\chi-1)[1-\alpha+\alpha(1-\eta)u'(q_s)]}. \tag{59}$$

Lastly, substitute (56) to (59) and collecting terms yields

$$f'(z_s) = \frac{\chi\left[1-\alpha+\alpha(1-\eta)\left(1+\frac{i\chi}{\alpha[1+i(1-\chi)]}\right)\right]}{(1+i)\left\{1+(\chi-1)\left[1-\alpha+\alpha(1-\eta)\left(1+\frac{i\chi}{\alpha[1+i(1-\chi)]}\right)\right]\right\}}.$$

■

**Proof of Proposition 4.** I divide three period cycles into two cases.
Case 1: Let there exists a three-period cycle satisfying $z_1 < z_s < p^* < z_2 < z_3$. Since $z_2, z_3 > p^*$, we have $z_2 = (1+i)z_1$, $z_3 = (1+i)z_2 = (1+i)^2 z_1$. Using (40) with $z_1 < \bar{p}_1 < p^*$ gives

$$\chi = \frac{(1+i)^3 \alpha L(z_1)}{\{(1+i)^3 - 1\}\{1+\alpha L(z_1)\}} \tag{60}$$

This three-period cycle should satisfy $z_1 < z_s < p^*$ and $z_2 = (1+i)z_1 > p^*$. First one can be easily shown using $z_2 = (1+i)z_1 > p^*$. First one can be easily shown using

$$0 = L(p^*) < L(\bar{p}_s) = \frac{i}{\alpha[1+i(1-\chi)]}\chi < \frac{(1+i)^3 - 1}{\alpha[1+\{(1+i)^3-1\}(1-\chi)]}\chi = L(\bar{p}_1)$$

since we have $L'(\cdot) < 0$. Because $\partial\bar{p}_1/\partial\chi < 0$ and $\partial\bar{p}_1/\partial z_1 > 0$, the latter one, $z_1 > p^*/(1+i)$, is held when

$$0 < \chi < \frac{(1+i)^3 \alpha L\left(\frac{p^*}{1+i}\right)}{\{(1+i)^3 - 1\}\left\{1+\alpha L\left(\frac{p^*}{1+i}\right)\right\}}.$$

Case 2: Let there exists a three-period cycle satisfying $z_1 < z_2 < p^* \leq z_3$. Since

$z_3 > p^*$, we have $z_3 = z_2(1+i)$ and $(z_2, z_1)$ solves (61)-(62).

$$z_1 = f(z_2) = \{1 + i_d(z_2)\}\Big\{1 + \alpha L\big(z_2 + i_d(z_2)z_2\big)\Big\}\frac{z_2}{1+i} \tag{61}$$

$$z_2 \equiv \tilde{f}(z_1) = \{1 + i_d(z_1)\}\Big\{1 + \alpha L\big(z_1 + i_d(z_1)z_1\big)\Big\}\frac{z_1}{(1+i)^2}. \tag{62}$$

These functions satisfies $f(x) > x$ for $x < z_s$, $f(x) < x$ for $x > z_s$, $\tilde{f}(x) > x$ for $x < \tilde{z}$ and $\tilde{f}(x) < x$ for $x > \tilde{z}$ where $\tilde{z}$ solves $\tilde{z} = \tilde{f}(\tilde{z})$. One can easily show $\tilde{z} < z_s$. Therefore any intersection between $z_1 = f(z_2)$ and $z_2 = \tilde{f}(z_1)$ satisfies $z_1 > z_2$ which contradicts to our initial conjecture $z_1 < z_2$. This implies there is no three-period cycles satisfying $z_1 < z_2 < p^* \leq z_3$.

Now, we can conclude that a three-period cycle exist when

$$0 < \chi < \frac{(1+i)^3 \alpha L\left(\frac{p^*}{1+i}\right)}{\{(1+i)^3 - 1\}\left\{1 + \alpha L\left(\frac{p^*}{1+i}\right)\right\}}.$$

The existence of a three-cycle implies the existence of cycles of all orders and chaotic dynamics by the Sarkovskii theorem (Sharkovskii, 1964) and the Li-Yorke theorem (Li and Yorke, 1975). ■

**Proof of Corollary 1:** Proposition 4 shows that at least one periodic point satisfies $z_t < z_s < p^*$ in 3- period cycles. Two period cycles satisfies $z_1 < z_s < z_2$ also implies at least one periodic point satisfies $z_t < z_s < p^*$ in 2-period cycles since $z_1 < z_s < p^*$. This result holds for any $n$-periodic cycles. Let $z_1 < z_2 < ...z_n$ be the periodic points of a $n$-cycle. Suppose $z_j > z_s$ for all $j = 1, 2, ..n$. By the definition of a $n$-period cycle, $z_1 = f(z_n) < z_n$ since $f(z) < z$ for $z > z_s$.

$$z_n = f(z_{n-1}) < z_{n-1} = f(z_{n-2}) < z_{n-2}... < z_1.$$

which shows the contradiction implying at least one periodic point satisfies $z_t < z_s < p^*$. ■

**Proof of Proposition 5.** By definition, if there exists $(\zeta_1, \zeta_2)$ satisfying

$$z^1 = \zeta_1 f(z^1) + (1 - \zeta_1) f(z^2) \tag{63}$$

$$z^2 = (1 - \zeta_2) f(z^1) + \zeta_2 f(z^2) \tag{64}$$

with $\zeta_1, \zeta_2 < 1$, then there exists a proper sunspot equilibrium. Because $z^1$ and $z^2$ are weighted averages of $f(z^1)$ and $f(z^2)$, where $f(z^1) > z^1$ and$f(z^2) < z^2$, by the uniqueness of the positive steady state, necessary and sufficient conditions for (63) and (64) are

$$f(z^2) < z_1 < f(z^1) \text{ and } f(z^2) < z_2 < f(z^1).$$

Since $z^1 < z^2$, above conditions are reduce to

$$z^2 < f(z^1) \text{ and } z^1 > f(z^2). \tag{65}$$

When $\chi < \chi_m$, there exists $(z^1, z^2)$ that satisfies (65). Rewrite (63) and (64) as

$$\zeta_1 + \zeta_2 = \frac{z^1 - f(z^2) - z^2 + f(z^1)}{f(z^1) - f(z^2)} = \frac{z^1 - z^2}{f(z^1) - f(z^2)} + 1 < 1 \tag{66}$$

since $z^2 < z^1$ and $f(z^1) > f(z^2)$. Therefore, when $\chi < \chi_m$, a stationary sunspot equilibrium exists.

Now consider the case with $f'(z_s) < -1$. Since $f'(z_s) < 0$, there is an interval $[z_s - \varepsilon_1, z_s + \varepsilon_2]$, which satisfy $\varepsilon_1, \varepsilon_2 > 0$ and $f(z^1) > f(z^2)$ for $z^1 \in [z_s - \varepsilon_1, z_s)$ and $z^2 \in (z_s, z_s + \varepsilon_2]$.

$$\frac{z^2 - z_s}{z_s - z^1} < -f'(z_s) < \frac{z_s - z^1}{z^2 - z_s}$$

Since $f'(z_s) < -1$, the above condition can be reduce to $-f'(z_s) < \frac{z_s - z^1}{z^2 - z_s} = \frac{\varepsilon_1}{\varepsilon_2}$. There exist multiple solutions, $(\varepsilon_1, \varepsilon_2)$, satisfying $-f'(z_s)\varepsilon_2 < \varepsilon_1$ given $-f'(z_s) > 1$ and $\varepsilon_1, \varepsilon_2 > 0$. These solutions satisfy (66). Therefore, if $f'(z_s) < -1$, there exists a stationary sunspot cycle. ■

**Proof of Proposition 6.** First, a two-period cycle result is presented and three-period case will follow. Let there exists a two-period cycle satisfying $a_1 < q^* < a_2$ where $a_j = z_j + \bar{b}_j$. Since $a_2 > q^*$, by (47) and (48), we have $z_2 = (1 + i)z_1$ and $\bar{b}_2 = (1 + \rho)\bar{b}_1 - \alpha\mu\sigma S(q^*)$ where $q_1$, $\bar{b}_1$, and $z_1$ solve

$$u'(q_1) = 1 + \frac{i\chi(i + 2)}{\alpha\{i^2(1 - \chi) + 2i(1 - \chi) + 1\}}$$
$$\bar{b}_1 = [(1 + \rho)^2 - 1]^{-1} \left\{\chi\mu \left[1 - (1 + i)^2\right] z_1 + \mu\alpha\sigma[S(q_1) + (1 + \rho)S(q^*)]\right\}$$

and $q_1 = z_1[1 + \iota_d(z_1)] + \bar{b}_1$.

This two-period cycle should satisfy $q_1 < q^*$ and $a_2 = (1 + i)z_1 + (1 + \rho)\bar{b}_1 -$

$\mu\alpha\sigma S(q^*) > q^*$. For given $i > 0$ and $\chi > 0$, the first one can be easily shown using

$$1 = u'(q^*) < u'(q_s) = 1 + \frac{i\chi}{\alpha[1 + i(1-\chi)]} < 1 + \frac{i\chi(i+2)}{\alpha\{i^2(1-\chi) + 2i(1-\chi) + 1\}} = u'(q_1)$$

since we have $u''(\cdot) < 0$. Now we also can check the latter using the below conditions

$$\begin{aligned} (1+\rho)q_1 >& (1+i)z_1 + (1+\rho)\bar{b}_1 - \mu\alpha\sigma S(q^*) = a_2 > q^* > q_1 = [1 + \iota_d(z_1)]z_1 + \bar{b}_1 \quad \text{if } \rho > i \\ (1+i)q_1 >& (1+i)z_1 + (1+\rho)\bar{b}_1 - \mu\alpha\sigma S(q^*) = a_2 > q^* > q_1 = [1 + \iota_d(z_1)]z_1 + \bar{b}_1 \quad \text{if } i > \rho. \end{aligned}$$

The sufficient conditions to have $a_2 > q^*$ is $q_1 > q^*/(1+\rho)$ for $\rho > i$ and $q_1 > q^*/(1+i)$ for $i > \rho$. Since we have $dq_1/d\chi < 0$, there exist a two-period cycle $a_1 < q_1 < q_s < q^* < a_2$ when

$$0 < \chi < \frac{(1+\iota)^2\alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]}{\{(1+\iota)^2 - 1\}\left\{1 + \alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]\right\}}.$$

where $\iota = \max\{i, \rho\}$.

Now, let there exists a three-period cycle satisfying $a_1 < q_1 < q_s < q^* < a_2 < a_3$ where $a_j = z_j + \bar{b}_j$. Since $a_3$, $a_2 > q^*$, by (47) and (48), we have $z_2 = (1+i)z_1$, $z_3 = (1+i)^2 z_1$, $\bar{b}_2 = (1+\rho)\bar{b}_1 - \mu\alpha\sigma S(q^*)$ and $\bar{b}_3 = (1+\rho)^2\bar{b}_1 - (2+\rho)\mu\alpha\sigma S(q^*)$ where $q_1$, $\bar{b}_1$, and $z_1$ solve

$$\begin{aligned} u'(q_1) &= 1 + \frac{i\chi(i^2 + 3i + 3)}{\alpha\{i^3(1-\chi) + 3i^2(1-\chi) + 3i(1-\chi) + 1\}} \\ \bar{b}_1 &= [(1+\rho)^3 - 1]^{-1}\left\{\chi\mu\left[1 - (1+i)^2\right]z_1 + \mu\alpha\sigma[S(q_1) + (1+\rho)S(q^*) + (1+\rho)^2 S(q^*)]\right\} \end{aligned}$$

and $q_1 = z_1[1 + \iota_d(z_1)] + \bar{b}_1$. This three-period cycle should satisfy $q_1 < q_s < q^*$ and $a_2 = (1+i)z_1 + (1+\rho)\bar{b}_1 - \mu\alpha\sigma S(q^*) > q^*$. For given $i > 0$ and $\chi > 0$, first one can be easily shown using $1 = u'(q^*) < u'(q_s)$ and

$$u'(q_s) = 1 + \frac{i\chi}{\alpha[1 + i(1-\chi)]} < 1 + \frac{i\chi(i^2 + 3i + 3)}{\alpha\{i^3(1-\chi) + 3i^2(1-\chi) + 3i(1-\chi) + 1\}} = u'(q_1)$$

since we have $u''(\cdot) < 0$. Now we also can check the latter using below conditions

$$\begin{aligned} (1+\rho)q_1 >& (1+i)z_1 + (1+\rho)\bar{b}_1 - \mu\alpha\sigma S(q^*) = a_2 > q^* > q_1 = [1 + \iota_d(z_1)]z_1 + \bar{b}_1 \quad \text{if } \rho > i \\ (1+i)q_1 >& (1+i)z_1 + (1+\rho)\bar{b}_1 - \mu\alpha\sigma S(q^*) = a_2 > q^* > q_1 = [1 + \iota_d(z_1)]z_1 + \bar{b}_1 \quad \text{if } i > \rho. \end{aligned}$$

The sufficient conditions to have $a_2 > q^*$ is $q_1 > q^*/(1+\rho)$ for $\rho > i$ and $q_1 > q^*/(1+i)$ for $i > \rho$. Because $\partial \bar{p}_1/\partial \chi < 0$ and $\partial \bar{p}_1/\partial z_1 > 0$, there exist a three-period cycle $a_1 < q_1 < q_s < q^* < a_2 < a_3$ when

$$0 < \chi < \frac{(1+\iota)^3 \alpha [u'\left(\frac{q^*}{1+\iota}\right) - 1]}{\{(1+\iota)^3 - 1\}\left\{1 + \alpha [u'\left(\frac{q^*}{1+\iota}\right) - 1]\right\}}.$$

where $\iota = \max\{i, \rho\}$. Again, the existence of a three-cycle implies the existence of cycles of all orders and chaotic dynamics by the Sarkovskii theorem and the Li-Yorke theorem. ■

# Appendix B Empirical Appendix

This section provides robustness checks for empirical results. To check the sensitivity of the results, Table 5 and 6 repeat all the empirical analysis, reported in Table 1 and 2, using quarterly series instead of annual data. The results are similar to the benchmark analysis shown in Table 1 and 2.

**Table 5:** Effect of Required Reserve Ratio: Robustness Check (Quarterly)

| Price level | CPI | | Core CPI | | PCE | | Core PCE | |
|---|---|---|---|---|---|---|---|---|
| Dependent variable: $\sigma_t^{Roll}$ | OLS (1) | CCR (2) | OLS (3) | CCR (4) | OLS (5) | CCR (6) | OLS (7) | CCR (8) |
| $\chi$ | $-0.282^{***}$ | $-0.452^{***}$ | $-0.266^{***}$ | $-0.400^{***}$ | $-0.305^{***}$ | $-0.485^{***}$ | $-0.306^{***}$ | $-0.476^{***}$ |
| | (0.016) | (0.001) | (0.014) | (0.003) | (0.015) | (0.000) | (0.014) | (0.006) |
| `ffr` | | $-0.050^{***}$ | | $-0.058^{***}$ | | $-0.015^{***}$ | | $-0.047^{***}$ |
| | | (0.000) | | (0.002) | | (0.000) | | (0.005) |
| Constant | $0.074^{***}$ | $0.085^{***}$ | $0.070^{***}$ | $0.079^{***}$ | $0.074^{***}$ | $0.089^{***}$ | $0.073^{***}$ | $0.086^{***}$ |
| | (0.002) | (0.000) | (0.002) | (0.000) | (0.002) | (0.000) | (0.002) | (0.001) |
| Obs. | 196 | 196 | 196 | 196 | 196 | 196 | 196 | 196 |
| $adj R^2$ | 0.696 | 0.240 | 0.725 | 0.263 | 0.737 | 0.222 | 0.761 | 0.268 |
| $\lambda_{trace}(r=0)$ | **9.496** | 31.950 | **11.045** | 33.808 | **10.930** | 34.481 | **12.103** | 35.951 |
| 5% CV | 15.41 | 29.68 | 15.41 | 29.68 | 15.41 | 29.68 | 15.41 | 29.68 |
| $\lambda_{trace}(r=1)$ | 1.677 | **11.162** | 1.959 | **12.266** | 1.938 | **12.094** | 1.887 | **12.485** |
| 5% CV | 3.76 | 15.41 | 3.76 | 15.41 | 3.76 | 15.41 | 3.76 | 15.41 |

Note: For (1), OLS estimates are reported, and Newey-West standard errors with lag 1 are reported in parentheses. For (2) and (3), first-stage long-run variance estimations for CCR are based on the quadratic spectral kernel and Bayesian information criterion. The bandwidth selection is based on Newey-West fixed lag, $4\times(T/100)^{2/9}$; `ffr` denotes federal funds rates and $\sigma_t^{Roll}$ denotes cyclical volatility of real inside money balances. ***, **, and * denotes significance at the 1, 5, and 10 percent levels, respectively.

**Table 6:** Unit Root Tests: Robustness Check (Quarterly)

| | | Phillips-Perron test | | ADF test |
|---|---|---|---|---|
| | | $Z(\rho)$ | $Z(t)$ | $Z(t)$ w/ lag 1 |
| `ffr` | | $-8.611$ | $-1.956$ | $-2.183$ |
| $\chi$ | | $-1.335$ | $-1.145$ | $-1.199$ |
| $\sigma_t^{Roll}$ | (CPI) | $-4.320$ | $-2.062$ | $-1.554$ |
| $\sigma_t^{Roll}$ | (Core CPI) | $-4.388$ | $-2.201$ | $-1.924$ |
| $\sigma_t^{Roll}$ | (PCE) | $-3.822$ | $-1.946$ | $-1.868$ |
| $\sigma_t^{Roll}$ | (Core PCE) | $-3.565$ | $-1.928$ | $-2.023$ |
| $\Delta$`ffr` | | $-139.701^{***}$ | $-10.792^{***}$ | $-10.288^{***}$ |
| $\Delta\chi$ | | $-163.796^{***}$ | $-12.272^{***}$ | $-9.909^{***}$ |
| $\Delta\sigma_t^{Roll}$ | (CPI) | $-23.132^{***}$ | $-2.604^{*}$ | $-3.576^{***}$ |
| $\Delta\sigma_t^{Roll}$ | (Core CPI) | $-30.423^{***}$ | $-3.544^{***}$ | $-4.894^{***}$ |
| $\Delta\sigma_t^{Roll}$ | (PCE) | $-24.507^{***}$ | $-2.874^{*}$ | $-4.362^{***}$ |
| $\Delta\sigma_t^{Roll}$ | (Core PCE) | $-28.054^{***}$ | $-3.373^{**}$ | $-5.138^{***}$ |

Note: `ffr` denotes federal funds rates, $\chi$ denotes required reserve ratio, and $\sigma_t^{Roll}$ denotes cyclical volatility of real inside money balances. ***, **, and * denotes significance at the 1, 5, and 10 percent levels, respectively.

This section also provides robustness checks using time-series before 2008. Table 7 and 8 repeat the analysis using time-series before 2008. Again, the results are similar to the benchmark analysis shown in Table 1 and 2.

**Table 7:** Effect of Required Reserve Ratio: Robustness Check (pre-2008)

| Price level | CPI | | Core CPI | | PCE | | Core PCE | |
|---|---|---|---|---|---|---|---|---|
| Dependent | OLS | CCR | OLS | CCR | OLS | CCR | OLS | CCR |
| variable: $\sigma_t^{Roll}$ | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| $\chi$ | $-0.266^{***}$ | $-0.297^{***}$ | $-0.266^{***}$ | $-0.268^{***}$ | $-0.307^{***}$ | $-0.288^{***}$ | $-0.305^{***}$ | $-0.277^{***}$ |
| | (0.030) | (0.001) | (0.030) | (0.001) | (0.032) | (0.002) | (0.029) | (0.002) |
| `ffr` | | $-0.107^{***}$ | | $-0.124^{***}$ | | $-0.189^{***}$ | | $-0.210^{***}$ |
| | | (0.001) | | (0.001) | | (0.002) | | (0.002) |
| Constant | $0.070^{***}$ | $0.080^{***}$ | $0.070^{***}$ | $0.076^{***}$ | $0.074^{***}$ | $0.082^{***}$ | $0.072^{***}$ | $0.080^{***}$ |
| | (0.004) | (0.000) | (0.004) | (0.000) | (0.004) | (0.000) | (0.004) | (0.002) |
| Obs. | 43 | 43 | 43 | 43 | 43 | 43 | 43 | 43 |
| $adjR^2$ | 0.727 | 0.659 | 0.727 | 0.710 | 0.739 | 0.708 | 0.759 | 0.734 |
| $\lambda_{trace}(r=0)$ | **8.373** | 32.228 | **7.438** | 31.299 | **7.661** | 31.867 | **6.897** | 31.250 |
| 5% CV | 15.41 | 29.68 | 15.41 | 29.68 | 15.41 | 29.68 | 15.41 | 29.68 |
| $\lambda_{trace}(r=1)$ | 1.504 | **9.554** | 1.125 | **8.428** | 1.146 | **8.603** | 0.938 | **7.693** |
| 5% CV | 3.76 | 15.41 | 3.76 | 15.41 | 3.76 | 15.41 | 3.76 | 15.41 |

Note: For (1), OLS estimates are reported, and Newey-West standard errors with lag 1 are reported in parentheses. For (2) and (3), first-stage long-run variance estimations for CCR are based on the quadratic spectral kernel and Bayesian information criterion. The bandwidth selection is based on Newey-West fixed lag, $4 \times (T/100)^{2/9}$; `ffr` denotes federal funds rates and $\sigma_t^{Roll}$ denotes cyclical volatility of real inside money balances. ***, **, and * denotes significance at the 1, 5, and 10 percent levels, respectively.

**Table 8:** Unit Root Tests: Robustness Check (pre-2008)

| | | Phillips-Perron test | | ADF test |
|---|---|---|---|---|
| | | $Z(\rho)$ | $Z(t)$ | $Z(t)$ w/ lag 1 |
| `ffr` | | $-9.476$ | $-2.258$ | $-2.868^{**}$ |
| $\chi$ | | $-0.768$ | $-0.660$ | $-0.877$ |
| $\sigma_t^{Roll}$ | (CPI) | $-2.966$ | $-1.738$ | $-1.770$ |
| $\sigma_t^{Roll}$ | (Core CPI) | $-2.860$ | $-1.641$ | $-1.495$ |
| $\sigma_t^{Roll}$ | (PCE) | $-2.662$ | $-1.515$ | $-1.627$ |
| $\sigma_t^{Roll}$ | (Core PCE) | $-2.412$ | $-1.371$ | $-1.400$ |
| $\Delta$`ffr` | | $-25.378^{***}$ | $-4.773^{***}$ | $-5.833^{***}$ |
| $\Delta\chi$ | | $-28.208^{***}$ | $-4.594^{***}$ | $-3.658^{***}$ |
| $\Delta\sigma_t^{Roll}$ | (CPI) | $-25.627^{***}$ | $-4.281^{***}$ | $-3.813^{***}$ |
| $\Delta\sigma_t^{Roll}$ | (Core CPI) | $-25.836^{***}$ | $-4.329^{***}$ | $-3.764^{***}$ |
| $\Delta\sigma_t^{Roll}$ | (PCE) | $-24.420^{***}$ | $-4.101^{***}$ | $-3.594^{**}$ |
| $\Delta\sigma_t^{Roll}$ | (Core PCE) | $-23.848^{***}$ | $-4.034^{***}$ | $-3.464^{**}$ |

Note: `ffr` denotes federal funds rates, $\chi$ denotes required reserve ratio, and $\sigma_t^{Roll}$ denotes cyclical volatility of real inside money balances. ***, **, and * denotes significance at the 1, 5, and 10 percent levels, respectively.

Lastly, this section provides robustness checks using alternative data set. Table 9 and 10 repeat the analysis using the alternative data, M1J from Lucas and Nicolini (2015). Instead of the total checkable deposit, this analysis uses the deposit component of M1J. To get the deposit component of M1J, I subtract the currency component of M1 from M1J. The required reserve ratio is calculated by dividing required reserves by the deposit component of M1J. Again, the results are similar to the benchmark analysis shown in Table 1 and 2.

**Table 9:** Effect of Required Reserve Ratio: Alternative Data

| Price level | CPI | | Core CPI | | PCE | | Core PCE | |
|---|---|---|---|---|---|---|---|---|
| Dependent | OLS | CCR | OLS | CCR | OLS | CCR | OLS | CCR |
| variable: $\sigma_t^{Roll}$ | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| $\chi$ | −0.032 | −0.251*** | −0.011 | −0.181*** | −0.045 | −0.257*** | −0.037 | −0.211*** |
| | (0.100) | (0.06) | (0.099) | (0.066) | (0.100) | (0.059) | (0.099) | (0.062) |
| `ffr` | | 0.869*** | | 0.797*** | | 0.853*** | | 0.789*** |
| | | (0.088) | | (0.099) | | (0.087) | | (0.092) |
| Constant | 0.059*** | 0.015** | 0.055*** | 0.013* | 0.058*** | 0.014** | 0.056*** | 0.014** |
| | (0.010) | (0.006) | (0.010) | (0.007) | (0.010) | (0.006) | (0.010) | (0.007) |
| Obs. | 49 | 49 | 49 | 49 | 49 | 49 | 49 | 49 |
| $adjR^2$ | -0.018 | 0.456 | -0.021 | 0.465 | -0.016 | 0.446 | -0.017 | 0.428 |
| $\lambda_{trace}(r=0)$ | **14.840** | 51.914 | **14.793** | 51.550 | **14.806** | 52.093 | **14.737** | 51.761 |
| 5% CV | 15.41 | 29.68 | 15.41 | 29.68 | 15.41 | 29.68 | 15.41 | 29.68 |
| $\lambda_{trace}(r=1)$ | 3.487 | **14.870** | 3.179 | **14.798** | 3.391 | **14.754** | 3.194 | **14.640** |
| 5% CV | 3.76 | 15.41 | 3.76 | 15.41 | 3.76 | 15.41 | 3.76 | 15.41 |

Note: For (1), OLS estimates are reported, and Newey-West standard errors with lag 1 are reported in parentheses. For (2) and (3), first-stage long-run variance estimations for CCR are based on the quadratic spectral kernel and Bayesian information criterion. The bandwidth selection is based on Newey-West fixed lag, $4 \times (T/100)^{2/9}$; `ffr` denotes federal funds rates and $\sigma_t^{Roll}$ denotes cyclical volatility of real inside money balances. ***, **, and * denotes significance at the 1, 5, and 10 percent levels, respectively.

**Table 10:** Unit Root Tests: Alternative Data

| | | Phillips-Perron test | | ADF test |
|---|---|---|---|---|
| | | $Z(\rho)$ | $Z(t)$ | $Z(t)$ w/ lag 1 |
| `ffr` | | −6.766 | −1.704 | −2.362 |
| $\chi$ | | −1.011 | −0.889 | −1.075 |
| $\sigma_t^{Roll}$ | (CPI) | −3.758 | −1.431 | −1.928 |
| $\sigma_t^{Roll}$ | (Core CPI) | −3.788 | −1.424 | −1.965 |
| $\sigma_t^{Roll}$ | (PCE) | −3.833 | −1.444 | −1.994 |
| $\sigma_t^{Roll}$ | (Core PCE) | −3.882 | −1.445 | −2.026 |
| $\Delta$`ffr` | | −28.373*** | −5.061*** | −6.357*** |
| $\Delta\chi$ | | −12.018* | −2.514 | −2.514 |
| $\Delta\sigma_t^{Roll}$ | (CPI) | −12.018*** | −2.514*** | −2.470 |
| $\Delta\sigma_t^{Roll}$ | (Core CPI) | −12.004* | −2.529 | −2.565 |
| $\Delta\sigma_t^{Roll}$ | (PCE) | −12.235* | −2.538 | −2.555 |
| $\Delta\sigma_t^{Roll}$ | (Core PCE) | −12.346* | −2.561 | −2.666* |

Note: `ffr` denotes federal funds rates, $\chi$ denotes required reserve ratio, and $\sigma_t^{Roll}$ denotes cyclical volatility of real inside money balances. ***, **, and * denotes significance at the 1, 5, and 10 percent levels, respectively.

# Appendix C Closed Form Solution for Calibration

This section provides closed-form solutions that are used for the calibration. The utility functions and the DM cost function for the parameterization are

$$U(X) = B\log(X), \qquad u(q) = \frac{Cq^{1-\eta}}{1-\eta}, \qquad c(q) = q.$$

The stationary equilibrium satisfies

$$u'(q) = Cq^{-\eta} = 1 + \frac{i\chi}{\alpha[1+i(1-\chi)]}$$

and the DM consumption is

$$q = \left(\frac{1}{C}\left\{1 + \frac{i\chi}{\alpha[1+i(1-\chi)]}\right\}\right)^{-\frac{1}{\eta}}.$$

The real balance of aggregate money $z$ is

$$z = \frac{q}{1+(1-\chi)i}$$

and the real balance $z$ to output ratio is given by

$$Z \equiv \frac{z}{y} = \frac{z}{\sigma\alpha q + B}.$$

A derivative of $z$ with respect to $i$ is

$$\frac{\partial z}{\partial i} = \left\{\frac{\partial q}{\partial i} - (1-\chi)z\right\}\frac{1}{1+(1-\chi)i}$$

where

$$\frac{\partial q}{\partial i} = \frac{-\chi}{\alpha\eta C[1+i(1-\chi)]^2}\left(\frac{1}{C}\left\{1 + \frac{i\chi}{\alpha[1+i(1-\chi)]}\right\}\right)^{-\frac{1+\eta}{\eta}}.$$

The semi-elasticity of aggregate money with respect to $i$ is given by

$$\frac{\partial\log(Z)}{\partial i} = \frac{1}{Z}\frac{\partial Z}{\partial i}.$$

When $\mu = 0$, the semi-elasticity of $Z$ with respect to $i$ is

$$\frac{\partial\log(Z)}{\partial i} = \frac{1}{Z}\frac{\partial Z}{\partial i} = \frac{\sigma\alpha q + B}{z}\left[\frac{\partial q}{\partial i}\left\{\frac{-\sigma\alpha z}{(\sigma\alpha q + B)^2}\right\} + \frac{\partial z}{\partial i}\frac{1}{\sigma\alpha q + B}\right].$$

Then elasticity of aggregate money with respect to $i$ is

$$\frac{\partial Z}{\partial i}\frac{i}{Z} = i \times \frac{\partial \log(Z)}{\partial i}.$$

When $\mu > 0$, unsecured credit limit is determined by

$$\bar{b} = \Omega(\bar{b}) = \frac{\mu\sigma\alpha}{\rho}[u(\tilde{q}) - \tilde{q}] + \frac{i\mu\chi}{\rho}(\tilde{q} - \bar{b}) = \left\{1 + \frac{i\mu\chi}{\rho}\right\}^{-1} \left\{\frac{\mu\sigma\alpha}{\rho}[u(\tilde{q}) - \tilde{q}] + \frac{i\mu\chi}{\rho}\tilde{q}\right\}$$

and the derivative of unsecured credit with respect to $i$ is

$$\begin{aligned}\frac{\partial \bar{b}}{\partial i} &= \left\{1 + \frac{i\mu\chi}{\rho}\right\}^{-1} \left\{\frac{\mu\sigma\alpha}{\rho}[u'(q) - 1]\frac{\partial q}{\partial i} - \frac{i\mu\chi}{\rho}\frac{\partial q}{\partial i} - \frac{\mu\chi}{\rho}q\right\} \\ &\quad - \frac{\mu\chi}{\rho}\left\{1 + \frac{i\mu\chi}{\rho}\right\}^{-2} \left\{\frac{\mu\sigma\alpha}{\rho}[u(\tilde{q}) - \tilde{q}] + \frac{i\mu\chi}{\rho}\tilde{q}\right\}.\end{aligned}$$

When $\mu > 0$, the real balance of aggregate money is

$$z = \frac{q - \bar{b}}{1 + (1 - \chi)i}$$

and the derivative of aggregate money real balances with respect to $i$ is

$$\frac{\partial z}{\partial i} = \left\{\frac{\partial q}{\partial i} - \frac{\partial \bar{b}}{\partial i} - (1 - \chi)z\right\} \frac{1}{1 + (1 - \chi)i}.$$

The aggregate money real balances $z$ to output ratio is

$$Z \equiv \frac{z}{y} = \frac{z}{\sigma\alpha q + B}$$

and the semi-elasticity of aggregate money with respect to $i$ is

$$\frac{\partial \log(Z)}{\partial i} = \frac{1}{Z}\frac{\partial Z}{\partial i} = \frac{\sigma\alpha q + B}{z}\left[\frac{\partial q}{\partial i}\left\{\frac{-\sigma\alpha z}{(\sigma\alpha q + B)^2}\right\} + \frac{\partial z}{\partial i}\frac{1}{\sigma\alpha q + B}\right].$$

The elasticity of aggregate money with respect to $i$ is

$$\frac{\partial Z}{\partial i}\frac{i}{Z} = i \times \frac{\partial \log(Z)}{\partial i}.$$