On the Instability of Fractional Reserve Banking*

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Abstract

This paper develops a dynamic general equilibrium model to study the (in)stability of the fractional reserve banking system. The model shows that the fractional reserve banking system can endanger stability in that equilibrium is more prone to exhibit endogenous cyclic, chaotic, and stochastic dynamics under lower reserve requirements, although it can increase welfare in the steady-state. Introducing endogenous unsecured credit to the baseline model does not change the main results. This paper also provides empirical evidence that is consistent with the prediction of the model.

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> Motivated partly by a desire to avoid such [excessive] price-level fluctuations and possible Wicksellian price-level indeterminacy, quantity theorists have advocated legal restrictions on private intermediation. ... Thus, for example, Friedman (1959, p. 21) ... has advocated 100 percent reserves against bank liabilities called demand deposit. **Sargent and Wallace (1982)**

1 Introduction

There have been claims that fractional reserve banking is an important cause of boom-bust cycles, based on the notion that banks create excess credit under fractional reserve banking. (e.g., Fisher, 1935; Von Mises, 1953; Minsky, 1957; Minsky, 1970). For instance, Fisher (1935) views fractional reserve banking as one of several important factors in explaining economic fluctuations. Others believe that this is a primary cause of boom-bust cycles. According to Von Mises (1953), the overexpansion of bank credit as a result of fractional reserve banking is the root cause of business cycles. Minsky (1970) claims that economic booms and structural characteristics of the financial system, such as fractional reserve banking, can result in an economic collapse even when fundamentals remain unchanged.

This idea leads to policy debates on fractional reserve banking. Earlier examples include Peel's Banking Act of 1844 and the Chicago plan of banking reform with a 100% reserve requirement proposed by Irving Fisher, Paul Douglas, and others in 1939. Later, Friedman (1959) supported this banking reform, whereas Becker (1956) took the opposite position of supporting free banking with 0% reserve requirement.[1] Recently in 2018, Switzerland had a referendum of 100% reserve banking, which was rejected by 75.72% of the voters. The referendum aimed at making money safe from crisis by constructing full-reserve banking.[2] Whereas the debate on whether a fractional reserve banking system is inherently unstable has been an important policy discussion since a long time ago, the debate has never stopped.

This paper examines the instability of fractional banking by answering the following

[1] Sargent (2011) provides a novel review of the historical debates between narrow banking and free banking as tensions between stability versus efficiency.

[2] The official title of the referendum was *the Swiss federal popular initiative "for crisis-safe money: money creation by the National Bank only! (Sovereign Money Initiative)"* and also titled as "*debt-free money.*"

questions: (i) Can fractional reserve banking be inherently volatile even if we shut down the stochastic component of the economy? (ii) If so, under what condition can fractional reserve banking generate endogenous cycles without the presence of exogenous shocks and changes in fundamentals? To assess the claim that fractional reserve banking causes business cycles, this paper constructs a model of money and banking that captures the role of fractional reserve banking.

In the model, each agent faces an idiosyncratic liquidity shock. Banks accept deposits and extend loans to provide risk-sharing among the depositors whereas the bank's lending is constrained by the reserve requirement. The real balance of money is determined by two factors: storage value and liquidity premium. The storage value is increasing in the future value of money. However, the liquidity premium, the marginal value of its liquidity function, is decreasing if the money becomes more abundant. When the liquidity premium dominates the storage value, the economy can exhibit endogenous fluctuations. Fractional reserve banking amplifies the liquidity premium because it allows the bank to create inside money through lending. Due to this amplified liquidity premium, the fractional reserve banking system is more prone to endogenous cycles.

In the baseline model, lowering the reserve requirement increases welfare in the steady state. However, lowering the reserve requirements can induce two-period cycles as well as three-period cycles, which implies the existence of periodic cycles of all order and chaotic dynamics. This also implies it can induce sunspot cycles. This result holds in the extended model with unsecured credit. The model also can deliver a self-fulfilling bubble burst. It is worth noting that the full reserve requirement does not necessarily exclude the possibility of endogenous cycles. However, the economy will be more susceptible to cycles with lower reserve requirement.[3]

This paper departs from previous works in two ways. First, in contrast to the previous works on banking instability, which mostly focus on bank runs following the seminal model by Diamond and Dybvig (1983), this paper focuses on the volatility of real balances of inside money. It is another important focal point of banking instability because recurring boom-bust cycles associated with banking are probably be more prevalent than bank runs. Second, the approach here differs from a traditional approach to economic fluctuations with financial frictions. To understand economic fluctuations, there are two major points of view. The first one is that economic fluctuations are

[3] Gu, Monnet, Nosal and Wright (2019) show that introducing banks to the economy could induce instability in various settings which is in line with this result.

driven by exogenous shocks disturbing the dynamic system, and the effects of exogenous shocks shrink over time as the system goes back to its balanced path or steady-state. The second one is that they instead reflect an endogenous mechanism that produces boom-bust cycles. While there has been a lot of work on the role of financial friction in the business cycles including Kiyotaki and Moore (1997), Bernanke, Gertler and Gilchrist (1999), and Gertler and Karadi (2011), most of them focused on the first approach, in which all economic fluctuations are caused by exogenous shocks and the financial sectors only serve as an amplifier. This paper, however, takes the second approach and focuses on whether the endogenous cycles arise in the absence of the stochastic components of the economy.

To evaluate the main prediction from the theory that fractional reserve banking induces excess volatility, I test the relationship between the required reserves ratio and the volatilty in real balance using cointegrating regression. A significant negative relationship between the two variables are found, and the results are robust to different measures of inflation and different frequency of time series. Both theoretical and empirical evidence indicate a link between the reserve requirement and the (in)stability.

Related Literature This paper builds on Berentsen, Camera and Waller (2007), who introduce financial intermediaries with enforcement technology to Lagos and Wright (2005) framework. The approach to introduce unsecured credit to the monetary economy is related to Lotz and Zhang (2016) and Gu, Mattesini and Wright (2016) which are based on the earlier work by Kehoe and Levine (1993).

This paper is related to the large literature on fractional reserve banking. Freeman and Huffman (1991) and Freeman and Kydland (2000) develop general equilibrium models that explicitly capture the role of fractional reserve banking. Using those models, they explain the observed relationships between key macroeconomic variables over business cycles. Chari and Phelan (2014) study the condition under which fractional reserve banking can be socially useful by preventing bank runs in the cash-in-advance framework. More recently, Andolfatto, Berentsen and Martin (2020) integrates Diamond (1997) into Lagos and Wright (2005) to provide a model in which fractional reserve banking emerges endogenously and a central bank can prevent bank panic as a lender of last resort. Whereas many previous work on instability focuses on bank runs, this paper studies a different type of instability in the sense that fractional reserve banking induces endogenous monetary cycles.

This paper is also related to the large literature on endogenous fluctuations, chaotic dynamics, and indeterminacy that have been surveyed by Brock (1988), Baumol and Benhabib (1989), Boldrin and Woodford (1990), Scheinkman and Woodford (1994) and Benhabib and Farmer (1999). For a model of bilateral trade, Gu, Mattesini, Monnet and Wright (2013) show that credit markets can be susceptible to endogenous fluctuations due to limited commitment. Gu et al. (2019) show that introducing financial intermediaries to an economy can engender instability in four distinct setups that capture various functions of banking. The model in this paper is closely related to Gu et al. (2019), whereas the model here is extended to incorporate fractional reserve banking.

The rest of the paper is organized as follows. Section 2 constructs the baseline search-theoretic monetary model. Section **??** provides main results. Section 2.5 introduces unsecured credit. Section 3 discusses the empirical evaluation of the model. Section 4 concludes.

2 Theory

First, I simply add reserve requirement to Berentsen et al. (2007) type of banks. Second, I extend the first model by

2.1 Basic assumptions

The model is based on Lagos and Wright (2005) with a financial intermediary as in Berentsen et al. (2007). Time is discrete and infinite. In each period, three markets convene sequentially. First, a centralized financial market (FM), followed by a decentralized goods market (DM), and finally a centralized goods market (CM). The FM and CM are frictionless. The DM is subject to search frictions, anonymity, and limited commitment. Therefore, a medium of exchange is needed to execute trades.

There is a continuum of agents who produce and consume perishable goods. At the beginning of the FM, a preference shock is realized: With probability σ, an agent will be a buyer in the following DM and with probability $1-\sigma$, she will be a seller. The buyers and the sellers randomly meet and trade bilaterally in the DM. Agents discount their utility each period by β. Within-period utility is represented by

$$\mathcal{U} = U(X) - H + u(q) - c(q),$$

where X is the CM consumption, H is the CM disutility from production, and q is the DM consumption. As standard U', u', $c' > 0$, U'', $u'' < 0$, $c'' \geq 0$, and $u(0) = c(0) = 0$. The CM consumption good X is produced one-for-one with H, implying the real wage is 1. The efficient consumption in CM and DM is X^* and q^* that solve $U'(X^*) = 1$ and $u'(q^*) = c'(q^*)$, respectively.

The terms of trade in the DM are determined by an abstract mechanism that is studied in Gu and Wright (2016). The buyer must pay $p = v(q)$ to the seller to get q where $v(q)$ is some payment function satisfying $v'(q) > 0$ and $v(0) = 0$. As shown in Gu and Wright (2016), if the trading protocol satisfies four common axioms, then the terms of trade can be written in the following form:

$$p_t = \begin{cases} \bar{p}_t & \text{if } \bar{p}_t < p^* \\ p^* & \text{if } \bar{p}_t \geq p^* \end{cases} \qquad q_t = \begin{cases} v^{-1}(\bar{p}_t) & \text{if } \bar{p}_t < p^* \\ q^* & \text{if } \bar{p}_t \geq p^*, \end{cases} \tag{1}$$

where $\bar{p}_t$ is the liquidity position of the buyer, and p^* is the payment required to get efficient consumption q^*. Many standard mechanisms, such as Kalai and generalized Nash bargaining, are consistent with this specification.

- type buyers and seller. σ
- meeting probability α

There is a representative bank who accepts deposits and lends loans in the FM. In the FM, the agent can borrow money from the bank for a promise to repay money in the subsequent CM at nominal lending rate i_l. The agent can also deposit money to the bank and receive money in the subsequent CM at nominal deposit rate i_d. The banking market is perfectly competitive. The bank can enforce the repayment of loans at no cost. Last, there is a central bank that controls the money supply M_t. Let γ be the growth rate of the money stock. Changes in money supply are accomplished by lump-sum transfer if $\gamma > 0$ and by lump-sum tax if $\gamma < 0$.

2.2 Model 1 without unsecured credit

Let W_t, G_t, and V_t denote the agent's value function in the CM, FM, and DM, respectively, in period t. There are two payment instruments for the DM transaction: fiat money (outside money) and loans from the bank (inside money). I will allow the agents to use unsecured credit as a means of payment in the next section. An agent entering the CM with nominal balance m_t, deposit d_t, and loan ℓ_t, solves the following problem:

$$\begin{aligned} W_t(m_t, d_t, \ell_t) = & \max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta G_{t+1}(\hat{m}_{t+1}) \\ \text{s.t. } \phi_t \hat{m}_{t+1} + X_t = & H_t + T_t + \phi_t m_t + (1 + i_{d,t})\phi_t d_t - (1 + i_{l,t})\phi_t \ell_t, \end{aligned} \tag{2}$$

where T_t is the lump-sum transfer (or tax if it is negative), $i_{d,t}$ is the deposit interest rate, $i_{l,t}$ is the loan interest rate, ϕ_t is the price of money in terms of the CM goods, and $\hat{m}_{t+1}$ is the money balance carried to the FM where banks take deposits and makes loans. The first-order conditions (FOCs) result in $X_t = X^*$ and

$$\phi_t = \beta G'_{t+1}(\hat{m}_{t+1}), \tag{3}$$

where $G'_{t+1}(\hat{m}_{t+1})$ is the marginal value of an additional unit of money taken into the FM of period $t + 1$. The envelope conditions are

$$\frac{\partial W_t}{\partial m_t} = \phi_t, \qquad \frac{\partial W_t}{\partial d_t} = \phi_t(1 + i_{d,t}), \qquad \frac{\partial W_t}{\partial \ell_t} = -\phi_t(1 + i_{l,t}),$$

implying W_t is linear in m_t, d_t, and ℓ_t.

The value function of an agent at the beginning of FM is

$$G_t(m) = \sigma G_{b,t}(m) + (1 - \sigma)G_{s,t}(m), \tag{4}$$

where $G_{j,t}$ is the value function of type j agent in the FM. Agents choose their deposit balance d_j and loan ℓ_j based on the realization of their types in the following DM. The value function $G_{j,t}$ can be written as

$$G_{j,t}(m) = \max_{d_{j,t}, \ell_{j,t}} V_{j,t}(m - d_{j,t} + \ell_{j,t}, d_{j,t}, \ell_{j,t}) \quad \text{s.t} \quad d_{j,t} \leq m, \tag{5}$$

where $V_{j,t}$ is the value function of type j agent in the DM. The FOCs are

$$\frac{\partial V_{j,t}}{\partial \ell_{j,t}} \leq 0, \qquad \frac{\partial V_{j,t}}{\partial d_{j,t}} - \lambda_d \leq 0$$

where λ_d is the Lagrange multiplier for $d_{j,t} \leq m$.

With probability α, a buyer meets a seller in the DM while a seller meets a buyer with probability α_s. Since the CM value function is linear, the DM value function for the buyer can be written as

$$V_{b,t}(m_t - d_{b,t} + \ell_{b,t}, d_{b,t}, \ell_{b,t}) = \alpha[u(q_t) - p_t] + W(m_t - d_{b,t} + \ell_{b,t}, d_{b,t}, \ell_{b,t}), \tag{6}$$

where $p_t \leq \phi_t(m_t - d_{b,t} + \ell_{b,t})$. Assuming interior solution, differentiating $V_{b,t}$ yields

$$\frac{\partial V_{b,t}}{\partial m} = \phi_t[\alpha\lambda(q_t) + 1], \qquad \frac{\partial V_{b,t}}{\partial d} = \phi_t[-\alpha\lambda(q_t) + i_{d,t}], \qquad \frac{\partial V_{b,t}}{\partial \ell} = \phi_t[\alpha\lambda(q_t) - i_{l,t}],$$

where $\lambda(q) = u'(q)/v'(q) - 1$ if $p^* > z$ and $\lambda(q) = 0$ if $z \geq p^*$. Combining the buyer's FOCs in the FM and the derivatives of V_b yields

$$\phi i_{d,t} - \phi\alpha\lambda(q_t) - \lambda_d \leq 0, \text{``} = \text{''} 0 \text{ iff } d_{b,t} > 0 \tag{7}$$

$$-\phi i_{l,t} + \phi\alpha\lambda(q) \leq 0, \text{``} = \text{''} 0 \text{ iff } \ell_{b,t} > 0. \tag{8}$$

A seller's DM value function is

$$V_{s,t}(m_t - d_{s,t} + \ell_{s,t}, d_{s,t}, \ell_{s,t}) = \alpha_s[p_t - c(q_t)] + W_t(m_t - d_{s,t} + \ell_{s,t}, d_{s,t}, \ell_{s,t}). \tag{9}$$

Differentiating $V_{s,t}$ yields

$$\frac{\partial V_{s,t}}{\partial m_t} = \phi_t, \qquad \frac{\partial V_{s,t}}{\partial d_t} = \phi_t i_{d,t}, \qquad \frac{\partial V_{s,t}}{\partial \ell_t} = -\phi_t i_{l,t}.$$

Similar to the buyer's case, combining the seller's FOCs in the FM and the first-order derivatives of $V_{s,t}$ yields

$$\phi_t i_{d,t} - \lambda_d \leq 0, \text{``} = \text{''} 0 \text{ iff } d_{s,t} > 0 \tag{10}$$

$$-\phi_t i_{l,t} \leq 0, \text{``} = \text{''} 0 \text{ iff } \ell_{s,t} > 0. \tag{11}$$

One can show that buyers do not deposit and sellers always deposit whereas buyers always borrow loans but sellers do not. This is because the buyer needs liquidity to trade for q in the DM but the seller does not. Formally, for $m > 0$, we have

$\partial V_{b,t}/\partial d_{b,t} < \partial V_{s,t}/\partial d_{s,t} = 0$ and $\partial V_{s,t}/\partial \ell_{s,t} < \partial V_{b,t}/\partial \ell_{b,t} = 0$ because

$$0 = \overbrace{i_{d,t} - \lambda_d/\phi_t}^{\partial V_{s,t}/\partial d_{s,t}} > \overbrace{i_{d,t} - \lambda_d/\phi_t - \alpha\lambda(q_t)}^{\partial V_{b,t}/\partial d_{b,t}} \tag{12}$$

$$0 = \underbrace{-\phi_t i_{l,t} + \phi_t \alpha\lambda(q_t)}_{\partial V_{b,t}/\partial \ell_{b,t}} > \underbrace{-\phi_t i_{l,t}}_{\partial V_{s,t}/\partial \ell_{s,t}} \tag{13}$$

implying $i_{l,t} = \alpha\lambda(q_t)$, $d_{s,t} = m$, $d_{b,t} = 0$, $\ell_{s,t} = 0$, and $\ell_b > 0$ as long as $\lambda(q_t) > 0$.

Using the above results, we can rewrite the value functions in the FM as follows:

$$G_{b,t}(m_t) = \alpha[u(q_t) - p_t] + W(m_t + \ell_{b,t}, 0, \ell_{b,t}) \tag{14}$$

$$G_{s,t}(m_t) = \alpha_s[p_t - c(q_t)] + W(m_t - d_{s,t}, d_{s,t}, 0) \tag{15}$$

where $q_t = v^{-1}(p_t)$ and $p_t = \min\{p^*, \phi(m_t + \ell_{b,t})\}$. Take derivative of $G_{j,t}(m_t)$ with respect to m_t to get

$$G'_{b,t}(m_t) = \phi_t + \phi_t \alpha\lambda(q_t), \text{ and } G'_{s,t}(m_t) = \phi_t + \phi_t i_{d,t}.$$

Since $G'_t(m_t) = \sigma G'_{b,t}(m_t) + (1-\sigma)G'_{s,t}(m_t)$, we have the following:

$$G'_t(m_t) = \phi_t \sigma[1 + \alpha\lambda(q_t)] + \phi_t(1-\sigma)(1 + i_{d,t}). \tag{16}$$

Combine (34) and (52) to get the Euler equation

$$\phi_t = \begin{cases} \phi_{t+1}\beta\left[\sigma\left\{1 + \alpha\lambda(q_{t+1})\right\} + (1-\sigma)(1 + i_{d,t+1})\right] & \text{if } z_{t+1} < p^* \\ \phi_{t+1}\beta & \text{if } z_{t+1} \geq p^*, \end{cases} \tag{17}$$

where $q_{t+1} = v^{-1}(z_{t+1})$ and $z_{t+1} = \phi_{t+1}(m_{t+1} + \ell_{b,t+1})$

Consider a bank's problem. A representative bank accepts deposits d and makes loans ℓ. The depositors are paid at the nominal interest rate i_d by the bank, and the borrowers need to repay their borrowing with a nominal interest rate i_l. The central bank sets reserve requirement χ. The representative bank solves the following profit maximization problem.

$$\max_{d,\ell} \quad (i_l \ell - i_d d) \quad s.t. \quad \chi\ell \leq d \tag{18}$$

The FOCs for the bank's problem are

$$0 = i_l - \lambda_L, \quad 0 = -i_d + \lambda_L/\chi$$

where λ_L is the Lagrange multiplier with respect to the bank's lending constraint. For $\lambda_L > 0$, we have

$$i_l = \chi i_d \tag{19}$$

while $\lambda_L = 0$ implies $i_d = i_l = 0$.

The next step is to characterize the equilibrium. With binding bank's lending constraint, the equilibrium lending satisfies $\ell_t = (1-\sigma)m_t/\chi$ and $\ell_{b,t} = (1-\sigma)m_t/(\sigma\chi)$. Combine equations (40), (53) and (19), and use equilibrium condition $m_{t+1} = M_{t+1}$ to get

$$\phi_t = \begin{cases} \phi_{t+1}\beta \left[\dfrac{1-\sigma+\sigma\chi}{\chi}\alpha\lambda \circ v^{-1}(z_{t+1}) + 1\right] & \text{if } z_{t+1} < p^* \\ \phi_{t+1}\beta & \text{if } z_{t+1} \geq p^*, \end{cases} \tag{20}$$

where $z_{t+1} = \phi_{t+1}M_{t+1}(1-\sigma+\sigma\chi)/\sigma\chi$. Then multiplying both sides of (20) by $M_t(1-\sigma+\sigma\chi)/\sigma\chi$ allows us to reduce the equilibrium condition to one difference equation of real balances z:

$$z_t = f(z_{t+1}) \equiv \frac{z_{t+1}}{1+i}\left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_{t+1}) + 1\right], \tag{21}$$

where $(1+i) \equiv \gamma/\beta$ and $L(z) \equiv \lambda \circ v^{-1}(z)$ is the liquidity premium.[4] The dynamics of monetary equilibrium is characterized by equation (21).

Consider a stationary equilibrium, which is a fixed point that satisfies $z = f(z)$. There always exists an non-monetary equilibrium with $z = 0$. Given $i \in [0, \bar{\iota})$ and $\chi \in (0, 1]$, where $\bar{\iota} = \alpha(1-\sigma+\sigma\chi)L(0)/\chi$,[5] an unique stationary monetary equilibrium exists and satisfies

$$\chi i = (1-\sigma+\sigma\chi)\alpha L(z_s).$$

Since $L'(z) < 0$ (see Gu and Wright, 2016), the following result holds:

Proposition 1. *In the stationary equilibrium, lowering i or lowering χ increases q.*

[4] In the stationary equilibrium, $i = \gamma/\beta - 1$ is the nominal interest rate.

[5] Nash and Kalai bargaining provides simple examples for $\bar{\iota}$. Under the Inada condition $u'(0) = \infty$, with Kalai, $\bar{\iota} = \theta\alpha(1-\sigma+\sigma\chi)/\chi(1-\theta)$; whereas with Nash bargaining, $\bar{\iota} = \infty$.

2c_figures/cycle21.pdf

2c_figures/cycle22.pdf

Figure 1: A Two-period Cycle under Fractional Reserve Banking

We can derive the condition that the economy exhibits a two-period cycle that satisfies $z_1 < z_s \leq p^* < z_2$.

Proposition 2. *There exists a two-period cycle with $z_1 < z_s \leq p^* < z_2$,*

$$\frac{(1+i)^2 - 1}{\alpha(1-\sigma+\sigma\chi)}\chi = L(z_1), \text{ and } z_2 = (1+i)z_1$$

if $\chi \in (0, \bar{\chi}_m)$, where

$$\bar{\chi}_m \equiv \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^2 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)},$$

Proposition 2 shows that lowering the reserve requirement can induce a two-period cycle. We can also check whether it also increases the volatility. Consider the difference between peak and trough $z_2 - z_1 = (1+i)z_1$. Since $dz_1/d\chi = \frac{L(z_1)}{L'(z_1)}\left(\frac{1-\sigma}{\chi^2}\right) / \left(\frac{1-\sigma}{\chi} + 1\right) < 0$, lowering reserve requirement increases the endogenous volatility of the real balances. However, the difference in proportions is independent of the reserve requirement because $z_2/z_1 = (1+i)$.

Proposition 2 shows that lowering the reserve requirement can induce a two-period cycle under the general trading mechanism. However, in general, a two period cycle with $z_1 < z_s < z_2$, could be either $z_2 > p^*$ or $z_2 < p^*$. Following the standard textbook method (see Azariadis, 1993), we can show that if $f'(z_s) < -1$, there exists a two-period cycle with $z_1 < z_s < z_2$. Consider a special case where $-qu''(q)/u'(q) = \eta$, $c(q) = q$ and the buyer makes take-it-or-leave-it (TIOLI) offer. The following proposition says that there exists a two-period cycle if χ is low.

Proposition 3. *Assume $-qu''(q)/u'(q) = \eta$, $c(q) = q$, and the buyer makes take-it-or-leave-it offer to seller. If $\chi \in (0, \chi_m)$, where*

$$\chi_m \equiv \frac{\alpha\eta(1-\sigma)}{\eta(1-\alpha\sigma) + (2-\eta)(1+i)}, \tag{22}$$

then $f'(z_s) < -1$.

Whereas (22) is written in terms of χ, this condition can be written in terms of i, as follows:

$$0 < i < \frac{\eta[\alpha(1-\sigma) - \chi(1-\alpha\sigma)]}{\chi(2-\eta)} \tag{23}$$

The role of i on cycles depends on η. By (23), if $\eta < 2$, lowering either χ or i can induce a cycle. If $2/(\alpha\sigma) > \eta > 2$, χ_m is negative when $i > \frac{\eta\alpha\sigma-2}{2-\eta}$ and positive when $i < \frac{\eta\alpha\sigma-2}{2-\eta}$. In this case, setting i higher than $\frac{\eta\alpha\sigma-2}{2-\eta}$ eliminates cyclic equilibria. If $\eta \geq 2/(\alpha\sigma)$, χ_m is negative for all i, implying the cycle does not exist. When $\eta = 2$, χ_m is constant, implying that the i has no effect on the cycle in this case.

To interpret the results, recall $f(z_{t+1})$ from equation (21). The first term, $z_{t+1}/(1+i)$ on the right-hand side, reflects the store of value, which is monotonically increasing in z_{t+1}. The second term $(1-\sigma+\sigma\chi)\alpha L(z_{t+1})/\chi + 1$, reflecting the liquidity premium, is decreasing in z_{t+1}. Because $f'(z_{t+1})$ depends on both terms, $f(z_{t+1})$ is nonmonotone in general. If the liquidity premium dominates the storage value, we can have $f'(\cdot) < -1$, which is a standard condition for the existence of cyclic equilibria. Lowering the reserve requirement amplifies the liquidity premium because it allows the bank to create more liquidity through lending. This amplification of liquidity generates endogenous cycles.

In addition to the condition for two-period cycles, the next result provides the condition for three-period cycles under the general trading mechanism. The existence of three period-cycles implies cycles of all orders as well as chaotic dynamics (see Sharkovskii, 1964 and Li and Yorke, 1975).

Proposition 4 (**Three-period Monetary Cycle and Chaos**). *A three-period cycle with $z_1 < z_2 < p^* < z_3$ does not exist. There exists a three-period cycle with $z_1 < p^* < z_2 < z_3$ if $\chi \in (0, \hat{\chi}_m)$, where*

$$\hat{\chi}_m \equiv \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^3 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)}.$$

The following corollary is a direct result from proposition **??**.

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2c_figures/cycle32.pdf

Figure 2: A Three-period Cycle under Fractional Reserve Banking

Corollary 1 (**Binding Liquidity Constraint**). *In any n-period cycle, the liquidity constraint binds, $z_t < p^*$, at least one periodic point over the cycle.*

The model can also generate sunspot cycles. Consider a Markov sunspot variable $S_t \in \{1, 2\}$. This sunspot variable is not related to fundamentals but may affect equilibrium. Let $\Pr(S_{t+1} = 1|S_t = 1) = \zeta_1$ and $\Pr(S_{t+1} = 2|S_t = 2) = \zeta_2$. The sunspot is realized in the FM. Let W_t^S be the CM value function in state S in period t, then

$$\begin{aligned} W_t^S(m_t, d_t, \ell_t) = \max_{X_t, H_t, \hat{m}_{t+1}} \; & U(X_t) - H_t + \beta \left[\zeta_s G_{t+1}^S(\hat{m}_{t+1}) + (1-\zeta_s) G_{t+1}^{-S}(\hat{m}_{t+1})\right] \\ \text{s.t. } & \phi_t^S \hat{m}_{t+1} + X_t = H_t + T_t + \phi_t^S m_t + (1+i_{d,t})\phi_t^S d_t - (1+i_{l,t})\phi_t^S \ell_t. \end{aligned}$$

The FOC can be written as

$$-\phi_t^S + \beta \zeta_s G_{t+1}'^S(\hat{m}_{t+1}) + \beta(1-\zeta_s) G_{t+1}'^{-S}(\hat{m}_{t+1}) = 0. \tag{24}$$

Solving the FM problem results in

$$G_{t+1}'^S(m_{t+1}^S) = \phi_{t+1}^S \left[\frac{1-\sigma+\sigma\chi}{\chi} \alpha L(z_{t+1}^S) + 1\right]. \tag{25}$$

We substitute (70) into (69) and use the money market clearing condition $m_{t+1} = M_{t+1}$ to get the Euler equation.

$$\phi_t^S = \beta \zeta_s \phi_{t+1}^S \left[\frac{1-\sigma+\sigma\chi}{\chi} \alpha L(z_{t+1}^S) + 1\right] + \beta(1-\zeta_s)\phi_{t+1}^{-S} \left[\frac{1-\sigma+\sigma\chi}{\chi} \alpha L(z_{t+1}^{-S}) + 1\right].$$

where $z_{t+1}^S = \phi_{t+1}^S M_{t+1}(1-\sigma+\sigma\chi)/\sigma\chi$. Then multiply both sides of the Euler equation by $M_t(1-\sigma+\sigma\chi)/\sigma\chi$ to reduce the equilibrium condition into one difference equation of real balances z_{t+1}^S:

$$\begin{aligned} z_t^S &= \frac{\zeta_s z_{t+1}^S}{1+i}\left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_{t+1}^S)+1\right] + \frac{(1-\zeta_s) z_{t+1}^{-S}}{1+i}\left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_{t+1}^{-S})+1\right] \\ &= \zeta_s f(z_{t+1}^S) + (1-\zeta_s) f(z_{t+1}^{-S}). \end{aligned} \tag{26}$$

We define a sunspot equilibrium as follows:

Definition 1 (**Proper Sunspot Equilibrium**). *A proper sunspot equilibrium consists of the sequences of real balances $\{z_t^S\}_{t=0,S=1,2}^{\infty}$ and probabilities (ζ_1,ζ_2), solving (71) for all t.*

Consider stationary sunspot equilibria with $z^1 < z^2$ that only depend on the state, not the time. The liquidity constraint is binding in state $S=1$. By the standard approach (see again Azariadis, 1993 for the textbook treatment), the condition for two-period cycles is also sufficient and necessary for two-state sunspot equilibrium. If $f'(z_s) < -1$, there exists $(\zeta_1,\zeta_2) \in (0,1)^2$, $\zeta_1+\zeta_2<1$, such that the economy has a proper sunspot equilibrium in the neighborhood of z_s.

Proposition 5 (**Stationary Sunspot Equilibria**). *The stationary sunspot equilibrium exists if either $\chi < \chi_m$ or $f'(z_s) < -1$.*

2.3 Model 1 with unsecured credit

Consider an alternative payment instrument in the DM - unsecured credit. The buyer can pay for DM goods using unsecured credit that will be redeemed to the seller in the following CM and she can borrow up to her debt limit, $\bar{b}_t$. For simplicity, I assume that the buyer makes a TIOLI offer to the seller in the DM, which means the buyer maximizes her surplus subject to the seller's participation constraint. The DM cost function is $c(q)=q$. Suppose the buyer has issued b_t units of unsecured debt in the previous DM. The CM value function is

$$\begin{aligned} W_t(m_t,d_t,\ell_t,-b_t) = &\max_{X_t,H_t,\hat{m}_{t+1}} U(X_t) - H_t + \beta G_{t+1}(\hat{m}_{t+1}) \\ \text{s.t. } \phi_t\hat{m}_{t+1}+X_t = &H_t+T_t+\phi_t m_t+(1+i_{d,t})\phi_t d_t-(1+i_{l,t})\phi_t\ell_t-b_t, \end{aligned} \tag{27}$$

which is the same as before except that the agent needs to pay or collect the debt. The agent's FM problem is identical to the previous section. Then, $1-\sigma$ fraction of agents will deposit $\hat{m}_{t+1}$, and σ fraction of agents will borrow loan from the bank. The DM value function is

$$V_t^b(m_t+\ell_t,0,\ell_t)=\alpha[u(q_t)-q_t]+W_t(m_t+\ell_t,0,\ell_t,0),$$

where $q_t=\min\{q^*,\bar{b}_t+\phi_t(m_t+\ell_t)\}$. Given $\bar{b}_t$, solving equilibrium yields

$$z_t=\begin{cases}\dfrac{z_{t+1}}{1+i}\left\{\dfrac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u'(z_{t+1}+\bar{b}_{t+1})-1\right]+1\right\} & \text{if } z_{t+1}+\bar{b}_{t+1}<q^* \\ \dfrac{z_{t+1}}{1+i} & \text{if } z_{t+1}+\bar{b}_{t+1}\geq q^*,\end{cases} \tag{28}$$

where $z_{t+1}=(1-\sigma+\sigma\chi)\phi_{t+1}M_{t+1}/(\sigma\chi)$.

Next, I am going to endogenize the debt limit. The buyer cannot commit to pay back the debt. If the buyer reneges she is captured with probability μ. The punishment for a defaulter is permanent exclusion from the DM trade but she can still produce for herself in the CM. The value of autarky is $\underline{\mathrm{W}}(0,0,0,0)=[U(X^*)-X^*+T]/(1-\beta)$. The incentive condition for voluntary repayment is

$$\underbrace{-b_t+W_t(m_t,d_t,\ell_t,0)}_{\text{value of honoring debts}}\geq\underbrace{(1-\mu)W_t(m_t,d_t,\ell_t,0)+\mu\underline{\mathrm{W}}(m_t,d_t,\ell_t,0)}_{\text{value of not honoring debts}}.$$

One can write the debt limit $\bar{b}_t$ as $b_t\leq\bar{b}_t\equiv\mu W_t(0,0,0,0)-\mu\underline{\mathrm{W}}(0,0,0,0)$. Recall the CM value function. Using the solution of FM, we can rewrite the buyer's CM value function as

$$\begin{aligned}W_t(0,0,0,0)&=U(X^*)-X^*+T_t+\beta W_{t+1}(0,0,0,0)\\&\quad+\max_{\hat{m}_{t+1}}\{-\phi_t\hat{m}_{t+1}+\beta\alpha\sigma[u(q_{t+1})-q_{t+1}]+\beta\phi_{t+1}\hat{m}_{t+1}\},\end{aligned}$$

where $q_{t+1}=\min\{q^*,z_{t+1}+\bar{b}_{t+1}\}$. Substituting $W_t(0,0,0,0)=\bar{b}_t/\mu+\underline{\mathrm{W}}(0,0,0,0)$ yields

$$\frac{\bar{b}_t}{\mu}=-\phi_tM_{t+1}+\beta\alpha\sigma[u(z_{t+1}+\bar{b}_{t+1})-z_{t+1}-\bar{b}_{t+1}]+\frac{\beta\bar{b}_{t+1}}{\mu}+\beta\phi_{t+1}M_{t+1},$$

where M_{t+1} and z_{t+1} solve (72). Rearranging terms yields

$$\bar{b}_t = \begin{cases} \beta\bar{b}_{t+1} + \dfrac{\chi\mu\sigma[-\gamma z_t + \beta z_{t+1}]}{1-\sigma+\sigma\chi} + \beta\alpha\mu\sigma S(z_{t+1}+\bar{b}_{t+1}) & \text{if } z_{t+1}+\bar{b}_{t+1} < q^* \\ \beta\bar{b}_{t+1} + \dfrac{\chi\mu\sigma[-\gamma z_t + \beta z_{t+1}]}{1-\sigma+\sigma\chi} + \beta\alpha\mu\sigma S(q^*) & \text{if } z_{t+1}+\bar{b}_{t+1} \geq q^*, \end{cases} \tag{29}$$

where $S(z_{t+1}+\bar{b}_{t+1}) \equiv [u(z_{t+1}+\bar{b}_{t+1}) - z_{t+1} - \bar{b}_{t+1}]$ is the buyer's trade surplus. The equilibrium can be collapsed into a dynamic system satisfying (72)-(73).

In the stationary equilibrium, (72) becomes

$$-\frac{i\chi}{\alpha(1-\sigma+\sigma\chi)} + u'(q) \leq 0, = \text{ if } z > 0 \tag{30}$$

and (73) becomes

$$(1-\beta)\bar{b} = \begin{cases} \dfrac{\chi\mu\sigma[\beta-\gamma]z}{1-\sigma+\sigma\chi} + \beta\alpha\mu\sigma[u(z+\bar{b}) - z - \bar{b}] & \text{if } z+\bar{b} < q^* \\ \dfrac{\chi\mu\sigma[\beta-\gamma]z}{1-\sigma+\sigma\chi} + \beta\alpha\mu\sigma[u(q^*) - q^*] & \text{if } z+\bar{b} \geq q^*, \end{cases} \tag{31}$$

where $q = \min\{z+\bar{b}, q^*\}$. The stationary equilibrium solves the above two equations, and it falls into one of the three cases: the pure money equilibrium, the pure credit equilibrium, and the money-credit equilibrium. First, if no one can capture the buyer after she reneges, $\mu = 0$, the unsecured credit is not feasible, $\bar{b} = 0$. In this case, the equilibrium will be the pure money equilibrium. Second, when $\bar{b}$ solving (75) satisfies $u'(\bar{b}) < i\chi/[\alpha(1-\sigma+\sigma\chi)]$ then money is not valued, $z = 0$. We have the pure credit equilibrium in this case. Third, if solutions of (74)-(75), $(z, \bar{b})$ are strictly positive then we have the money-credit equilibrium.

The debt limit at the stationary equilibrium, $\bar{b}$, is a fixed point satisfying $\bar{b} = \Omega(\bar{b})$ where

$$\Omega(\bar{b}) = \begin{cases} \dfrac{\mu\sigma\alpha}{r}[u(\tilde{q}) - \tilde{q}] - \dfrac{i\mu\sigma\chi}{r(1-\sigma+\sigma\chi)}(\tilde{q} - \bar{b}) & \text{if } \tilde{q} > \bar{b} \geq 0 \\ \dfrac{\mu\sigma\alpha}{r}[u(\bar{b}) - \bar{b}] & \text{if } q^* > \bar{b} \geq \tilde{q} \\ \dfrac{\mu\sigma\alpha}{r}[u(q^*) - q^*] & \text{if } \bar{b} \geq q^* \end{cases} \tag{32}$$

where $\tilde{q}$ solves $u'(\tilde{q}) = 1 + i\chi/[\alpha(1-\sigma+\sigma\chi)]$ and $r \equiv 1/\beta - 1$. The DM consumption

q_s is determined by $q_s = \min\{q^*, \max\{\tilde{q}, \bar{b}\}\}$. Money and credit coexist if and only if $0 < \bar{b} < \tilde{q}$, which holds when $0 < \mu < \min\{1, \tilde{\mu}\}$, where

$$\tilde{\mu} \equiv \frac{r(1-\sigma+\sigma\chi)}{\alpha\sigma[u(\tilde{q})/\tilde{q}-1](1-\sigma+\sigma\chi)-2i\sigma\chi}.$$

The DM consumption is decreasing in i in the stationary monetary equilibrium.

Consider the dynamics of equilibria where money and credit coexist. I claim the main results from Section **??** - lowering the reserve requirement can induce endogenous cycles - still hold even after unsecured credit is introduced. For compact notation, let $\iota \equiv \max\{i, r\}$ and $w_j \equiv z_j + \bar{b}_j$. The following proposition establishes the conditions for two-period cycles, three-period cycles, and chaotic dynamics.

Proposition 6 (**Monetary Cycles with Unsecured Credit**). *There exists a two-period cycle of money and credit with $w_1 < q^* < w_2$ if $\chi \in (0, \chi_c)$, where*

$$\chi_c \equiv \frac{(1-\sigma)\alpha\left[u'\left(\frac{q^*}{1+\iota}\right)-1\right]}{(1+i)^2-1-\sigma\alpha\left[u'\left(\frac{q^*}{1+\iota}\right)-1\right]}.$$

There exists a three-period cycle of money and credit with $w_1 < q^* < w_2 < w_3$, if $\chi \in (0, \hat{\chi}_c)$, where

$$\hat{\chi}_c \equiv \frac{(1-\sigma)\alpha\left[u'\left(\frac{q^*}{1+\iota}\right)-1\right]}{(1+i)^3-1-\sigma\alpha\left[u'\left(\frac{q^*}{1+\iota}\right)-1\right]}.$$

2.4 Model 2 without unsecured credit

Let W_t, G_t, and V_t denotes the agent's value function in the CM, FM, and DM, respectively, in period t. There are two payment instruments for the DM transaction: fiat money (outside money) and demand deposit issued by the bank (inside money). I will allow the agents to use unsecured credit as a means of payment in the next section. An agent entering the CM with fiat money nominal balance m_t, demand deposit d_t, saving deposit s_t, and loan ℓ_t, solves the following problem:

$$\begin{aligned} W_t(m_t, d_t, s_t, \ell_t) &= \max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta G_{t+1}(\hat{m}_{t+1}) \\ \text{s.t. } \phi_t\hat{m}_{t+1} + X_t &= H_t + T_t + \phi_t m_t + (1+i_{d,t})\phi_t d_t + (1+i_{s,t})\phi_t s_t - (1+i_{l,t})\phi_t \ell_t, \end{aligned} \tag{33}$$

where T_t is the lump-sum transfer (or tax if it is negative), $i_{d,t}$ is the demand deposit interest rate, $i_{s,t}$ is the saving deposit interest rate, $i_{l,t}$ is the loan interest rate, ϕ_t is the price of money in terms of the CM goods, and $\hat{m}_{t+1}$ is the money balance carried to the FM where banks take deposits and makes loans. The first-order conditions (FOCs) result in $X_t = X^*$ and

$$\phi_t = \beta G'_{t+1}(\hat{m}_{t+1}), \tag{34}$$

where $G'_{t+1}(\hat{m}_{t+1})$ is the marginal value of an additional unit of money taken into the FM of period $t+1$. The envelope conditions are

$$\frac{\partial W_t}{\partial m_t} = \phi_t, \quad \frac{\partial W_t}{\partial d_t} = \phi_t(1+i_{d,t}), \quad \frac{\partial W_t}{\partial s_t} = \phi_t(1+i_{s,t}), \quad \frac{\partial W_t}{\partial \ell_t} = -\phi_t(1+i_{l,t}),$$

implying W_t is linear in m_t, d_t, s_t, and ℓ_t.

The value function of an agent at the beginning of FM is

$$G_t(m) = \sigma G_{b,t}(m) + (1-\sigma) G_{s,t}(m), \tag{35}$$

where $G_{j,t}$ is the value function of type j agent in the FM. Agents choose their money holding m_j, demand deposit balance d_j, saving deposit balance s_j, and loan ℓ_j based on the realization of their types in the following DM. The value function $G_{j,t}$ can be written as

$$\begin{aligned} G_{j,t}(m_t) = & \max_{m_{j,t}, d_{j,t}, s_{j,t}, \ell_{j,t}} V_{j,t}(m_{j,t}, d_{j,t}, s_{j,t}, \ell_{j,t}) \\ & \text{subject to} \quad m_{j,t} + d_{j,t} + s_{j,t} \leq m_t + \ell_{j,t} \end{aligned} \tag{36}$$

where $V_{j,t}$ is the value function of type j agent in the DM. The FOCs are

$$\frac{\partial V_{j,t}}{\partial m_{j,t}} - \lambda_{j,m} \leq 0, \quad \frac{\partial V_{j,t}}{\partial d_{j,t}} - \lambda_{j,m} \leq 0, \quad \frac{\partial V_{j,t}}{\partial s_{j,t}} - \lambda_{j,m} \leq 0, \quad \frac{\partial V_{j,t}}{\partial \ell_{j,t}} + \lambda_{j,m} \leq 0 \tag{37}$$

where $\lambda_{j,m}$ is the Lagrange multiplier for $m_{j,t} + d_{j,t} + s_{j,t} \leq m + \ell_{j,t}$.

With probability α, a buyer meets a seller in the DM while a seller meets a buyer with probability α_s. Since the CM value function is linear, the DM value function for the seller can be written as

$$V_{s,t}(m_{s,t}, d_{s,t}, s_{s,t}, \ell_{s,t}) = \alpha_s[p_t - c(q_t)] + W_t(m_{s,t}, d_{s,t}, s_{s,t}, \ell_{s,t}). \tag{38}$$

Differentiating $V_{s,t}$ yields

$$\frac{\partial V_{s,t}}{\partial m_{s,t}} = \phi_t, \quad \frac{\partial V_{s,t}}{\partial d_{s,t}} = \phi_t(1 + i_{d,t}), \quad \frac{\partial V_{s,t}}{\partial s_{s,t}} = \phi_t(1 + i_{s,t}), \quad \frac{\partial V_{s,t}}{\partial \ell_t} = -\phi_t(1 + i_{l,t}).$$

Combining the seller's FOCs in the FM and the derivatives of $V_{s,t}$ yields

$$\phi_t - \lambda_{m,s} \leq 0, \text{“} = \text{”} 0 \text{ iff } m_{s,t} > 0 \tag{39}$$

$$\phi_t(1 + i_{d,t}) - \lambda_{m,s} \leq 0, \text{“} = \text{”} 0 \text{ iff } d_{s,t} > 0. \tag{40}$$

$$\phi_t(1 + i_{s,t}) - \lambda_{m,s} \leq 0, \text{“} = \text{”} 0 \text{ iff } s_{s,t} > 0 \tag{41}$$

$$-\phi_t(1 + i_{l,t}) + \lambda_{m,s} \leq 0, \text{“} = \text{”} 0 \text{ iff } \ell_{s,t} > 0. \tag{42}$$

Since the seller does not need to carry liquidity to the DM, the DM terms of trade (p_t, q_t) is independent of $m_{s,t}$, $d_{s,t}$ and $s_{s,t}$. Then the seller's choices on m_t, d_t and s_t only depend on their returns. For i_d, $i_s > 0$, the budget constraint is binding, $\lambda_{s,m} > 0$, and the seller does not hold $m_{s,t}$. If $i_s > i_d$, the seller deposits all the money balance to saving deposit $s_{s,t} = m_t$, and if $i_s = i_d$ the seller is indifferent between depositing to demand deposit and saving deposit. Holding saving deposit is weakly preferred to holding demand deposit in this case. The seller does not have a strict incentive to borrow loans from the bank.

A buyer's DM value function is

$$V_{b,t}(m_{b,t}, d_{b,t}, s_{b,t}, \ell_{b,t}) = \alpha[u(q_t) - p_t] + W(m_{b,t}, d_{b,t}, s_{b,t}, \ell_{b,t}), \tag{43}$$

where $p_t \leq \bar{p}_t = \phi_t \left\{m_{b,t} + (1 + i_{d,t})d_{b,t}\right\}$. Assuming interior solutions, differentiating $V_{b,t}$ yields

$$\frac{\partial V_{b,t}}{\partial m} = \phi_t[\alpha\lambda(q_t) + 1], \qquad \frac{\partial V_{b,t}}{\partial d} = \phi_t(1 + i_{d,t})[\alpha\lambda(q_t) + 1],$$
$$\frac{\partial V_{b,t}}{\partial s} = \phi_t(1 + i_{s,t}), \qquad \frac{\partial V_{b,t}}{\partial \ell} = -\phi_t(1 + i_{l,t}),$$

where $\lambda(q) = u'(q)/v'(q) - 1$ if $p^* > \bar{p}$ and $\lambda(q) = 0$ if $\bar{p} \geq p^*$. Similar to the seller's case, combining the buyer's FOCs in the FM and the first-order derivatives of $V_{b,t}$ yields

$$\phi_t[\alpha\lambda(q_t)+1]-\lambda_{m,b}\leq 0, \text{“}=\text{”}0 \text{ iff } m_{b,t}>0 \tag{44}$$

$$\phi_t(1+i_{d,t})[\alpha\lambda(q_t)+1]-\lambda_{m,b}\leq 0, \text{“}=\text{”}0 \text{ iff } d_{b,t}>0 \tag{45}$$

$$\phi_t(1+i_{s,t})-\lambda_{m,b}\leq 0, \text{“}=\text{”}0 \text{ iff } s_{b,t}>0 \tag{46}$$

$$-\phi_t(1+i_{l,t})+\lambda_{m,b}\leq 0, \text{“}=\text{”}0 \text{ iff } \ell_{b,t}>0. \tag{47}$$

When $i_d>0$, the buyer does not hold $m_{b,t}=0$ and deposits all the fiat money balance to her demand deposit account, $d_{b,t}=m_t$. When $i_d=0$, the buyer is indifferent between holding $m_{b,t}$ and $d_{b,t}$. Holding the demand deposit is weakly preferred to holding the cash. The buyer does not have a strict incentive to deposit her balance to the saving deposit. With the binding constraint $\lambda_{m,b}>0$, we have

$$1+i_{l,t}=(1+i_{d,t})[\alpha\lambda(q_t)+1]$$

which implies that the choice of $\ell_{b,t}$ equates the marginal cost and the marginal gain from borrowing loans from the bank. Since $i_{s,t}=i_{l,t}$, we have $p_t=v(q_t)=\phi_t(1+i_{d,t})d_{b,t}=\phi_t(1+i_{d,t})(m_t+\ell_{b,t})$,

Using the above results, we can rewrite the value functions in the FM as follows:

$$G_{b,t}(m_t)=\alpha[u(q_t)-p_t]+W(0,m_t+\ell_{b,t},0,\ell_{b,t}) \tag{48}$$

$$G_{s,t}(m_t)=\alpha_s[p_t-c(q_t)]+W(0,0,m_t,0) \tag{49}$$

where $q_t=v^{-1}(p_t)$ and $p_t=\min\{p^*,\phi_t(1+i_{d,t})(m_t+\ell_{b,t})\}$. Take derivative of $G_{j,t}(m_t)$ with respect to m_t to get

$$G'_{b,t}(m_t)=\phi_t(1+i_{d,t})[1+\alpha\lambda(q_t)] \tag{50}$$

$$G'_{s,t}(m_t)=\phi_t(1+i_{s,t}). \tag{51}$$

Since $G'_t(m_t)=\sigma G'_{b,t}(m_t)+(1-\sigma)G'_{s,t}(m_t)$, we have the following:

$$G'_t(m_t)=\phi_t\sigma(1+i_{d,t})[1+\alpha\lambda(q_t)]+\phi_t(1-\sigma)\left(1+i_{s,t}\right) \tag{52}$$

Combine (34) and (52) to get the Euler equation

$$\phi_t = \phi_{t+1}\beta\left[\sigma(1+i_{d,t+1})\left\{1+\alpha\lambda(q_{t+1})\right\}+(1-\sigma)(1+i_{s,t+1})\right] \tag{53}$$

where $v(q_{t+1}) = \phi_{t+1}(1+i_{d,t+1})(m_{t+1}+\ell_{b,t+1})$.

A representative bank makes loans ℓ_t and issues demand deposits d_t and saving deposits s_t. The depositors are paid at the nominal interest rates, $i_{d,t}$ and $i_{s,t}$, by the bank, and the borrowers need to repay their borrowing with a nominal interest rate $i_{l,t}$. The bank also decides the reserve balance r_t and its fiat money holdings $m_{k,t}$. The central bank sets the reserve requirement χ. The representative bank solves the following profit maximization problem.

$$\begin{aligned} \max_{d_t,s_t,\ell_t,m_{k,t},r_t} \quad & (1+i_{l,t})\ell_t-(1+i_{d,t})d_t-(1+i_{s,t})s_t+m_{k,t} \\ \text{subject to} \quad & m_{k,t}+\ell_t=d_t+s_t, \quad \chi d\leq r, \quad r_t\leq m_{k,t} \end{aligned} \tag{54}$$

The first constraint is a balance sheet identity and the second constraint is the reserve requirement constraint. The last constraint says the bank's reserve cannot exceed its fiat money holdings. The FOCs for the bank's problem are

$$(1+i_{l,t})-\lambda^b\leq 0, \text{“}=\text{''}0 \text{ iff } \ell_t>0 \tag{55}$$

$$-(1+i_{s,t})+\lambda^b\leq 0, \text{“}=\text{''}0 \text{ iff } s_t>0. \tag{56}$$

$$-(1+i_{d,t})+\lambda^b-\chi\lambda^r\leq 0, \text{“}=\text{''}0 \text{ iff } d_t>0 \tag{57}$$

$$-\lambda^b+\lambda^r-\lambda^m\leq 0, \text{“}=\text{''}0 \text{ iff } r_t>0 \tag{58}$$

$$1+\lambda^m\leq 0, \text{“}=\text{''}0 \text{ iff } m_{k,t}>0 \tag{59}$$

where λ^b, λ^r, and λ^m denotes the Lagrange multiplier of the first constraint, the second constraint, and the last constraint in (54), respectively. Assuming interiority, we have

$$i_{l,t}=i_{s,t} \tag{60}$$

$$i_{d,t}=i_{s,t}(1-\chi) \tag{61}$$

The next step is to characterize the equilibrium. Assuming interior solutions, combining buyers' FOCs (44)-(47), sellers' FOCs (39)-(42) and bank's FOCs (60)-(61)

gives

$$\phi_t(1+i_{s,t}) = \phi_t(1+i_{d,t})\{1+\alpha\lambda(q_t)\} \tag{62}$$

$$\phi_t(1+i_{d,t}) = \phi_t\frac{\chi}{\chi+(\chi-1)\alpha\lambda(q_t)} \tag{63}$$

and when $i_{d,t} = i_{s,t} = 0$, $q_t = q^*$. Combine equations (53), (62), and (63), and use equilibrium condition $m_{t+1} = M_{t+1}$ to get

$$\phi_t = \beta\phi_{t+1}\frac{\chi\{1+\alpha\lambda(q_{t+1})\}}{\chi+(\chi-1)\alpha\lambda(q_{t+1})}. \tag{64}$$

Given $i_d \geq 0$, any equilibrium $\{\phi_t\}_{t=1}^{+\infty}$ satisfying (64) must satisfy either (1) $\lambda(q) < \chi/\{\alpha(1-\chi)\}$ and $\phi_t M_t > 0$ or (2) $q_t = \phi_t = 0$.

Since $i_{d,t+1} = \frac{\chi}{\chi+(\chi-1)\alpha\lambda(q_{t+1})} - 1$, we can rewrite (64) as

$$\phi_t = \beta\phi_{t+1}(1+i_{d,t+1})\{1+\alpha L(z_{t+1})\} \tag{65}$$

where $L(\cdot) \equiv \lambda \circ v^{-1}(\cdot)$. Define $z_t \equiv \phi_t\{m_t + (1+i_{d,t})d_t\}$ and multiplying both sides of (65) by M_t/χ allows us to reduce the equilibrium condition to one difference equation of real balances z:

$$\frac{\chi+(\chi-1)\alpha L(z_t)}{\chi}z_t = \frac{z_{t+1}}{1+i}\{1+\alpha L(z_{t+1})\} \tag{66}$$

where $i \equiv \gamma/\beta - 1$.[6]. Let $x_t \equiv \frac{z_{t+1}}{1+i}\{1+\alpha L(z_{t+1})\}$ implying $\frac{\chi+(\chi-1)\alpha L(z_t)}{\chi}z_t = x_t$. Since

$$\frac{dz_t}{dx_t} = \frac{\chi}{\chi+(\chi-1)\alpha L(z_t)+(\chi-1)\alpha L'(z_t)z_t} > 0,$$

solving (66) yields unique z_t given z_{t+1}. Then we can define z_t as a function of z_{t+1}

$$z_t \equiv g(z_{t+1}) \tag{67}$$

Consider a stationary equilibrium, which is a fixed point that satisfies $z = g(z)$. There always exists an non-monetary equilibrium with $z = 0$. Given $i \in [0, \bar{\iota})$ and $\chi \in (0,1)$, where $\bar{\iota} = \alpha\lambda(\hat{q})/\{\alpha\lambda(\hat{q})(\chi-1)+\chi\}$, an unique stationary monetary

[6] 2.2 In the stationary equilibrium, $i = \gamma/\beta - 1$ is the nominal interest rate.

equilibrium exists and satisfies

$$i\chi = \{1 + i(1-\chi)\}\, \alpha L(z_s). \tag{68}$$

Nash and Kalai bargaining provides simple examples for $\bar{\iota}$. Under the Inada condition $u'(0) = \infty$, with Nash bargaining, $\bar{\iota} = \infty$ while with Kalai, $\bar{\iota} = \infty$ when $\frac{\chi}{\alpha(1-\chi)} \leq \frac{\theta}{1-\theta}$ and $\bar{\iota} = \frac{\alpha\theta}{\alpha\theta(\chi-1)+\chi(1-\theta)}$ otherwise.

Since $L'(\cdot) < 0$ (see Gu and Wright, 2016), the following result holds:

Proposition 7. *In the stationary equilibrium, lowering i or lowering χ increases q.*

The dynamics of monetary equilibrium is characterized by equation (21). We can derive the condition that the economy exhibits a two-period cycle that satisfy $z_1 < z_s \leq p^* < z_2$.

Proposition 8 (**Two-period Monetary Cycle**)**.** *There exists a two-period cycle with $z_1 < z_s \leq p^* < z_2$ that solves*

$$\alpha L(z_1) = \frac{i(2+i)\chi}{(1-\chi)(1+i)^2 + \chi}, \qquad \textit{and} \qquad z_2 = \frac{z_1}{1+i}\{1 + \alpha L(z_1)\},$$

if $\chi \in (0, \chi_m)$, where $\chi_m \equiv \frac{(1+i)^2}{i(2+i)}$.

It is straightforward to show that a decrease in interest rate lowers the threshold.

$$\frac{\partial \hat{\chi}}{\partial i} = -\frac{2(1+i)}{i^2(2+i)^2} < 0$$

Proposition 8 shows that lowering the reserve requirement can induce a two-period cycle. We can also check whether it also increases the volatility. Consider the difference in proportions between peak and trough $z_2/z_1 = \frac{\{1+\alpha L(z_1)\}}{1+i}$. Since $\frac{\partial(z_2/z_1)}{\partial\chi} = \frac{\partial z_1}{\partial\chi}\frac{\alpha L'(z_1)}{1+i} < 0$, lowering reserve requirement increases difference in proportions. Given this result, it is straightforward to show that lowering the reserve requirement increases the difference in level as well i.e., $\frac{\partial(z_2-z_1)}{\partial\chi} < 0$. Therefore we can conclude that lowering the reserve requirement increases the volatility.

Proposition 8 shows that lowering the reserve requirement can induce a two-period cycle under the general trading mechanism. However, in general, a two period cycle with $z_1 < z_s < z_2$, could be either $z_2 > p^*$ or $z_2 < p^*$. Following the standard textbook method (see Azariadis, 1993), we can show that if $f'(z_s) < -1$, there exists a two-period cycle in the neighborhood of z_s which includes $z_2 < p^*$ case. Consider a special case where $-qu''(q)/u'(q) = \eta$, $c(q) = q$ and the buyer makes take-it-or-leave-it (TIOLI) offer. The following proposition provides a closed-form expression of $g'(z_s)$.

Proposition 9. *Assume $-qu''(q)/u'(q) = \eta$ and $c(q) = q$. Then, we have*

$$g'(z_s) = \frac{\chi\left[1-\alpha+\alpha(1-\eta)\left(1+\frac{i\chi}{\alpha[1+i(1-\chi)]}\right)\right]}{(1+i)\left\{1+(\chi-1)\left[1-\alpha+\alpha(1-\eta)\left(1+\frac{i\chi}{\alpha[1+i(1-\chi)]}\right)\right]\right\}}$$

There exists a two-period cycle when either $f'(z_s) < -1$ or $\chi < \chi_m$. To interpret the results, consider equation (**??**). The first term, $z_{t+1}/(1+i)$ on the right-hand side, reflects the store of value, which is monotonically increasing in z_{t+1}. The second term, $\chi\{1+\alpha L(\cdot)\}/[\chi+(\chi-1)\alpha L(\cdot)]$, reflecting the liquidity premium, is decreasing in q_{t+1} but non-monotone in z_{t+1}. Because $f'(z_{t+1})$ depends on both terms, $f(z_{t+1})$ is non-monotone in general. If the liquidity premium dominates the storage value at $z = z_s$, we can have $f'(z_s) < -1$, which is a standard condition for the existence of cyclic equilibria. Similarly, when $\chi < \chi_m$, the liquidity premium dominates the storage value at $z = z_1$ where $z_1 < z_s$. Lowering the reserve requirement amplifies the liquidity premium because it allows the bank to create more liquidity through lending. This amplification of liquidity generates endogenous cycles.

It is worth noting that the 100% reserve requirement does not necessarily rule out endogenous cycles. Even if $\chi = 1$, the condition of endogenous cycles around z_s, $f'(z_s) < -1$, still can hold when

$$\eta > \frac{2(1+i)}{\alpha+i}.$$

This implies that the 100% reserve requirement can not rule out endogenous cycles when the agents are highly risk averse. Similarly, if $\alpha L\left(\frac{p^*}{1+i}\right) \geq (1+i)^2 - 1$, the 100% reserve requirement can not rule out endogenous cycles as well because $\chi_m \geq 1$ in this case.

Whereas the condition in Proposition 8 is written in terms of χ, this condition is not independent of i. Taking derivative with respect of i gives the following

$$\frac{\partial \chi_m}{\partial i} = -\frac{\alpha\left\{2\alpha(1+i)\left[L\left(\frac{p^*}{1+i}\right)\right]^2 + p^* i(i+2)L'\left(\frac{p^*}{1+i}\right) + 2(1+i)L\left(\frac{p^*}{1+i}\right)\right\}}{i^2(i+2)^2\left[1+\alpha L\left(\frac{p^*}{1+i}\right)\right]^2}$$

the effect of i on χ_m is, however, ambiguous in general.

In addition to the condition for two-period cycles, the next result provides the condition for three-period cycles under the general trading mechanism. The existence of three period-cycles implies cycles of all orders as well as chaotic dynamics (see Sharkovskii, 1964 and Li and Yorke, 1975).

Proposition 10 (Three-period Monetary Cycle and Chaos). *There exists a two-period cycle with $z_1 < z_s \leq p^* < z_2 < z_3$ that solves*

$$\alpha L(z_1) = \frac{i\chi(i^2+3i+3)}{i^3(1-\chi)+3i^2(1-\chi)+3i(1-\chi)+1}, \quad z_2 = \frac{z_1}{1+i}\{1+\alpha L(z_1)\}$$

and $z_3 = \frac{z_2}{1+i}$ if $\chi \in (0, \bar{\chi})$, where $\bar{\chi} = \frac{1+i(i^2+3i+3)}{i(i^2+3i+3)}$

$$\frac{\partial(z_3/z_1)}{\partial\chi} = \frac{\partial z_1}{\partial\chi}\frac{\alpha L'(z_1)}{(1+i)^2} < 0$$

The model can also generate sunspot cycles. Consider a Markov sunspot variable $S_t \in \{1, 2\}$. This sunspot variable is not related to fundamentals but may affect equilibrium. Let $\Pr(S_{t+1} = 1|S_t = 1) = \zeta_1$ and $\Pr(S_{t+1} = 2|S_t = 2) = \zeta_2$. The sunspot is realized in the FM. Let W_t^S be the CM value function in state S in period t, then W_t^S can be expressed as

$$\begin{aligned} W_t^S(m_t, d_t, s_t, \ell_t) &= \max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta\left[\zeta_s G_{t+1}^S(\hat{m}_{t+1}) + (1-\zeta_s)G_{t+1}^{-S}(\hat{m}_{t+1})\right] \\ \text{s.t. } \phi_t^S\hat{m}_{t+1} + X_t &= H_t + T_t + \phi_t^S m_t + (1+i_{d,t})\phi_t^S d_t + (1+i_{s,t})\phi_t^S s_t - (1+i_{l,t})\phi_t^S \ell_t. \end{aligned}$$

The FOC can be written as

$$-\phi_t^S + \beta\zeta_s G_{t+1}'^S(\hat{m}_{t+1}) + \beta(1-\zeta_s)G_{t+1}'^{-S}(\hat{m}_{t+1}) = 0. \tag{69}$$

Solving the FM problem results in

$$G_{t+1}'^S(m_{t+1}^S) = \phi_{t+1}^S\left[\sigma(1+i_{d,t+1}^S)\{1+\alpha\lambda(q_{t+1}^S)\} + (1-\sigma)(1+i_{s,t+1}^S)\right]. \tag{70}$$

We substitute (70) into (69) and use the money market clearing condition $m_{t+1} = M_{t+1}$

to get the Euler equation.

$$\begin{aligned}\phi_t^S =& \beta\zeta_s\phi_{t+1}^S\left\{1+i_d(z_{t+1}^S)\right\}\left\{1+\alpha L\big(z_{t+1}^S+i_d(z_{t+1}^S)z_{t+1}^S\big)\right\}\\ &+\beta(1-\zeta_s)\phi_{t+1}^{-S}\left\{1+i_d(z_{t+1}^{-S})\right\}\left\{1+\alpha L\big(z_{t+1}^{-S}+i_d(z_{t+1}^{-S})z_{t+1}^{-S}\big)\right\}\end{aligned}$$

where $z_{t+1}^S = \phi_{t+1}^S\{m_{b,t}^S + d_{b,t}^S\}$. Then multiply both sides of the Euler equation by M_t/χ to reduce the equilibrium condition into one difference equation of real balances z_{t+1}^S:

$$\begin{aligned}z_t^S &= \frac{\zeta_s z_{t+1}^S}{1+i}\left\{1+i_d(z_{t+1}^S)\right\}\left\{1+\alpha L\big(z_{t+1}^S+i_d(z_{t+1}^S)z_{t+1}^S\big)\right\}\\ &\quad+\frac{(1-\zeta_s)z_{t+1}^{-S}}{1+i}\left\{1+i_d(z_{t+1}^{-S})\right\}\left\{1+\alpha L\big(z_{t+1}^{-S}+i_d(z_{t+1}^{-S})z_{t+1}^{-S}\big)\right\}\\ &=\zeta_s f(z_{t+1}^S)+(1-\zeta_s)f(z_{t+1}^{-S}).\end{aligned}\tag{71}$$

Define a sunspot equilibrium as follows:

Definition 2 (**Proper Sunspot Equilibrium**). *A proper sunspot equilibrium consists of the sequences of real balances $\{z_t^S\}_{t=0,S=1,2}^\infty$ and probabilities (ζ_1,ζ_2), solving (71) for all t.*

Consider stationary sunspot equilibria with $z^1 < z^2$ that only depend on the state, not the time. The liquidity constraint is binding in state $S = 1$. By the standard approach (see again Azariadis, 1993 for the textbook treatment), the conditions for two-period cycles also suffices for two-state sunspot equilibrium. If $f'(z_s) < -1$, there exists $(\zeta_1,\zeta_2) \in (0,1)^2$, $\zeta_1+\zeta_2 < 1$, such that the economy has a proper sunspot equilibrium in the neighborhood of z_s. In addition to that, if $\chi < \chi_m$, there exists $(\zeta_1,\zeta_2) \in (0,1)^2$, $\zeta_1+\zeta_2 < 1$, such that the economy has a proper sunspot equilibrium satisfying $z^1 < p^* < z^2$.

Proposition 11 (**Stationary Sunspot Equilibria**). *The stationary sunspot equilibrium exists if either $\chi < \chi_m$ or $g'(z_s) < -1$.*

2.5 Model 2 with unsecured credit

Consider an alternative payment instrument in the DM - unsecured credit. The buyer can pay for DM goods using unsecured credit that will be redeemed to the seller in the following CM and she can borrow up to her debt limit, $\bar{b}_t$. For simplicity, I assume that the buyer makes a TIOLI offer to the seller in the DM, which means the buyer maximizes her surplus subject to the seller's participation constraint. The DM cost function is $c(q) = q$. Suppose the buyer has issued b_t units of unsecured debt in the previous DM. The CM value function is

$$W_t(m_t, d_t, s_t, \ell_t, -b_t) = \max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta G_{t+1}(\hat{m}_{t+1})$$
$$\text{s.t. } \phi_t \hat{m}_{t+1} + X_t = H_t + T_t + \phi_t m_t + (1 + i_{d,t})\phi_t d_t + (1 + i_{s,t})\phi_t s_t - (1 + i_{l,t})\phi_t \ell_t - b_t,$$

which is the same as before except that the agent needs to pay or collect the debt. The agent's FM problem is identical to the previous section. Then, the seller's DM value function is

$$V_t^s(0, 0, s_{s,t}, 0) = W_t(0, 0, s_{s,t}, 0, 0)$$

The buyer's DM value function is

$$V_t^b(0, d_{b,t}, 0, \ell_{b,t}) = \alpha[u(q_t) - q_t] + W_t(0, d_{b,t}, 0, \ell_{b,t}, 0),$$

where $q_t = \min\{q^*, \bar{b}_t + \phi_t(1 + i_{d,t})d_{b,t}\}$.

Similar to Lemma **??**, there exist a positive unique fixed point $\hat{\iota}_{d,t}$ satisfying

$$\hat{\iota}_{d,t} = \frac{\chi}{\chi + (\chi - 1)\alpha[u'(z_t(1 + \hat{\iota}_{d,t}) + \bar{b}_t) - 1]} - 1$$

given $\bar{b}_t$ and $z_t \in (\hat{p} - \bar{b}_t, p^* - \bar{b}_t)$. Define $\iota_d(z, \bar{b})$ as function of z and $\bar{b}$ where $\iota_d(z, \bar{b}) = \hat{\iota}_{d,t}(z)$ if $p^* > z + \bar{b} > \hat{p}$ and $i_d(z) = 0$ if $z + \bar{b} \geq p^*$.

Given $\bar{b}_t$, solving equilibrium yields

$$z_t = \frac{z_{t+1}}{1+i}\left\{1 + \iota_d(z_{t+1}, \bar{b}_{t+1})\right\}\left\{1 + \alpha[u'\big(z_{t+1} + \iota_d(z_{t+1}, \bar{b}_{t+1})z_{t+1} + \bar{b}_{t+1}\big) - 1]\right\} \tag{72}$$

Next, I am going to endogenize the debt limit. The buyer cannot commit to pay back the debt. If the buyer reneges she is captured with probability μ. The punishment for a defaulter is permanent exclusion from the DM trade but she can still produce for herself in the CM. The value of autarky is $\underline{\mathrm{W}}(0, 0, 0, 0) = [U(X^*) - X^* + T]/(1 - \beta)$. The incentive condition for voluntary repayment is

$$\underbrace{-b_t + W_t(m_t, d_t, s_t, \ell_t, 0)}_{\text{value of honoring debts}} \geq \underbrace{(1 - \mu)W_t(m_t, d_t, s_t, \ell_t, 0) + \mu\underline{\mathrm{W}}(m_t, d_t, s_t, \ell_t, 0)}_{\text{value of not honoring debts}}.$$

One can write the debt limit $\bar{b}_t$ as $b_t \leq \bar{b}_t \equiv \mu W_t(0,0,0,0) - \mu \underline{\mathrm{W}}(0,0,0,0)$. Recall the CM value function. Using the solution of FM, we can rewrite the buyer's CM value function as

$$\begin{aligned} W_t(0,0,0,0,0) = & \, U(X^*) - X^* + T_t + \beta W_{t+1}(0,0,0,0,0) \\ & + \max_{\hat{m}_{t+1}} \Big\{ -\phi_t \hat{m}_{t+1} + \beta\alpha\sigma[u(q_{t+1}) - q_{t+1}] + \beta\phi_{t+1}\hat{m}_{t+1} \Big\} \end{aligned}$$

where $q_{t+1} = \min\{q^*, z_{t+1}[1 + \iota_d(z_{t+1}, \bar{b}_{t+1})] + \bar{b}_{t+1}\}$. Substituting $W_t(0,0,0,0,0) = \bar{b}_t/\mu + \underline{\mathrm{W}}(0,0,0,0,0)$ yields

$$\begin{aligned} \frac{\bar{b}_t}{\mu} = & \frac{\beta\bar{b}_{t+1}}{\mu} - \phi_t M_{t+1} + \beta\phi_{t+1}M_{t+1} \\ & + \beta\alpha\sigma\Big[u\big(\bar{b}_{t+1} + z_{t+1}\{1 + i_d(z_{t+1})\}\big) - \bar{b}_{t+1} - z_{t+1}\{1 + i_d(z_{t+1})\}\Big] \end{aligned}$$

where $M_{t+1} = \hat{m}_{t+1}$ and $\phi_{t+1}M_{t+1} = \chi z_{t+1}$ and z_{t+1} solves (72). Rearranging terms yields

$$\bar{b}_t = \begin{cases} \beta\bar{b}_{t+1} + \chi\mu[-\gamma z_t + \beta z_{t+1}] + \beta\alpha\mu\sigma S(q_{t+1}) & \text{if } z_{t+1} + \bar{b}_{t+1} < q^* \\ \beta\bar{b}_{t+1} + \chi\mu[-\gamma z_t + \beta z_{t+1}] + \beta\alpha\mu\sigma S(q^*) & \text{if } z_{t+1} + \bar{b}_{t+1} > q^* \end{cases} \tag{73}$$

where $S(q) \equiv u(q) - q$ is the buyer's trade surplus and $q_{t+1} = \min\{q^*, z_{t+1}[1 + \iota_d(z_{t+1}) + \bar{b}_{t+1}]\}$. The equilibrium can be collapsed into a dynamic system satisfying (72)-(73).

In the stationary equilibrium, (72) becomes

$$-1 - \frac{i\chi}{\alpha[1 + (1-\chi)i]} + u'(q) \leq 0, = \text{ if } z > 0 \tag{74}$$

and (73) becomes

$$(1-\beta)\bar{b} = \begin{cases} \chi\mu[\beta - \gamma]z + \beta\alpha\mu\sigma S(z + \bar{b}) & \text{if } z + \bar{b} < q^* \\ \chi\mu[\beta - \gamma]z + \beta\alpha\mu\sigma S(q^*) & \text{if } z + \bar{b} \geq q^*, \end{cases} \tag{75}$$

where $q = \min\{q^*, z + \bar{b}\}$. The stationary equilibrium solves the above two equations, and it falls into one of the three cases: the pure money equilibrium, the pure credit equilibrium, and the money-credit equilibrium. First, if no one can capture the buyer after she reneges, $\mu = 0$, the unsecured credit is not feasible, $\bar{b} = 0$. In this case, the equilibrium will be the pure money equilibrium. Second, when $\bar{b}$ solving (75) satisfies

$u'(\bar{b}) < i\chi/\{\alpha[1+(1-\chi)i]\}$ then money is not valued, $z = 0$. We have the pure credit equilibrium in this case. Third, if the solutions of (74)-(75), $(z, \bar{b})$ are strictly positive then money and credit coexist, which is the money-credit equilibrium.

The debt limit at the stationary equilibrium, $\bar{b}$, is a fixed point satisfying $\bar{b} = \Omega(\bar{b})$ where

$$\Omega(\bar{b}) = \begin{cases} \dfrac{\mu\sigma\alpha}{\rho}[u(\tilde{q}) - \tilde{q}] + \dfrac{i\mu\chi}{\rho}(\tilde{q} - \bar{b}) & \text{if } \tilde{q} > \bar{b} \geq 0 \\ \dfrac{\mu\sigma\alpha}{\rho}[u(\bar{b}) - \bar{b}] & \text{if } q^* > \bar{b} \geq \tilde{q} \\ \dfrac{\mu\sigma\alpha}{\rho}[u(q^*) - q^*] & \text{if } \bar{b} \geq q^* \end{cases} \tag{76}$$

and $\tilde{q}$ solves $u'(\tilde{q}) = 1 + i\chi/[\alpha\{1+(1-\chi)i\}]$ and $\rho \equiv 1/\beta - 1$. The DM consumption q_s is determined by $q_s = \min\{q^*, \max\{\tilde{q}, \bar{b}\}\}$. Money and credit coexist if and only if $0 < \bar{b} < \tilde{q}$, which holds when $0 < \mu < \min\{1, \tilde{\mu}\}$, where

$$\tilde{\mu} \equiv \frac{\rho\tilde{q}}{\alpha\sigma[u(\tilde{q}) - \tilde{q}]}.$$

Consider the dynamics of equilibria where money and credit coexist. I claim the main results from Section **??** - lowering the reserve requirement can induce endogenous cycles - still hold even after unsecured credit is introduced. For compact notation, let $\iota \equiv \max\{i, r\}$ and $a_j \equiv z_j + \bar{b}_j$. The following proposition establishes the conditions for two-period cycles, three-period cycles, and chaotic dynamics.

Proposition 12 (**Monetary Cycles with Unsecured Credit**). *There exists a two-period cycle of money and credit with $a_1 < q^* < a_2$ if $\chi \in (0, \chi_c)$, where*

$$\chi_c \equiv \frac{(1+\iota)^2\alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]}{\{(1+\iota)^2 - 1\}\left\{1 + \alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]\right\}}.$$

There exists a three-period cycle of money and credit with $a_1 < q^* < a_2 < a_3$, if $\chi \in (0, \hat{\chi}_c)$, where

$$\hat{\chi}_c \equiv \frac{(1+\iota)^3\alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]}{\{(1+\iota)^3 - 1\}\left\{1 + \alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]\right\}}.$$

scatter.pdf

Figure 3: Scatter Plot for Inside Money Volatility and Required Reserve Ratio

3 Empirical Evaluation: Inside Money Volatility

In the previous sections, the theoretical results show that lowering the required reserve ratio can induce instability. To evaluate the model prediction, I examine whether the required reserve ratio is associated with the cyclical volatility of the real balance of the inside money.

Following Jaimovich and Siu (2009) and Carvalho and Gabaix (2013), I measure the cyclical volatility in quarter t as the standard deviation of a filtered log real total checkable deposit during a 41-quarter (10-year) window centered around quarter t. Total checkable deposits are from the H.6 Money Stock Measures published by the Federal Reserve Board and converted to real value using the Consumer Price Index (CPI). Seasonally adjusted series are used to smooth the seasonal fluctuation. I adopt the Hodrick-Prescott (HP) filter with a 1600 smoothing parameter as standard. To construct an annual series, quarterly observations are averaged for each year. The sample period is from 1960:I to 2018:IV so that there are annual series from 1965 to 2013. To check whether the results are sensitive to different measures of the price level, I also use the core CPI, the Personal Consumption Expenditures (PCE), and the core PCE to transform the total checkable deposit into real value.

The legal reserve requirement for the demand deposits had been 10% from April 2, 1992, to March 25, 2020. However, the Federal Reserve imposes different reserve requirements depending on the size of a bank's liability. These criteria have changed over time. For example, during 1992:Q1-2019:Q4, this changed 27 times. To consider these changes, I divide the required reserves by total checkable deposits to compute

Table 1: Effect of Required Reserve Ratio

Price level	CPI		Core CPI		PCE		Core PCE	
Dependent	OLS	CCR	OLS	CCR	OLS	CCR	OLS	CCR
variable: σ_t^{Roll}	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
χ	-0.283^{***}	-0.245^{***}	-0.267^{***}	-0.221^{***}	-0.306^{***}	-0.227^{***}	-0.307^{***}	-0.220^{***}
	(0.027)	(0.002)	(0.027)	(0.003)	(0.029)	(0.004)	(0.027)	(0.005)
`ffr`		-0.109^{***}		-0.125^{***}		-0.187^{***}		-0.207^{***}
		(0.002)		(0.003)		(0.004)		(0.004)
Constant	0.074^{***}	0.074^{***}	0.070^{***}	0.071^{***}	0.074^{***}	0.075^{***}	0.073^{***}	0.073^{***}
	(0.003)	(0.000)	(0.004)	(0.000)	(0.004)	(0.000)	(0.004)	(0.000)
Obs.	49	49	49	49	49	49	49	49
$adjR^2$	0.700	0.621	0.728	0.648	0.740	0.650	0.764	0.665
$\lambda_{trace}(r = 0)$	**9.807**	35.688	**9.120**	35.145	**9.109**	35.367	**8.593**	35.028
5% CV	15.41	29.68	15.41	29.68	15.41	29.68	15.41	29.68
$\lambda_{trace}(r = 1)$	3.324	**10.682**	2.839	**10.065**	2.723	**9.894**	2.417	**9.345**
5% CV	3.76	15.41	3.76	15.41	3.76	15.41	3.76	15.41

Note: For (1), (3), (5) and (7), OLS estimates are reported, and Newey-West standard errors with lag 1 are reported in parentheses. For (2), (4), (6), and (8), first-stage long-run variance estimations for CCR are based on the quadratic spectral kernel and Bayesian information criterion. The bandwidth selection is based on Newey-West fixed lag, $4 \times (T/100)^{2/9}$; χ denotes the required reserve ratio, `ffr` denotes federal funds rates and σ_t^{Roll} denotes the cyclical volatility of real inside money balances. ***, **, and * denotes significance at the 1, 5, and 10 percent levels, respectively.

Table 2: Unit Root Tests

		Phillips-Perron test		ADF test
		$Z(\rho)$	$Z(t)$	$Z(t)$ w/ lag 1
`ffr`		-6.766	-1.704	-2.362
χ		-1.492	-1.173	-1.341
σ_t^{Roll}	(CPI)	-4.708	-2.191	-2.090
σ_t^{Roll}	(Core CPI)	-4.681	-2.189	-1.978
σ_t^{Roll}	(PCE)	-4.329	-2.038	-2.047
σ_t^{Roll}	(Core PCE)	-4.076	-1.954	-1.930
Δ`ffr`		-28.373^{***}	-5.061^{***}	-6.357^{***}
$\Delta\chi$		-31.818^{***}	-4.802^{***}	-3.693^{***}
$\Delta\sigma_t^{Roll}$	(CPI)	-24.905^{***}	-3.416^{**}	-2.942^{**}
$\Delta\sigma_t^{Roll}$	(Core CPI)	-24.758^{***}	-3.509^{**}	-2.942^{**}
$\Delta\sigma_t^{Roll}$	(PCE)	-23.691^{***}	-3.330^{**}	-2.842^{*}
$\Delta\sigma_t^{Roll}$	(Core PCE)	-22.826^{***}	-3.296^{**}	-2.768^{*}

Note: `ffr` denotes federal funds rates, χ denotes required reserve ratio, and σ_t^{Roll} denotes cyclical volatility of real inside money balances. ***, **, and * denotes significance at the 1, 5, and 10 percent levels, respectively.

the required reserve ratio.

Figure 3 presents a scatter plot of the cyclical volatility of the real inside money balance and the required reserve ratio. Column (1) of Table 1 reports its regression estimates with Newey-West standard errors. The plot and estimates show a negative relationship between the cyclical volatility of the real inside money balance and the required reserve ratio with statistically significant regression coefficients. However, this result can be driven by a spurious regression. Table 2 provides unit root test results for the federal funds rate, the required reserve ratio, and the cyclical volatility of inside money. Both augmented Dickey-Fuller tests and Phillips-Perron tests fail to reject the null hypotheses of unit roots for these series, whereas they reject the null hypotheses of unit roots at their first differences. In addition to that, the Johansen cointegration test in Column (1), suggests that there is no cointegration relationship between two variables. So it is hard to rule out that Column (1)'s results are driven by a spurious regression.

To overcome this issue, I adopt the cointegrating regression with an additional variable, the federal funds rate. Column (2) of Table 1 provides Johansen cointegration test results for the federal funds rate, the required reserves, and the cyclical volatility of inside money. The trace test suggests a cointegration relationship among these three variables, which is consistent with the theoretical result: The instability depends on the reserve requirement and the interest rate. With the cointegration relationship, we may not have to worry about a spurious relationship. Column (2) of Table 1 reports the estimates for the cointegrating relationship. Because of the potential bias from long-run variance, I estimate a canonical cointegrating regression (CCR). The estimates are statistically significant with a sizeable level.

To check the sensitivity of the results, I redo all the analyses using the core CPI, the Personal Consumption Expenditures (PCE), and the core PCE to transform the total checkable deposit into real value. Columns (3), (5), and (7) of Table 1 regress required reserve ratio on the inside money volatility and report its Newey-West standard errors. They also report the trace test statistics of Johansen cointegration test between these two variables. The results are consistent with the benchmark case in Column (1). Columns (4), (6), and (8) of Table 1 report CCR estimates regressing the required reserve ratio and federal funds rate on the inside money volatility and the trace test statistics of Johansen cointegration test between these three variables. All the results are consistent with the benchmark case in Column (2).

Appendix B includes more sensitivity analyses: (1) Using quarterly series instead of annual series; (2) Using time series before 2008; (3) Using alternative data proposed by Lucas and Nicolini (2015). All the results are not sensitive with respect to different frequencies, time periods, and alternative data.

4 Conclusion

The goal of this paper is to examine the (in)stability of fractional reserve banking. To that end, this paper builds a simple monetary model of fractional reserve banking that can capture the conditions for (in)stability under different specifications. Lowering the reserve requirement increases the welfare at the steady state. However, it can induce instability. The baseline model and its extension establish the conditions for endogenous cycles and chaotic dynamics. The model also features stochastic cycles and self-fulfilling boom and burst under explicit conditions. The model shows that fractional reserve banking can endanger stability in the sense that equilibrium is more prone to exhibit cyclic, chaotic, and stochastic dynamics under lower reserve requirements. This is due to the amplified liquidity premium. This result holds in the extended model with unsecured credit.

This paper also provides some empirical evidence that is consistent with the prediction of the model. I test the association between the required reserves ratio and the real inside money volatility using cointegrating regression. I find a significant negative relationship between the two variables. Both theoretical and empirical evidence find a link between the reserve requirement policy and (in)stability.

References

Andolfatto, David, Aleksander Berentsen, and Fernando M Martin, "Money, Banking, and Financial Markets," *The Review of Economic Studies*, 2020, *87* (5), 2049–2086.

Azariadis, Costas, *Intertemporal Macroeconomics*, Blackwell Publishing, Cambridge, MA, 1993.

Baumol, William J and Jess Benhabib, "Chaos: Significance, Mechanism, and Economic Applications," *Journal of Economic Perspectives*, 1989, *3* (1), 77–105.

Becker, Gary, "Free Banking," Unpublished note, 1956.

Benhabib, Jess and Roger E.A. Farmer, "Indeterminacy and sunspots in macroeconomics," in J. B. Taylor and M. Woodford, eds., *Handbook of Macroeconomics*, Vol. 1 of *Handbook of Macroeconomics*, Elsevier, 1999, chapter 6, pp. 387–448.

Berentsen, Aleksander, Gabriele Camera, and Christopher Waller, "Money, Credit and Banking," *Journal of Economic Theory*, 2007, *135* (1), 171–195.

Bernanke, Ben S, Mark Gertler, and Simon Gilchrist, "The Financial Accelerator in a Quantitative Business Cycle Framework," *Handbook of Macroeconomics*, 1999, *1*, 1341–1393.

Boldrin, Michele and Michael Woodford, "Equilibrium Models Displaying Endogenous Fluctuations and Chaos: A Survey," *Journal of Monetary Economics*, 1990, *25* (2), 189–222.

Brock, William A, "Nonlinearity and Complex Dynamics in Economics and Finance," in Philip W. Anderson, Kenneth J. Arrow, and David Pines, eds., *The Economy as an Evolving Complex System*, Vol. 5, CRC Press, Boca Raton, 1988, pp. 77–97.

Carvalho, Vasco and Xavier Gabaix, "The Great Diversification and Its Undoing," *American Economic Review*, 2013, *103* (5), 1697–1727.

Chari, VV and Christopher Phelan, "On the Social Usefulness of Fractional Reserve Banking," *Journal of Monetary Economics*, 2014, *65*, 1–13.

Diamond, Douglas W, “Liquidity, Banks, and Markets,” *Journal of Political Economy*, 1997, *105* (5), 928–956.

_ **and Philip H Dybvig**, “Bank Runs, Deposit Insurance, and Liquidity,” *Journal of Political Economy*, 1983, *91* (3), 401–419.

Fisher, Irving, *100% money*, Adelphi Company, New York, 1935.

Freeman, Scott and Finn E Kydland, “Monetary Aggregates and Output,” *American Economic Review*, 2000, *90* (5), 1125–1135.

_ **and Gregory W Huffman**, “Inside Money, Output, and Causality,” *International Economic Review*, 1991, pp. 645–667.

Friedman, Milton, *A Program for Monetary Stability*, Fordham University Press, New York, 1959.

Gertler, Mark and Peter Karadi, “A Model of Unconventional Monetary Policy,” *Journal of monetary Economics*, 2011, *58* (1), 17–34.

Gu, Chao and Randall Wright, “Monetary Mechanisms,” *Journal of Economic Theory*, 2016, *163*, 644–657.

_ **, Cyril Monnet, Ed Nosal, and Randall Wright**, “On the Instability of Banking and Financial Intermediation,” Working Papers 1901, Department of Economics, University of Missouri April 2019.

_ **, Fabrizio Mattesini, and Randall Wright**, “Money and Credit Redux,” *Econometrica*, 2016, *84* (1), 1–32.

_ **,** _ **, Cyril Monnet, and Randall Wright**, “Endogenous Credit Cycles,” *Journal of Political Economy*, 2013, *121* (5), 940–965.

Jaimovich, Nir and Henry E Siu, “The Young, the Old, and the Restless: Demographics and Business Cycle Volatility,” *American Economic Review*, 2009, *99* (3), 804–826.

Kehoe, Timothy J and David K Levine, “Debt-Constrained Asset Markets,” *The Review of Economic Studies*, 1993, *60* (4), 865–888.

Kiyotaki, Nobuhiro and John Moore, "Credit cycles," *Journal of political economy*, 1997, *105* (2), 211–248.

Lagos, Ricardo and Randall Wright, "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 2005, *113* (3), 463–484.

Li, Tien-Yien and James A Yorke, "Period Three Implies Chaos," *The American Mathematical Monthly*, 1975, *82* (10), 985–992.

Lotz, Sebastien and Cathy Zhang, "Money and Credit as Means of Payment: A New Monetarist Approach," *Journal of Economic Theory*, 2016, *164*, 68–100.

Lucas, Robert E and Juan Pablo Nicolini, "On the Stability of Money Demand," *Journal of Monetary Economics*, 2015, *73*, 48–65.

Minsky, Hyman P, "Monetary Systems and Accelerator Models," *American Economic Review*, 1957, *47* (6), 860–883.

_ , "Financial Instability Revisited: The Economics of Disaster," 1970. Available from `https://fraser.stlouisfed.org/files/docs/historical/federa%20reserve%20history/discountmech/fininst_minsky.pdf`.

Mises, Ludwig Von, *The Theory of Money and Credit*, Yale University Press, 1953.

Sargent, Thomas J, "Where to Draw Lines: Stability Versus Efficiency," *Economica*, 2011, *78* (310), 197–214.

_ **and Neil Wallace**, "The Real-Bills Doctrine versus the Quantity Theory: A Reconsideration," *Journal of Political Economy*, 1982, *90* (6), 1212–1236.

Scheinkman, Jose A and Michael Woodford, "Self-Organized Criticality and Economic Fluctuations," *American Economic Review*, 1994, *84* (2), 417–421.

Sharkovskii, AN, "Cycles Coexistence of Continuous Transformation of Line in Itself," *Ukr. Math. Journal*, 1964, *26* (1), 61–71.

Appendix

Appendix A Proofs

Proof of Proposition 1. ■

Proof of Proposition 2. Let there exists a two-period cycle satisfying $z_1 < z_s < p^* < z_2$. Since $z_2 > p^*$, we have $z_2 = (1+i)z_1$. Using (21) with $z_1 < p^*$ gives

$$\chi = \frac{(1-\sigma)\alpha L(z_1)}{(1+i)^2 - 1 - \sigma\alpha L(z_1)} \tag{77}$$

This two-period cycle should satisfy $z_1 < z_s < p^*$ and $z_2 = (1+i)z_1 > p^*$. First one can be easily shown using

$$0 = L(p^*) < L(z_s) = \frac{i}{\alpha(1-\sigma+\sigma\chi)}\chi < \frac{(1+i)^2-1}{\alpha(1-\sigma+\sigma\chi)}\chi = L(z_1)$$

since we have $L'(\cdot) < 0$. Because $dz_1/d\chi < 0$, the latter one, $z_1 > p^*/(1+i)$, is held when

$$0 < \chi < \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^2 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)}.$$

■

Proof of the Existence of a Two-period Monetary Cycle where $f'(z) < -1$. Let $f^2(z) = f \circ f(z)$. With given the unique steady state, $f(z) > z$ for $z < z_s$ and $f(z) < z$ for $z > z_s$. Because $f(z)$ is linear increasing function for $z > p^*$, there exist a $\tilde{z} > p^*$ s.t $f(\tilde{z}) > p^*$. Since $\tilde{z} > p^*$ and $f(\tilde{z}) < \tilde{z}$, $\tilde{z}$ satisfies $f^2(\tilde{z}) < f(\tilde{z}) < \tilde{z}$. We can write slope of $f^2(z)$ as follows.

$$\frac{\partial f^2(z)}{\partial z} = f'[f(z)]f'(z) = f'(z)f'(z) = [f'(z)]^2$$

which implies $\partial f^2(z)/\partial z > 1$ when $f(z) < -1$. And it is easy to show $\partial f^2(0)/\partial z > 0$. With given $i > 0$ and $\chi > 0$, there exist a (z_1, z_2), satisfying $0 < z_1 < z_s < z_2$ which are fix points for $f^2(z)$ ■

Proof of Proposition 3. When DM trade is based on take-it-or-leave-it offer from buyer to seller with $c(q) = q$ and $-qu''(q)/u' = \eta$, f' can be written as

$$f'(q) = \frac{1}{1+i}\left\{\frac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u''(q)q + u'(q) - 1\right] + 1\right\} < -1$$

Using $u''(q)q = -\eta u'(q)$ gives

$$\frac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u'(q)(1-\eta) - 1\right] + 1 < -(1+i)$$

where $u'(q) = 1 + \frac{i\chi}{\alpha(1-\sigma+\sigma\chi)}$. Substituting $u'(q)$ and rearranging terms give

$$0 < \chi < \frac{\alpha\eta(1-\sigma)}{\eta(1-\alpha\sigma) + (2-\eta)(1+i)}$$

■

Proof of Proposition 4. I divide three period cycles into two cases.
Case 1: Let there exists a three-period cycle satisfying $z_1 < z_s < p^* < z_2 < z_3$. Since $z_2, z_3 > p^*$, we have $z_2 = (1+i)z_1$, $z_3 = (1+i)z_2 = (1+i)^2 z_1$. Using (21) with $z_1 < p^*$ gives

$$\chi = \frac{(1-\sigma)\alpha L(z_1)}{(1+i)^3 - 1 - \sigma\alpha L(z_1)} \tag{78}$$

This three-period cycle should satisfy $z_1 < z_s < p^*$ and $z_2 = (1+i)z_1 > p^*$. First one can be easily shown using

$$0 = L(p^*) < L(z_s) = \frac{i}{\alpha(1-\sigma+\sigma\chi)}\chi < \frac{(1+i)^3 - 1}{\alpha(1-\sigma+\sigma\chi)}\chi = L(z_1)$$

since we have $L'(\cdot) < 0$. Because $dz_1/d\chi < 0$, the latter one, $z_1 > p^*/(1+i)$, is held when

$$0 < \chi < \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^3 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)}.$$

Case 2: Let there exists a three-period cycle satisfying $z_1 < z_2 < p^* \leq z_3$. Since $z_3 > p^*$, we have $z_3 = z_2(1+i)$ and (z_2, z_1) solves (79)-(80).

$$z_1 = f(z_2) = \left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_2) + 1\right]\frac{z_2}{1+i} \tag{79}$$

$$z_2 \equiv \tilde{f}(z_1) = \left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_1) + 1\right]\frac{z_1}{(1+i)^2}. \tag{80}$$

These functions satisfies $f(x) > x$ for $x < z_s$, $f(x) < x$ for $x > z_s$, $\tilde{f}(x) > x$ for $x < \tilde{z}$

2c_figures_2/example1.pdf

2c_figures_2/example2.pdf

Figure 4: Intersection of $\tilde{f}(z)$ and $f(z)$

and $\tilde{f}(x) < x$ for $x > \tilde{z}$ where $\tilde{z}$ solves $\tilde{z} = \tilde{f}(\tilde{z})$. One can easily show $\tilde{z} < z_s$. Therefore any intersection between $z_1 = f(z_2)$ and $z_2 = \tilde{f}(z_1)$ satisfies $z_1 > z_2$ which contradicts to our initial conjecture $z_1 < z_2$. This implies there is no three-period cycles satisfying $z_1 < z_2 < p^* \leq z_3$. Therefore we can conclude that a three-period cycle exist when

$$0 < \chi < \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^3 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)}.$$

The existence of a three-cycle implies the existence of cycles of all orders and chaotic dynamics by the Sarkovskii theorem (Sharkovskii, 1964) and the Li-Yorke theorem (Li and Yorke, 1975). ■

Proof of Corollary 1: Proposition **??** shows that at least one periodic point satisfies $z_t < z_s < p^*$ in 3- period cycles. Two period cycles satisfies $z_1 < z_s < z_2$ also implies at least one periodic point satisfies $z_t < z_s < p^*$ in 2-period cycles since $z_1 < z_s < p^*$. This result holds for any n-periodic cycles. Let $z_1 < z_2 < ...z_n$ be the periodic points of a n-cycle. Suppose $z_j > z_s$ for all $j = 1, 2, ..n$. By the definition of a n-period cycle, $z_1 = f(z_n) < z_n$ since $f(z) < z$ for $z > z_s$.

$$z_n = f(z_{n-1}) < z_{n-1} = f(z_{n-2}) < z_{n-2}... < z_1.$$

which shows the contradiction implying at least one periodic point satisfies $z_t < z_s < p^*$. ■

Proof of the Existence of the Proper Sunspot Equilibrium. Since $f'(z_s) < 0$, there is an interval $[z_s - \epsilon_1, z_s + \epsilon_2]$, with $\epsilon_1, \epsilon_2 > 0$, such that $f(z_1) > f(z_2)$ for $z_1 \in [z_s - \epsilon_1, z_s)$, $z_2 \in (z_s, z_s + \epsilon_2]$. By definition (z_1, z_2) is a proper sunspot equilibrium if there exists (ζ_1, ζ_2), with $\zeta_1, \zeta_2 < 1$, such that

$$z_1 = \zeta_1 f(z_1) + (1 - \zeta_1) f(z_2) \tag{81}$$

$$z_2 = (1 - \zeta_2) f(z_1) + \zeta_2 f(z_2). \tag{82}$$

One can rewrite (81) and (82) as

$$\zeta_1 + \zeta_2 = \frac{z_1 - f(z_2) - z_2 + f(z_1)}{f(z_1) - f(z_2)} = \frac{z_1 - z_2}{f(z_1) - f(z_2)} + 1 < 1,$$

since $f(z_1) - f(z_2) > 0$ and $z_1 - z_2 < 0$. Therefore $\zeta_1 + \zeta_2 < 1$.

Because z_1 and z_2 are weighted averages of $f(z_1)$ and $f(z_2)$, where $f(z_1) > z_1$ and $f(z_2) < z_2$, by the uniqueness of the positive steady state, necessary and sufficient conditions for (81) and (82) are

$$f(z_2) < z_1 < f(z_1) \quad \text{and} \quad f(z_2) < z_2 < f(z_1)$$

We can reduce this to

$$z_2 < f(z_1) \quad \text{and} \quad z_1 > f(z_2).$$

because $z_1 < z_2$. The above inequalities can be written as

$$\frac{z_2 - z_s}{z_s - z_1} < -f'(z_s) < \frac{z_s - z_1}{z_2 - z_s}$$

If $-f'(z_s) < \frac{z_s - z_1}{z_2 - z_s}$ holds, we have $\frac{z_2 - z_s}{z_s - z_1} < -f'(z_s)$ since $-f'(z_s) > 1$. Therefore, any solution (z_1, z_2) on $[z_s - \epsilon_1, z_s + \epsilon_2]$ satisfies $-f'(z_s) < \frac{z_s - z_1}{z_2 - z_s}$ can be a proper sunspot and it is straightforward that multiple solutions exist.

■

Proof of Proposition 6. A two period cycle result is presented and three-period case will follow. Let there exists a two-period cycle satisfying $w_1 < q^* < w_2$ where $w_j = z_j + \bar{b}_j$. Since $w_2 > q^*$, we have $z_2 = (1 + i) z_1$ and $\bar{b}_2 = (1 + r) \bar{b}_1$ where q_1, $\bar{b}_1$, and z_1

solve

$$u'(q_1) = 1 + \chi \frac{(1+i)^2 - 1}{\alpha(1-\sigma+\sigma\chi)}$$
$$\bar{b}_1 = [(1+r)^2 - 1]^{-1} \left\{ \frac{i\mu\sigma\chi}{1-\sigma+\sigma\chi} \left[1 - \frac{(1+i)^2}{\beta} \right] z_1 + \mu\alpha\sigma[u(q_1) - q_1] \right\}$$

and $z_1 = q_1 - \bar{b}_1$. This two-period cycle should satisfy $q_1 < q^*$ and $w_2 = (1+i)z_1 + (1+r)\bar{b}_1 > q^*$. For given $i > 0$ and $\chi > 0$, first one can be easily shown using

$$1 = u'(q^*) < u'(q_s) = 1 + \frac{i}{\alpha(1-\sigma+\sigma\chi)}\chi < 1 + \frac{(1+i)^2 - 1}{\alpha(1-\sigma+\sigma\chi)}\chi = u'(q_1)$$

since we have $u''(\cdot) < 0$. Now we also can check the latter using the below conditions

$$\begin{aligned} (1+r)q_1 >& (1+i)z_1 + (1+r)\bar{b}_1 = w_2 > q^* > q_1 = z_1 + \bar{b}_1 \quad \text{if } r > i \\ (1+i)q_1 >& (1+i)z_1 + (1+r)\bar{b}_1 = w_2 > q^* > q_1 = z_1 + \bar{b}_1 \quad \text{if } i > r. \end{aligned}$$

The sufficient conditions to have $w_2 > q^*$ is $q_1 > q^*/(1+r)$ for $r > i$ and $q_1 > q^*/(1+i)$ for $i > r$. Since we have $dq_1/d\chi < 0$, there exist a three period cycle $q_1 = w_1 < q_s < q^* < w_2 < w_3$ when

$$0 < \chi < \frac{(1-\sigma)\alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]}{(1+i)^2 - 1 - \sigma\alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]}$$

where $\iota = \max\{i, r\}$. Now, let there exists a three-period cycle satisfying $q_1 = w_1 < q_s < q^* < w_2 < w_3$ where $w_j = z_j + \bar{b}_j$. Since w_3, $w_2 > q^*$, we have $z_2 = (1+i)z_1$, $z_3 = (1+i)^2 z_1$, $\bar{b}_2 = (1+r)\bar{b}_1$ and $\bar{b}_3 = (1+r)^2\bar{b}_1$ where q_1, $\bar{b}_1$, and z_1 solve

$$u'(q_1) = 1 + \chi \frac{(1+i)^3 - 1}{\alpha(1-\sigma+\sigma\chi)}$$
$$\bar{b}_1 = [(1+r)^3 - 1]^{-1} \left\{ \frac{i\mu\sigma\chi}{1-\sigma+\sigma\chi} \left[1 - \frac{(1+i)^2}{\beta} \right] z_1 + \mu\alpha\sigma[u(q_1) - q_1] \right\}$$

and $z_1 = q_1 - \bar{b}_1$. This three-period cycle should satisfy $q_1 < q_s < q^*$ and $w_2 = (1+i)z_1 + (1+r)\bar{b}_1 > q^*$. For given $i > 0$ and $\chi > 0$, first one can be easily shown using

$$1 = u'(q^*) < u'(q_s) = 1 + \frac{i}{\alpha(1-\sigma+\sigma\chi)}\chi < 1 + \frac{(1+i)^3 - 1}{\alpha(1-\sigma+\sigma\chi)}\chi = u'(q_1)$$

since we have $u''(\cdot) < 0$. Now we also can check the latter using below conditions

$$\begin{aligned}(1+r)q_1 >&(1+i)z_1 + (1+r)\bar{b}_1 = w_2 > q^* > q_1 = z_1 + \bar{b}_1 \quad \text{ if } r > i\\ (1+i)q_1 >&(1+i)z_1 + (1+r)\bar{b}_1 = w_2 > q^* > q_1 = z_1 + \bar{b}_1 \quad \text{ if } i > r.\end{aligned}$$

The sufficient conditions to have $w_2 > q^*$ is $q_1 > q^*/(1+r)$ for $r > i$ and $q_1 > q^*/(1+i)$ for $i > r$. Since we have $dq_1/d\chi < 0$, there exist a three period cycle $q_1 = w_1 < q_s < q^* < w_2 < w_3$ when

$$0 < \chi < \frac{(1-\sigma)\alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]}{(1+i)^3 - 1 - \sigma\alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]}$$

where $\iota = \max\{i, r\}$. Again, the existence of a three-cycle implies the existence of cycles of all orders and chaotic dynamics by the Sarkovskii theorem and the Li-Yorke theorem. ■

Proof of Proposition 7. ■

Proof of Proposition 8. ■

Proof of Proposition 9. Recall (21)

$$\begin{aligned}f(z) =&\frac{z}{1+i}\{1 + i_d(z)\}\Big\{1 + \alpha L\big(z + i_d(z)z\big)\Big\}\\ =&\frac{q}{1+i}\left\{1 + \alpha\lambda\big(q\big)\right\}\end{aligned}$$

and take a derivative with respect to z

$$f'(z) = \frac{\partial q}{\partial z}\left[1 + \alpha\lambda(q) + q\alpha\lambda'(q)\right]\frac{1}{1+i}.$$

Since $-qu''(q)/u'(q) = \eta$ and $c(q) = q$, we have

$$f'(z) = \left[\frac{1 - \alpha + \alpha u'(q)(1-\eta)}{1+i}\right]\frac{\partial q}{\partial z}.$$

We want to get $f'(z)|_{z=z_s}$ which is

$$f'(z_s) = \left[\frac{1-\alpha+\alpha u'(q_s)(1-\eta)}{1+i}\right] \times \frac{\partial q}{\partial z}\Big|_{z=z_s}.$$

Using $-qu''(q)/u'(q) = \eta$ and $c(q) = q$, rewrite equation (68) as

$$u'(q_s) = 1 + \frac{i\chi}{\alpha\left\{1+i(1-\chi)\right\}}. \tag{83}$$

Then we have

$$f'(z_s) = \left[\frac{1-\alpha+\alpha(1-\eta)\left\{1+\frac{i\chi}{\alpha\{1+i(1-\chi)\}}\right\}}{1+i}\right] \times \frac{\partial q}{\partial z}\Big|_{z=z_s}. \tag{84}$$

Recall

$$q = (1+i_d)z = \frac{z\chi}{\chi+(\chi-1)\alpha[u'(q)-1]}$$

and applying the implicit function theorem gives

$$\frac{\partial q}{\partial z}\Big|_{z=z_s} = \frac{\chi}{\chi+(1-\chi)\alpha+(\chi-1)\alpha(1-\eta)u'(q_s)}. \tag{85}$$

Combining (84) and (85) gives

$$f'(z_s) = \left[\frac{1-\alpha+\alpha(1-\eta)u'(q_s)}{1+i}\right]\frac{\chi}{1+(\chi-1)[1-\alpha+\alpha(1-\eta)u'(q_s)]}. \tag{86}$$

Lastly, substitute (83) to (86) and collecting terms yields

$$f'(z_s) = \frac{\chi\left[1-\alpha+\alpha(1-\eta)\left(1+\frac{i\chi}{\alpha[1+i(1-\chi)]}\right)\right]}{(1+i)\left\{1+(\chi-1)\left[1-\alpha+\alpha(1-\eta)\left(1+\frac{i\chi}{\alpha[1+i(1-\chi)]}\right)\right]\right\}}.$$

■

Proof of Proposition 10. ■

Proof of Corollary ??. ■

Proof of Proposition 12. ■

Appendix B Empirical Appendix

This section provides robustness checks for empirical results. To check the sensitivity of the results, Table 3 and 4 repeat all the empirical analysis, reported in Table 1 and 2, using quarterly series instead of annual data. This section also provides robustness checks using time-series before 2008. Table 5 and 6 repeat the analysis using time-series before 2008. All the results are similar to the benchmark analysis shown in Table 1 and 2.

Table 3: Effect of Required Reserve Ratio: Robustness Check (Quarterly)

Price level	CPI		Core CPI		PCE		Core PCE	
Dependent variable: σ_t^{Roll}	OLS (1)	CCR (2)	OLS (3)	CCR (4)	OLS (5)	CCR (6)	OLS (7)	CCR (8)
χ	-0.282^{***}	-0.452^{***}	-0.266^{***}	-0.400^{***}	-0.305^{***}	-0.485^{***}	-0.306^{***}	-0.476^{***}
	(0.016)	(0.001)	(0.014)	(0.003)	(0.015)	(0.000)	(0.014)	(0.006)
`ffr`		-0.050^{***}		-0.058^{***}		-0.015^{***}		-0.047^{***}
		(0.000)		(0.002)		(0.000)		(0.005)
Constant	0.074^{***}	0.085^{***}	0.070^{***}	0.079^{***}	0.074^{***}	0.089^{***}	0.073^{***}	0.086^{***}
	(0.002)	(0.000)	(0.002)	(0.000)	(0.002)	(0.000)	(0.002)	(0.001)
Obs.	196	196	196	196	196	196	196	196
$adjR^2$	0.696	0.240	0.725	0.263	0.737	0.222	0.761	0.268
$\lambda_{trace}(r=0)$	**9.496**	31.950	**11.045**	33.808	**10.930**	34.481	**12.103**	35.951
5% CV	15.41	29.68	15.41	29.68	15.41	29.68	15.41	29.68
$\lambda_{trace}(r=1)$	1.677	**11.162**	1.959	**12.266**	1.938	**12.094**	1.887	**12.485**
5% CV	3.76	15.41	3.76	15.41	3.76	15.41	3.76	15.41

Note: For (1), OLS estimates are reported, and Newey-West standard errors with lag 1 are reported in parentheses. For (2) and (3), first-stage long-run variance estimations for CCR are based on the quadratic spectral kernel and Bayesian information criterion. The bandwidth selection is based on Newey-West fixed lag, $4 \times (T/100)^{2/9}$; `ffr` denotes federal funds rates and σ_t^{Roll} denotes cyclical volatility of real inside money balances. ***, **, and * denote significance at the 1, 5, and 10 percent levels, respectively.

Table 4: Unit Root Tests: Robustness Check (Quarterly)

		Phillips-Perron test		ADF test
		$Z(\rho)$	$Z(t)$	$Z(t)$ w/ lag 1
`ffr`		−8.611	−1.956	−2.183
χ		−1.335	−1.145	−1.199
σ_t^{Roll}	(CPI)	−4.320	−2.062	−1.554
σ_t^{Roll}	(Core CPI)	−4.388	−2.201	−1.924
σ_t^{Roll}	(PCE)	−3.822	−1.946	−1.868
σ_t^{Roll}	(Core PCE)	−3.565	−1.928	−2.023
Δ`ffr`		−139.701***	−10.792***	−10.288***
$\Delta\chi$		−163.796***	−12.272***	−9.909***
$\Delta\sigma_t^{Roll}$	(CPI)	−23.132***	−2.604*	−3.576***
$\Delta\sigma_t^{Roll}$	(Core CPI)	−30.423***	−3.544***	−4.894***
$\Delta\sigma_t^{Roll}$	(PCE)	−24.507***	−2.874*	−4.362***
$\Delta\sigma_t^{Roll}$	(Core PCE)	−28.054***	−3.373**	−5.138***

Note: `ffr` denotes federal funds rates, χ denotes required reserve ratio, and σ_t^{Roll} denotes cyclical volatility of real inside money balances. ***, **, and * denote significance at the 1, 5, and 10 percent levels, respectively.

Table 5: Effect of Required Reserve Ratio: Robustness Check (pre-2008)

Price level	CPI		Core CPI		PCE		Core PCE	
Dependent	OLS	CCR	OLS	CCR	OLS	CCR	OLS	CCR
variable: σ_t^{Roll}	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
χ	−0.266***	−0.297***	−0.266***	−0.268***	−0.307***	−0.288***	−0.305***	−0.277***
	(0.030)	(0.001)	(0.030)	(0.001)	(0.032)	(0.002)	(0.029)	(0.002)
`ffr`		−0.107***		−0.124***		−0.189***		−0.210***
		(0.001)		(0.001)		(0.002)		(0.002)
Constant	0.070***	0.080***	0.070***	0.076***	0.074***	0.082***	0.072***	0.080***
	(0.004)	(0.000)	(0.004)	(0.000)	(0.004)	(0.000)	(0.004)	(0.002)
Obs.	43	43	43	43	43	43	43	43
$adj R^2$	0.727	0.659	0.727	0.710	0.739	0.708	0.759	0.734
$\lambda_{trace}(r=0)$	**8.373**	32.228	**7.438**	31.299	**7.661**	31.867	**6.897**	31.250
5% CV	15.41	29.68	15.41	29.68	15.41	29.68	15.41	29.68
$\lambda_{trace}(r=1)$	1.504	**9.554**	1.125	**8.428**	1.146	**8.603**	0.938	**7.693**
5% CV	3.76	15.41	3.76	15.41	3.76	15.41	3.76	15.41

Note: For (1), OLS estimates are reported, and Newey-West standard errors with lag 1 are reported in parentheses. For (2) and (3), first-stage long-run variance estimations for CCR are based on the quadratic spectral kernel and Bayesian information criterion. The bandwidth selection is based on Newey-West fixed lag, $4 \times (T/100)^{2/9}$; `ffr` denotes federal funds rates and σ_t^{Roll} denotes cyclical volatility of real inside money balances. ***, **, and * denote significance at the 1, 5, and 10 percent levels, respectively.

Table 6: Unit Root Tests: Robustness Check (pre-2008)

		Phillips-Perron test		ADF test
		$Z(\rho)$	$Z(t)$	$Z(t)$ w/ lag 1
`ffr`		-9.476	-2.258	-2.868^{**}
χ		-0.768	-0.660	-0.877
σ_t^{Roll}	(CPI)	-2.966	-1.738	-1.770
σ_t^{Roll}	(Core CPI)	-2.860	-1.641	-1.495
σ_t^{Roll}	(PCE)	-2.662	-1.515	-1.627
σ_t^{Roll}	(Core PCE)	-2.412	-1.371	-1.400
Δ`ffr`		-25.378^{***}	-4.773^{***}	-5.833^{***}
$\Delta\chi$		-28.208^{***}	-4.594^{***}	-3.658^{***}
$\Delta\sigma_t^{Roll}$	(CPI)	-25.627^{***}	-4.281^{***}	-3.813^{***}
$\Delta\sigma_t^{Roll}$	(Core CPI)	-25.836^{***}	-4.329^{***}	-3.764^{***}
$\Delta\sigma_t^{Roll}$	(PCE)	-24.420^{***}	-4.101^{***}	-3.594^{**}
$\Delta\sigma_t^{Roll}$	(Core PCE)	-23.848^{***}	-4.034^{***}	-3.464^{**}

Note: `ffr` denotes federal funds rates, χ denotes required reserve ratio, and σ_t^{Roll} denotes cyclical volatility of real inside money balances. ***, **, and * denote significance at the 1, 5, and 10 percent levels, respectively.