On the Instability of Fractional Reserve Banking*

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Abstract

This paper develops a dynamic monetary model to study the (in)stability of the fractional reserve banking system. The model shows that the fractional reserve banking system can endanger stability in that equilibrium is more prone to exhibit endogenous cyclic, chaotic, and stochastic dynamics under lower reserve requirements, although it can increase consumption in the steady state. Introducing endogenous unsecured credit to the baseline model does not change the main results. The calibrated exercise suggests that this channel could be another source of economic fluctuations. This paper also provides empirical evidence that is consistent with the prediction of the model.

JEL Classification Codes: E42, E51, G21
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> Motivated partly by a desire to avoid such [excessive] price-level fluctuations ..., quantity theorists have advocated legal restrictions on private intermediation. ... Thus, for example, Friedman (1959, p. 21) ... has advocated 100 percent reserves against bank liabilities called demand deposit.
>
> **Sargent and Wallace (1982)**

1 Introduction

There have been claims that fractional reserve banking is an important cause of boom-bust cycles, based on the notion that banks create excess credit under fractional reserve banking. (e.g., Fisher, 1935; Von Mises, 1953; Minsky, 1957; Minsky, 1970). For instance, Fisher (1935) views fractional reserve banking as one of several important factors in explaining economic fluctuations. Others believe that this is a primary cause of boom-bust cycles. According to Von Mises (1953), the overexpansion of bank credit as a result of fractional reserve banking is the root cause of business cycles. Minsky (1970) claims that economic booms and structural characteristics of the financial system, such as fractional reserve banking, can result in an economic collapse even when fundamentals remain unchanged.

This idea leads to policy debates on fractional reserve banking. Earlier examples include Peel's Banking Act of 1844 and the Chicago plan of banking reform with a 100% reserve requirement proposed by Irving Fisher, Paul Douglas, and others in 1939. Later, Friedman (1959) supported this banking reform, whereas Becker (1956) took the opposite position of supporting free banking with 0% reserve requirement.[1] Recently in 2018, Switzerland had a referendum of 100% reserve banking, which was rejected by 75.72% of the voters. The referendum aimed at making money safe from crisis by constructing full-reserve banking.[2] Whereas the debate on whether a fractional reserve banking system is inherently unstable has been an important policy discussion since a long time ago, the debate has never stopped.

This paper examines the instability of fractional banking by answering the following questions: (i) Can fractional reserve banking be inherently volatile even if we shut

[1] Sargent (2011) provides a novel review of the historical debates between narrow banking and free banking as tensions between stability versus efficiency.

[2] The official title of the referendum was *the Swiss federal popular initiative "for crisis-safe money: money creation by the National Bank only! (Sovereign Money Initiative)"* and also titled as "*debt-free money.*"

down the stochastic component of the economy? (ii) If so, under what condition can fractional reserve banking generate endogenous cycles without the presence of exogenous shocks and changes in fundamentals? To assess the claim that fractional reserve banking causes business cycles, this paper constructs a model of money and banking that captures the role of fractional reserve banking.

In the model, each agent faces an idiosyncratic liquidity shock. Banks accept and issue deposits and extend loans to provide risk-sharing among the depositors. The bank makes loans by creating deposit money, and its lending deposit money creation is constrained by the reserve requirement. At equilibrium, the real balance of money is determined by two factors: storage value and liquidity premium. The storage value increases with the future value of money. However, the liquidity premium, which is the marginal value of money's liquidity function, decreases as money becomes more abundant. When the liquidity premium dominates the storage value, the economy can exhibit endogenous fluctuations. Fractional reserve banking amplifies the liquidity premium because it allows the bank to create inside money through lending. Due to this amplified liquidity premium, the fractional reserve banking system is more prone to endogenous cycles.

In the baseline model, lowering the reserve requirement increases consumption in the steady state. However, lowering the reserve requirements can induce two-period cycles as well as three-period cycles, which implies the existence of periodic cycles of all order and chaotic dynamics. This also implies it can induce sunspot cycles. This result holds in the extended model with unsecured credit. The model also can deliver a self-fulfilling bubble burst. It is worth noting that the full reserve requirement does not necessarily exclude the possibility of endogenous cycles. However, the economy will be more susceptible to cycles with lower reserve requirement.[3] The calibrated exercise suggests that this channel could be another source of economic fluctuations.

This paper departs from previous works in two ways. First, in contrast to the previous works on banking instability, which mostly focus on bank runs following the seminal model by Diamond and Dybvig (1983), this paper focuses on the volatility of real balances of money. It is another important focal point of banking instability because recurring boom-bust cycles associated with banking are probably be more prevalent than bank runs. Second, the approach here differs from a traditional approach to economic fluctuations with financial frictions. To understand economic fluctuations,

[3] Gu, Monnet, Nosal and Wright (2023) show that introducing banks to the economy could induce instability in various settings which is in line with this result.

there are two major points of view. The first one is that economic fluctuations are driven by exogenous shocks disturbing the dynamic system, and the effects of exogenous shocks shrink over time as the system goes back to its balanced path or steady state. The second one is that they instead reflect an endogenous mechanism that produces boom-bust cycles. While there has been a lot of work on the role of financial friction in the business cycles including Kiyotaki and Moore (1997), Bernanke, Gertler and Gilchrist (1999), and Gertler and Karadi (2011), most of them focused on the first approach, in which all economic fluctuations are caused by exogenous shocks and the financial sectors only serve as an amplifier. This paper, however, takes the second approach and focuses on whether the endogenous cycles arise in the absence of the stochastic components of the economy.[4]

To evaluate the main prediction from the theory that fractional reserve banking induces excess volatility, I test the relationship between the required reserve ratio and the volatilty in real balance using cointegrating regression. A significant negative relationship between the two variables are found, and the results are robust to different measures of inflation and different frequency of time series. Both theoretical and empirical evidence indicate a link between the reserve requirement and the (in)stability.

Related Literature This paper builds on Berentsen, Camera and Waller (2007), who introduce financial intermediaries with enforcement technology to Lagos and Wright (2005) framework. The approach to introduce unsecured credit to the monetary economy is related to Lotz and Zhang (2016) and Gu, Mattesini and Wright (2016) which are based on the earlier work by Kehoe and Levine (1993).

This paper is related to the large literature on fractional reserve banking. Freeman and Huffman (1991) and Freeman and Kydland (2000) develop general equilibrium models that explicitly capture the role of fractional reserve banking. Using those models, they explain the observed relationships between key macroeconomic variables over business cycles. Chari and Phelan (2014) study an economy where private agents have incentives to establish fractional reserve banking as an alternative payment system. This alternative system is inherently fragile because it is susceptible to socially costly bank runs. They study the conditions under which the social benefits of fractional

[4]The view that macroeconomic fluctuations reflect strong internal propagation mechanisms is not new. This perspective dates back to early contributions in economics, including Le Corbeiller (1933)'s work published in the first volume of *Econometrica*, as well as subsequent works such as Kalecki (1937) and Kaldor (1940).

reserve banking can exceed its social costs which crucially depend on communication technologies. For recent work, Sanches (2016) and Wipf (2020) studies the welfare implications of fractional reserve banking in a New Monetarist economy. Wipf (2020) studies a model of imperfect competition and identifies the conditions under which fractional reserve banking can be welfare-improving compared to narrow banking. In Sanches (2016), incentive-feasible arrangements within the banking sector that preserves the safety of bank liabilities as a store of value but also achieves a better allocation of resources via investment opportunities. Andolfatto, Berentsen and Martin (2020) integrate Diamond (1997) into Lagos and Wright (2005) to provide a model in which fractional reserve banking emerges endogenously and a central bank can prevent bank panic as a lender of last resort. Whereas many previous work on instability focuses on bank runs or societal value at the steady state, this paper studies a different type of instability in the sense that fractional reserve banking induces endogenous monetary cycles.

This paper is also related to the large literature on endogenous fluctuations, chaotic dynamics, and indeterminacy that have been surveyed by Brock (1988), Baumol and Benhabib (1989), Boldrin and Woodford (1990), Scheinkman and Woodford (1994) and Benhabib and Farmer (1999). For a model of bilateral trade, Gu, Mattesini, Monnet and Wright (2013) show that credit markets can be susceptible to endogenous fluctuations due to limited commitment. Using a continuous-time New Monetarist economy, Rocheteau and Wang (2023) show that asset liquidity can be a source of price volatility when assets have a non-positive intrinsic value. Altermatt, Iwasaki and Wright (2023) study economies with multiple liquid assets and show that liquidity considerations could imply endogenous fluctuations as self-fulfilling prophecies. Gu et al. (2023) show that introducing financial intermediaries to an economy can engender instability in the sense that endogenous cycles are more likely to emerge in the presence of financial intermediaries. They demonstrate this in four distinct setups that capture various functions of banking. The model in this paper is closely related to that of Gu et al. (2023) in the sense that both papers study the role of banking in instability. However, this paper goes further by focusing on the fractional reserve banking system. One focal point is that, rather than treating banks as mere intermediaries, in this paper, the bank creates deposit money (inside money) by making loans through the fractional reserve banking. Furthermore, this paper establishes exact thresholds of reserve requirements under which the equilibrium can exhibit endogenous cycles and chaotic dynamics.

The rest of the paper is organized as follows. Section 2 constructs the baseline search-theoretic monetary model. Section 3 provides main results. Section 4 introduces unsecured credit. Section 5 calibrates the model to quantify the theory. Section 6 discusses the empirical evaluation of the model's prediction. Section 7 concludes.

2 Model

The model is based on Lagos and Wright (2005) with banking as in Berentsen et al. (2007). Time is discrete and infinite. In each period, three markets convene sequentially. First, a centralized financial market (FM), followed by a decentralized goods market (DM), and finally a centralized goods market (CM). The FM and CM are frictionless. The DM is subject to search frictions, anonymity, and limited commitment. Therefore, a medium of exchange is needed to execute trades.

There is a continuum of agents who produce and consume perishable goods. At the beginning of the FM, a preference shock is realized: With probability σ, an agent will be a buyer in the following DM and with probability $1-\sigma$, she will be a seller. The buyers and the sellers randomly meet and trade bilaterally in the DM. Agents discount their utility each period by β. Within-period utility is represented by

$$\mathcal{U} = U(X) - H + u(q) - c(q),$$

where X is the CM consumption, H is the CM disutility from production, and q is the DM consumption. As standard U', u', $c' > 0$, U'', $u'' < 0$, $c'' \geq 0$, and $u(0) = c(0) = 0$. The CM consumption good X is produced one-for-one with H, implying the real wage is 1. The efficient consumption in CM and DM is X^* and q^* that solve $U'(X^*) = 1$ and $u'(q^*) = c'(q^*)$, respectively.

There is a representative bank that accepts and issues deposits and lends loans in the FM. In the FM, an agent can borrow money from the bank with a promise to repay the money in the subsequent CM at a nominal lending rate i_l. There are two kinds of deposits: demand deposits and savings deposits. The agent can deposit her fiat money into the bank's savings deposit and receive money in the subsequent CM at a nominal deposit rate. The agent can also hold her balance in a demand deposit. The demand deposit does not pay interest, while the interest rate on savings deposits is i_s. In addition to that, the demand deposit can be used as a means of payment in

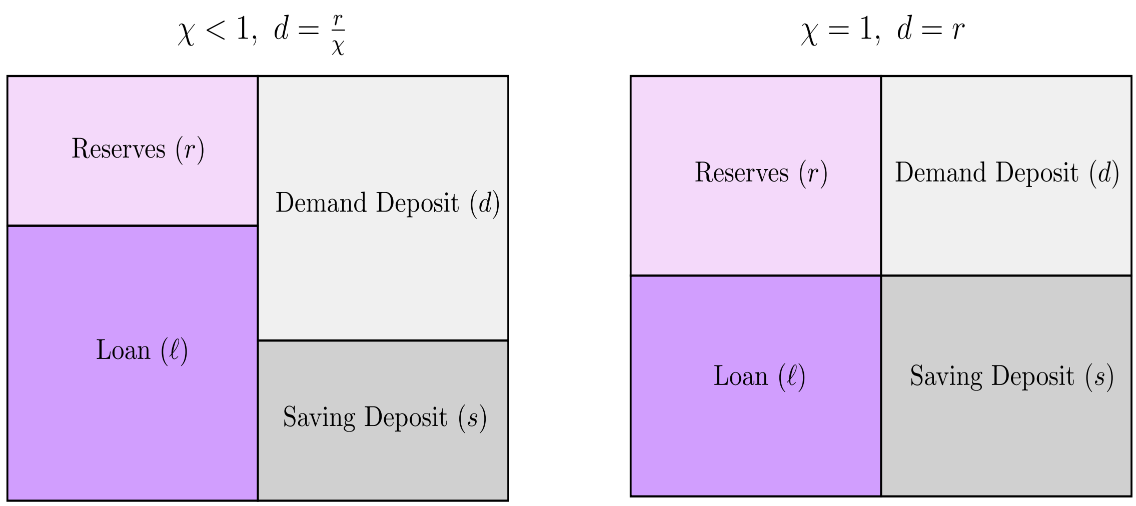


Figure 1: Bank's Balance Sheet: fractional reserve banking vs. full reserve banking

DM trade, whereas the savings deposit cannot. When the bank lends loans, it creates demand deposits, and the bank's issuance of demand deposits is subject to a reserve requirement, χ. Figure 1 illustrates the bank's balance sheet identity given the reserve requirement. The banking market is perfectly competitive, and the bank can enforce the repayment of loans at no cost. Lastly, there is a central bank that controls the fiat money supply M_t. Let γ be the growth rate of the fiat money stock. Changes in the fiat money supply are accomplished by lump-sum transfers if $\gamma > 0$ and by lump-sum taxes if $\gamma < 0$.

2.1 Agent's Problem

Let W_t, G_t, and V_t denote the agent's value function in the CM, FM, and DM, respectively, in period t. There are two payment instruments for the DM transaction: fiat money and demand deposits. However, buyers and sellers do not discriminate between these instruments in the DM transaction because agents treat them as the same 'money.' The agent's state variables in the CM are a_t, s_t, and ℓ_t, where $a_t = m_t + d_t$, s_t is a savings deposit, ℓ_t is a loan borrowed from the bank, m_t is fiat money (outside money) issued by the central bank, and d_t is a demand deposit (inside money) issued by the bank. The state variable a_t represents the agent's nominal balance of liquid assets, which can be used for transactions in the DM. I will allow the agents to use unsecured credit as a means of payment in the next section. An agent entering the CM with nominal balance a_t, a savings deposit s_t, and a loan ℓ_t solves the following problem:

$$
\begin{aligned}
W_t(a_t, s_t, \ell_t) = & \max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta G_{t+1}(\hat{m}_{t+1}) \\
\text{s.t. } \phi_t \hat{m}_{t+1} + X_t = & H_t + T_t + \phi_t a_t + (1 + i_{s,t})\phi_t s_t - (1 + i_{l,t})\phi_t \ell_t,
\end{aligned}
\tag{1}
$$

where T_t is the lump-sum transfer (or tax if it is negative), $i_{s,t}$ is the savings deposit interest rate, $i_{l,t}$ is the loan interest rate, ϕ_t is the price of money in terms of the CM goods, and $\hat{m}_{t+1}$ is the money balance carried to the FM where the bank takes deposits and makes loans. The first-order conditions (FOCs) result in $X_t = X^*$ and

$$
\phi_t = \beta G'_{t+1}(\hat{m}_{t+1}), \tag{2}
$$

where $G'_{t+1}(\hat{m}_{t+1})$ is the marginal value of an additional unit of money taken into the FM of period $t+1$. The envelope conditions are

$$
\frac{\partial W_t}{\partial a_t} = \phi_t, \qquad \frac{\partial W_t}{\partial s_t} = \phi_t(1 + i_{s,t}), \qquad \frac{\partial W_t}{\partial \ell_t} = -\phi_t(1 + i_{l,t}),
$$

implying W_t is linear in m_t, s_t, and ℓ_t.

The value function of an agent at the beginning of the FM is

$$
G_t(m) = \sigma G_{b,t}(m) + (1 - \sigma) G_{s,t}(m), \tag{3}
$$

where $G_{j \in \{b,s\},t}$ is the value function of a type j agent in the FM. Agents choose their deposit balances d_j, s_j and loans ℓ_j based on the realization of their types in the following DM, and they can acquire demand deposits by borrowing loans from the bank.

The value function $G_{j,t}$ can be written as

$$
\begin{aligned}
G_{j,t}(m) = & \max_{d_{j,t}, s_{j,t}, \ell_{j,t}} V_{j,t}(m + d_{j,t} - s_{j,t}, s_{j,t}, \ell_{j,t}) \\
\text{s.t.} \quad & s_{j,t} \le m, \text{ and } d_{j,t} = \ell_{j,t}
\end{aligned}
\tag{4}
$$

where $V_{j,t}$ is the value function of a type j agent in the DM. The FOCs are

$$\frac{\partial V_{j,t}}{\partial \ell_{j,t}} \leq 0 \tag{5}$$

$$\frac{\partial V_{j,t}}{\partial s_{j,t}} - \lambda_s \leq 0 \tag{6}$$

where λ_s is the Lagrange multiplier for $s_{j,t} \leq m$.

With probability α, a buyer meets a seller in the DM, while a seller meets a buyer with probability α_s. Upon meeting, the buyer and seller choose a quantity q and a payment p, subject to the constraint $p \leq \mathcal{L}$, where $\mathcal{L}$ represents the buyer's liquidity position. For the terms of trade in the DM, denoted by (p, q), I focus on the bilateral trading mechanisms that take the form of

$$p = \Gamma_p = \begin{cases} \mathcal{L} & \text{if } \mathcal{L} < p^* \\ p^* & \text{if } \mathcal{L} \geq p^* \end{cases} \qquad q = \Gamma_q = \begin{cases} v^{-1}(\mathcal{L}) & \text{if } \mathcal{L} < p^* \\ q^* & \text{if } \mathcal{L} \geq p^*, \end{cases} \tag{7}$$

where $v(q)$ is a payment function satisfying $v'(q) > 0$ and $v(0) = 0$, and p^* is the payment required to achieve the efficient consumption level q^*. Gu and Wright (2016) show that if a trading protocol satisfies the following four axioms, then the terms of trade can be written as (7):

Axiom 1 (Feasibility). *For all $\mathcal{L}$, $0 \leq \Gamma_p(\mathcal{L}) \leq \mathcal{L}$, and $0 \leq \Gamma_q(\mathcal{L})$.*

Axiom 2 (Individual Rationality). *For all $\mathcal{L}$, $u \circ \Gamma_q(\mathcal{L}) \geq \Gamma_p(\mathcal{L})$ and $\Gamma_p(\mathcal{L}) \geq c \circ \Gamma_q(\mathcal{L})$.*

Axiom 3 (Monotonicity). *$\Gamma_p(\mathcal{L}_2) > \Gamma_p(\mathcal{L}_1)$ if and only if $\Gamma_q(\mathcal{L}_2) > \Gamma_q(\mathcal{L}_1)$.*

Axiom 4 (Bilateral Efficiency). *For all $\mathcal{L}$, there does not exist a pair (p', q') with $p' \leq \mathcal{L}$ such that $u(q') - p' > u \circ \Gamma_q(\mathcal{L}) - \Gamma_p(\mathcal{L})$ and $p' - c(q') > \Gamma_p(\mathcal{L}) - c \circ \Gamma_q(\mathcal{L})$.*

Many standard mechanisms, including those based on Kalai and generalized Nash bargaining solutions, satisfy this specification.

Since the CM value function is linear, the DM value function for the buyer can be written as

$$V_{b,t}(m_t + d_{b,t} - s_{b,t}, s_{b,t}, \ell_{b,t}) = \alpha[u(q_t) - p_t] + W(m_t + d_{b,t} - s_{b,t}, s_{b,t}, \ell_{b,t}), \tag{8}$$

where $p_t \le (m_t - s_{b,t} + d_{b,t})\phi_t$. Assuming an interior solution, differentiating $V_{b,t}$ yields

$$\frac{\partial V_{b,t}}{\partial m} = \frac{\partial V_{b,t}}{\partial d} = \phi_t[\alpha\lambda(q_t)+1], \qquad \frac{\partial V_{b,t}}{\partial s} = \phi_t[-\alpha\lambda(q_t)+i_{s,t}], \qquad \frac{\partial V_{b,t}}{\partial \ell} = \phi_t[\alpha\lambda(q_t)-i_{l,t}],$$

where $\lambda(q) = u'(q)/v'(q) - 1$ if $p^* > \mathcal{L}$ and $\lambda(q) = 0$ if $\mathcal{L} \ge p^*$. Combining the buyer's FOCs in the FM and the derivatives of V_b yields

$$\phi i_{s,t} - \phi\alpha\lambda(q_t) - \lambda_s \le 0, \text{``}=\text{''}0 \text{ iff } s_{b,t} > 0 \tag{9}$$

$$-\phi i_{l,t} + \phi\alpha\lambda(q) \le 0, \text{``}=\text{''}0 \text{ iff } \ell_{b,t} > 0. \tag{10}$$

A seller's DM value function is

$$V_{s,t}(m_t + d_{s,t} - s_{s,t}, s_{s,t}, \ell_{s,t}) = \alpha_s[p_t - c(q_t)] + W_t(m_t + d_{s,t} - s_{s,t}, s_{s,t}, \ell_{s,t}). \tag{11}$$

where $d_{s,t} = \ell_{s,t}$. Differentiating $V_{s,t}$ after substituting the constraint yields

$$\frac{\partial V_{s,t}}{\partial m_t} = \frac{\partial V_{s,t}}{\partial d_t} = \phi_t, \qquad \frac{\partial V_{s,t}}{\partial s_t} = \phi_t i_{s,t}, \qquad \frac{\partial V_{s,t}}{\partial \ell_t} = -\phi_t i_{l,t}.$$

Similar to the buyer's case, combining the seller's FOCs in the FM and the first-order derivatives of $V_{s,t}$ yields

$$\phi_t i_{s,t} - \lambda_s \le 0, \text{``}=\text{''}0 \text{ iff } s_{s,t} > 0 \tag{12}$$

$$-\phi_t i_{l,t} \le 0, \text{``}=\text{''}0 \text{ iff } \ell_{s,t} > 0. \tag{13}$$

One can show that buyers do not deposit their money into savings deposits and sellers always deposit their money into savings deposits, whereas buyers always borrow loans, but sellers do not. We can rewrite the value functions in the FM as follows:

$$G_{b,t}(m_t) = \alpha[u(q_t) - p_t] + W(m_t + d_{b,t}, 0, \ell_{b,t}) \tag{14}$$

$$G_{s,t}(m_t) = \alpha_s[p_t - c(q_t)] + W(0, m_t, 0) \tag{15}$$

where $q_t = v^{-1}(p_t)$, $d_{b,t} = \ell_{b,t}$ and $p_t = \min\{p^*, (m_t + d_{b,t})\phi_t\}$. Since $G'_t(m_t) = \sigma G'_{b,t}(m_t) + (1-\sigma)G'_{s,t}(m_t)$, we have the following:

$$G'_t(m_t) = \phi_t\sigma[1 + \alpha\lambda(q_t)] + \phi_t(1-\sigma)\left(1 + i_{s,t}\right). \tag{16}$$

Combining (2) and (16) gives the Euler equation

$$\phi_t = \begin{cases} \phi_{t+1}\beta\left[\sigma\left\{1+\alpha\lambda(q_{t+1})\right\}+(1-\sigma)(1+i_{s,t+1})\right] & \text{if } z_{t+1} < p^* \\ \phi_{t+1}\beta & \text{if } z_{t+1} \geq p^*, \end{cases} \tag{17}$$

where $q_{t+1} = v^{-1}(z_{t+1})$ and $z_{t+1} = (m_{t+1} + d_{b,t+1})\phi_{t+1}$

2.2 Bank's Problem and Equilibrium

A representative bank accepts savings deposits s_t, issues demand deposits d_t, and makes loans ℓ_t. The bank is required to hold reserves r_t equal to χd_t. The bank pays a nominal interest rate of $i_{s,t}$ to depositors on their savings deposits, while borrowers must repay their loans at a nominal interest rate of $i_{l,t}$. Demand deposits do not pay interest. The central bank sets the reserve requirement χ. Given these conditions, the representative bank solves the following profit maximization problem.

$$\begin{aligned} \max_{r_t,d_t,\ell_t,s_t} \quad & (1+i_{l,t})\ell_t + r_t - d_t - (1+i_{s,t})s_t \\ \text{s.t.} \quad & \ell_t + r_t = d_t + s_t, \quad d_t = \ell_t \quad \text{and} \quad r_t \geq \chi d_t \end{aligned} \tag{18}$$

In the first constraint, the balance sheet identity, the left-hand side represents assets, which include reserves and loans, while the right-hand side represents liabilities, which include demand deposits and savings deposits. The second constraint simply states that banks only make loans by issuing demand deposits, and vice versa. The last constraint is the reserve requirement constraint. By substituting the two constraints, the bank's problem can be rewritten as:

$$\max_{\ell_t,r_t} \quad (1+i_{l,t})\ell_t + r_t - \ell_t - (1+i_{s,t})r_t \quad \text{s.t. } r_t \geq \chi\ell_t \tag{19}$$

The FOCs for the bank's problem are

$$0 = i_{l,t} - \lambda_\chi\chi \tag{20}$$

$$0 = -i_{s,t} + \lambda\chi \tag{21}$$

With the binding reserve requirement constraint, we have

$$i_{l,t} = \chi i_{s,t}. \tag{22}$$

Given the bank's problem and the agent's problem, we can define an equilibrium as follows:

Definition 1. *Given (γ, χ), a monetary equilibrium consists of sequences of prices $\{\phi_t, i_{l,t}, i_{s,t}\}_{t=0}^{\infty}$, quantities $\{m_t, \ell_{b,t}, \ell_{s,t}, d_{b,t}, d_{s,t}, s_{b,t}, s_{s,t}\}_{t=0}^{\infty}$, and allocations $\{q_t, X_t\}_{t=0}^{\infty}$ satisfying the following:*

- *Agents solve CM, FM and DM problems: (1) and (4)*
- *The terms of trade in the DM satisfy (7), (8) and (11)*
- *A representative bank solves its profit maximization problem: (18)*
- *Markets clear in every period:*
  1. *Deposit Markets: $\sigma d_{b,t} + (1-\sigma)d_{s,t} = d_t$ and $\sigma s_{b,t} + (1-\sigma)s_{s,t} = s_t$*
  2. *Loan Market: $\sigma \ell_{b,t} + (1-\sigma)\ell_{s,t} = \ell_t$*
  3. *Money Market: $m_t = M_t$*
- *Transversality condition: $\lim_{t\to\infty} \beta^t \phi_t m_t = 0$, and $\phi_t > 0$.*

The next step is to characterize the equilibrium. Given the agents' and the bank's optimization problems, we combine equations (10), (17), and (22) to obtain

$$\phi_t = \begin{cases} \phi_{t+1}\beta \left[\dfrac{1-\sigma+\sigma\chi}{\chi}\alpha\lambda(q_{t+1}) + 1\right] & \text{if } z_{t+1} < p^* \\ \phi_{t+1}\beta & \text{if } z_{t+1} \geq p^*, \end{cases} \tag{23}$$

Multiplying both sides of (23) by $\frac{1-\sigma+\sigma\chi}{\sigma\chi} M_t$ allows us to reduce the equilibrium condition to one difference equation of real balances z:

$$z_t = f(z_{t+1}) \equiv \frac{z_{t+1}}{1+i}\Bigg[\underbrace{\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_{t+1})}_{\text{liquidity premium}} + 1\Bigg], \tag{24}$$

where $(1+i) \equiv \gamma/\beta$ and $L(z) \equiv \lambda \circ v^{-1}(z)$.[5] When $z_{t+1} \geq p^*$, $q_{t+1} = q^*$ because the buyer has sufficient liquidity to buy q^*. In this case, liquidity is abundant, and the liquidity premium and $L(\cdot)$ become zero. When $z_{t+1} < p^*$, $q_{t+1} = v^{-1}(z_{t+1}) < q^*$ because the buyer does not have enough liquidity to buy q^*. In this case, liquidity is scarce, and the liquidity premium and $L(\cdot)$ are strictly positive.

[5] In the stationary equilibrium, $i = \gamma/\beta - 1$ is the nominal interest rate.

To explain the equilibrium mechanism that leads to equation (24), I summarize the equilibrium behavior as follows. In the CM, all agents are *ex ante* identical and each holds M_{t+1} units of fiat money at the end of the period. In the FM, once agents' types are revealed, sellers deposit all their fiat money with the bank because the sellers do not have a trade opportunity to use that fiat money, while buyers obtain funds by borrowing from the bank in the form of demand deposits because they lack money to consume in the DM.

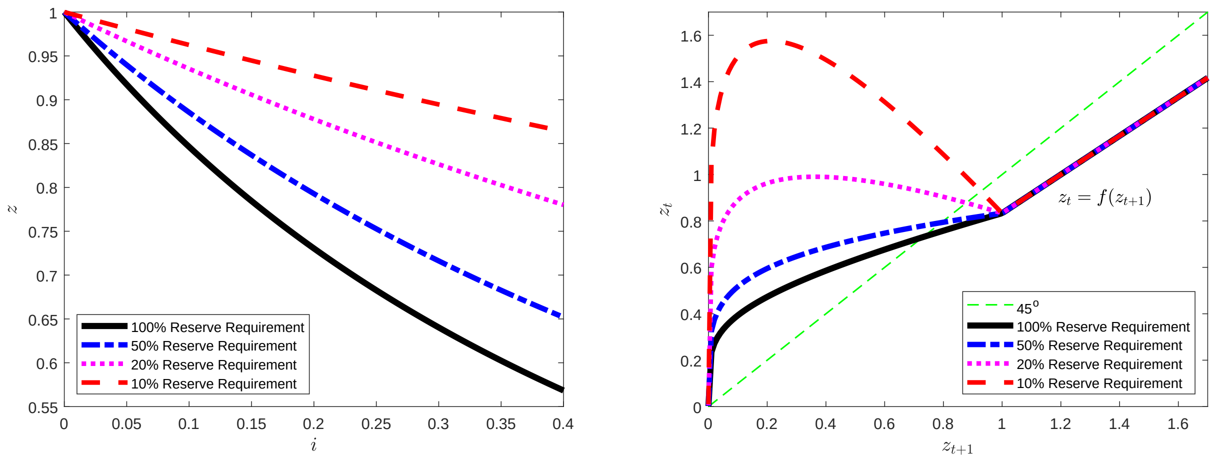
Since the measure of sellers is $1-\sigma$, the total fiat money deposited with the bank equals $(1-\sigma)M_{t+1}$ in savings deposits. The bank then creates demand deposits by extending loans, subject to the reserve requirement. Using the sellers' savings deposits as reserves, the bank can create a total of $\frac{(1-\sigma)M_{t+1}}{\chi}$ in demand deposits. With σ buyers in the economy, each buyer enters the DM holding $d_{b,t} = \frac{(1-\sigma)M_{t+1}}{\sigma\chi}$ units of demand deposits. Combined with their initial fiat money holdings M_{t+1}, each buyer's total real balance is

$$z_{t+1} = (M_{t+1} + d_{b,t+1})\phi_{t+1} = \frac{1-\sigma+\sigma\chi}{\sigma\chi}\phi_{t+1}M_{t+1} \tag{25}$$

We define the aggregate money supply as the sum of currency in circulation plus demand deposits, yielding an aggregate money supply of

$$\frac{1-\sigma+\sigma\chi}{\chi}M_{t+1}. \tag{26}$$

The term $\frac{1-\sigma+\sigma\chi}{\chi}$ represents the money multiplier and deserves further discussion. When $\chi = 1$ (100% reserve requirement), the multiplier equals 1 and $z_{t+1} = \phi_{t+1}M_{t+1}$. In this case, each buyer holds m_{t+1} in fiat money plus $(1-\sigma)M_{t+1}/\sigma$ in demand deposits. Each seller deposits all of her fiat money as savings deposits, so $s_{b,t+1} = m_{t+1}$, which yields $s_{t+1} = (1-\sigma)m_{t+1}$ and $d_{t+1} = (1-\sigma)m_{t+1}$. Under a 100% reserve requirement, the bank serves merely as an intermediary, providing risk-sharing against idiosyncratic consumption risk in the DM by redistributing sellers' idle balances to buyers. This result under $\chi = 1$ parallels the banking model in Berentsen et al. (2007), where agents can either borrow to overcome cash constraints or deposit idle balances to earn interest. When $\chi < 1$, however, the money multiplier becomes greater than 1 and the bank functions as an intermediary while simultaneously creating money through lending in fractional reserve banking. This allows the buyers to use more money in DM trade compared to the case where the bank functions as a mere intermediary.



(a) Stationary Monetary Equilibrium

(b) Dynamic Monetary Equilibrium

Figure 2: Monetary Equilibrium

3 Results

This section establishes key results. Before starting a discussion on dynamics, consider stationary equilibria which are defined as fixed points that satisfy $z = f(z)$. There always exists a non-monetary equilibrium with $z = 0$. A unique solution of monetary stationary equilibrium $z_s > 0$ exists and solves

$$\chi i = (1 - \sigma + \sigma\chi)\alpha L(z_s) \tag{27}$$

when $i < \bar{\imath}$ where $\bar{\imath} = \alpha(1 - \sigma + \sigma\chi)L(0)/\chi$. Nash and Kalai bargaining provide simple examples for $\bar{\imath}$. Under the Inada condition $u'(0) = \infty$, with Kalai bargaining, $\bar{\imath} = \theta\alpha(1 - \sigma + \sigma\chi)/\chi(1 - \theta)$ where θ is the buyer's bargaining power. With Nash bargaining, we have $\bar{\imath} = \infty$. In the remaining sections of this paper, I focus on the monetary equilibrium where money is valued $\phi_t > 0$.

Since $L'(z) < 0$ and $v'(q) > 0$ (see Gu and Wright, 2016), the following result holds:

Proposition 1. *In the stationary equilibrium, lowering i or lowering χ increases q.*

Proof. See Appendix A. ■

Figure 2a plots z against i. It shows downward-sloping money demand in the stationary equilibrium given the reserve requirement. Lowering the reserve requirement increases z because it allows a bank to create more liquidity in the economy which increases q as well.

Now we will examine the dynamics of monetary equilibrium where money is valued each period. The dynamics of monetary equilibrium are characterized by $f(z_{t+1})$ from equation (24). The first term, $z_{t+1}/(1+i)$ on the right-hand side, reflects the store of value, which is monotonically increasing in z_{t+1}. The second term $(1-\sigma+\sigma\chi)\alpha L(z_{t+1})/\chi+1$, reflecting the liquidity premium, is decreasing in z_{t+1}. Because $f'(z_{t+1})$ depends on both terms, $f(z_{t+1})$ is nonmonotone in general.

Figure 2b provides an example. In this example, as the reserve requirement decreases, the equation (24) is more likely to have the backward bending feature. Lowering the reserve requirement amplifies the liquidity premium, as it enables banks to create more liquidity through lending. This amplification of liquidity enhances the backward-bending feature, potentially leading to endogenous cycles.

The standard treatment for showing the existence of an endogenous cycle is $f'(z_s) < -1$ (see Azariadis, 1993). In this case, the economy can exhibit a two-period cycle with $z_1 < z_s < z_2$ which can be either $z_2 < p^*$ or $z_2 \geq p^*$. However, without further assumptions, we cannot determine the conditions under which this can occur. For illustration, let's take the derivative of (24) with respect to z_{t+1} and evaluate it at $z_{t+1} = z_s$. We obtain the following expression:

$$f'(z_s) = \frac{1}{1+i}\left[\frac{\alpha(1-\sigma+\sigma\chi)}{\chi}\left\{L'(z_s)z_s + L(z_s)\right\} + 1\right] \tag{28}$$

As $L'(z_s)$ is not explicitly defined here, we are unable to establish the conditions under which the standard condition of cycles, $f'(z_s) < -1$, would hold under the general bilateral trading mechanism.

We can show the existence of an endogenous cycle without relying on the standard treatment of $f'(\cdot) < -1$. To establish a sufficient condition for an endogenous cycle, consider a two-period cycle with $z_1 < z_s < p^* \leq z_2$. Since $z_2 \geq p^*$, this cycle satisfies $z_1 = \frac{z_2}{1+i} < z_s < p^*$, where z_1 solves

$$L(z_1) = \frac{(1+i)^2 - 1}{\alpha(1-\sigma+\sigma\chi)}\chi.$$

It is straightforward to show that $z_1 < z_s$ because $L'(\cdot) < 0$, and z_s solves (27). By checking the condition $z_2 = z_1(1+i) > p^*$, we can derive the condition under which the economy exhibits a two-period cycle that satisfies $z_1 < z_s < p^* \leq z_2$.

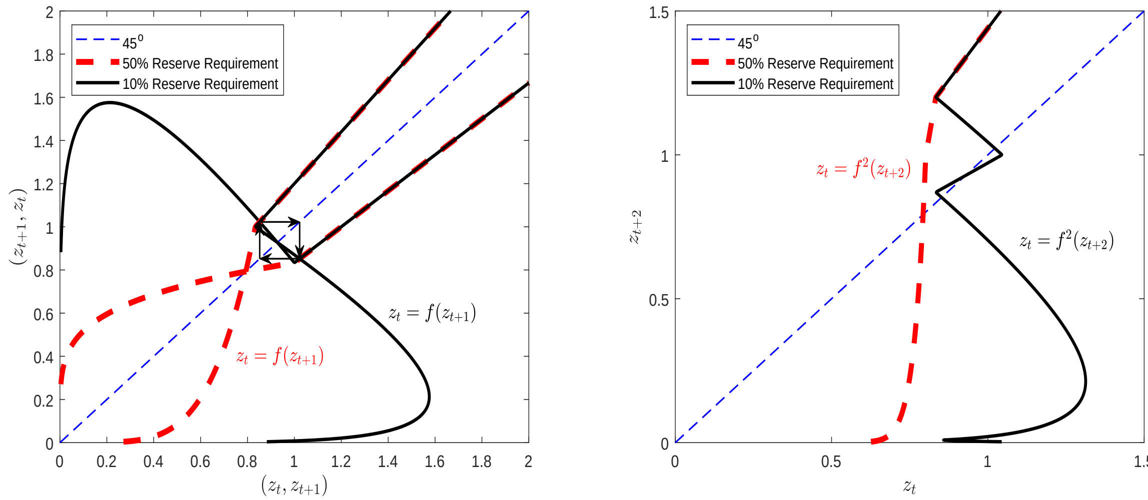


Figure 3: A Two-period Cycle under Fractional Reserve Banking

Proposition 2 (**Two-period Monetary Cycle**). *There exists a two-period cycle with* $z_1 < z_s < p^* \leq z_2$ *if* $\chi \in (0, \bar{\chi}_m]$, *where*

$$\bar{\chi}_m \equiv \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^2 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)}.$$

When this type of two-period cycle exists, lowering χ *increases the difference between peak and trough,* $z_2 - z_1$.

Proof. See Appendix A. ■

Proposition 2 shows that, under the general trading mechanism, lowering the reserve requirement can induce a two-period cycle and increase the volatility of the real balances. By lowering χ, the liquidity premium dominates the storage value. Consequently, $f(\cdot)$ is more likely to exhibit the backward bending feature, which can lead to an endogenous cycle. Figure 3 shows an example of this case.

This two-period cycle emerges as follows. If you believe the value of money tomorrow will decrease so that your money holdings cannot cover the required money holdings for efficient quantity, you will want to acquire more money today. Due to the increase in demand for money, the value of money will go up. This gives $z_2 = \frac{z_1}{1+i}\left\{\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_1) + 1\right\}$ where $z_1 < q^*$ and $z_2 > z_1$. Tomorrow, you believe the value of money the next day will go up to cover the level of money holdings required to buy the efficient quantity. The buyer is willing to hold their additional currency because the rate of return on money is exactly equal to i. This gives $z_1 = \frac{z_2}{1+i}$. A lower reserve requirement increases today's marginal value of money, given your belief about tomorrow's money value, because more inside money (demand deposits) is created through fractional reserve banking. In other words, given a value of z_1, lowering χ increases z_2 because of the money multiplier effect. Therefore, under a low reserve requirement, there is a tendency for real money balances to go up and down, which can generate a two-period cycle. This tendency also leads the economy to feature higher-order cycles and chaotic dynamics.

Now, let's introduce some additional assumptions to determine the condition for χ such that $f'(z_s) < -1$. Consider a special case where $-qu''(q)/u'(q) = \eta$, $c(q) = q$, and the buyer makes a take-it-or-leave-it (TIOLI) offer. In this case, as $L(z_s) = u'(q) - 1$ and $L'(z_s)z_s = z_s u''(z_s) = -\eta u'(z_s)$, we can rewrite (28) as follows:

$$f'(z_s) = \frac{1}{1+i}\left\{\frac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u'(z_s)(1-\eta)-1\right]+1\right\} < -1 \tag{29}$$

where $u'(z_s) = 1 + \frac{i\chi}{\alpha(1-\sigma+\sigma\chi)}$. Solving (29) for χ yields the following proposition.

Proposition 3. *Assume $-qu''(q)/u'(q) = \eta$, $c(q) = q$, and the buyer makes take-it-or-leave-it offer to the seller. If $\chi \in (0, \chi_m)$, where*

$$\chi_m \equiv \frac{\alpha\eta(1-\sigma)}{\eta(1-\alpha\sigma)+(2-\eta)(1+i)}, \tag{30}$$

then $f'(z_s) < -1$.

Proof. See Appendix A. ■

Since $\chi < \chi_m$ implies $f'(z_s) < -1$, following the standard textbook method (see Azariadis, 1993), we can show that if $\chi < \chi_m$, there exists a two-period cycle with $z_1 < z_s < z_2$. While condition (30) is expressed in terms of χ, we can also express it in terms of i as follows:

$$i_m \equiv \frac{\eta[\alpha(1-\sigma)-\chi(1-\alpha\sigma)]}{\chi(2-\eta)} - 1$$

The effect of i on cycles depends on η. If $\eta < 2$, a cycle exists when $i < i_m$. The two conditions $i < i_m$ and $\chi < \chi_m$ are equivalent. χ_m is positive for all i, and is decreasing in i. Thus, lowering either χ or i can induce a cycle. When $\eta = 2$, χ_m is constant and i_m doesn't exist, so i has no effect on the existence of cycles. If $2 < \eta < \frac{2}{\alpha\sigma}$, then χ_m

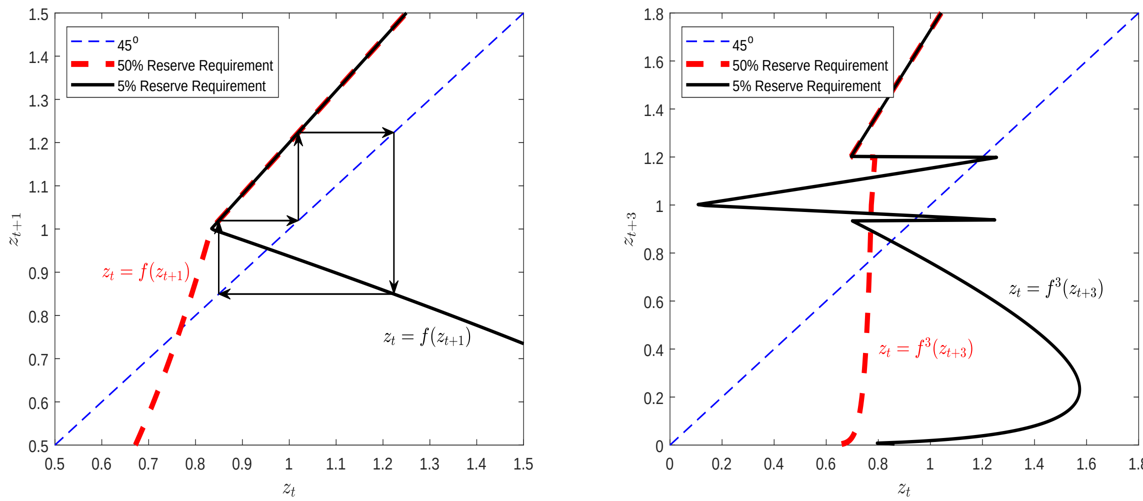


Figure 4: A Three-period Cycle under Fractional Reserve Banking

is negative when $i > \frac{2-\eta\alpha\sigma}{\eta-2}$ and positive when $i < \frac{2-\eta\alpha\sigma}{\eta-2}$. In this case, setting i above $\frac{2-\eta\alpha\sigma}{\eta-2}$ eliminates cyclic equilibria. If $\eta \geq \frac{2}{\alpha\sigma}$, then both χ_m and i_m are negative for all i, implying that cycles do not occur.

In addition to the conditions for a two-period cycle, the next result provides the condition for a three-period cycle under the general trading mechanism. The existence of three period-cycles implies cycles of all orders as well as chaotic dynamics (see Sharkovskii, 1964 and Li and Yorke, 1975).

Proposition 4 (**Three-period Monetary Cycle and Chaos**). *A three-period cycle with $z_1 < z_2 < p^* \leq z_3$ does not exist. There exists a three-period cycle with $z_1 < p^* \leq z_2 < z_3$ if $\chi \in (0, \hat{\chi}_m]$, where*

$$\hat{\chi}_m \equiv \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^3 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)}.$$

When this type of three-period cycle exists, lowering χ increases the difference between peak and trough, $z_3 - z_1$.

Proof. See Appendix A. ■

The following corollary is a direct result from Proposition 4.

Corollary 1 (**Binding Liquidity Constraint**). *In any n-period cycle, the liquidity constraint binds, $z_t < p^*$, at least one periodic point over the cycle.*

Proof. See Appendix A. ■

The model can also generate sunspot cycles. Consider a Markov sunspot variable $S_t \in \{1, 2\}$. This sunspot variable is not related to fundamentals but may affect equilibrium. Let $\Pr(S_{t+1} = 1|S_t = 1) = \zeta_1$ and $\Pr(S_{t+1} = 2|S_t = 2) = \zeta_2$. The sunspot is realized in the FM. Let W_t^S be the CM value function in state S in period t, then

$$W_t^S(m_t, d_t, \ell_t) = \max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta \left[\zeta_s G_{t+1}^S(\hat{m}_{t+1}) + (1 - \zeta_s) G_{t+1}^{-S}(\hat{m}_{t+1})\right]$$
$$\text{s.t. } \phi_t^S \hat{m}_{t+1} + X_t = H_t + T_t + \phi_t^S m_t + (1 + i_{d,t})\phi_t^S d_t - (1 + i_{l,t})\phi_t^S \ell_t.$$

The FOC can be written as

$$-\phi_t^S + \beta\zeta_s G_{t+1}'^S(\hat{m}_{t+1}) + \beta(1 - \zeta_s) G_{t+1}'^{-S}(\hat{m}_{t+1}) = 0. \tag{31}$$

Solving the FM problem results in

$$G_{t+1}'^S(m_{t+1}^S) = \phi_{t+1}^S \left[\frac{1 - \sigma + \sigma\chi}{\chi} \alpha L(z_{t+1}^S) + 1\right]. \tag{32}$$

We substitute (32) into (31) and use the money market clearing condition $m_{t+1} = M_{t+1}$ to get the Euler equation.

$$\phi_t^S = \beta\zeta_s\phi_{t+1}^S \left[\frac{1 - \sigma + \sigma\chi}{\chi} \alpha L(z_{t+1}^S) + 1\right] + \beta(1 - \zeta_s)\phi_{t+1}^{-S} \left[\frac{1 - \sigma + \sigma\chi}{\chi} \alpha L(z_{t+1}^{-S}) + 1\right].$$

where $z_{t+1}^S = \phi_{t+1}^S M_{t+1}(1 - \sigma + \sigma\chi)/\sigma\chi$. Then multiply both sides of the Euler equation by $M_t(1 - \sigma + \sigma\chi)/\sigma\chi$ to reduce the equilibrium condition into one difference equation of real balances z_{t+1}^S:

$$\begin{aligned} z_t^S &= \frac{\zeta_s z_{t+1}^S}{1 + i} \left[\frac{1 - \sigma + \sigma\chi}{\chi} \alpha L(z_{t+1}^S) + 1\right] + \frac{(1 - \zeta_s) z_{t+1}^{-S}}{1 + i} \left[\frac{1 - \sigma + \sigma\chi}{\chi} \alpha L(z_{t+1}^{-S}) + 1\right] \\ &= \zeta_s f(z_{t+1}^S) + (1 - \zeta_s) f(z_{t+1}^{-S}). \end{aligned} \tag{33}$$

We define a sunspot equilibrium as follows:

Definition 2 (**Proper Sunspot Equilibrium**). *A proper sunspot equilibrium consists of the sequences of real balances $\{z_t^S\}_{t=0,S=1,2}^{\infty}$ and probabilities (ζ_1, ζ_2), solving (33) for all t.*

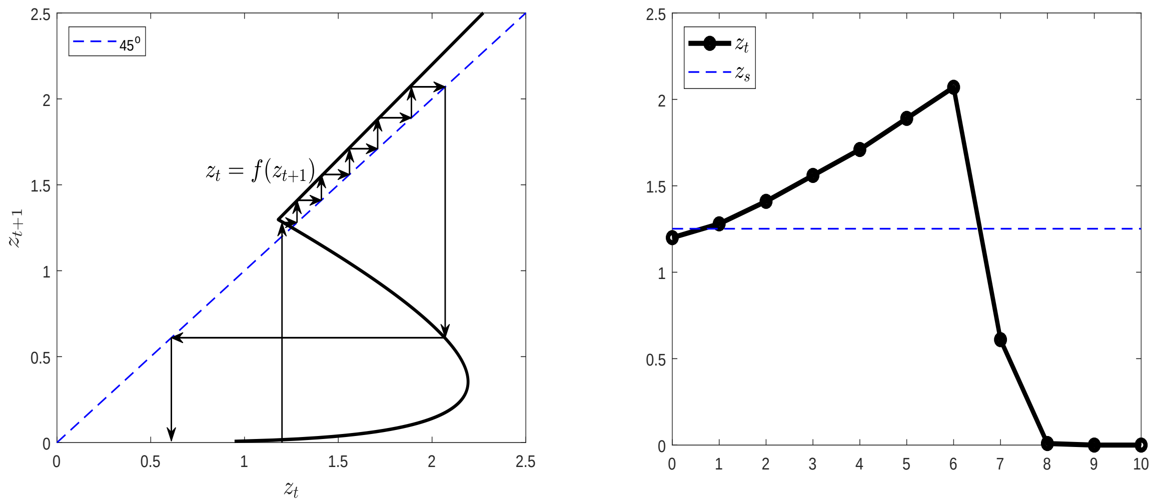


Figure 5: Self-Fulfilling Bubble and Burst Equilibrium

Consider stationary sunspot equilibria with $z^1 < z^2$ that only depend on the state, not the time. The liquidity constraint is binding in state $S = 1$. By the standard approach (see again Azariadis, 1993 for the textbook treatment), the condition for two-period cycles is also sufficient and necessary for two-state sunspot equilibrium. If $f'(z_s) < -1$ or $\chi < \bar{\chi}_m$, there exists $(\zeta_1, \zeta_2) \in (0,1)^2$, $\zeta_1 + \zeta_2 < 1$, such that the economy has a proper sunspot equilibrium in the neighborhood of z_s.

Proposition 5 (Stationary Sunspot Equilibrium). *The stationary sunspot equilibrium exists if either $\chi < \bar{\chi}_m$ or $f'(z_s) < -1$.*

Proof. See Appendix A. ■

In addition to the deterministic and stochastic cycles, the model also features the equilibria where real balance increases above the steady state until certain time, T, and crashes to zero. Consider a sequence of real balances $\{z_t\}_{t=0}^{\infty}$ with $z_T \equiv \max\{z_t\}_{t=0}^{\infty} > z_s$ (bubble) that crashes to 0 (burst) as $t \to \infty$, where $T \geq 1$ and $z_T > z_0$. We refer to this equilibrium as a self-fulfilling bubble and burst equilibrium:

Definition 3 (Self-Fulfilling Bubble and Burst Equilibrium). *For initial real balance $z_0 > 0$, a self-fulfilling bubble and burst equilibrium is a sequence of $\{z_t\}_{t=0}^{\infty}$ satisfying (24) and $0 < z_s < z_T$, $\lim_{t\to\infty} z_t = 0$ where $z_T = \max\{z_t\}_{t=0}^{\infty}$ with $T \geq 1$.*

Figure 5 illustrates an example. In Figure 5, f is not monotone, so f^{-1} is a correspondence. When f is not monotone, there are multiple equilibrium paths for $\{z_{t+1}\}$

over some range for z_t. This example starts at z_0, which is lower than z_s, and then increases, surpassing z_s, repeatedly rising until it reaches z_6. Afterward, it crashes and eventually converges to 0. During the bubble, the return on money is equal to $1/(1+i)$, and liquidity is abundant. However, the real balances cannot continue to increase indefinitely otherwise it would violate the transversality condition. The real balance increases until it reaches a certain point, after which the economy crashes and moves toward a non-monetary equilibrium. The timing of these crashes is indeterminate.

The next step is to examine the conditions under which this type of equilibrium can occur. When $z_s > \bar{z}$, where $\bar{z}$ satisfies $f'(\bar{z}) = 0$, multiple equilibria exist. This implies that solving $z_t = f(z_{t+1})$ given z_t yields multiple solutions for z_{t+1}. If $f(\bar{z}) \geq q^*$, the self-fulfilling bubble and burst equilibrium exists. Assuming $-qu''(q)/u'(q) = \eta$, $c(q) = q$, and a buyer makes a TIOLI offer to the seller, Proposition 6 shows that lowering the reserve requirement can induce this type of equilibrium.

Proposition 6 (**Existence of Self-Fulfilling Bubble and Burst Equilibrium**). *Assume $-qu''(q)/u'(q) = \eta$, $c(q) = q$, and the buyer makes take-it-or-leave-it offer to the seller. There exist a self-fulfilling bubble and burst equilibrium, if*

$$0 < \chi < \min\left\{\frac{(1-\sigma)\alpha\eta(1+i)}{(1-\eta)^2 q^* + (1+i)[(1-\eta)(3+i-\eta) - \alpha\sigma\eta]}, \frac{\alpha\eta(1-\sigma)}{1+i-\eta(i+\alpha\sigma)}\right\}$$

Proof. See Appendix A. ■

4 Money and Unsecured Credit

Consider an alternative payment instrument in the DM - unsecured credit. The buyer can pay for DM goods using unsecured credit that will be redeemed to the seller in the following CM and she can borrow up to her debt limit, $\bar{b}_t$. For simplicity, I assume that the buyer makes a TIOLI offer to the seller in the DM, which means the buyer maximizes her surplus subject to the seller's participation constraint. The DM cost function is $c(q) = q$. Suppose the buyer has issued b_t units of unsecured debt in the previous DM (or, if $b_t < 0$, the seller has extended unsecured loans to the buyer from the previous DM). The CM value function is

$$\begin{aligned} W_t(a_t, s_t, \ell_t, -b_t) = &\max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta G_{t+1}(\hat{m}_{t+1}) \\ \text{s.t. } &\phi_t \hat{m}_{t+1} + X_t = H_t + T_t + \phi_t a_t + (1+i_{s,t})\phi_t s_t - (1+i_{l,t})\phi_t \ell_t - b_t, \end{aligned} \tag{34}$$

which is the same as before except that the agent needs to pay or collect the debt. The agent's FM problem is identical to the previous section. Then, $1-\sigma$ fraction of agents will deposit $\hat{m}_{t+1}$, and σ fraction of agents will borrow loan from the bank. The DM value function is

$$V_t^b(m_t+d_t,0,\ell_t)=\alpha[u(q_t)-q_t]+W_t(m_t+d_t,0,\ell_t,0),$$

where $q_t=\min\{q^*,\bar{b}_t+(m_t+d_t)\phi_t\}$ and $d_t=\ell_t$. Given $\bar{b}_t$, solving equilibrium yields

$$z_t=\begin{cases}\dfrac{z_{t+1}}{1+i}\left\{\dfrac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u'(z_{t+1}+\bar{b}_{t+1})-1\right]+1\right\} & \text{if } z_{t+1}+\bar{b}_{t+1}<q^* \\ \dfrac{z_{t+1}}{1+i} & \text{if } z_{t+1}+\bar{b}_{t+1}\geq q^*,\end{cases} \tag{35}$$

where $z_{t+1}=(1-\sigma+\sigma\chi)\phi_{t+1}M_{t+1}/(\sigma\chi)$.

Next, I am going to endogenize the debt limit. The buyer cannot commit to pay back the unsecured debt bt while she cannot default on the bank debt. If the buyer reneges, she is captured with probability μ. The punishment for a defaulter is permanent exclusion from the DM trade, but she can still produce for herself in the CM. The value of autarky is $\underline{\mathrm{W}}(0,0,0,0)=[U(X^*)-X^*+T]/(1-\beta)$. The incentive condition for voluntary repayment is

$$\underbrace{-b_t+W_t(a_t,d_t,\ell_t,0)}_{\text{value of honoring debts}}\geq\underbrace{(1-\mu)W_t(a_t,d_t,\ell_t,0)+\mu\underline{\mathrm{W}}(a_t,d_t,\ell_t,0)}_{\text{value of not honoring debts}}.$$

One can write the debt limit $\bar{b}_t$ as $b_t\leq\bar{b}_t\equiv\mu W_t(0,0,0,0)-\mu\underline{\mathrm{W}}(0,0,0,0)$. Recall the CM value function. Using the solution of FM, we can rewrite the buyer's CM value function as

$$\begin{aligned}W_t(0,0,0,0)=&\,U(X^*)-X^*+T_t+\beta W_{t+1}(0,0,0,0)\\&+\max_{\hat{m}_{t+1}}\{-\phi_t\hat{m}_{t+1}+\beta\alpha\sigma[u(q_{t+1})-q_{t+1}]+\beta\phi_{t+1}\hat{m}_{t+1}\},\end{aligned}$$

where $q_{t+1}=\min\{q^*,z_{t+1}+\bar{b}_{t+1}\}$. Substituting $W_t(0,0,0,0)=\bar{b}_t/\mu+\underline{\mathrm{W}}(0,0,0,0)$ and $\hat{m}_{t+1}=M_{t+1}$ yields

$$\frac{\bar{b}_t}{\mu}=-\phi_tM_{t+1}+\beta\alpha\sigma[u(z_{t+1}+\bar{b}_{t+1})-z_{t+1}-\bar{b}_{t+1}]+\frac{\beta\bar{b}_{t+1}}{\mu}+\beta\phi_{t+1}M_{t+1},$$

where M_{t+1} and z_{t+1} solve (35). Rearranging terms yields

$$\bar{b}_t = \beta\bar{b}_{t+1} + \frac{\chi\mu\sigma[-\gamma z_t + \beta z_{t+1}]}{1-\sigma+\sigma\chi} + \beta\alpha\mu\sigma S(z_{t+1} + \bar{b}_{t+1}) \tag{36}$$

where $S(\cdot)$ is the buyer's trade surplus and defined as

$$S(z_{t+1} + \bar{b}_{t+1}) \equiv \begin{cases} u(z_{t+1} + \bar{b}_{t+1}) - z_{t+1} - \bar{b}_{t+1} & \text{if } z_{t+1} + \bar{b}_{t+1} < q^* \\ u(q^*) - q^* & \text{if } z_{t+1} + \bar{b}_{t+1} \geq q^*. \end{cases}$$

The equilibrium can be collapsed into a dynamic system satisfying (35)-(36).

Stationary Equilibrium In the stationary equilibrium, (35) becomes

$$-\frac{i\chi}{\alpha(1-\sigma+\sigma\chi)} + u'(q) - 1 \leq 0, = \text{ if } z > 0 \tag{37}$$

and (36) becomes

$$(1-\beta)\bar{b} = \begin{cases} \dfrac{\chi\mu\sigma[\beta-\gamma]z}{1-\sigma+\sigma\chi} + \beta\alpha\mu\sigma[u(z+\bar{b}) - z - \bar{b}] & \text{if } z + \bar{b} < q^* \\ \dfrac{\chi\mu\sigma[\beta-\gamma]z}{1-\sigma+\sigma\chi} + \beta\alpha\mu\sigma[u(q^*) - q^*] & \text{if } z + \bar{b} \geq q^*, \end{cases} \tag{38}$$

where $q = \min\{z + \bar{b}, q^*\}$. The stationary equilibrium solves the above two equations, and it falls into one of the three cases: the pure money equilibrium, the pure credit equilibrium, and the money-credit equilibrium. First, if no one can capture the buyer after she reneges, $\mu = 0$, the unsecured credit is not feasible, $\bar{b} = 0$. In this case, the equilibrium will be the pure money equilibrium. Second, when $\bar{b}$ solving (38) satisfies $u'(\bar{b}) < 1 + i\chi/[\alpha(1-\sigma+\sigma\chi)]$ then money is not valued, $z = 0$. We have the pure credit equilibrium in this case. Third, if solutions of (37)-(38), $(z, \bar{b})$ are strictly positive then we have the money-credit equilibrium.

The debt limit at the stationary equilibrium, $\bar{b}$, is a fixed point satisfying $\bar{b} = \Omega(\bar{b})$

where

$$\Omega(\bar{b})=\begin{cases}\frac{\mu\sigma\alpha}{\rho}[u(\tilde{q})-\tilde{q}]-\frac{i\mu\sigma\chi}{\rho(1-\sigma+\sigma\chi)}(\tilde{q}-\bar{b}) & \text{if } \tilde{q}>\bar{b}\geq 0\\ \frac{\mu\sigma\alpha}{\rho}[u(\bar{b})-\bar{b}] & \text{if } q^{*}>\bar{b}\geq\tilde{q}\\ \frac{\mu\sigma\alpha}{\rho}[u(q^{*})-q^{*}] & \text{if } \bar{b}\geq q^{*}\end{cases} \tag{39}$$

where $\tilde{q}$ solves $u'(\tilde{q})=1+i\chi/[\alpha(1-\sigma+\sigma\chi)]$ and $\rho\equiv 1/\beta-1$.

The DM consumption q_s is determined by $q_s=\min\{q^*,\max\{\tilde{q},\bar{b}\}\}$. Money and credit coexist if and only if $0<\bar{b}<\tilde{q}$, which holds when $0<\mu<\min\{1,\tilde{\mu}\}$, where

$$\tilde{\mu}\equiv\frac{\rho}{\alpha\sigma[u(\tilde{q})/\tilde{q}-1]}.$$

The DM consumption is decreasing in i in the stationary monetary equilibrium.

Cycles Consider the dynamics of equilibria where money and credit coexist. I claim the main results from Section 3 - lowering the reserve requirement can induce endogenous cycles - still hold even after unsecured credit is introduced. It is clear that the standard treatment $f'(z_s)<-1$ from Azariadis (1993) cannot be used here because now the equilibrium consists of a system of equations. Instead, I apply the approach used in Proposition 2 and 4. For compact notation, let $w_j\equiv z_j+\bar{b}_j$. The following proposition establishes the conditions for a two-period cycle, a three-period cycle, and chaotic dynamics.

Proposition 7 (**Monetary Cycles with Unsecured Credit**). *There exists a two-period cycle of money and credit with $w_1<q^*<w_2$ if $\chi\in(0,\bar{\chi}_c)$ where $(\bar{\chi}_c,\bar{q},z_1)$ solves*

$$\bar{\chi}_c=\frac{(1-\sigma)\alpha\left[u'\left(\bar{q}\right)-1\right]}{(1+i)^2-1-\sigma\alpha\left[u'\left(\bar{q}\right)-1\right]} \tag{40}$$

$$\{q^*-(1+i)z_1\}=\frac{\sigma\alpha\mu\{\beta^2S(q^*)+\beta S(\bar{q})\}+\frac{\bar{\chi}_c\mu\sigma z_1\{1-(1+i)^2\}}{\beta(1-\sigma+\sigma\bar{\chi}_c)}}{(1-\beta^2)} \tag{41}$$

$$z_1(i-\rho)+(1+\rho)\bar{q}-\sigma\alpha\mu S(q^*)=q^* \tag{42}$$

There exists a three-period cycle of money and credit with $w_1<q^*<w_2<w_3$, if

$\chi \in (0, \hat{\chi}_c)$ *where* $(\hat{\chi}_c, \bar{q}, z_1)$ *solves*

$$\hat{\chi}_c = \frac{(1-\sigma)\alpha\left[u'\left(\bar{q}\right)-1\right]}{(1+i)^3 - 1 - \sigma\alpha\left[u'\left(\bar{q}\right)-1\right]} \tag{43}$$

$$\{q^* - (1+i)z_1\} = \frac{\sigma\alpha\mu\{(\beta^2+\beta^3)S(q^*) + \beta S(\bar{q})\} + \frac{\hat{\chi}_c\mu\sigma z_1\{1-(1+i)^3\}}{\beta(1-\sigma+\sigma\hat{\chi}_c)}}{(1-\beta^3)} \tag{44}$$

$$z_1(i-\rho) + (1+\rho)\bar{q} - \sigma\alpha\mu S(q^*) = q^* \tag{45}$$

Proof. See Appendix A. ■

5 Calibrated Examples

5.1 Parameters and Targets

In this section, I calibrate the model with unsecured credit from Section 4 using U.S. data from 1976-2008. For monetary aggregates, I use M1 adjusted for retail sweep accounts, following Aruoba, Waller and Wright (2011) and Venkateswaran and Wright (2014). Following Krueger and Perri (2006) and Bethune, Choi and Wright (2020), I use the revolving consumer credit series as unsecured credit.

The following functional forms are used for parametrization. The utility functions are

$$U(X) = B\log(X), \qquad u(q) = \frac{q^{1-\eta}}{1-\eta}$$

implying $X^* = B$ and the DM cost function is given as $c(q) = q$. I assume the buyer makes a take-it-or-leave-it offer to the seller in the DM trade, implying $\lambda(q) = q^{-\eta} - 1$. The matching function in the DM is $\mathcal{M}(\mathcal{B}, \mathcal{S}) = \frac{\mathcal{B}\mathcal{S}}{\mathcal{B}+\mathcal{S}}$, where $\mathcal{B}$ and $\mathcal{S}$ denote the measure of buyers and sellers, respectively. This implies $\alpha = \mathcal{M}(\sigma, 1-\sigma)/\sigma = 1-\sigma$ and $\alpha_s = \mathcal{M}(\sigma, 1-\sigma)/(1-\sigma) = \sigma$.

First, as standard, I calibrate the model at an annual frequency. The discount rate is set to $\beta = 0.9709$ so that the real annual interest rate ρ is 3%. The other targets are calculated using US data for 1976-2008. Based on the average required reserve-to-deposit ratio for 1976-2008, I set the benchmark required reserve ratio to 8.24%.[6] The benchmark nominal interest rate i is set to 0.0579, matching the average

[6] This ratio is computed by dividing the required reserves by the deposit component of sweep-

annualized rate of 5.79%. For normalization, I set the fraction of buyers to $\sigma = 0.5$. The parameters (B, η) are calibrated to match money demand. In the model, money demand is expressed as real balances of money relative to output:

$$Z \equiv \frac{z}{Y} = \frac{z}{B + \sigma\alpha q}$$

where Y represents real output. The elasticity of z/Y with respect to i is:

$$\frac{\partial \log(Z)}{\partial \log(i)} = \frac{i}{Z}\frac{\partial Z}{\partial i}.$$

The parameter B is calibrated to match the average money stock-to-GDP ratio of 0.1473, while η is set to match the elasticity of z/Y with respect to i. This target elasticity is estimated by:

$$\log(Z_t) = \beta_0 + \beta_1 \log(i_t) + \varepsilon_t,$$

which gives $\beta_1 = -0.0692$.[7] The monitoring probability μ is calibrated to match the unsecured credit-to-output ratio of 0.0428, represented in the model as $\sigma\alpha b/Y$. I also recalibrate B and σ for a quarterly frequency. The benchmark quarterly nominal interest rate is set to $i = 0.0142$, corresponding to the annual rate of 5.79% through $(1.0579)^{\frac{1}{4}} - 1 = 0.0142$. The quarterly real interest rate is set to 0.74%, corresponding to the annual rate of 3% through $(1.03)^{\frac{1}{4}} - 1 = 0.0074$. I set B to one-quarter of its annual value B_y, maintain the same η and μ, and recalibrate σ to match 0.5907 (4×0.1477). Given calibrated parameters and the benchmark interest rate, we can report $\bar{\chi}_c$ and $\hat{\chi}_c$ as defined in (40) and (43). Table 1 presents the calibration results. At the benchmark interest rate, the thresholds are $\bar{\chi}_c = 1.80\%$ and $\hat{\chi}_c = 1.16\%$ in the annual model. In the quarterly model $\bar{\chi}_c = 5.93\%$ and $\hat{\chi}_c = 3.91\%$ which are below the benchmark reserve requirement. While χ at or below the threshold generates cycles, values above the threshold do not necessarily exclude the possibility of cycles, as this is not a necessary and sufficient condition for their occurrence.

The DM curvature parameter η deserves some discussion. In monetary search models, higher-order cycles typically emerge as the curvature parameter increases. Many

adjusted M1, averaging 0.0824 for 1976-2008.

[7] This OLS estimate is similar to other estimates from the literature. Under similar specifications, Kejriwal, Perron and Yu (2022) report -0.0786 using 1976:Q2–2010:Q4 data, and Mogliani and Urga (2018) report -0.11 using 1976-2013 data, respectively

Table 1: Model parametrization

Description	Value	Data	Model	Target/Reason
Annual				
Rate of time discount, ρ_y	0.03			Annual real interest rate of 3%
Prob. of being a buyer, σ	0.5			Normalization
CM utility level, B_y	2.7302	0.1477	0.1477	Avg. z/Y_y
Monitoring probability, μ	0.1364	0.0428	0.0428	Avg. $\sigma\alpha b/Y_y$
Parameter of $u(\cdot)$, η	0.3123	-0.0692	-0.0692	Elasticity of z/Y_y wrt i
Thresholds				
$\bar{\chi}_c$	1.80%			
$\hat{\chi}_c$	1.16%			
Quarterly				
Rate of discount, ρ	0.0074			$(1+\rho_y)^{1/4}-1$
Prob. of being a buyer, σ	0.0714	0.5907	0.5907	Avg. z/Y
CM utility level, B	0.6825			$B = B_y/4$
Monitoring probability, μ	0.1364			Annual calibration
Parameter of $u(\cdot)$, η	0.3123			Annual calibration
Thresholds				
$\bar{\chi}_c$	5.93%			
$\hat{\chi}_c$	3.91%			

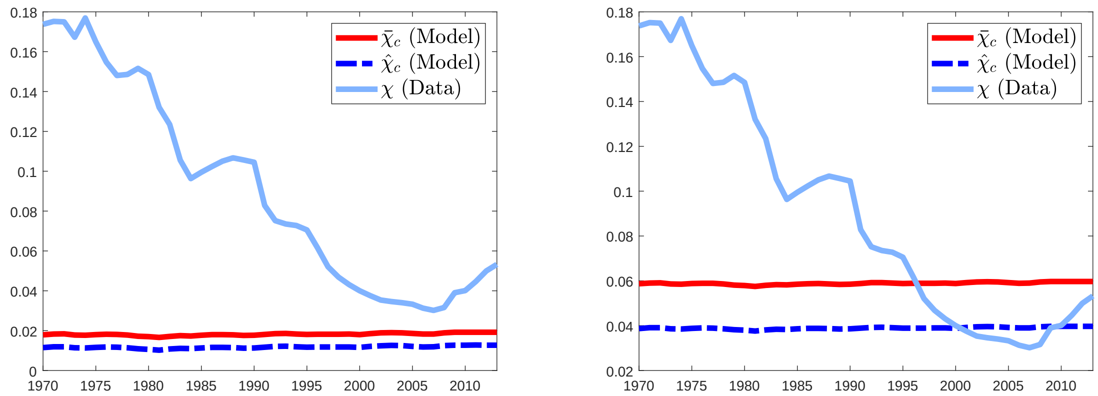
Note: To distinguish CM utility level parameter B and discount rate ρ between annual model and quarterly model, I denote their annual values as B_y and ρ_y

cases exhibiting endogenous boom-bust dynamics have curvature parameters greater than 1[8]. However, calibrated values of the DM curvature parameter in monetary search models usually fall below 1. Lagos and Wright (2005) reports values between 0.27 and 0.48 using 1959-2000 data. Venkateswaran and Wright (2014) finds values of 0.63 and 0.39, both yielding similar quantitative results. Berentsen, Huber and Marchesiani (2015) documents values of 0.377 and 0.389, while Bethune et al. (2020) reports 0.67 under DM price posting and 0.58 under DM sequential search. This paper finds a value of 0.3123, which is consistent with the literature. When χ is below the threshold, the model can exhibit endogenous boom-bust dynamics at reasonable parameter values when we incorporate the fractional reserve banking.

Given the parameterization, we compare the thresholds with actual data.[9] Figure 6 presents the model-implied thresholds for cycles. Figures 6a and 6b show thresholds for two-period cycles ($\bar{\chi}_c$) and three-period cycles with chaotic dynamics ($\hat{\chi}_c$) using annual and quarterly parameterization, respectively. The annual model shows that $\bar{\chi}_c$ falls below χ after 2000. In the quarterly model, $\bar{\chi}_c$ has remained below χ since 1981, while $\hat{\chi}_c$ dropped below χ in 1991. These findings suggest the economy could

[8]See He, Wright and Zhu (2015), Gu, Han and Wright (2019) and Altermatt et al. (2023) for related numerical exercises and discussion.

[9]To compute χ, I divide required reserves by the deposit component of sweep-adjusted M1.



(a) Annual Model

(b) Quarterly Model

Figure 6: Model Implied Thresholds

exhibit endogenous fluctuations and chaotic dynamics that stem from fractional reserve banking, independent of exogenous shocks and fundamental changes. Incorporating fractional reserve banking allows the model to feature cyclical behavior, indicating that this volatility channel should be considered alongside traditional economic fluctuations induced by exogenous shocks.

5.2 Endogenous Cycles and Limit Cycles

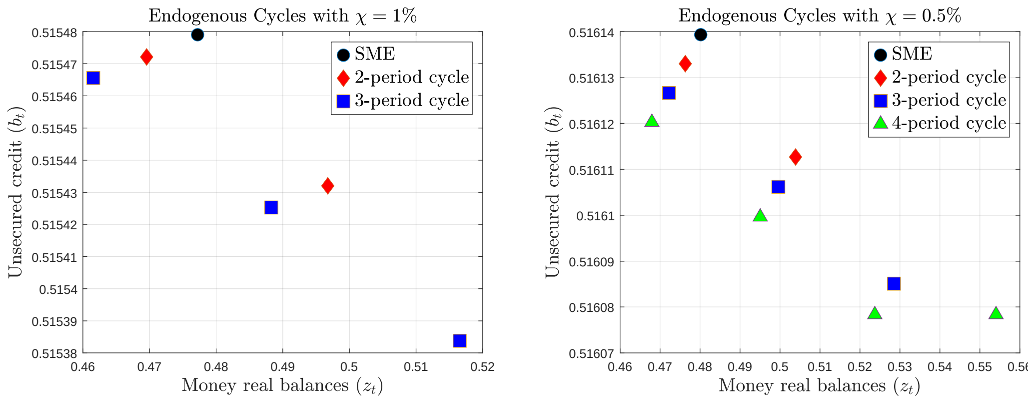
5.2.1 Endogenous Cycles

Recall that the equilibrium is a dynamic system satisfying (35)-(36):

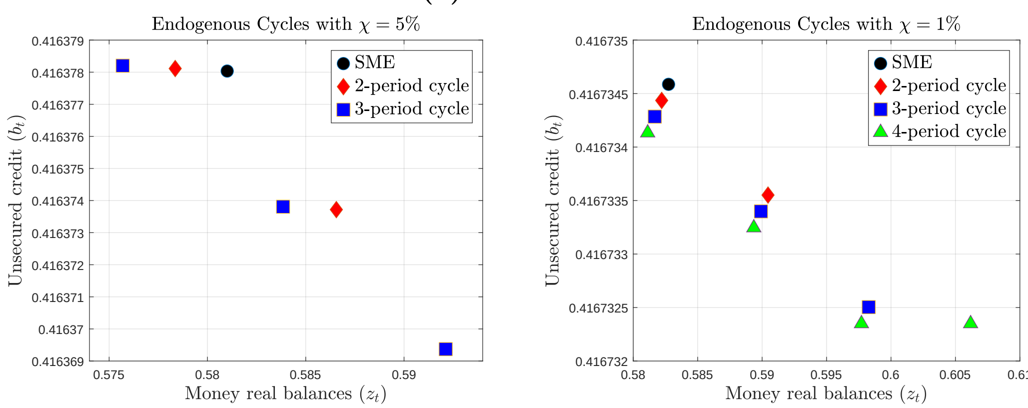
$$z_t = \begin{cases} \frac{z_{t+1}}{1+i}\left\{\frac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u'(z_{t+1}+\bar{b}_{t+1})-1\right]+1\right\} & \text{if } z_{t+1}+\bar{b}_{t+1}<q^* \\ \frac{z_{t+1}}{1+i} & \text{if } z_{t+1}+\bar{b}_{t+1}\geq q^*, \end{cases}$$

$$\bar{b}_t = \beta\bar{b}_{t+1} + \frac{\chi\mu\sigma[-\gamma z_t+\beta z_{t+1}]}{1-\sigma+\sigma\chi} + \beta\alpha\mu\sigma S(z_{t+1}+\bar{b}_{t+1})$$

Given calibrated parameters, different types of cycles can emerge depending on the reserve requirement level. Figure 7 provides examples of these endogenous cycles, with Figure 7a showing the annual model and Figure 7b showing the quarterly model. In the annual model, χ needs to be set below the empirical reserve requirement to generate endogenous cycles, while the quarterly model can generate such cycles even with the



(a) Annual Model



(b) Quarterly Model

Figure 7: Calibrated Examples I

historical value of χ. This suggests that the inherent instability of fractional reserve banking may contribute to economic fluctuations at quarterly frequency but not at annual frequency. In all examples, the lowest values (z_1, b_1) are lower than their steady state values (z_s, b_s), while all the others are greater than their steady state values. These example cycles feature fluctuations around the steady state.

One notable observation is that the fluctuations in b_t are very small compared to the fluctuations in z_t. To understand this, one can take the derivative of (36) with respect to b_{t+1}, evaluated at the steady state, assuming money balances are given:

$$\left.\frac{\partial b_t}{\partial b_{t+1}}\right|_{b_{t+1}=b_s} = \beta\{1 + \sigma\alpha\mu[u'(q_s) - 1]\} = \beta\left\{1 - \sigma\alpha\mu + \sigma\alpha\mu u'(q_s)\right\} > 0 \tag{46}$$

where $\chi i = (1 - \sigma + \sigma\chi)\alpha[u'(q_s) - q_s]$. To generate endogenous credit cycles solely

Table 2: Comparison of three different economies.

Type	$\mathbf{z_t} =$
Fractional Reserve Banking	$\frac{z_{t+1}}{1+i}\left\{\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_{t+1}+b_{t+1})+1\right\}$
100% Reserve Banking	$\frac{z_{t+1}}{1+i}\{\alpha L(z_{t+1}+b_{t+1})+1\}$
No Banking	$\frac{z_{t+1}}{1+i}\{\sigma\alpha L(z_{t+1}+b_{t+1})+1\}$
Type	$\mathbf{b_t} =$
Fractional Reserve Banking	$\beta\bar{b}_{t+1}+\frac{\chi\mu\sigma[-\gamma z_t+\beta z_{t+1}]}{1-\sigma+\sigma\chi}+\beta\sigma\alpha\mu S(z_{t+1}+\bar{b}_{t+1})$
100% Reserve Banking	$\beta\bar{b}_{t+1}+\mu\sigma[-\gamma z_t+\beta z_{t+1}]+\beta\sigma\alpha\mu S(z_{t+1}+\bar{b}_{t+1})$
No Banking	$\beta\bar{b}_{t+1}+\mu[-\gamma z_t+\beta z_{t+1}]+\beta\sigma\alpha\mu S(z_{t+1}+\bar{b}_{t+1})$

Note: When the buyer makes a take-it-or-leave-it offer to the seller in the DM trade, we have $L(q) = u'(q) - 1$ and $S(q) = u(q) - q$.

from the credit market side, we need $\partial b_t/\partial b_{t+1}|_{b_{t+1}=b_s} < -1$.[10] However, this cannot happen in this model. Therefore, the cycles in this model are either from the money real balance dynamics channel or from more complicated joint dynamics of money and credit.

To better understand how these cycles work, we can compare with other examples: 100% reserve banking and the economy without banking. 100% reserve banking is a special case with $\chi = 1$, which can be written as:

$$z_t = \begin{cases} \frac{z_{t+1}}{1+i}\left\{\alpha\left[u'(z_{t+1}+\bar{b}_{t+1})-1\right]+1\right\} & \text{if } z_{t+1}+\bar{b}_{t+1} < q^* \\ \frac{z_{t+1}}{1+i} & \text{if } z_{t+1}+\bar{b}_{t+1} \geq q^*, \end{cases} \tag{47}$$

$$\bar{b}_t = \beta\bar{b}_{t+1} + \mu\sigma[-\gamma z_t + \beta z_{t+1}] + \beta\alpha\mu\sigma S(z_{t+1}+\bar{b}_{t+1}) \tag{48}$$

We can also consider a model without banking à la Gu et al. (2016). More details of

[10]See Gu et al. (2013) and Gu (2023) for the assessment of endogenous credit cycles and Azariadis (1993) for textbook treatment on endogenous cycles.

the model environment and the agents' problem can be found in Appendix B.[11] The equilibrium without banks solves (49)-(50)

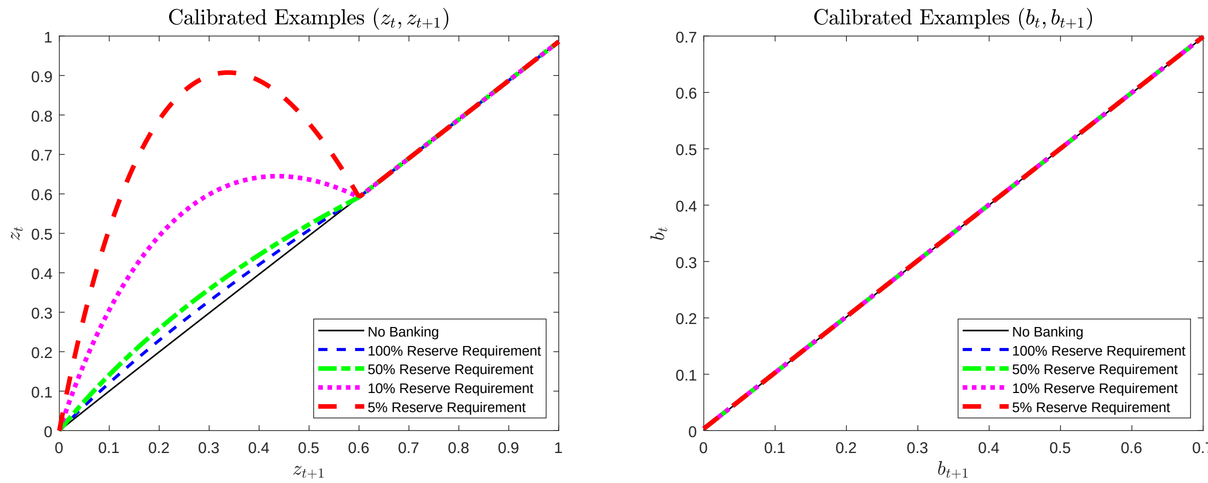
$$z_t = \begin{cases} \frac{z_{t+1}}{1+i}\left\{\sigma\alpha\left[u'(z_{t+1}+\bar{b}_{t+1})-1\right]+1\right\} & \text{if } z_{t+1}+\bar{b}_{t+1}<q^* \\ \frac{z_{t+1}}{1+i} & \text{if } z_{t+1}+\bar{b}_{t+1}\geq q^*, \end{cases} \tag{49}$$

$$\bar{b}_t = \beta\bar{b}_{t+1} + \mu[-\gamma z_t + \beta z_{t+1}] + \beta\mu\sigma\alpha S(z_{t+1}+\bar{b}_{t+1}) \tag{50}$$

Table 2 summarizes the model comparison. In terms of real balance dynamics, the key differences can be captured by multipliers on $L(\cdot)$. In the model with banking, it is multiplied by $\sigma\alpha$ but under 100% reserve banking it is multiplied by α. This is because having banking and inside money in the environment allows agents to offset the idiosyncratic risk σ. Without banking, agents would hold idle balances due to idiosyncratic risk σ, but 100% reserve banking allows agents to insure against this idiosyncratic risk through inside money and reallocation of liquidity. This is a case similar to Berentsen et al. (2007) except that this model has unsecured credit. However, under fractional reserve banking, $L(\cdot)$ is multiplied by $\alpha\frac{1-\sigma+\sigma\chi}{\chi}$. Lowering χ increases this multiplier so that it increases the aggregate money supply more than in the 100% reserve banking case because it creates more inside money.

While monetary economies often exhibit different kinds of dynamic equilibria stemming from the self-fulfilling nature of money, N-periodic cycles cannot be found numerically in cases of no banking, 100% reserve banking, and fractional reserve banking with high reserve requirements. Figure 8 illustrates the differences between each case. Figure 8a provides examples of (35) where b_t is fixed at 0.4, and Figure 8b provides examples of (36) where z_t and z_{t+1} are fixed at 0.4. Under calibrated parameters, Figure 8a remains similar to Figure 2b of Section 3 even after introducing unsecured credit. It is nonmonotonic in general. Adding 100% reserve banking to the no-banking case or lowering the reserve requirement shifts up the nonlinear branch in Figure 8a. This occurs because adding banks or lowering the reserve requirement amplifies the liquidity premium, as it enables banks to generate more liquidity through lending. This amplification enhances the backward-bending feature, leading to endogenous cycles. Unlike Figure 8a, the dynamics of b_t in Figure 8b are virtually invariant to χ and almost

[11] There are some differences compared to Gu et al. (2016): (1) In Gu et al. (2016), agents' types in DM are fixed, but not here; (2) In addition to the buyers' problem of whether they honor their repayment of credit, Gu et al. (2016) also consider whether they honor their public (tax) obligations.



(a) Money Real Balance Dynamics

(b) Unsecured Credit Dynamics

Figure 8: Calibrated Examples II

linear, showing no features related to endogenous cycles.

This result relates to previous work showing that higher-order cycles typically emerge as the curvature parameter increases in monetary search models. In discrete-time monetary search models, endogenous cycles usually emerge due to the backward-bending feature of nonlinear difference equations. Increasing the curvature parameter of DM utility enhances this backward-bending feature, making cycles more likely. However, as shown above, lowering the reserve requirement has a similar effect. Consequently, endogenous cycles can emerge even without a high curvature parameter. While both enhance the backward-bending features, increasing η and lowering χ have different effects in general.

5.2.2 Limit Cycles

This section examines limit cycles and their cyclical volatility using the quarterly model.[12] To study limit cycles, I solve the model with 10,000 iterations and a 9,700-period transient phase. Figures 9 and 10 illustrate limit cycles under 8% and 4% reserve requirements, respectively. Figure 9 shows that the limit cycle under an 8% reserve requirement does not exhibit significant fluctuations over time. In contrast, Figure 10 shows that with a 4% reserve requirement, the limit cycle exhibits complex dynamics of z_t with substantial fluctuations over time. Unsecured credit does not show significant

[12]Recent applications of limit cycles and bifurcation theory include Beaudry, Galizia and Portier (2020), Asano, Shibata and Yokoo (2024), and Gu, Wang and Wright (2024).

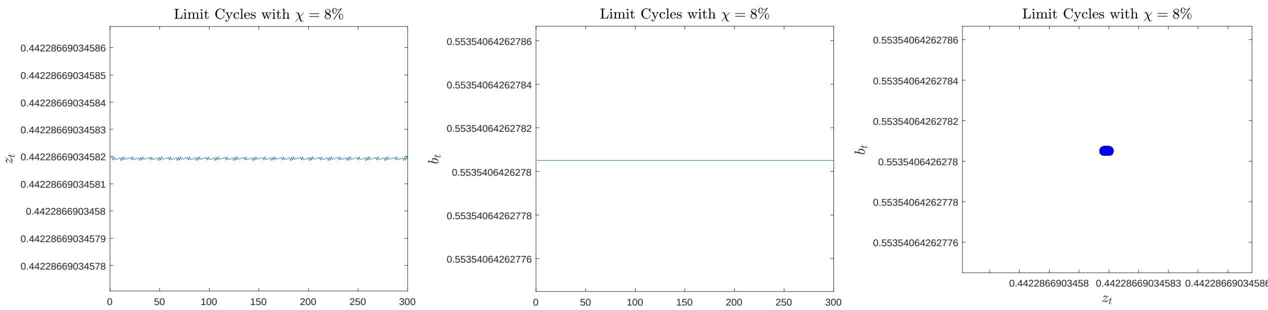


Figure 9: Limit Cycles under 8% Reserve Requirement

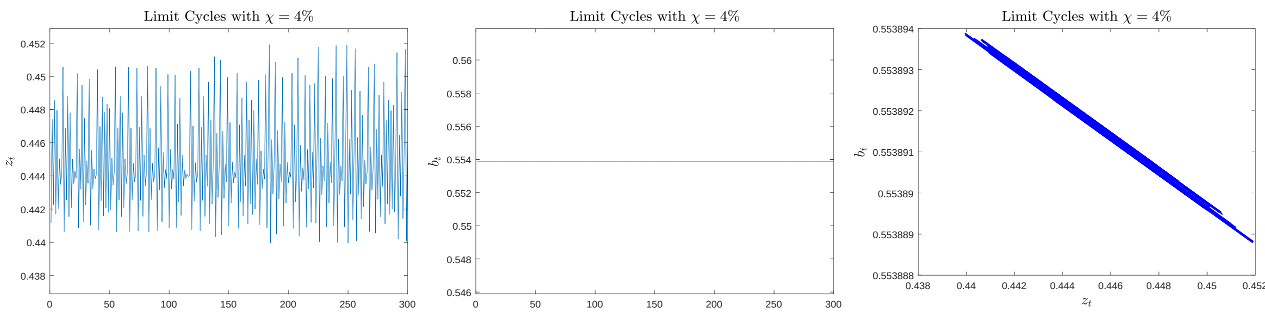


Figure 10: Limit Cycles under 4% Reserve Requirement

fluctuations under both reserve requirements.

These examples illustrate two important findings about fractional reserve banking: first, fractional reserve banking can generate limit cycles with varying magnitudes of fluctuation, and second, lower reserve requirements tend to produce larger economic fluctuations. Given these observations, the natural question becomes how to quantify these endogenous fluctuations systematically, beyond comparing limited numbers of illustrative examples. To assess the volatility of endogenous fluctuations under different reserve requirements, I fix all parameters and the central bank's policy except reserve requirement, and plot its bifurcation diagrams. Similar as before, given the parameters and policies, I compute the limit cycles by solving the model with 10,000 iterations and 9,000 transients.

Figure 11 presents the bifurcation diagrams. The left panel of Figure 11 plots a bifurcation diagram of real money balances, and the right panel of Figure 11 plots a bifurcation diagram of unsecured credit. It shows that when the reserve requirement is higher than 6%, real money balances and the economy tend to converge to a steady state (or stationary equilibrium). When the reserve requirement is around 6%, evident limit cycles emerge. When the reserve requirement is lower than 6%, lowering the reserve

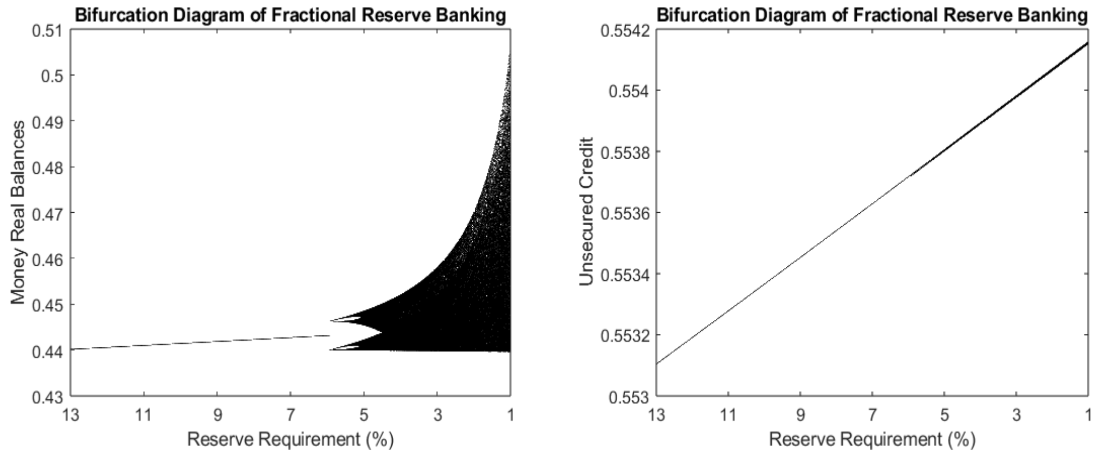


Figure 11: Bifurcation Diagram of Fractional Reserve Banking

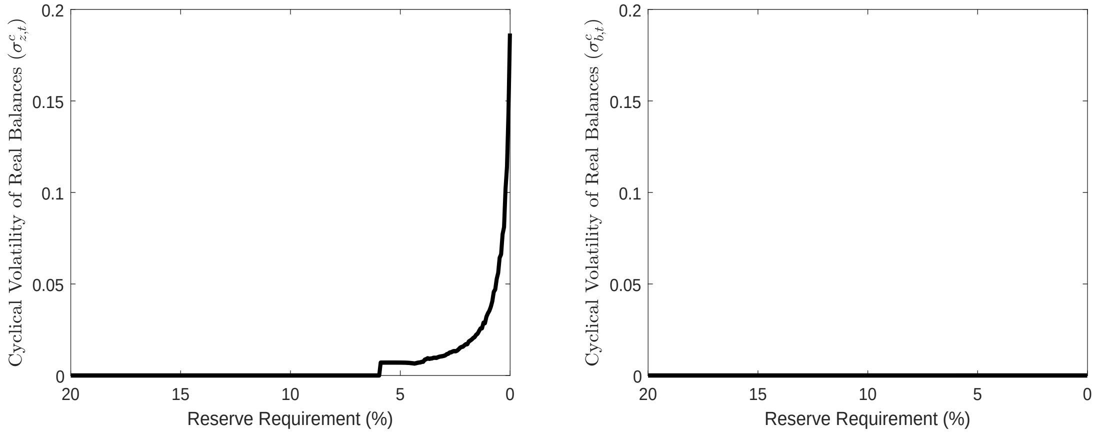


Figure 12: Cyclical Volatility of Limit Cycles

requirement monotonically increases the cyclical volatility of real money balances.

Given the bifurcation diagrams of Figure 11, it is clear that lowering the reserve requirement increases the magnitude of the limit cycles of real money balances. Unlike real money balances, the bifurcation diagrams do not clearly show whether lowering reserve requirements increases the volatility of unsecured credit fluctuations. To quantify how reserve requirements affect money and credit fluctuations, I compute the cyclical volatility of limit cycles under different reserve requirements. I measure cyclical volatility by calculating the standard deviation of filtered logarithms—both for real money balances and the real value of unsecured credit. For filtering, I use the standard Hodrick-Prescott (HP) filter with a smoothing parameter of 1600, as I use the quarterly model.

Figure 12 plots the cyclical volatility of limit cycles for each given reserve requirement. The cyclical volatility of real money balances is zero or almost zero when χ is greater than 6%. When χ is lower than 6%, one can observe cyclical volatility of the limit cycles, and lowering χ monotonically increases the volatility. However, the cyclical volatility of unsecured credit is zero or almost zero whether χ is greater than 6% or lower than 6%.

Interest rate fluctuations over cycles As pointed out by a referee, I acknowledge three points: (1) The (gross) inflation ϕ_t/ϕ_{t+1} equals γ only in stationary equilibrium; (2) Similarly, the (gross) nominal interest rate equals γ/β only in stationary equilibrium; (3) The actual nominal interest rate fluctuates during cycles. To examine how the nominal interest rate fluctuates during cycles, recall (24) and rewrite as below using $i \equiv \gamma/\beta - 1$ and $\gamma = M_{t+1}/M_t$:

$$\frac{\phi_t}{\beta\phi_{t+1}} = \left\{ \frac{1-\sigma+\sigma\chi}{\chi} \alpha L(z_{t+1} + b_{t+1}) + 1 \right\}$$

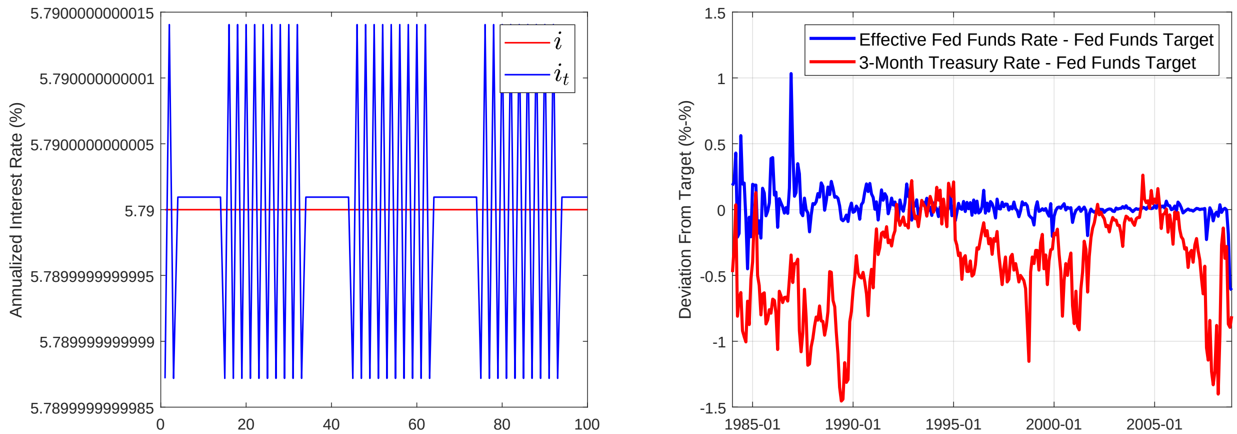
In nonstationary equilibria, the nominal interest rate on (fictitious) illiquid bonds i_t equals $\frac{\phi_t}{\beta\phi_{t+1}} - 1$. Thus, we can solve for the sequence of nominal interest rates:

$$i_t = \left\{ \frac{1-\sigma+\sigma\chi}{\chi} \alpha L(z_{t+1} + b_{t+1}) + 1 \right\} - 1$$

Figure 13a shows an example of nominal interest rate fluctuations from the limit cycle in the quarterly model under $\chi = 6\%$. The actual nominal interest rate in the model, i_t, fluctuates around the long-run target interest rate, i, during cycles.[13] In the model economy, pegging $\frac{\phi_t}{\beta\phi_{t+1}}$ fixes all allocations over time and eliminates endogenous fluctuations. Similar to this, Altermatt et al. (2023) show that a nominal interest rate peg eliminates endogenous fluctuations when money is the only liquid asset, but also show this does not hold with multiple liquid assets.

The model economy in this paper considers a policy of pegging money growth rate, which allows interest rate dynamics around the longer-term target rate. Allowing the fluctuation of nominal interest rates over cycles is not inconsistent with actual data. Figure 13b plots the differences between the federal funds rate and its target, and between the 3-month treasury rate and the federal funds target rate. The short-term

[13]Please note that this model is not intended to match interest rate dynamics quantitatively. Rather, it just demonstrates that, qualitatively, interest rates fluctuate over time around the target rates in both the model and data.



(a) Interest Rate Dynamics (Model) **(b)** Interest Rate Dynamics (Data)

Figure 13: Nominal Interest Rate Dynamics

rates fluctuate around the target rather than being strictly pegged to it.

5.3 A Fundamental Shock and Stochastic Cycles

Consider the linear CM production technology with aggregate productivity A_t. The CM consumption good X is produced by the technology $X_t = A_t N_t$ where N_t is labor input in the CM production. With the labor market clearing condition, $H_t = N_t$ where H_t is the agent's labor supply in CM, we have the real wage A_t instead of 1. Given this modification, we can rewrite the model as

$$\frac{z_t}{A_t} = E_t\left\{\frac{z_{t+1}}{A_{t+1}(1+i)}\left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L\left(\frac{z_{t+1}}{A_{t+1}}+\frac{\bar{b}_{t+1}}{A_{t+1}}\right)+1\right]\right\} \tag{51}$$

$$\frac{\bar{b}_t}{A_t} = \frac{\chi\sigma\mu}{1-\sigma+\sigma\chi}\left[-\gamma\frac{z_t}{A_t}+E_t\left\{\frac{\beta z_{t+1}}{A_{t+1}}\right\}\right]+\beta E_t\left\{\frac{\bar{b}_{t+1}}{A_{t+1}}+\mu\sigma\alpha S\left(\frac{z_{t+1}}{A_{t+1}}+\frac{\bar{b}_{t+1}}{A_{t+1}}\right)\right\} \tag{52}$$

where A_t be a sequence of independent and identically distributed (i.i.d.) random variables following a log-normal distribution with a mean value of 1 i.e., $\log A_t \sim N(0, \sigma_A)$ so that $E_t[A_{t+1}] = 1$. Appendix C provides the derivation of the stochastic equilibrium (51)-(52). Given the random shock A_t, we can simulate the time series of the dynamic system (51)-(52).

Figure 14 shows examples using the modified quarterly model where $\sigma_A = 0.03$. The

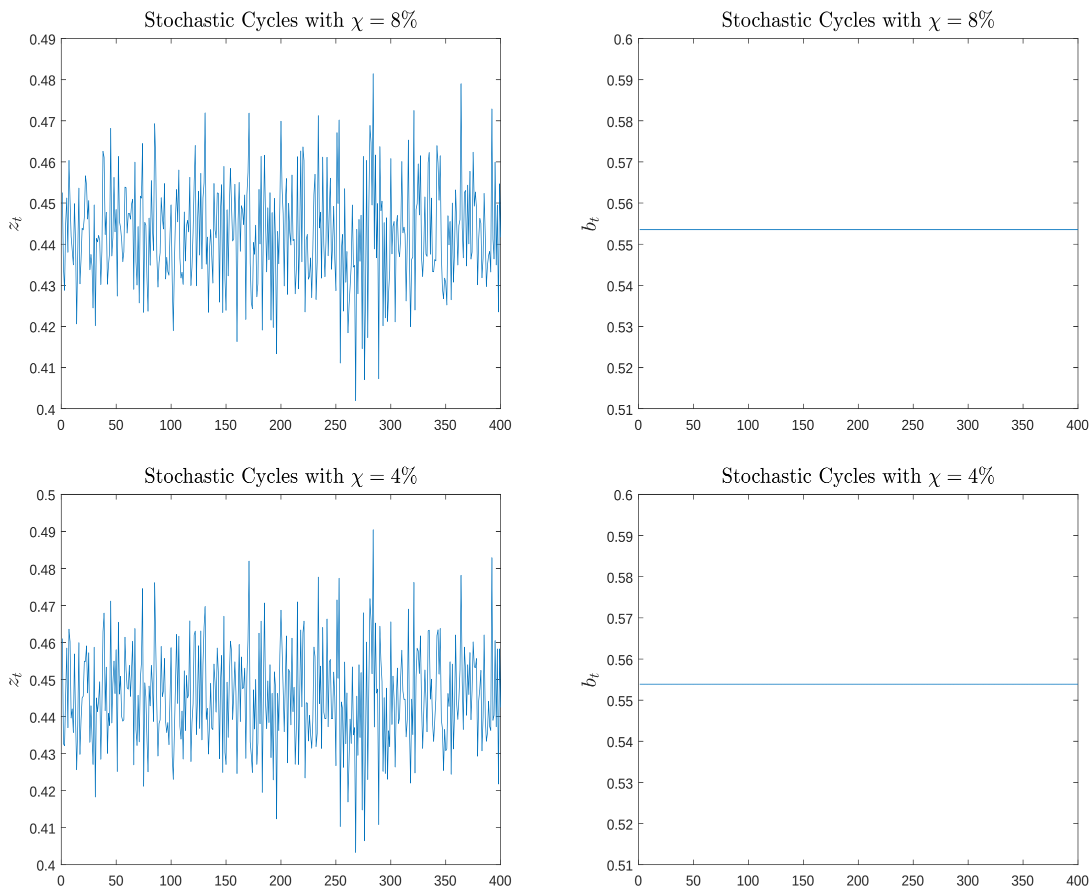


Figure 14: Stochastic Cycle: Examples

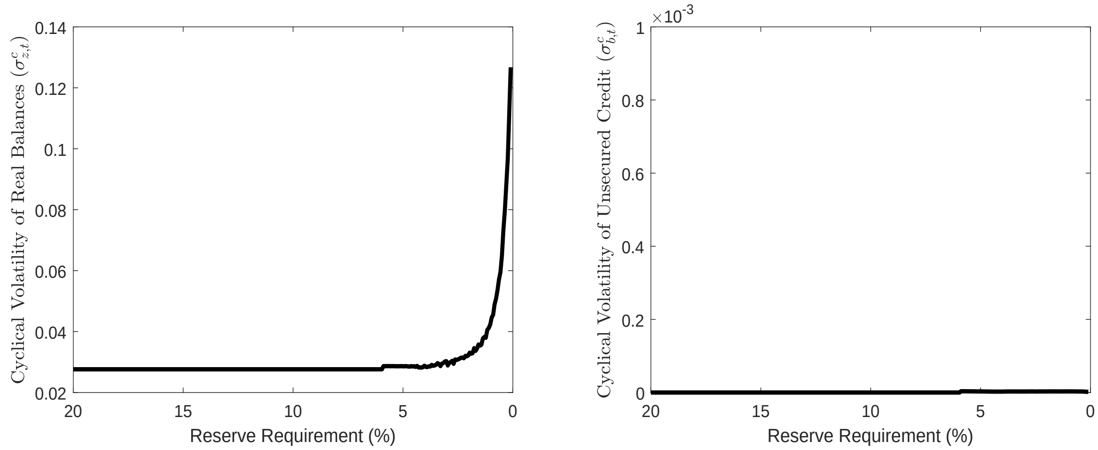


Figure 15: Cyclical Volatility of Stochastic Cycles

top panels present the result with $\chi = 8\%$ and the bottom panels provide the example with $\chi = 4\%$. When $\chi = 8\%$, money real balances z_t exhibit large fluctuations while credit remains stable. In contrast, when $\chi = 4\%$, z_t shows larger variations than in the $\chi = 8\%$ case. This example shows that reducing the reserve requirements not only generates cycles but also amplifies their magnitude.

Figure 15 plots the cyclical volatility of limit cycles for each given reserve requirement. The cyclical volatility of real money balances is almost constant when χ is greater than 6%. When χ is lower than 6%, one can observe an increase in cyclical volatility, and lowering χ monotonically increases the volatility. However, as with limit cycles, the cyclical volatility of unsecured credit is zero or almost zero regardless of whether χ is greater than 6% or not.

Given the numerical examples provided above, one can ask whether there is any possibility of distinguishing between dynamics driven by external shocks and those driven by the system's inherent instability. Motivated by Beaudry et al. (2020), we can perform a simple exercise to separate these two sources of dynamics as follows: First, we treat the limit cycles without external shocks as the system's inherent instability. Second, given the dynamic system, we can choose the parameter of the external shock to match the volatility of economic fluctuations. This allows us to quantify the contribution of each source to the overall economic fluctuations.

More specifically, we can choose σ_A to match the cyclical volatility from the data. Given the parameters, we can compute the cyclical volatility of limit cycles without external shocks. Then one can compute what fraction of fluctuations can be explained by

Table 3: Cyclical Volatility of Cycles

Period	χ	i	Cyclical Volatility of Real Balances (σ_z^c) Data	Model w/o shock	Model with shock	σ_A	$\frac{\sigma_z^c \text{ without shocks}}{\sigma_z^c \text{ with shock}}$
1971-2010	9.14%	5.51%	0.0315	7.8563e-14	0.0315	0.0327	0.00%
1971-1980	16.11%	6.80%	0.0269	7.9601e-14	0.0269	0.0280	0.00%
1981-1990	10.81%	8.43%	0.0402	7.9229e-14	0.0402	0.0418	0.00%
1991-2000	6.19%	4.69%	0.0288	7.8446e-14	0.0288	0.0299	0.00%
2001-2010	3.47%	2.13%	0.0290	0.0037	0.0290	0.0299	12.68%
Numerical Examples	10%	2.13%		7.8217e-14	0.0288	0.0299	0.00%
	5%	2.13%		0.0026	0.0290	0.0299	8.97%
	1%	2.13%		0.0131	0.0313	0.0299	41.85%
	0.1%	2.13%		0.0529	0.0620	0.0299	85.32%

Note: Cyclical volatility from data is calculated using quarterly sweep-adjusted M1 with the Hodrick-Prescott filter. The reserve requirement χ is calculated by dividing required reserves by the deposit component of sweep-adjusted M1.

the inherent instability of the system. These assessments are based on 1971-2010, and its sub-periods (1971-1980, 1981-1990, 1991-2000, and 2001-2010). Table 3 summarizes the results.

Except for 2001-2010, the percentage of fluctuations explained by limit cycles is close to zero. But for 2001-2010, it accounts for 12.68% of the cyclical volatility, which is significant. The differences are mainly due to the lower reserve requirement. Table 3 also provides some numerical experiments that fix σ_A and i, and compute volatility under 10%, 5%, 1%, and 0.1% reserve requirements. The results show that lowering the reserve requirement increases volatility both with and without shocks. Also, the contribution of the inherent instability of the system increases as the reserve requirement decreases.

5.4 News Shocks

This section explores the role of reserve requirements in the dynamics resulting from news about future changes in monetary policy. To analyze the impact of such news, I follow Gu et al. (2019) who study the effect of news in the economy where liquidity plays a role. Gu et al. (2019) show that the response to the announcement can be complicated, and it highly depends on parameters. Using the calibrated quarterly model, this section examines how reducing the reserve requirement can complicate the effects of the monetary policy announcements.

Let the central bank change i permanently, from i_0 to i_T.[14] Suppose news about

[14]The permanent change in i implies a permanent change in the rate of monetary expansion, γ,

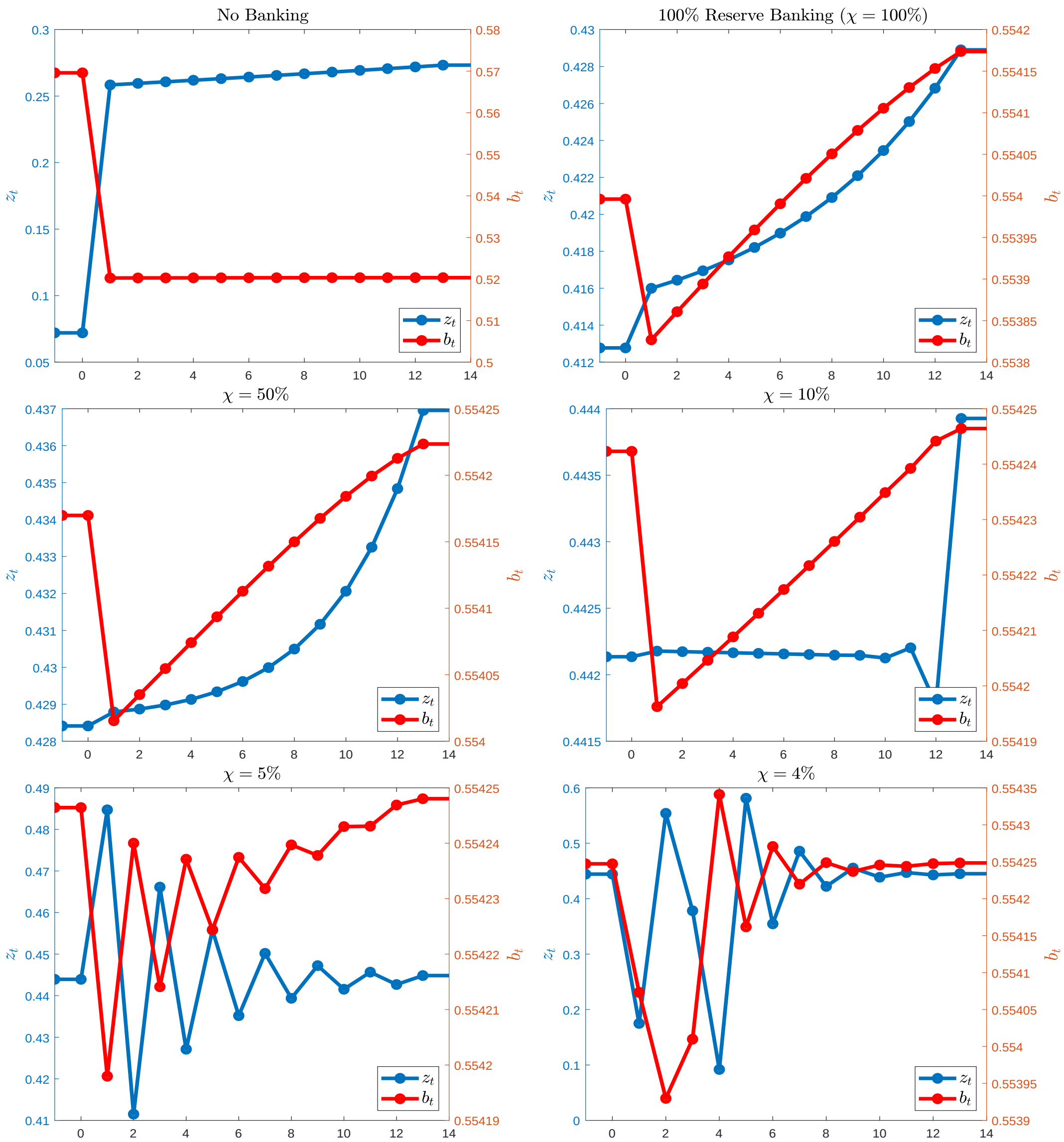


Figure 16: Phase Dynamics and Transition Paths for Known Policy Change

changes in monetary policy at T is announced at time 0. As in Gu et al. (2019), I focused on the unique transition consistent with stationarity after information shock. Initially, the economy is in its unique stationary equilibrium for a given i_0. At $t = 0$ it is announced that i will change to $i = i_T$ at $t = T$ and stay at i_T permanently. Therefore the stationary equilibrium z_T for a given i_T is a fixed terminal condition that pins down the transition by backward induction.

Consider the dynamics of real balances and unsecured credit. The permanent

from $\gamma_0 = \beta(1 + i_0)$ to $\gamma_T = \beta(1 + i_T)$.

change from i_0 to i_T implies a shift of equations (35) and (36), from $z_t = \Phi_0(z_{t+1}, \bar{b}_{t+1}; i_0, \chi)$ and $\bar{b}_t = \Gamma_0(z_{t+1}, \bar{b}_{t+1}; i_0, \chi)$ to $z_t = \Phi_T(z_{t+1}, \bar{b}_{t+1}; i_T, \chi)$ and $\bar{b}_t = \Gamma_T(z_{t+1}, \bar{b}_{t+1}; i_T, \chi)$, respectively. Then, starting in steady state with $(z_0, \bar{b}_0)$ and ending in steady state with $(z_T, \bar{b}_T)$, the transitional dynamics of the equilibrium can be solved by backward induction.

$$\begin{array}{lllll} z_T = \Phi_T(z_T, \bar{b}_T), & z_{T-1} = \Phi_0(z_T, \bar{b}_T), & z_{T-2} = \Phi_0(z_{T-1}, \bar{b}_{T-1}), & \ldots & z_0 = \Phi_0(z_1, \bar{b}_1) \\ \bar{b}_T = \Gamma_T(z_T, \bar{b}_T), & \bar{b}_{T-1} = \Gamma_0(z_T, \bar{b}_T), & \bar{b}_{T-2} = \Gamma_0(z_{T-1}, \bar{b}_{T-1}), & \ldots & \bar{b}_0 = \Gamma_0(z_1, \bar{b}_1) \end{array}$$

Consider a case with $T = 13$. In Figure 16, we begin with a stationary equilibrium with $i = 0.04$. At time 0, an announcement is made that there will be a permanent change in i to 0.02 at time $T = 13$. The stationary equilibrium with $i = 0.02$ is a terminal condition for this dynamic system.

For comparison, we first examine cases with no banking and full reserve requirements. In the no-banking case, real money balances and credit converge monotonically to the new equilibrium. Most policy impacts are reflected as soon as the announcement is made. Under a 100% reserve requirement, the transitional dynamics of real money balances are monotonic, and a significant portion of the effect is reflected when the announcement is released. However, the transitional dynamics of credit become non-monotonic. Credit drops first and eventually converges to the new equilibrium gradually.

Fractional reserve banking cases show more complicated dynamics. Under a 50% reserve requirement, when the announcement is released, only negligible changes are reflected in real balances. When $\chi = 10\%$, both transition dynamics become non-monotonic. They can fluctuate considerably when reserve requirements are very low ($\chi = 5\%$ and $\chi = 4\%$). News about monetary policy induces complicated dynamics in z and b when reserve requirements are low. Figure 16 also shows that there is an asymmetry in the transitional dynamics when the reserve requirement is low. When $\chi = 5\%$, news leading to an increase in z_t at $t = 1$ tends to lower b_t at $t = 1$ because higher z_t raises the equilibrium payoff of holding money, which in turn reduces the use of credit.

As shown in Figure 16, transitions display various patterns depending on χ, with lower reserve requirements more likely to induce cyclic and boom-bust responses. There is perfect foresight about the event, and the transition is uniquely determined. However, it can display a wide range of patterns depending on reserve requirements. Under

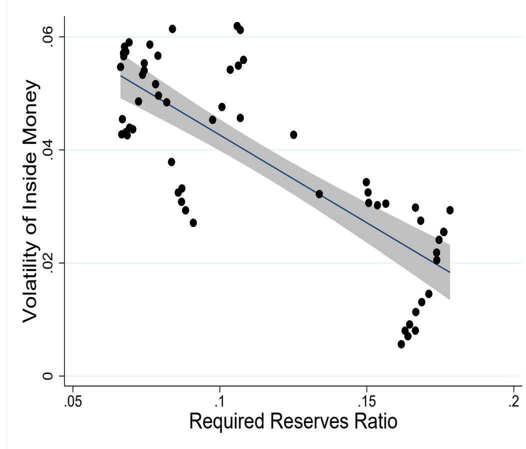


Figure 17: Scatter Plot for Inside Money Volatility and Required Reserve Ratio

the no-banking case and the 100% reserve requirement case, z_t increases monotonically towards z_T. However, with low reserve requirements, the transition path is non-monotonic, and the economy is more likely to experience cyclic and boom-bust responses following a monetary announcement.

6 Empirical Evaluation

In the previous sections, the theoretical results show that lowering the required reserve ratio can induce instability, as endogenous cycles are more likely to emerge with low reserve requirements. When endogenous cycles exist, lowering the reserve requirement increases the amplitude of the cycles. Also, the calibrated example shows that lowering reserve requirements increases the volatility of transition dynamics as the economy is more likely to induce cyclic and boom-bust responses from information (news) shocks. Whether the economic fluctuation occurs from an exogenous shock or arises endogenously, the fractional reserve system increases the volatility of inside money real balances

To evaluate the model's prediction, I examine whether the required reserve ratio is associated with the cyclical volatility of real inside money balances. Following Jaimovich and Siu (2009) and Carvalho and Gabaix (2013), I measure the cyclical volatility in quarter t as the standard deviation of filtered log real total checkable deposits during a 41-quarter (10-year) rolling window. As in Carvalho and Gabaix (2013), end periods use uncentered one-sided standard deviation. Total checkable deposits are from the H.6 Money Stock Measures published by the Federal Reserve Board and converted to real value using the Consumer Price Index (CPI). Seasonally adjusted series are used to smooth seasonal fluctuation. I adopt the Hodrick-Prescott (HP) filter with a 1600 smoothing parameter as standard. To construct an annual series, quarterly observations are averaged for each year. The sample period is from 1960:I to 2018:IV. We can also compute the volatility of unsecured credit. The credit data are converted to real value using the Consumer Price Index (CPI). The sample period for unsecured credit is from 1968:I to 2018:IV.

To compute the required reserve ratios, I divide the required reserves by the deposit component of M1 instead of using the official legal reserve requirement. The reason for this approach is as follows. The legal reserve requirement for demand deposits was 10% from April 2, 1992, to March 25, 2020. However, the Federal Reserve imposed different reserve requirements depending on the size of a bank's liabilities. For example, from December 29, 2011, to December 26, 2012, the Fed had a reserve requirement exemption for liabilities up to $11.5 million. For liabilities between $11.5 million and $71.0 million, the Fed imposed a 3% reserve requirement. These criteria have changed over time. On December 27, 2012, the Fed increased the exemption threshold to $12.4 million and raised the low reserve tranche from $71.0 million to $79.5 million. During 1992:I-2019:IV, there were 27 changes in these thresholds. Dividing the required reserves by the deposit component of M1 allows us to track these changes as well.

Figure 17 presents a scatter plot of the cyclical volatility of the real inside money balance and the required reserve ratio. Columns (1) and (3) of Table 4 reports its regression estimates with Newey-West standard errors. The plot and estimates show a negative relationship between the cyclical volatility of the real inside money balance and the required reserve ratio with statistically significant regression coefficients. However, this result can be driven by a spurious regression. Table 5 provides unit root test results for the federal funds rate, the required reserve ratio, and the cyclical volatility of inside money. Both augmented Dickey-Fuller tests and Phillips-Perron tests fail to reject the null hypotheses of unit roots for these series, whereas they reject the null hypotheses of unit roots at their first differences. In addition to that, the Johansen cointegration test in Columns (1) and (3), suggests that there is no cointegration relationship between the two variables. So it is hard to rule out that the results of Columns (1) and (3) are driven by a spurious regression.

To overcome this issue, I adopt cointegrating regression with an additional variable,

Table 4: Effect of Required Reserve Ratio

Dependent variable	$\sigma_{z,t}^{Roll}$				$\sigma_{b,t}^{Roll}$			
	Yearly		Quarterly		Yearly		Quarterly	
	OLS	CCR	OLS	CCR	OLS	CCR	OLS	CCR
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
χ	-0.310^{***}	-0.380^{***}	-0.309^{***}	-0.422^{***}	1.103^{***}	0.883^{***}	1.116^{***}	0.791^{***}
	(0.038)	(0.029)	(0.019)	(0.041)	(0.146)	(0.015)	(0.074)	(0.148)
`ffr`		0.161^{***}		0.255^{***}		0.359^{***}		0.357^{***}
		(0.028)		(0.030)		(0.007)		(0.134)
Constant	0.074^{***}	0.072^{***}	0.074^{***}	0.073^{***}	-0.053^{***}	-0.055^{***}	-0.053^{***}	-0.034^{***}
	(0.005)	(0.003)	(0.002)	(0.005)	(0.012)	(0.002)	(0.006)	(0.016)
Obs.	59	59	236	236	56	56	223	223
$adjR^2$	0.622	0.637	0.622	0.594	0.706	0.587	0.723	0.149
Johansen Tests for Cointegration (No trend)								
$\lambda_{\text{max rank}}(r=0)$	**4.13**	29.10	**3.85**	24.09	**6.66**	22.27	**7.25**	24.70
5% CV	11.44	17.89	11.44	17.89	11.44	17.89	11.44	17.89
$\lambda_{\text{max rank}}(r=1)$	1.54	**6.01**	2.30	**5.21**	3.09	**7.83**	3.39	**6.59**
5% CV	3.84	11.44	3.84	11.44	3.84	11.44	3.84	11.44

Note: For (1), (3), (5) and (7), OLS estimates are reported, and Newey-West standard errors (using a lag order of 1) are reported in parentheses. For (2), (4), (6), and (8), first-stage long-run variance estimations for CCR are based on the quadratic spectral kernel and Bayesian information criterion. The bandwidth selection is based on Newey-West fixed lag, $4 \times (T/100)^{2/9}$; χ denotes the required reserve ratio, `ffr` denotes federal funds rates, $\sigma_{b,t}^{Roll}$ denotes the cyclical volatility of real inside money balances, and $\sigma_{d,t}^{Roll}$ denotes the cyclical volatility of real unsecured credit. ***, **, and * denotes significance at the 1, 5, and 10 percent levels, respectively.

Table 5: Unit Root Tests

		Phillips-Perron test		ADF test
		$Z(\rho)$	$Z(t)$	$Z(t)$
Yearly	`ffr`	-7.322	-1.899	-2.615
	χ	-1.450	-1.067	-1.216
	$\sigma_{d,t}^{Roll}$	-4.050	-1.922	-1.911
	$\sigma_{b,t}^{Roll}$	-3.166	-1.211	-1.330
	Δ`ffr`	-35.123^{***}	-5.580^{***}	-6.963^{***}
	$\Delta\chi$	-38.181^{***}	-5.214^{***}	-3.960^{***}
	$\Delta\sigma_{d,t}^{Roll}$	-30.677^{***}	-4.524^{***}	-3.953^{***}
	$\Delta\sigma_{b,t}^{Roll}$	-20.795^{***}	-3.948^{***}	-4.009^{***}
Quarterly	`ffr`	-9.416	-2.145	-2.371
	χ	-1.265	-1.046	-1.039
	$\sigma_{d,t}^{Roll}$	-3.337	-1.901	-2.264
	$\sigma_{b,t}^{Roll}$	-1.949	-0.961	-0.979
	Δ`ffr`	-167.648^{***}	-11.799^{***}	-11.256^{***}
	$\Delta\chi$	-201.808^{***}	-13.615^{***}	-10.854^{***}
	$\Delta\sigma_{d,t}^{Roll}$	-52.488^{***}	-5.276^{***}	-6.115^{***}
	$\Delta\sigma_{b,t}^{Roll}$	-94.047^{***}	-7.884^{***}	-6.634^{***}

Note: `ffr` denotes federal funds rates, χ denotes required reserve ratio, and σ_t^{Roll} denotes cyclical volatility of real inside money balances. All series are demeaned before implementing the unit root test following to Elliott and Müller (2006) and Harvey, Leybourne and Taylor (2009), because the magnitude of the initial value can be problematic. Let ***, **, and * denotes significance at the 1, 5, and 10 percent levels, respectively.

the federal funds rate. Columns (2) and (4) of Table 4 provide Johansen cointegration test results for the federal funds rate, the required reserve ratio, and the cyclical volatility of inside money. The Johansen cointegration test suggests a cointegration relationship among these three variables, which is consistent with the theoretical result: The instability depends on the reserve requirement and the interest rate. With the cointegration relationship, we may not have to worry about a spurious relationship. Columns (2) and (4) of Table 4 report the estimates for the cointegrating relationship. Because of the potential bias from long-run variance, I estimate a canonical cointegrating regression (CCR). The estimates are statistically significant with a sizable effect. The cointegration analysis confirms that a lower reserve requirement is associated with higher volatility of inside money real balances, consistent with the model's prediction.

We can also assess the relationship between credit volatility and the required reserve ratio. Columns (5) and (7) of Table 4 report regression estimates with Newey-West standard errors. The plot and estimates show a positive relationship between the cyclical volatility of unsecured credit and the required reserve ratio, with statistically significant regression coefficients. However, this result could also be driven by a spurious regression. The unit root test results in Table 5 support this concern. Columns (6) and (8) of Table 4 provide Johansen cointegration test results for the federal funds rate, the required reserve ratio, and the cyclical volatility of unsecured credit. The Johansen cointegration test suggests a cointegration relationship among these three variables. Columns (6) and (8) of Table 4 report the estimates for the cointegrating relationship, which are statistically significant with a sizable effect. However, the theory in this paper does not feature rich dynamics of unsecured credit and cannot generate this observation. Extending the model to incorporate richer credit dynamics that are consistent with this observation could be another research avenue for the future.

7 Conclusion

The goal of this paper is to examine the (in)stability of fractional reserve banking. To that end, this paper builds a simple monetary model of fractional reserve banking that can capture the conditions for (in)stability under different specifications. Lowering the reserve requirement increases consumption at the steady state. However, it can induce instability. The baseline model and its extension establish the conditions for endogenous cycles and chaotic dynamics. The model also features stochastic cycles and self-fulfilling boom and burst under explicit conditions. The model shows that fractional reserve banking can endanger stability in the sense that equilibrium is more prone to exhibit cyclic, chaotic, and stochastic dynamics under lower reserve requirements. This is due to the amplified liquidity premium. This result holds in the extended model with unsecured credit.

The calibrated exercise suggests that this channel could be another source of economic fluctuations. This paper also provides some empirical evidence that is consistent with the prediction of the model. I test the association between the required reserve ratio and real money volatility using cointegrating regression. I find a significant negative relationship between the two variables. Both theoretical and empirical evidence find a link between reserve requirement policy and (in)stability.

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Online Appendix

Appendix A Proofs

Proof of Proposition 1. Recall (27)

$$\chi i = (1-\sigma+\sigma\chi)\alpha L(z_s).$$

Since $L'(\cdot) < 0$, we have the following:

$$\frac{\partial z_s}{\partial i} = \frac{\chi}{(1-\sigma+\sigma\chi)\alpha L'(z_s)} < 0, \quad \text{and} \quad \frac{\partial z_s}{\partial \chi} = \frac{i-\sigma\alpha L(z_s)}{(1-\sigma+\sigma\chi)\alpha L'(z_s)} < 0.$$

Since $z = v(q)$ and $v'(q) > 0$, it is straightforward to show that lowering i or lowering χ increases q. ■

Proof of Proposition 2. Let there exists a two-period cycle satisfying $z_1 < z_s < p^* \le z_2$. Since $z_2 \ge p^*$, we have $z_2 = (1+i)z_1$. Using (24) with $z_1 < p^*$ gives

$$\chi = \frac{(1-\sigma)\alpha L(z_1)}{(1+i)^2 - 1 - \sigma\alpha L(z_1)} \tag{53}$$

This two-period cycle should satisfy $z_1 < z_s < p^*$ and $z_2 = (1+i)z_1 \ge p^*$. The first one can be easily shown using

$$0 = L(p^*) < L(z_s) = \frac{i}{\alpha(1-\sigma+\sigma\chi)}\chi < \frac{(1+i)^2-1}{\alpha(1-\sigma+\sigma\chi)}\chi = L(z_1)$$

since we have $L'(\cdot) < 0$. Because $dz_1/d\chi < 0$, the latter one, $z_1 \ge p^*/(1+i)$, is held when

$$0 < \chi \le \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^2 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)}.$$

This equilibrium solves

$$\frac{(1+i)^2-1}{\alpha(1-\sigma+\sigma\chi)}\chi = L(z_1), \text{ and } z_2 = (1+i)z_1$$

We can check if lowering the reserve requirement also increases the volatility. Consider the difference between peak and trough $z_2 - z_1 = iz_1$. Since

$$\frac{\partial z_1}{\partial \chi} = \frac{\alpha(1-\sigma)}{\chi\{(1+i)^2 - 1\}} \frac{\{L(z_1)\}^2}{L'(z_1)} < 0,$$

reducing the reserve requirement increases the difference between peak and trough. ■

Proof of the Existence of a Two-period Monetary Cycle where $f'(z) < -1$. Let $f^2(z) = f \circ f(z)$. With given the unique steady state, $f(z) > z$ for $z < z_s$ and $f(z) < z$ for $z > z_s$. Because $f(z)$ is linear increasing function for $z > p^*$, there exist a $\tilde{z} > p^*$ s.t $f(\tilde{z}) > p^*$. Since $\tilde{z} > p^*$ and $f(\tilde{z}) < \tilde{z}$, $\tilde{z}$ satisfies $f^2(\tilde{z}) < f(\tilde{z}) < \tilde{z}$. We can write slope of $f^2(z)$ as follows.

$$\frac{\partial f^2(z)}{\partial z} = f'[f(z)]f'(z) = f'(z)f'(z) = [f'(z)]^2$$

which implies $\partial f^2(z)/\partial z > 1$ when $f'(z) < -1$. And it is easy to show $\partial f^2(0)/\partial z > 0$. With given $i > 0$ and $\chi > 0$, there exist a (z_1, z_2), satisfying $0 < z_1 < z_s < z_2$ which are fix points for $f^2(z)$. ■

Proof of Proposition 3. When DM trade is based on take-it-or-leave-it offer from buyer to seller with $c(q) = q$ and $-qu''(q)/u'(q) = \eta$, $f'(q)$ can be written as

$$f'(q) = \frac{1}{1+i}\left\{\frac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u''(q)q + u'(q) - 1\right] + 1\right\} < -1$$

Using $u''(q)q = -\eta u'(q)$ gives

$$\frac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u'(q)(1-\eta) - 1\right] + 1 < -(1+i)$$

where $u'(q) = 1 + \frac{i\chi}{\alpha(1-\sigma+\sigma\chi)}$. Substituting $u'(q)$ yields

$$\left\{1 + \frac{i\chi}{\alpha(1-\sigma+\sigma\chi)}\right\}(1-\eta) - 1 < -\frac{\chi(2+i)}{\alpha(1-\sigma+\sigma\chi)}.$$

Then rearranging terms gives

$$0 < \chi < \frac{\alpha\eta(1-\sigma)}{\eta(1-\alpha\sigma) + (2-\eta)(1+i)}.$$

■

Proof of Proposition 4. I divide three period cycles into two cases.

Case 1: Let there exists a three-period cycle satisfying $z_1 < z_s < p^* \leq z_2 < z_3$. Since $z_2, z_3 \geq p^*$, we have $z_2 = (1+i)z_1$, $z_3 = (1+i)z_2 = (1+i)^2 z_1$. Using (24) with $z_1 < p^*$ gives

$$\chi = \frac{(1-\sigma)\alpha L(z_1)}{(1+i)^3 - 1 - \sigma\alpha L(z_1)} \tag{54}$$

This three-period cycle should satisfy $z_1 < z_s < p^*$ and $z_2 = (1+i)z_1 \geq p^*$. First one can be easily shown using

$$0 = L(p^*) < L(z_s) = \frac{i}{\alpha(1-\sigma+\sigma\chi)}\chi < \frac{(1+i)^3 - 1}{\alpha(1-\sigma+\sigma\chi)}\chi = L(z_1)$$

since we have $L'(\cdot) < 0$. Because $dz_1/d\chi < 0$, the latter one, $z_1 \geq p^*/(1+i)$, is held when

$$0 < \chi \leq \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^3 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)}.$$

Case 2: Let there exists a three-period cycle satisfying $z_1 < z_2 < p^* \leq z_3$. Since $z_3 > p^*$, we have $z_3 = z_2(1+i)$ and (z_2, z_1) solves (55)-(56).

$$z_1 = f(z_2) = \left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_2) + 1\right]\frac{z_2}{1+i} \tag{55}$$

$$z_2 = \tilde{f}(z_1) \equiv \left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_1) + 1\right]\frac{z_1}{(1+i)^2}. \tag{56}$$

These functions satisfies $f(x) > x$ for $x < z_s$, $f(x) < x$ for $x > z_s$, $\tilde{f}(x) > x$ for $x < \tilde{z}$ and $\tilde{f}(x) < x$ for $x > \tilde{z}$ where $\tilde{z}$ solves $\tilde{z} = \tilde{f}(\tilde{z})$. One can easily show $\tilde{z} < z_s$. Therefore any intersection between $z_1 = f(z_2)$ and $z_2 = \tilde{f}(z_1)$ satisfies $z_1 > z_2$ which contradicts to our initial conjecture $z_1 < z_2$. This implies there is no three-period cycle satisfying $z_1 < z_2 < p^* \leq z_3$. Therefore we can conclude that a three-period cycle exists when

$$0 < \chi \leq \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^3 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)}.$$

This equilibrium solves

$$\frac{(1+i)^3 - 1}{\alpha(1-\sigma+\sigma\chi)}\chi = L(z_1), \quad z_2 = (1+i)z_1, \quad \text{and } z_3 = (1+i)z_2.$$

We can check if lowering the reserve requirement also increases the volatility. Con-

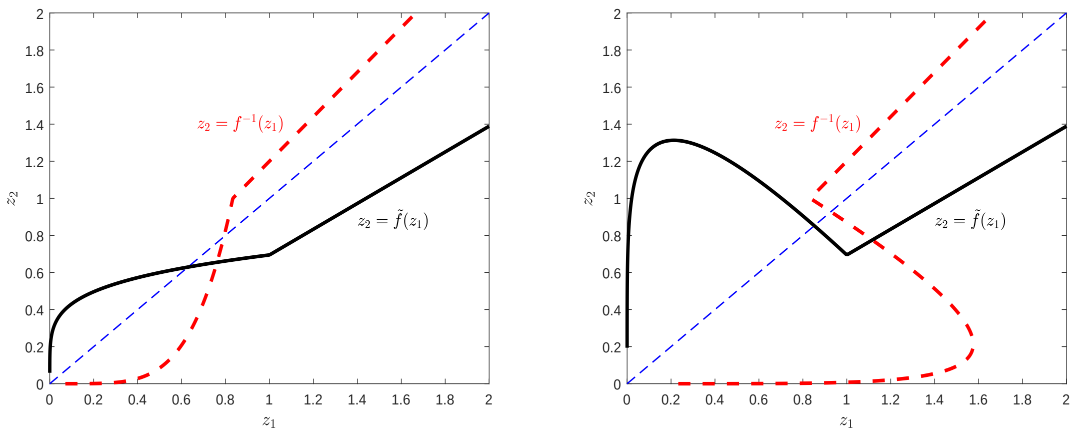

Figure 18: Intersection of $\tilde{f}(z)$ and $f(z)$

sider the difference between peak and trough $z_3 - z_1 = (i^2 + 2i)z_1$. Since

$$\frac{\partial z_1}{\partial \chi} = \frac{\alpha(1-\sigma)}{\chi\{(1+i)^3 - 1\}} \frac{\{L(z_1)\}^2}{L'(z_1)} < 0,$$

reducing the reserve requirement increases the difference between peak and trough.

The existence of a three-cycle implies the existence of cycles of all orders and chaotic dynamics by the Sarkovskii theorem (Sharkovskii, 1964) and the Li-Yorke theorem (Li and Yorke, 1975). ■

Proof of Corollary 1: Proposition 4 shows that at least one periodic point satisfies $z_t < z_s < p^*$ in 3- period cycles. Two period cycles satisfies $z_1 < z_s < z_2$ also implies at least one periodic point satisfies $z_t < z_s < p^*$ in 2-period cycles since $z_1 < z_s < p^*$. This result holds for any n-periodic cycles.

Consider n-period cycles, where $n \in \mathbb{N}$. I do not assume a specific temporal order among the values $z_1, z_2, \ldots, z_n$. Instead, I impose an ordering by value, $z_1 < z_2 < \cdots < z_n$, purely for notational convenience and without loss of generality. Since a periodic cycle is invariant under relabeling, this value-based ordering simplifies the analysis without altering the substance of the result. Under this ordering, I claim that $z_1 = f(z_k) < z_k \leq z_n$ for some k. Since z_1 is the smallest value in the cycle and $z_1 = f(z_k)$, it must be that $z_k > z_1$. The equilibrium satisfies the following conditions:

$$z_t = f(z_{t+1}) > z_{t+1} \quad \text{for } z_{t+1} < z_s, \tag{57}$$

$$z_t = f(z_{t+1}) < z_{t+1} \quad \text{for } z_{t+1} > z_s. \tag{58}$$

By the ordering and conditions (57)–(58), it follows that $z_k > z_s$. Therefore, by the properties of f in the domain $z_{t+1} > z_s$, we have:

$$f(z_k) < z_k,$$

and thus

$$z_1 = f(z_k) < z_k \leq z_n.$$

These properties imply that the function f crosses the 45° line from above at z_s, and therefore any nontrivial cycle (i.e., not a fixed point) must oscillate around z_s, containing values both above and below the steady state. In particular, this shows that the smallest element in the cycle, z_1, must be generated by a point above the steady state and must lie strictly below it. That is,

$$z_1 = f(z_k) < z_s.$$

Since $z_s < p^*$, it follows that $z_1 < p^*$, and hence the liquidity constraint $z_t < p^*$ binds at z_1. Therefore, in any n-period cycle, at least one periodic point must satisfy the constraint $z_t < p^*$. ■

Proof of Proposition 5. By definition, if there exists (ζ_1, ζ_2) satisfying

$$z^1 = \zeta_1 f(z^1) + (1 - \zeta_1) f(z^2) \tag{59}$$

$$z^2 = (1 - \zeta_2) f(z^1) + \zeta_2 f(z^2) \tag{60}$$

with $\zeta_1, \zeta_2 < 1$, then there exists a proper sunspot equilibrium. Because z^1 and z^2 are weighted averages of $f(z^1)$ and $f(z^2)$, where $f(z^1) > z^1$ and $f(z^2) < z^2$, by the uniqueness of the positive (monetary) steady state, necessary and sufficient conditions for (59) and (60) are

$$f(z^2) < z_1 < f(z^1) \text{ and } f(z^2) < z_2 < f(z^1).$$

Since $z^1 < z^2$, above conditions are reduce to

$$z^2 < f(z^1) \text{ and } z^1 > f(z^2). \tag{61}$$

When $\chi < \bar{\chi}_m$, there exists (z^1, z^2) that satisfies (61). Rewrite (59) and (60) as

$$\zeta_1 + \zeta_2 = \frac{z^1 - f(z^2) - z^2 + f(z^1)}{f(z^1) - f(z^2)} = \frac{z^1 - z^2}{f(z^1) - f(z^2)} + 1 < 1 \tag{62}$$

since $z^2 < z^1$ and $f(z^1) > f(z^2)$. Therefore, when $\chi < \bar{\chi}_m$, a stationary sunspot equilibrium exists.

Now we consider the case with $f'(z_s) < -1$. The first order Taylor expansion of (61) around z_s gives

$$z_s + z^2 - z_s < f(z_s) + f'(z_s)(z^1 - z_s)$$
$$z_s + z^1 - z_s > f(z_s) + f'(z_s)(z^2 - z_s)$$

Since $f(z_s) = z_s$, we can rewrite as below

$$z^2 - z_s < f'(z_s)(z^1 - z_s) \Rightarrow z^2 - z_s < -f'(z_s)(z_s - z^1)$$
$$z^1 - z_s > f'(z_s)(z^2 - z_s) \Rightarrow z_s - z^1 < -f'(z_s)(z^2 - z_s)$$

For compact notation, let $\varepsilon_1 \equiv z_s - z^1$ and $\varepsilon_2 \equiv z^2 - z_s$, and rewrite as:

$$\frac{\varepsilon_2}{\varepsilon_1} < -f'(z_s) \text{ and } \frac{\varepsilon_1}{\varepsilon_2} < -f'(z_s) \tag{63}$$

Since $-f'(z_s) > 1$, there exist multiple solutions $\varepsilon_1, \varepsilon_2 > 0$ that satisfy (63). These solutions satisfy (62). Therefore, if $f'(z_s) < -1$, there exists a stationary sunspot cycle. ■

Proof of Proposition 6. Consider $z_t = f(z_{t+1})$. If $z_s > \bar{z}$ where $\bar{z}$ solves $f'(\bar{z}) = 0$. In this case, there exist multiple equilibria. If $q^* \leq f(\bar{z})$, then there exist equilibria $\{z_t\}_{t=0}^{\infty}$ with $z_T \equiv \max\{z_t\}_{t=0}^{\infty} > q^*$ (bubble) which crashes to 0 (burst) as $t \to \infty$, where $T \geq 1$ and $z_T > z_0$. Then there exist equilibria with bubble-burst as a self-fulfilling crisis. Conditions for this case are shown as below. Similar to Corollary 3, consider take-it-leave-it offer with $-qu''/u' = \eta$ and $c(q) = q$. Then we have following difference equation:

$$z_t = f(z_{t+1}) \begin{cases} \dfrac{z_{t+1}}{1+i}\left\{\dfrac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u'(z_{t+1})-1\right]+1\right\} & \text{if } z_{t+1} < q^* \\ \dfrac{z_{t+1}}{1+i} & \text{if } z_{t+1} \geq q^* \end{cases} \tag{64}$$

Step 1: [Multiplicity i.e., $z_s > \bar{z}$ where $\bar{z}$ solves $f'(\bar{z}) = 0$] Consider the following condition.

$$f'(\bar{z}) = \frac{1}{1+i}\left\{\frac{\alpha(1-\sigma+\sigma\chi)}{\chi}[u'(\bar{z})(1-\eta)-1]+1\right\} = 0$$

Since $z_s > \bar{z} \rightarrow u'(z_s) < u'(\bar{z})$, we have

$$u'(z_s) = 1 + \frac{i\chi}{\alpha(1-\sigma+\sigma\chi)} < \frac{1}{1-\eta}\left\{1-\frac{\chi}{\alpha(1-\sigma+\sigma\chi)}\right\} = u'(\bar{z}).$$

This can be reduced as

$$\chi < \frac{\alpha\eta(1-\sigma)}{1+i-\eta(i+\alpha\sigma)}$$

Step 2: [Show $q^* \leq f(\bar{z})$] It is straightforward to show that $q^* < f(\bar{z})$ holds when

$$\chi < \frac{(1-\sigma)\alpha\eta(1+i)}{(1-\eta)^2 q^* + (1+i)[(1-\eta)(3+i-\eta)-\alpha\sigma\eta]}$$

Therefore, when

$$0 < \chi < \min\left\{\frac{(1-\sigma)\alpha\eta(1+i)}{(1-\eta)^2 q^* + (1+i)[(1-\eta)(3+i-\eta)-\alpha\sigma\eta]}, \frac{\alpha\eta(1-\sigma)}{1+i-\eta(i+\alpha\sigma)}\right\}$$

there exist $\{z_t\}_{t=0}^{\infty}$ satisfying $z_T \equiv \max\{z_t\}_{t=0}^{\infty} > q^*$ and $\lim_{t\to\infty} z_t = 0$, where $T \geq 1$ and $z_T > z_0 > q^*/(1+i)$. ■

Proof of Proposition 7. I first present a two period cycle result and three-period case will follow. Let there exists a two-period cycle satisfying $w_1 < q^* < w_2$ where $w_j = z_j + \bar{b}_j$. Since $w_2 > q^*$, we have $z_2 = (1+i)z_1$ and $\bar{b}_2 = (1+\rho)\bar{b}_1 - \sigma\alpha\mu S(q^*)$

where q_1, $\bar{b}_1$, $\bar{b}_2$, and z_1 solve

$$\begin{aligned} u'(q_1) &= 1 + \chi\frac{(1+i)^2-1}{\alpha(1-\sigma+\sigma\chi)} \\ \bar{b}_2 &= \beta\bar{b}_1 + \frac{\chi\mu\sigma z_1\{1-(1+i)^2\}}{\beta(1-\sigma+\sigma\chi)} + \beta\alpha\mu\sigma S(q_1) \\ \bar{b}_1 &= \beta\bar{b}_2 + \beta\alpha\mu\sigma S(q^*) \end{aligned}$$

and $z_1 = q_1 - \bar{b}_1$. This two-period cycle should satisfy $q_1 < q^*$ and

$$w_2 = (1+i)z_1 + (1+\rho)\bar{b}_1 - \sigma\alpha\mu S(q^*) > q^*.$$

For given $i > 0$ and $\chi > 0$, first one can be easily shown using

$$1 = u'(q^*) < u'(q_s) = 1 + \frac{i}{\alpha(1-\sigma+\sigma\chi)}\chi < 1 + \frac{(1+i)^2-1}{\alpha(1-\sigma+\sigma\chi)}\chi = u'(q_1)$$

since we have $u''(\cdot) < 0$. Now we also can check the latter. Consider $(\bar{\chi}_c, z_1, b_1, q_1)$ solving

$$\bar{\chi}_c = \frac{(1-\sigma)\alpha\left[u'(q_1)-1\right]}{(1+i)^2 - 1 - \sigma\alpha\left[u'(q_1)-1\right]} \tag{65}$$

$$(1+i)z_1 + (1+\rho)b_1 - \sigma\alpha\mu S(q^*) = q^* \tag{66}$$

$$(1+\rho)b_1 - \sigma\alpha\mu S(q^*) = \frac{\sigma\alpha\mu\{\beta^2 S(q^*) + \beta S(q_1)\} + \frac{\bar{\chi}_c\mu\sigma z_1\{1-(1+i)^2\}}{(1-\sigma+\sigma\bar{\chi}_c)\beta}}{(1-\beta^2)} \tag{67}$$

and $q_1 = z_1 + b_1$ given i. In this case, we have both conditions: $w_2 = z_2 + b_2 > q^*$ and $q_1 < q_s < q^*$. Therefore we can say, there exists a two-period cycle when $\chi = \bar{\chi}_c$. Since we have $u''(\cdot) < 0$ and $S = u(q) - q$, one can show that there exist a two-period cycle when χ is lower than $\bar{\chi}_c$, given i.

Now, let there exists a three-period cycle satisfying $q_1 = w_1 < q_s < q^* < w_2 < w_3$ where $w_j = z_j + \bar{b}_j$. Since w_3, $w_2 > q^*$, we have $z_2 = (1+i)z_1$, $z_3 = (1+i)^2 z_1$, $\bar{b}_2 = (1+\rho)\bar{b}_1 - \sigma\alpha\mu S(q^*)$ and $\bar{b}_3 = (1+\rho)^2\bar{b}_1 - (2+\rho)\sigma\alpha\mu S(q^*)$ where q_1, $\bar{b}_1$, and z_1 solve

$$
\begin{aligned}
u'(q_1) &= 1 + \chi \frac{(1+i)^3 - 1}{\alpha(1-\sigma+\sigma\chi)} \\
\bar{b}_3 &= \beta \bar{b}_1 + \frac{\chi\mu\sigma z_1\{1-(1+i)^3\}}{\beta(1-\sigma+\sigma\chi)} + \beta\alpha\mu\sigma S(q_1) \\
\bar{b}_1 &= \beta \bar{b}_2 + \beta\alpha\mu\sigma S(q^*) \\
\bar{b}_2 &= \beta \bar{b}_3 + \beta\alpha\mu\sigma S(q^*)
\end{aligned}
$$

and $z_1 = q_1 - \bar{b}_1$. This three-period cycle should satisfy $q_1 < q_s < q^*$ and $w_2 = (1+i)z_1 + (1+\rho)\bar{b}_1 - \sigma\alpha\mu S(q^*) > q^*$. For given $i > 0$ and $\chi > 0$, first one can be easily shown using

$$
1 = u'(q^*) < u'(q_s) = 1 + \frac{i}{\alpha(1-\sigma+\sigma\chi)}\chi < 1 + \frac{(1+i)^3 - 1}{\alpha(1-\sigma+\sigma\chi)}\chi = u'(q_1)
$$

since we have $u''(\cdot) < 0$. Now we also can check the latter. Consider $(\hat{\chi}_c, z_1, b_1, q_1)$ solving

$$
\hat{\chi}_c = \frac{(1-\sigma)\alpha\left[u'(q_1) - 1\right]}{(1+i)^3 - 1 - \sigma\alpha\left[u'(q_1) - 1\right]} \tag{68}
$$

$$
(1+i)z_1 + (1+\rho)b_1 - \sigma\alpha\mu S(q^*) = q^* \tag{69}
$$

$$
(1+\rho)b_1 - \sigma\alpha\mu S(q^*) = \frac{\sigma\alpha\mu\{(\beta^2+\beta^3)S(q^*) + \beta S(q_1)\} + \frac{\hat{\chi}_c\mu\sigma z_1\{1-(1+i)^3\}}{\beta(1-\sigma+\sigma\hat{\chi}_c)}}{(1-\beta^3)} \tag{70}
$$

and $q_1 = z_1 + b_1$ given i. In this case, we have both conditions: $w_2 = z_2 + b_2 > q^*$ and $q_1 < q_s < q^*$. Therefore we can say, there exists a three-period cycle when $\chi = \hat{\chi}_c$. Since we have $u''(\cdot) < 0$ and $S = u(q) - q$, one can show that there exist a three-period cycle when χ is lower than $\hat{\chi}_c$, given i.

Since we have $u''(\cdot) < 0$ and $S = u(q) - q$, one can show that there exist a two-period cycle when χ is lower than $\bar{\chi}_c$, given i. Again, the existence of a three-cycle implies the existence of cycles of all orders and chaotic dynamics by the Sarkovskii theorem and the Li-Yorke theorem. ■

Appendix B Model without banking,

This section documents a model without banking à la Gu et al. (2016). Consider an economy same to Section 4 except there is no FM between CM and DM. The agent solves the following problem in the CM.

$$\begin{aligned} W_t(m_t, -b_t) = &\max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta V_{t+1}(\hat{m}_{t+1}) \\ &\text{s.t. } \phi_t \hat{m}_{t+1} + X_t = H_t + T_t + \phi_t m_t - b_t, \end{aligned} \tag{71}$$

The DM value function is

$$V_t^b(m_t) = \sigma\alpha[u(q_t) - q_t] + W_t(m_t, 0),$$

where $q_t = \min\{q^*, \bar{b}_t + \phi_t m_t\}$. Given $\bar{b}_t$, solving equilibrium yields

$$z_t = \begin{cases} \frac{z_{t+1}}{1+i}\left\{\sigma\alpha\left[u'(z_{t+1} + \bar{b}_{t+1}) - 1\right] + 1\right\} & \text{if } z_{t+1} + \bar{b}_{t+1} < q^* \\ \frac{z_{t+1}}{1+i} & \text{if } z_{t+1} + \bar{b}_{t+1} \geq q^*, \end{cases} \tag{72}$$

where $z_{t+1} = \phi_{t+1} M_{t+1}$. Now we endogenize the debt limit. The buyer cannot commit to pay back the debt. If the buyer reneges she is captured with probability μ. The punishment for a defaulter is permanent exclusion from the DM trade but she can still produce for herself in the CM. The value of autarky is $\underline{W}(0,0) = [U(X^*) - X^* + T]/(1-\beta)$. The incentive condition for voluntary repayment is

$$\underbrace{-b_t + W_t(m_t, 0)}_{\text{value of honoring debts}} \geq \underbrace{(1-\mu)W_t(m_t, 0) + \mu\underline{W}(m_t, 0)}_{\text{value of not honoring debts}}.$$

One can write the debt limit $\bar{b}_t$ as $b_t \leq \bar{b}_t \equiv \mu W_t(0,0) - \mu\underline{W}(0,0)$. Recall the CM value function. Using the solution of FM, we can rewrite the buyer's CM value function as

$$\begin{aligned} W_t(0,0) = &U(X^*) - X^* + T_t + \beta W_{t+1}(0,0) \\ &+ \max_{\hat{m}_{t+1}} \{-\phi_t \hat{m}_{t+1} + \beta\sigma\alpha[u(q_{t+1}) - q_{t+1}] + \beta\phi_{t+1}\hat{m}_{t+1}\}, \end{aligned}$$

where $q_{t+1} = \min\{q^*, z_{t+1} + \bar{b}_{t+1}\}$. Substituting $W_t(0,0) = \bar{b}_t/\mu + \underline{W}(0,0)$ and $\hat{m}_{t+1} =$

M_{t+1} yields

$$\frac{\bar{b}_t}{\mu} = -\phi_t M_{t+1} + \beta\alpha\sigma[u(z_{t+1} + \bar{b}_{t+1}) - z_{t+1} - \bar{b}_{t+1}] + \frac{\beta\bar{b}_{t+1}}{\mu} + \beta\phi_{t+1}M_{t+1},$$

where M_{t+1} and z_{t+1} solve (72). Rearranging terms yields

$$\bar{b}_t = \beta\bar{b}_{t+1} + \mu\{-\gamma z_t + \beta z_{t+1}\} + \beta\sigma\alpha\mu S(z_{t+1} + \bar{b}_{t+1}) \tag{73}$$

where $S(\cdot)$ is the buyer's trade surplus and defined as

$$S(z_{t+1} + \bar{b}_{t+1}) \equiv \begin{cases} u(z_{t+1} + \bar{b}_{t+1}) - z_{t+1} - \bar{b}_{t+1} & \text{if } z_{t+1} + \bar{b}_{t+1} < q^* \\ u(q^*) - q^* & \text{if } z_{t+1} + \bar{b}_{t+1} \geq q^*. \end{cases}$$

The equilibrium can be collapsed into a dynamic system satisfying (72)-(73).

Stationary Equilibrium In the stationary equilibrium, (72) becomes

$$-\frac{i}{\sigma\alpha} + u'(q) - 1 \leq 0, = \text{ if } z > 0 \tag{74}$$

and (73) becomes

$$(1-\beta)\bar{b} = \begin{cases} \mu\{-\gamma + \beta\}z + \beta\sigma\alpha\mu[u(z + \bar{b}) - z - \bar{b}] & \text{if } z + \bar{b} < q^* \\ \mu\{-\gamma + \beta\}z + \beta\sigma\alpha\mu[u(q^*) - q^*] & \text{if } z + \bar{b} \geq q^*, \end{cases} \tag{75}$$

where $q = \min\{z + \bar{b}, q^*\}$.

Appendix C Derivation of Stochastic Equilibrium

This section presents a modified model from Section 4 by introducing a stochastic component to the model. Consider the linear CM production technology with aggregate productivity A_t. The CM consumption good X is produced by the technology $X_t = A_t N_t$, where N_t is the labor input in the CM production. With the labor market clearing condition, $H_t = N_t$, where H_t is the agent's labor supply in the CM, we have the real wage A_t instead of 1. Given this modification, we can rewrite the CM problem as below:

$$
\begin{aligned}
W_t(a_t, s_t, \ell_t, -b_t) &= \max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta G_{t+1}(\hat{m}_{t+1}) \\
\text{s.t. } \phi_t \hat{m}_{t+1} + X_t &= A_t H_t + T_t + \phi_t a_t + (1 + i_{s,t})\phi_t s_t - (1 + i_{l,t})\phi_t \ell_t - b_t,
\end{aligned}
\tag{76}
$$

where A_t be a sequence of independent and identically distributed (i.i.d.) random variables following a log-normal distribution with a mean value of 1 i.e., $\log A_t \sim N(0, \sigma_A)$ so that $E_t[A_{t+1}] = 1$.

The envelope conditions are

$$
\frac{\partial W_t}{\partial a_t} = \frac{\phi_t}{A_t}, \qquad \frac{\partial W_t}{\partial s_t} = \frac{\phi_t}{A_t}\phi_t(1 + i_{s,t}), \qquad \frac{\partial W_t}{\partial \ell_t} = \frac{\phi_t}{A_t}\phi_t(1 + i_{l,t}).
$$

and the DM value function is

$$
\begin{aligned}
V_t^b(m_t + d_t, 0, \ell_t) &= \alpha[u(q_t) - q_t] + W_t(m_t + d_t, 0, \ell_t, 0) \\
&= \alpha[u(q_t) - q_t] + \frac{\phi_t}{A_t}\{m_t + d_t - (1 + i_{l,t})\ell_t\} + W_t(m_t + d_t, 0, \ell_t, 0)
\end{aligned}
$$

where $q_t = \min\left\{q^*, \frac{\bar{b}_t + (m_t + d_t)\phi_t}{A_t}\right\}$ and $d_t = \ell_t$. Given $\bar{b}_t$, solving equilibrium yields

$$
\frac{z_t}{A_t} = \begin{cases} \dfrac{z_{t+1}}{A_{t+1}(1+i)} \left\{ \dfrac{1 - \sigma + \sigma\chi}{\chi} \alpha \left[u'(z_{t+1} + \bar{b}_{t+1}) - 1 \right] + 1 \right\} & \text{if } z_{t+1} + \bar{b}_{t+1} < q^* \\ \dfrac{z_{t+1}}{1+i} & \text{if } z_{t+1} + \bar{b}_{t+1} \geq q^*, \end{cases}
\tag{77}
$$

where $z_{t+1} = (1 - \sigma + \sigma\chi)\phi_{t+1}M_{t+1}/(\sigma\chi)$. $\underline{\mathrm{W}}(0, 0, 0, 0) = [U(X^*) - X^* + T]/(1 - \beta)$. The incentive condition for voluntary repayment is

$$
\underbrace{-\frac{b_t}{A_t} + W_t(a_t, d_t, \ell_t, 0)}_{\text{value of honoring debts}} \geq \underbrace{(1 - \mu)W_t(a_t, d_t, \ell_t, 0) + \mu \underline{\mathrm{W}}(a_t, d_t, \ell_t, 0)}_{\text{value of not honoring debts}}.
$$

One can write the debt limit $\bar{b}_t$ as $b_t \leq \bar{b}_t \equiv \{W_t(0, 0, 0, 0) - \underline{\mathrm{W}}(0, 0, 0, 0)\}\mu A_t$. Recall the CM value function. Using the solution of FM, we can rewrite the buyer's

CM value function as

$$W_t(0,0,0,0) = U(X^*) - X^* + T_t + \beta W_{t+1}(0,0,0,0) + \max_{\hat{m}_{t+1}} \left\{ -\frac{\phi_t \hat{m}_{t+1}}{A_t} + \beta\alpha\sigma[u(q_{t+1}) - q_{t+1}] + \frac{\beta\phi_{t+1}\hat{m}_{t+1}}{A_{t+1}} \right\},$$

Substituting $W_t(0,0,0,0) = \frac{\bar{b}_t}{\mu A_t} + \underline{\mathrm{W}}(0,0,0,0)$ and $\hat{m}_{t+1} = M_{t+1}$ yields

$$\frac{\bar{b}_t}{\mu A_t} = -\frac{\phi_t M_t}{A_t} + \beta\alpha\sigma\left[u\left(\frac{z_{t+1} + \bar{b}_{t+1}}{A_{t+1}}\right) - \frac{z_{t+1} + \bar{b}_{t+1}}{A_{t+1}}\right] + \frac{\beta\bar{b}_{t+1}}{\mu A_{t+1}} + \frac{\beta\phi_{t+1}M_{t+1}}{A_{t+1}},$$

where M_{t+1} and z_{t+1} solve (77). Rearranging terms yields

$$\frac{\bar{b}_t}{A_t} = \frac{\chi\sigma\mu}{1-\sigma+\sigma\chi}\left[-\gamma\frac{z_t}{A_t} + E_t\left\{\frac{\beta z_{t+1}}{A_{t+1}}\right\}\right] + \beta E_t\left\{\frac{\bar{b}_{t+1}}{A_{t+1}} + \mu\sigma\alpha S\left(\frac{z_{t+1}}{A_{t+1}} + \frac{\bar{b}_{t+1}}{A_{t+1}}\right)\right\}.$$