On the Instability of Fractional Reserve Banking*

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Abstract

This paper develops a dynamic general equilibrium model to study the (in)stability of the fractional reserve banking system. The model shows that the fractional reserve banking system can endanger stability in the sense that equilibrium is more prone to exhibit endogenous cyclic, chaotic, and stochastic dynamics under lower reserve requirements, although it can increase welfare in the steady state. Introducing endogenous unsecured credit to the baseline model does not change the main results.

JEL Classification Codes: E42, E51, G21
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> Motivated partly by a desire to avoid such [excessive] price-level fluctuations and possible Wicksellian price-level indeterminacy, quantity theorists have advocated legal restrictions on private intermediation. ... Thus, for example, Friedman (1959, p.21) ... has advocated 100 percent reserves against bank liabilities called demand deposit. **Sargent and Wallace (1982)**

1 Introduction

Is fractional reserve banking systems inherently unstable and susceptible to endogenous fluctuations? There have been long-lasting debates on whether the fractional reserve banking induces the instability. Prominent examples are the Peel's Banking Act of 1844 and the Chicago plan of banking reform with 100% reserve requirement proposed by Irving Fisher, Paul Douglas, and others in 1939. Later, Friedman (1959) supported this banking reform while Becker (1956) took the opposite position of free banking with 0% reserve requirement.[1] Switzerland had a referendum of 100% reserve banking in 2018, which was rejected with 75.72% of objections. While the debates on the instability of fractional reserve banking started from a long time ago, the debate has never stopped.

The purpose of this paper is to provide a model of money and banking that captures the role of fractional reserve banking and can shed light on its (in)stability. The model is simple and tractable to capture the conditions for cycles and chaotic dynamics explicitly. In the model, each agent faces an idiosyncratic liquidity shock. Banks accept deposits and extend loans to provide risk-sharing among the depositors. The bank's lending is constrained by the reserve requirement. In the baseline model lowering the reserve requirement can induce two-period cycles as well as three-period cycles, which implies the existence of periodic cycles of all order and chaos. It can also induce sunspot cycles. This result holds in the extended model with credit. The model also can deliver self-fulling bubble and burst arise from fractional reserve banking. However, lowering the reserve requirement increases the welfare at the steady state.

Related Literature This paper builds on Berentsen, Camera and Waller (2007), which introduces financial intermediaries with enforcement technology to Lagos and Wright (2005) framework. Similarly Gu, Mattesini, Monnet and Wright (2013a) study the environment where banks take deposits and their liabilities facilitate third-party transactions. The ap-

[1] Sargent (2011) provides a novel review for the historical debates between narrow banking and free banking as tensions between stability versus efficiency.

proach to introduce unsecured credit to the monetary economy is related to Lotz and Zhang (2016) and Gu, Mattesini and Wright (2016) which incorporate the agents' default decision similar to Kehoe and Levine (1993).

This paper is related to the large literature on fractional reserve banking. Freeman and Huffman (1991) and Freeman and Kydland (2000) develop general equilibrium models that explicitly capture the role of fractional reserve banking. Using those models, they explain the observed relationships between key macroeconomic variables over business cycles. Chari and Phelan (2014) study the condition under which fractional reserve banking can be socially useful by preventing bank runs in the cash-in-advance framework. For recent study, Andolfatto, Berentsen and Martin (2019) integrate Diamond (1997) to the Lagos and Wright (2005) framework and provide a model in which the fractional reserve banking emerges endogenously and a central bank can prevent bank panic as a lender of last resort. While many of previous works on the instability fractional reserve banking focus on bank runs, this paper studies the instability in the sense that fractional reserve banking induces endogenous monetary cycles.

This paper is also related to the large literature on endogenous fluctuations, chaotic dynamics, and indeterminacy which are surveyed by Baumol and Benhabib (1989), Azariadis (1993) and Benhabib and Farmer (1999). For a model of bilateral trade, Gu, Mattesini, Monnet and Wright (2013b) show that credit markets can be susceptible to endogenous fluctuations. Gu, Monnet, Nosal and Wright (2019a) shows that introducing financial intermediaries to an economy can engender the instability in four distinct setups that capture various functions of banking. The model in this paper is closely related to Gu et al. (2019a) while the model here is extended to incorporate fractional reserve banking.

The rest of the paper is organized as follows. Section 2 constructs the baseline search-theoretic monetary model. Section 3 provides main results. Section 4 introduces unsecured credit. Section 5 establishes the condition for self-fulling boom and burst equilibria. Section 6 calibrates the model to quantify the theory. Section 7 discusses the empirical evaluation of the model. Section 8 concludes.

2 Model

The model is based on Lagos and Wright (2005) with a financial intermediary as in Berentsen et al. (2007). Time is discrete and infinite. In each period, three markets convene sequentially. First, a centralized financial market (FM), followed by a decentralized goods market (DM) and finally a centralized goods market (CM). The FM and CM are frictionless. The DM is subject to search frictions, anonymity, and limited commitment. Therefore, a medium of exchange is needed to execute trade.

There is a continuum of agents who produce and consume perishable goods. At the beginning of the FM, a preference shock is realized: with probability σ, an agent will be a buyer in following DM and with probability $1-\sigma$, she will be the seller. Agents discount their utility each period by β. Within-period utilty is represented by

$$\mathcal{U} = U(X) - H + u(q) - c(q)$$

where X is the CM consumption, H is the CM disutility from production, q is the DM consumption. As standard U', u', $c' > 0$, U'', $u'' < 0$, $c'' \geq 0$ and $u(0) = c(0) = 0$. One unit of H produces one unit of X in the CM. The efficient consumptions in CM and DM are X^* and q^* that solve $U'(X^*) = 1$ and $u'(q^*) = c'(q^*)$, respectively.

A representative bank accepts deposits and lends loans. The banking market is perfectly competitive. The bank can enforce the repayment of loans at no cost. Lastly, there is a central bank that controls money supply M_t. Let γ be the growth rate of the money stock. Changes in money supply are accomplished by lump-sum transfer if $\gamma > 0$ and by lump-sum tax if $\gamma < 0$.

2.1 Agent's Problem

Let W_t, G_t, and V_t denote agent's value function in the CM, FM, and DM, respectively, in period t. An agent entering the CM with the nominal balance m_t, the deposit d_t and the loan ℓ_t, solves the following problem:

$$\begin{aligned} W_t(m_t, d_t, \ell_t) = & \max_{X_t, H_t, \hat{m}_{t+1}} U(X_t) - H_t + \beta G_{t+1}(\hat{m}_{t+1}) \\ \text{s.t. } \phi_t \hat{m}_{t+1} + X_t = & H_t + T_t + \phi_t m_t + (1+i_{d,t})\phi_t d_t - (1+i_{l,t})\phi_t \ell_t \end{aligned} \tag{1}$$

where T_t is the lump-sum transfer (or tax if it is negative), $i_{d,t}$ is the deposit interest rate, $i_{l,t}$ is the loan interest rate and $\hat{m}_{t+1}$ is the money balance carried to the FM. The first-order conditions are $U'(X_t) = 1$ and

$$\phi_t = \beta G'_{t+1}(\hat{m}_{t+1}) \tag{2}$$

where $G'_{t+1}(\hat{m}_{t+1})$ is the marginal value of an additional unit of money taken into the FM of period $t+1$. The envelope conditions are

$$\frac{\partial W_t}{\partial m_t} = \phi_t, \qquad \frac{\partial W_t}{\partial d_t} = \phi_t(1+i_{d,t}), \qquad \frac{\partial W_t}{\partial \ell_t} = -\phi_t(1+i_{l,t})$$

implying W_t is linear in m_t, d_t and ℓ_t.

The value function of an agent at the beginning of FM is

$$G_t(m) = \sigma G_{b,t}(m) + (1-\sigma) G_{s,t}(m) \tag{3}$$

where $G_{j\in\{b,s\},t}$ is value function of type j agent in the FM. Agents choose their deposit balance d_j and loan ℓ_j based on the realization of their types in the following DM. The value function $G_{j,t}$ can be written as

$$G_{j,t}(m) = \max_{d_{j,t},\ell_{j,t}} V_{j,t}(m - d_{j,t} + \ell_{j,t}, d_{j,t}, \ell_{j,t}) \quad \text{s.t} \quad d_{j,t} \le m \tag{4}$$

where $V_{j,t}$ is value function of type j agent in the DM. The first order conditions are

$$\frac{\partial V_{j,t}}{\partial \ell_{j,t}} \le 0 \tag{5}$$

$$\frac{\partial V_{j,t}}{\partial d_{j,t}} - \lambda_d \le 0 \tag{6}$$

where λ_d is the Lagrange multiplier for $d_{j,t} \le m$.

The terms of trade in the DM are determined by an abstract mechanism that was studied in Gu and Wright (2016). The buyer must pay $p = v(q)$ to seller to get q. As shown in Gu and Wright (2016), this bilateral trade satisfies following

$$p = \begin{cases} z & \text{if } z < p^* \\ p^* & \text{if } z \ge p^* \end{cases} \qquad q = \begin{cases} v^{-1}(z) & \text{if } z < p^* \\ q^* & \text{if } z \ge p^* \end{cases} \tag{7}$$

where $v(q)$ is some payment function satisfying $v'(q) > 0$ and p^* is the payment required to get efficient consumption q^*.

With probability α, a buyer meets a seller in the DM while a seller meets a buyer with probability α_s. In the DM, buyer can use the cash earned in the previous CM and also can use the loan borrowed from the bank in the previous FM. Since the CM value function is linear, the DM value function for the buyer can be written as

$$V_{b,t}(m_t - d_{b,t} + \ell_{b,t}, d_{b,t}, \ell_{b,t}) = \alpha[u(q_t) - p_t] + W(m_t - d_{b,t} + \ell_{b,t}, d_{b,t}, \ell_{b,t})$$

where $p_t \leq m_t - d_{b,t} + \ell_{b,t}$. Assuming interior solution, differentiating $V_{b,t}$ yields

$$\frac{\partial V_{b,t}}{\partial m} = \phi_t[\alpha\lambda(q_t) + 1], \qquad \frac{\partial V_{b,t}}{\partial d} = \phi_t[-\alpha\lambda(q_t) + i_d], \qquad \frac{\partial V_{b,t}}{\partial \ell} = \phi_t[\alpha\lambda(q_t) - i_l]$$

where $\lambda(q) = u'(q)/v'(q) - 1$ if $p^* > z$ and $\lambda(q) = 0$ if $z \geq p^*$. Combining the buyer's first order conditions in the FM, (5)-(6), with the derivatives of V_b yields

$$\partial G_b/\partial d_b = \phi i_d - \phi\alpha\lambda(q) - \lambda_d \leq 0, = 0 \text{ iff } d_b > 0 \tag{8}$$

$$\partial G_b/\partial \ell_b = -\phi i_l + \phi\alpha\lambda(q) \leq 0, = 0 \text{ iff } \ell_b > 0. \tag{9}$$

A seller's DM value function can be written as

$$V_{s,t}(m_t - d_{s,t} + \ell_{s,t}, d_{s,t}, \ell_{s,t}) = \alpha_s[p_t - c(q_t)] + W_t(m_t - d_{s,t} + \ell_{s,t}, d_{s,t}, \ell_{s,t}).$$

Differentiating $V_{s,t}$ yields

$$\frac{\partial V_{s,t}}{\partial m_t} = \phi_t, \qquad \frac{\partial V_{s,t}}{\partial d} = \phi_t(1 + i_{d,t}), \qquad \frac{\partial V_{s,t}}{\partial \ell} = -\phi_t(1 + i_{l,t}).$$

Similar to the buyer's case, combining the seller's first conditions in the FM, (5)-(6), with the first order derivatives of V_s yields

$$\partial G_s/\partial d_s = \phi_t i_d - \lambda_d \leq 0, = 0 \text{ iff } d_s > 0 \tag{10}$$

$$\partial G_s/\partial \ell_s = -\phi_t i_l \leq 0, = 0 \text{ iff } \ell_s > 0. \tag{11}$$

2.2 Bank's Problem

A representative bank accepts deposits d and makes loan ℓ. The depositors are paid with nominal interest rate i_d by the bank, and the borrowers need to repay their borrowing with nominal interest rate i_l. The central bank sets reserve requirement χ. The banking market is competitive. The representative bank solves the following profit maximization problem

$$\max_{d,\ell} \quad (i_l\ell - i_d d) \quad s.t. \quad \chi\ell \leq d \tag{12}$$

The first order conditions for the bank's problem are

$$0 = i_l - \lambda_L \tag{13}$$

$$0 = -i_d + \lambda_L/\chi \tag{14}$$

where λ_L is the Lagrangian multiplier with respect to the bank's lending constraint. For $\lambda_L > 0$, we have

$$i_l = \chi i_d \tag{15}$$

while $\lambda_L = 0$ implies $i_d = i_l = 0$. Given the bank's problem and agent's problem, we can define an equilibrium as follows:

Definition 1. *Given (γ, χ), an equilibrium consists of the sequences of prices $\{\phi_t, i_{l,t}, i_{d,t}\}_{t=0}^{\infty}$, real balances $\{m_t, \ell_{b,t}, \ell_{s,t}, d_{b,t}, d_{s,t}\}_{t=0}^{\infty}$, and allocations $\{q_t, X_t, \ell_t\}_{t=0}^{\infty}$ satisfying the following:*

- *Agents solve CM and FM problems: (1) and (4)*
- *A representative bank solves its profit maximization problem: (12)*
- *Markets clear in every period:*

  1. *Deposit Market: $\sigma d_{b,t} + (1-\sigma)d_{s,t} = d_t$*
  2. *Loan Market: $\sigma\ell_{b,t} + (1-\sigma)\ell_{s,t} = \ell_t$*
  3. *Money Market: $m_t = M_t$*

Given the definition of the equilibrium, we have the following result:

Proposition 1. *Given (γ, χ), an equilibrium can be summarizes into the following difference equation:*

$$z_t = f(z_{t+1}) \equiv \frac{z_{t+1}}{1+i}\left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_{t+1}) + 1\right] \tag{16}$$

where $1+i \equiv \gamma/\beta$, $z_t = \phi_t m_t(1-\sigma+\sigma\chi)/\sigma\chi$, and $L(z) \equiv \lambda \circ v^{-1}(z)$ is liquidity premium.

Proof. See Appendix A. ■

3 Theoretical Results

This section establishes key results on the instability of banking. Consider a stationary equilibrium. A stationary equilibrium is a fixed point that satisfies $z = f(z)$. There always exists an non-monetary equilibrium with $z = 0$. Given $i \in [0, \bar{\iota})$ and $\chi \in (0, 1]$ with $\bar{\iota} = \alpha(1 - \sigma + \sigma\chi)L(0)/\chi$, an unique stationary monetary equilibrium exists satisfying

$$\chi i = (1 - \sigma + \sigma\chi)\alpha L(z_s)$$

where $z_s = v(q_s)$. The Nash and Kalai bargaining provides simple examples for $\bar{\iota}$. Under the Inada condition $u'(0) = \infty$, with the Kalai solution we have $\bar{\iota} = \theta\alpha(1 - \sigma + \sigma\chi)/\chi(1 - \theta)$ while with the Nash bargaining we have $\bar{\iota} = \infty$. Since $\lambda'(q) < 0$, lowering i or lowering χ increases the DM consumption q_s at the stationary equilibrium.

The dynamics of monetary equilibrium is characterized by the first order difference equation (16). Following the standard textbook method (Azariadis, 1993), we can show that if $f'(z_s) < -1$, there exists a two-period cycle.

Proposition 2 (**Monetary Cycle**). *If $f'(z_s) < -1$, there is a two-period cycle with $z_1 < z_s < z_2$.*

Proof. See Appendix A. ■

The first order derivative of $f(z_{t+1})$, evaluated at $z_s < p^*$, can be written as

$$f'(z_{t+1})\big|_{z_{t+1}=z_s} = \frac{1}{1+i}\left\{\frac{1 - \sigma + \sigma\chi}{\chi}\alpha\left[L'(z_s)z_s + L(z_s)\right] + 1\right\}. \tag{17}$$

Now consider a special case with some assumptions on $u(q)$ and $c(q)$. Assume buyer makes take-it-or-leave-it offer. Then above condition for two period cycles, $f'(z_s) < -1$, holds when χ is lower than some threshold. The results can be expressed as below:

Corollary 1. *Assume $-qu''(q)/u' = \eta$ and $c(q) = q$. If $\chi \in (0, \chi_m)$, where*

$$\chi_m \equiv \frac{\alpha\eta(1 - \sigma)}{\eta(1 - \alpha\sigma) + (2 - \eta)(1 + i)} \tag{18}$$

then $f'(z_s) < -1$.

Proof. See Appendix A. ■

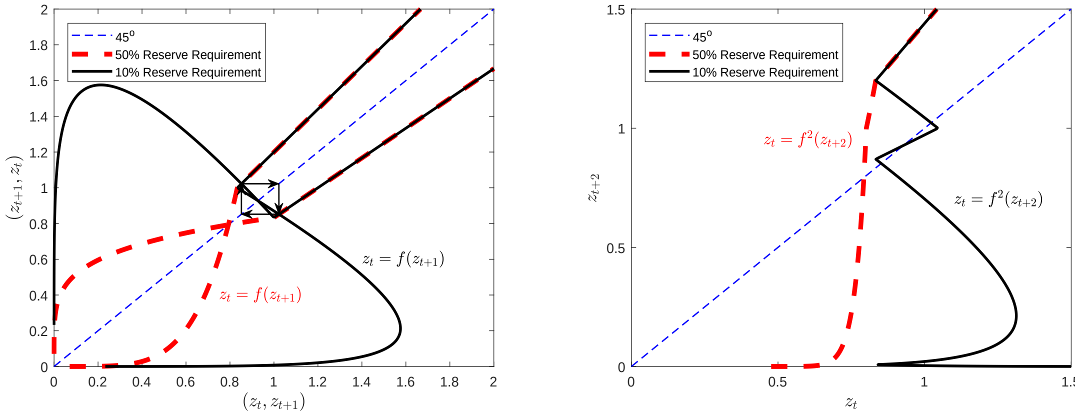

Figure 1: A Two-period Cycle under Fractional Reserve Banking

Corollary 1 implies that if the reserve requirement is lower than χ_m, the monetary economy exhibits endogenous cycles. While condition (18) is written in terms of χ, this condition can be written in terms of i as follows:

$$0 < i < \frac{\eta[\alpha(1-\sigma) - \chi(1-\alpha\sigma)]}{\chi(2-\eta)} \tag{19}$$

Condition (19) implies that not only lowering reserve requirement can induce cycles, but also lowering interest rate can induce cycles in this case, as long as $\eta < 2$.

To interpret the results, recall $f(z_{t+1})$ from equation (16).

$$f(z_{t+1}) = \frac{z_{t+1}}{1+i}\left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_{t+1}) + 1\right]$$

The first term, $z_{t+1}/(1+i)$, reflects the store of value, which is monotonically increasing in z_{t+1}. The second term $(1-\sigma+\sigma\chi)\alpha L(z_{t+1})/\chi + 1$, reflecting the liquidity premium, is decreasing in z_{t+1}. Since $f'(z_{t+1})$ depends on both terms, $f(z_{t+1})$ is generally nonmonotone. If the second term dominates the first term, we can have $f'(\cdot) < -1$ which is a standard condition for the existence of cyclic equilibria. Lowering the reserve requirement amplifies the second term because the absolute value of second term is increasing in χ.

In addition to the condition for two-period cycles, the next result provides the condition for three-period cycles under the abstract mechanism. The existence of three period cycles implies periodic cycles of all orders and chaotic dynamics by Sharkovskii (1964) and Li and Yorke (1975).

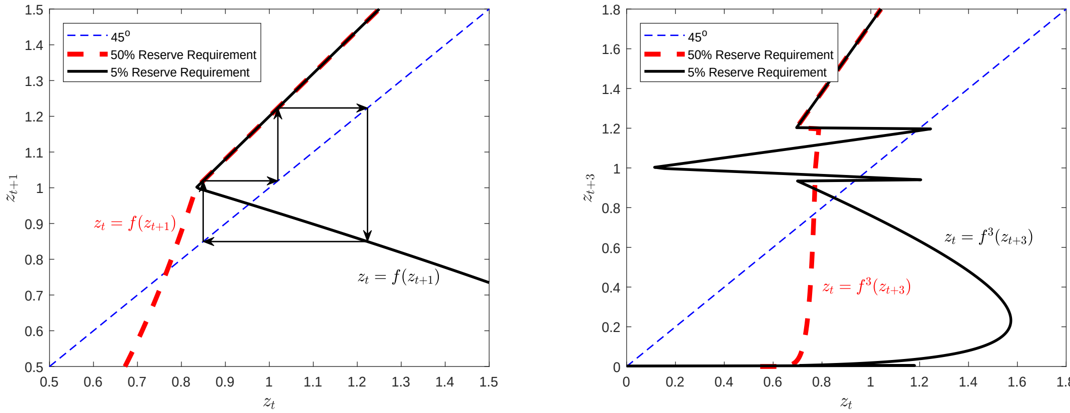

Figure 2: A Three-period Cycle under Fractional Reserve Banking

Proposition 3 (**Three-period Monetary Cycle and Chaos**). *There exists a three-period cycle with $z_1 < z_2 < z_3$ if $\chi \in (0, \hat{\chi}_m)$, where*

$$\hat{\chi}_m \equiv \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^3 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)}$$

Proof. See Appendix A. ■

In any periodic cycle, at some point over the cycle satisfies $z_t < p^*$.

Corollary 2 (**Binding Liquidity Constraint**). *In any n-period cycle, at least one periodic point has the liquidity constraint binding, $z_t < z_s < p^*$.*

Proof. See Appendix A. ■

The model can also generate sunspot cycles. Consider a Markov sunspot variable $S_t \in \{1, 2\}$. This sunspot variable is not related with fundamentals, but may affect equilibrium. Let $\Pr(S_{t+1} = 1 | S_t = 1) = \zeta_1$ and $\Pr(S_{t+1} = 2 | S_t = 2) = \zeta_2$. The sunspot is realized in the CM. Let W_t^S be the CM value function in state S in period t then

$$\begin{aligned} W_t^S(m_t, d_t, \ell_t) = \max_{X_t, H_t, \hat{m}_{t+1}} & U(X_t) - H_t + \beta\left[\zeta_s G_{t+1}^S(\hat{m}_{t+1}) + (1-\zeta_s) G_{t+1}^{-S}(\hat{m}_{t+1})\right] \\ \text{s.t. } & \phi_t^S \hat{m}_{t+1} + X_t = H_t + T_t + \phi_t^S m_t + (1 + i_{d,t})\phi_t^S d_t - (1 + i_{l,t})\phi_t^S \ell_t. \end{aligned}$$

The first order condition can be written as

$$-\phi_t^S + \beta\zeta_s G_{t+1}^{\prime S}(\hat{m}_{t+1}) + \beta(1-\zeta_s) G_{t+1}^{\prime -S}(\hat{m}_{t+1}) = 0. \tag{20}$$

Solving the FM problem gives similar results with (33) except for the sunspot variable, S.

$$G'^{S}_{t+1}(m^{S}_{t+1}) = \phi^{S}_{t+1}\left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z^{S}_{t+1})+1\right] \tag{21}$$

Substituting (21) into (20) and multiplying $(1-\sigma+\sigma\chi)m_{t+1}/(\sigma\chi)$ to the both sides yield

$$\begin{aligned} z^{S}_{t} &= \frac{\zeta_s z^{S}_{t+1}}{1+i}\left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z^{S}_{t+1})+1\right] + \frac{(1-\zeta_s)z^{-S}_{t+1}}{1+i}\left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z^{-S}_{t+1})+1\right] \\ &= \zeta_s f(z^{S}_{t+1}) + (1-\zeta_s) f(z^{-S}_{t+1}) \end{aligned} \tag{22}$$

where $z^{S}_{t} = (1-\sigma+\sigma\chi)\phi^{S}_{t} m_t/(\sigma\chi)$. We define a sunspot equilibrium as follows:

Definition 2 (**Proper Sunspot Equilibrium**). *A proper sunspot equilibrium consists of the sequences of real balances $\{z^{S}_{t}\}^{\infty}_{t=0,S=1,2}$, where z_1 is not equal to z_2, and probabilities (ζ_1, ζ_2), solving (22) for all t.*

Similar to the baseline model, we can show the existence of sunspot equilibrium when $f'(z_s) < -1$. The following proposition summarizes the result:

Proposition 4 (**Existence of Proper Sunspot Equilibrium**). *If $f'(z_s) < -1$, there exist (ζ_1, ζ_2), $\zeta_1 + \zeta_2 < 1$, such that the economy has a proper sunspot equilibrium in the neighborhood of z_s.*

Proof. See Appendix A. ■

4 Endogenous Credit Limits

This section introduces unsecured credit. For simplicity, I assume that the buyer makes a take-it-or-leave-it offer to the seller in the DM, which means the buyer maximizes her surplus subject to the seller's participation constraint. The DM cost function is $c(q) = q$. Let the CM value function be the same as before. Given debt limit $\bar{b}_t$, the DM value function for t period is

$$V_t^b(m_t + \ell_t, 0, \ell_t) = \alpha[u(q_t) - q_t] + W_t(m_t + \ell_t, 0, \ell_t)$$

where $q_t = \min\{q^*, \phi_t(m_t + \ell_t) + \bar{b}_t\}$. Given $\bar{b}_t$, solving equilibrium yields

$$z_t = \begin{cases} \dfrac{z_{t+1}}{1+i}\left\{\dfrac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u'(z_{t+1}+\bar{b}_{t+1}) - 1\right] + 1\right\} & \text{if } z_{t+1} + \bar{b}_{t+1} < q^* \\ \dfrac{z_{t+1}}{1+i} & \text{if } z_{t+1} + \bar{b}_{t+1} \geq q^*. \end{cases} \tag{23}$$

where $z_{t+1} = (1 - \sigma + \sigma\chi)\phi_{t+1}m_{t+1}/(\sigma\chi)$. The equilibrium credit limit, $\bar{b}_t$, is determined so that the buyer voluntarily repays her debt. The punishment for a defaulter is permanent exclusion from the DM trade. The buyer can be captured with the probability μ if the buyer reneges. The value of autarky is $\underline{\mathrm{W}}(0,0,0) = \{U(X^*) - X^* + T\}/(1-\beta)$. The incentive condition for voluntary repayment is

$$\underbrace{-b_t + W_t(0,0,0)}_{\text{value of honoring debts}} \geq \underbrace{(1-\mu)W_t(0,0,0) + \mu\underline{\mathrm{W}}(0,0,0)}_{\text{value of not honoring debts}}.$$

One can write the debt limit $\bar{b}_t$ as $b_t \leq \bar{b}_t = \mu W_t(0,0,0) - \mu\underline{\mathrm{W}}(0,0,0)$. Recall the CM value function,

$$W_t(0,0,0) = U(X^*) - X^* + T_t - \phi_t m_{t+1} + \beta\alpha\sigma[u(q_{t+1}) - q_{t+1}] + \beta\phi_{t+1}m_{t+1} + \beta W_{t+1}(0,0,0)$$

where $q_t = \min\{q^*, z_t + \bar{b}_t\}$. Substituting $W_t(0,0,0) = \bar{b}_t/\mu + \underline{\mathrm{W}}(0,0,0)$ yields

$$\frac{\bar{b}_t}{\mu} = -\phi_t m_{t+1} + \beta\alpha\sigma[u(q_{t+1}) - q_{t+1}] + \frac{\beta\bar{b}_{t+1}}{\mu} + \beta\phi_{t+1}m_{t+1}$$

where $q_{t+1} = z_{t+1} + \bar{b}_{t+1}$. Rearranging terms gives

$$\bar{b}_t = \begin{cases} \beta\bar{b}_{t+1} + \dfrac{\chi\mu\sigma[-\gamma z_t + \beta z_{t+1}]}{1-\sigma+\sigma\chi} + \beta\alpha\mu\sigma S(z_{t+1}+\bar{b}_{t+1}) & \text{if } z_{t+1} + \bar{b}_{t+1} < q^* \\ \beta\bar{b}_{t+1} + \dfrac{\chi\mu\sigma[-\gamma z_t + \beta z_{t+1}]}{1-\sigma+\sigma\chi} + \beta\alpha\mu\sigma S(q^*) & \text{if } z_{t+1} + \bar{b}_{t+1} \geq q^* \end{cases} \tag{24}$$

where $S(z_{t+1}+\bar{b}_{t+1}) \equiv [u(z_{t+1}+\bar{b}_{t+1}) - z_{t+1} - \bar{b}_{t+1}]$. The equilibrium can be collapsed in to a dynamic system satisfying (23)-(24).

The stationary equilibrium falls in one of the three cases: the pure monetary equilibrium, the pure credit equilibrium, and the money-credit equilibrium. The debt limit at the stationary equilibrium, $\bar{b}$, is a fixed point satisfying $\bar{b} = \Omega(\bar{b})$ where

$$\Omega(\bar{b}) = \begin{cases} \frac{\mu\sigma\alpha}{r}[u(\tilde{q}) - \tilde{q}] - \frac{i\mu\sigma\chi}{1-\sigma+\sigma\chi}[\tilde{q} - \bar{b}] & \text{if } \tilde{q} > \bar{b} \geq 0 \\ \frac{\mu\sigma\alpha}{r}[u(\bar{b}) - \bar{b}] & \text{if } q^* > \bar{b} \geq \tilde{q} \\ \frac{\mu\sigma\alpha}{r}[u(q^*) - q^*] & \text{if } \bar{b} \geq q^* \end{cases} \tag{25}$$

where $\tilde{q}$ solves $u'(\tilde{q}) = 1 + i\chi/[\alpha(1-\sigma+\sigma\chi)]$. The DM consumption q_s is determined by $q_s = \min\{q^*, \max\{\tilde{q}, \bar{b}\}\}$. Money and credit coexist if and only if $0 < \mu < \min\{1, \tilde{\mu}\}$, where $\tilde{\mu} \equiv \sigma\left\{i\chi[(1-\sigma+\sigma\chi)/\tilde{q} - 1] + (\alpha/r)(1-\sigma+\sigma\chi)^2[u(\tilde{q})/\tilde{q} - 1]\right\}$, since they coexist when $\bar{b} < \tilde{q}$. The DM consumption is decreasing in i in the monetary equilibrium.

Consider the dynamics of equilibria where money and credit coexist. I claim the main results from the Section 3 - lowering the reserve requirement can induce endogenous cycles - still hold even after unsecured credit is introduced. We can establish the conditions for two-period cycles, three period cycles and chaotic dynamics. For compact notation, define $\iota = \max\{i, r\}$. The following proposition summarizes the results:

Proposition 5 (**Monetary Cycles with Unsecured Credit**). *There exist two period cycles of money and credit with $w_1 < q^* < w_2$ if $\chi \in (0, \chi_c)$, where $w_j = z_j + \bar{b}_j$ and*

$$\chi_c \equiv \frac{(1-\sigma)\alpha\left[u'\left(\frac{q^*}{1+\iota}\right) - 1\right]}{(1+i)^2 - 1 - \sigma\alpha\left[u'\left(\frac{q^*}{1+\iota}\right) - 1\right]}.$$

There exist three period cycles of money and credit with $w_1 < q^* < w_2 < w_3$, if $\chi \in (0, \hat{\chi}_c)$, where

$$\hat{\chi}_c \equiv \frac{(1-\sigma)\alpha\left[u'\left(\frac{q^*}{1+\iota}\right) - 1\right]}{(1+i)^3 - 1 - \sigma\alpha\left[u'\left(\frac{q^*}{1+\iota}\right) - 1\right]}.$$

Proof. See Appendix A. ■

5 Self-Fulfilling Bubble and Burst

This section focuses on the equilibria where real balance increases above the steady state until certain time, T, and crashes to zero. Similar to Section 4, I assume that the buyer makes a take-it-or-leave-it offer to the seller and the DM utility function and the cost function satisfies $-qu''(q)/u'(q) = \eta$ and $c(q) = q$, respectively. Consider a sequence of real balance $\{z_t\}_{t=0}^{\infty}$ with $z_T \equiv \max\{z_t\}_{t=0}^{\infty} > q^*$ (bubble) that crashes to 0 (burst) as $t \to \infty$, where $T \geq 1$ and $z_T > z_0$. I refer to this equilibrium as a self-fulfilling bubble and burst equilibria:

Definition 3 (**Self-Fulfilling Bubble and Burst Equilibria**). *For initial level of real balance $z_0 > 0$, a self-fulfilling bubble and burst is a set of sequence $\{z_t\}_{t=0}^{\infty}$ satisfying (26)*

$$z_t = \frac{z_{t+1}}{1+i}\left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha[u'(z_{t+1})-1]+1\right] \tag{26}$$

where $0 < z_s < z_T$, $\lim_{t\to\infty} z_t = 0$, $z_T = \max\{z_t\}_{t=0}^{\infty}$ with $T \geq 1$.

The next step is to check under which condition this type of equilibria can occur. When $z_s > \bar{z}$, where $\bar{z}$ solves $f'(\bar{z}) = 0$, there exist multiple equilibria. Then, if $f(\bar{z}) \geq q^*$, the self-fulfilling bubble and burst equilibria exist. Lowering the reserve requirement can induce this type of equilibria. The following proposition summarizes the results:

Proposition 6 (**Existence of Self-Fulfilling Bubble and Burst Equilibria**). *There exist self-fullfilling bubble and burst equilibria, $\{z_t\}_{t=0}^{\infty}$ if*

$$0 < \chi < \min\left\{\frac{(1-\sigma)\alpha\eta(1+i)}{(1-\eta)^2q^* + (1+i)[(1-\eta)(3+i-\eta)-\alpha\sigma\eta]}, \frac{(1-\sigma)\alpha\eta}{2+i(2-\eta)-\alpha\sigma\eta}\right\}$$

Proof. See Appendix A. ■

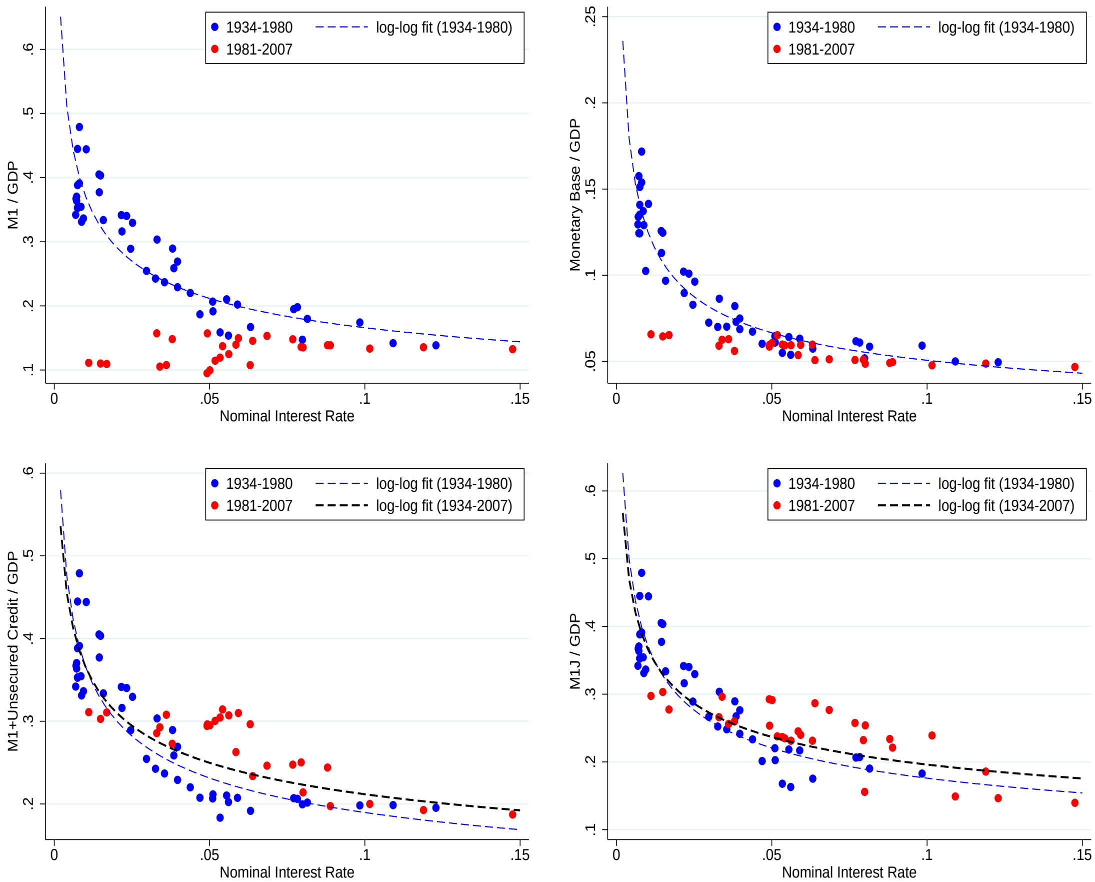

Figure 3: US Money Demand and Credit

Note: GDP and nominal interest rate series are from Alvarez and Lippi (2014). M1 series are from Ireland (2009) for 1934 to 1960 and remaining series are from H.6 Money Stock Measures release by published by the Federal Reserve Board. Monetary base series are from the Federal Reserve Bank of St. Louis. For unsecured credit, following to Krueger and Perri (2006), I use the revolving consumer credit series which are from G.19 Consumer Credit release published by the Federal Reserve Board. M1J series are from Bethune, Choi and Wright (2020).

6 Calibrated Examples

In this section, I calibrate the model to match the US data. As is well-known, the money demand in the U.S. have been unstable since around 80s and 90s. There are two different strategies to match the money demand with the data. The first strategy uses M1J, which is proposed by Lucas and Nicolini (2015), as an alternative measure of M1. The second strategy divides the money-output ratio into two sub-period and calibrate separately. This section adopts both approaches and compares the results.

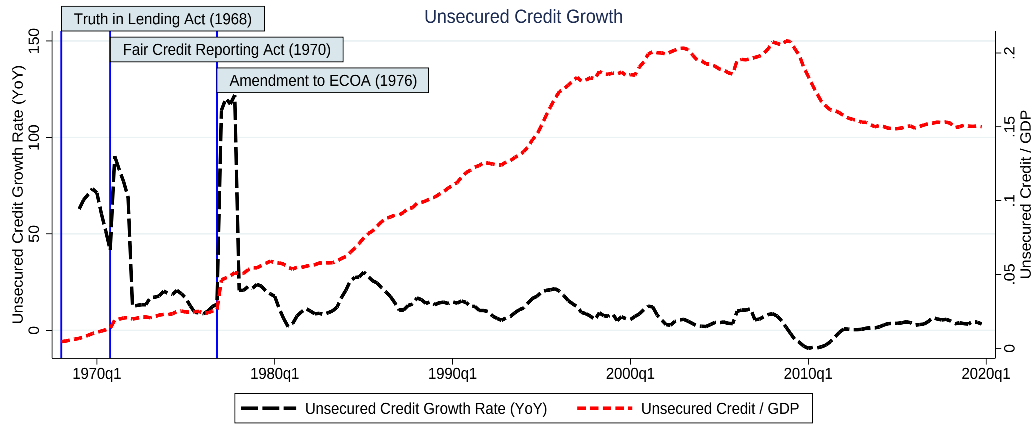

Figure 4: Institutional Changes and Unsecured Credit Growth

6.1 A Digression on the Two Different Approach

Examples of the first strategy includes Wang, Wright and Liu (2020) and Bethune et al. (2020). Lucas and Nicolini (2015) propose a new measure of M1: the sum of conventional M1 and money-market deposit account (MMDA). The idea is that the MMDA became as liquid as conventional M1 after the regulatory changes in the early 1980s. This new measure of M1 recovers the stable money demand relationship.

Examples of second strategy includes Ireland (2009) and Alvarez and Lippi (2014). They divide the sample into pre- and post-1980 subsamples. Ireland (2009) points out the shift of money demand coincides with the time when Paul Volcker started his term at the Federal Reserve Board and the Depository Institutions Deregulation and Monetary Control Act was implemented. Berentsen, Menzio and Wright (2011) and Berentsen, Huber and Marchesiani (2015) split the sample into pre- and post 1990 subsamples and interpret the downward shift in money demand as due to innovations in payments, such as the introduction of credit cards, ATMs, and sweep accounts. While the previous works identify the year of structural break differently, the common idea is that financial innovations or institutional changes increase the access to credit and thus shift the money demand.

However, as shown in Gu et al. (2016), an increase in credit could merely crowd out the money balance while keeping the total liquidity the same. This is consistent with the bottom left panel of Figure 3 which recovers stability by adding the unsecured credit to M1. Figure 4 shows the time-plot of the unsecured credit growth rate (left-axis) and the unsecured credit to GDP ratio (right-axis). Figure 4 also shows several legislative changes during huge unsecured credit growth. This unsecured credit series started in 1968 when Truth in Lending Act was passed. The Act required lenders to disclose the annual percentage rate (APR).

Table 1: Annual Model (1934-2007)

Parameter	Model 1	Model 2	Target
DM utility curvature, η	0.179	0.129	elasticity of z/y wrt i
CM utility level, B	1.653	0.952	avg. z/y
fraction of buyers, σ	0.771	0.790	avg. m/y
monitoring probability, μ	-	0.402	avg. b/y

The unsecured credit had drastically increased with the enactment of a series of laws. For example, the Fair Credit Reporting Act of 1970 increased transparency and accountability of credit-rating agencies. The Equal Credit Opportunity Act (ECOA) of 1974 initially banned credit discrimination based on sex or marital status, and was amended in 1976 to include race. In the same year, an amendment to the Fair Credit Reporting Act of 1970 required credit rating agencies to keep records on married women.[2]

While the jury is still out on the source of a structural shift in money demand, by adapting both strategies, we can examine whether these different approaches lead to different results.

6.2 Parameters and Targets

The model calibration is based on the data from 1934 to 2007. In the model, the fitted lines in Figure 3 correspond to the money demand in the stationary equilibrium. Assuming there is no structural break given M1J, I calibrate the model without unsecured credit. I refer the first calibrated model without unsecured credit as Model 1. In addition to Model 1, assuming there is no structural break, the other calibration uses the model with unsecured credit. I divide the sample into two sub-sample and the monitoring probability parameter μ captures the structural break. I refer the second calibrated model with unsecured credit and the structural break as Model 2.

I set the discount rate $\beta = 0.9709$ to match the real annual interest rate of 3%. Using the average required reserve to total checkable deposit ratio for 1959-2007, the reserve requirements is set at $\chi = 0.12$. Utility functions for this parameterization are

$$u(q) = \frac{q^{1-\eta}}{1-\eta}, \qquad U(X) = B\log(X)$$

implying $X^* = B$. The number of matches in the DM is given by following matching function, $\mathcal{M}(\mathcal{B}, \mathcal{S}) = \frac{\mathcal{B}\mathcal{S}}{\mathcal{B}+\mathcal{S}}$ where $\mathcal{B}$ and $\mathcal{S}$ denotes the measure of buyers and sellers, respectively. This

[2]While only a few laws are discussed here, there had been a lot of amendment on consumer credit from 1968 to 1982. See Ryan, Trumbull and Tufano (2011) for a brief history of the consumer credit development including a series of institutional changes during this period.

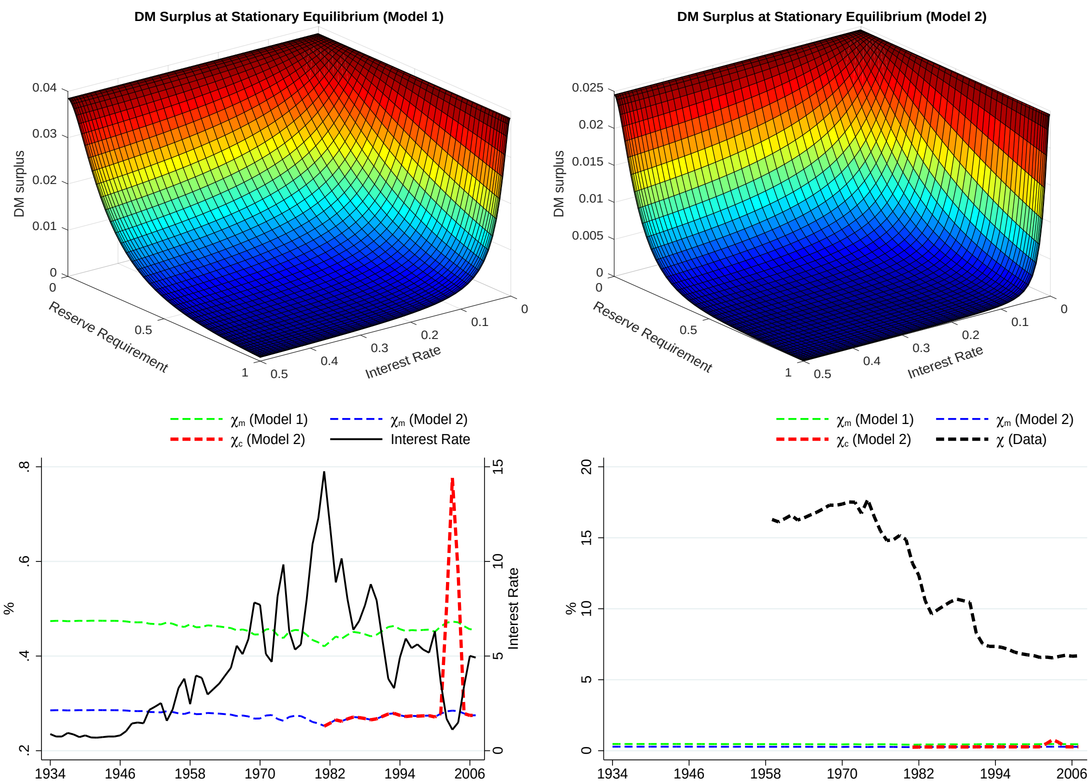

Figure 5: Calibrated Examples

implies $\alpha = \mathcal{M}(\sigma, 1-\sigma)/\sigma$ and $\alpha_s = \mathcal{M}(\sigma, 1-\sigma)/(1-\sigma)$. In the DM, the buyers make take-it-of-leave-it offer to seller. For Model 1, the parameters (B, η, σ) are set to match the three main targets of money demand during 1934-2007: (i) the elasticity of q/y with respect to i, -0.27; (ii) the average M1 to GDP ratio, 0.28 and; (iii) the average monetary base to GDP ratio, 0.08. For Model 2, to capture the shift of money demand around 1980, I set $\mu_0 = 0$ for 1934-1980 and (B, η, σ) are set to match the three main targets of the money demand during 1934-1980: (i) the elasticity of q/y with respect to i, -0.35; (ii) the average M1 to GDP ratio, 0.29 and; (iii) the average monetary base to GDP ratio, 0.09. Then, μ is calibrated to match with the average unsecured credit to output ratio during 1981-2007, 0.14 with given (B, η, σ).

6.3 Results

Table 1 provides a summary of the calibrated parameters. Top panel of Figure 5 shows the DM surplus in the stationary equilibrium for models 1 and 2. As illustrated in Section 3, lowering interest rate or lowering reserve requirement increases DM trade surplus. They also increase the instability since the equilibrium is more prone to endogenous cycle. The bottom

Table 2: Quarterly Model (1934-2007)

Parameter	Model 1	Model 2	Target
DM utility curvature, η	0.179	0.129	elasticity of z/y wrt i
CM utility level, B	0.007	0.024	avg. z/y
fraction of buyers, σ	0.805	0.917	avg. m/y
monitoring probability, μ	-	0.474	avg. b/y

panels of Figure 5 shows the model implied χ_m and χ_c. If we compare the critical values for monetary cycle, χ_m, Model 2 shows the lower critical values than Model 1. After the unsecured credit is introduced in 1980, χ_c in Model 2 had been lower than χ_m in Model 1 but χ_c become far bigger than χ_m around mid 2000s when $i < r = 0.03$. These values are far lower than the actual required reserve ratios, in both models.

6.4 News Shocks

This section considers discrete-time dynamics resulting from news. Following to Gu, Han and Wright (2019b) and Burdett, Trejos and Wright (2017), news on changes in the monetary policy at T is announced at time 0. First, consider the dynamics without unsecured credit i.e., $\mu = 0$. This transition can be solved by backward induction as below.

$$z_T = f_T(z_T), \quad z_{T-1} = f_0(z_T), \quad z_{T-2} = f_0(z_{T-1}), \quad \dots \quad z_0 = f_0(z_0)$$

Let equations (23) and (24) be $z_t = \Phi(z_{t+1}, \bar{b}_{t+1})$ and $\bar{b}_t = \Gamma(z_{t+1}, \bar{b}_{t+1})$. The transitional dynamics of the equilibrium with unsecured credit also can be solved by backward induction.

$$\begin{aligned} z_T &= \Phi_T(z_T, \bar{b}_T), \quad z_{T-1} = \Phi_0(z_T, \bar{b}_T), \quad z_{T-2} = \Phi_0(z_{T-1}, \bar{b}_{T-1}), \quad \dots \quad z_0 = \Phi_0(z_0, \bar{b}_0) \\ \bar{b}_T &= \Gamma_T(z_T, \bar{b}_T), \quad \bar{b}_{T-1} = \Gamma_0(z_T, \bar{b}_T), \quad \bar{b}_{T-2} = \Gamma_0(z_{T-1}, \bar{b}_{T-1}), \quad \dots \quad \bar{b}_0 = \Gamma_0(z_0, \bar{b}_0) \end{aligned}$$

To obtain parameters for higher frequency, I transform the data to make them quarterly and recalibrate the parameters B, σ and μ while I fix η the same as in the annual models. Table 2 shows recalibrated parameters. Consider a case with $T = 9$. In Figures 6 and 7, I start with a stationary equilibrium with $i = 0.1$. At time 0, it is announced that $i = 0.02$ at the time $T = 9$. The stationary equilibrium with $i = 0.02$ is a terminal condition for this dynamics system. While real money balance converges to the new equilibrium smoothly when $\chi = 0.12$, real money balances fluctuate a lot when reserve requirements are very low.

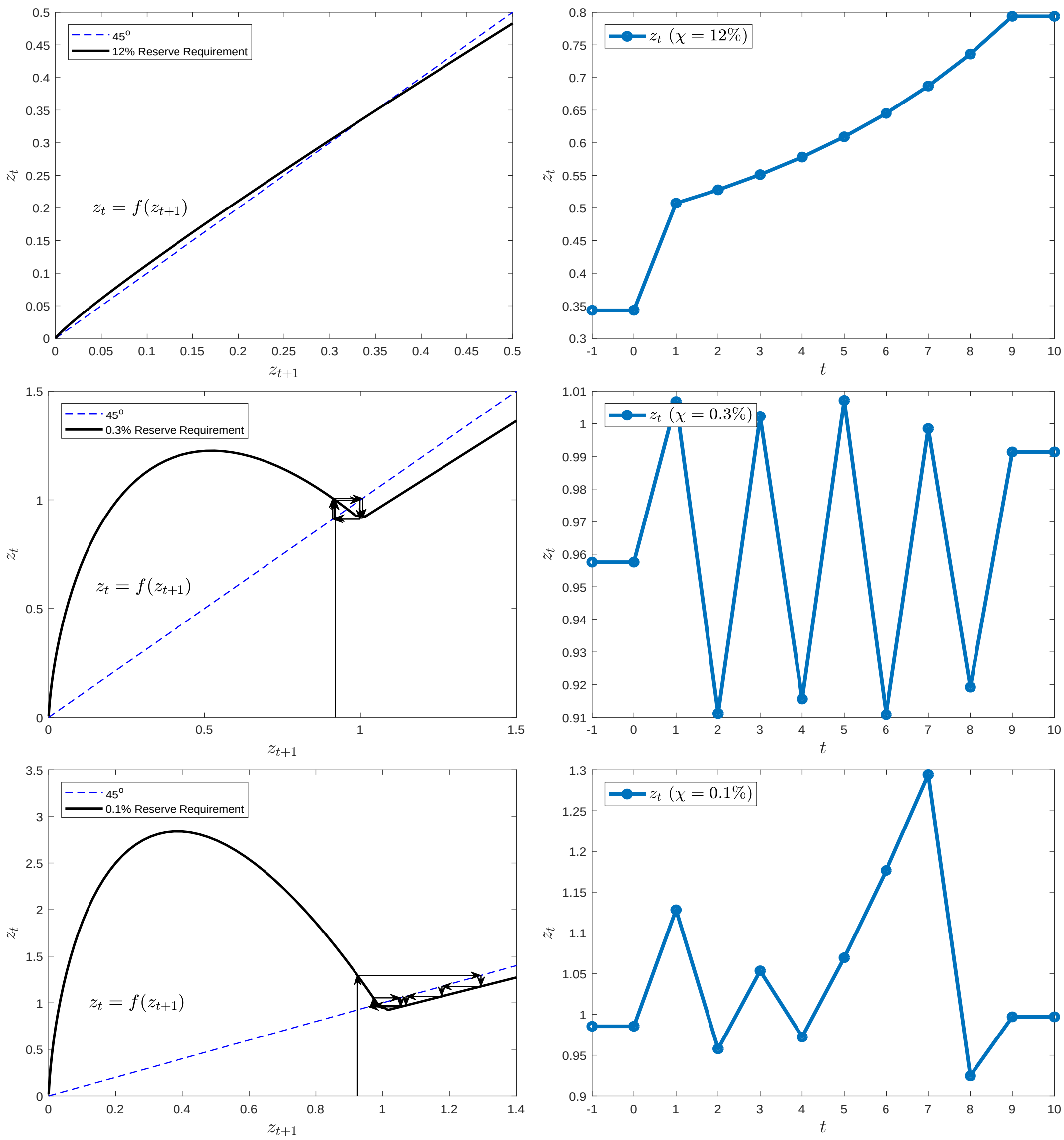

Figure 6: Phase Dynamics and Transition Paths for Known Policy Change: Model 1

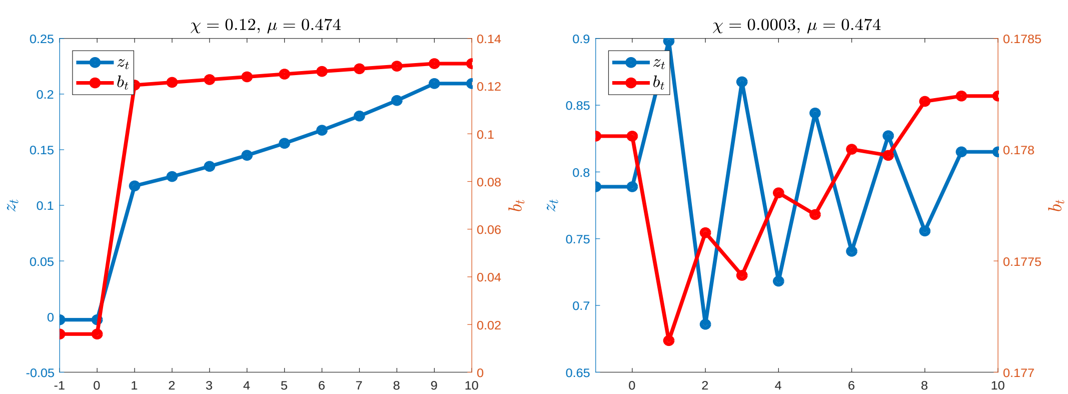

Figure 7: Transition Paths for Known Policy Change: Model 2

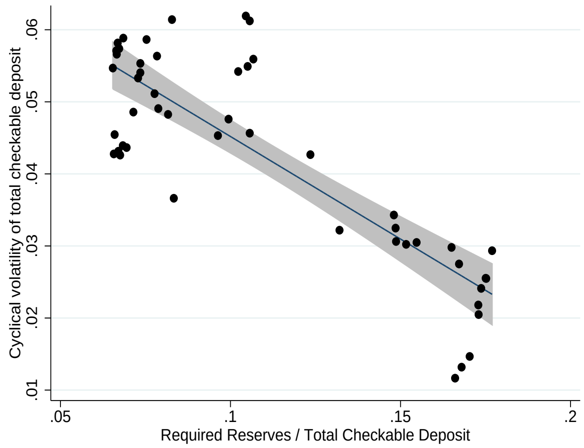

Figure 8: Scatter Plot for Inside Money Volatility and Required Reserve Ratio

7 Empirical Evaluation: Inside Money Volatility

In previous sections, the theoretical results and the quantitative results show that lowering the required reserve ratio can induce the instability. To evaluate the model prediction, I examine whether the required reserve ratio is associated with the cyclical volatility of the inside money real balance.

Following Jaimovich and Siu (2009) and Carvalho and Gabaix (2013), I measure the cyclical volatility in quarter t as the standard deviation of filtered log real total checkable deposit during a 41-quarter (10-year) window centered around quarter t. Total checkable deposits are from H.6 Money Stock Measures release published by the Federal Reserve Board and converted to real value using Consumer Price Index (CPI). Seasonally adjusted series are used to smooth the seasonal fluctuation. I adopt the Hodrick-Prescott (HP) filter with 1600 smoothing parameter as standard. To construct annual series, quarterly observations are averaged for each year. Sample period is from 1960:I to 2017:IV so that we have annual series from 1965 to 2012. As a robustness check, I redo all the analyses using the core CPI, the Personal Consumption Expenditures (PCE), the core PCE to transform the total checkable deposit into real value. I also redo all the analyses using the quarterly series. Appendix C contains the additional empirical results including the robustness results using different measures of the price level and different frequency.

The legal reserve requirement for the demand deposits has been 10% since April 2, 1992. However, the Federal Reserves imposed different reserve requirements depending on their size of liabilities. These criteria have changed over time. For example, during 1992:Q1-2019:Q4, this had changed 27 times. To consider these changes, I divide the required reserves by total

Table 3: Empirical Evaluation

(a) Effect of Require Reserve Ratio

	OLS	CCR	FMOLS
	(1)	(2)	(3)
χ	-0.283^{***}	-0.245^{***}	-0.211^{***}
	(0.031)	(0.002)	(0.003)
`ffr`		-0.109^{***}	-0.248^{***}
		(0.002)	(0.003)
Constant	0.074^{***}	0.074^{***}	0.078^{***}
	(0.004)	(0.000)	(0.000)
Obs.	49	49	49
R^2	0.706	0.637	0.144

(b) Unit Root Test

	Phillips-Perron test		ADF test
	$Z(\rho)$	$Z(t)$	$Z(t)$ w/ lag 1
`ffr`	-6.766	-1.704	-2.362
χ	-1.518	-1.199	-1.363
σ_t^{Roll}	-4.708	-2.191	-2.090
Δ`ffr`	-28.373^{***}	-5.061^{***}	-6.357^{***}
$\Delta\chi$	-31.783^{***}	-4.794^{***}	-3.682^{***}
$\Delta\sigma_t^{Roll}$	-24.905^{***}	-3.416^{**}	-2.942^{**}

(c) Johansen Test for Cointegration

Max rank	$\lambda_{trace}(r)$	5% CV	1% CV
0	35.6880	29.68	35.65
1	**10.6820**	15.41	20.04
2	4.5391	3.76	6.65
Max rank	$\lambda_{max}(r, r+1)$	5% CV	1% CV
0	25.0060	20.97	25.52
1	**6.1429**	14.07	18.63
2	4.5391	3.76	6.65

Notes: For (1), OLS estimates are reported and Newey-West standard errors with lag 1 are reported in parentheses. For (2) and (3), first stage long-run variance estimations for CCR and FMOLS are based quadratic spectral kernel and Bayesian information criterion. The bandwidth selection is based on Newey-West fixed lag, $4 \times (T/100)^{2/9}$ and `ffr` denotes federal funds rates and σ_t^{Roll} denotes cyclical volatility of real inside money balances. ***, **, and * denote significance at the 1, 5, and 10 percent levels, respectively. Sensitivity analysis are included in Appendix C.

checkable deposits to compute the required reserve ratio.

Figure 8 presents a scatter plot of the cyclical volatility of the real inside money balance and the required reserve ratio. Column (1) of Table 7a reports its regression estimates with Newey-West standard errors. The plot and estimates show a negative relationship between the cyclical volatility of the real inside money balance and the required reserve ratio with statistically significant regression coefficients. However, this result can be driven by a spurious regression. Table 7b provides unit root test results for federal funds rates, required reserve ratio, cyclical volatility of inside money. I fail to reject the unit root hypothesis for these series. So we cannot distinguish whether Column (1)'s results are driven by a spurious regression or not.

To overcome this spurious relationship issue, I adopt the cointegrating regression. Table 3c provides Johansen cointegration test results for federal funds rates, required reserves, and cyclical volatility of inside money. Trace test and max test both suggest a cointegration relationship among these three variables which is consistent to the theoretical result: the instability depends on reserve requirement and interest rate. With the cointegration relationship, we may not have to worry about a spurious relationship. Columns (2)-(3) of Table 7a report the estimates for cointegrating relationship. Because of the potential bias from long-run variance, I estimate a canonical cointegrating regression (CCR) and a Fully Modified OLS (FMOLS) for Columns (2)-(3) of Table 7a, respectively. The estimates are statistically significant with sizeable level and consistent with the prediction from the model.

8 Conclusion

The goal of this paper is to examine the (in)stability of fractional reserve banking. To that end, this paper builds a simple monetary model of fractional reserve banking that can capture the conditions for (in)stability under different specifications. The baseline model and its extension establish the conditions for the endogenous cycles and chaotic dynamics. The model also features stochastic cycles and self-fulling boom and bust under explicit conditions. The model shows that the fractional reserve banking can endanger stability in the sense that equilibrium is more prone to exhibit endogenous cyclic, chaotic, and stochastic dynamics under the lower reserve requirements. This result holds in the extended model with credit. However, lowering the reserve requirement increases the welfare at the steady state.

To evaluate the main predictions from the theory, I test the association between the required reserves ratio and real inside money volatility using cointegrating regression. I find a significant negative relationship between two variables and the results are robust with respect to different measures of inflation. Both theoretical and empirical evidence find a link between the reserve requirement policy and the (in)stability.

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Appendix

Appendix A Proofs

Proof of Proposition 1. For $m > 0$ and $\lambda_d > 0$, we have $\partial G_b/\partial d_b < \partial G_s/\partial d_s = 0$ since

$$0 = i_d - \lambda_d/\phi > i_d - \lambda_d/\phi - \alpha\lambda(q) \tag{27}$$
$$0 = -\phi i_l + \phi\alpha\lambda(q) > -\phi_t i_l \tag{28}$$

implying $i_l = \alpha\lambda(q)$, $\ell_b = m(1-\sigma)/\sigma\chi > 0$, $\ell_s = 0$, $\ell = (1-\sigma)\frac{m}{\chi}$, $d_s = m$, $(1-\sigma)m = d$. First order conditions from buyers' and sellers' problems give

$$i_l = \alpha\lambda(q), \qquad \frac{\lambda_d}{\phi} = [-\alpha\lambda(q) + i_d].$$

Now we can rewrite value functions at the FM for buyers and sellers as below:

$$G_b(m) = \alpha[u(q) - p] + W(m + \ell_b, 0, \ell_b) \tag{29}$$
$$G_s(m) = \alpha_s[p - c(q)] + W(m - d, d, 0). \tag{30}$$

where $q = v^{-1}(p)$ and $p = \min\{p^*, \phi(m + \ell_b)\}$. Derivatives $G_{j,t}(m)$ of $j \in \{b, s\}$ are

$$G'_{b,t}(m) = \phi_t - \phi_t i_{l,t}\frac{1-\sigma}{\sigma\chi} + \phi_t\frac{1-\sigma+\sigma\chi}{\sigma\chi}\alpha\lambda(q_t) \tag{31}$$
$$G'_{s,t}(m) = \phi_t + \phi_t i_{d,t} \tag{32}$$

Since $G'_t(m_t) = \sigma G'_{b,t}(m_t) + (1-\sigma)G'_{s,t}(m_t)$, we have below expression:

$$G'_t(m_t) = \phi_t\frac{1-\sigma+\sigma\chi}{\chi}\alpha\lambda(q_t) + \phi_t + \phi_t(1-\sigma)\left(i_{d,t} - \frac{i_{l,t}}{\chi}\right)$$

Substituting (15) and FOCs of buyers' and sellers' decision yields

$$G'_t(m_t) = \phi_t\left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha\lambda(q_t) + 1\right] \tag{33}$$

Substituting (33) into (2) and simplifying, we get

$$\phi_t = \begin{cases} \phi_{t+1}\beta\left[\dfrac{1-\sigma+\sigma\chi}{\chi}\alpha\lambda\circ v^{-1}(z_{t+1}) + 1\right] & \text{if } z_{t+1} < p^* \\ \phi_{t+1}\beta & \text{if } z_{t+1} \geq p^* \end{cases} \tag{34}$$

where $z_{t+1} = \phi_{t+1}m_{t+1}(1-\sigma+\sigma\chi)/\sigma\chi$. Let $1+i$ nominal interest rate at the steady state.[3] Then multiplying $m_t(1-\sigma+\sigma\chi)/\sigma\chi$ to both side of (34) allows us to collapse the conditions

[3] The nominal interest rate. For given the target interest rate

in 1 into one difference equation for real balances:

$$z_t = f(z_{t+1}) \equiv \frac{z_{t+1}}{1+i}\left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_{t+1})+1\right] \tag{35}$$

where $(1+i) \equiv \gamma/\beta$ and $L(z) \equiv \lambda \circ v^{-1}(z)$ is liquidity premium. ■

Proof of Proposition 2. Let $f^2(z) = f \circ f(z)$. With given the unique steady state, $f(z) > z$ for $z < z_s$ and $f(z) < z$ for $z > z_s$. Because $f(z)$ is linear increasing function for $z > p^*$, there exist a $\tilde{z} > p^*$ s.t $f(\tilde{z}) > p^*$. Since $\tilde{z} > p^*$ and $f(\tilde{z}) < \tilde{z}$, $\tilde{z}$ satisfies $f^2(\tilde{z}) < f(\tilde{z}) < \tilde{z}$. We can write slope of $f^2(z)$ as follows.

$$\frac{\partial f^2(z)}{\partial z} = f'[f(z)]f'(z) = f'(z)f'(z) = [f'(z)]^2$$

which implies $\partial f^2(z)/\partial z > 1$ when $f(z) < -1$. And it is easy to show $\partial f^2(0)/\partial z > 0$. With given $i > 0$ and $\chi > 0$, there exist a (z_1, z_2), satisfying $0 < z_1 < z_s < z_2$ which are fix points for $f^2(z)$ ■

Proof of Corollary 1. When DM trade is based on take-it-or-leave-it offer from buyer to seller with $c(q) = q$ and $-qu''(q)/u' = \eta$, f' can be written as

$$f'(q) = \frac{1}{1+i}\left\{\frac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u''(q)q+u'(q)-1\right]+1\right\} < -1$$

Using $u''(q)q = -\eta u'(q)$ gives

$$\frac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u'(q)(1-\eta)-1\right]+1 < -(1+i)$$

where $u'(q) = 1 + \frac{i\chi}{\alpha(1-\sigma+\sigma\chi)}$. Substituting $u'(q)$ and rearranging terms give

$$0 < \chi < \frac{\alpha\eta(1-\sigma)}{\eta(1-\alpha\sigma)+(2-\eta)(1+i)}$$

■

Proof of Corollary 2: Let $z_1 < z_2 < ...z_n$ be the periodic points of a n-cycle. Suppose $z_j > z_s$ for all $j = 1, 2, ..n$. By the definition of a n-period cycle, $z_1 = f(z_n) < z_n$ since $f(z) < z$ for $z > z_s$.

$$z_n = f(z_{n-1}) < z_{n-1} = f(z_{n-2}) < z_{n-2}... < z_1.$$

which shows the contradiction implying at least one periodic point satisfies $z_t < z_s < p^*$. ■

Proof of Proposition 3. I divide three period cycles into two cases.
Case 1: Let there exists a three-period cycle satisfying $z_1 < z_s < p^* < z_2 < z_3$. Since

$z_2, z_3 > p^*$, we have $z_2 = (1+i)z_1$, $z_3 = (1+i)z_2 = (1+i)^2 z_1$. Using (16) with $z_1 < p^*$ gives

$$\chi = \frac{(1-\sigma)\alpha L(z_1)}{(1+i)^3 - 1 - \sigma\alpha L(z_1)} \tag{36}$$

This three-period cycle should satisfy $z_1 < z_s < p^*$ and $z_2 = (1+i)z_1 > p^*$. First one can be easily shown using

$$0 = L(p^*) < L(z_s) = \frac{i}{\alpha(1-\sigma+\sigma\chi)}\chi < \frac{(1+i)^3-1}{\alpha(1-\sigma+\sigma\chi)}\chi = L(z_1)$$

since we have $L'(\cdot) < 0$. Since $dz_1/d\chi < 0$ the latter one, $z_1 > p^*/(1+i)$, is held when

$$0 < \chi < \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^3 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)}.$$

Case 2: Let there exists a three-period cycle satisfying $z_1 < z_2 < p^* \leq z_3$. If this type of

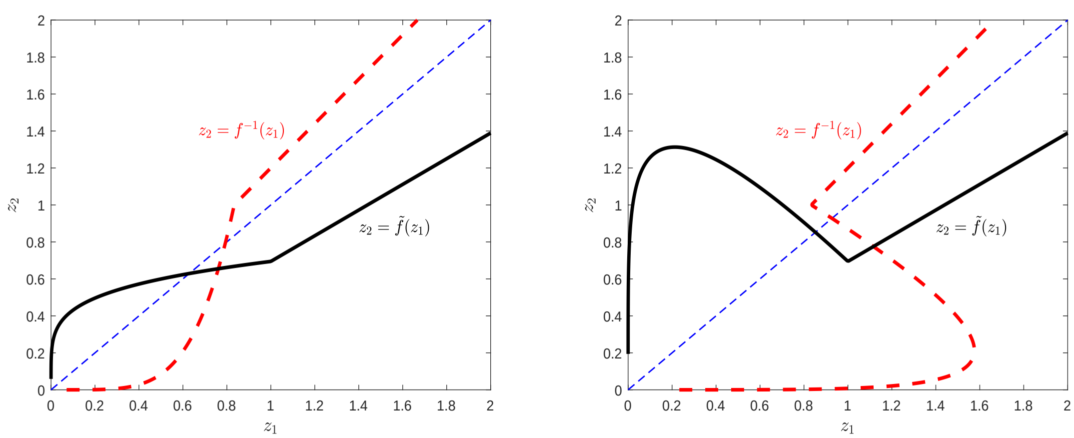

Figure 9: Intersection of $\tilde{f}(z)$ and $f(z)$

three-period cycle exist. Since $z_3 > p^*$, we have $z_3 = z_2(1+i)$ and (z_2, z_1) solves (37)-(38).

$$z_1 = f(z_2) = \left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_2) + 1\right]\frac{z_2}{1+i} \tag{37}$$

$$z_2 \equiv \tilde{f}(z_1) = \left[\frac{1-\sigma+\sigma\chi}{\chi}\alpha L(z_1) + 1\right]\frac{z_1}{(1+i)^2}. \tag{38}$$

These functions satisfies $f(x) > x$ for $x < z_s$, $f(x) < x$ for $x > z_s$, $\tilde{f}(x) > x$ for $x < \tilde{z}$ and $\tilde{f}(x) < x$ for $x > \tilde{z}$ where $\tilde{z}$ solves $\tilde{z} = \tilde{f}(\tilde{z})$. One can easily show $\tilde{z} < z_s$. Therefore any intersection between $z_1 = f(z_2)$ and $z_2 = \tilde{f}(z_1)$ satisfies $z_1 > z_2$ which contradicts to our initial conjecture $z_1 < z_2$. This implies there is no three-period cycles satisfying

$z_1 < z_2 < p^* \le z_3$. Therefore we can conclude that a three-period cycle exist when

$$0 < \chi < \frac{(1-\sigma)\alpha L\left(\frac{p^*}{1+i}\right)}{(1+i)^3 - 1 - \sigma\alpha L\left(\frac{p^*}{1+i}\right)}.$$

The existence of a three-cycle implies the existence of cycles of all orders and chaotic dynamics by the Sarkovskii theorem (Sharkovskii, 1964) and the Li-Yorke theorem (Li and Yorke, 1975). ■

Proof of Proposition 4. Since $f'(z_s) < 0$, there is an interval $[z_s - \epsilon_1, z_s + \epsilon_2]$, with $\epsilon_1, \epsilon_2 > 0$, such that $f(z_1) > f(z_2)$ for $z_1 \in [z_s - \epsilon_1, z_s)$, $z_2 \in (z_s, z_s + \epsilon_2]$. By definition (z_1, z_2) is a proper sunspot equilibrium if there exists (ζ_1, ζ_2), with $\zeta_1, \zeta_2 < 1$, such that

$$z_1 = \zeta_1 f(z_1) + (1-\zeta_1) f(z_2) \tag{39}$$

$$z_2 = (1-\zeta_2) f(z_1) + \zeta_2 f(z_2). \tag{40}$$

One can rewrite (39) and (40) as

$$\zeta_1 + \zeta_2 = \frac{z_1 - f(z_2) - z_2 + f(z_1)}{f(z_1) - f(z_2)} = \frac{z_1 - z_2}{f(z_1) - f(z_2)} + 1 < 1,$$

since $f(z_1) - f(z_2) > 0$ and $z_1 - z_2 < 0$. Therefore $\zeta_1 + \zeta_2 < 1$.

Because z_1 and z_2 are weighted averages of $f(z_1)$ and $f(z_2)$, where $f(z_1) > z_1$ and $f(z_2) < z_2$, by the uniqueness of the positive steady state, necessary and sufficient conditions for (39) and (40) are

$$f(z_2) < z_1 < f(z_1) \quad \text{and} \quad f(z_2) < z_2 < f(z_1)$$

We can reduce this to

$$z_2 < f(z_1) \quad \text{and} \quad z_1 > f(z_2).$$

because $z_1 < z_2$. The above inequalities can be written as

$$\frac{z_2 - z_s}{z_s - z_1} < -f'(z_s) < \frac{z_s - z_1}{z_2 - z_s}$$

If $-f'(z_s) < \frac{z_s - z_1}{z_2 - z_s}$ holds, we have $\frac{z_2 - z_s}{z_s - z_1} < -f'(z_s)$ since $-f'(z_s) > 1$. Therefore, any solution (z_1, z_2) on $[z_s - \epsilon_1, z_s + \epsilon_2]$ satisfies $-f'(z_s) < \frac{z_s - z_1}{z_2 - z_s}$ can be a proper sunspot and it is straightforward that multiple solutions exist.

■

Proof of Proposition 5. A two period cycle result is presented and three-period case will follow. Let there exists a two-period cycle satisfying $w_1 < q^* < w_2$ where $w_j = z_j + \bar{b}_j$. Since $w_2 > q^*$, we have $z_2 = (1+i)z_1$ and $\bar{b}_2 = (1+r)\bar{b}_1$ where q_1, $\bar{b}_1$, and z_1 solve

$$u'(q_1) = 1 + \chi\frac{(1+i)^2 - 1}{\alpha(1-\sigma+\sigma\chi)}, \quad \bar{b}_1 = [(1+r)^2 - 1]^{-1}\left\{\frac{i\mu\sigma\chi}{1-\sigma+\sigma\chi}\left[1 - \frac{(1+i)^2}{\beta}\right] z_1 + \mu\alpha\sigma[u(q_1) - q_1]\right\}$$

and $z_1 = q_1 - \bar{b}_1$. This two-period cycle should satisfy $q_1 < q^*$ and $w_2 = (1+i)z_1 + (1+r)\bar{b}_1 > q^*$. For given $i > 0$ and $\chi > 0$, first one can be easily shown using

$$1 = u'(q^*) < u'(q_s) = 1 + \frac{i}{\alpha(1-\sigma+\sigma\chi)}\chi < 1 + \frac{(1+i)^2-1}{\alpha(1-\sigma+\sigma\chi)}\chi = u'(q_1)$$

since we have $u''(\cdot) < 0$. Now we also can check the latter using below conditions

$$\begin{aligned}(1+r)q_1 >&(1+i)z_1 + (1+r)\bar{b}_1 = w_2 > q^* > q_1 = z_1 + \bar{b}_1 \quad \text{if } r > i \\ (1+i)q_1 >&(1+i)z_1 + (1+r)\bar{b}_1 = w_2 > q^* > q_1 = z_1 + \bar{b}_1 \quad \text{if } i > r.\end{aligned}$$

The sufficient conditions to have $w_2 > q^*$ is $q_1 > q^*/(1+r)$ for $r > i$ and $q_1 > q^*/(1+i)$ for $i > r$. Since we have $dq_1/d\chi < 0$, there exist a three period cycle $q_1 = w_1 < q_s < q^* < w_2 < w_3$ when

$$0 < \chi < \frac{(1-\sigma)\alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]}{(1+i)^2 - 1 - \sigma\alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]}$$

where $\iota = \max\{i, r\}$. Now, let there exists a three-period cycle satisfying $q_1 = w_1 < q_s < q^* < w_2 < w_3$ where $w_j = z_j + \bar{b}_j$. Since w_3, $w_2 > q^*$, we have $z_2 = (1+i)z_1$, $z_3 = (1+i)^2 z_1$, $\bar{b}_2 = (1+r)\bar{b}_1$ and $\bar{b}_3 = (1+r)^2\bar{b}_1$ where q_1, $\bar{b}_1$, and z_1 solve

$$u'(q_1) = 1 + \chi\frac{(1+i)^3-1}{\alpha(1-\sigma+\sigma\chi)}, \quad \bar{b}_1 = [(1+r)^3 - 1]^{-1}\left\{\frac{i\mu\sigma\chi}{1-\sigma+\sigma\chi}\left[1 - \frac{(1+i)^2}{\beta}\right]z_1 + \mu\alpha\sigma[u(q_1) - q_1]\right\}$$

and $z_1 = q_1 - \bar{b}_1$. This three-period cycle should satisfy $q_1 < q_s < q^*$ and $w_2 = (1+i)z_1 + (1+r)\bar{b}_1 > q^*$. For given $i > 0$ and $\chi > 0$, first one can be easily shown using

$$1 = u'(q^*) < u'(q_s) = 1 + \frac{i}{\alpha(1-\sigma+\sigma\chi)}\chi < 1 + \frac{(1+i)^3-1}{\alpha(1-\sigma+\sigma\chi)}\chi = u'(q_1)$$

since we have $u''(\cdot) < 0$. Now we also can check the latter using below conditions

$$\begin{aligned}(1+r)q_1 >&(1+i)z_1 + (1+r)\bar{b}_1 = w_2 > q^* > q_1 = z_1 + \bar{b}_1 \quad \text{if } r > i \\ (1+i)q_1 >&(1+i)z_1 + (1+r)\bar{b}_1 = w_2 > q^* > q_1 = z_1 + \bar{b}_1 \quad \text{if } i > r.\end{aligned}$$

The sufficient conditions to have $w_2 > q^*$ is $q_1 > q^*/(1+r)$ for $r > i$ and $q_1 > q^*/(1+i)$ for $i > r$. Since we have $dq_1/d\chi < 0$, there exist a three period cycle $q_1 = w_1 < q_s < q^* < w_2 < w_3$ when

$$0 < \chi < \frac{(1-\sigma)\alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]}{(1+i)^3 - 1 - \sigma\alpha[u'\left(\frac{q^*}{1+\iota}\right) - 1]}$$

where $\iota = \max\{i, r\}$. Again, the existence of a three-cycle implies the existence of cycles of all orders and chaotic dynamics by the Sarkovskii theorem and the Li-Yorke theorem. ■

Proof of Proposition 6. Consider $z_t = f(z_{t+1})$. If $z_s > \bar{z}$ where $\bar{z}$ solves $f'(\bar{z}) = 0$. In this case, there exist multiple equilibria. If $q^* \leq f(\bar{z})$, then there exist equilibria $\{z_t\}_{t=0}^{\infty}$ with $z_T \equiv \max\{z_t\}_{t=0}^{\infty} > q^*$ (bubble) which crashes to 0 (burst) as $t \to \infty$, where $T \geq 1$ and

$z_T > z_0$. Then there exist equilibria with bubble-burst as a self-fulfilling crisis. Conditions for this case are shown as below. Similar to Corollary 1, consider take-it-leave-it offer with $-qu''/u' = \eta$ and $c(q)$. Then we have following difference equation:

$$z_t = f(z_{t+1}) \begin{cases} \dfrac{z_{t+1}}{1+i}\left\{\dfrac{1-\sigma+\sigma\chi}{\chi}\alpha\left[u'(z_{t+1})-1\right]+1\right\} & \text{if } z_{t+1} < q^* \\ \dfrac{z_{t+1}}{1+i} & \text{if } z_{t+1} \geq q^* \end{cases} \tag{41}$$

Step 1: [Concavity of f, $f'' < 0$] For $z_{t+1} < q^*$ and $\eta \in (0,1)$, $f''(z_{t+1}) < 0$ since

$$\begin{aligned} f''(z) &= \frac{\alpha(1-\sigma+\sigma\chi)}{(1+i)\chi}u'' + \frac{z}{1+i}\frac{\alpha(1-\sigma+\sigma\chi)}{\chi}\alpha[-u''\eta z - u'\eta] \\ &= \frac{\alpha(1-\sigma+\sigma\chi)}{(1+i)\chi}u''[1+q-\eta q] < 0 \end{aligned}$$

Step 2: [Multiplicity i.e., $z_s > \bar{z}$ where $\bar{z}$ solves $f'(\bar{z}) = 0$] Consider the following condition.

$$f'(\bar{z}) = \frac{\alpha(1-\sigma+\sigma\chi)}{\chi}[u'(\bar{z})(1-\eta)-1] = 0$$

Since $z_s > \bar{z} \to u'(z_s) < u'(\bar{z})$, we have

$$u'(z_s) = 1 + \frac{i\chi}{\alpha(1-\sigma+\sigma\chi)} < 1 - \frac{(2+i)\chi}{\alpha(1-\sigma+\sigma\chi)} = u'(\bar{z}).$$

This can be reduced as

$$\chi < \frac{(1-\sigma)\alpha\eta}{2+i(2-\eta)-\alpha\sigma\eta}$$

Step 3: [Show $q^* \leq f(\bar{z})$] It is straightforward to show that $q^* < f(\bar{z})$ holds when

$$\chi < \frac{(1-\sigma)\alpha\eta(1+i)}{(1-\eta)^2q^* + (1+i)[(1-\eta)(3+i-\eta)-\alpha\sigma\eta]}$$

Therefore, when

$$0 < \chi < \min\left\{\frac{(1-\sigma)\alpha\eta(1+i)}{(1-\eta)^2q^* + (1+i)[(1-\eta)(3+i-\eta)-\alpha\sigma\eta]}, \frac{(1-\sigma)\alpha\eta}{2+i(2-\eta)-\alpha\sigma\eta}\right\}$$

there exist $\{z_t\}_{t=0}^{\infty}$ satisfying $z_T \equiv \max\{z_t\}_{t=0}^{\infty} > q^*$ and $\lim_{t\to\infty} z_t = 0$, where $T \geq 1$ and $z_T > z_0 > q^*/(1+i)$. ■

Appendix B Closed Form Solutions for Calibration

$$q(i) = \left\{ \frac{1}{A} \left(1 + \frac{i\chi}{\alpha[1-\sigma+\sigma\chi]} \right) \right\}^{-\frac{1}{\eta}}$$

$$q'(i) = \frac{-\chi}{\alpha A[1-\sigma+\sigma\chi]\eta} \left\{ \frac{1}{A} \left(1 + \frac{i\chi}{\alpha[1-\sigma+\sigma\chi]} \right) \right\}^{-\frac{1+\eta}{\eta}}$$

$$u(q) = A\frac{q^{1-\eta}}{1-\eta}$$

$$\bar{b} = \frac{1-\sigma+\sigma\chi}{1-\sigma+\sigma\chi-i\mu\sigma\chi} \left\{ \frac{\mu\sigma\alpha}{r}[u(q)-q] - \frac{i\mu\sigma\chi}{1-\sigma+\sigma\chi}q \right\}$$

$$\begin{aligned} \bar{b}'(i) =& \frac{i\chi\mu\sigma(1-r)q'(i)}{r(1-\sigma+\sigma\chi-i\mu\sigma\chi)} \\ &- \frac{(1-\sigma+\sigma\chi)(\mu\sigma\chi)^2}{(1-\sigma+\sigma\chi-i\mu\sigma\chi)^2} \left\{ \frac{u(q)-q}{r} - \frac{iq}{1-\sigma+\sigma\chi} \right\} \\ &- \frac{\mu\sigma\chi q}{1-\sigma+\sigma\chi-i\mu\sigma\chi} \end{aligned}$$

$$z'(i) = q'(i) - b'(i), \quad z = q - \bar{b}$$

$$L(i) = \frac{z+\bar{b}}{z+\bar{b}+B}$$

$$Z(i) = \frac{z}{z+\bar{b}+B}, \quad D(i) = \frac{\bar{b}}{z+\bar{b}+B}$$

the elasticity of q/Y with respect to i

$$\frac{\partial L(i)}{\partial i}\frac{i}{L(i)} = \frac{Biq'(i)}{[B+q(i)]q(i)}$$

Appendix C Empirical Appendix

This section provides robustness check for empirical result. To check sensitivity of the results, following results repeat all the empirical analysis using different measure of price level: Consumer Price Index Excluding Food and Energy: (Core CPI), Personal Consumption Expenditures: Chain-type Price Index (PCE) and Personal Consumption Expenditures Excluding Food and Energy: Chain-Type Price Index (Core PCE). This section also provides robustness checks using quarterly series instead of annual data.

Table 4: Effect of Require Reserve Ratio: Robustness Check

(a) Benchmark: CPI

	OLS (1)	CCR (2)	FMOLS (3)
χ	-0.283^{***}	-0.245^{***}	-0.211^{***}
	(0.031)	(0.002)	(0.003)
`ffr`		-0.109^{***}	-0.248^{***}
		(0.002)	(0.003)
Constant	0.074^{***}	0.074^{***}	0.078^{***}
	(0.004)	(0.000)	(0.000)
Obs.	49	49	49
R^2	0.706	0.637	0.144

(b) Core CPI

	OLS (1)	CCR (2)	FMOLS (3)
χ	-0.267^{***}	-0.221^{***}	-0.192^{***}
	(0.027)	(0.003)	(0.003)
`ffr`		-0.125^{***}	-0.248^{***}
		(0.003)	(0.004)
Constant	0.070^{***}	0.071^{***}	0.074^{***}
	(0.004)	(0.000)	(0.000)
Obs.	49	49	49
R^2	0.734	0.663	0.133

(c) PCE

	OLS (1)	CCR (2)	FMOLS (3)
χ	-0.306^{***}	-0.227^{***}	-0.189^{***}
	(0.029)	(0.004)	(0.005)
`ffr`		-0.187^{***}	-0.350^{***}
		(0.004)	(0.005)
Constant	0.074^{***}	0.075^{***}	0.079^{***}
	(0.004)	(0.000)	(0.000)
Obs.	49	49	49
R^2	0.746	0.664	0.121

(d) Core PCE

	OLS (1)	CCR (2)	FMOLS (3)
χ	-0.307^{***}	-0.220^{***}	-0.182^{***}
	(0.027)	(0.005)	(0.005)
`ffr`		-0.207^{***}	-0.362^{***}
		(0.004)	(0.006)
Constant	0.073^{***}	0.073^{***}	0.077^{***}
	(0.004)	(0.000)	(0.001)
Obs.	49	49	49
R^2	0.769	0.680	-0.042

Notes: For (1), OLS estimates are reported and Newey-West standard errors with lag 1 are reported in parentheses. For (2) and (3), first stage long-run variance estimations for CCR and FMOLS are based quadratic spectral kernel and Bayesian information criterion. The bandwidth selection is based on Newey-West fixed lag, $4 \times (T/100)^{2/9}$. ***, **, and * denote significance at the 1, 5, and 10 percent levels, respectively.

Table 5: Unit Root Test: Robustness Check

(a) Benchmark: CPI

	Phillips-Perron test		ADF test
	$Z(\rho)$	$Z(t)$	$Z(t)$ w/ lag 1
σ_t^{Roll}	-4.708	-2.191	-2.090
$\Delta\sigma_t^{Roll}$	-24.905^{***}	-3.416^{**}	-2.942^{**}

(b) Core CPI

	Phillips-Perron test		ADF test
	$Z(\rho)$	$Z(t)$	$Z(t)$ w/ lag 1
σ_t^{Roll}	-4.681	-2.189	-1.978
$\Delta\sigma_t^{Roll}$	-24.758^{***}	-3.509^{***}	-2.942^{***}

(c) PCE

	Phillips-Perron test		ADF test
	$Z(\rho)$	$Z(t)$	$Z(t)$ w/ lag 1
σ_t^{Roll}	-4.329	-2.038	-2.047
$\Delta\sigma_t^{Roll}$	-23.691^{***}	-3.330^{***}	-2.842^{**}

(d) Core PCE

	Phillips-Perron test		ADF test
	$Z(\rho)$	$Z(t)$	$Z(t)$ w/ lag 1
σ_t^{Roll}	-4.076	-1.954	-1.930
$\Delta\sigma_t^{Roll}$	-22.826^{***}	-3.296^{**}	-2.768^{**}

Table 6: Johansen Test for Cointegration: Robustness Check

(a) Benchmark: CPI

Max rank	$\lambda_{trace}(r)$	5% CV	1% CV
0	35.6880	29.68	35.65
1	**10.6820**	15.41	20.04
2	4.5391	3.76	6.65
Max rank	$\lambda_{max}(r, r+1)$	**5% CV**	**1% CV**
0	25.0060	20.97	25.52
1	**6.1429**	14.07	18.63
2	4.5391	3.76	6.65

(b) Core CPI

Max rank	$\lambda_{trace}(r)$	5% CV	1% CV
0	35.1449	29.68	35.65
1	**10.0645**	15.41	20.04
2	4.2011	3.76	6.65
Max rank	$\lambda_{max}(r, r+1)$	**5% CV**	**1% CV**
0	25.0804	20.97	25.52
1	**5.8635**	14.07	18.63
2	4.2011	3.76	6.65

(c) PCE

Max rank	$\lambda_{trace}(r)$	5% CV	1% CV
0	35.3667	29.68	35.65
1	**9.8942**	15.41	20.04
2	3.9605	3.76	6.65
Max rank	$\lambda_{max}(r, r+1)$	**5% CV**	**1% CV**
0	25.4725	20.97	25.52
1	**5.9337**	14.07	18.63
2	3.9605	3.76	6.65

(d) Core PCE

Max rank	$\lambda_{trace}(r)$	5% CV	1% CV
0	35.0280	29.68	35.65
1	**9.3450**	15.41	20.04
2	3.6465	3.76	6.65
Max rank	$\lambda_{max}(r, r+1)$	**5% CV**	**1% CV**
0	25.6830	20.97	25.52
1	**5.6986**	14.07	18.63
2	3.6465	3.76	6.65

Table 7: Robustness Check: Quarterly Series

(a) Effect of Require Reserve Ratio

	OLS	CCR	FMOLS
	(1)	(2)	(3)
χ	-0.286^{***}	-0.405^{***}	-0.464^{***}
	(0.016)	(0.000)	(0.000)
`ffr`		-0.120^{***}	-0.279^{***}
		(0.000)	(0.000)
Constant	0.074^{***}	0.080^{***}	0.077^{***}
	(0.002)	(0.000)	(0.000)
Obs.	192	192	192
R^2	0.719	0.403	0.081

(b) Unit Root Test

	Phillips-Perron test		ADF test
	$Z(\rho)$	$Z(t)$	$Z(t)$ w/ lag 1
`ffr`	-8.900	-1.989	-2.219
χ	-1.263	-1.092	-1.150
σ_t^{Roll}	-3.946	-2.372	-2.227
Δ`ffr`	-136.820^{***}	-10.679^{***}	-10.179^{***}
$\Delta\chi$	-160.164^{***}	-12.130^{***}	-9.804^{***}
$\Delta\sigma_t^{Roll}$	-40.319^{***}	-4.515^{**}	-5.627^{**}

(c) Johansen Test for Cointegration

Max rank	$\lambda_{trace}(r)$	5% CV	1% CV
0	35.5243	29.68	35.65
1	**15.2586**	15.41	20.04
2	4.0275	3.76	6.65
Max rank	$\lambda_{max}(r, r+1)$	5% CV	1% CV
0	20.2657	20.97	25.52
1	11.2311	14.07	18.63
2	4.0275	3.76	6.65

Notes: For (1), OLS estimates are reported and Newey-West standard errors with lag 1 are reported in parentheses. For (2) and (3), first stage long-run variance estimations for CCR and FMOLS are based quadratic spectral kernel and Bayesian information criterion. The bandwidth selection is based on Newey-West fixed lag, $4 \times (T/100)^{2/9}$ and `ffr` denotes federal funds rates and σ_t^{Roll} denotes cyclical volatility of real inside money balances. ***, **, and * denote significance at the 1, 5, and 10 percent levels, respectively.