Exact Hybrid Covariance Thresholding for Joint Graphical Lasso

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Overview

we design a hybrid algorithm to achieve non-uniform thresholding for Group Graphical Lasso problem by

- Proposing a novel hybrid covariance thresholding algorithm that can effectively identify zero entries in the precision matrices and split a large joint graphical lasso problem into many small subproblems.
- Establishing necessary and sufficient conditions for our hybrid covariance thresholding algorithm.

Advantage: Our hybrid covariance thresholding method can split group graphical lasso into much smaller subproblems, each of which can be solved very fast. Experimental results on both synthetic and real data validate the superior performance of our thresholding method over the others.

Joint Graphical Lasso

Gaussian Graphical Model Joint Graphical Lasso A typical joint graphical lasso is formulated as the following optimization problem:

$$\min \sum_{k=1}^{K} L(\mathbf{\Theta}^{(k)}) + P(\mathbf{\Theta}) \tag{1}$$

Where $\mathbf{\Theta}^{(k)} \succ 0$ is the precision matrix $(k = 1, \ldots, K)$ and $\mathbf{\Theta}$ represents the set of $\mathbf{\Theta}^{(k)}$. The negative log-likelihood $L(\mathbf{\Theta}^{(k)})$ and the regularization $P(\mathbf{\Theta})$ are defined as follows.

$$L(\mathbf{\Theta}^{(k)}) = -\log\det(\mathbf{\Theta}^{(k)}) + \mathrm{tr}(\mathcal{S}^{(k)}\mathbf{\Theta}^{(k)}) \tag{2}$$

$$P(\mathbf{\Theta}) = \lambda_1 \sum_{k=1}^{K} \|\mathbf{\Theta}^{(k)}\|_1 + \lambda_2 J(\mathbf{\Theta}) \tag{3}$$

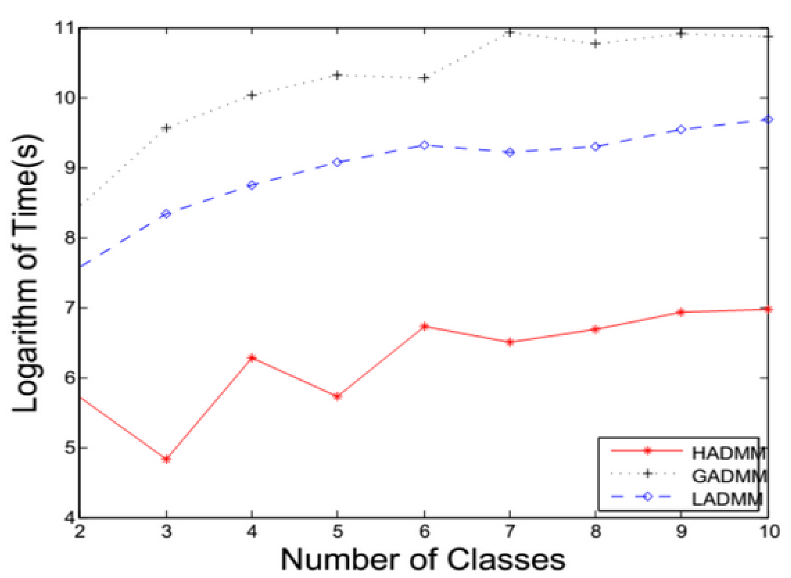
Here $\lambda_1 > 0$ and $\lambda_2 > 0$ and $J(\mathbf{\Theta})$ is some penalty function used to encourage similarity (of the structural patterns) among the K classes. In this paper, we focus on group graphical lasso. That is,

$$J(\mathbf{\Theta}) = 2 \sum_{1 \le i < j \le p} \sqrt{\sum_{k=1}^{K} (\mathbf{\Theta}_{i,j}^{(k)})^2} \tag{4}$$

Experiments

- Synthetic data

Experiment (Synthetic)

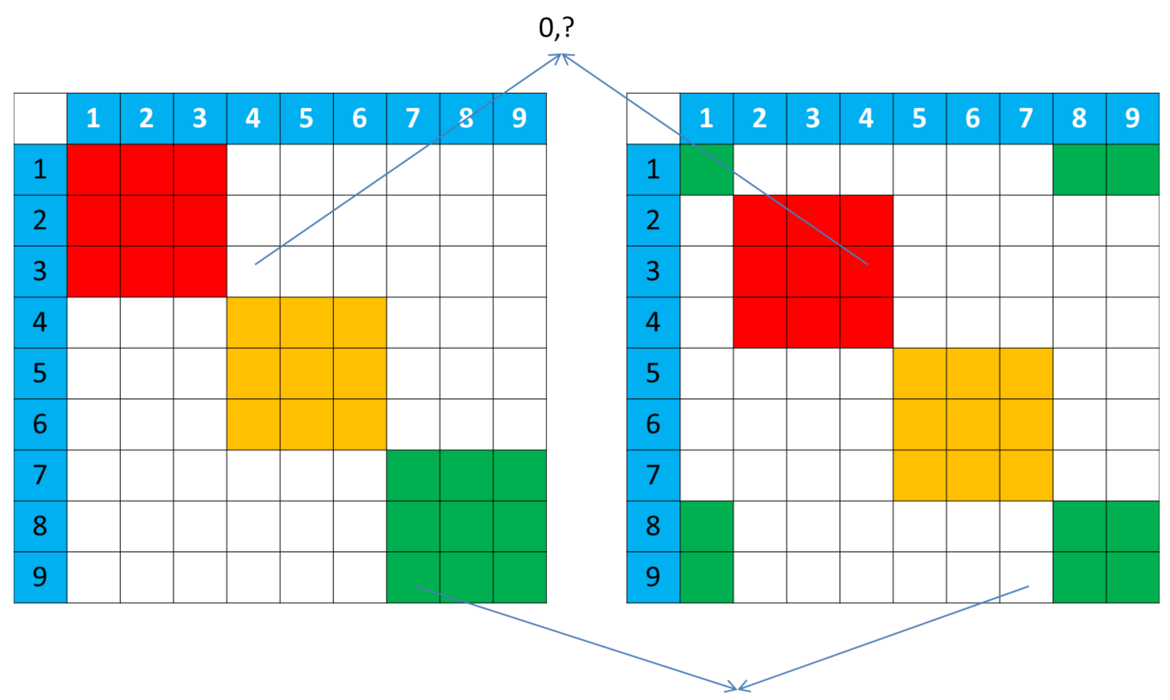


- Real data

Method	Setting 1	Setting 2	Setting 3	Setting 4	Setting 5
HADMM	3.46	8.23	3.9	1.71	1.11
LADMM	> 12	> 12	> 12	3.72	1.98
GADMM	4.2	> 12	> 12	11.04	6.93

Non-uniform Thresholding

Non-uniform thresholding generates a non-uniform feasible partition by thresholding the K empirical covariance matrices separately. In a non-uniform partition, two variables of the same group in one class may belong to different groups in another class. Figure **??** shows an example of non-uniform partition. In this example, all the matrix elements in white color are set to 0 by non-uniform thresholding. Except the white color, each of the other colors indicates one group. The 7$^{\text{th}}$ and 9$^{\text{th}}$ variables belong to the same group in the left matrix, but not in the right matrix. Similarly, the 3$^{\text{rd}}$ and 4$^{\text{th}}$ variables belong to the same group in the right matrix, but not in the left matrix.



Algorithm 1 Hybrid Covariance Screening Algorithm

for $k = 1$ *to* K **do**
  Initialize $\mathbf{I}_{i,j}^{(k)} = \mathbf{I}_{j,i}^{(k)} = 1, \forall 1 \le i < j \le p$
  Set $\mathbf{I}_{i,j}^{(k)} = 0$, if $|\boldsymbol{S}_{i,j}^{(k)}| \le \lambda_1$ and $i \ne j$
  Set $\mathbf{I}_{i,j}^{(k)} = 0$, if $\sum_{k=1}^{K}(|\boldsymbol{S}_{i,j}^{(k)}| - \lambda_1)_+^2 \le \lambda_2^2$ and $i \ne j$
end for
for $k = 1$ *to* K **do**
  Construct a graph $\boldsymbol{G}^{(k)}$ for $\mathcal{V}$ from $\boldsymbol{I}^{(k)}$
  Find connected components of $G^{(k)}$
  for $\forall(i,j)$ *in the same component of* $\boldsymbol{G}^{(k)}$ **do**
    Set $\boldsymbol{I}_{i,j}^{(k)} = \boldsymbol{I}_{j,i}^{(k)} = 1$
  end for
end for
repeat
  Search for triple (x, i, j) satisfying the following condition:
  $\boldsymbol{I}_{i,j}^{k} = 0, |\boldsymbol{S}_{i,j}^{(x)}| > \lambda_1$ and $\exists s$, s.t. $\boldsymbol{I}_{i,j}^{(s)} = 1$
  if $\exists(x, i, j)$ satisfies the condition above **then**
    merge the two components of $\boldsymbol{G}^{(x)}$ that containing variable i and j into new component;
    for $\forall(m, n)$ in this new component **do**
      Set $\boldsymbol{I}_{m,n}^{(x)} = \boldsymbol{I}_{n,m}^{(x)} = 1$;
    end for
  end if
until No such kind of triple.
return the connected components of each graph which define the non-uniform feasible solution

Supplemental and Code

Code is available at www.harryyang.xyz