THE EE EPP TEAM

ELECTRICAL ENGINEERING PRINCIPLES & PRACTICE II

ELECTRICAL ENGINEERING

NATIONAL UNIVERSITY OF SINGAPORE

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Contents

Introduction to AC Circuit

DC versus AC

When you learnt about voltage and current in EPP1, the power sources were battery or PV. When a load is connected to such a source, the current flows in one particular direction depending on the polarity of the source which is fixed. These sources are examples of **DC** (direct current) sources.

On the contrary, the polarity of an **AC** source keeps alternating with time. If a load is connected, the direction of current also alternates. The power supply that we get from the utility companies is an AC source.

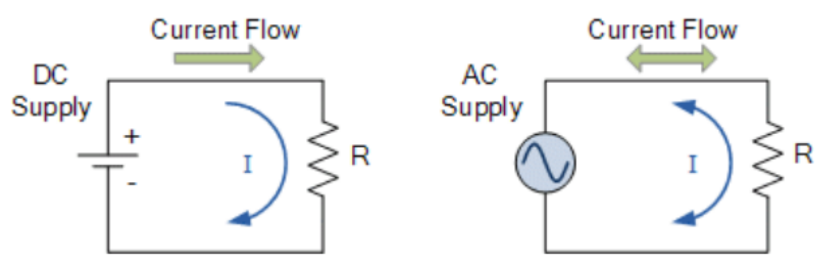


Figure 1: Current flows in one direction in a DC circuit but the direction alternates in an AC circuit.

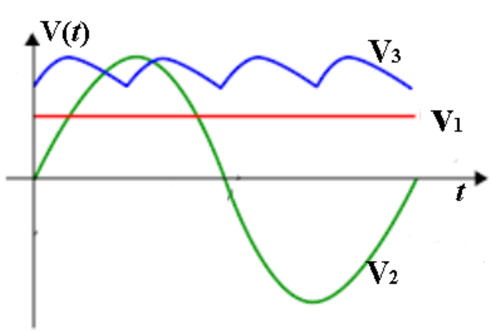


Figure 2: AC voltage or current is always a time-varying quantity (V_2 is AC voltage). DC voltage or current can be either constant or time-varying. V_1 and V_3 are both DC; V_1 is constant while V_3 is time varying.

If you observe a DC voltage using an oscilloscope, you will not see any change in polarity (sign) of the voltage. But when you observe an AC voltage, the polarity keeps alternating with time.

Waveform

Waveform is the plot or graph of a time-varying quantity (voltage or current) versus time (Figure 2). A waveform is **periodic** if it repeats the exact same shape again and again.
Sinusoidal waveform is by far the most common waveform used in AC circuit analysis.

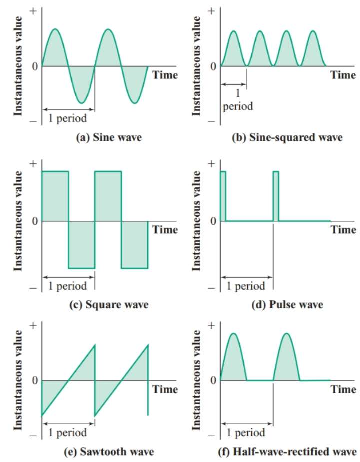


Figure 3: Some common periodic waves (Source of the image: electricalacademia.com)

> **Why Sinusoidal Waveform?**
>
> - Voltage provided by the utility companies is sinusoidal. Generators at power station, a rotating machine, produces sinusoidal voltage.
> - Any periodic waveform can be reconstructed by adding a set of sinusoidal waves (Fourier series). You will learn more about this in higher level modules such as EE2023. A simple explanation of this concept is given using graphical illustration at the end of this chapter.

Sinusoidal variation of any variable, *e.g.*, voltage or current in ac circuits, can be represented using either the sine function or the cosine function, for example,

$$
\begin{aligned}
v(t) &= V_m \sin(\omega t + \theta) \\
i(t) &= I_m \cos(\omega t + \theta).
\end{aligned}
$$

The parameters ω, θ, V_m, and I_m are explained in the next few sub-sections.

Waveform parameters: Period and Frequency

A periodic waveform repeats the exact same shape again and again. The interval of repetition is the **period** of the waveform. The period of the waveform shown in Figure 4 is 0.04 second or 40 milliseconds.

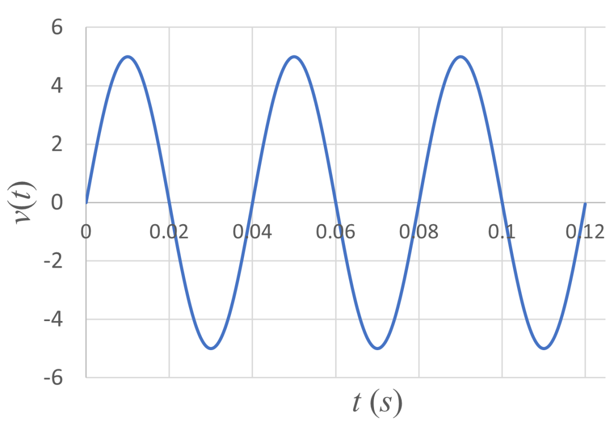


Figure 4: A periodic waveform with period of 40 millisecond or 0.04 s

Frequency f is the number of repetitions in unit time. If T is the period, then the frequency is

$$f = \frac{1}{T}.$$

Taking *second* as the unit of time, the unit of frequency is *cycles per second* which is also known as **hertz (Hz)**. The frequency of the waveform shown in Figure 4 is

$$f = \frac{1}{40 \times 10^{-3}} = 25Hz.$$

Waveform parameter: Angular frequency

The functions $\sin(x)$ and $\cos(x)$ repeat themselves every 2π radians or $360°$, which is the period of these two functions. The unit for the variable x is radians.

Time-varying sinusoidal functions $\sin(\omega t)$ and $\cos(\omega t)$ repeat themselves every T seconds where T is the period of the waveform. So,

$$\begin{aligned} \omega T &= 2\pi, \\ \omega &= \frac{2\pi}{T}. \end{aligned}$$

ω is the angular frequency of the sinusoidal waveform, and its unit is rad/s. Since $f = \frac{1}{T}$, the angular frequency can also be expressed in terms of f:

$$\omega = 2\pi f.$$

Points to note:

- Period and frequency are parameters of any periodic waveform regardless of its shape.
- Angular frequency is a parameter used only for a sinusoidal waveform.

Waveform parameter: Amplitude

The parameter V_m of the sinusoidal waveform

$$V_m \sin(\omega t)$$

is the **amplitude** of the sinusoidal waveform. It represents the maximum deviation from the mean. The amplitude is 5 for the waveform shown in Figure 4.

For a DC-shifted waveform

$$v(t) = 2 + 5\sin(\omega t)$$

shown in Figure 5, though the variable v varies between 7 and -3, the amplitude is 5. The mean of this waveform is 2.

Waveform parameter: Phase

In both Figure 4 and Figure 5,

$$v(t)|_{t=0} = V_{mean}.$$

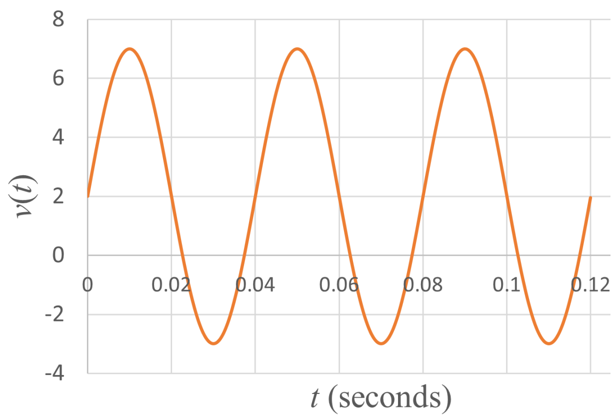


Figure 5: DC-shifted sinusoidal waveform

In other words,

$$\begin{aligned} v(0) &= 5\sin(\omega \times 0) \quad \textit{for the first one,} \\ v(0) &= 2 + 5\sin(\omega \times 0) \quad \textit{for the 2nd one,} \end{aligned}$$

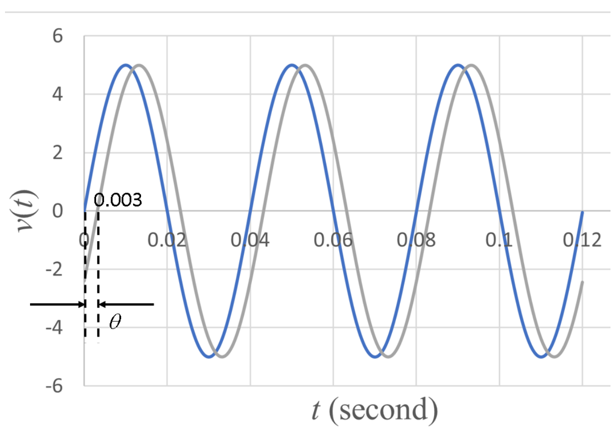


Figure 6: DC-shifted sinusoidal waveform

The parameter "**phase**" is needed to describe a sinusoidal waveform that shows an offset in time-axis. In Figure 6, the blue colored graph represents the function $5\sin(\omega t)$. The grey colored graph is also a sine wave but with an offset in time-axis. The phase θ in

$$V_m \sin(\omega t + \theta)$$

describes the offset in time-axis.

For the grey-colored waveform in Figure 6, $\sin(0)$ occurs at $t = 0.003$ s.

So,

$$\begin{aligned} 5\sin(0) &= 5\sin(\omega \times 0.003 + \theta), \\ \omega \times 0.003 + \theta &= 0, \\ \theta &= -0.003\omega, \\ \theta &= \frac{-0.003}{T} \times 2\pi. \end{aligned}$$

Use this eqn if its lagging. If its leading use $A\sin(\omega t - \phi)$

The grey-colored waveform can be expressed as

$$5\sin(\omega t - \frac{0.003}{T} \times 2\pi) = 5\sin(\omega t - 0.47).$$

This waveform has a negative phase and is said to be **lagging** the graph of $\sin(\omega t)$ by 0.47 rad.

If you plot the waveform

$$5\sin(\omega t + 0.5),$$

you will notice that this one appears earlier in time with respect to the waveform $5\sin(\omega t)$. The waveform $5\sin(\omega t + 0.5)$ is said to be **leading** the graph of $5\sin(\omega t)$ by 0.5 rad.

The unit of ω is rad/s and hence the unit of ωt is rad. In the two examples shown here, the unit of phase is also rad. However, in many books and published literature, the phase is expressed in degrees ($\circ$).

Sine or Cosine?

Sinusoidal variation can be expressed using either *sine* or *cosine* as these two functions have identical shape but with offset in time axis that results in a phase shift of $\frac{pi}{2}$ rad or 90°.

$$\begin{aligned} \cos(\omega t - \frac{\pi}{2}) &= \cos(\omega t)\cos(\frac{\pi}{2}) + \sin(\omega t)\sin(\frac{\pi}{2}) \\ &= \sin(\omega t). \end{aligned}$$

The waveform $\sin(\omega t)$ lags the waveform $\cos(\omega t)$ by $\frac{\pi}{2}$ rad.

$$\begin{aligned} \sin(\omega t + \frac{\pi}{2}) &= \sin(\omega t)\cos(\frac{\pi}{2}) + \cos(\omega t)\sin(\frac{\pi}{2}) \\ &= \cos(\omega t). \end{aligned}$$

The waveform $\cos(\omega t)$ leads the waveform $\sin(\omega t)$ by $\frac{\pi}{2}$ rad.

Summary

The general expression of a sinusoidal waveform is

$$v(t) = V_m \sin(\omega t + \theta)$$

or

$$v(t) = V_m \cos(\omega t + \theta)$$

- The **amplitude** V_m represents the magnitude of the maximum deviation from the mean

- ω is the angular frequency which is related to the frequency (in Hz) and the period of the waveform:

$$\omega = 2\pi f = \frac{2\pi}{T}.$$

- The **phase** θ represents offset in time-axis with respect to another waveform.
  - If θ is positive, the waveform is said to be leading with respect to the reference waveform.
  - If θ is negative, the waveform is said to be lagging with respect to the reference waveform.

RMS voltage and RMS current

Although the voltage at the power socket varies with time, we do not use a time-varying function when we refer to these voltages. Instead, we say 230 V, 50 Hz (in Singapore, UK, Australia etc.) or 120 V, 60 Hz (in the USA or Canada) power supply. → RMS value

Similarly, when an equipment is connected to such socket, the current drawn is also a time-varying quantity but we say 15 A or 3 A to refer to the current drawn.

These constant numbers used to describe the magnitude of AC voltages or currents are their **RMS** (root mean square) values.

As the name suggests, the RMS value is the square **root** of the **mean** of the **square** of a time-varying quantity.

In case of a set of N discrete values $(x_1, x_2, x_3, \cdots x_N)$,

$$X_{rms} = \sqrt{\frac{x_1^2 + x_2^2 + x_3^2 + \cdots + x_N^2}{N}}.$$

For a continuous function $x(t)$,

$$X_{rms} = \lim_{T \to \infty} \sqrt{\frac{1}{T} \int_0^T x^2(t)dt}.$$

If the continuous function $x(t)$ is periodic with period T_p then,

$$X_{rms} = \sqrt{\frac{1}{T_p} \int_0^{T_p} x^2(t)dt}.$$

AC Circuit Analysis

The circuit laws and device characteristics, *i.e.*.

- KVL,

- KCL,
- Ohm's law etc.

are same in both DC circuit and AC circuit. What make the AC circuit analysis difficult and often challenging are

1. time-varying nature of the voltage and current, and
2. properties of capacitor and inductor involve rate of change of voltage and current, respectively.

Let's consider the simple cases where only one circuit element (a resistor, an inductor, or a capacitor) is connected to a sinusoidal voltage source.

Load is a resistor

Instantaneous voltage ($v_R(t)$) across the resistor is same as the source voltage:

$$v_R(t) = V_m \sin(\omega t).$$

Applying Ohm's law, the instantaneous current through the resistor is

$$i_R(t) = \frac{V_m \sin(\omega t)}{R} = I_m \sin(\omega t),$$

where, the amplitude of the current waveform is

$$I_m = \frac{V_m}{R}.$$

- The current waveform and the voltage waveform have same frequency.
- The current waveform is in-phase with the voltage waveform (zero phase difference between them).
- RMS values of the voltage and the current are

$$V_{R,rms} = \frac{V_m}{\sqrt{2}}, \quad I_{R,rms} = \frac{I_m}{\sqrt{2}}.$$

- Ratio of the peak voltage to the peak current, or the RMS voltage to the RMS current is equal to R:

$$\frac{V_m}{I_m} = R, \quad \frac{V_{R,rms}}{I_{R,rms}} = R.$$

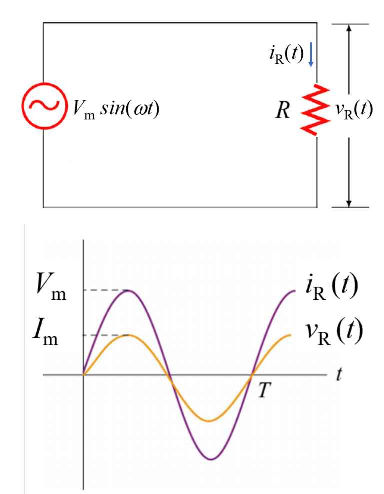


Figure 7: Top: A resistor connected to a sinusoidal voltage source, Bottom: Voltage waveform and current waveform

Load is an inductor

Instantaneous voltage ($v_L(t)$) across the inductor:

$$v_L(t) = V_m \sin(\omega t).$$

For an inductor (L Henry),

$$v_L(t) = L\frac{di_L}{dt}.$$

For the circuit shown,

$$\frac{di_L}{dt} = \frac{V_m}{L}\sin(\omega t),$$

By integrating,

$$i_L = \frac{V_m}{R}\int \sin(\omega t)dt = -\frac{V_m}{\omega L}\cos(\omega t).$$

Using the trigonometric identity

$$\sin(\omega t - \frac{\pi}{2}) = -\cos(\omega t),$$

we get

$$i_L = \frac{V_m}{\omega L}\sin(\omega t - \frac{\pi}{2}) = I_m \sin(\omega t - \frac{\pi}{2}).$$

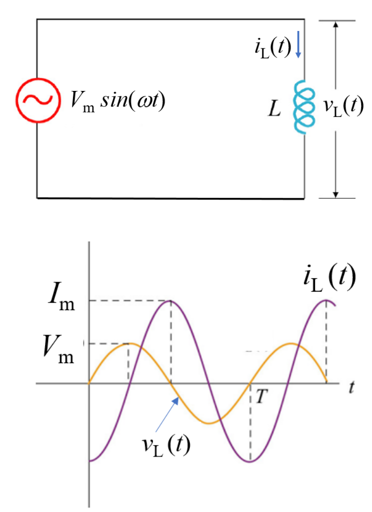


Figure 8: Top: An inductor connected to a sinusoidal voltage source, Bottom: Voltage waveform and current waveform

- The current waveform and the voltage waveform have same frequency.
- Voltage and current are not in-phase. Inductor current waveform lags the inductor voltage waveform by $\frac{\pi}{2}$ rad.
- Ratio of the peak voltage to the peak current, or the RMS voltage to the RMS current is equal to $X_L = \omega L$:

$$\frac{V_m}{I_m} = \omega L, \quad \frac{V_{L,rms}}{I_{L,rms}} = \omega L,$$

$X_L = \omega L$ is called the **inductive reactance**.

The SI units of X_L is **ohm** (Ω), similar to the unit of resistance. However, unlike resistance, the value of X_L varies linearly with varying angular frequency ω. Thus, for an inductor, the resistance to current flow increases with increasing frequency of the source voltage.

Load is a capacitor

Instantaneous voltage ($v_C(t)$) across the capacitor:

$$v_C(t) = V_m \sin(\omega t).$$

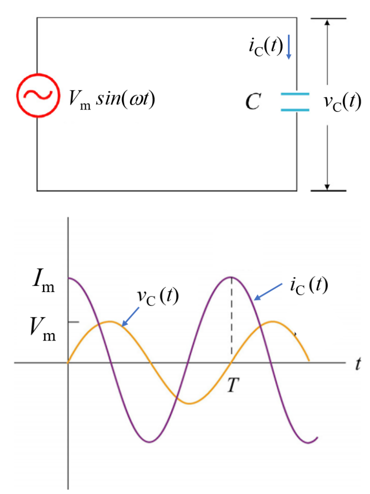


Figure 9: Top: A capacitor connected to a sinusoidal voltage source, Bottom: Voltage waveform and current waveform

For capacitor (C Farad),

$$i_C(t) = C\frac{dv_C}{dt}.$$

For the circuit shown,

$$i_C = C\frac{d}{dt}V_m \sin(\omega t) = \omega C V_m \cos(\omega t).$$

Using the trigonometric identity

$$\sin(\omega t + \frac{\pi}{2}) = \cos(\omega t),$$

we get

$$i_C = \omega C V_m \sin(\omega t + \frac{\pi}{2}) = I_m \sin(\omega t + \frac{\pi}{2}).$$

- The current waveform and the voltage waveform have same frequency.
- Voltage and current are not in-phase. Capacitor current waveform leads the capacitor voltage waveform by $\frac{\pi}{2}$ rad.
- Ratio of the peak voltage to the peak current, or the RMS voltage to the RMS current is equal to $X_L = \frac{1}{\omega C}$:

$$\frac{V_m}{I_m} = \frac{V_{C,rms}}{I_{C,rms}} = \frac{1}{\omega C},$$

$X_C = \frac{1}{\omega C}$ is called the **capacitive reactance**.

The SI units of X_C is **ohm** (Ω). The value of X_C is inversely proportional to the angular frequency ω. Thus, for a capacitor, the resistance to current flow decreases with increasing frequency of the source voltage.

When only one type of component is connected to a sinusoidal voltage source, the resulting current can be found easily.

If the load is a network of components but all are of the same type, then the network can be replaced by one equivalent resistance, or inductance, or capacitance.

1. The frequency of the current waveform is the same as the frequency of the voltage waveform.
2. Amplitude of the current waveform:
   - Resistor: $I_m = \frac{V_m}{R}$
   - Inductor: $I_m = \frac{V_m}{X_L}$
   - Capacitor: $I_m = \frac{V_m}{X_C}$
3. Phase of the current waveform:
   - Resistor: current and voltage are in-phase
   - Inductor: current lags voltage by $\frac{\pi}{2}$ rad
   - Capacitor: current leads voltage by $\frac{\pi}{2}$ rad

V_C lags V_R
V_L leads V_R

Circuit with multiple components of different types

First, we consider circuits with two elements - an RL circuit and an RC circuit. Finally, this chapter will be concluded with the case of an RLC circuit.

RC circuit with sinusoidal voltage source

Applying KVL to the circuit shown in Figure 10,

$$v_R(t) + v_C(t) = V_m \sin(\omega t).$$

But

$$v_R = Ri = RC\frac{dv_C}{dt}.$$

So,

$$RC\frac{dv_C}{dt} + v_C(t) = V_m \sin(\omega t),$$

$$\frac{dv_C}{dt} + \frac{1}{RC}v_C(t) = \frac{V_m}{RC}\sin(\omega t).$$

You can solve this non-homogeneous first-order ODE to find the unknown voltages and current in this circuit. However, solution of ODE is not main focus of this chapter. Instead of the solution, different voltages and current are shown in Figure 11.

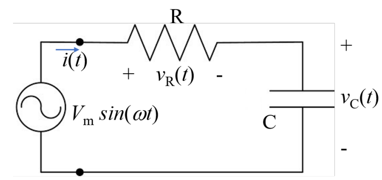


Figure 10: Series RC circuit with sinusoidal voltage source.

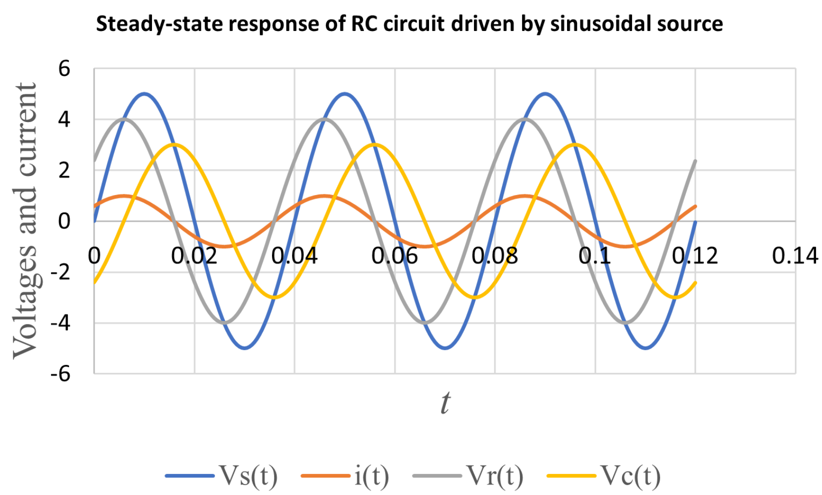


Figure 11: Steady-state voltages and current in an RC circuit driven by a sinusoidal voltage source.

Points to note:

1. Frequency is same for all voltages and current
2. $i(t)$ is in-phase with $V_r(t)$
3. $i(t)$ leads $V_c(t)$ by 90° or $\frac{\pi}{2}$ rad
4. $i(t)$ leads $V_s(t)$ but the phase angle is less than 90° or $\frac{\pi}{2}$ rad

RL circuit with sinusoidal voltage source

Applying KVL to the circuit shown in Figure 12,

$$v_R(t) + v_L(t) = V_m \sin(\omega t).$$

But

$$v_R = Ri,$$

and

$$v_L = L\frac{di}{dt}.$$

So,

$$\begin{aligned} Ri + L\frac{di}{dt} &= V_m \sin(\omega t), \\ \frac{di}{dt} + \frac{R}{L}i &= V_m \sin(\omega t). \end{aligned}$$

Different voltages and current are shown in Figure 13.

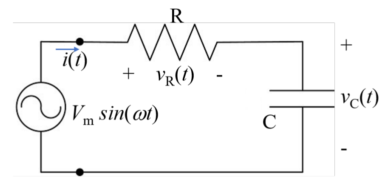


Figure 12: Series RC circuit with sinusoidal voltage source.

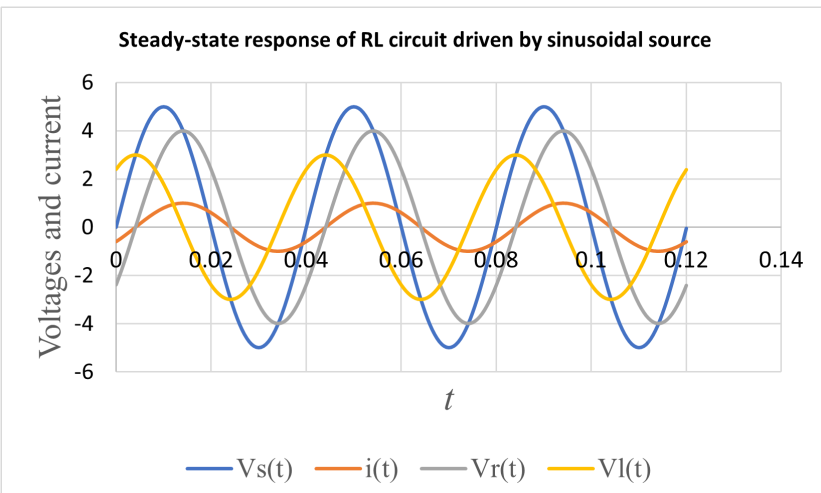


Figure 13: Steady-state voltages and current in an RC circuit driven by a sinusoidal voltage source.

Points to note:

1. Frequency is same for all voltages and current
2. $i(t)$ is in-phase with $V_r(t)$
3. $i(t)$ lags $V_L(t)$ by 90° or $\frac{\pi}{2}$ rad
4. $i(t)$ lags $V_s(t)$ but the phase angle is less than 90° or $\frac{\pi}{2}$ rad

RLC circuit with sinusoidal voltage source

Consider a series RLC circuit driven by a sinusoidal voltage source as shown in Figure 14. A time-domain equation that relates the source voltage to the current is derived here, but the solution of the equation is not shown.

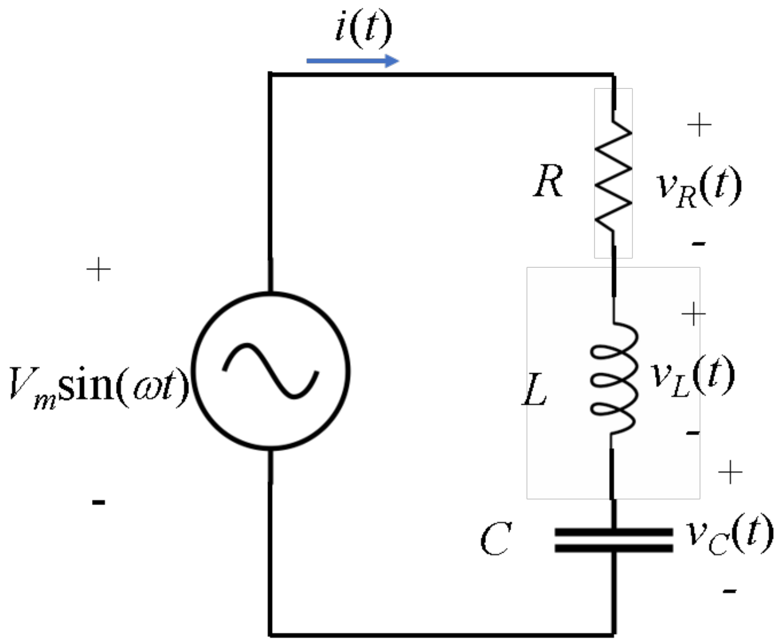


Figure 14: Series RLC circuit driven by a sinusoidal source.

Applying KVL around the loop,

$$v_R + v_L + v_C = V_m \sin(\omega t).$$

Properties of the individual components:

$$v_R = Ri, \quad v_L = L\frac{di}{dt}, \quad i = C\frac{dv_C}{dt}.$$

$$Ri + L\frac{di}{dt} + v_C = V_m \sin(\omega t).$$

Taking derivative,

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{dv_C}{dt} = \omega V_m \cos(\omega t).$$

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = \omega V_m \cos(\omega t).$$

Reorganizing the terms,

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt}\frac{1}{LC}i = \omega V_m \cos(\omega t).$$

We can solve this non-homogeneous second-order ODE to find the time-domain expression for the current $i(t)$. Refer to the appendix at the end of EPP I notes on how to solve second order ODE.

Points to Note:

- Solution of a non-homogeneous ODE has twp parts:
  1. Complementary function gives the transient response
  2. Particular integral is the steady-state response
- The chapter **AC circuit analysis: Part 2** presents another method using **complex phasor** and **impedance** for finding steady-state solution of ac circuit driven by a sinusoidal source.
  - The time domain circuit equation involving ODE is transformed into an algebraic problem when we use phasor and impedance.

Appendix

A-1: RMS value of sine function

$$x(t) = A\sin(\omega t)$$

If the period of the waveform is T seconds then

$$\omega T = 2\pi.$$

The RMS value of $x(t)$:

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T A^2 \sin^2(\omega t) dt}.$$

But,

$$\sin^2(\omega t) = \frac{1-\cos 2\omega t}{2}.$$

So,

$$\begin{aligned} X_{rms} &= \sqrt{\frac{1}{T}\int_0^T \frac{A^2}{2}(1-\cos(2\omega t)dt} \\ &= \sqrt{\frac{A^2}{2T}\int_0^T (1-\cos(2\omega t)dt} \\ &= \sqrt{\frac{A^2}{2T}(\int_0^T dt - \int_0^T \cos(2\omega t)dt)} \\ &= \sqrt{\frac{A^2}{2T}((T-0)-(\frac{\sin 2\omega T - \sin 0}{2\omega}))} \\ &= \frac{A}{\sqrt{2}}. \end{aligned}$$

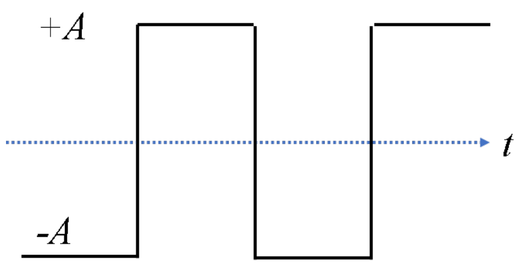
$\sin 2\omega T = \sin 4\pi = 0.$

RMS value depends on the amplitude of the waveform and not its frequency or phase. The functions $A\sin(100t)$, $A\cos(100t)$ and $A\sin(100t + 0.2\pi)$ have the same RMS value which is $\frac{A}{\sqrt{2}}$.

A-2: RMS value of non-sinusoidal periodic waveform

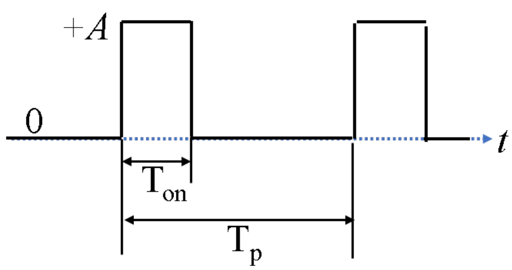
RMS values of some non-sinusoidal periodic functions are given below.

Bipolar square wave



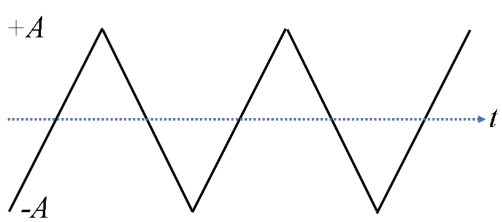
$$RMS\ value = A.$$

Series of square pulses



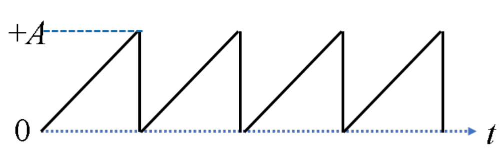
$$RMS\ value = A\sqrt{\frac{T_{on}}{T_p}}.$$

Triangular wave



$$RMS\ value = \frac{A}{\sqrt{3}}.$$

Saw-tooth wave



$$RMS\ value = \frac{A}{\sqrt{3}}.$$

AC Circuit Analysis using Phasor

You have seen in the chapter *Introduction to AC Circuit* that KVL or KCL formulation leads to ordinary differential equations. That means, analysis of ac circuits involves solution of ordinary differential equations. The order of ODE and hence the complexity increases with the number of branches with reactive elements, *i.e.*, capacitor and inductor.

Finding the **steady-state solution** of a constant coefficient ODE with sinusoidal forcing function, for example, the 2nd-order case

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = V_m \sin(\omega t + \phi)$$

can be simplified by replacing the sinusoidal function with complex exponential $e^{j\omega t}$. This is true for ODE of any order. However, the sinusoidal sources are not complex exponential functions; they are either sin or cos functions.

According to **Euler's formula:**

$$e^{j\theta} = \cos\theta + j\sin\theta,$$

where

$$j = \sqrt{-1}$$

is the imaginary number. Therefore,

In mathematics, you might have used i as the symbol for the imaginary number. Since, i is used to represent current in electrical engineering, the symbol j is used for imaginary number.

$$\begin{aligned} A\cos(\omega t + \phi) &= \Re(Ae^{j(j\omega t + \theta)}), \\ A\sin(\omega t + \phi) &= \Im(Ae^{j(j\omega t + \theta)}). \end{aligned}$$

If the ac circuit is driven by a sinusoidal source,

- We substitute the sinusoidal function with complex exponential in the KVL/KCL equation so that the steady-state solution can be obtained by solving algebraic equations instead of solving ODEs.
  - However, the solution obtained is a complex exponential function.
- If the driving function is a cos function, the real part of the complex exponential solution is the steady-state solution.
- If the driving function is a sin function, the imaginary part of the complex exponential solution is the steady-state solution.

Phasor

Phasor analysis is a technique to find the steady-state response of AC circuits when the input is a sinusoidal function.

You have seen in the chapter **Introduction to AC Circuits**, that when a circuit element (R, L or C) is driven by a sinusoidal voltage source, the resulting steady-state current is also sinusoidal having the same frequency as the voltage's. That means, in a linear AC circuit, all variables (voltages and currents) will be sinusoidal functions of identical frequency.

The time domain KVL/KCL equations involve trigonometric functions sine and cosine. In general, arithmetic operations of sinusoidal functions are not convenient as they involve using trigonometric identities. An abstraction of sinusoidal functions as real (or imaginary) part of a rotating vector in the complex plane makes their arithmetic operations more convenient. The abstraction used is the **phasor** representations of the sinusoidal variables.

Consider the sinusoidal function

$$x(t) = A\sin(\omega t + \phi) \tag{1}$$

which can be expressed using complex exponential as

$$\begin{aligned} x(t) &= \Im A e^{j(\omega t+\phi)}, \\ &= \Im A e^{j\phi} e^{j\omega t}, \\ &= \Im X e^{j\omega t}. \end{aligned}$$

- The sinusoidal function $x(t)$ is factored into a time-independent part $\bar{X} = Ae^{j\phi}$ and a time-varying part $e^{j\omega t}$.

- The time-varying part contains the frequency information that in same in the voltages and currents at all nodes and branches of a circuit.

- The time-independent part $Ae^{j\phi}$ is the **phasor** representation of the sinusoidal function. This factor is different at different parts of the circuit.

  - For finding the steady-state voltages and currents in an ac circuit, we need to deal with the phasor only.

Extracting phasor from the time-domain waveform:

$$x(t) = A\sin(\omega t + \phi) \quad \Rightarrow \quad \bar{X} = Ae^{j\phi}.$$

The phasor $Ae^{j\phi}$ is a complex number with magnitude A and argument ϕ.

$$\begin{aligned} Ae^{j\phi} &= A(\cos\phi + j\sin\phi), \\ |A| &= A\sqrt{\cos^2\phi + \sin^2\phi} = A, \\ \angle A &= \tan^{-1}\frac{A\sin\phi}{A\cos\phi} = \phi. \end{aligned}$$

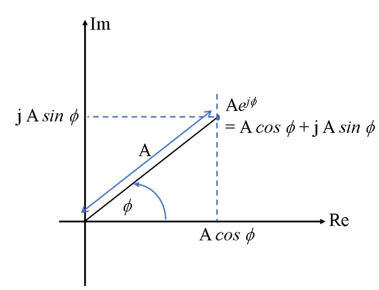


Figure 15: Argand diagram of the phasor $\bar{X} = Ae^{j\phi}$: its imaginary part is $A\sin\phi$ which is $x_i(t) = A\sin(\omega t + \phi)$ at $t = 0$ regardless of the value of ω. Similarly, the real part is the value of $x_r(t) = A\cos(\omega t + \phi)$ at $t = 0$.

Three different notations are used interchangeably for the phasor $\bar{X}$:

$$\boxed{Ae^{j\phi} = A\cos\phi + jA\sin\phi = A\angle\phi.}$$

What is the significance of $e^{j\omega t}$?

$$
\begin{aligned}
e^{j\omega t} &= A(\cos\omega t + j\sin\omega t),\\
|e^{j\omega t}| &= \sqrt{\cos^2\omega t + \sin^2\omega t} = 1,\\
\angle e^{j\omega t} &= \tan^{-1}\frac{A\sin\omega t}{A\cos\omega t} = \omega t.
\end{aligned}
$$

Its magnitude is 1 and the argument (ωt) increases linearly with as time passes. In the argand diagram, this is the locus of a point rotating counter-clockwise (CCW) around the origin at a distance of 1 from it.

Multiplying the phasor $\bar{X} = Ae^{j\phi}$ by $e^{j\omega t}$ results in a rotating vector with initial position $A\angle\phi$ at $t = 0$ and rotating CCW in the complex plane at the rate of ω radians per second. Plotting the imaginary part of this rotating vector versus time will generate the plot of

Associate Phasor with sine wave

$$A\sin(\omega t + \phi)$$

and plotting the real part will give

$$A\cos(\omega t + \phi).$$

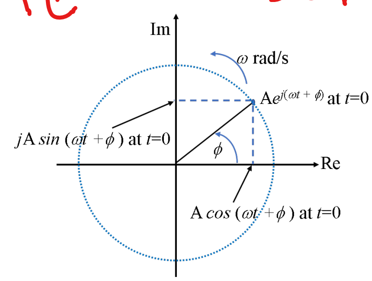


Figure 16: The argand diagram of $Ae^{j(\omega t+\phi)}$ is a rotating vector rotating counter clockwise at the rate of ω rad/s. The phasor $Ae^{j\phi}$ is that rotating vector at $t = 0$.

Note: The same phasor $Ae^{j\phi}$ can be associated with both cos and sin waveform in time-domain depending whether we want to use real part of the complex exponential or the imaginary part. **In this note, we will use** sin **function to represent the sinusoidal waveform**. So, the phasor convention used in this note is:

$$A\sin(\omega t + \phi) \quad \Longleftrightarrow \quad = Ae^{j\phi} = A\angle\phi.$$

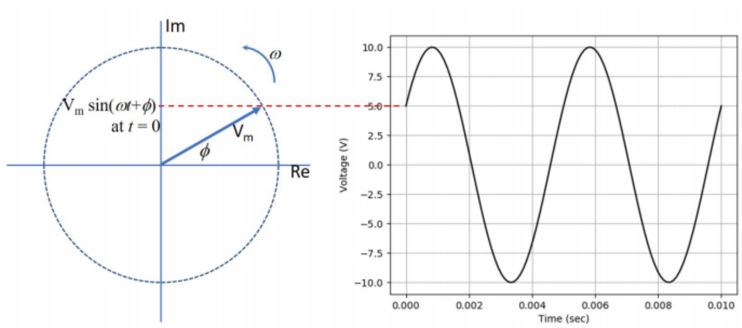


Figure 17: The phasor of $v(t) = V_m\sin(\omega t + \phi)$ is shown on the left and the corresponding waveform on the right for $V_m = 10$ and $\phi = \pi/6$. The phasor itself does not tell what the frequency is. From the waveform, you can determine the frequency which is $f = 1/0.005 = 200\ Hz$ or $\omega \approx 1257$ rad/s.

- All voltages and currents in a linear AC circuit driven by a sinusoidal source are also sinusoidal having the same frequency and, therefore, can be represented by corresponding phasors.
- The rotating vectors corresponding to all voltages and currents will be rotating at the same rate.

- If observed in a reference frame which itself is rotating at the same rate, all rotating vectors become standing still, *i.e.*, we can simply drop the time-varying component $e^{j\omega t}$, and represent the functions in terms of their phasors capturing the amplitude and phase.

- **As a result, the time-varying problem of AC circuit is transformed into a constant value (DC) problem.**

Example 1: Use of phasor to find sum of sinusoidal waveform

Consider three voltage sources

$$v_1 = 6\sin\omega t, \quad v_2 = 12\sin(\omega t + \pi/2), \quad v_3 = 4\sin(\omega t - pi/2)$$

connected in series. What is the overall voltage of the combination?
Solution:

$$v(t) = 6\sin\omega t + 12\sin(\omega t + \pi/2) + 4\sin(\omega t - pi/2).$$

- The actual problem is in time-domain.

Representing the voltages by phasors,

$$\bar{V} = 6\angle 0 + 12\angle \pi/2 + 4\angle - \pi/2,$$

$$\bar{V} = (6 + j0) + (0 + j12) + (0 - j4) \;=\; 6 + j8.$$

$$\bar{V} = \sqrt{6^2 + 8^2}\angle \tan^{-1}(8/6) = 10\angle 0.92.$$

- Numerical manipulations are done in phasor-domain.

Converting back to time-domain:

$$v(t) = \Im 10\angle 0.92 = 10\sin(\omega t + 0.92).$$

- Appropriate phasor-to-time domain transformation gives the solution in time-domain.

- In the example above, the numbers are converted into Cartesian form before adding them. Similarly, product and division are done easily in polar coordinate representation. However, most scientific calculators nowadays can do complex number operations directly in any form of number representation.

- The imaginary part of $10\angle 0.92$ is found in the above example. However, the time-domain expression can be written straight from the phasor. Its magnitude is the amplitude of the sin function and argument is the phase:

$$10\angle 0.92 \Rightarrow 10\sin(\omega t + 0.92).$$

Example 2: Either sine or cosine function can be used while defining the phasor: Find

$$v(t) = 6\sin(\omega t + \pi/4) + 8\cos(\omega t - \pi/6).$$

Solution #1: If $V_m\angle\phi$ is the phasor of $V_m\sin(\omega t + \phi)$. Then

$$v_1 = 6\sin(\omega t + \pi/4) \;\; \Rightarrow \;\; \bar{V}_1 = 6\angle\pi/4.$$

However, $8\cos(\omega t - \pi/6)$ must be expressed using the sine function before getting the phasor:

$$v_2 = 8\cos(\omega t - \pi/6) = 8\sin(\omega t + \pi/3) \;\; \Rightarrow \;\; \bar{V}_2 = 8\angle\pi/3.$$

$\cos(x) = \sin(90 + x)$

$$\bar{V} = \bar{V}_1 + \bar{V}_2 = 6\angle\pi/4 + 8\angle\pi/3 = 13.33\angle 0.94.$$

$$v(t) = 13.33\sin(\omega t + 0.94).$$

Solution #2: If $V_m\angle\phi$ is the phasor of $V_m\cos(\omega t + \phi)$. Then

In this case, $6\sin(\omega t + \pi/4)$ must be expressed using the cosine function before getting the phasor:

$$v_1 = 6\sin(\omega t + \pi/4) = 8\cos(\omega t - \pi/4) \;\; \Rightarrow \;\; \bar{V}_1 = 6\angle - \pi/4.$$

$$v_2 = 8\cos(\omega t - \pi/6) \;\; \Rightarrow \;\; \bar{V}_2 = 6\angle - \pi/6.$$

$$\bar{V} = \bar{V}_1 + \bar{V}_2 = 6\angle - \pi/4 + 8\angle - \pi/6 = 13.33\angle - 0.63.$$

$$v(t) = 13.33\cos(\omega t - 0.63).$$

Both approaches lead to the same waveform.

$$13.33\cos(\omega t - 0.63) = 13.33\sin(\omega t - 0.63 + \pi/2) = 13.33\sin(\omega t + 0.94).$$

In this note, we have chosen trigonometric function $A\sin(\omega t + \phi)$ to represent the time-domain waveform and the corresponding phasor is $A\angle\phi$. In some books, the cosine function is used. As illustrated in the example above, both will give the same result.

Complex Impedance

Let $\bar{V}$ and $\bar{I}$ be the phasors of the sinusoidal voltage across a component/load and the resulting sinusoidal current through it. Then the complex impedance of the component/ load is

$$Z = \frac{\bar{V}}{\bar{I}}.$$

$\dfrac{\text{Voltage Phasor}}{\text{current Phasor}}$

In the following subsections, we first consider simple one element loads, *i.e.*, resistor, inductor, and capacitor.

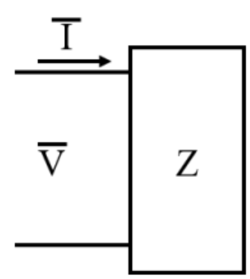


Figure 18: Impedance of a load is the ratio of voltage to current, both expressed as phasor.

1. Impedance of ideal resistor

For a resistor, the voltage waveform and current waveform (Figure 19) are in-phase:

$$\begin{aligned} v_R(t) &= V_m \sin(\omega t), \\ i_R(t) &= \frac{V_m}{R} \sin(\omega t), \\ &= I_m \sin(\omega t). \end{aligned}$$

Corresponding phasors are:

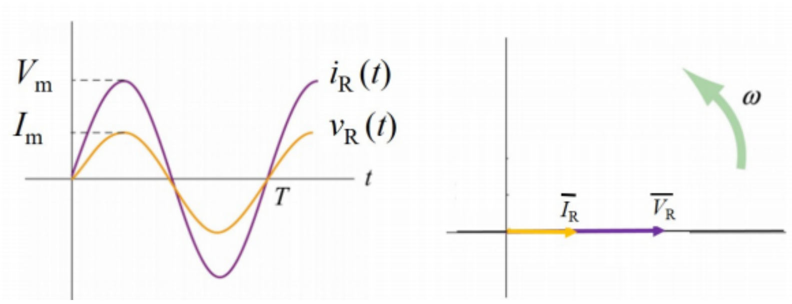


Figure 19: Voltage waveform and current waveform of a resistor (left) and the phasor diagram (right)

$$\bar{V}_R = V_m \angle 0, \quad \bar{I}_R = I_m \angle 0.$$

The impedance is

$$Z_R = \frac{V_m \angle 0}{I_m \angle 0} = \frac{V_m}{I_m} = R.$$

Impedance of R: $Z_R = R\angle 0 = R + j0$

It is a real number and is equal to the value of resistance. The voltage phasor and the current phasor of a resistor are shown in the **phasor diagram** in Figure 19 (right).

2. Impedance of ideal inductor

For ideal inductor, voltage waveform and current waveform are out of phase by 90° or $\frac{\pi}{2}$ radians, with **the current lagging the voltage**:

$$\begin{aligned} v_L(t) &= V_m \sin(\omega t), \\ i_L(t) &= -\frac{V_m}{\omega L} \cos(\omega t), \\ &= I_m \sin(\omega t - \pi/2). \end{aligned}$$

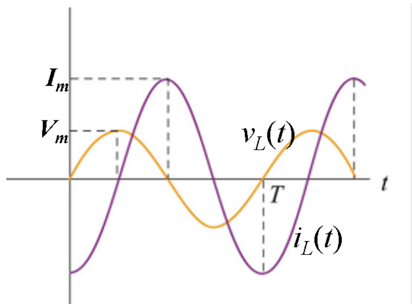
The phasors are:

$$\bar{V}_L = V_m \angle 0, \quad \bar{I}_L = I_m \angle -\frac{\pi}{2}.$$

Impedance of L: $Z_L = \omega L \angle \frac{\pi}{2} = 0 + j\omega L$

The impedance of ideal inductor is

$$Z_L = \frac{V_m \angle 0}{I_m \angle -\frac{\pi}{2}} = \frac{V_m}{I_m} \angle \frac{\pi}{2} = \omega L \angle \frac{\pi}{2}.$$



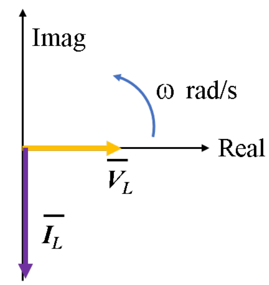


Figure 20: Voltage waveform and current waveform of an inductor (left) and the phasor diagram (right)

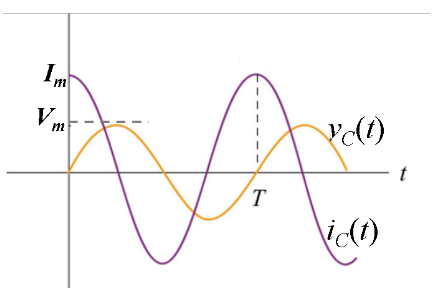
It is a positive imaginary number with magnitude equal to ωL, the reactance of the inductor. The voltage phasor and the current phasor of an inductor are shown in the phasor diagram in Figure 20 (right).

3. Impedance of a capacitor

Voltage waveform and current waveform are out of phase by $90°$ or $\frac{\pi}{2}$ radians with **the current leading the voltage**:

$$\begin{aligned} v_C(t) &= V_m \sin(\omega t), \\ i_C(t) &= \omega C V_m \cos(\omega t), \\ &= I_m \sin(\omega t - \pi/2). \end{aligned}$$

The phasors are:



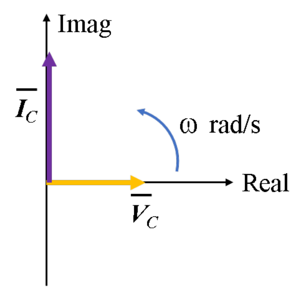


Figure 21: Voltage waveform and current waveform of a capacitor (left) and the phasor diagram (right)

$$\bar{V}_C = V_m \angle 0, \quad \bar{I}_C = \omega C V_m \angle -\frac{\pi}{2}.$$

Impedance of C:
$Z_C = \frac{1}{\omega C} \angle - \frac{\pi}{2} = 0 - j\frac{1}{\omega C}$

The impedance of a capacitor is

$$Z_C = \frac{V_m \angle 0}{I_m \angle \frac{\pi}{2}} = \frac{V_m}{I_m} \angle -\frac{\pi}{2} = \frac{1}{\omega C} \angle -\frac{\pi}{2}.$$

It is a negative imaginary number with its magnitude equal to $\frac{1}{\omega C}$, the reactance of the capacitor. The voltage phasor and the current phasor of a capacitor are shown in the phasor diagram in Figure 21 (right).

Impedance of load consisting of multiple components

- When two or more components are connected in series, the equivalent impedance is equal to the sum of individual impedances:

$$Z_{eq} = Z_1 + Z_2 + Z_3 + \cdots\cdots\cdots + Z_n.$$

- When components are connected in parallel, the equivalent impedance is:

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots\cdots\cdots + \frac{1}{Z_n}.$$

1. Impedance of R-L and R-C

Impedance of a series R-L branch is

$$Z_{RL} = Z_R + Z_L = R + j\omega L.$$

In polar form:

$$\begin{aligned} Z_{RL} &= \sqrt{R^2 + (\omega L)^2} \angle \tan^{-1} \frac{\omega L}{R}, \\ &= |Z_{RL} \angle \theta_{RL}. \end{aligned}$$

Since, ω, R, and L are all non-zero, positive real numbers, $\theta_{RL} = \tan^{-1} \frac{\omega L}{R}$ varies between 0 and $+\pi/2$:

$$0 < \theta_{RL} < +\frac{\pi}{2}.$$

Can you show that the condition $0 < \theta_{RL} < +\frac{\pi}{2}$ is also true when R and L are connected in parallel?

Impedance of a series R-C branch is

$$Z_{RC} = Z_R + Z_C = R - j\frac{1}{\omega C}.$$

In polar form:

$$\begin{aligned} Z_{RC} &= \sqrt{R^2 + \frac{1}{(\omega C)^2}} \angle \tan^{-1} \frac{-\frac{1}{\omega C}}{R}, \\ &= |Z_{RC} \angle \theta_{RC}. \end{aligned}$$

Since, ω, R, and C are all non-zero, negative real numbers, $\theta_{RC} = \tan^{-1} \frac{-\frac{1}{\omega C}}{R}$ varies between 0 and $-\pi/2$:

$$0 > \theta_{RC} > -\frac{\pi}{2}.$$

Can you show that the condition $0 > \theta_{RC} < -\frac{\pi}{2}$ is also true when R and C are connected in parallel?

Impedance of an R-L load or R-C load is a complex number in the form of

$$Z_{RL} = R + j\omega L = |Z_{RL}| \angle \theta_{RL}, \quad 0 < \theta_{RL} < \frac{\pi}{2},$$

$$Z_{RC} = R - j\frac{1}{\omega C} = |Z_{RC}| \angle \theta_{RC}, \quad 0 > \theta_{RC} > -\frac{\pi}{2}.$$

Impedance of R-L-C

Consider a series R-L-C load:

$$Z_{RLC} = R + j\omega L - j\frac{1}{\omega C} = R + jX_L - jX_C.$$

Depending on the relative values of X_L and X_C, there are three possible characteristics of the equivalent impedance:

1. **Inductive** ($X_L > X_C$):
$$Z_{RLC} = R + jX,$$
where $X = X_L - X_C$.

2. **Capacitive** ($X_L < X_C$):
$$Z_{RLC} = R - jX,$$
where $X = X_L - X_C$.

3. **Resistive** ($X_L = X_C$):
$$Z_{RLC} = R.$$
This condition is called **resonance**. At resonance, a series RLC load behaves like a resistor. If the amplitude of the input voltage is kept constant but the frequency is varied, the current through the series-RLC will have maximum amplitude at the resonant frequency.

Can you determine the characteristics of an RLC circuit when the three components are connected in parallel?

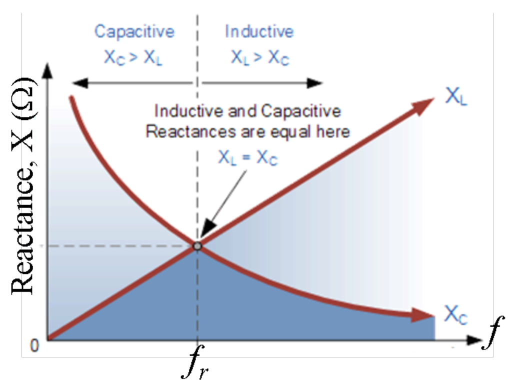


Figure 22: Reactance of a capacitor decreases with increasing frequency but the reactance of an inductor increases.

At the resonant frequency ($\omega_r = 2\pi f_r$) of an RLC circuit,

$$X_L = X_C \quad \Rightarrow \quad \omega_r L = \frac{1}{\omega_r C}.$$

$$\omega_r = \frac{1}{\sqrt{LC}},$$

$$\boxed{f_r = \frac{1}{2\pi\sqrt{LC}}}.$$

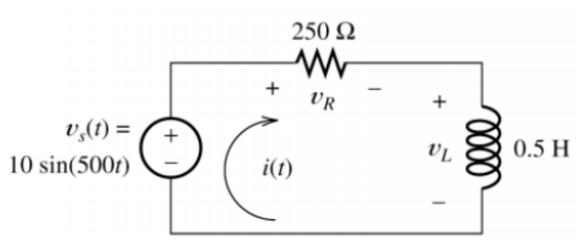
The impedance of an ac circuit varies with frequency. For a given input voltage amplitude, the current through the circuit will be different if the frequency of the input voltage is changed. You will learn more about this later when you study about **filters**.

AC circuit analysis using phasor and impedance

1. Replace the time-domain description of the sources with the corresponding phasor
$$V_m \sin(\omega t + \phi) \quad \Rightarrow \quad V_m \angle \phi.$$
   - You can do this for all sources provided their frequencies are the same.
   - If there are sources with different frequencies, analyze the circuit for one frequency at a time and then apply principle of superposition.
2. Replace each component with its complex impedance.
3. Because of these steps mentioned above, the ac circuit is transformed into a time-independent circuit with constant-valued sources. Analyze the circuit using any technique you have learnt for DC circuit analysis, but you need to perform calculations using complex numbers.
4. The solution you get will be in the phasor-domain.
   - If you are interested to know the magnitude of voltage/ current, and its phase shift with respect to the source, you can get it directly from the phasor.
$$A \angle \phi \quad \Rightarrow \quad Amplitude = A, \;\; Phase = \gamma.$$
   - If the time-domain expression is desired, transform the phasor into trigonometric function.
$$A \angle \gamma \quad \Rightarrow \quad A \sin(\omega t + \gamma).$$

Example 3: AC circuit analysis using phasor and impedance?

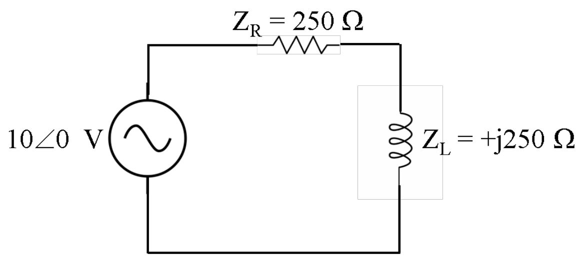
Find the RMS current through the circuit shown below and the phase of the current. Also find the trigonometric expressions for the voltages v_R and v_L.



Solution:

$$\begin{aligned}
v_S(t) &= 10 \sin(500t) \quad \Rightarrow \quad \bar{V}_S = 10 \angle 0 \;\; volt, \\
\omega &= 500 \; rad/s, \\
Z_R &= 250 \, \Omega, \\
Z_L &= +j\omega L = +j250 \, \Omega.
\end{aligned}$$

Redraw the circuit with phasors and impedances:



Total impedance:

$$Z = 250 + j250 = 250 \times \sqrt{2}\angle\frac{\pi}{4}\ \Omega$$

.

Phasor of the current:

$$\bar{I} = \frac{10\angle 0\ \ volt}{250 \times \sqrt{2}\angle\frac{\pi}{r}\ \ \Omega} = 28.3 \times 10^{-3}\angle - \frac{\pi}{4}\ \ ampere.$$

$$I_m = 28.3 \times 10^{-3}\ ampere.$$

$$I_{rms} = \frac{28.3 \times 10^{-3}}{\sqrt{2}} = 20.0 \times 10^{-3}\ ampere.$$

RMS current is **20.0 mA** and current waveform lags input voltage waveform by $\frac{\pi}{4}$ radians.

$$\bar{V}_R = \bar{I} \times Z_R = 28.3 \times 10^{-3}\angle - \frac{\pi}{4} \times 250 = 7.07\angle - \frac{\pi}{4}\ volt.$$

$$v_R(t) = 7.07\sin(500t - \frac{\pi}{4})\ volt.$$

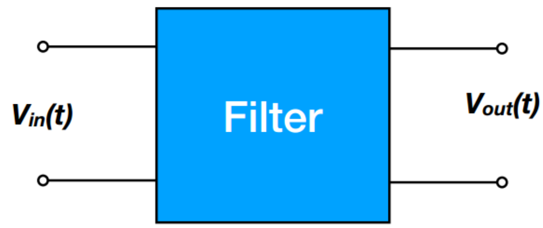
$$\bar{V}_L = \bar{I} \times Z_L = 28.3 \times 10^{-3}\angle - \frac{\pi}{4} \times 250\angle\frac{\pi}{2} = 7.07\angle + \frac{\pi}{4}\ volt.$$

$$v_L(t) = 7.07\sin(500t + \frac{\pi}{4})\ volt.$$

Filters

What is a filter

A filter is a 2-port device that takes in a signal $V_{in}(t)$ at the input port, modifies it in some way, and sends out the modified signal $V_{out}(t)$ on the output port. Signals are often simply time-varying voltages.



Potential divider as a filter

Let us start with a very simple example. Consider the potential divider circuit shown below as a filter: The output V_{out} is simply related to the input V_{in} as:

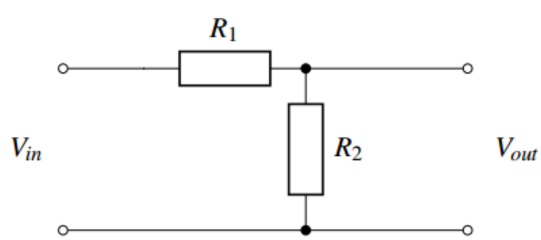


Figure 23: Very simple filter

$$V_{out} = \frac{R_2}{R_1 + R_2} V_{in}. \tag{2}$$

So the output signal is simply a scaled version of the input signal. the input signal is time-varying, the output signal will also vary with time:

$$V_{out}(t) = \frac{R_2}{R_1 + R_2} V_{in}(t). \tag{3}$$

The scaling factor G is what we call the *Gain* of the filter.

$$V_{out}(t) = G V_{in}(t), \tag{4}$$

$$G = \frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2}. \tag{5}$$

The gain G here is independent of the input signal $V_{in}(t)$. This is not generally the case for other circuits, as we see next.

RC low-pass filter

Let us next consider a circuit that is very similar to the potential divider we just studied, but replace one of the resistors with a capacitor. If the input

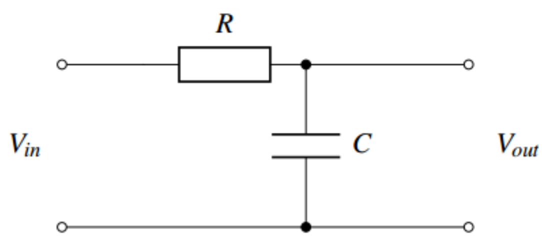


Figure 24: RC low-pass filter

voltage V_{in} is not time-varying, the capacitor acts as an open circuit, and the output voltage $V_{out} = V_{in}$ in the steady-state. The gain G of this circuit is then just 1.

However, things get more interesting with a time-varying input voltage $V_{in}(t)$. We already know how to analyze linear AC circuits, where the voltages are sinusoidally time-varying with constant frequency f. To simplify notation, we write our analysis in terms of the angular frequency $\omega = 2\pi f$.

Let the time-varying input voltage be represented by $V_{in}\angle 0$ where V_{in} is the amplitude of the sinusoid of angular frequency ω and phase angle is 0 (we choose time origin suitably). The time-varying output voltage $V_{out}\angle\phi_{out}$ then has an amplitude V_{out} and phase angle ϕ_{out}. To analyze this circuit, we use the impedance $Z_R = R$ and $Z_C = \frac{1}{j\omega C}$ for resistor R and capacitor C, respectively. The gain G now depends on ω:

$$G = \frac{V_{out}}{V_{in}} = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}. \tag{6}$$

This circuit has a different gain G for different frequencies! To study this further, let us plot $|G|$ against f for some example values of R (1 k Ω) and C (5 nF) which is shown in Figure 25.

We see that this filter has a gain of $\approx$ 1 at low frequencies, and the gain reduces with increasing frequency. By 50 kHz, the gain is 0.5, and by 200 kHz, the gain drops to less than 0.2. This filter allows low frequencies to pass through, but attenuates high frequencies. Such a filter is known as a *low-pass filter*.

It is customary to plot the gain and frequency on a logarithmic scale (Figure 26). The units for gain in the logarithmic scale is called decibel (dB), and is defined as $\boxed{G_{dB} = 20\log_{10}(|G|)}$.

The frequency at which the response drops to -3 dB (0.707 in linear scale)

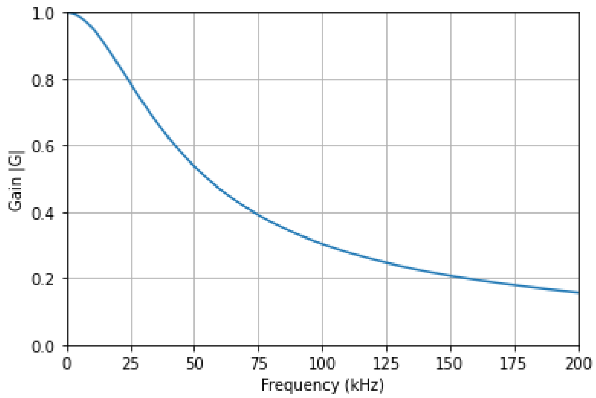


Figure 25: Gain versus frequency plot for RC low pass filter

is known as the *cutoff frequency*. For a RC low-pass filter, this is given by

$$f_{cutoff} = \frac{1}{2\pi RC}.$$

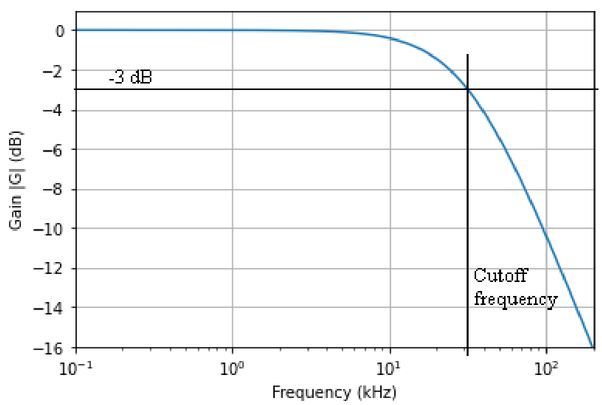


Figure 26: Frequency response of RC low pass filter: Gain (dB) versus frequency plot

RC high-pass filter

What happens if we take the same circuit as above, but exchange the positions of the resistor and capacitor? If the input voltage V_{in} is not time-varying, the capacitor acts as an open circuit, and the output voltage $V_{in} = 0$. The gain G of this circuit is then just 0. As the frequency increases, however, the capacitor starts to conduct and the gain increases. Let us analyze the circuit as before. We again use the impedance $Z_R = R$ and $\frac{1}{j\omega C}$ for resistor R and capacitor C, respectively. The gain G depends on ω:

$$G(\omega) = \frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_C + Z_R} = \frac{R}{\frac{1}{j\omega C} + R = \frac{j\omega RC}{1+j\omega RC}}.$$

$$|G(\omega)| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \tag{7}$$

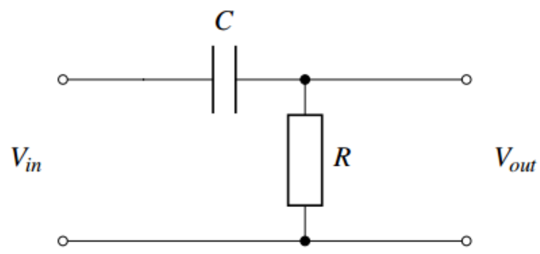


Figure 27: RC high-pass filter

Let us plot $|G(\omega)|$ against f for the same example values of R (1 kΩ) and C (5 nF) (Figure 28).

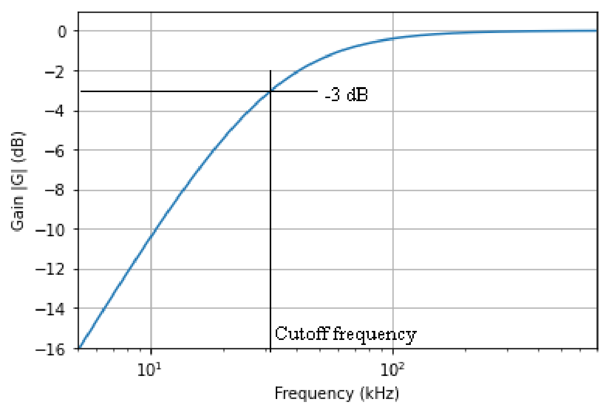


Figure 28: Frequency response of RC high pass filter: Gain (dB) versus frequency plot

We now see that this forms a high-pass filter, allowing high frequencies to pass through, and attenuating the low frequencies. The cutoff frequency of this filter is the same $f_{cutoff} = \frac{1}{2\pi RC}$.

Bandpass filter

Once we know how the basic building blocks of a RC filter work, we can combine high- and low-pass filters to create more complex filters. For example, if we want a filter that only allows frequencies between f_1 and f_2 to pass through, we can combine a RC high-pass filter with a RC low-pass filter. Such filters are called *bandpass* filters. The difference $f_2 - f_1$ is known as the *bandwidth* of the filter. The *frequency response* of the above filter,

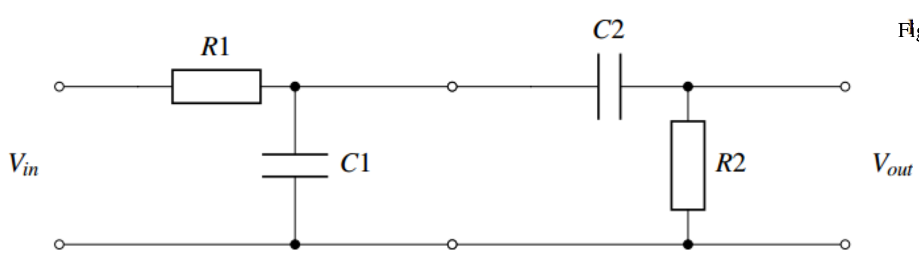


Figure 29: RC band-pass filter

i.e., change in gain of the filter as function of frequency, for $R1 = 1\ k\Omega$, $C1 = 0.5\ nF$, $R2 = 10\ k\Omega$ and $C2 = 5\ nF$ is shown in Figure 30.

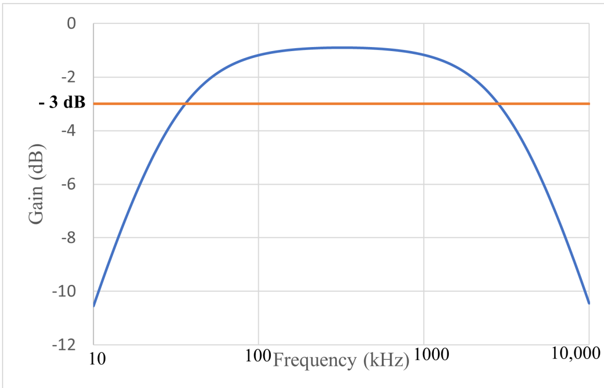


Figure 30: Frequency response of a band pass filter

- Note that the lower and higher cutoff frequencies of the RC bandpass filter are not simply the cutoff frequencies of the high-pass and low-pass sections of the filter. Why is that? Can you analyze the RC bandpass filter circuit to compute it's gain?
- Can you think of what other types of filters could you create by cascading various RC filters?
- Can you think of how to build a band-pass filter using one resistor, one capacitor and one inductor?

What if my signal is not sinusoidal?

So far, we have studies how our filters respond to time-invariant signals (DC voltages), and to sinusoidally time-varying signals. But what if my signal is time-varying but not sinusoidal? Surely real world signals are not always just sinusoidal!

You're right, real-world signals can have arbitrary time variations. But the mathematics of **Fourier series** tells us that any signal can be represented as a sum of many sinusoidal signals with different frequencies, different amplitudes and different phases. Since our filter circuit is linear, if we know how it behaves for each of the component frequencies, we also know how it behaves for the sum (simply sum up the outputs). Hence it is sufficient to know the frequency response of the filter to know how exactly it'll modify any input signal! This is a very powerful idea, and the basis of most signal processing techniques.

Note: You will learn more about Fourier series in **EE2023 Signals and Systems**.

Applications

Okay, so now we understand how to build filters – but what are they used for?

Filters appear everywhere in Engineering. Take an example of some sensor (say the antenna of your cellphone) that generates a tiny voltage that we are interested in measuring. Chances are that the output of the sensor is contaminated by a 50 Hz noise being picked up from power transmission lines that are oscillating with a AC voltage at 50 Hz and radiating electromagnetic waves. This noise may even be much larger in amplitude than the signal from the sensor we want to measure! So what do we do? Create a filter that'll filter away the 50 Hz noise! Your cellphone probably operates at the 900 MHz band, and so a high-pass filter with cutoff frequency just below this frequency can easily filter away the 50 Hz noise.

Take another example of a sensor – say a light sensor that is measuring ambient light to automatically switch on your porch light at night. Again, if its output is contaminated by the 50 Hz power-line noise, you don't want it to turn your porch light on/off many times a second. We know that the ambient light does not vary rapidly, so we can setup a low-pass filter with a cutoff frequency of 0.1 Hz (or even lower) at the output of your sensor to filter off the 50 Hz noise. This will give you a stable output that can be used to drive a relay to turn on/off your porch light.

- Do you recall the AC/DC coupling option on the Oscilloscope channel? What kind of a filter do you think that uses?

How do systems sense their environment?

All engineering systems have sensors. Without sensors, a system would not have any knowledge about its environment, and therefore would not be very useful.

Take an air-conditioning unit as an example. It requires a temperature sensor to know the temperature in the room. If it is much warmer than the desired temperature (also known as a *set-point*), the air-conditioner has to blow cold air to cool the room. If the temperature is cooler than the set-point, the air-conditioner should stop blowing cold air, and simply monitor the temperature. In addition to a temperature sensor, most air-conditioners will have other sensors. For example, they may have buttons or touch sensors to allow the user to change the set-point. Since buttons allow the system to sense when the user is pressing them, they are sensors too.

In this chapter, we will learn about various kinds of sensors, how they work, and how they are typically connected to an engineering system.

How sensors work?

Sensors come in various shapes and forms, ranging from simple circuit elements to integrated circuits (ICs) to sophisticated modules that can be directly interfaced using standard ports (e.g. USB port, serial port).

Figure 31: TM temperature sensor from papouch.com.

A digital temperature sensor

Let us begin by exploring an off-the-shelf digital temperature sensor (TM) from papouch.com that can be connected to the RS232 serial port of a computer.

If you have a desktop computer or an older laptop, it'll probably have one or more RS232 serial ports (often called COM ports) that looks something like the one shown in Figure 32.

Newer laptops often only come with USB ports, but one can get a USB-to-RS232 dongle that will provide a RS232 serial port to connect to. In either case, once we have a RS232 serial port on a computer, we simply connect the sensor to that port and run a serial terminal software (e.g. termite) to obtain the data from the sensor:

Figure 32: RS232 serial port on desktop PC.

```
+025.3C
+025.4C
+025.3C
```

This sensor automatically sends out a temperature measurement every 10 seconds, once it is connected to a RS232 serial port.

In practice, we may want to connect our sensor to an embedded computer (most embedded computers have serial ports), and read the temperature measurements from our own embedded software. While the details of the software would depend on the detailed of the embedded computer, it is not hard to write software that reads the sensor measurements. For example, if the embedded computer runs our software in Python on Linux, we may write something like this:

```
import serial
with serial.Serial('/dev/ttyS1', 9600) as port:  # open serial port @ 9600 bps
    s = port.readline()                           # read sensor output as string
    s = s[:-1]                                    # remove the trailing 'C'
    temperature = float(s)                        # convert string to float
```

Okay, so that wasn't so hard!

Understanding sensor specifications

If we search for "temperature sensor" on the web, we will find many! So how do we select one? Are all of them the same, and we simply pick one? The answer is no! To choose one, we have to carefully look at the *technical specifications* of the sensor (typically in the *data sheet* for that sensor), and see which ones meet our application needs. Then other factors such as price, size, power consumption, etc may help narrow down the choice further.

Technical parameters

Measurable range	–55 to +125 °C
Accuracy	±0.5 °C within range from –10 °C to +85 °C and ±2 °C outside of this range
Resolution	0,1°C
Operating temperature of electronic	–40 to +85 °C
Communication	ASCII, described below
Measurement speed	the first measurement within 1 sec, subsequently once per 10 sec ±2 %
Communication line	RS232 (simplified)
Communication parameters	9600 Bd, 8 bits, 1 stop-bit, parity – none

Figure 33: Temperature sensor specifications from the data sheet.

- What is the difference between accuracy and resolution?
- How can it be that the operating temperature range of the electronics is narrower than the measurable range of the sensor? Wouldn't the electronics get damaged if we tried to measure a temperature of, say, 100°C?

Let us take a look at the technical specifications for the temperature sensor that we explored in the previous section (see Figure 33). The specifications tell us what is the temperature range that the sensor can measure, to what precision (resolution), to what accuracy, what interface the sensor reports its measurements over, and how often.

How does a digital sensor work?

We used a digital temperature sensor that gave us temperature data over a serial port. But how does it really measure temperature? And how does the measurement get converted to the ASCII text that is sent over the serial port?

Let us first assume that we have an *analog sensor* that produces a voltage that changes with the quantity we wish to measure. In the digital temperature sensor module above, we will perhaps find an analog temperature sensor that produces a voltage output that is dependent on the temperature:

$$V_o = f(T) \tag{8}$$

where V_o is the voltage output in Volts, T is the temperature in °C, and $f(\cdot)$ is some monotonic function.

- Why do we require that $f(\cdot)$ be monotonic? What would happen if it was not?

The output of this analog temperature sensor will be a voltage. If we have a way to measure this voltage, then we have a working sensor. Microcontrollers typically have analog-to-digital convertors (ADCs) that can measure voltage, and also have many standard ports. So our temperature sensor module would need a microcontroller with an ADC and a RS232 serial port:

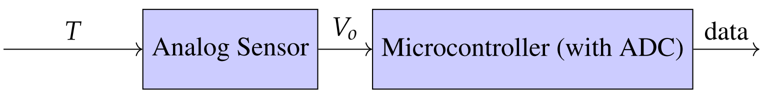


Figure 34: Functional breakdown of a typical digital sensor measuring T.

The microcontroller would run a simple program to convert the voltage into temperature:

$$T = g(V_o) \tag{9}$$

where $g \approx f^{-1}$. This value can then be sent as ASCII text over the RS232 serial port.

How does an analog sensor work?

In the previous section, we assumed that we had an analog sensor that produces a voltage depending on the parameter that we measure. In some cases, it is easy to find such a device or material. For example:

- *Photovoltaic materials* (used in solar cells) directly convert light into electricity at the atomic level. The atoms absorb photons of light and release electrons that can flow in an electric circuit.

- *Piezoelectric materials* naturally produces a voltage depending on the force applied on the material.

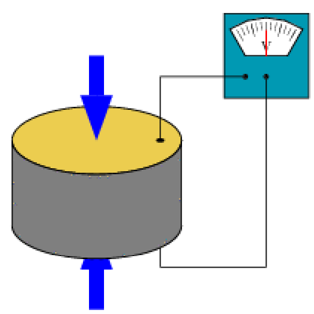


Figure 35: A piezoelectric material produces a voltage depending on the force applied on it.

- Can you find a device whose output voltage changes in response to a magnetic field? What is it called?

However, quite often it is not possible to find such materials or devices directly. It is more common to find circuit elements whose electrical properties (e.g. resistance, capacitance, inductance) change with the quantity to be measured. We will call such circuit elements *sensing elements*.

For example, a *thermistor* is a type of resistor whose resistance R_s is strongly dependent[1]]Standard resistors exhibit a weak temperature dependence. on temperature T:

[1][

$$\log R_s = \frac{\beta}{T} + \alpha \tag{10}$$

for some constants α, β. We can incorporate the thermistor (or other sensing elements) into an electrical circuit that generates a voltage that can be measured:

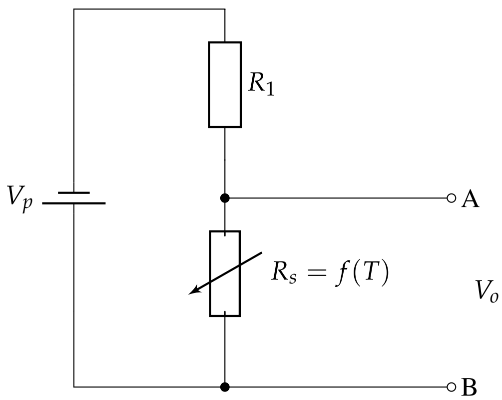


Figure 36: Simple electrical circuit to convert temperature T to voltage V_o that can be measured.

In this circuit, V_p is a voltage source of our choice (perhaps a battery), R_1 is a known resistance and R_s is our sensing element (thermistor). By analyzing the circuit, we easily see that:

$$V_o = \frac{R_s}{R_s + R_1} V_p. \tag{11}$$

Since V_p and R_1 are known, we can calculate R_s once we measure V_o:

$$R_s = \frac{V_o}{V_p - V_o} R_1. \tag{12}$$

The temperature T is easily obtained once R_s is known:

$$T = \frac{\beta}{\log R_s - \alpha}. \tag{13}$$

If we find other sensing elements whose electrical properties depend on other quantities of interest, we can use the same technique to make other sensors.

- Can you find a device whose resistance depends on the intensity of light incident on it? What is it called?
- In the circuit shown in Figure 36, how would you choose the values of V_p and R_1?

How do you measure the output voltage?

Once we know how to build an analog sensor, we can ignore the detailed circuit and simply think of it as a voltage source V_o with an output resistance R_o:

- In modeling an analog sensor as a simple voltage source with an output resistance, are we making an approximation?

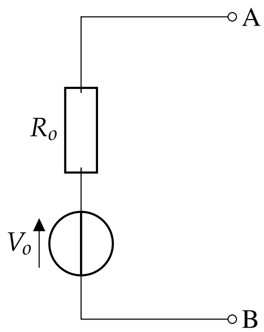


Figure 37: Equivalent circuit of an analog sensor.

For the circuit in Figure 36, we can compute R_o using Thevenin's theorem:

$$R_o = \frac{R_1 R_s}{R_1 + R_s}. \tag{14}$$

We can measure the output voltage between nodes A and B using a voltmeter or an oscilloscope, or in the case of the digital sensor, using an ADC. When we connect the leads of our measuring instrument to the nodes A and B, a tiny amount of current has to flow through the measuring instrument for it to sense the voltage. So the measuring instrument can be thought of as having an input resistance (aka *input impedance*) R_i. The instrument measures voltage V_{AB} between nodes A and B:

$$V_{AB} = \frac{R_i}{R_i + R_o} V_o. \tag{15}$$

Clearly this is not the same as V_o, unless $R_o = 0$ or $R_i = \infty$. In order for our measurement to be as close to V_o as possible, we require that $R_o \ll R_i$.

- Should we choose R_1 to be small or large for $R_o \ll R_i$?
- What is the effect of this choice of R_1 on the power consumed by the sensor?

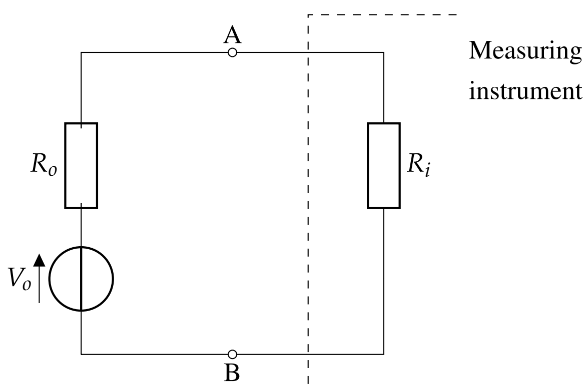


Figure 38: Equivalent circuit of an analog sensor and measuring instrument.

How do active sensors work?

So far, the sensors we have studied are *passive sensors*. They sense their environment without transmitting any energy. In some cases, sensors actively transmit energy in order to sense their environment. For example, a flash on a camera actively emits light. The light reflects off the subject and is received by the camera's CCD sensor to generate an image. The camera + flash system is therefore considered to be an *active sensor*. Typically, this means that active sensors require an energy source to operate, but passive sensors may operate with or without one.

Many active sensors are commonly found in engineering systems:

- Ultrasound ranging sensors, commonly used in reverse sensors in cars, emit a high frequency sound that echoes off obstacles. A measurement of the time taken for the sound to reach the obstacle and return back provides an estimate of the range to the obstacle.
- Drones commonly use ultrasound rangefinders to measure their altitude above the ground. An underwater version of ultrasound rangefinders are known as depth sounders, and are very commonly found on boats and ships.
- Laser rangefinders work on a similar principle, but use laser light instead of ultrasound to find range. Handheld laser rangefinders are commonly used in the construction and renovation industry, and by military.
- Active infrared proximity sensors are commonly used as touch-less switches for taps and towel dispensers in public washrooms.

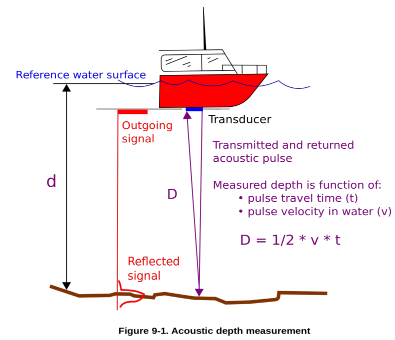


Figure 39: Principle of operation of a depth sounder.

Supplementary Reading

- How do photovoltaics work?
- Piezoelectric sensors
- Hall effect sensor
- Photoresistor

DC Power Supply

You learnt in EPP I about battery and photo-voltaic (PV) cells as source of electrical power. These sources are DC sources. However, most of electricity generation today is AC, which won the famous battle of the currents in the late 1880s and early 1890s. The transmission and distribution of electricity is done in an AC power network. The AC power network (utility grid) provides a sinusoidally time-varying voltage of fixed RMS value and frequency. For Singapore, the utility at home and offices is 230V/50Hz.

There are many loads like computers and LED lights that require DC voltage which can be obtained from the utility grid with the help of a DC power supply. Portable equipment and appliances are often powered by rechargeable batteries, and charging of batteries require DC source. In this chapter, you will learn the basics of how to make a DC power supply.

Building a DC supply from AC supply

Three essential stages of making a DC power supply:

1. Change in amplitude: the voltage at the utility power point is large (230 V RMS) compared to the amplitude of DC voltage required. This is done using a *transformer*.
2. Polarity: the polarity of AC voltage continuously alternates causing the current through any component flowing in both directions but the DC supply has a fixed polarity. A *rectifier* is used to do this.
3. Time-varying versus constant: AC voltage is time-varying but DC voltage is constant. A *filter* made with a capacitor is used to produce a nearly constant DC voltage.

Transformer

Transformers are used in AC power systems to change the voltage and current levels as shown in Figure 40. Power generated at power generating stations need to be transported over long distance using transmission cables.

There will be transmission loss equal to I^2R, where R is the resistance of the transmission line. For the same amount of power $P = V \times I$ transmitted,

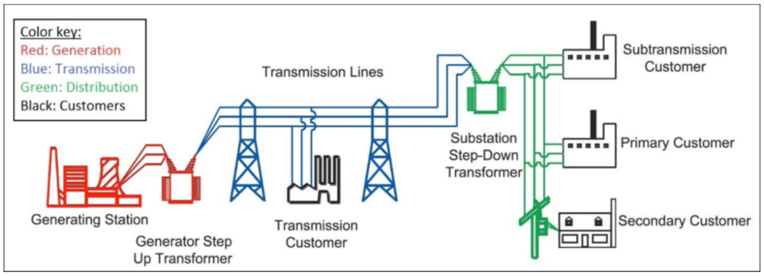


Figure 40: Power system schematic showing generation, transmission and distribution (Source: en.wikipedia.org)

increasing the voltage V would reduce the current I and thereby the transmission loss. Increasing the voltage level at the generator before transmission is done using a *step-up* transformer. However, at the other end of the transmission line, the voltage level must be lowered to safe level before distributing to the consumers which is done using a *step-down* transformer. Such step-up and step-down transformers are essential part of every power grid.

Ideal transformer

A transformer is a device that couples two AC circuits magnetically rather than through any direct conduction. A transformer consists of two or more coils of wire used to transfer electrical energy by means of a changing magnetic field as shown in Figure 41.

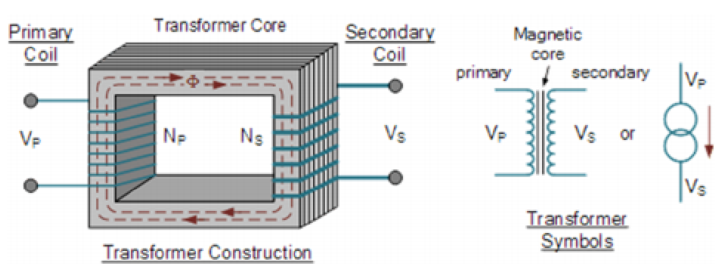


Figure 41: Basic structure of a transformer and circuit symbol used to represent a transformer

The AC voltage connected to the primary winding produces a time-varying magnetic field in the common magnetic core. Voltage is induced in the secondary winding as per *Faraday's law* of magnetic induction.

When a sinusoidal voltage is connected to the primary, *e.g.*, $V_p = V_m \sin \omega t$, it produces a flux in the magnetic core such that:

$$\begin{aligned} V_p &= V_m \sin \omega t, \\ &= \frac{dN_p \phi}{dt}, \\ &= N_p \frac{d\phi}{dt}, \end{aligned}$$

where, N_p is the number of turns in the primary side coil and ϕ is the flux produced by each turn of the coil.

$$\frac{d\phi}{dt} = \frac{V_m}{N_p} \sin \omega t.$$

Assuming that flux produced by the primary coil is fully linked to the secondary coil, the time varying flux passing through the secondary coil is also ϕ. Then the voltage induced in the secondary coil:

$$\begin{aligned} V_s &= \frac{dN_s\phi}{dt}, \\ &= N_s \frac{V_m}{N_p} \sin \omega t, \\ &= \frac{N_s}{N_p} V_m \sin \omega t, \\ &= \frac{N_s}{N_p} V_p. \end{aligned}$$

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}. \tag{16}$$

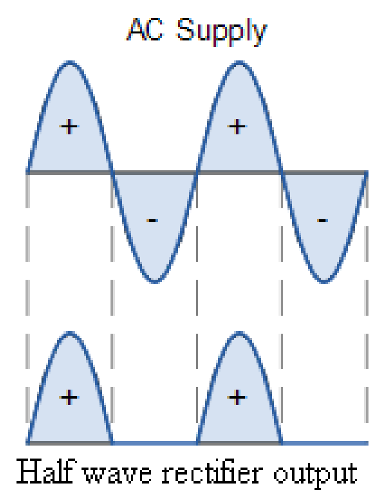
Here the assumption is that all the flux created by the primary winding are limited to the magnetic core and thus link the secondary coil.

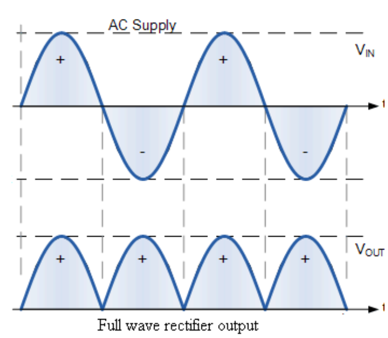
The ratio $\frac{N_s}{N_p}$, known as the **turns-ratio** of the transformer decides if the transformer is a step-up or a step-down transformer.

Assuming no power is lost in the winding and the core, then power is fully transferred from the primary side to the secondary side:

$$\begin{aligned} V_p \times I_p &= V_s \times I_s, \\ \frac{V_s}{V_p} &= \frac{I_p}{I_s}. \end{aligned}$$

$$\frac{I_s}{I_p} = \frac{N_p}{N_s}. \tag{17}$$





AC-DC rectifier

A rectifier converts the AC supply voltage to a DC voltage which can be connected to a load that requires DC supply. Since the DC power supply has a fixed polarity, the rectifier is expected to do one of the following:

- Allow power flow to the load when AC voltage has one polarity (let's say positive) and block power flow during the other polarity (negative). Such rectifier is known as *half wave rectifier*. (Top waveform in the margin)
- Allow power flow to the load during both positive half and negative half of the AC voltage but channel the current in the same direction through the load. This type of rectifier is known as *full wave rectifier*. (Bottom waveform in the margin)

Diode

You have experienced in EPP I that an LED is activated only if the voltage is applied with correct polarity. LED is a special type of diode that emits light when current flows through it.

All diodes including LED are semiconductor devices with a PN junction. A diode conducts in one direction only when it is forward-biased and blocks flow of current when it is reverse-biased. This property of diode makes it suitable for a rectifier.

Half wave rectifier

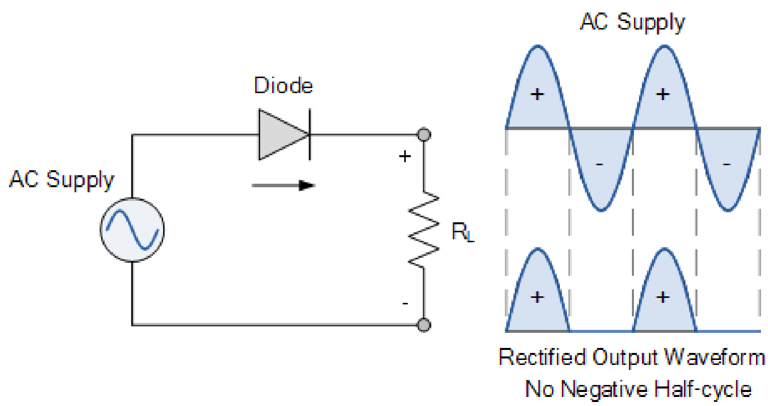


Figure 42: In half-wave rectifier, only one diode is used. As current flows in one direction only, the load does not get power during half of the cycle of AC voltage.

The power output and, therefore, rectification efficiency is quite low. This is due to the fact that power is delivered only during one-half cycle of the input alternating voltage.

Although the voltage appearing across the load resistor has one polarity, the voltage is not constant. The **ripple voltage**, *i.e.*, the fluctuation in the DC voltage is equal to the amplitude (V_m) of the AC voltage. The ripple can be reduced by adding a capacitor at the output of the rectifier (parallel to the load). The capacitor acts as a filter by not allowing fast-changing voltage at the output of the rectifier.

How capacitor filter helps to reduce ripple is explained later in the context of full wave rectification. However, the same explanation is valid for half wave rectifier.

Full wave rectifier: Diode bridge rectifier

Diode bridge rectifier is one common way to implement full wave rectification. In this rectifier, four diodes are connected to form a bridge. At any time, two diodes will be forward biased while the other two diodes will be reverse biased.

Diode bridge is one of the ways to make a full wave rectifier. There are other ways which are not discussed here.

During the positive half-cycle of the AC supply, diodes D1 and D2 conduct and AC supply is directly connected to the load (positive end of the supply to the positive end of the load and the negative to negative) as shown in Figure 44 (left). During the negative half cycle, diodes D3 and D4 conduct, effectively swapping the terminals of the AC supply and hence the load sees

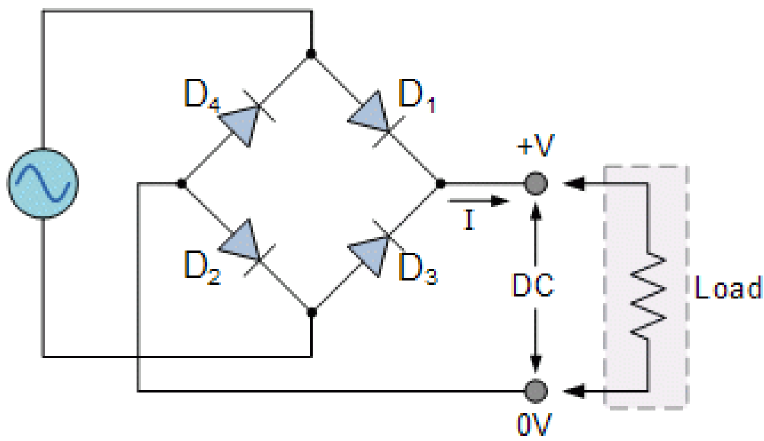


Figure 43: When the resistor is connected, current will flow in both cycles of the AC voltage. However, in both cycles, the flow of current is in the same direction as shown by the arrow.

a positive voltage as shown in Figure 44 (right). Note that the voltage at the

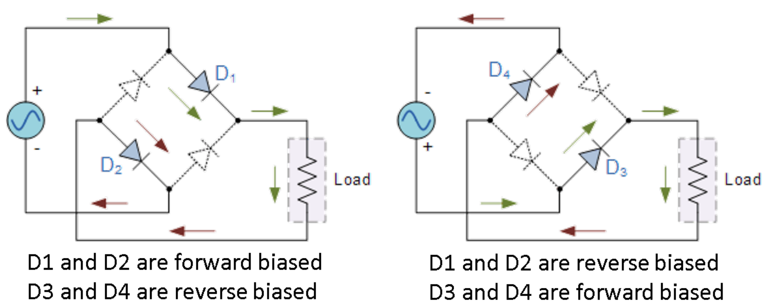


Figure 44: Flow of current during the positive half cycle (left) and negative half cycle (right) of AC voltage.

load is unidirectional but not constant. The output voltage has a large 'ripple', which can be minimized by using a capacitor filter.

Average voltage of the FWR output:

Let the AC voltage be

$$v_{in}(t) = V_m \sin \omega t.$$

Assuming the diodes to be ideal, *i.e.*, zero diode voltage during conduction, the average voltage of the FWR output:

$$\begin{aligned} V_{o,avg} &= \frac{1}{T/2}\int_0^{T/2} V_m \sin \omega t dt, \\ &= \frac{2}{T}\frac{1}{\omega}[-V_m \cos \omega t]_0^{T/2}, \\ &= \frac{2V_m}{\omega T}[\cos(0) - \cos(\omega T/2)], \\ &= \frac{V_m}{\pi}[1+1], \\ &= \frac{2V_m}{\pi}. \end{aligned}$$

Diode bridge rectifier with capacitor filter

If a capacitor is connected across the resistor as shown in Figure 45, the output voltage becomes smoother.

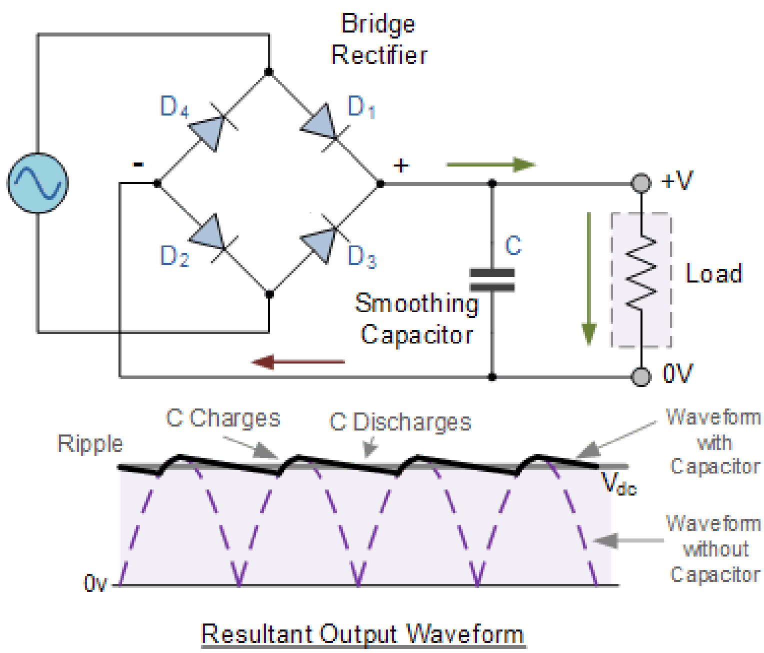


Figure 45: Diode bridge rectifier (FWR) with smoothing capacitor.

When the input AC voltage is more than the capacitor voltage, one pair of diodes conduct; the load receives power directly from the AC supply, and the capacitor gets charged, *i.e.*, energy is stored in the capacitor. When supply voltage falls below the capacitor voltage, the diodes stop conducting and the capacitor supplies energy to the load. As the capacitor supplies energy, its voltage starts decreasing. The fluctuation of the capacitor voltage, *i.e.*, the *peak-to-peak ripple voltage* of the rectifier output depends on the capacitance, the resistance, output voltage, and the period of the AC supply.

When the capacitor supplies energy to the resistor,

$$C\frac{dv}{dt} \approx \frac{V_0}{R}.$$

Assuming the discharge time to be approximately equal to $\frac{T}{2}$, change in capacitor voltage during this time:

$$\frac{\Delta V}{T/2} \approx \frac{V_0}{RC}, \tag{18}$$

$$\boxed{\Delta V = \frac{V_0}{2fRC},} \tag{19}$$

where, ΔV is the fluctuation of capacitor voltage known as the *ripple voltage*, T is the period of the AC waveform, and f is its frequency.

Introduction to DC Motor

Motors produce mechanical power from electrical power, and are used in many practical applications ranging from small appliances to large industrial drives.

Although the term *motor* is more widely associated with machines capable of producing rotational motion, there are motors that produce linear motion which are known as *linear motors*.

- Input power: electrical $Power = Voltage \times Current$
- Output power: mechanical
  1. Rotational motion: $Power = Torque \times Angularvelocity$
  2. Linear motion: $Power = Force \times Velocity$

Depending on the type of electrical supply used, the motors are classified as

1. DC motors: powered by a DC supply, and
2. AC motors: powered by an AC supply.

Each class is further classified depending on how the motor works and how it is built.

A summary of different motors is given at the end of this chapter. In EPP, you will learn about DC motors only.

Energy conversion in electromechanical systems

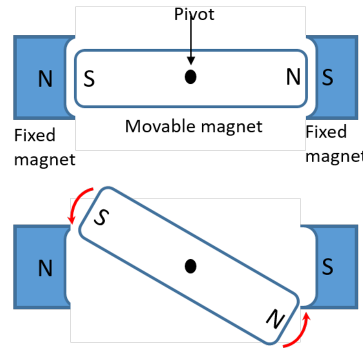
In any electromechanical system, *e.g.*, motor and generator, interaction between an electrical system and a mechanical system takes place through an electromagnetic field that is common to both systems. The interaction is governed by two laws:

- A current carrying conductor placed in a magnetic field experiences a force exerted on it. This is exploited to make a motor.
- An electromotive force (voltage) is developed across a conductor moving inside a magnetic field such that the flux linkages with the conductor is changed. In a generator, externally applied mechanical power is used to move the conductors in a magnetic field.

As in any energy conversion process, there are losses in electromechanical systems as well. Some causes of energy loss are:

- friction in the mechanical system,
- power loss (I^2R) in the conductors of the electrical system,
- eddy current loss and hysteresis loss in the ferromagnetic materials of the magnetic system.

How does a DC motor work?



Consider a simple arrangement of magnets shown in the margin figure in which the magnet in the middle is pivoted at the center and is free to rotate. Without any external force applied, the inner magnet will remain aligned with the two outer poles that are fixed. If you rotate the center magnet and then release, a magnetic force will make the inner magnet rotate.

However, the movement of the inner magnet will not be continuous if the poles remain fixed as shown. But if electromagnet is used, the polarity can be made to alternate so that continuous motion is produced. In DC motors, at least one set of magnets must be electromagnet. The second set of magnets can be either permanent magnet (**PMDC** motor) or electromagnet (**Wound field DC** motor).

Stator and Rotor

DC motors include two key components:

1. a stator - the stationary part, and
2. a rotor - the rotating part.

A simple DC motor has a stationary set of magnets (permanent magnet or electromagnet) in the stator. In the rotor, a coil of insulated wire with a current running through it generates an electromagnetic field aligned with the centre of the coil. One or more windings of insulated wire are wrapped around the core of the motor to concentrate the magnetic field.

Structure of a DC motor

The cross-section of a motor with two fixed poles is shown in Figure 46. The central magnetic core carries current-carrying conductors and is also known as the **armature**.

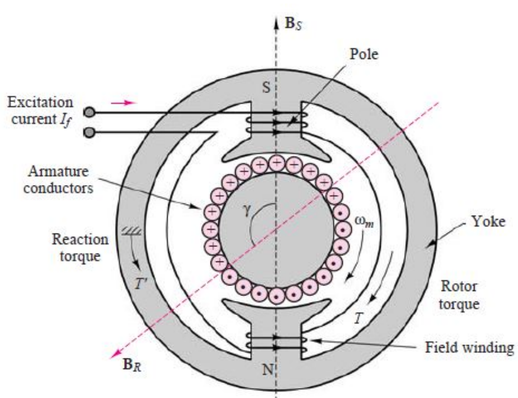


Figure 46: Cross-section of a two-pole DC motor. The grey circle in the middle is cross-section of the iron core of rotor. Small circles around the periphery of the core are conductors that form loops around the core. Half of these conductors are indicated with a x sign meaning current through them flows into the paper, while current flows out of the paper through the remaining conductors which are marked with a dot (.). When current flows through the **armature** conductors, a torque is produced. In this drawing, the second magnet (poles indicated using N and S) is also electromagnet. They are permanent magnets in a PMDC motor.

Motor torque

When current flows through a conductor placed in magnetic field, a force is exerted on the conductor. The magnitude of the force (unit: Newton) is

$$F = BIL,$$

where, B is the magnetic flux density (unit: Tesla), I is the current (ampere) and L is the length (unit: meter) of the conductor within the magnetic field. The direction of the force can be found using **Fleming's left hand rule**.

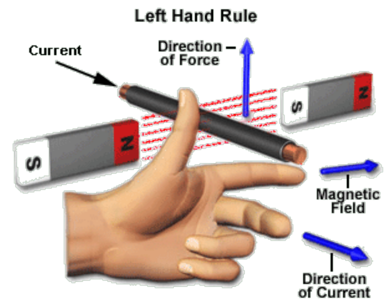


Figure 47: Fleming's left hand rule.

Consider a conductor loop placed in the magnetic field (Figure 48) having current (I) flowing through it. Force of equal magnitude will be exerted on two sides of the loop as the magnitudes of B, I and L are identical for both. However, the forces act in opposite direction giving rise to a moment. In a DC motor, many such loops are wound around the rotor core and the sum of all individual moments is the torque produced by the motor.

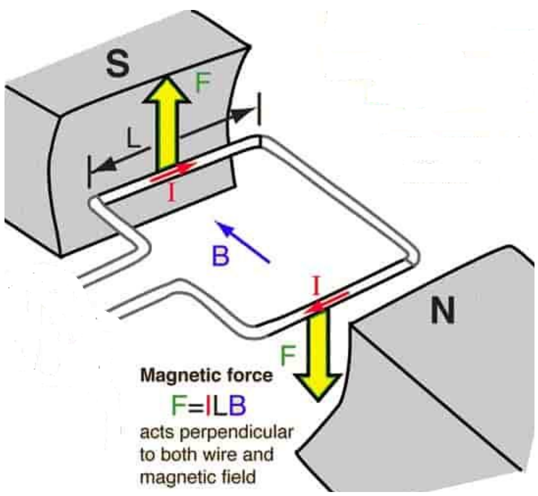
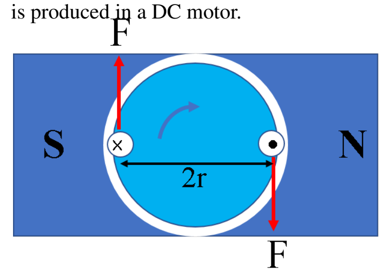
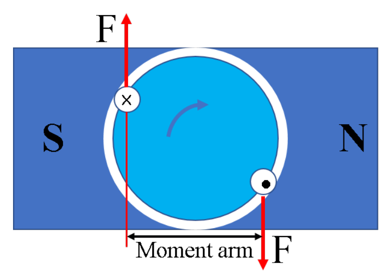
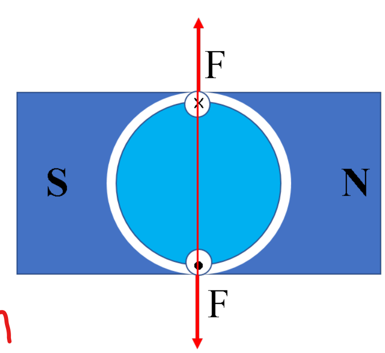


Figure 48: Simple illustration of how torque is produced in a DC motor.





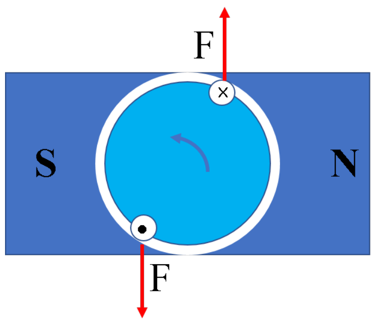


For ease of sketching, a two-dimensional drawing is used with $\times$ and dot (.) representing currents in opposite directions. Then the first sketch shown in the margin represents the motor of Figure 48 that has a rotor with a single loop. If the radius of the rotor core is r then the resulting torque is

$$\begin{aligned} T_{1loop} &= BIL \times 2r, \\ &= 2BILr. \end{aligned} \tag{20}$$

This torque will force the conductor loop as well as the rotor to rotate, and as it rotates, the perpendicular distance between the forces becomes smaller resulting in decrease in torque (the sketch in the middle in the margin). Torque produced by a single loop will continuously decrease and eventually, the loop will reach a position where the torque is zero (the sketch at the bottom).

Though the torque becomes zero, the loop will continue to rotate due to inertia. Once it goes past the zero-torque position, the torque will become non-zero again (Figure below) but the direction will be reversed.



Unless something is done, the loop will oscillate around the zero-torque position and eventually get locked. The sub-section *Commutator* explains how the torque in DC motor is maintained in one direction.

It is mentioned earlier that the torque produced by a single loop varies with position of the loop. In a DC motor, many conductor loops are present around the circumference of the rotor core (Figure 46). Though the torque from one loop varies, the sum of torques remains constant.

Conductor length L and rotor radius r are fixed quantity for a given motor. The number of conductor loops is also fixed. So if B is assumed constant, the electro-magnetic torque (T_{em}) is proportional to the armature current (I_a), also called the rotor current,

The stator field (B) in some wound-field DC motors vary with operating condition. You will learn more about them in the section *Different types of DC motors*.

$$\boxed{\begin{aligned} T_{em} &\propto I_a, \\ &= K_t I_a. \end{aligned}} \qquad (21)$$

The constant K_t is known as **torque constant** and is one of the motor parameters that characterizes a DC motor.

Not always true!!

Electromagnetic torque and Shaft torque

The mechanical power produced by the motor is

$$\boxed{P_{mechanical} = T_{em} \times \omega,}$$

where, ω is the angular speed measured in radian per second. A more common unit of motor speed is **RPM** or revolutions per minute. *What is the relation between RPM and ω?*

If the shaft makes N revolutions per minute, *i.e.*, $\frac{N}{60}$ revolutions per second, then the angular speed is

Converting RPM to rad/s

$$\boxed{\omega = 2\pi \frac{N}{60}.} \qquad (22)$$

In each revolution, any point on the shaft's circumference goes through an angular change of 2π radians.

The torque T_{em} cannot be fully used to drive a mechanical load. The shaft of the motor is mounted with bearings. Part of the power developed is wasted as heat due to friction, which is known as **friction loss** (P_{fric}) or **rotational loss** (P_{rot}). Power available at the motor shaft is

$$\begin{aligned} P_{sh} &= T_{em}\omega - P_{fric}, \\ T_{sh}\omega &= T_{em}\omega - T_{fric}\omega. \end{aligned}$$

The torque

$$T_{sh} = T_{em} - T_{fric}$$

is known as the **shaft torque**.

Commutator

As a conductor goes from facing the south pole to facing the north pole and vice versa, the current must be reversed to maintain continuous torque. This reversal of current can be done mechanically using **commutator**.

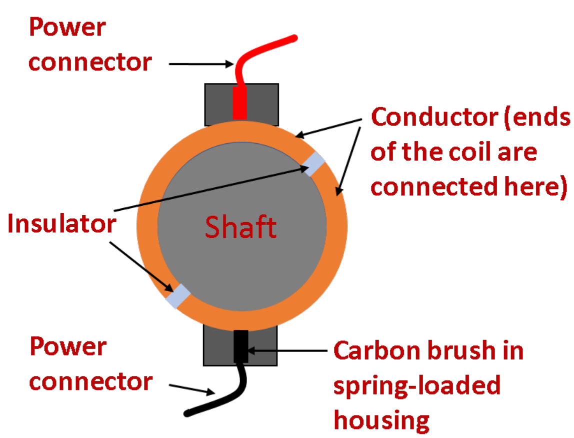
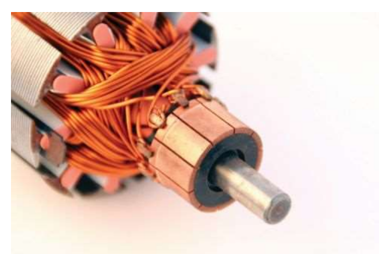


Figure 49: Construction of the commutator: as the shaft rotates, the commutator ring rotates with it. Carbon brushes or metallic brushes at the end of the power connectors are pressed agains the commutator using a spring-loaded housing. As the rotor spins, the conducting segments of the commutator ring gets connected alternately to positive and negative sides of the power supply.

Commutator is a metal ring segmented by insulator and is wrapped around the motor shaft. A commutator with two conductor segments is shown in Figure 49. This is also known as **split-ring commutator**. In a practical motor, the commutator comes with several segments.



Back EMF

When armature conductors rotate inside the magnetic field produced by the stator, a voltage is induced between the ends of the conductor loop. This induced voltage, known as **back emf** (electromotive force) or *counter emf*, opposes the flow of current set by the external power supply which is producing the torque.

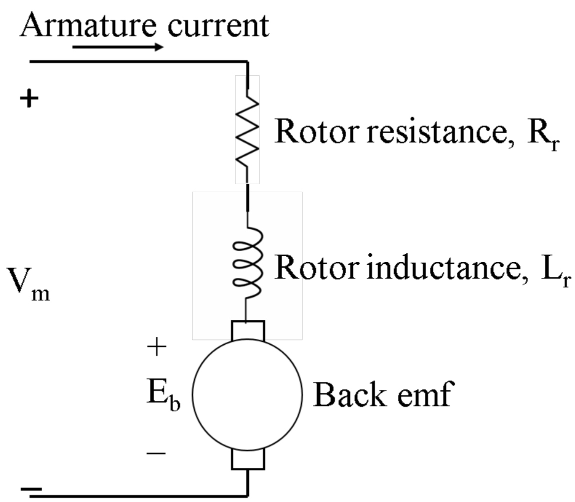
The back emf E_b is proportional to the speed of rotation. Faster the rotor rotates, higher is the magnitude of the back emf.

$$\boxed{\begin{aligned} E_b &\propto \omega, \\ &= K_e\omega. \end{aligned}} \tag{23}$$

The constant K_e is known as **back emf constant** and is another parameter used for characterizing a motor.

DC motor characterization

In motor characterization, we study the effect of applied voltage and load demand on motor speed. Characterization of PMDC motor is shown in this section. The circuit model of the rotor of a PMDC motor is shown below.

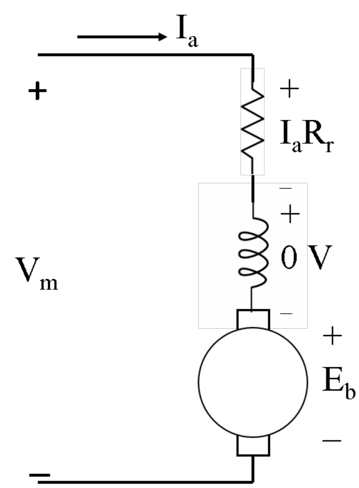


The stator is not shown here. In steady-state, the armature current is constant and hence

$$v_{L,ss} = L_r \frac{dI_a}{dt} = 0.$$

This is shown in the margin figure. Writing KVL around the rotor circuit,

$$\boxed{E_b = V_S - I_a R_r.} \tag{24}$$



The torque constant (K_t) and the back emf constant (K_e) do not change with operating speed and torque for a PMDC motor. So,

$$T_{em} = K_t I_a, \qquad E_b = K_e \omega.$$

Substituting these in equation 24,

$$\boxed{\begin{aligned} K_e \omega &= V_S - \frac{T_{em}}{K_t} R_r, \\ \omega &= \frac{V_S}{K_e} - \frac{R_r}{K_t K_e} T_{em}. \end{aligned}} \tag{25}$$

Speed versus applied voltage

We can conclude from equation 25 that

for a fixed torque demand, the speed of the motor can be increased by increasing applied voltage.

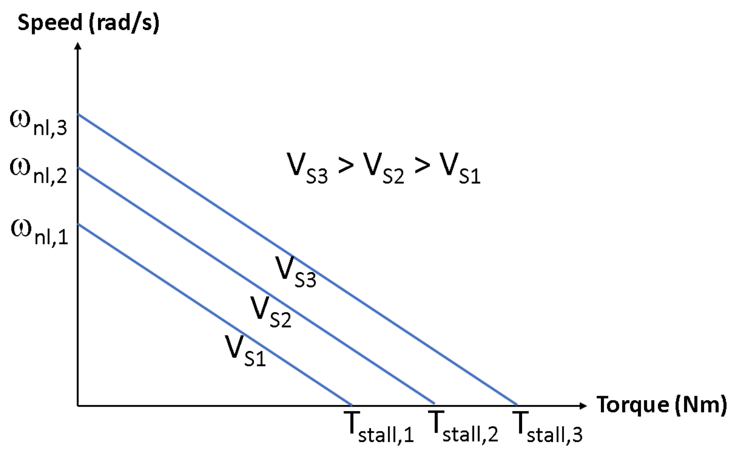


Figure 50: Speed versus torque graph of PMDC motor for three different supply voltages V_{S1}, V_{S2}, V_{S2}.

Speed versus torque

Similarly, we can conclude that

for a given supply voltage, the speed of the motor decreases with increasing load demand.

No-load speed

This is the speed of a DC motor when the torque T_{em} is zero. Substituting $T_{em} = 0$ in equation 25,

$$\omega_{nl} = \frac{V_S}{K_e}.$$

When the torque required is zero, armature current becomes zero making the back emf $E_b = V_S$.

This is the maximum theoretical speed the motor can run at for a given supply voltage. In reality, motor will never run at this speed as a small amount of torque is always required to overcome the friction torque of bearings.

$V_s = V_a + E_b$
$V_s = I_a R_a + k_e \omega$
$V_s = \frac{T_{em}}{K_t} R_a + k_e W$

Stall torque

This is the torque produced by the motor when the speed ω is zero. Substituting $\omega = 0$ in equation 25,

$$T_{stall} = K_t \frac{V_S}{R_r}.$$

When the speed is zero, $E_b = 0$ making armature current $I_a = \frac{V_S}{R_r}$.

This is the maximum torque the motor can produce for a given supply voltage. In practice, it is advisable not to run a motor in this condition as large amount of $I_a^2 R_r$ loss makes the motor hot. The motor may be damaged if the heat is not adequately dissipated.

$$\boxed{E_b I_a - P_{friction} = T\omega}$$

Power developed

The electro-magnetic power developed by the motor is

$$\boxed{P_{dev} = T_{em} \times \omega.}$$

$$\boxed{P_{dev} = E_b * I_a}$$

No-load speed (ω_{nl}) at a given input voltage is the maximum speed the motor can spin at that voltage. However, the torque produced is zero and, therefore, the power developed is also zero. Similarly, the power developed is zero when the motor is stalled. The speed versus torque graph and the power versus torque graph of a small PMDC motor operating at 6V are shown in Figure 51.

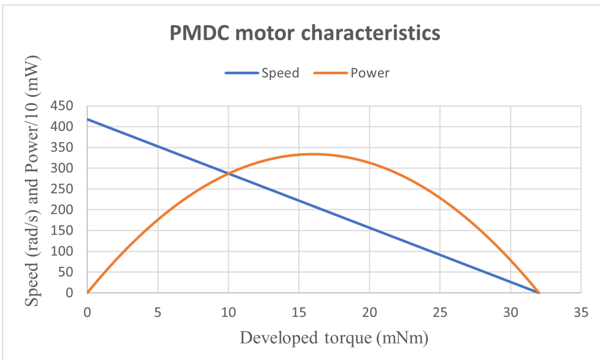


Figure 51: Speed decreases with increasing torque produced. Power developed is zero at the two extreme points of the speed-torque graph. For PMDC motor, the power is maximum at the an operating point where the torque is half of the stall torque and speed is half of the no-load speed. The power plot is scaled by a factor of 10. The maximum power developed by this motor is 3340 mW or 3.34 W.

***

Motor dynamics

When voltage is applied to a DC motor or the magnitude of the voltage is changed, the speed is not changed instantaneously. Inductance of the coils and velocity-dependent friction are two of the causes that make motor a dynamic system.

Electrical time constant

The change in the armature current is described by the first-order differential equation

$$L_r \frac{dI_a}{dt} + R_r I_a = V_S - E_b. \tag{26}$$

The **electrical time constant**

$$\tau_e = \frac{L_r}{R_r}$$

determines the time taken for the armature current to reach a constant value. The steady-state current is

$$V_s = V_a + E_b$$

$$\begin{aligned} I_a &= \frac{V_S - E_b}{R_a}, \\ &= \frac{V_S}{R_a} - \frac{K_e}{R_a}\omega. \end{aligned} \tag{27}$$

Mechanical time constant

At equilibrium point,

$T_{em} = T_L$

motor torque = Load torque

If the net torque is

$$T_{net} = T_{em} - T_{fric} - T_L$$

then according to Newton's second law:

$$J\frac{d\omega}{dt} = T_{em} - T_{fric} - T_L,$$

where J is the moment of inertia. Assuming that the friction torque varies linearly with, *i.e.*, $T_{fric} = K_f\omega$

$$J\frac{d\omega}{dt} + K_f\omega = K_t I_a - T_L. \tag{28}$$

This is also a first-order differential equation with time constant

$$\tau_m = \frac{J}{K_f},$$

where, τ_m is called **mechanical time constant**.

The mechanical time constant τ_m is usually order of magnitude higher than the electrical time constant τ_e. The time taken by armature current to reach its steady-state is much smaller than the time taken by the speed to reach its steady-state.

DC motor has a self-regulating behaviour.

- If the load torque is increased, the motor needs to produce higher torque (T_{em}) to bring the system described by equation 28 to equilibrium. Then the motor will draw higher armature current (I_a) to produce the extra torque. As a consequence, the back emf

  $$E_b = V_S - I_a R_r$$

  will decrease. Decrease in back emf means decrease in the speed

  $$\omega = \frac{E_b}{K_e}.$$

- If the load torque is decreased, the motor needs to produce less (T_{em}) to bring the system to equilibrium. The motor draws less armature current (I_a) and, as a consequence, the back emf will increase. Increased back emf means higher speed.

Different types of DC motors

In PMDC motor, the stator field is produced by permanent magnets. As a result, we can analyse motor characteristics by considering the rotor circuit only with the assumption that K_t and K_e remain constant regardless of the operating condition.

In wound-field DC motors, the stator field is generated by electromagnets. So, the motor characteristics depend on the stator current as well. Instead of assuming $T_{em} = K_t I_a$ and $E_b = K_e \omega$, we should include the magnetic flux produced by the stator in these expressions.

Let ϕ be the stator flux which is proportional to the current (I_f) flowing through the stator coil, then the flux density

$O = k(I_f)$

$$B \propto \phi.$$

Since the remaining motor parameters, *i.e.*, rotor diameter, length, and number of conductor remain constant, the torque and back emf expressions become

$$\begin{aligned} T_{em} &= K\phi I_a, \\ E_b &= K\phi\omega. \end{aligned} \tag{29}$$

Depending on the way the stator is energized (Figure 52), the wound-field motors are classified into five types:

1. Separately excited motor (Figure 52 (a)),
2. Shunt motor (Figure 52 (b)),
3. Series motor (Figure 52 (c)),
4. Compound motor:
   (a) Short-shunt compound motor (Figure 52 (d)), and
   (b) Long-shunt compound motor (Figure 52 (e)).

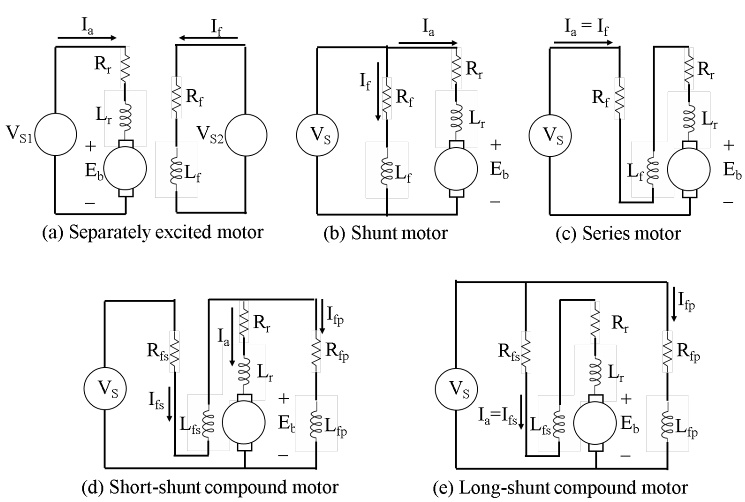


Figure 52: Types of wound-field DC motors

Separately excited motor: The speed-torque characteristics of this motor for a given stator current is like that of PMDC motor. However, unlike in PMDC motor, there is flexibility in varying the stator flux by changing the stator current I_f. It gives greater flexibility in regulation of speed and mechanical torque as two separate voltage sources are used.

Shunt motor: The speed-torque characteristics is again like that of shunt motor. The stator current can be varied by inserting a variable resistor in series with the stator coil. Any change in the stator circuit does not affect the rotor circuit.

Stator and rotor windings are parallel. ⇒ I_f & I_a are diff.

Series motor: The armature current (I_a) and the field current (I_f) are the same.

↳ because stator an rotor windings are in series

DC motor specifications