CS544 MP2

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1 Problem Statement

We are given set of unordered points P: (0, 0, 1); (0, 1/2, 0); (0, 1, 1); (1/2, 0, 0); (1/2, 1/2, 1); (1/2, 1, 0); (1, 0, 1); (1, 1/2, 0); (1, 1, 1). That we will use for setting up constraints in constraint optimization problem.

We define create sparse matrix L that refers to x, y coordinates of P, and C that refers to z coordinates of P.

2 Part 1.

In part 1, we minimize smoothness cost, defined by norm of height map gradients, that interpolates given points, using Augmented Lagrangian method.

We first define smoothness term as L2 Norm of height map gradients in both x and y coordinates.

$$||\nabla H||_2 = \sqrt{(\delta_x H)^2 + (\delta_y H)^2}$$

where H is a vectorized height map.

For simplicity, we use squared L2 Norm instead of L2 norm as our objective function.

$$F = (\delta_x H)^2 + (\delta_y H)^2$$

We obtain gradient of height map using finite differences:

$$\delta_x H \simeq \frac{H_{i,j} - H_{i+\Delta x,j}}{\Delta x}$$
$$\delta_y H \simeq \frac{H_{i,j} - H_{i,j+\Delta y}}{\Delta y}$$

In real implementation, we create sparse gradient matrices $D_x, and D_y$ for efficient gradient calculation. Then, our smoothness term becomes:

$$
\begin{aligned}
F &= (D_xH)^2 + (D_yH)^2 \\
&= (D_xH)^T \cdot (D_xH) + (D_yH)^T \cdot (D_yH) \\
&= H^T D_x^T D_x H + H^T D_y^T D_y H \\
&= H^T (D_x^T D_x + D_y^T D_y) H \\
&= H^T M H
\end{aligned}
$$

We apply constraints of interpolating points as:

$$LH - C = 0$$

We finally solve our ALM equation:

$$f(H) = H^T M H + \lambda^T (LH - C) + \frac{\sigma}{2} ||LH - C||^2$$

Final results look like:

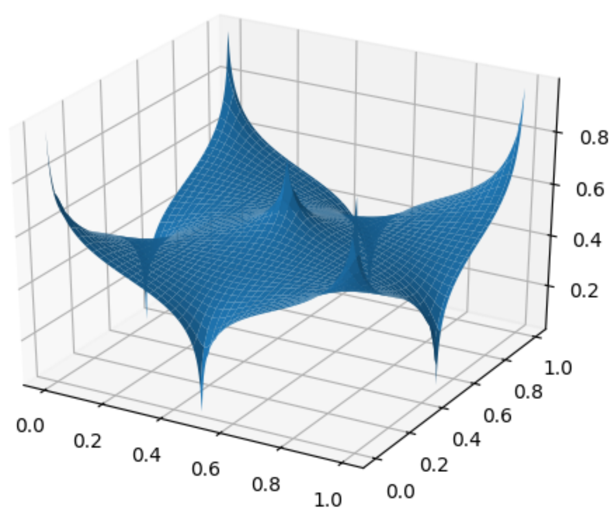

Figure 1: Part 1 solution, solved by ALM

3 Part 2.

In part 2, we minimize smoothness cost, defined by norm of height map gradients using linear equations solver.

We first formulate system of linear equation:

$$\begin{vmatrix} M & L^T \\ L & 0 \end{vmatrix} = \begin{vmatrix} \mathbf{0} \\ C \end{vmatrix}$$

Solving the linear system yields:

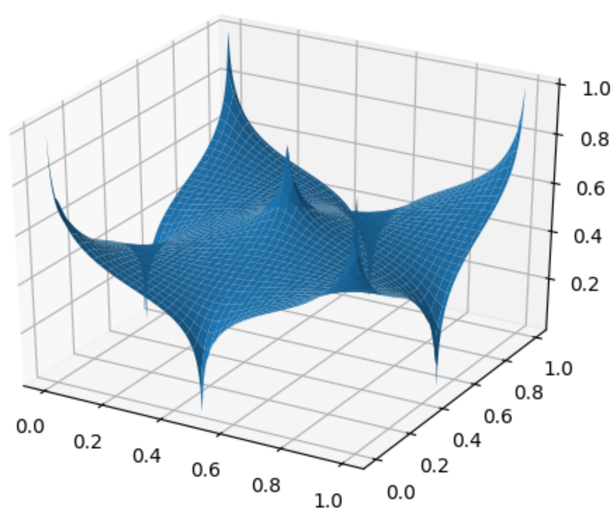

Figure 2: Part 2 solution. Obtained from least squares solution

4 Part 3.

In part 3, we minimize surface area that interpolates points and lines that interpolates those points on grid cell.

We first define surface area assuming that the surface is parametric:

$$\begin{aligned}
S(H) &= \sum \sqrt{(\delta_x H_{i,j})^2 + (\delta_y H_{i,j})^2 + A} \\
&= \sum_{i,j} \sqrt{(\frac{H_{i,j} - H_{i+\Delta x,j}}{\Delta x})^2 + (\frac{H_{i,j} - H_{i,j+\Delta y}}{\Delta y})^2 + (\Delta x \cdot \Delta y)} \\
&\simeq \sum \sqrt{((D_x H)_{i,j})^2 + ((D_y H)_{i,j})^2 + (\Delta x \cdot \Delta y)}
\end{aligned}$$

where A refers to area of single grid cell. Then, we have:

$$\begin{aligned}
\frac{\sigma}{\sigma H_{i,j}} S(H) &= \frac{\sigma}{\sigma H_{i,j}} \sum_{i,j} \sqrt{((D_x H)_{i,j})^2 + ((D_y H)_{i,j})^2 + (\Delta x \cdot \Delta y)} \\
&= \sum_{i,j} \frac{\sigma}{\sigma H_{i,j}} \sqrt{((D_x H)_{i,j})^2 + ((D_y H)_{i,j})^2 + (\Delta x \cdot \Delta y)} \\
&= \sum_{i,j} \frac{\frac{\sigma}{\sigma H_{i,j}} (((D_x H)_{i,j})^2 + ((D_y H)_{i,j})^2 + (\Delta x \cdot \Delta y))}{2\sqrt{((D_x H)_{i,j})^2 + ((D_y H)_{i,j})^2 + (\Delta x \cdot \Delta y)}} \\
&= \frac{(D_x^2 H)_{i,j} + (D_y^2 H)_{i,j}}{\sqrt{((D_x H)_{i,j})^2 + ((D_y H)_{i,j})^2 + (\Delta x \cdot \Delta y)}} \\
&= \frac{(D_x^T D_x + D_y^T D_y) H}{\sqrt{(D_x H)^2 + (D_y H)^2 + (\Delta x \cdot \Delta y)}}[i,j] \\
&= \frac{MH}{\sqrt{(D_x H)^2 + (D_y H)^2 + (\Delta x \cdot \Delta y)}}[i,j]
\end{aligned}$$

We define new constraint L' and C' as x, y coordinates of points that interpolates P on grid cell lines, and z coordinates of points that interpolates P on grid cell lines, respectively.

Finally, we solve following ALM equation:

$$f(H) = S(H) + \lambda^T (L'H - C') + \frac{\sigma}{2} ||L'H - C'||^2$$

The results looks like:

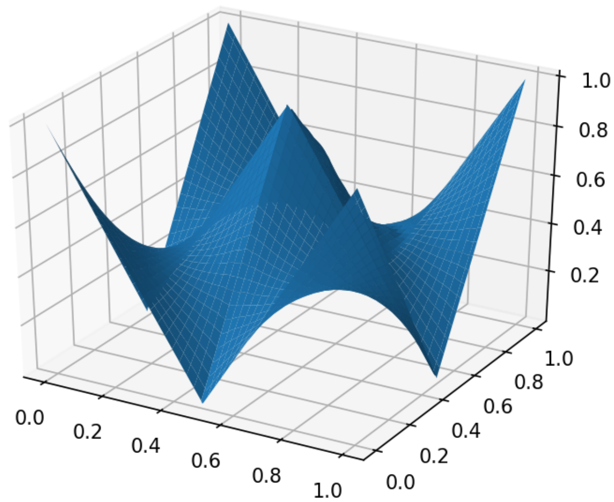

Figure 3: Part 3 solution obtained from ALM

5 Part 4.

In part 4, we minimize the surface area of the interpolating surface that interpolates only the vertices.

Similarly, as part 3, first define surface area assuming that the surface is parametric

$$S(H) = \sum \sqrt{(\delta_x H_{i,j})^2 + (\delta_y H_{i,j})^2 + A}$$

Then similar to part 1, we define the constraints as

$$LH - C = 0$$

We then solve the equations using Augmented Lagrangrian Method as

$$f(H) = S(H) + \lambda^T (LH - C) + \frac{\sigma}{2} ||LH - C||^2$$

Solving the system yields:

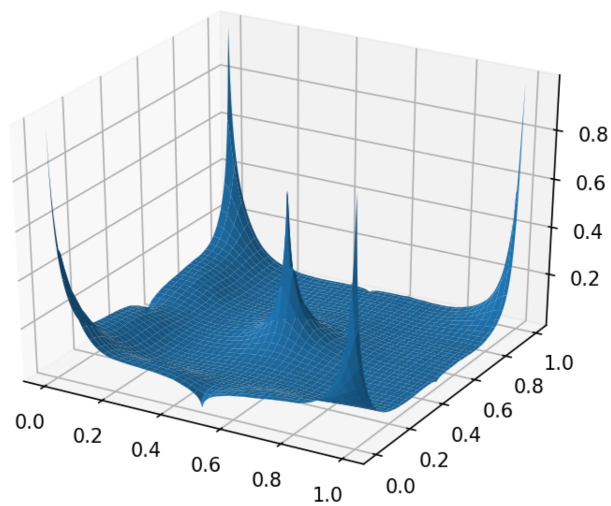

Figure 4: Part 4 solution obtained from ALM

6 Misc.

We used scipy's $fmin\_l\_bfgs\_b$ with factr set to 1e+12 to run each ALM iteration. Default values for H were set to 0, except for interpolating points, which had z value of interpolating point. Default values for λ and σ was set to 1, except for part 3's σ which had to be set to $\frac{1}{32}$ in order to get good results. This is due to solution falling into wrong local minimum caused by high initial multipliers. We have ran 10 iterations of LBFGS, with average of 12 seconds running time for each problems.