MAS 241 Analysis – HW #4 20190497 이충서

#16. If $f(0)=0$ or $f(1)=1$, we are done.

Suppose not.

Define $g: I \to \mathbb{R}$ as $x \mapsto f(x)-x$.

Then, $g(0)>0$ and $g(1)<0$.

$\Rightarrow$ By Thm 4.23 (IVT), $\exists\ 0<c<1$ s.t. $g(c)=0$.

Take such c.

$\Rightarrow g(c)=f(c)-c=0 \Rightarrow f(c)=c$ $\therefore$ we are done.

#17. (a) f is continuous at every irrational point.

pf) Let $a \in \mathbb{R}\setminus\mathbb{Q}$ and $\varepsilon>0$ be given.

Take $N=\lceil \frac{1}{\varepsilon} \rceil + 1$.

For each $n \in \mathbb{N}$, define m_n as the integer s.t.

$$\frac{m_n}{n} < a < \frac{m_n+1}{n}.$$

Define m, M as

$$m = \max\left\{ \frac{m_n}{n} : 1 \le n \le N \right\}$$
$$M = \min\left\{ \frac{m_n+1}{n} : 1 \le n \le N \right\}.$$

finite sets $\Rightarrow$ max/min well-defined.

Clearly, $m<a<M$.

Take $\delta = \min\{a-m, M-a\}$.

For each $x \in N_\delta(a)$,

① If $x \in \mathbb{R}\setminus\mathbb{Q}$, $f(x)=0 \Rightarrow |f(x)-f(a)|=0<\varepsilon$.

② If $x \in \mathbb{Q}$, $x=\frac{q}{p}$ for $p \in \mathbb{N}, q \in \mathbb{Z}$, p, q = relative prime.

$\Rightarrow$ By definition of δ, $p>N$.

$\Rightarrow |f(x)-f(a)| = |f(x)| = \frac{1}{p} < \frac{1}{N} = \frac{1}{\lceil \frac{1}{\varepsilon} \rceil + 1} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$.

$\therefore f$ is continuous at every irrational point.

(b) f has simple discontinuity at every rational point.

pf) Let $a \in \mathbb{Q}$ and $\varepsilon > 0$ be given.

$a = \frac{m}{n}$ for some $n \in \mathbb{N}$, $m \in \mathbb{Z}$, n, m have no common divisor.

Define N to be $N = \lceil \frac{1}{\varepsilon} \rceil + 1$.

Let $\delta = \frac{1}{nN}$.

For every rational number $\frac{q}{p} \in N_\delta(a)$, ($p, q$ = rel. prime, $p \in \mathbb{N}$, $q \in \mathbb{Z}$).

$\left|\frac{q}{p} - \frac{m}{n}\right| = \frac{1}{np}|mp - nq| < \frac{1}{nN}$

$\Rightarrow \frac{1}{np} < \frac{1}{nN} \Rightarrow p > N$. ---(*)

For each $x \in N_\delta(a)$,

① If $x \in \mathbb{Q}$, $|f(x) - 0| = |f(x)| < \frac{1}{N}$ ($\because$ (*))

$= \frac{1}{\lceil \frac{1}{\varepsilon} \rceil + 1} < 1/\frac{1}{\varepsilon} = \varepsilon$.

② If $x \notin \mathbb{Q}$, $|f(x) - 0| = |f(x)| = 0 < \varepsilon$.

$\therefore \lim_{x \to a} f(x) = 0$.

$\Rightarrow \lim_{x \to a} f(x) = 0 \neq \frac{1}{n} = f(a)$

$\therefore f$ has simple discontinuity at a.

#18. Define a function $\rho : X \to \mathbb{R}_{\geq 0}$ as

$\rho(x) := \inf_{y \in F} d(x, y)$.

① $\rho(x) = 0 \Leftrightarrow x \in F$.

pf) If $x \notin F$, $x \in F^c$

$\Rightarrow \exists \varepsilon > 0$ s.t. $d(x, y) < \varepsilon \to y \notin F$

$\Rightarrow \rho(x) \geq \varepsilon > 0$.

If $x \in F$, $\forall \varepsilon > 0, \exists y \in F$ s.t. $d(x, y) < \varepsilon$

$\Rightarrow 0 \leq \rho(x) \leq \varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow \rho(x) = 0$.

② $|\rho(x) - \rho(y)| \leq d(x, y)$.

pf 1) If $x, y \in F$, $d(x, y) \geq 0 = |\rho(x) - \rho(y)|$

2) If $x \in F$, $y \notin F$,

$|\rho(x) - \rho(y)| = |\rho(y)| = \inf_{z \in F} d(y, z) \leq d(x, y)$.

3) If $x, y \notin F$

$\rho(x) \le d(x, z) \le d(x,y) + d(y,z)$ for $\forall z \in F$.

$\forall \varepsilon > 0$, $\exists z \in F$ s.t.

$\rho(x) \le d(x, y) + \underline{d(y, z)} \le d(x,y) + \underline{\rho(y) + \varepsilon}$.

$\Rightarrow \rho(x) - \rho(y) \le d(x,y)$.

Similarly, $\rho(y) - \rho(x) \le d(x,y)$.

$\therefore |\rho(x) - \rho(y)| \le d(x,y)$. $\square$

By 2), ρ is uniformly continuous on X.

Since ρ is continuous on K and K is compact,

$\rho(K)$ is compact ($\Rightarrow$ closed & bounded)

By ①, $0 \notin \rho(K) \Rightarrow 0 \notin \rho(K)$, $0 \notin \rho(K)'$

$\Rightarrow \exists \delta > 0$ s.t. $\forall x \in (0-\delta, 0+\delta)$, $x \notin \rho(K)$.

Choose $p \in K$, $q \in F \Rightarrow d(p,q) \in \rho(K) \Rightarrow d(p,q) > \delta$. ■

If K, F are not compact,

Take $K = \{ n + \frac{1}{n} : n > 2 \}$

$F = \mathbb{N}$.

$\Rightarrow$ F is closed, K, F are not compact

And we can take p, q ($p \in K$, $q \in F$) s.t. $d(p,q) < \frac{1}{n}$ $\forall n > 2$.

$\Rightarrow \nexists \delta > 0$ s.t. $\left(d(p,q) < \delta \;\; \forall p \in K, q \in F \right)$.

#19. Let $x, y \in \mathbb{R}$ with $x \neq y$

$$\left|\frac{f(x)-f(y)}{x-y}\right| = \frac{|f(x)-f(y)|}{|x-y|} \leq \frac{|x-y|^2}{|x-y|} = |x-y|$$

$$\Rightarrow |f'(x)| = \lim_{y\to x} \frac{|f(x)-f(y)|}{|x-y|} \leq \lim_{y \to x} |x-y| = 0.$$

$\therefore f'(x) = 0 \quad \forall x \in \mathbb{R}.$

$\therefore$ f is constant function.

#20. Take $\varepsilon > 0$ s.t. $\varepsilon < \frac{1}{M}$.

$$\Rightarrow f'(x) = \lim_{t\to x} \frac{x-t}{x-t} + \lim_{t \to x} \frac{\varepsilon g(x) - \varepsilon g(t)}{x-t} = 1 + \varepsilon g'(x).$$

$$\Rightarrow \quad 0 < 1-\varepsilon M < f'(x) < 1+\varepsilon M < 2.$$

$\Rightarrow f'(x) > 0 \quad \forall x \in \mathbb{R}.$

$\Rightarrow$ f is increasing (strictly) on $\mathbb{R}$.

$\Rightarrow x \neq y \Leftrightarrow x < y$ or $x > y$

$\Leftrightarrow f(x) < f(y)$ or $f(x) > f(y) \Leftrightarrow f(x) \neq f(y)$

$\therefore x = y \Leftrightarrow f(x) = f(y)$

$\therefore$ f is one-to-one