

## Homework 5.

(Due May. 14)

**Problem 21** (Exercise 5.15). Suppose  $f$  is defined in a neighborhood of  $x$ , and suppose  $f''(x)$  exists. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if  $f''(x)$  does not exist.

**Problem 22** (Exercise 6.5). Suppose  $f$  is a bounded real function on  $[a, b]$ , and  $f^2 \in \mathcal{R}$  on  $[a, b]$ . Does it follow that  $f \in \mathcal{R}$ ? Does the answer change if we assume that  $f^3 \in \mathcal{R}$ ?

**Problem 23** (Exercise 6.6). Let  $P$  be the Cantor set constructed in Sec. 2.44. Let  $f$  be a bounded real function on  $[0, 1]$  which is continuous at every point outside  $P$ . Prove that  $f \in \mathcal{R}$  on  $[0, 1]$ .

**Problem 24** (Exercise 6.11). Let  $\alpha$  be a fixed increasing function on  $[a, b]$ . For  $u \in \mathcal{R}(\alpha)$ , define

$$\|u\|_2 = \left( \int_a^b |u|^2 d\alpha \right)^{1/2}.$$

Suppose  $f, g, h \in \mathcal{R}(\alpha)$ , and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

**Problem 25** (Exercise 6.15). Suppose  $f$  is a real, continuously differentiable function on  $[a, b]$ ,  $f(a) = f(b) = 0$ , and

$$\int_a^b f^2(x) dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x) dx > \frac{1}{4}.$$