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#6. We know that open interval in $\mathbb{R}^1$ is open.
Consider $\{C_i\}_{i\in\mathbb{N}}$ where $C_i = \left(\frac{1}{i+2}, \frac{1}{i}\right)$.
Then each of $C_i$ is open.

Claim 1. $\{C_i\}_{i\in\mathbb{N}}$ is an open cover.

pf) To show that, we should show $(0,1) \subseteq \bigcup_i C_i$.
Let $x \in (0,1)$.
Then, $\exists\, m \in \mathbb{N}$ s.t. $\frac{1}{m+1} \leq x < \frac{1}{m}$.
Since $\frac{1}{m+2} < \frac{1}{m+1} < \frac{1}{m}$, $\left[\frac{1}{m+1}, \frac{1}{m}\right) \subseteq C_m$
so that $x \in C_m$.
$\therefore x \in \bigcup_i C_i$.
Since $x \in (0,1) \Rightarrow x \in \bigcup_i C_i$, $(0,1) \subseteq \bigcup_i C_i$.

Claim 2. There are no finite subcover of $\{C_i\}_{i\in\mathbb{N}}$.

pf) Let $S = \{C_{i_1}, C_{i_2}, \cdots, C_{i_m}\}$ be finite subset of $\{C_i\}$.
Without loss of generality, let $i_1 < i_2 < \cdots < i_m$.
Then, let $x = \frac{1}{i_m+3}$.
Then, $x \in (0,1)$ but $x \notin \bigcup_{C_{i_j}\in S} C_{i_j}$.
$\therefore$ S is not a open cover.
$\therefore$ There are no finite subcover of $\{C_i\}_{i\in\mathbb{N}}$.

By Claim 1, 2 done.

#7. Let $C$ be cantor set.

Each of $x \in C$ is rational and $C$ is perfect.

Now, define $S \subseteq \mathbb{R}^1$ as $S = \{x + \sqrt{2} : x \in C\}$.

Since $\sqrt{2}$ is irrational, $\mathbb{Q} \cap S = \emptyset$.

Define $D_n$ as $D_n := \{x + \sqrt{2} : x \in E_n\}$ where $E_n$ is defined same as in p. 41 of textbook.

Then, $S = \bigcap_{n=1}^{\infty} D_n$ and each of $D_n$ is compact

$\Rightarrow S$ is clearly compact.

<u>Claim</u> $S$ is perfect.

<u>pf</u> Let $x \in S$. For each $n = 1, 2, \cdots$ let $I_n$ be the interval of $D_n$ that contains $x$.

Let $M$ be any segment containing $x$.

Then, for sufficiently large $n$, $I_n \subseteq M$.

Let $x_n$ be the end point of $I_n$ s.t. $x_n \neq x$.

Then, $x_n \in S$.

$\Rightarrow x$ is limit point of $S$

$\Rightarrow S$ is perfect.

#8. ① Closures of connected sets : connected.

pf) Let $A, B$ be connected.

Then, $\exists p \in A \cap \bar{B}$ or $\exists q \in \bar{A} \cap B$

$\Rightarrow \exists p \in (A \cup A') \cap \bar{B}$ or $\exists q \in \bar{A} \cap (B \cup B')$

$\Rightarrow \exists p \in \bar{A} \cap \bar{B}$ or $\exists q \in \bar{A} \cap \bar{B}$

$\Rightarrow \bar{A} \cap \bar{B} \neq \emptyset$. $\Rightarrow$ connected.

② interior points of connected sets : may not be connected.

pf) We will construct an example.

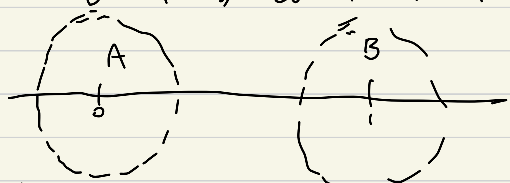
Consider $E := \{(x,0) : 0 < x < 1\} \cup N_{1/4}((0,0)) \cup N_{1/4}((1,0)) \subseteq \mathbb{R}^2$.

Then since segment $\{(x,0) : x \in (0,1)\}$ does not have interior point, interior of $E$ is $N_{1/4}((0,0)) \cup N_{1/4}((1,1))$.

Let $A = N_{1/4}((0,0))$ and $B = N_{1/4}((1,1))$.

Then, $\bar{A} = \{(x,y) : d((x,y),(0,0)) \leq \frac{1}{4}\}$ and

$\bar{B} = \{(x,y) : d((x,y),(1,1)) \leq \frac{1}{4}\}$.




$\Rightarrow$ It is clear that $A, B$ are separated. //

#9. It is clear that $0 < s_n \ \forall n$.

① It is bounded above by 2.

pf) For base case, $s_1 = \sqrt{2} < 2$.

Suppose that $0 < s_n < 2$.

$\Rightarrow 0 < s_{n+1} = \sqrt{2+\sqrt{s_n}} < \sqrt{2+\sqrt{2}} < \sqrt{2+2} = 2$.

$\therefore$ By Mathematical Induction, done.

② It is monotonically increasing.

pf) Use strong induction on $n$.

For base case, $s_1 = \sqrt{2} < \sqrt{2+\sqrt{2}} = s_2$.

Now, suppose that $s_{i-1} < s_i \quad \forall 1 \le i \le n$.

Then, $s_{n+1} = \sqrt{2+\sqrt{s_n}}$

$> \sqrt{2+\sqrt{s_{n-1}}} \quad (\because s_{n-1} < s_n)$

$> s_n$.

$\therefore$ By Mathematical Induction, done.

$\therefore$ By ① & ②, and Monotone Convergence Thm (#3.14) done.

#10. Let $A_m = \sum_{n=1}^{m} \frac{\sqrt{a_n}}{n}$.

① $A_m$ is monotonically increasing.

$A_m - A_{m-1} = \frac{\sqrt{a_m}}{m} \geq 0$ since $a_m \geq 0$.

② $A_m$ is bounded above.

It is clear that $\sum_{n=1}^{m} a_n$ is monotonic increasing and converges.

$\Rightarrow$ By thm 3-14, it has an upper bound $M$.

Also, $\sum_{n=1}^{m} \frac{1}{n^2}$ is monotonic increasing and converges

$\Rightarrow$ By thm 3-14, it has an upper bound $K$.

For $m \in \mathbb{N}$, $A_m = \sum_{n=1}^{m} \left( \frac{1}{n} \sqrt{a_n} \right) \leq \sqrt{\sum_{n=1}^{m} a_n \sum_{n=1}^{m} \frac{1}{n}} \leq \sqrt{MK}$

from Cauchy-Schwarz inequality $\therefore$ $A_m$ is bounded above

From ①, ② and thm 3-14, $A_m$ converges.