MAS241 - Analysis HW#6 20190497 이종서

#26. ① Let $\{f_n\}$ converges uniformly to $f$ and $\{g_n\}$ converges uniformly to $g$.

Let arbitrary $\varepsilon > 0$ be given.

$\Rightarrow \exists N_1 \in \mathbb{N}$ s.t. $n > N_1 \Rightarrow \|f_n(x) - f(x)\| < \frac{\varepsilon}{2}$

$\exists N_2 \in \mathbb{N}$ s.t. $n > N_2 \Rightarrow \|g_n(x) - g(x)\| < \frac{\varepsilon}{2}$

Let $N := \max\{N_1, N_2\}$

$\Rightarrow$ if $n > N$, $\|f_n(x) + g_n(x) - (f(x) + g(x))\| \leq \|f_n(x) - f(x)\| + \|g_n(x) - g(x)\|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

$\therefore \{f_n + g_n\}$ uniformly converges.

② Let $\{f_n\}$ converges uniformly to $f$ and $\{g_n\}$ converges uniformly to $g$.

Let $\|f_n(x)\|, \|f(x)\| < M_1$ and $\|g_n(x)\|, \|g(x)\| < M_2$.

Let arbitrary $\varepsilon > 0$ be given.

$\exists N_1 \in \mathbb{N}$ s.t. $n > N_1 \Rightarrow \|f_n(x) - f(x)\| < \frac{\varepsilon}{2M_2}$

$\exists N_2 \in \mathbb{N}$ s.t. $n > N_2 \Rightarrow \|g_n(x) - g(x)\| < \frac{\varepsilon}{2M_1}$

Let $N = \max\{N_1, N_2\}$.

$\Rightarrow$ if $n > N$, $\|f_n(x) g_n(x) - f(x) g(x)\| = \|g_n(x)(f_n(x) - f(x)) + f(x)(g_n(x) - g(x))\|$

$\leq \|g_n(x)\| \|f_n(x) - f(x)\|$

$+ \|f(x)\| \|g_n(x) - g(x)\|$

$< M_2 \cdot \frac{\varepsilon}{2M_2} + M_1 \cdot \frac{\varepsilon}{2M_1}$

$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

$\therefore \{f_n g_n\}$ uniformly converges.

#27. Let $f_n(x) = |x|$, $g_n(x) = \frac{1}{n}$. $\Rightarrow f_n g_n(x) = \frac{1}{n}|x|$. $\Rightarrow \lim_{n\to\infty} f_n g_n(x) = 0$.

It is obvious that $\{f_n\}$, $\{g_n\}$ uniformly converge.

Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be given.

Let $x = (n^2+n)\varepsilon + 1$.

$|f_n(x) - f_{n+1}(x)| = \frac{|x|}{n^2+n} > \frac{n^2+n}{n^2+n}\varepsilon = \varepsilon.$

$\Rightarrow$ Cauchy Criterion does not hold

$\Rightarrow$ Not uniformly converge.

#28. Let $f_k(x) = \sum_{n=1}^{k} (-1)^n \frac{x^2+n}{n^2}$.

① $\{f_n\}$ uniformly converge in closed interval $[a, b]$.

First, since we can rearrange finite number of clauses,
write $f_k(x) = \sum_{n=1}^{k} (-1)^n \frac{1}{n} + \left(\sum_{n=1}^{k} (-1)^n \frac{1}{n^2}\right) \cdot x^2$.

Let $\varepsilon > 0$ be given.
Since $\sum (-1)^n \frac{1}{n}$ converges by alternating series test,
by Cauchy criterion, $\exists N \in \mathbb{N}$ s.t. $\left|\sum_{n=k+1}^{\ell} (-1)^n \frac{1}{n}\right| < \frac{\varepsilon}{2}$ $\forall\, \ell > k > N$.
Also, since $\sum (-1)^n \frac{1}{n^2}$ converges from alternating series test,
by Cauchy criterion, $\exists M \in \mathbb{N}$ s.t. $\left|\sum_{n=k+1}^{\ell} (-1)^n \frac{1}{n^2}\right| < \frac{\varepsilon}{2\max\{a^2, b^2, 1\}}$ $\forall\, \ell > k > M$.

Then, $\forall\, \ell > k > \max\{N, M\}$,

$$|f_\ell(x) - f_k(x)| = \left|\sum_{n=k+1}^{\ell} (-1)^n \frac{1}{n} + \left(\sum_{n=k+1}^{\ell} (-1)^n \frac{1}{n^2}\right) \cdot x^2\right|$$
$$\leq \left|\sum_{n=k+1}^{\ell} (-1)^n \frac{1}{n}\right| + |x^2| \left|\sum_{n=k+1}^{\ell} (-1)^n \frac{1}{n^2}\right|$$
$$< \frac{\varepsilon}{2} + \max\{a^2, b^2\} \cdot \frac{\varepsilon}{2\max\{a^2, b^2, 1\}}$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ By Cauchy criterion, it converges in $[a, b]$.

② It does not absolutely converge.

$$\sum \left|(-1)^n \frac{x^2+n}{n^2}\right| = \sum \left|\frac{x^2+n}{n^2}\right| \geq \sum \left|\frac{n}{n^2}\right| = \sum \frac{1}{n} > 0$$

and $\sum \frac{1}{n}$ diverges so that by comparison test it diverges.

#29. Let $\varepsilon > 0$ be given arbitrarily.
Since $\sum f_n$ has uniformly bounded partial sums,
$\exists M > 0$ s.t. $\left|\sum_{n=1}^{m} f_n(x)\right| < M$ for all $m \in \mathbb{N}$, $x \in E$.
Since $g_n$ uniformly converge to $0$,
$\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|g_n(x)| < \frac{\varepsilon}{3M}$ for every $x \in E$.
Now, define $F_m(x)$ to be $F_m(x) = \sum_{n=1}^{m} f_n(x) g_n(x)$.
Suppose $\ell > k-1 > N$

$$\Rightarrow |F_\ell(x) - F_{k-1}(x)| = \left|\sum_{m=k}^{\ell} f_m(x) g_m(x)\right| = \left|\left(\sum_{m=k}^{\ell} f_m(x)\right) g_{\ell+1}(x) - \sum_{m=k}^{\ell}\left(\left(\sum_{n=k}^{m} f_n(x)\right)(g_{m+1}(x) - g_m(x))\right)\right|$$
$$< M \left|g_{\ell+1}(x) - \sum_{m=k}^{\ell}(g_{m+1}(x) - g_m(x))\right| \quad \Leftarrow \text{telescoping sum}$$
$$= M\left(|g_{\ell+1}(x)| + |g_{\ell+1}(x)| + |g_k(x)|\right)$$
$$< M \cdot 3 \cdot \frac{\varepsilon}{3M} = \varepsilon. \quad (\text{for all } x \in E)$$

$\therefore$ By Cauchy Criterion, $\sum f_n g_n$ converges uniformly.

#30. Let $\varepsilon > 0$ be given.

Since $\{f_n\}$ is equicontinuous, $\exists\, r > 0$ s.t.

$d(x,y) < r \ (x, y \in K) \Rightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$. --- ①

Now define open cover $\{C_p\}_{p \in K}$ of $K$ where $C_p = N_r(p)$.

Since $K$ is compact, $\exists\, i = 1, 2, \cdots, m \ (< \infty)$ s.t.

$p_i \in K$ and $C_{p_1} \cup C_{p_2} \cup \cdots \cup C_{p_K} = K$.

Now, from pointwise convergence of $\{f_n\}$,

let $x$ be a fixed element of $K$, then,

$N_x \in \mathbb{N}$ s.t. $n, m > N \Rightarrow |f_n(x) - f_m(x)| < \frac{\varepsilon}{3}$. --- ②

Now, let $x \in K$ be given.

Let $N = \max\{N_{p_1}, N_{p_2}, \cdots, N_{p_K}\}$ (as defined in ②)

Among $p_1, p_2, \cdots, p_m$, $\exists\, p_i$ s.t. $x \in p_i$.


$\Rightarrow$ By ①, ②, for all $n, m > N$,

$|f_n(x) - f_n(p_i)| < \frac{\varepsilon}{3}$

$|f_n(p_i) - f_m(p_i)| < \frac{\varepsilon}{3}$

$|f_m(p_i) - f_m(x)| < \frac{\varepsilon}{3}$.

Therefore, $|f_n(x) - f_m(x)| \le |f_n(x) - f_n(p_i)| + |f_n(p_i) - f_m(p_i)| + |f_m(p_i) - f_m(x)|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

$\therefore$ By Cauchy criterion,

$\{f_n\}$ uniformly converge on $K$.