Homework 4.

(Due Apr. 30)

Problem 16 (Exercise 4.14)**.** Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Problem 17 (Exercise 4.18)**.** Every rational x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}^1$ by

$$f(x) = \begin{cases} 0 & (x \text{ irrational}), \\ \dfrac{1}{n} & \left(x = \dfrac{m}{n}\right). \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Problem 18 (Exercise 4.21)**.** Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$, $q \in F$. Show that the conclusion may fail for two disjoint closed sets if neither is compact.

Problem 19 (Exercise 5.1)**.** Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Prove that f is constant.

Problem 20 (Exercise 5.3)**.** Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$, and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissible values of ϵ can be determined which depends only on M.)