MAS241 HW#5

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#21 ① As $h \to 0$, $h^2 \to 0$ and $f(x+h) + f(x-h) - 2f(x) \to 0$,

since f'' exists at x, f' exists in a neighbor of x

$\Rightarrow$ f is continuous at a neighbor of x $\ni$ $f(x+h) \to f(x)$ and $f(x-h) \to f(x)$ as $h \to 0$.

$\therefore$ By L'Hôpital's rule,

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} \quad \text{---}(*)$$

Note that

$$\lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h} = f''(x) = \lim_{h\to 0} \frac{f'(x)-f'(x-h)}{h}$$

$$\Rightarrow f''(x) = \frac{1}{2}f''(x) + \frac{1}{2}f''(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{2h} + \lim_{h\to 0} \frac{f'(x)-f'(x-h)}{2h}$$

$$= \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h}$$

$$= \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} \quad (\because (*))$$

So we are done.

② If $f(x) = \begin{cases} \frac{1}{2}x^2 & x \geq 0 \\ -\frac{1}{2}x^2 & x < 0 \end{cases}$,

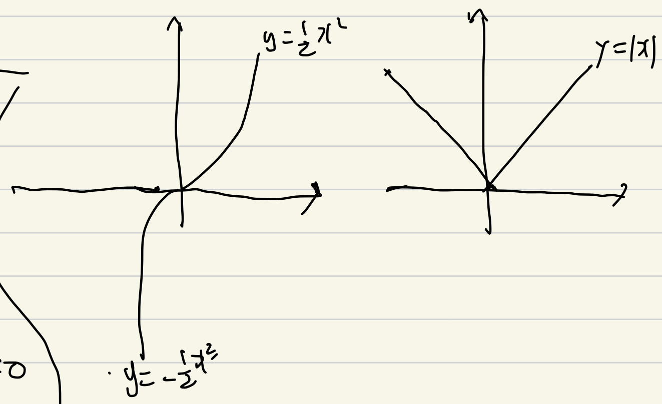
and $x = 0$,

$f(x+h) + f(x-h) = 0$,

$f(x) = 0$

$\Rightarrow \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = 0$

but $f''(0)$ does not exist.



#22 ① No.

Define $f(x)$ to be

$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{Q}^c \end{cases}$.

Then, let partition $P = \{x_0 < x_1 < x_2 < \cdots < x_n\}$ be given.

$\Rightarrow U(P, f) = \sum (x_i - x_{i-1}) \cdot 1 = x_n - x_0$.

$L(P, f) = \sum (x_i - x_{i-1}) \cdot (-1) = x_0 - x_n$

$\therefore \sup L(P, f) = -(x_n - x_0) < 0 < x_n - x_0 = \inf(U(P, f))$.

$\therefore f \notin \mathcal{R}$

However, f^2 is constant function, so that $f^2 \in \mathcal{R}$

② Yes.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be $x \mapsto \sqrt[3]{x}$.

Then, ϕ is continuous in $\mathbb{R}$.

By Thm 6.11, since $f^3 \in \mathcal{R}$, $f = \phi \circ f^3 \in \mathcal{R}$.

#23. Let $m \le f(x) \le M$ $\forall x \in [0,1]$. (let $m < M$).

Let arbitrary $\varepsilon > 0$ be given.

Take sufficiently large $n \in \mathbb{N}$ so that $\frac{2^{n+1}-1}{3^n} < \frac{\varepsilon}{2} \frac{1}{M-m}$ --- ($*$)

Let $P = \{0 < \frac{1}{3^n} < \frac{2}{3^n} < \cdots < \frac{3^n-1}{3^n} < 1\}$.

Among subintervals $I_k = [\frac{k}{3^n}, \frac{k+1}{3^n}]$,

exactly $2^{n+1}-1$ of them does have a point that belongs to Cantor set.

Let $X = \{k \in \{0,1,\cdots,3^n-1\} : I_k \cap C = \emptyset\}$ where C = cantorset.

For each $k \in X$, since f is continuous on I_k,

$\exists$ partition P_k of I_k s.t. $U(P_k, f) - L(P_k, f) < \frac{1}{3^n - 2^{n+1}+1} \cdot \frac{\varepsilon}{2}$.

Let $P' = P \cup (\cup_{k \in X} P_k)$.

Then,
$$
\begin{aligned}
U(P', f) - L(P', f) &= \sum_{k \in X} U(P_k, f) - L(P_k, f) + \sum_{k \notin X} \left(\left(\sup_{x \in I_k} f(x) - \inf_{x \in I_k} f(x)\right) \cdot \frac{k+1-k}{3^n} \right) \\
&< (3^n - (2^{n+1}-1)) \cdot \frac{1}{3^n - 2^{n+1}+1} \cdot \frac{\varepsilon}{2} + \quad // \\
&= \frac{\varepsilon}{2} + \quad // \\
&\le \frac{\varepsilon}{2} + (2^{n+1}-1) \cdot \frac{1}{3^n} \cdot (M-m) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad (\because (*)) \\
&= \varepsilon.
\end{aligned}
$$

$\therefore$ By Thm 6.6, f is integrable on $[0,1]$.

#24. Let $f, g \in \mathcal{R}(\alpha)$.

$$
\begin{aligned}
\|f+g\|_2^2 &= \int_a^b |f+g|^2 \, d\alpha \le \int_a^b \big| |f| + |g| \big|^2 d\alpha \quad (\because \text{ triangle ineq. of } |\cdot|) \\
&= \int_a^b |f|^2 d\alpha + \int_a^b |g|^2 d\alpha + 2 \int_a^b |fg| \, d\alpha \\
&\le \int_a^b |f|^2 d\alpha + \int_a^b |g|^2 d\alpha + 2 \sqrt{\int_a^b |f| d\alpha \int_a^b |g| d\alpha} \quad (\because \text{ Cauchy-Schwartz for integral}) \\
&= \|f\|_2^2 + \|g\|_2^2 + 2\|f\|_2 \|g\|_2 \\
&= (\|f\| + \|g\|)^2
\end{aligned}
$$

See ~~last~~ page of HW

$\therefore$ $\|f+g\| \le \|f\| + \|g\|$.

Now, replacing f by $f-g$ and g by $g-h$, we are done.

#5.

① $1 = \int_a^b 1 \cdot f^2(x)\,dx = \cancel{\left[x f^2(x)\right]_a^b} - \int_a^b 2x f(x) f'(x)\,dx$

0 ($\because b f^2(b) = a f^2(a) = 0$)

$\therefore \int_a^b x f(x) f'(x)\,dx = -\frac{1}{2}$.

② $\int_a^b (f'(x))^2\,dx \int_a^b x^2 f^2(x)\,dx \geq \left(\int_a^b x f(x) f'(x)\,dx\right)^2 = \frac{1}{4}$.

($\because$ ① & Cauchy-Schwartz for integral).

Since $\nexists\, t \in \mathbb{R}$ s.t. ($x f(x) = t f'(x)$ $\forall x \in [a,b]$), equality not holds.

$\therefore \int_a^b (f'(x))^2\,dx \int_a^b x^2 f^2(x)\,dx > \left(\int_a^b x f(x) f'(x)\,dx\right)^2 = \frac{1}{4}$.

Cauchy-Schwartz for integral.

Suppose $f, g : [a,b] \to \mathbb{R}$ is given where $f, g \in \mathcal{R}$.

Consider $f - tg$ for $t \in \mathbb{R}$.

$0 \leq \int_a^b (f - tg)^2(x)\,dx = \int_a^b f(x)^2\,dx - 2t \int_a^b (fg)(x)\,dx + t^2 \int_a^b g(x)^2\,dx$

$\Rightarrow D/4 = \left(\int_a^b f(x) g(x)\,dx\right)^2 - \left(\int_a^b f(x)^2\,dx\right)\left(\int_a^b g(x)^2\,dx\right) \leq 0$,

equality holds when $\exists\, t$ s.t. $(f - tg)(x) = 0$ $\forall x \in [a,b]$.

$\therefore$ done.