MAS241 - Analysis 20190497 이종서

#1. Let $\alpha = \inf A$.

We claim that $\sup(-A) = -\alpha$.

Since $\forall x \in A$, $x \geq \alpha$, we know that $\forall x \in -A$, $x \leq -\alpha$.

so that $-A$ is bounded above.

From least upper bound property,

$\exists\ \beta = \sup(-A)$.

Since $-\alpha$ is an upper bound of $-A$, $\beta \leq -\alpha$

Suppose to the contrary that $\beta < -\alpha$.

Then, $\exists\ \varepsilon > 0$ s.t. $\beta + \varepsilon < -\alpha$.

Then, $\forall x \in -A$, $x \leq \beta + \varepsilon \ (< \alpha)$

$\Rightarrow \forall x \in A$, $x \geq -\beta - \varepsilon$

So that $-\beta - \varepsilon$ is a lower bound of A.

However, $-\beta - \varepsilon > \alpha$, and it contradicts "$\alpha$ is a greatest lower bound of A."

Therefore, $\beta \leq -\alpha$ and not $\beta < -\alpha$

$\Rightarrow \beta = -\alpha \Rightarrow \sup(-A) = -\alpha$ and we are done.

#2. Suppose such order exists.

Then, $i \neq 0$ and $i^2 = -1 < 0$. ($-1 < 0$ because $1 > 0$ and $[x > 0 \Rightarrow -x < 0]$)

However, from Prop. 1.18, $x \neq 0 \Rightarrow x^2 > 0$ holds in every ordered field,

but it contradicts $i^2 < 0$.

$\therefore$ order can't be defined in complex field that makes it into ordered field.

#3. Answer : uncountably infinite.

We know that $\mathbb{R}\setminus\mathbb{Q}$ is an infinite set.

It means $\mathbb{R}\setminus\mathbb{Q}$ is either countably infinite or uncountably infinite.

Suppose to the contrary that $\mathbb{R}\setminus\mathbb{Q}$ is countable.

Since $\mathbb{Q}$ is countable, $\mathbb{R}=(\mathbb{R}\setminus\mathbb{Q})\cup\mathbb{Q}$ is also countable from 2.12. (Let $E_1=\mathbb{R}\setminus\mathbb{Q}$; $E_2, E_3, \cdots = \mathbb{Q}$ and apply theorem.)

However, we know that $\mathbb{R}$ is uncountable.

$\Rightarrow$ contradiction

$\therefore$ $\mathbb{R}\setminus\mathbb{Q}$ is uncountably infinite.

#4. Let $A=\{x+\frac{1}{y} : x\in\{0,1,2\} \text{ and } y\in\{2,3,4,\cdots\}\}$.

① If $p\in\{0,1,2\}$, p is a limit point of A.

Let $r>0$ be given.

From Archimedean property, we may take $n\in\mathbb{N}$ s.t. $n>\max\{1,\frac{1}{r}\}$.

Then, $d(p+\frac{1}{n}, p)<r$ so that $p+\frac{1}{n}\in N_r(p)$.

Also, $p+\frac{1}{n}\in A$.

$\therefore$ p is a limit point of A.

② If $p\in\mathbb{R}\setminus\{0,1,2\}$, then p is not a limit point of A.

If $p<0$, $N_{|p|/2}(p)\cap A$ is clearly $\emptyset$ so that we are done.

If $p\geq 3$, $N_{|p-(2+1/2)|/2}(p)\cap A$ is clearly $\emptyset$ " " "

Now, suppose $0\leq p<3$ and $p\neq 0,1,2$.

If $p\in A$, let $p=n+\frac{1}{m}$ where $n\in\{0,1,2\}$ and $m\in\{2,3,4,\cdots\}$.

Then, $N_{\min(\frac{1}{m}-\frac{1}{m+1}, \frac{1}{m-1}-\frac{1}{m})/2}(p)\cap A=\{p\}$ so that we are done.

Otherwise, let $p=u+v$ where $u\in\{0,1,2\}$ and $0<v<1$.

Then, let $n+\frac{1}{m}$, $n\in\{0,1,2\}$ and $m\in\{2,3,4,\cdots\}$ be

the greatest element of A smaller then p.
Then, $N_{\min(v-\frac{1}{n}, \frac{1}{n-1}-v)}(p) \cap A = \emptyset$ so that we are done.
Since $\exists\, r>0$ s.t. $(N_r(p) \cap A) \setminus \{p\} = \emptyset$ for every $p \in \mathbb{R} \setminus \{0,1,2\}$,
p is not a limit point of A for every $p \in \mathbb{R} \setminus \{0,1,2\}$

By ①, ②, A has exactly 3 limit points. Also, it is clear that A is bounded so that we are done.

#5. (a) In case of open set: Prove
Let E be an open set and $p(x,y) \in E$.
Then, since p is an interior point of E,
$\exists\, r>0$ s.t. $N_r(p) \subseteq E$.
Let $p'(x+r/2, y)$. Then, $p' \neq p$ and $p' \in N_r(p)$
$\Rightarrow p' \in N_r(p) \cap E$. $\therefore$ p is a limit point of E
$\therefore$ done.

(b). In case of closed set: Disprove.
Let $E = \{(x+\frac{1}{y}, 0) : x \in \{0,1,2\}$ and $y \in \{2,3,4,\dots\}\}$.

<u>Lem</u> For (x,y) s.t. $y \neq 0$ is not a limit point of E.
pf) Consider $N_{|y|/2}((x,y))$.
Then, $N_{|y|/2}((x,y)) \cap E = \emptyset$ since $\forall (u,v) \in E$, $v=0$. $\therefore$ (x,y) is not a limit point of E.

By <u>Lem</u> and ①, ② of #4, $(0,0), (1,0), (2,0)$ are the only limit points of E and belongs to E so that E is closed.
However, from ② of #4, $(1/3, 0) \in E$ but it is not a limit point of E.
$\therefore$ This is a counter example