MAT 499
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Grouping Hypergroups with Associated Association Schemes

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Groups

Every undergraduate pure mathematics department requires that students study abstract algebra. Abstract algebra is the study of some fundamental structures called groups and rings. Among these, an analogue of groups are our particular research interest. More specifically, we have studied parallel results between interesting theory that arises from groups and the more abstract concept of hypergroups. First, we define what a group is.

Definition

A group $\langle G, * \rangle$ is a binary algebraic structure G with an operation $* : G \times G \Rightarrow G$ such that

1. For any $x, y, z \in G$, we have $(x * y) * z = x * (y * z)$.
2. There exists an element $e \in G$ such that for any $g \in G$, $e * g = g = g * e$.
3. For any element $g \in G$, there exists an element $g^{-1} \in G$ such that $g * g^{-1} = e = g^{-1} * g$.

For example, consider the integers as the collection of elements with the operation of addition. Then addition is associative, so condition (i) is satisfied. Since 0 added to any other integer is the integer itself, 0 is the element described in (ii). Lastly, any integer added to its negative is 0, so every element satisfies condition (iii). Results from group theory also show that groups are essentially the study of symmetry.

Hypergroups

Since our primary research interest, however, is the study of hypergroups, we define what a hypergroup is.

Definition

A hypergroup is a set H equipped with a hyperproduct operation $* : H \times H \to \mathcal{P}(H) \setminus \{\varnothing\}$. The following axioms must hold

1. $(pq)r = p(qr)$ for all $p, q, r \in H$
2. There exists an element $1 \in H$ such that $p1 = \{p\} = 1p$ for any $p \in H$
3. For each $p \in H$, there is an element $p^* \in H$ such that if $r \in pq$, then $q \in p^*r$ and $p \in rq^*$ for any $p, q, r \in H$

In essence, applying the operation of a hypergroup gives as the output a set of elements, indicating that there are different possibilities for the outcome of applying an operation to two elements of the hypergroup. For example, if a hypergroup is the set of all positive integers with an operation $*$ such that $2 * 2 = \{2, 4\}$, heuristically it is possible that in some instances $2 * 2$ results in 2 and in other instances $2 * 2$ results in 4. Groups can then be seen as a specific example of a hypergroup where the result of applying the operation is a set of one element.

Permutations and Association Schemes

We mentioned earlier that groups can be used to study symmetry. This result was proved by English mathematician Arthur Cayley, motivating our current definition of groups. He found that given our definition of groups, every group is isomorphic, i.e. has the same structure, to a subset of a group we call the Symmetric Group on a set X. Given a set X, the symmetric group is the set of all possible permutations of X under function composition.

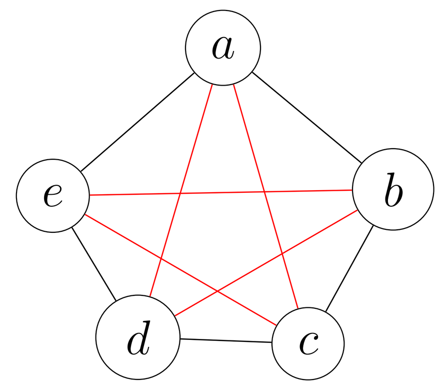
Since groups and hypergroups are similar, our research involves exploring the parallel results of groups and permutations to hypergroups and structures we call association schemes.

Definition

An association scheme on a set X is a set S of nonempty relations that partition $X \times X$. The following axioms must hold

1. $1 \in S$ where $1 := \{(x, x) \,|\, x \in X\}$
2. If $p \in S$, then $p^* := \{(y, x) \in X \times X \,:\, (x, y) \in p\} \in S$.
3. For any $p, q, r \in S$, there is an $a^r_{pq} \in \mathbb{N}$, called structural constants, such that if $(x, y) \in r$, there are a^r_{pq} elements $z \in X$ such that $(x, z) \in p$ and $(z, y) \in q$.

For example, we consider an association scheme $S = \{1, 2, 3\}$ on a set of five points $a, b, c, d, e \in X$. We can use graph theory to visualize these schemes where X is the set of vertices, or points, and S is the set of edges, or lines between points. We let black lines represent $2 \in S$ and red lines represent $3 \in S$.



Notice then that the pairs outlining the pentagon are all contained in relation 2, e.g. (a, b). We also have then that all the other pairs are in relation 3, e.g. (a, c). By axiom (i), we have that a pair of the same elements, e.g. (a, a), is in relation $1 \in S$. Notice that these lines are not directed, indicating that $(a, b) \in 2$ and $(b, a) \in 2$. Then $2^* = 2$, in accordance with axiom (ii). Lastly, we show an example of a structural constant, a^3_{22}. Notice that $(a, c) \in 3$ and there is only one vertex, b, such that $(a, b) \in 2$ and $(b, c) \in 2$. As a result, we have that $a^3_{22} = 1$.

These graphs are then a very intuitive method to visualize association schemes in all of their complexities. As previously, our research aims to explore the parallel results of Cayley's Theorem to hypergroups and association schemes. However, the literature finds that there are in fact hypergroups that cannot be realized, or do not have the same structure, as any association schemes. Thus Cayley's Theorem for hypergroups is not true. We take the next step and ask which hypergroups can be realized as association schemes and which ones cannot.

Results

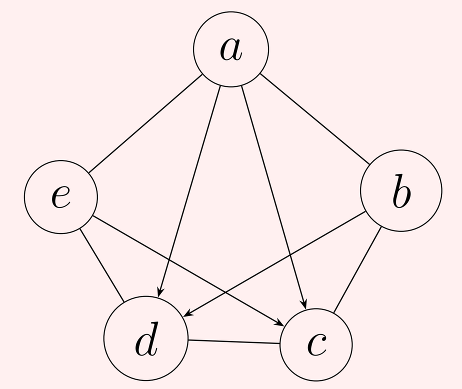
Due to the large number of hypergroups that exist, we only explored the hypergroups with 4 elements and extensively examined the 37 nonsymmetric hypergroups. We found that there are at least 11 hypergroups that can be realized as finite association schemes and 20 that cannot be realized as finite association schemes. Among these, there are 10 association schemes that can be realized as infinite association schemes and 10 that cannot be realized as infinite schemes. There are 6 association schemes for which we do not complete proofs. We present a proof that one of the hypergroups cannot be realized as an association scheme.

Theorem

The hypergroup in table 1 cannot be realized as an association scheme.

Proof.

Let (X, S) be a scheme realizing the hypergroup in table 1. Let $a \in X$ be arbitrary and choose $c \in X$ such that $(a, c) \in p$. Since $p \in pr$, there exists $d \in X$ such that $(a, d) \in p$ and $(d, c) \in r$. Also, since $p \in rr$, there exist $b, e \in X$ such that $(a, b) \in r$, $(b, c) \in r$ and $(a, e) \in r$, $(e, d) \in r$. Notice $(b, c) \in r$, $(c, d) \in r$ and $(b, a) \in r$, $(a, d) \in p$. Since $rr \cap rp = \{p\}$, $(b, d) \in p$. Similarly, $(e, c) \in r$, $(d, c) \in r$ and $(e, a) \in r$, $(a, c) \in p$. Since $rr \cap rp = \{p\}$, $(e, c) \in p$. However, notice that $(e, a) \in r$, $(a, b) \in r$ and $(e, d) \in r$, $(d, b) \in p^*$, and $(e, c) \in p$, $(c, b) \in r$. Since $rr \cap rp^* \cap pr = \{\emptyset\}$, this contradicts the association scheme being a partition of X.



	1	p	p^*	r
1	1	p	p^*	r
p	p	p	$1, p, p^*, r$	p, r
p^*	p^*	$1, p, p^*, r$	p^*	p^*, r
r	r	p, r	p^*, r	$1, p, p^*$

Table 1: Hypermultiplication

In the diagram above, directed edges represent p and solid edges represent r. Thus this hypergroup cannot be realized as an infinite scheme.

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