Realizing hypergroups as finite association schemes.

Jun Taek Lee
Grinnell College

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Definition (Groups)

A **group** $\langle G, * \rangle$ is a set G, closed under a binary operation $*$, such that the following axioms are satisfied:

1. For all $a, b, c \in G$, we have $(a * b) * c = a * (b * c)$.
2. There is an element $e \in G$ such that for all $x \in G$, $e * x = x * e = x$.
3. Corresponding to each $a \in G$, there is an element $a' \in G$ such that $a * a' = a' * a = e$.

Definition (Permutation)

A **permutation of a set** A is a bijective function $\phi : A \Rightarrow A$.

Background

Theorem (Cayley's Theorem)

Every group is isomorphic to a group of permutations.

Figure: Group of moves on a Rubik's cube

Motivated the modern definition of groups.
Can we extend the results?

Background

Can we extend the results?

- Groups $\Rightarrow$ Hypergroups
- Permutations $\Rightarrow$ Association Schemes

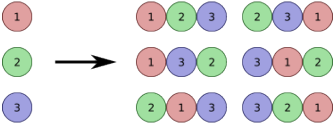

Figure: Permutations

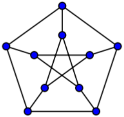

Figure: Association Scheme

Groups $\cong$ Group of Permutations

Hypergroups $\overset{?}{\cong}$ Association Schemes

Definition (Hypergroups)

A **hypergroup** is a set H equipped with a hyperproduct operation $* : H \times H \Rightarrow P(H) \setminus \{\emptyset\}$ defined as
$p * q = \{h_i \mid \text{ for any } i \in I \subseteq H\}$,
$P * Q = \{h_i \mid h_i \in p * q \text{ for any } p \in P, q \in Q\}$,
$p * Q = \{h_i \mid \text{ for any } p * q \text{ for any } q \in Q\}$, and
$P * q = \{h_i \mid \text{ for any } p * q \text{ for any } p \in P\}$, where $p, q \in H$ and $P, Q \in P(H)$. The following axioms must hold:

1. $(pq)r = p(qr)$ for all $p, q, r \in H$.
2. There exists an element $1 \in H$ such that $p \cdot 1 = \{p\} = 1 \cdot p$ for any $p \in H$.
3. For each $p \in H$, there is an element $p^* \in H$ such that if $r \in pq$, then $q \in p^* r$ and $p \in rq^*$ for any $p, q, r \in H$.

Association Schemes

Definition (Association Scheme)

An **association scheme** on a set X is a set S of nonempty relations that partitions $X \times X$. We denote the association scheme by (X, S). The following axioms must hold:

1. $1 \in S$ where $1 = \{(x, x) : x \in X\}$.
2. If $p \in S$, then $p^* = \{(y, x) \in X \times X : (x, y) \in p\} \in S$.
3. If $p, q \in S$, then $\sigma_p \sigma_q$ is a linear combination of elements in S, i.e. for all $p, q, r \in S$, there are $a_{pq}^r \in \mathbb{N}$ such that

$$\sigma_p \sigma_q = \sum_{r \in S} a_{pq}^r \sigma_r.$$

We refer to the number of elements in X as **order** and the number of elements in S as **rank**.

Example

- $X = \{0, 1, 2, 3, 4\}$
- $S = \{1, p, q\}$
  - $1 = \{(x, y) \in X \times X : |x -_5 y| = 0\}$,
  - $p = \{(x, y) \in X \times X : |x -_5 y| = 1$ or $|x -_5 y| = 4\}$,
  - $q = \{(x, y) \in X \times X : 1 < |x -_5 y| < 4\}$.

	0	1	2	3	4
0	1	p	q	q	p
1	p	1	p	q	q
2	q	p	1	p	q
3	q	q	p	1	p
4	p	q	q	p	1

Figure: Relation Table

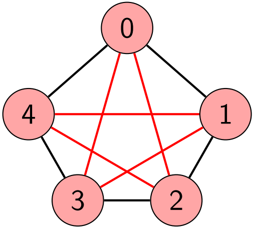

Figure: (X, S) graph

Example

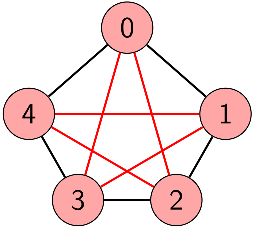

Figure: (X, S) graph

	1	p	q
1	1	p	q
p	p	$1, q$	p, q
q	q	p, q	$1, p$

Figure: Hypermultiplication Table

Definition (Symmetric and Nonsymmetric Hypergroups)

A hypergroup H is symmetric if for all $h \in H$, $h^* = h$. If H is not symmetric, we say that H is nonsymmetric.

Structural Constants

Definition (Structural Constants)

Let (X, S) be a scheme with $x, y \in X$ and $p, q, r \in S$. If $(x, y) \in r$, there are a^r_{pq} elements $z \in X$ such that $(x, z) \in p$ and $(z, y) \in q$.

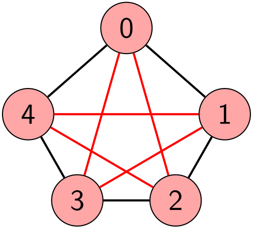

Figure: (X, S) graph

a^z_{xy}	1	p	q
11	1	0	0
$1p$	0	1	0
$1q$	0	0	1
$p1$	0	1	0
pp	2	0	1
pq	0	1	1
$q1$	0	0	1
qp	0	1	1
qq	2	1	0

Figure: Structural Constants

Structural Constants

Remark (Valency of p)

Let (X, S) be a scheme with $p \in S$. For each $x \in X$, there are exactly $a^1_{pp^*}$ elements $y \in X$ such that $(x, y) \in p$ for any $p \in S$.

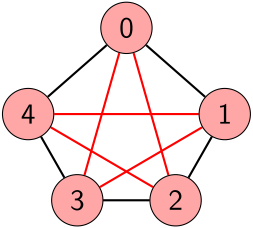

Figure: (X, S) graph

a^z_{xy}	1	p	q
11	**1**	0	0
$1p$	0	1	0
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pp	**2**	0	1
pq	0	1	1
$q1$	0	0	1
qp	0	1	1
qq	**2**	1	0

Figure: Structural Constants

Structural Constants

Definition (Product of Relations)

Let (X, S) be a scheme with $p, q \in S$. The product of relations is defined by

$$pq = \{r \in S : a^r_{pq} > 0\}.$$

	1	p	q
1	1	p	q
p	p	$1, q$	p, q
q	q	p, q	$1, p$

Figure: Hypermultiplication Table

a^z_{xy}	1	p	q
11	1	0	0
$1p$	0	1	0
$1q$	0	0	1
$p1$	0	1	0
pp	2	0	1
pq	0	1	1
$q1$	0	0	1
qp	0	1	1
qq	2	1	0

Association Schemes and Hypergroups

Theorem

For all association schemes (X, S), there exists a hypergroup H such that S realizes the hypermultiplication table of H.

- Axiom 1: Product of relations is associative.
- Axiom 2: Identity relation exists.
- Axiom 3: Every relation has a "star"/"inverse"

Groups of permutations on a set X determine groups
Association schemes on a set X determine hypergroups

To what degree does Cayley's theorem hold for hypergroups and association schemes?

Questions

Research Questions:

- Which hypergroups can be realized as finite association schemes?
- Which hypergroups cannot be realized as finite association schemes?
  - Can they be realized as infinite association schemes?
  - If not, can we prove that they cannot be realized as infinite association schemes?

Hypergroups of rank 4:

- 139 symmetric
- 37 nonsymmetric

Questions

Research Questions:

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- **37 nonsymmetric**

Hypergroups as Finite AS

Classification of association schemes with small vertices

(*Izumi Miyamoto and Akihide Hanaki*)
You can see some partial result here.

order	association schemes	finite groups	primitive	noncommutative	non Schurian	character tables
1	1	1	1	0	0	
2	1	1	1	0	0	
3	2	1	2	0	0	order 3 to 10
4	4	2	1	0	0	
5	3	1	3	0	0	
6	8	2	1	1	0	
7	4	1	4	0	0	
8	21	5	1	2	0	
9	12	2	2	0	0	
10	13	2	2	2	0	
11	4	1	4	0	0	same as Schurian
12	59	5	1	12	0	order 12
13	6	1	6	0	0	same as Schurian
14	16	2	1	2	0	order 14, 15
15	25	1	3	1	1	
16	222	14	6	49	16	order 16
17	5	1	5	0	0	same as Schurian
18	95	5	1	22	2	order 18
19	7	1	7	0	1	same as Schurian
20	95	5	1	22	0	order 20
21	32	2	3	3	0	order 21
22	16	2	1	2	0	order 22
23	22	1	22	0	18	same as Schurian
24	750	15	1	242	81	order 24

Miyamoto & Hanaki (2003)
11 as finite AS

Hypergroups not realizable as Finite AS

Theorem

Let H be a hypergroup. If $pp = \{p\}$ for any $p \in H$, H cannot be realized as a finite association scheme.

Proof: Let (X, S) be a finite scheme realizing H such that $pp = \{p\}$. Suppose $n_p = a$ where $a \in \mathbb{Z}^+$. Let $x \in X$ be arbitrary. Choose $y_1, y_2, \cdots, y_a, z \in H$ such that $(x, y_i) \in p$ for all $i \in \{1, \cdots a\}$ and $(x, z) \in p^*$. Then $(z, x) \in p$ by definition of p^*. Notice then that $(z, y_i) \in p$ for all $i \in \{1, \cdots, a\}$ since $(z, x) \in p$, $(x, y_i) \in p$ and $pp = \{p\}$. Then since $x \neq y_i$ for any $i \in \{1, \cdots, a\}$, there are $a + 1$ vertices such that $z \in X$ paired with each such vertex is in p. Thus $n_p \geq a + 1$. Since we also have that $n_p = a$, there is a contradiction. Thus n_p is not finite. Therefore, H cannot be realized as a finite association scheme.

Hypergroups not realizable as Finite AS

Theorem

Let H be a hypergroup. If $pp = \{p\}$ for any $p \in H$, H cannot be realized as a finite association scheme.

- 7 hypergroups, H, such that there exists an element $h \in H$ with $hh = \{h\}$.

Conclusion

Nonsymmetric Hypergroups of rank 4:

- 11 hypergroups as finite association schemes
- 20 hypergroups not realizable as finite association schemes
- 6 hypergroups unknown

Acknowledgements and References

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References:

- Paul-Hermann Zieschang, *Theory of Association Schemes*. Springer Monographs in Mathematics, Berlin Heidelberg New York, 2005
- Izumi Miyamoto, Akihide Hanaki. Classification of association schemes with small vertices http://math.shinshu-u.ac.jp/ hanaki/as/ (updated July 1, 2014)