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Realizing Hypergroups as Association Schemes

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1 INTRODUCTION

Abstract algebra introduces the notion of group theory that builds on binary algebraic structures with a binary operation that is associative, has an identity element, and each element of the structure has an inverse with respect to the binary operation. The binary operation with the given axioms give rise to famous theorems such as Cayley's Theorem, Lagrange's Theorem, and Cauchy's Theorem. However abstract as groups may seem, in the 20th century, mathematicians introduced the, more abstract, idea of hypergroups.

Definition 1. *A hypergroup is a set H equipped with a hyper product operation $* : H \times H \to \mathscr{P}(H) \setminus \{\varnothing\}$ defined as $p * q = \{h_i \mid i \in I \subseteq H\}$, $P * Q = \{h_i \mid h_i \in p * q \text{ for any} p \in P, q \in Q\}$, $p * Q = \{h_i \mid p * q \text{ for any} q \in Q\}$, and $P * q = \{h_i \mid p * q \text{ for any} p \in P\}$, where $p, q \in H$ and $P, Q \in \mathscr{P}(H)$. For any elements $p, q \in H$, we write pq or $p \cdot q$ for the product of p, q. The following axioms must hold*

1. $(pq)r = p(qr)$ *for all* $p, q, r \in H$.
2. *There exists an element* $1 \in H$ *such that* $p \cdot 1 = \{p\} = 1 \cdot p$ *for any* $p \in H$.
3. *For each* $p \in H$*, there is an element* $p^* \in H$ *such that if* $r \in pq$*, then* $q \in p^* r$ *and* $p \in rq^*$ *for any* $p, q, r \in H$.

Thus, similar to groups, hypergroups require a set with an operation with three axioms analogous to the group-axioms of associativity (1), existence of identity (2), and existence of inverses for each element (3). Notice, however, that the hyperproduct of two elements in a hypergroup H is a subset of H. Relating groups to hypergroups, groups are in fact hypergroups in which the hypermultiplication of any two elements is a singleton set. We refer to such hypergroups as *thin hypergroups*.

While groups have been explored extensively resulting in the notable theorems as mentioned earlier, hypergroups have been studied to a far lesser extent. However, since groups can be viewed as a specific case of hypergroups, we can use results from group theory to motivate further research in hypergroups. In fact, the axioms of groups were famously motivated and solidified in the 18th century after English mathematician Arthur Cayley proved his famous theorem, Cayley's Theorem. Every group could then be realized as a permutation group leading group theory to become synonymous to *symmetry study.* Analogously, mathematicians have found similar results to hypergroups using association schemes, a theory that first arose in statistics.

Definition 2. *Given a nonempty relation p on a set $X = \{x_1, x_2, \cdots, x_n\}$ for some $n \in \mathbb{Z}^+$, we define the relation matrix of p, σ_p by*

$$(\sigma_p)_{ij} = \begin{cases} 1 & (x_i, x_j) \in p \\ 0 & \mathit{otherwise} \end{cases}$$

Definition 3. *An association scheme on a set X is a set S of nonempty relations that partition $X \times X$, i.e. every pair $(x, y) \in X \times X$ belongs to exactly one relation in S. The following axioms must hold*

1. $1 \in S$ *where* $1 := \{(x,x) \mid x \in X\}$.
2. *If $p \in S$, then $p^* := \{(y,x) \in X \times X : (x,y) \in p\} \in S$.*
3. *If $p, q \in S$, then $\sigma_p \sigma_q$ is a linear combination of elements in S, i.e. for all $p, q, r \in S$, there are $a^r_{pq} \in \mathbb{N}$ such that*

$$\sigma_p \sigma_q = \sum_{r \in S} a^r_{pq} \sigma_r.$$

We denote this scheme (X, S). We refer to the number of elements in X as the order and the number of elements in S as the rank. We also denote a^r_{pq} for any combination of $p, q, r \in S$ as the structural constants of (X, S). The numbers a^r_{pq} can also be given a combinatorial definition.

Definition 4. *Let (X, S) be a scheme with $x, y \in X$. If $(x, y) \in r$, there are a^r_{pq} elements $z \in X$ such that $(x, z) \in p$ and $(z, y) \in q$.*

This definition can then be used to define products of relations.

Definition 5. *Let (X, S) be a scheme with $p, q \in S$. We define the product of relations as follows*

$$pq = \{r \in S \mid a^r_{pq} > 0\}.$$

This definition of the product of relations using structural constants will be used in other sections.

In group theory, given a set X of order n, we can find all the permutation groups on X. Similarly, we can find all the association schemes on X for a given order n.

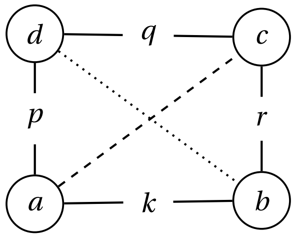
Theorem 6. *For all association schemes (X, S), there exists a hypergroup H such that S realizes the hypermultiplication table of H.*

Proof. We show that axioms 1,2,3 of hypergroups hold for any association scheme (X, S).

Axiom 1. Let $k \in (pq)r$ be arbitrary for some $p, q, r \in S$. Let

$$\begin{aligned} pq &= \{x_1, x_2, \cdots, x_m\} && \text{for some } x_1, x_2, \cdots, x_m \in S, \\ qr &= \{y_1, y_2, \cdots, y_n\} && \text{for some } y_1, y_2, \cdots, y_n \in S. \end{aligned}$$

Then by definition of pq, $a_{pq}^{x_i} > 0$ for all $i \in \{1, \cdots, m\}$. Then $a_{x_i r}^{k} > 0$ for some $i \in \{1, \cdots, m\}$. Let $a \in X$ be arbitrary and choose $b \in X$ such that $(a, b) \in k$. Then there exists $c \in X$ such that $(a, c) \in x_i$ and $(c, b) \in r$. Also, since $a_{pq}^{x_i} > 0$ for all $i \in \{1, \cdots, m\}$ and $(a, c) \in x_i$, there exists $d \in X$ such that $(a, d) \in p$ and $(d, c) \in q$. Since $(d, c) \in q$ and $(c, b) \in r$, there exists $j \in \{1, 2, \cdots, n\}$ such that $(d, b) \in y_j$. Lastly, since $(a, d) \in p$, $(d, b) \in y_j$, and $(a, b) \in k$, we have $k \in py_j \subseteq p(qr)$. Thus $(pq)r \subseteq p(qr)$. Pictorially, we have



where the dashed line indicates relation x_i and the dotted line indicates relation y_j. By similar argument, we can show that $p(qr) \subseteq (pq)r$. Thus axiom 1 holds.

Axiom 2. Let $p \in S$ be arbitrary. By definition of the product of relations, we have

$$1 \cdot p = \{r \in S \mid a_{1p}^{r} > 0\}.$$

We show that a_{1p}^{r} is greater than 0 if $p = r$ and 0 otherwise. Suppose $p \neq r$. Let $(x, y) \in r$ where $x, y \in X$. Then there are a_{1p}^{r} values of $z \in X$ such that $(x, z) \in 1$ and $(z, y) \in p$. By definition of 1, $(x, z) \in 1$ implies $z = x$. Thus $a_{1p}^{r} \leq 1$. Notice also $(z, y) = (x, y) \in p$. We have $p \neq r$ and $(x, y) \in r$. Thus this contradicts $p \cap r = \{\varnothing\}$, so $a_{1p}^{r} = 0$. Next, we suppose $p = r$. Let $(x, y) \in r$ where $x, y \in X$. Suppose $z = x$ where $z \in X$. By definition of 1, $(x, z) \in 1$. Since $(x, y) \in r$, we have $(z, y) \in r$. Also, since $p = r$, we have$(z, y) \in p$. Thus $a_{1p}^{r} > 0$. By similar argument, we can show that a_{p1}^{r} is greater than 0 if $p = r$ and 0 otherwise. Thus axiom 2 holds.

Axiom 3. Let $p \in qr$ be arbitrary where $q, r \in S$. We then have that $a_{qr}^{p} > 0$. Let $x \in X$ be arbitrary and choose $y \in X$ such that $(x, y) \in p$. Then there exists $z \in X$ such that $(x, z) \in q$ and $(z, y) \in r$. By axiom 2 of association schemes, we have that $(z, x) \in q^*$ and $(y, z) \in r^*$. Notice then that since there exists $y \in X$ such that $(x, y) \in p$ and $(y, z) \in r^*$ given $(x, z) \in q$, we have that $a_{pr^*}^{q} > 0$, so $q \in pr^*$. Similarly, since there exists $x \in X$ such that $(z, x) \in q^*$ and $(x, y) \in p$ given $(z, y) \in r$, we have $a_{q^* p}^{r} > 0$, so $r \in q^* p$. Thus axiom 3 holds.

Thus, all association schemes realize hypergroups. □

We now consider examples of schemes. For a set $X = \{x_1, x_2, x_3, x_4, x_5\}$, there are two schemes that are not thin. We represent these schemes as tables where the i, j entry is the relation that $(x_i, x_j) \in X \times X$ belongs to. The tables corresponding to the two schemes on X are
Thus we see from Table 1.1 that for the first scheme on X, $(x_1, x_2) \in 2$, $(x_1, x_1) \in 1$, and so on and so forth. Notice that along the diagonal of Table 1.1 and 1.2, we have relation 1 because relation 1 is defined such that $(x_i, x_j) \in 1$ if $i = j$. Thus, for every scheme, we have the relation

	x_1	x_2	x_3	x_4	x_5
x_1	1	2	2	2	2
x_2	2	1	2	2	2
x_3	2	2	1	2	2
x_4	2	2	2	1	2
x_5	2	2	2	2	1

Table 1.1

	x_1	x_2	x_3	x_4	x_5
x_1	1	2	2	3	3
x_2	2	1	3	2	3
x_3	2	3	1	3	2
x_4	3	2	3	1	2
x_5	3	3	2	2	1

Table 1.2

1 along the diagonal for such tables. Then we also have $j = 1$, so $(x_j, x_i) \in 1$. Thus $1^* = 1$. Notice that for both schemes, we also have $(x_1, x_2) \in 2$ and $(x_2, x_1) \in 2$. So $2^* = 2$. Similarly, $3^* = 3$. Thus, we say that these two relations are *symmetric*. If there is a relation, p, such that $p^* \neq p$, we say that p is a *nonsymmetric* relation. A scheme is *nonsymmetric* is it has at least one nonsymmetric scheme.

Based on Tables 1.1 and 1.2, we can then determine the hypergroups that each scheme realizes. We consider the second scheme. Notice that $(x_1, x_3) \in 2$. Since $(x_1, x_2) \in 2$ and $(x_2, x_3) \in 3$, we then have that $2 \in 2 \cdot 3$. More generally, given $(x_i, x_k) \in r$, if there exists $x_j \in X$ such that $(x_i, x_j) \in p$, and $(x_j, x_k) \in q$, then $r \in pq$. i.e. for relations $p, q \in S$

$$pq = \{r \in S \mid a^r_{pq} > 0\}.$$

Using this method, we find that the two schemes realize the following two hypergroups in Table 1.3 and 1.4.

	1	2
1	1	2
2	2	1,2

Table 1.3

	1	2	3
1	1	2	3
2	2	1,3	2,3
3	3	2,3	1,2

Table 1.4

Thus, the hypergroups in Table 1.3 and 1.4 can be realized as finite association schemes.

Notice that in Table 1.3, $1 \cdot 2 = 2 \cdot 1$ and in Table 1.4, $1 \cdot 2 = 2 \cdot 1$, $1 \cdot 3 = 3 \cdot 1$, and $2 \cdot 3 = 3 \cdot 2$. Similar to group theory, we call this observation commutativity. i.e. a hypergroup H is commutative if for any $p, q \in H$, $pq = qp$. Thus, an association scheme (X, S) realizes a commutative hypergroup if for any $p, q \in S$, $pq = qp$. We will use commutativity and symmetry to organize and assist with our results in the following sections. Next, we build on the claim that groups are in fact hypergroups by redefining groups to be consistent with its original definition but also with the definition of a hypergroup, and show that every group can be realized as a hypergroup.

Definition 7. *A group is a hypergroup such that the product of $p, q \in G$ is a singleton set for any $p, q \in G$. We denote the identity element as $1 \in G$ and the inverse element as $p^* \in G$ for any $p \in G$.*

Theorem 8. *Every group G can be realized as an association scheme.*

Proof. We want to define (X, S), a scheme. Let $X = G$. For each $g \in G$, let $[g] := \{(g_1, g_2) \in$

$X \times X \mid g_2 = g_1 g\}$ and $S = \{[g] \subseteq G \times G\}$. We show that the axioms of schemes hold.

$$\begin{aligned}
[1] &= \{(g_1, g_2) \in X \times X \mid g_2 = g_1 1\} = 1, \\
[g]^* &= \{(g_1, g_2) \in X \times X \mid g_1 = g_2 g\} \\
&= \{(g_1, g_2) \in X \times X \mid g_2 = g_1 g^{-1}\} = [g^{-1}].
\end{aligned}$$

Given $[p], [q], [r] \in S$, we want $a_{[p][q]}^{[r]}$. Let $(g_1, g_2) \in [r]$ be arbitrary. Then $g_2 = g_1 r$. We count how many $g_3 \in G$ there are with $(g_1, g_3) \in [p]$ and $(g_3, g_2) \in [q]$. To get $(g_1, g_3) \in [p]$ and $(g_3, g_2) \in [q]$, we need

$$\begin{aligned}
g_3 &= g_1 p, \\
g_2 &= g_3 q.
\end{aligned}$$

Notice that there is at most one g_3 that satisfies these equations. Suppose there exists such an element. Then we have

$$\begin{aligned}
g_1 r = g_2 &= g_3 q, \\
g_1 r &= g_3 q, \\
g_1 r &= g_1 p q, && \text{substituting } g_3 = g_1 p \\
r &= p q.
\end{aligned}$$

Thus we have

$$a_{[p][q]}^{[r]} = \begin{cases} 0 & r \neq pq \\ 1 & r = pq \end{cases}$$

We then have that (X, S) is a scheme. Thus every group can be realized as a scheme. □

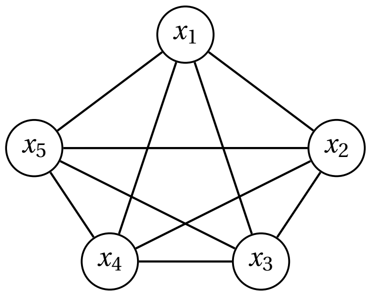
Corollary 9. *Every finite group G can be realized as a finite association scheme.*

Proof. Assuming that G is a finite group, we let $X = G$ and $|S| = |G|$. Notice that the same argument from theorem 8 hold and (X, S) is a finite scheme. □

Corollary 10. *Every infinite group G can be realized as an infinite association scheme.*

Proof. Assuming that G is an infinite group, we let $X = G$ and S has infinite cardinality since we defined $S = \{[g] \subseteq G \times G\}$ for all $g \in G$ and G is an infinite group. Similarly, notice that the argument from theorem 8 hold and (X, S) is an infinite scheme. □

We use graphs as a tool to visualize association schemes. Again, we consider the schemes represented in Table 1.1 and 1.2. We consider X to be the set of vertices and S as the set of edges. Since this scheme is symmetric, all the edges are bidirectional, resulting in an undirected graph. On the other hand, if for relation $p \in S$, $p^* \neq p$, then the edge representing p is directional, resulting in a partially or completely directed graph. We then have the following graphs representing the two schemes respectively



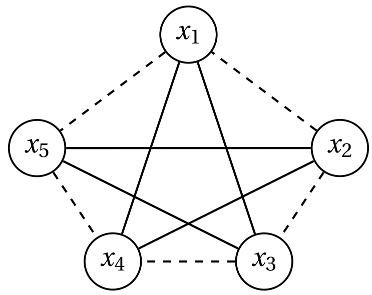


Figure 1.1: Scheme representing Table 1.1

Figure 1.2: Scheme representing Table 1.2

where the solid edges represent relation 2 and dashed edges represent relation 3. This schematic helps in visualizing the structure of each scheme. Notice that since association schemes require that S is a set of nonempty relations that partition X, every pair of vertices $(x, y) \in X \times X$ is in a relation in S.

Returning to the analogue of Cayley's Theorem for hypergroups, while all association schemes realize hypergroups, thorough investigation reveals that not all hypergroups can be realized as association schemes. As a result, the analogue of Cayley's Theorem fails to hold for these broader structures. On the other hand, this failure of the analogue implies more richness to hypergroup theory than group theory allowing more complex results to be studied.

In an effort to most efficiently produce results during the ten week duration of the Mentored Advanced Project studying hypergroups and association schemes, we studied all the association schemes of rank three as an introductory exploration. Next, we briefly explored the symmetric schemes of rank 4. Lastly, we spent the majority of our research exploring the nonsymmetric schemes of rank 4. More specifically, we proved whether hypergroups of rank 3 or 4 can be realized as finite association schemes or infinite association schemes, and whether hypergroups cannot be realized as finite association schemes or infinite association schemes. We present our results in the following sections.

2 HYPERGROUPS OF RANK 3

In this section, we use hypergroups of rank 3 to introduce lemmas, heuristics, and ideas that prove the results we are interested in and assist us in higher rank hypergroups.

Prior to our research, mathematicians have used computer codes to generate all the hypergroups of rank n for small values of n, and association schemes of order m. We find that there are exactly ten hypergroups of rank 3 based on these codes. We present all ten hypergroups.

	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

Table 2.1

	1	2	3
1	1	2	3
2	2	2,3	1,2,3
3	3	1,2,3	2,3

Table 2.2

	1	2	3
1	1	2	3
2	2	1	3
3	3	3	1,2

Table 2.3

	1	2	3
1	1	2	3
2	2	1	3
3	3	3	1,2,3

Table 2.4

	1	2	3
1	1	2	3
2	2	1,3	2
3	3	2	1,3

Table 2.5

	1	2	3
1	1	2	3
2	2	1,3	2,3
3	3	2,3	1,2

Table 2.6

	1	2	3
1	1	2	3
2	2	1,3	2,3
3	3	2,3	1,2,3

Table 2.7

	1	2	3
1	1	2	3
2	2	1,2	3
3	3	3	1,2,3

Table 2.8

	1	2	3
1	1	2	3
2	2	1,2,3	2,3
3	3	2,3	1,2,3

Table 2.9

	1	2	3
1	1	2	3
2	2	2	1,2,3
3	3	1,2,3	3

Table 2.10

First, we check whether there exists a finite association scheme realizing each of these hypergroups. Miyamoto & Hanaki (2003)[2] presented results of all association schemes of orders 5-30, 32-34, and 38 that do not realize thin hypergroups on their website

http://math.shinshu-u.ac.jp/ hanaki/as/

We can then use the algorithm presented in the introduction to generate the hypermultiplication table of a scheme. Due to the large number of schemes we needed to check, we coded a function in SageMath that uses a scheme as the input and gives a hypermultiplication table as its output. We present the pseudocode of the algorithm.

Algorithm 1 Generate Hypermultiplication Table from Scheme

```
procedure HYPERMULT(A)                          ▷ A is the matrix representing a scheme
    MaxRel ← max(A)          ▷ Assign MaxRel as the number of relations in the scheme
    Hypertable ← matrix(n × n)                                ▷ Create an n × n matrix
    for i ← 1, MaxRel do                                       ▷ Loop over relation i
        for j ← 1, MaxRel do                                   ▷ Loop over relation j
            for f ← 1, n do                                      ▷ Loop over vertex f
                for g ← 1, n do                                  ▷ Loop over vertex g
                    for h ← 1, n do                              ▷ Loop over vertex h
                        if A_{f,h} = i, A_{h,g} = j then   ▷ If (f,h) ∈ i and (h,g) ∈ j
                            Hypertable_{i,j} ← append(A_{f,g})  ▷ Add (f,g) relation to output
    return Hypertable                                             ▷ Return output
```

We can then use this algorithm to generate the hypermultiplication tables that the schemes realize, and use the hypermultiplication tables as a basis for existence proofs.

We find from Miyamoto & Hanaki's website there are schemes that realize hypergroups in Tables 2.2 through 2.9.

Hypergroup	Schemes
2.2	7.2, 11.2, 15.5, 17.3
2.3	4
2.4	6.2, 8.2, 10.2, 12.2, 14.2, 16.2
2.5	6.3, 8.3, 10.4, 12.5, 14.3, 16.7
2.6	5.2
2.7	10.3, 16.4
2.8	9.2, 12.3, 12.4, 15.2, 15.3, 16.3
2.9	9.3, 13.2, 15.4, 16.5, 16.6, 17.2

Table 2.11

Notice also that Table 2.1 presents a thin hypergroup, which by corollary 9 can be realized as a finite scheme. Thus we have proofs that the hypergroups in Tables 2.1 through 2.9 can be realized as finite association schemes. We show that the hypergroup in Table 2.10 cannot be realized as a finite association scheme and present an infinite association scheme that realizes Table 2.10. The hypermultiplication table of Table 2.10 is

	1	2	3
1	1	2	3
2	2	2	1,2,3
3	3	1,2,3	3

Remark 11. *Let (X,S) be a scheme. For each $x \in X$, there are exactly $a^1_{pp^*}$ elements $y \in X$ such that $(x,y) \in p$ for any $p \in S$.*

Proof. This remark follows directly from definition 4. We denote $a^1_{pp^*}$ as n_p for any relation $p \in S$ and refer to n_p as the valency of p. □

Proposition 12. *The hypergroup in Table 2.10 cannot be realized as a finite association scheme.*

Proof. Let (X,S) be a finite scheme. Let $x \in X$ be arbitrary. Then by the definition of the valency of $2 \in S$, there are elements $y_1, y_2, \cdots, y_{n_2} \in X$ such that $(x, y_i) \in 2$ for all $i \in \{1, 2 \cdots, n_2\}$. Notice then that there are $z_1, z_2, \cdots, z_{n_2} \in X$ such that $(y_1, z_i) \in 2$ for all $i \in \{1, 2, \cdots, n_2\}$. However, since $2 \neq 2^*$, $(y, x) \notin 2$. Since the number of remaining y_i is $n_2 - 1$, there is a distinct z_j where $j \in \{1, 2, \cdots, n_2\}$ such that $z_j \neq y_i$ and $z_j \neq x$ for all i. Notice however that since $2 \cdot 2 = \{2\}$, and $(x, y_1) \in 2$ and $(y_1, z_j) \in 2$, we have that $(x, z_j) \in 2$. Then $z_j = y_i$ for some i, but we also have that there is no i such that $y_i = z_j$. Thus, there is a contradiction. So this hypergroup cannot be realized as a finite scheme. □

Proposition 13. *The hypergroup in Table 2.10 can be realized by an infinite association scheme.*

Proof. Let $X = \mathbb{Q}$ and $S = \{1, 2, 3\}$. Let $x, y \in X$. We define relations $1, 2, 3$ as the following:

- $(x, y) \in 1$ if and only if $x = y$,
- $(x, y) \in 2$ if and only if $x < y$,
- $(x, y) \in 3$ if and only if $x > y$.

Notice that each pair of elements in $X \times X$ can only be contained in one relation, $1 \in S$ is the identity element and, $2^* = 3$. Thus, we have the first two axioms of associations are true of this structure. We show that the structural constant property holds.

a_{22}^p: Let $p \in S$ be arbitrary. Now let $x \in X$ be arbitrary. Suppose $y, z \in X$ such that $(x, y) \in 2$ and $(y, z) \in 2$. Notice then that $x < y$ and $y < z$. Then $x < z$. Thus $2 \cdot 2 = \{2\}$. We then have that $a_{22}^1 = 0 = a_{22}^3$. Notice then that if $(x, w) \in 2$ for some $w \in X$, there are infinitely many possible choices of $v \in X$ such that $(x, v) \in 2$ and $(v, w) \in 2$. Thus, we have $a_{22}^2 = \infty$.

a_{33}^p: Let $p \in S$ be arbitrary. Now let $x \in X$ be arbitrary and choose $y, z \in X$ such that $(x, y) \in 3$ and $(y, z) \in 3$. Notice then that $x > y$ and $y > z$. Then $x > z$. Thus $3 \cdot 3 = \{3\}$. We then have that $a_{33}^1 = 0 = a_{33}^2$. Notice then that if $(x, w) \in 2$ for some $w \in X$, there are infinitely many possible choices of $v \in X$ such that $(x, v) \in 2$ and $(v, w) \in 2$. Thus, we have $a_{22}^2 = \infty$. .

a_{23}^1: Let $x \in X$ be arbitrary. Then $(x, x) \in 1$. Now choose $y \in X$ such that $x < y$. Then $x < y$, so $(x, y) \in 2$, and $y > x$, so $(y, x) \in 3$. Since there are infinitely many possible choices of $y \in X$, we have $a_{23}^1 = \infty$.

a_{32}^1: Let $x \in X$ be arbitrary. Then $(x, x) \in 1$. Now choose $y \in X$ such that $x > y$. Then $x > y$, so $(x, y) \in 3$, and $y < x$, so $(y, x) \in 2$. Since there are infinitely many possible choices of $y \in X$, we have $a_{32}^1 = \infty$.

a_{23}^2: Let $x \in X$ be arbitrary and choose $y \in X$ such that $(x, y) \in 2$. Then $x < y$. Choose $z \in X$ such that $y < z$. Since $x < y$ and $y < z$, we have $x < z$. Then $(x, z) \in 2$. Since $z > y$, we have that $(z, y) \in 3$. Since there are infinitely many possible choices of $z \in X$, we have $a_{23}^2 = \infty$.

a_{23}^3: Let $x \in X$ be arbitrary and choose $y \in X$ such that $(x, y) \in 3$. Then $x > y$. Choose $z \in X$ such that $x < z$. Then we have $(x, z) \in 2$. Since $y < x < z$, we have that $z > y$. Thus $(z, y) \in 3$. Since there are infinitely many possible choices of $z \in X$, we have $a_{23}^3 = \infty$.

a_{32}^2: Let $x \in X$ be arbitrary and choose $y \in X$ such that $(x, y) \in 2$. Then $x < y$. Choose $z \in X$ such that $x > z$. Then we have $(x, z) \in 3$. Since $z < x < y$, we have that $z < y$. Thus $(z, y) \in 2$. Since there are infinitely many possible choices of $z \in X$, we have $a_{32}^2 = \infty$.

a_{32}^3: Let $x \in X$ be arbitrary and choose $y \in X$ such that $(x, y) \in 3$. Then $x > y$. Choose $z \in X$ such that $z < y$. Since $x > y$ and $z < y$, we have $x > z$. Thus $(x, z) \in 3$. Since $z < y$, we have $(z, y) \in 2$. Since there are infinitely many possible choices of $z \in X$, we have $a_{32}^3 = \infty$.

a_{1p}^q: We showed in theorem 6, axiom 2 that $a_{1p}^r > 0$ if $p = r$ and 0 otherwise. Thus if $p \neq q$, then $a_{1p}^q = 0$. Now suppose $p = q$. Let $x \in X$ be arbitrary and choose $y \in X$ such that $(x, y) \in p$. Since if $(x, z) \in 1$ for some $z \in X$, by definition of relation 1, $x = z$. Thus there is only one element $z \in X$ such that $(x, z) \in 1$ and $(z, y) \in p$. Thus

$$1 = a_{11}^1 = a_{12}^2 = a_{13}^3$$
$$0 = a_{12}^1 = a_{13}^1 = a_{12}^2 = a_{13}^2 = a_{12}^3 = a_{13}^3$$

a_{p1}^q: By similar argument as the the proof for a_{1p}^q, we can show that

$$1 = a_{11}^1 = a_{21}^2 = a_{31}^3$$
$$0 = a_{21}^1 = a_{31}^1 = a_{21}^2 = a_{31}^2 = a_{21}^3 = a_{31}^3$$

We have determined all the values for the structural constants. □

Thus, we have examined all hypergroups of rank 3 to find whether they can be realized as finite schemes or infinite schemes. Notice, however, that in addition to determining which hypergroups can and cannot be realized by finite or infinite schemes, Table 2.11 presents interesting, possible number theoretic results about the order of schemes realizing certain hypergroups. For example, Hypergroup 2.2 has schemes only of order $3 \mod 4$ and Hypergroup 2.4 only has schemes of even order. While we do not explore these proofs extensively in this paper, these number theoretic ideas are a possible direction for future research in studying hypergroups and association schemes.

In the next sections, we explore hypergroups of rank 4 and organize our results as in this current section: which hypergroups can be realized as finite association schemes? If not, can we prove that they cannot be realized as finite schemes? Can they be realized as infinite schemes? If not, can we prove that they cannot be realized as infinite schemes? What are some number theoretic results that arise based on the structures of the hypergroups? We present proofs and conjectures to the relevant questions for each hypergroup.

3 Hypergroups of rank 4 Realized as Finite Schemes

Using similar algorithms to find all ten hypergroups of rank 3, we find that there are 139 symmetric (S) and 37 nonsymmetric (NS) hypergroups of rank 4. In this section, we explore which of these hypergroups of rank 4 can be realized as finite association schemes. We follow the methodology in section 2 of calculating the hypermultiplication table that each association scheme of rank 4 in Miyamato & Hanaki [2] as a basis for an existence proof to show the hypergroups that can be realized as finite association schemes. By axiom 2 of Definition 1 of hypergroups, we know that $1 \cdot p = \{p\} = p \cdot 1$ for any element p of a hypergroup. Thus we omit the first row and first column of each hypermultiplication table in the following table presenting each hypergroup of rank 4 that can be realized as a scheme.

Hypergroup	NS or S	Hyperproduct	Schemes
1	NS	$\begin{matrix} 4 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 1 \end{matrix}$	Thin
18	NS	$\begin{matrix} 3 & 1 & 4 \\ 1 & 2 & 4 \\ 4 & 4 & 1,2,3 \end{matrix}$	6.4
19	NS	$\begin{matrix} 3 & 1 & 4 \\ 1 & 2 & 4 \\ 4 & 4 & 1,2,3,4 \end{matrix}$	9.4, 12.6, 15.6, 18.6, 21.5, 24.8, 27.379, 30.8, 33.4
20	NS	$\begin{matrix} 3 & 1,4 & 2 \\ 1,4 & 2 & 3 \\ 2 & 3 & 1 \end{matrix}$	6.6
21	NS	$\begin{matrix} 3 & 1,4 & 2 \\ 1,4 & 2 & 3 \\ 2 & 3 & 1,4 \end{matrix}$	9.6, 12.13, 15.11, 18.14, 21.10, 24.23, 27.387, 30.31, 33.8
24	NS	$\begin{matrix} 2,3 & 1,2,3 & 4 \\ 1,2,3 & 2,3 & 4 \\ 4 & 4 & 1,2,3 \end{matrix}$	14.6, 22.6, 30.25, 30.26, 38.7, 38.8, 38.9
25	NS	$\begin{matrix} 2,3 & 1,2,3 & 4 \\ 1,2,3 & 2,3 & 4 \\ 4 & 4 & 1,2,3,4 \end{matrix}$	21.8, 28.14, 33.7
26	NS	$\begin{matrix} 2,3 & 1,2,3,4 & 2 \\ 1,2,3,4 & 2,3 & 3 \\ 2 & 3 & 1 \end{matrix}$	14.5, 22.5, 30.13, 38.5, 38.6
27	NS	$\begin{matrix} 2,3 & 1,2,3,4 & 2 \\ 1,2,3,4 & 2,3 & 3 \\ 2 & 3 & 1,4 \end{matrix}$	21.7, 28.72, 33.6
30	NS	$\begin{matrix} 2,3,4 & 1,2,3 & 3 \\ 1,2,3 & 2,3,4 & 2 \\ 3 & 2 & 1 \end{matrix}$	8.6, 16.11, 24.14, 32.14
35	NS	$\begin{matrix} 2,3,4 & 1,2,3,4 & 2,3 \\ 1,2,3,4 & 2,3,4 & 2,3 \\ 2,3 & 2,3 & 1,4 \end{matrix}$	16.9

Hypergroup	NS or S	Hyperproduct	Schemes
46	S	$\begin{matrix} 1 & 3 & 4 \\ 3 & 1,2,3 & 4 \\ 4 & 4 & 1,2,3,4 \end{matrix}$	18.7, 24.10, 24.11, 30.9, 30.10, 32.7
47	S	$\begin{matrix} 1 & 3 & 4 \\ 3 & 1,2,3,4 & 3,4 \\ 4 & 3,4 & 1,2,3,4 \end{matrix}$	18.8, 26.14, 30.11, 32.9, 32.10, 34.4
48	S	$\begin{matrix} 1,4 & 4 & 2,3 \\ 4 & 1,4 & 2,3 \\ 2,3 & 2,3 & 1,4 \end{matrix}$	14.7, 22.7, 26.16, 30.27, 30.28, 30.29, 30.30, 32.22, 32.23, 32.24, 38.10, 38.11, 38.12, 38.13, 38.14, 38.15
56	S	$\begin{matrix} 1,4 & 3 & 2,4 \\ 3 & 1,2,4 & 3 \\ 2,4 & 3 & 1,2 \end{matrix}$	10.7
57	S	$\begin{matrix} 1,4 & 3 & 2,4 \\ 3 & 1,2,4 & 3 \\ 2,4 & 3 & 1,2,4 \end{matrix}$	20.13
58	S	$\begin{matrix} 1,4 & 3 & 2,4 \\ 3 & 1,2,3,4 & 3 \\ 2,4 & 3 & 1,2,4 \end{matrix}$	15.7, 20.11, 25.12, 30.14
60	S	$\begin{matrix} 1,4 & 3,4 & 2,3 \\ 3,4 & 1,2 & 2,4 \\ 2,3 & 2,4 & 1,3 \end{matrix}$	7.3
82	S	$\begin{matrix} 1,3,4 & 2 & 2 \\ 2 & 1,3,4 & 3,4 \\ 2 & 3,4 & 1,3,4 \end{matrix}$	26.17, 30.24, 32.14, 32.20, 32.21, 34.6
83	S	$\begin{matrix} 1,3,4 & 2 & 2,4 \\ 2 & 1,3 & 4 \\ 2,4 & 4 & 1,2,3 \end{matrix}$	15.10, 20.14, 25.16, 30.23
105	S	$\begin{matrix} 1,3,4 & 2,3,4 & 2,3,4 \\ 2,3,4 & 1,2,4 & 2,3,4 \\ 2,3,4 & 2,3,4 & 1,2,3 \end{matrix}$	13.3, 16.20, 16.21
109	S	$\begin{matrix} 1,2 & 4 & 3,4 \\ 4 & 1,3 & 2,4 \\ 3,4 & 2,4 & 1,2,3,4 \end{matrix}$	9.5, 12.12, 15.8, 16.12, 18.11, 20.12, 21.6, 24.16, 24.20, 25.13, 27.381, 28.15, 30.17, 30.21, 32.16, 33.5
110	S	$\begin{matrix} 1,2 & 4 & 3,4 \\ 4 & 1,3,4 & 2,3,4 \\ 3,4 & 2,3,4 & 1,2,3,4 \end{matrix}$	15.9, 24.12, 27.382, 27.383
111	S	$\begin{matrix} 1,2 & 3 & 4 \\ 3 & 1,2 & 4 \\ 4 & 4 & 1,2,3 \end{matrix}$	12.11, 16.13, 20.15, 24.22, 28.73, 32.25
112	S	$\begin{matrix} 1,2 & 3 & 4 \\ 3 & 1,2 & 4 \\ 4 & 4 & 1,2,3,4 \end{matrix}$	18.10, 24.15, 24.19, 30.15, 30.20, 32.15

Hypergroup	NS or S	Hyperproduct	Schemes
113	S	1,2 3 4 3 1,2,3 4 4 4 1,2,3	18.12, 24.18, 24.21, 30.18, 30.22, 32.17
114	S	1,2 3 4 3 1,2,3 4 4 4 1,2,3,4	27.380
115	S	1,2 3 4 3 1,2,3,4 3,4 4 3,4 1,2,3	27.384, 30.16
121	S	1,2 3,4 3,4 3,4 1,2,3,4 2,3,4 3,4 2,3,4 1,2,3,4	16.14, 16.15, 16.16, 16.17, 16.18, 16.19, 25.14, 25.15, 28.16-71
126	S	1,2,4 3 2,4 3 1,2,3,4 3 2,4 3 1,2,4	18.13, 27.385
141	S	1,2,4 3,4 2,3,4 3,4 1,2,3,4 2,3,4 2,3,4 2,3,4 1,2,3,4	27.386
144	S	1,2,3 2,3 4 2,3 1,2 4 4 4 1,2,3	32.18
145	S	1,2,3 2,3 4 2,3 1,2 4 4 4 1,2,3,4	30.19
170	S	1,2,3,4 2,3,4 2,3 2,3,4 1,2,3,4 2,3,4 2,3 2,3,4 1,3,4	21.9
176	S	1,2,3,4 2,3,4 2,3,4 2,3,4 1,2,3,4 2,3,4 2,3,4 2,3,4 1,2,3,4	25.17, 25.18, 28.74, 28.75

Notice then that there are at least 35 hypergroups that can be realized as finite association schemes. 11 of them are nonsymmetric hypergroups, and 24 are symmetric hypergroups. As in the hypergroups of rank 3, notice that there are many number theoretic conjectures we can propose based on what order schemes realize each hypergroups. In section 4, we prove some of these conjectures.

4 Additional Results from Hypergroups of rank 4 realized as Finite Schemes

We mentioned previously that there are 139 symmetric and 37 nonsymmetric hypergroups of rank 4. Due to the large number of hypergroups and limited amount of time available, we only explore the 37 nonsymmetric hypergroups of rank 4. Before diving into results about specific hypergroups, we present lemmas that will assist in proving our conjectures. Further, we change the notation of relations and elements of hypergroups being represented as numbers to $1, p, q, r$.

Lemma 14. *Let (X, S) be a finite scheme. Then $n_1 = 1$.*

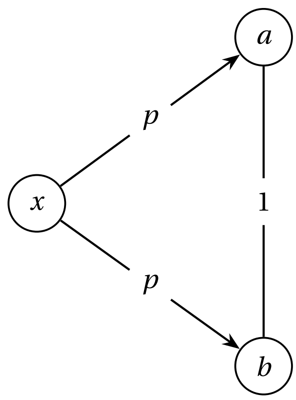
Proof. Let $x \in X$ be arbitrary and suppose $y \in X$ such that $(x, y) \in 1$ for all $i \in \{1, 2, \cdots, n_1\}$. By definition of $1 \in S$, $y = x$. Thus $n_1 = 1$. $\square$

Lemma 15. *Let (X, S) be a finite scheme. For any relation $p \in S$, we have that $n_p = n_{p^*}$.*

Proof. Let $p \in S$ be arbitrary. We count $|p|$. To count $|p|$, for any $(x, y) \in X \times X$ such that $(x, y) \in p$, there are $|X|$ such possibilities for $x \in X$ and n_p such possibilities for $y \in X$. Then $|p| = |X|n_p$. Notice then that $(y, x) \in p^*$ by definition of p^*. There are $|X|$ possibilities for $y \in X$ and n_{p^*} possibilities for $x \in X$. Then $|p^*| = |X|n_{p^*}$. Since p^* is defined to have the same cardinality as p, we have $|X|n_p = |p| = |p^*| = |X|n_{p^*}$. Thus $n_p = n_{p^*}$. $\square$

Lemma 16. *Let (X, S) be a scheme. If $p^* p = \{1\}$ for any $p \in S$, then $n_p = 1$.*

Proof. Suppose $n_p > 1$. Let $x \in X$ be arbitrary and choose $a, b \in X$ such that $(x, y_k) \in p$ since $n_p > 1$. Notice then $(a, x) \in p^*$ and $(x, b) \in p$. Thus $(a, b) \in p^* p$.



Since $p^* p = \{1\}$, $(a, b) \in 1$. By definition of the relation 1, then $a = b$. Thus we have a contradiction, so $n_p = 1$. $\square$

We now use these lemmas to prove claims about various schemes.

First, we explore hypergroup 1. Hypergroup 1 has the hypermultiplication table given by

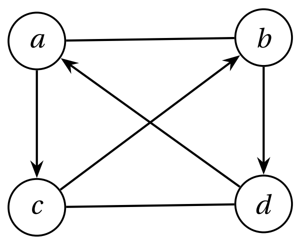
	1	p	p^*	r
1	1	p	p^*	r
p	p	r	1	p^*
p^*	p^*	1	r	p
r	r	p^*	p	1

Proposition 17. *There is a unique scheme realizing hypergroup 1.*

Proof. Notice that hypergroup 1 is a thin hypergroup. Thus we have that there exists a scheme realizing hypergroup 1 by corollary 9. We show that this scheme is the unique scheme realizing hypergroup 1. By lemma 15, we have that $n_p = n_{p^*}$. By lemma 16, we have that $n_p = 1 = n_{p^*}$. Since $r = r^*$ and $rr = \{1\}$, by lemma 16, we also have that $n_r = 1$. Thus the scheme realizing hypergroup 1 must have order 4. We can construct this unique scheme order 4 realizing hypergroup 1. This scheme has the following multiplication table

	x_1	x_2	x_3	x_4
x_1	1	r	p^*	p
x_2	r	1	p	p^*
x_3	p	p^*	1	r
x_4	p^*	p	r	1

Pictorially, we have



where solid edges represent relation r and directed edges represent relation p. We show that this graph represents a scheme. The following are the adjacency matrices for the relations p, p^*, r

$$\sigma_p = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \sigma_{p^*} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \sigma_r = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Notice that

$$\begin{aligned} \sigma_p \sigma_p &= \sigma_r, & \sigma_p \sigma_{p^*} &= \sigma_1, & \sigma_p \sigma_r &= \sigma_{p^*}, \\ \sigma_{p^*} \sigma_p &= \sigma_1, & \sigma_{p^*} \sigma_{p^*} &= \sigma_r, & \sigma_{p^*} \sigma_r &= \sigma_p, \\ \sigma_r \sigma_p &= \sigma_{p^*}, & \sigma_r \sigma_{p^*} &= \sigma_p, & \sigma_r \sigma_r &= \sigma_1. \end{aligned}$$

Thus all the structure constants hold. Also, $r = r^*$ and 1 is the identity relation, so this graph is in fact a scheme. We can find all the schemes of order 4 Using our algorithm, we find that

	x_1	x_2	x_3	x_4
x_1	1	p	p	p
x_2	p	1	p	p
x_3	p	p	1	p
x_4	p	p	p	1

Table 4.1

	x_1	x_2	x_3	x_4
x_1	1	p	q	q
x_2	p	1	q	q
x_3	q	q	1	p
x_4	q	q	p	1

Table 4.2

	x_1	x_2	x_3	x_4
x_1	1	p	q	r
x_2	p	1	r	q
x_3	q	r	1	p
x_4	r	q	p	1

Table 4.3

none of these schemes realize hypergroup 1. Thus, an exhaustive search of the four schemes of order 4 shows that this is the only scheme realizing hypergroup 1. $\square$

Next, notice that from the table in section 3, all the schemes realizing hypergroup 19 have order a multiple of 3 starting from 9. We show that every scheme realizing hypergroup 19 must have order a multiple of 3. The hypermultiplication table of hypergroup 19 is

	1	p	p^*	r
1	1	p	p^*	r
p	p	p^*	1	r
p^*	p^*	1	p	r
r	r	r	r	$1, p, p^*, r$

Lemma 18. *Let (X,S) be a scheme with $x, y \in X$, $p \in S$, and $p^* \neq p$ such that $pp = \{p^*\}$, $pp^* = \{1\}$, $p^*p = \{1\}$, and $p^*p^* = \{p\}$. We define the relation $\sim$ such that $x \sim y$ if and only if $(x, y) \in 1 \cup p \cup p^*$. Then $\sim$ is an equivalence relation.*

Proof. Let $x, y, z \in X$ be arbitrary. Since $(x, x) \in 1$ by definition of $1 \in S$, $x \sim x$. Thus $\sim$ is reflexive. Suppose $x \sim y$. Then $(x, y) \in 1 \cup p \cup p^*$. If $(x, y) \in 1$, then $x = y$. So $(y, x) \in 1$ and $y \sim x$. If $(x, y) \in p$, then $(y, x) \in p^*$, so $y \sim x$. Similarly, if $(x, y) \in p^*$, then $(y, x) \in p$, so $y \sim x$. Thus $\sim$ is symmetric. Suppose $x \sim y$ and $y \sim z$. Let $(x, y) \in s$ and $(y, z) \in t$ where $s, t \in \{1, p, p^*\}$. Then from the hypermultiplication table, we have that $st \subseteq \{1, p, p^*\}$. Thus $(x, z) \in 1 \cup p \cup p^*$. Thus $\sim$ is transitive. Thus $\sim$ is an equivalence relation. □

Lemma 19. *The equivalence classes determined by relation $\sim$ in lemma 18 have order 3.*

Proof. Let (X,S) be a scheme with $p \in S$ and $p^* \neq p$ such that $pp = \{p^*\}$, $pp^* = \{1\}$, $p^*p = \{1\}$, and $p^*p^* = \{p\}$. Since $pp^* = \{1\} = p^*p$, it follows that $n_p = 1 = n_{p^*}$ by lemma 16 and lemma 15. We also have that $n_1 = 1$ by lemma 14. Let $a \in X$ be arbitrary and choose $b, c \in X$ such that $(a, b) \in p$ and $(a, c) \in p^*$. Then $a \sim b$ and $a \sim c$. Thus if there exists $d \in X$ such that $(a, d) \in p$, $(a, d) \in p^*$, or $(a, d) \in 1$, then $a = d$, $b = d$, or $c = d$. Thus the equivalence classes determined by $\sim$ have order 3. □

Proposition 20. *Hypergroup 19 is finitely realizable and can only be realized as an association scheme with an order a multiple of 3 greater than or equal to 9.*

Proof. Notice that scheme 9.4 realizes hypergroup 19. Thus hypergroup is finitely realizable. Let (X,S) be a scheme realizing hypergroup 19. Since $pp = \{p^*\}$, $pp^* = \{1\}$, $p^*p = \{1\}$, and $p^*p^* = \{p\}$, we have by lemma 19 that the equivalence classes determined by relation $\sim$ have order 3. Notice, however that if there is a scheme of order 3 realizing hypergroup 19, we have a triple of elements in X in a cell realized as the equivalence class $\sim$ but no pair in relation 4. Further, an exhaustive search of all the schemes of order 6 show that the relation $\sim$ is only an equivalence relation in hypergroup 18, which is realized as scheme 6.4 in Miyamato & Hanaki's list of schemes. Thus, hypergroup 19 can only be realized as association schemes with an order a multiple of 3 and ≥ 9. □

Next, we show that there is only one scheme realizing hypergroup 18. The hypermultiplication table for hypergroup 18 is

	1	p	p^*	r
1	1	p	p^*	r
p	p	p^*	1	r
p^*	p^*	1	p	r
r	r	r	r	$1, p, p^*$

Proposition 21. *Hypergroup 18 is finitely realizable and there is a unique scheme (6.4) realizing hypergroup 18.*

Proof. Notice that scheme 6.4 realizes hypergroup 18. Thus, there exists a scheme realizing hypergroup 18. We now show that scheme 6.4 is the only scheme that realizes hypergroup 18. Since $pp = \{p^*\}$, $pp^* = \{1\}$, $p^*p = \{1\}$, and $p^*p^* = \{p\}$, we have by lemma 19 that the equivalence classes determined by relation $\sim$ have order 3. Now if there is a scheme realizing hypergroup 18 with order 3, then there is no pair of vertices in relation $r \in S$ since the equivalence classes have order 3. An exhaustive search of the schemes of order 6 show that only scheme 6.4 realizes hypergroup 18. Lastly, if there exists a scheme of order ≥ 9, then there are at least three equivalence classes. Let $x_1, x_2, x_3 \in X$ be vertices in each of the equivalence classes respectively. Notice then that since x_1, x_2, x_3 are in different equivalence classes, we have $(x_1, x_2) \in r$, $(x_1, x_3) \in r$, and $(x_2, x_3) \in r$. Since $(x_1, x_3) \in r$ and there is a vertex $x_2 \in X$ such that $(x_1, x_2) \in r$ and $(x_2, x_3) \in r$, we have that $a_{rr}^r > 0$. Then by definition 5 we have $r \in rr$. Since $r \notin rr$ in hypergroup 18, there are no schemes of order ≥ 9 that realize hypergroup 18. Thus, there is a unique scheme, 6.4, realizing hypergroup 18. □

The last two results we present in this section are parallel results to Proposition 20 and 21 about hypergroup 20 and 21. Hypergroup 20 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	p^*	$1, r$	p
p^*	p^*	$1, r$	p	p^*
r	r	p	p^*	1

Hypergroup 21 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	p^*	$1, r$	p
p^*	p^*	$1, r$	p	p^*
r	r	p	p^*	$1, r$

Lemma 22. *Let (X, S) be a scheme such that for $p, r \in S$, we have $pp^* = \{1, r\}$ and $pr = \{p\}$. Then $n_r = n_p - 1$.*

Proof. Let $x \in X$ be arbitrary and choose distinct $y_1, y_2, \cdots, y_{n_p} \in X$ such that $(x, y_k) \in p$ for all $k \in \{1, \cdots, n_p\}$. Let $i, j \in \{1, 2, \cdots, n_p\}$ be arbitrary such that $i \neq j$. Notice then $(y_i, x) \in p^*$ and $(x, y_j) \in p$. Since $pp^* = \{1, r\}$ and $y_i \neq y_j$, we have that $(y_i, y_j) \in r$. Notice that for y_i, we then have $n_p - 1$ values of y_l where $l \in \{1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_{n_p}\}$ such that $(y_i, y_l) \in r$. Thus $n_r \geq n_p - 1$. Suppose next that $(y_i, z) \in r$ for some $z \in X$ where $z \neq x$. Since $(x, y_i) \in p$ and $(y_i, z) \in r$, we have $(x, z) \in p$ since $pr = \{p\}$. Then we have that $z = y_j$ for some $j \in \{1, \cdots, n_p\}$. Thus, we have $n_r = n_p - 1$. □

Corollary 23. *Hypergroup 21 is finitely realizable and can only be realized as schemes with order a multiple of 3 and greater than or equal to 9.*

Proof. Notice that scheme 9.6 realizes hypergroup 21. Thus, hypergroup 21 is finitely realizable. Let (X, S) be a scheme realizing hypergroup 21. Since $pp^* = \{1, r\}$ and $pr = \{p\}$,

$n_r = n_p - 1$ by lemma 22. Since the order of a scheme is the sum of the valency numbers of all its relations, we have that the order is

$$\begin{aligned} n_1 + n_r + n_p + n_{p^*} &= 1 + n_p - 1 + n_p + n_p \qquad &&(\text{Since } n_1 = 1,\ n_p = n_{p^*},\ n_r = n_p - 1) \\ &= 3n_p. \end{aligned}$$

Notice that if $n_p = 1$, $n_r = 1-1 = 0$. Thus $n_p \neq 1$. The conditions that $n_p = n_{p^*}$ and $n_r = n_p - 1$ hold if $n_p = 2$. However, when $n_p = 2$, there is no $z \in X$ such that given $(x, y) \in r$, $(x, z) \in r$ and $(z, y) \in r$. Thus the condition that $n_p \geq 3$ must be true for any scheme realizing hypergroup 21. □

Proposition 24. *Hypergroup 20 is finitely realizable and there is a unique scheme (6.6) realizing hypergroup 20.*

Proof. Notice that scheme 6.6 realizes hypergroup 20. Thus, there exists a scheme realizing hypergroup 20. We now show that scheme 6.6 is the only scheme that realizes hypergroup 20. Since $pp^* = \{1, r\}$ and $pr = \{p\}$, we have $n_r = n_p - 1$ by lemma 22. We have from scheme 6.6 that $n_p = 2 = n_{p^*}$ and $n_r = 1$. However, now suppose $n_p > 2$ for a scheme (X, S) realizing hypergroup 20. Let $x \in X$ be arbitrary and choose distinct $y_1, y_2, y_{n_p} \in X$ such that $(x, y_i) \in p$ for all $i \in \{1, \cdots, n_p\}$. Without loss of generality, we consider y_1, y_2, y_3. We also have that $(y_1, x), (y_2, x), (y_3, x) \in p^*$. Since We then have that $(y_1, y_2), (y_1, y_3), (y_2, y_3) \in r$ since $pp^* = \{1, r\} = p^* p$ and $y_1 \neq y_2$, $y_2 \neq y_3$, $y_1 \neq y_3$. Notice then that $a_{rr}^r > 0$. However $r \notin rr$. Thus we have a contradiction. Thus $n_p \leq 2$. Since if $n_p = 1$, $n_r = 0$, but the valency of each relation is at least 1. So $n_p = 2$. Thus hypergroup 3.2 can only be realized as scheme 6.6. □

We have presented proofs for only a few of the patterns of the hypergroups that can be realized by finite association schemes. Notice, however, that there are many interesting patterns. For example, only schemes of order a multiple of 5 and greater than or equal to 15 realize hypergroups 83 and 58. Further, there are many hypergroups that are only realized by schemes of one order. In the next section, we explore the hypergroups that cannot be realized as finite association schemes.

5 Hypergroups of rank 4 not Realizable as Finite Groups

By studying all the schemes that Miyamoto & Hanaki [2] present in their classification of association schemes, we exhausted all the known finite schemes we could find. As a result, our next approach is less brute force and tries to make use of theoretical implications based on the features of each hypergroup to show that certain hypergroups cannot be realized by any finite association scheme.

Proposition 25. *Let H be a hypergroup. If $pp = \{p\}$ for any $p \in H$, H cannot be realized as a finite association scheme.*

Proof. Let (X, S) be a finite scheme realizing a hypergroup H such that $pp = \{p\}$. Now suppose $n_p = a$ where $a \in \mathbb{Z}^+$ is finite. Let $x \in X$ be arbitrary and choose distinct $y_1, y_2, \cdots, y_a, z \in X$ such that $(x, y_i) \in p$ for all $i \in \{1, \cdots a\}$ and $(x, z) \in p^*$. Then $(z, x) \in p$ by definition of p^*. Notice then that $(z, y_i) \in p$ for all $i \in \{1, \cdots, a\}$ since $(z, x) \in p$, $(x, y_i) \in p$ and $pp = \{p\}$. Then since $x \neq y_i$ for any $i \in \{1, \cdots, a\}$, there are $a+1$ vertices such that $z \in X$ paired with each such vertex is in p. Thus $n_p \geq a+1$. Since we also have that $n_p = a$, there is a contradiction. Thus n_p is not finite. Therefore, H cannot be realized as a finite association scheme. □

Since the argument in proposition 25 requires simply one condition, that $pp = \{p\}$ for any $p \in S$, we can find many hypergroups that cannot be realized as finite association schemes. Namely, each of hypergroups 4, 5, 6, 7, 8, 9, and 10 all satisfy the condition that $2 \cdot 2 = \{2\}$. Thus each of these hypergroups cannot be realized as finite association schemes. Notice, however, that if there is a scheme (X, S) such that $pp = \{p\}$, we have that $p \neq p^*$ since $1 \notin pp$. Thus, this argument only applies to nonsymmetric hypergroups.

Next, we consider hypergroup 2. Hypergroup 2 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	r	$1, r$	p, p^*
p^*	p^*	$1, r$	r	p, p^*
r	r	p, p^*	p, p^*	$1, r$

In proposition 28, we show that hypergroup 2 cannot be realized by a finite scheme. This proof, however, takes a vastly different approach–a combinatorial and number theoretic argument.

Lemma 26. *If $\gcd(a, b) = 1$, then $a \nmid b^2$ for any $a, b \in \mathbb{Z}^+$ with $a \geq 2$.*

Proof. Suppose $a | b^2$. Since $\gcd(a, b) = 1$, there exist $x, y \in \mathbb{Z}$ such that

$$ax + by = 1.$$

Multiplying both sides by b, we have

$$axb + b^2 y = b.$$

Notice that a divides axb and $b^2 y$ since $a | b^2$. Thus $a | b$. Since we also have that $\gcd(a, b) = 1$, we have a contradiction to $a > 1$. Thus $a \nmid b^2$. □

Lemma 27. *Let (X, S) be a scheme with $p \in P$. If $n_p = 1$, $pp^* = \{1, p\}$.*

Proof. Suppose $n_p = 1$. Let $x \in X$ be arbitrary. Since $n_p = 1$, there is only one element $y \in X$ such that $(x, y) \in p$. By lemma 15, we also have that $n_{p^*} = 1$, so there exists only one element $z \in X$ such that $(y, z) \in p^*$. Since by definition of p^*, $(y, x) \in p$, we have $z = x$. Since $(x, x) \in 1$, then $a^1_{pp^*} = 1$ and $1 \in pp^*$. However, since $y \in X$ is the only element such that $(x, y) \in p$ and $n_{p^*} = 1$, forcing $x \in X$ to be the only element such that $(y, x) \in p^*$, we have $pp^* = \{1\}$. □

Proposition 28. *Hypergroup 2 cannot be realized as a finite association scheme.*

Proof. Let (X, S) be a finite scheme realizing hypergroup 2. Let $x \in X$ be arbitrary and choose $y_1, y_2, \cdots, y_{n_r}, z \in X$ such that $(x, y_i) \in r$ for all $i \in \{1, \cdots, n_r\}$ and $(x, z) \in p$. We have by definition of $p^* \in S$ that $(z, x) \in p^*$. Since $(z, x) \in p^*$, $(x, y_i) \in r$ for all i, and $p^* r = \{p, p^*\}$, we have that $(z, y_i) \in p \cup p^*$ for all i. Suppose $w \in X$ such that $w \notin \{y_1, y_2, \cdots, y_{n_r}\}$ and $(z, w) \in p \cup p^*$. We show that this would result in a contradiction. Without loss of generality, we only consider x, y, z, w.

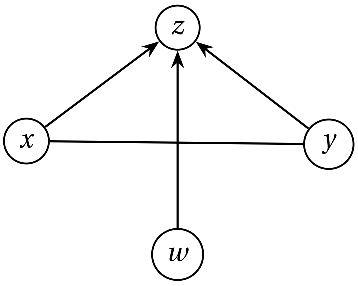


Figure 5.1: $(y, z) \in p$, $(w, z) \in p$

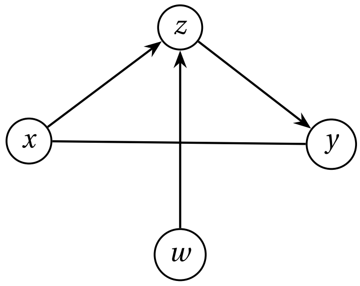


Figure 5.2: $(z, y) \in p$, $(w, z) \in p$

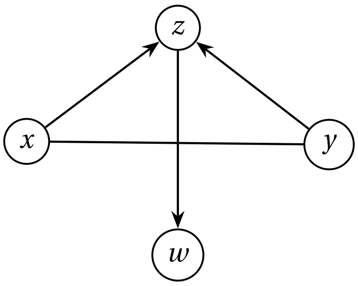


Figure 5.3: $(y, z) \in p$, $(z, w) \in p$

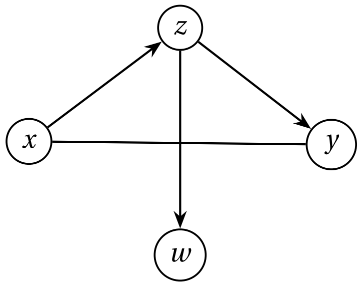


Figure 5.4: $(z, y) \in p$, $(z, w) \in p$

Notice that considering all the cases where $(z, w) \in p \cup p^*$, we have $(x, w) \in r$ which implies that $w = y_i$ for some $i \in \{1, \cdots, n_r\}$. Now we count the number of elements $u \in X$ such that $(z, u) \in p \cup p^*$ in two different ways. Since $(z, x), (z, y_j) \in p \cup p^*$ for all $j \in \{1, \cdots, n_r\}$, we have $n_r + 1$. Next, by definitions of the valencies of relations p, p^*, we also have $n_p + n_{p^*}$. Then we have that $n_p + n_{p^*} = n_r + 1$. Since $n_p = n_{p^*}$ by lemma 15, we have that $2n_p = n_r + 1$. Then we have $2n_p + (-1)n_r = 1$. Thus $\gcd(n_p, n_r) = 1$. Next, we consider a set J defined by

$$J = \{(x, a, b) : (x, a) \in p, (a, b) \in p\}.$$

Notice that $|J| = n_p^2$ since there are n_p values of a such that $(x, a) \in p$ and n_p values of b such that $(a, b) \in p$. On the other hand, since $pp = \{r\}$, if $(x, a, b) \in J$, we must have $(x, b) \in r$. There are n_r choices for b and for each choice of b, there are a^r_{pp} choices for a. Thus $|J| = a^r_{pp} n_r$. Thus $n_p^2 = a^r_{pp} n_r$. Then $n_r | n_p^2$. Since $pp^* \neq \{1\}$, by lemma 27 we have that $n_p > 1$. Since $\gcd(n_r, n_p) = 1$ and $n_p > 1$, by lemma 26, $n_r \nmid n_p^2$. Thus we have a contradiction. Thus hypergroup 2 cannot be realized as a finite association scheme. □

Next, we examine hypergroup 11. Hypergroup 11 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	p, r	$1, p, p^*$	p^*, r
p^*	p^*	$1, p, p^*$	p^*, r	p, r
r	r	p^*, r	p, r	$1, p, p^*$

Before the following proof, however, we use some of Paul-Hermann Zieschang's results from his text, *Theory of Association Schemes* [1] to prove some lemmas. The upcoming lemmas extensively use structural constants to prove results. Thus, we revert back to using numbers when referring to specific relations and elements of hypergroups.

Lemma 29. *Let (X, S) be a finite scheme with $p, q \in S$. For each $s \in S$, the following hold.*

1. *[1, Lemma 1.1.3 (ii)] We have*

$$a_{ps}^{q} n_q = a_{qs^*}^{p} n_p.$$

2. *[1, Lemma 1.1.1 (ii)] We have*

$$a_{pq}^{s} = a_{q^* p^*}^{s^*}.$$

Lemma 30. *Let (X, S) be a finite scheme with $p, q \in S$. The following hold.*

1. *[1, Lemma 1.1.3 (iii)] We have*

$$\sum_{s \in S} a_{ps}^{q} = n_p.$$

2. *[1, Lemma 1.1.3 (iv)] We have*

$$\sum_{s \in S} a_{pq}^{s} n_s = n_p n_q.$$

Lemma 31. *[1, Theorem 9.5.2] Let (X, S) be a finite scheme. If $|S| \leq 5$, then S is commutative.*

Notice that using lemma 29 and the definition of commutativity, we can find equalities between structural constants.

Corollary 32. *Let (X, S) be a finite scheme with $S = \{1, 2, 3, 4\}$ such that $2^* = 3$. Then $a_{22}^{2} = a_{23}^{2}$.*

Proof. We use lemma 29(1) where $p = 2$, $q = 2$, and $s = 2$. Notice then that

$$a_{22}^{2} n_2 = a_{23}^{2} n_2,$$
$$a_{22}^{2} = a_{23}^{2}.$$

Thus, $a_{22}^{2} = a_{23}^{2}$. □

Corollary 33. *Let* (X,S) *be a finite scheme where* $S=\{1,2,3,4\}$*,* $2^*=3$*, and* $4^*=4$*. Then*

$$\begin{aligned}
a_{22}^2 = a_{33}^3 = a_{23}^2 = a_{23}^3 = a_{32}^2 = a_{32}^3&,\\
a_{22}^3 = a_{33}^2&,\\
a_{22}^4 = a_{33}^4&,\\
a_{23}^4 = a_{32}^4&,\\
a_{24}^2 = a_{42}^2 = a_{34}^3 = a_{43}^3&,\\
a_{24}^3 = a_{42}^3 = a_{34}^2 = a_{43}^2&,\\
a_{24}^4 = a_{42}^4 = a_{34}^4 = a_{43}^4&,\\
a_{44}^2 = a_{44}^3&.
\end{aligned}$$

Proof. Since $|S|=4$, S is commutative by lemma 31. Notice that each result then follows by definition of commutativity, lemma 29(2), and/or corollary 32. □

Corollary 34. *Let* (X,S) *be a finite scheme where* $S=\{1,2,3,4\}$*,* $2^*=3$*, and* $4^*=4$*. Then*

$$\begin{aligned}
n_2 &= 1+2a_{22}^2+a_{24}^2,\\
n_2 &= a_{22}^3+a_{22}^2+a_{24}^3,\\
n_2 &= a_{22}^4+a_{23}^4+a_{24}^4,\\
n_4 &= a_{24}^2+a_{24}^3+a_{44}^2,\\
n_4 &= 1+2a_{24}^4+a_{44}^4,\\
(n_2)^2 &= a_{22}^2 n_2+a_{22}^3 n_2+a_{22}^4 n_4,\\
(n_2)^2 &= n_2+2a_{22}^2 n_2+a_{23}^4 n_4,\\
n_2 n_4 &= a_{24}^2 n_2+a_{24}^3 n_2+a_{24}^4 n_4,\\
(n_4)^2 &= n_4+2a_{44}^2+a_{44}^4 n_4.
\end{aligned}$$

Proof. Let $p=2$ and $q=2$. By lemma 30(1), we have

$$\begin{aligned}
n_2 &= a_{21}^2+a_{22}^2+a_{23}^2+a_{24}^2\\
&= 1+a_{22}^2+a_{23}^2+a_{24}^2 && \text{(given } (x,y)\in 1,\ x=y\text{)},\\
&= 1+2a_{22}^2+a_{24}^2. && (a_{22}^2=a_{23}^2 \text{ by corollary 32})
\end{aligned}$$

By lemma 30 (2), we have

$$\begin{aligned}
(n_2)^2 &= a_{22}^1 n_1+a_{22}^2 n_2+a_{22}^3 n_3+a_{22}^4 n_4\\
&= a_{22}^2 n_2+a_{22}^3 n_3+a_{22}^4 n_4 && (\text{since } 2^*\neq 2)\\
&= a_{22}^2 n_2+a_{22}^3 n_2+a_{22}^4 n_4. && (\text{by lemma 15})
\end{aligned}$$

Next, let $p=2$ and $q=3$. By lemma 30 (1), we have

$$\begin{aligned}
n_2 &= a_{21}^3+a_{22}^3+a_{23}^3+a_{24}^3\\
&= a_{22}^3+a_{23}^3+a_{24}^3 && (\text{since } 3\notin 2\cdot 1)\\
&= a_{22}^3+a_{22}^2+a_{24}^3. && (\text{by corollary 33})
\end{aligned}$$

By lemma 30(2), we have

$$\begin{aligned}
n_2 n_3 &= a_{23}^1 n_1 + a_{23}^2 n_2 + a_{23}^3 n_3 + a_{23}^4 n_4, \\
(n_2)^2 &= a_{23}^1 n_1 + a_{23}^2 n_2 + a_{23}^3 n_2 + a_{23}^4 n_4 && \text{(by lemma 15)} \\
&= n_2 + a_{23}^2 n_2 + a_{23}^3 n_2 + a_{23}^4 n_4 && \text{(by definition 11 and lemma 14)} \\
&= n_2 + 2a_{22}^2 n_2 + a_{23}^4 n_4. && \text{(by corollary 33)}
\end{aligned}$$

Next, let $p = 2$ and $q = 4$. By lemma 30(1), we have

$$\begin{aligned}
n_2 &= a_{21}^4 + a_{22}^4 + a_{23}^4 + a_{24}^4 \\
&= a_{22}^4 + a_{23}^4 + a_{24}^4. && \text{(since } 4 \not\in 2 \cdot 1\text{)}
\end{aligned}$$

By lemma 30(2), we have

$$\begin{aligned}
n_2 n_4 &= a_{24}^1 n_1 + a_{24}^2 n_2 + a_{24}^3 n_3 + a_{24}^4 n_4 \\
&= a_{24}^2 n_2 + a_{24}^3 n_3 + a_{24}^4 n_4 && \text{(since } 1 \not\in 2 \cdot 4\text{)} \\
&= a_{24}^2 n_2 + a_{24}^3 n_2 + a_{24}^4 n_4. && \text{(by lemma 15)}
\end{aligned}$$

Next, let $p = 4$ and $q = 2$. By lemma 30(1), we have

$$\begin{aligned}
n_4 &= a_{41}^2 + a_{42}^2 + a_{43}^2 + a_{44}^2 \\
&= a_{42}^2 + a_{43}^2 + a_{44}^2 && \text{(since } 2 \not\in 4 \cdot 1\text{)} \\
&= a_{24}^2 + a_{24}^3 + a_{44}^2. && \text{(by corollary 33)}
\end{aligned}$$

Lastly, let $p = 4$ and $q = 4$. By lemma 30(1), we have

$$\begin{aligned}
n_4 &= a_{41}^4 + a_{42}^4 + a_{43}^4 + a_{44}^4 \\
&= 1 + a_{42}^4 + a_{43}^4 + a_{44}^4 && \text{(given } (x, y) \in 1,\ x = y\text{)} \\
&= 1 + 2a_{24}^4 + a_{44}^4. && \text{(by corollary 33)}
\end{aligned}$$

By lemma 30(2), we have

$$\begin{aligned}
(n_4)^2 &= a_{44}^1 n_1 + a_{44}^2 n_2 + a_{44}^3 n_3 + a_{44}^4 n_4 \\
&= n_4 + a_{44}^2 n_2 + a_{44}^3 n_3 + a_{44}^4 n_4 && \text{(by definition 11 and lemma 14)} \\
&= n_4 + a_{44}^2 n_2 + a_{44}^3 n_2 + a_{44}^4 n_4 && \text{(by lemma 15)} \\
&= n_4 + 2a_{44}^2 n_2 + a_{44}^4 n_4. && \text{(by corollary 33)}
\end{aligned}$$

Thus, we have all nine equations. □

We use the preceding corollaries to show that hypergroup 11 cannot be realized by a finite scheme. The following is the hypermultiplication table for hypergroup 11

	1	p	p^*	r
1	1	p	p^*	r
p	p	p, r	$1, p, p^*$	p^*, r
p^*	p^*	$1, p, p^*$	p^*, r	p, r
r	r	p^*, r	p, r	$1, p, p^*$

Proposition 35. *Hypergroup 11 cannot be realized as a finite association scheme.*

Proof. Let (X,S) be a scheme realizing hypergroup 11. Notice first that hypergroup 11 is commutative, $S=\{1,2,3,4\}$, $2^*=3$, and $4^*=4$. Also, since $3\not\in 2\cdot 2$, $4\not\in 2\cdot 3$, $2\not\in 2\cdot 4$, and $4\not\in 4\cdot 4$, by corollary 33 we have that

$$\begin{aligned}
0 &= a_{22}^3 = a_{33}^2,\\
0 &= a_{23}^4 = a_{32}^4,\\
0 &= a_{24}^2 = a_{42}^2 = a_{34}^3 = a_{43}^3,\\
0 &= a_{44}^4.
\end{aligned}$$

Then by corollary 34, we have

$$\begin{aligned}
n_2 &= 1+2a_{22}^2,\\
n_2 &= a_{22}^2 + a_{24}^3,\\
n_2 &= a_{22}^4 + a_{24}^4,\\
n_4 &= a_{24}^3 + a_{44}^2,\\
n_4 &= 1+2a_{24}^4,\\
(n_2)^2 &= a_{22}^2 n_2 + a_{22}^4 n_4.
\end{aligned}$$

We then have

$$\begin{aligned}
a_{24}^3 &= n_2 - a_{22}^2,\\
a_{22}^4 &= n_2 - a_{24}^4,\\
a_{44}^2 &= n_4 - a_{24}^3 = n_4 - n_2 + a_{22}^2.
\end{aligned}$$

Substituting each of $a_{24}^3, a_{22}^4, a_{44}^2$ in the other equations, we get

$$(n_2)^2 = a_{22}^2 n_2 + n_2 n_4 - a_{24}^4 n_4.$$

Next, we can substitute for n_2, n_4 to get

$$1+4a_{22}^2+4(a_{22}^2)^2 = a_{22}^2+2(a_{22}^2)^2+1+2a_{22}^2+2a_{24}^4+4a_{22}^2a_{24}^4-a_{24}^4-2(a_{24}^4)^2.$$

Simplifying this expressions, we have

$$a_{22}^2+2(a_{22}^2)^2 = a_{24}^4+4a_{22}^2a_{24}^4-2(a_{24}^4)^2.$$

Solving for a_{22}^2, we get two solutions

$$a_{22}^2 = a_{24}^4, \qquad\qquad a_{22}^2 = \frac{1}{2}+a_{22}^2.$$

Notice, however, that the second possible solutions implies $\frac{1}{2} \in \mathbb{Z}$, so we have a contradiction. Thus, $a_{22}^2 = a_{24}^4$. We then have that

$$\begin{aligned} n_2 &= 1 + 2a_{22}^2, \\ n_4 &= 1 + 2a_{22}^2, \\ a_{44}^2 &= n_4 - n_2 + a_{22}^2 = a_{22}^2, \\ a_{34}^2 &= n_2 - a_{22}^2 = 1 + a_{22}^2. \end{aligned}$$

Now let $x \in X$ be arbitrary. Let

$$\begin{aligned} A_x &= \{y \in X \mid (x, y) \in 2\}, \\ B_x &= \{y \in X \mid (x, y) \in 4\}, \\ C_x &= \{y \in X \mid (x, y) \in 3\}. \end{aligned}$$

Then since $|A_x| = n_2$, $|B_x| = n_4$, and $|C_x| = n_3$ by definition of A_x, B_x, C_x. Thus, we have

$$1 + 2a_{22}^2 = |A_x| = |B_x| = |C_x|.$$

Notice that since $a_{43}^4 = a_{24}^4 = a_{22}^2$, for each vertex in B_x, there are $a_{42}^4 = a_{24}^4 = a_{22}^2$ values of vertices in B_x such that the pair are in relation 4. Further, for each vertex in A_x, there are $a_{44}^2 = a_{22}^2$ vertices in B_x such that the pairs are in relation 4 and $a_{34}^2 = 1 + a_{22}^2$ vertices in C_x such that y paired with the vertices are in relation 4. Now let $y \in A_x$ be arbitrary. We then have that there are $a_{22}^2 + 1$ vertices in C_x such that relation 4 contains the pair. Since $|C_x| = 1 + 2a_{22}^2$, we then have $|C_x \cap B_y| = a_{22}^2 + 1$. We also have that there are a_{22}^2 value of vertices in B_x such that y paired with the vertices are in relation 4. Since $|B_x| = 2a_{22}^2$, we then have $|B_x \cap B_y| = a_{22}^2$. Let $z \in B_y \cap C_x$ be arbitrary. Since $z \in B_y$, we have $(y, z) \in 4$. Also, since $a_{43}^4 = a_{22}^2$, there are a_{22}^2 values of B_y such that z paired the vertices are in relation 2. Notice then every pair of vertices must be in relation 2, but $2^* = 3$. Thus there must be at least one vertex, $w \in B_x \cap B_y$ such that $(z, w) \in 2$. Notice, however, that $(x, z) \in 3$, $(z, w) \in 2$, and $(x, w) \in 4$ since $w \in B_x$. Then $a32^4 > 0$, so $4 \in 3 \cdot 2$. However, $4 \notin 3 \cdot 2$. Thus there is a contradiction, so hypergroup 11 cannot be realized as a finite association scheme. $\square$

6 Hypergroups of rank 4 Realizable as Infinite Schemes

Unlike our existence proofs for hypergroups of rank 4 realizable as finite schemes, we do not have a list of infinite schemes that we can use to simply state that certain hypergroups are realizable as infinite schemes. As a result, we need to construct these association schemes much like the hypergroup of rank 3 shown in table 2.10. While the hypergroup in table 2.10 had relations that are straightforward, higher rank hypergroups become more complicated. Further, each construction of hypergroups requires that every structural constant be found. Notice that increasing the rank of a hypergroup then also increases the number of structural constants to find. However, the nature of these constructions then allows for creative constructions, making the process interesting. While we, with our collaborators, have constructed many infinite schemes realizing different hypergroups, our collaborators have constructed most of the infinite schemes. We present here the one infinite scheme we constructed realizing a hypergroup–hypergroup 37. Hypergroup 37 has the following hypermultiplication table

	1	2	3	4
1	1	2	3	4
2	2	2,3,4	1,2,3,4	2,3,4
3	3	1,2,3,4	2,3,4	2,3,4
4	4	2,3,4	2,3,4	1,2,3,4

Proposition 36. *Hypergroup 37 can be realized as an infinite association scheme.*

Proof. We show that the following structure is a scheme realizing hypergroup 37. Let $X = \mathbb{Q} \times \mathbb{Q}$ and $S = \{1, 2, 3, 4\}$. For any $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in X$ we have

- $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 1$ if and only if $a_1 = b_1$ and $a_2 = b_2$.
- $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 2$ if and only if any of the following is true:
  1. $0 \leq b_1 - a_1 < b_2 - a_2$.
  2. $b_2 - a_2 \leq -(b_1 - a_1) < 0$.
  3. $b_1 - a_1 \leq b_2 - a_2 < 0$.
- $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 3$ if and only if $\left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right) \in 2$. i.e. if and only if any of the following is true:
  1. $0 < b_2 - a_2 \leq b_1 - a_1$.
  2. $b_2 - a_2 < b_1 - a_1 \leq 0$.
  3. $0 < -(b_1 - a_1) \leq b_2 - a_2$.
- $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 4$ if and only if any of the following is true:
  1. $-(b_1 - a_1) < b_2 - a_2 \leq 0$.
  2. $0 \leq b_2 - a_2 < -(b_1 - a_1)$.

Notice that $1 \in S$ is the identity element, $2^* = 3$, and $4^* = 4$. We now show that the structural constant property holds.

a^p_{22} FOR ANY $p \in \{2,3,4\}$: Let $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in X$ be arbitrary and choose $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in X$ such that $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in p$.

CASE 1: Suppose $0 < b_1 - a_1$. Choose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ such that $b_2 - a_2 + b_1 - a_1 < x_2 - a_2$ and $0 = x_1 - a_1 \le x_2 - a_2$. Notice then $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \in 2$. Since $b_2 - a_2 + b_1 - a_1 < x_2 - a_2$, we have $b_2 - x_2 < -(b_1 - a_1)$. Since $0 = x_1 - a_1$, we have $a_1 = x_1$. Then $b_2 - x_2 < -(b_1 - x_1)$. Since $0 < b_1 - a_1 = b_1 - x_1$, we have $-(b_1 - x_1) < 0$. Then $b_2 - x_2 < -(b_1 - x_1) < 0$. Thus $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 2$.

CASE 2: Suppose $b_1 - a_1 < b_2 - a_2$ and $b_1 - a_1 \le 0$. Choose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ such that $b_1 - a_1 = x_1 - a_1 \le x_2 - a_2 < b_2 - a_2 < 0$. Notice then $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \in 2$. Since $b_1 - a_1 = x_1 - a_1$, we have $0 = b_1 - x_1$. Since $x_2 - a_2 < b_2 - a_2$, $0 < b_2 - x_2$. Then $0 = b_1 - x_1 < b_2 - x_2$. Thus $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 2$.

CASE 3: Suppose $b_1 - a_1 \le b_2 - a_2 < 0$. Choose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ such that $b_2 - a_2 < x_2 - a_2 \le -(x_1 - a_1) < 0$. Notice then $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \in 2$. Since $-(x_1 - a_1) < 0$, we have that $0 < x_1 - a_1$. Since $b_1 - a_1 \le b_2 - a_2 < x_2 - a_2 < x_1 - a_1$, we have that $b_1 - a_1 - (x_1 - a_1) < b_2 - a_2 - (x_2 - a_2)$. Notice that $b_1 - a_1 - (x_1 - a_1) = b_1 - x_1$ and $b_2 - a_2 - (x_2 - a_2) = b_2 - x_2$. Thus $b_1 - x_1 < b_2 - x_2$. Notice also that $b_2 - a_2 < x_2 - a_2$, which implies that $b_2 - x_2 < 0$. Thus $b_1 - x_1 < b_2 - x_2 < 0$, so $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 2$.

CASE 4: Suppose $b_1 - a_1 \le 0$ and $0 \le b_2 - a_2$. Choose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ such that $b_1 - a_1 = x_1 - a_1 \le x_2 - a_2 < 0$. Notice then $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \in 2$. Since $b_1 - a_1 = x_1 - a_1$, we have $b_1 - x_1 = 0$. Also, since $x_2 - a_2 < 0 \le b_2 - a_2$, we have $0 < b_2 - x_2$. Thus $0 = b_1 - x_1 < b_2 - x_2$, so $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 2$.

Since there are infinitely many possible choices of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ for each case, we have that $a^p_{22} = \infty$ for any $p \in \{2,3,4\}$. We then have that

$$\begin{aligned} a^2_{22} = a^3_{33} = a^2_{23} = a^3_{23} = a^2_{32} = a^3_{32} &= \infty, \\ a^3_{22} = a^2_{33} &= \infty, \\ a^4_{22} = a^4_{33} &= \infty. \end{aligned}$$

a^p_{23} FOR ANY $p \in S$: Let $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in X$ be arbitrary and choose $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in X$ such that $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in p$. Choose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ such that $x_1 - a_1 - (b_1 - a_1) + b_2 - a_2 < x_2 - a_2$ and $0 < b_1 - a_1 < x_1 - a_1 <$

$x_2 - a_2$. Notice then $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \in 2$. Since $x - 1 - a_1 - (b_1 - a_1) + b_2 - a_2 < x_2 - a_2$, we have $b_2 - x_2 < b_1 - x_1$. Since $b_1 - a_1 < x_1 - a_1$, we have $b_1 - x_1 < 0$. Then $b_2 - x_2 < b_1 - x_1 < 0$. Thus $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 3$. Since there are infinite possible choices of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$, we have a_{23}^p for any $p \in S$. We then have $a_{23}^1 = a_{32}^1 = a_{23}^4 = a_{32}^4 = \infty$.

a_{24}^p FOR ANY $p \in \{2,3,4\}$: Let $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in X$ be arbitrary and choose $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in X$ such that $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in p$.

CASE 1: Suppose $b_1 - a_1 < b_2 - a_2$ and $0 < b_2 - a_2$. Choose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ such that $b_1 - a_1 < x_1 - a_1 < b_2 - a_2$ and $0 \leq x_1 - a_1 < x_2 - a_2 = b_2 - a_2$. Notice then $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \in 2$. Since $b_1 - a_1 < x_1 - a_1$, we have $b_1 - x_1 < 0$. Then $0 < -(b_1 - x_1)$. Since $x_2 - a_2 = b_2 - a_2$, we have $0 = b_2 - x_2$. Then $0 = b_2 - x_2 < -(b_1 - x_1)$. Thus $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 4$.

CASE 2: Suppose $b_2 - a_2 < b_1 - a_1$ and $0 < b_2 - a_2$. Choose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ such that $x_1 - a_1 < b_1 - a_1$ and $0 \leq x_1 - a_1 < x_2 - a_2 = b_2 - a_2$. Notice then $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \in 2$. Since $x_2 - a_2 = b_2 - a_2$, we have $0 = b_2 - x_2$. Since $x_1 - a_1 < b_1 - a_1$, $0 < b_1 - x_1$. Then $-(b_1 - x_1) < 0$. We then have $-(b_1 - x_1) < b_2 - x_2 = 0$. Thus $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 4$.

CASE 3: Suppose $b_2 - a_2 < 0$. Choose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ such that $x_1 - a_1 \leq x_2 - a_2 = b_2 - a_2 < 0$ and $x_1 - a_1 < b_1 - a_1$. Notice then $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 2$. Since $x_1 - a_1 < b_1 - a_1$, we have $0 < b_1 - x_1$. Then $-(b_1 - x_1) < 0$. Since $x_2 - a_2 = b_2 - a_2$, we have $b_2 - x_2 = 0$. Then $-(b_1 - x_1) < b_2 - x_2 = 0$. Thus $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 4$.

CASE 4: Suppose $b_1 - a_1 < b_2 - a_2 = 0$. Choose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ such that $x_2 - a_2 = -(x_1 - a_1) < 0$. Notice then $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \in 2$. First, notice that since $-(x_1 - a_1) < 0$, we have $x_1 - a_1 \in \mathbb{Q}^+$. Since $x_2 - a_2 = -(x_1 - a_1)$, we have that $x_1 - a_1 = -(x_2 - a_2)$. Since $b_2 - a_2 = 0$, $b_2 - x_2 = b_2 - a_2 - (x_2 - a_2) = x_1 - a_1$. Notice that $b_1 - a_1 < 0$, so $0 < -(b_1 - a_1)$. Then $x_1 - a_1 < -(b_1 - a_1) + x_1 - a_1 = -(b_1 - x_1)$. Since $x_1 - a_1 = b_2 - x_2$ and $x_1 - a_1 \in \mathbb{Q}^+$, we have that $0 < b_2 - x_2 < -(b_1 - x_1)$. Thus $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 4$.

Since there are infinitely many possible choices of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ for each case, we have that a_{24}^p

for any $p \in S$. We then have that

$$a^2_{24} = a^2_{42} = a^3_{34} = a^3_{43} = \infty,$$
$$a^3_{24} = a^3_{42} = a^2_{34} = a^2_{43} = \infty,$$
$$a^4_{24} = a^4_{42} = a^4_{34} = a^4_{43} = \infty.$$

a^p_{44} FOR ANY $p \in S$: Let $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in X$ be arbitrary and choose $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in X$ such that $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in p$.

CASE 1: Suppose $0 \leq b_2 - a_2$. Choose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ such that $0 \leq x_2 - a_2 = b_2 - a_2 < -(x_1 - a_1)$ and $x_1 - a_1 < b_1 - a_1$. Since $0 \leq x_2 - a_2 < -(x_1 - a_1)$, we have $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \in 4$. Since $x_2 - a_2 = b_2 - a_2$, we have $b_2 - x_2 = 0$. Also, since $x_1 - a_1 < b_1 - a_1$, we have $x_1 - b_1 = -(b_1 - x_1) < 0$. Since $-(b_1 - x_1) < b_2 - a_2 = 0$, we have $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 4$.

CASE 2: Suppose $b_2 - a_2 < 0$. Choose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ such that $-(x_1 - a_1) < x_2 - a_2 = b_2 - a_2 \leq 0$ and $b_1 - a_1 < x_1 - a_1$. Notice then that $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \in 4$. Since $x_2 - a_2 = b_2 - a_2$, we have $b_2 - a_2 = 0$. Also, since $b_1 - a_1 < x_1 - a_1$, we have $0 < x_1 - b_1 = -(b_1 - x_1)$. Then, we have $0 = b_2 - x_2 < -(b_1 - x_1)$. Thus $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in 4$.

Since there are infinitely many possibilities for $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$ for each case, we have $a^p_{44} = \infty$ for all $p \in S$. We then have

$$a^1_{44} = a^2_{44} = a^3_{44} = a^4_{44} = \infty.$$

Notice that

$$a^1_{11} = a^1_{12} = a^1_{13} = a^1_{14} = 0,$$
$$a^1_{21} = a^1_{31} = a^1_{41} = 0,$$
$$a^2_{13} = a^2_{14} = a^2_{31} = a^2_{41},$$
$$a^3_{12} = a^3_{14} = a^3_{21} = a^3_{41},$$
$$a^4_{12} = a^4_{13} = a^4_{21} = a^4_{31},$$
$$a^2_{12} = a^2_{21} = 1,$$
$$a^3_{13} = a^3_{31} = 1,$$
$$a^4_{14} = a^4_{41} = 1.$$

These follow based on the definition of $1 \in S$. Also, since $2^* = 3$ and $4^* = 4$, we have that 1 is only contained in $2 \cdot 3$, $3 \cdot 2$, and $4 \cdot 4$. Thus

$$a^1_{22} = a^1_{33} = 0,$$
$$a^1_{24} - a^1_{42} = a^1_{34} = a^1_{43} = 0.$$

Thus, we have determined the values for each of the structural constants. We then have that our structure is a scheme realizing hypergroup 37. $\square$

7 Hypergroups of rank 4 not Realizable as Infinite Schemes

For the remaining hypergroups that cannot be realized by finite schemes or for which we do not have a constructed infinite scheme realizing the hypergroup, we ask the last possible question: can we prove that these hypergroups cannot be realized as infinite schemes? To prove such claims about hypergroups we can assume a scheme realizes the hypergroup, **construct a subset of the scheme**, and show that the hypermultiplication table of the scheme forces a pair of vertices to be in a relation inconsistent with the hypergroup the scheme realizes. In fact, we can use a brute force method using codes that test all possible subsets of n vertices that must exist within the scheme realizing the hypergroup. We present the algorithm for such a program that tests for subsets of 5 vertices.

Algorithm 2 Subroutines

```
function PRODUCTS(Relation,Hypmult,Maxrel,Vector)
    for x ← 2, Maxrel do
        for y ← 2, Maxrel do
            if Relation ∈ Hypmult_{x,y} then
                Vector ← append((x, y))
    return Vector

function INVERSEENTRY(Multtable,Vertices)
    for x ← 1, Vertices do
        for y ← 1, Vertices do
            if Multtable_{x,y} = 2 then
                Multtable_{y,x} = 3
            else if Multtable_{x,y} = 3 then
                Multtable_{y,x} = 2
            else if Multtable_{x,y} = 4 then
                Multtable_{y,x} = 4
            return Multtable
```

Algorithm 3 All Subsets of 5 vertices

```
function PENTAS(Hypmult)
    Multtable ← I_5
    Maxrel ← max(Hypmult)
    for i ← 2, Maxrel do
        Multtable_{1,2} ← i
        ab ← vector()
        Multtable ← PRODUCTS(i, Hypmult, Maxrel, ab)
        for j ← 1, length(ab) do
            Multtable_{1,3} ← ab_{j_1}
            Multtable_{3,2} ← ab_{j_2}
            for h ← 2, Maxrel do
                Multtable_{1,5} ← j
                ae ← vector()
                Multtable ← PRODUCTS(h, Hypmult, Maxrel, ae)
                for k ← 1, length(ae) do
                    Multtable_{1,4} ← ae_{k_1}
                    Multtable_{4,5} ← ae_{k_2}
                    Multtable ← INVERSEENTRY(Multtable,5)
                    cd ← Hypmult_{Multtable_{3,1}, Multtable_{1,4}}
                    for l ← 1, length(cd) do
                        Multtable_{3,4} ← cd_l
                        bd1 ← Hypmult_{Multtable_{2,3}, Multtable_{3,4}}
                        bd2 ← Hypmult_{Multtable_{2,1}, Multtable_{2,4}}
                        bd ← bd1 ∩ bd2
                        checking ← vector()
                        for m ← 1, length(bd) do
                            Multtable_{2,4} ← bd_m
                            ce1 ← Hypmult_{Multtable_{3,4}, Multtable_{4,5}}
                            ce2 ← Hypmult_{Multtable_{3,1}, Multtable_{1,5}}
                            ce ← ce1 ∩ ce2
                            for n ← 1, length(ce) do
                                Multtable_{3,5} ← ce_n
                                be1 ← Hypmult_{Multtable_{2,1}, Multtable_{1,5}}
                                be2 ← Hypmult_{Multtable_{2,3}, Multtable_{3,5}}
                                be3 ← Hypmult_{Multtable_{2,4}, Multtable_{4,5}}
                                be ← be1 ∩ be2 ∩ be3
                                if ce = {∅} then
                                    checking← append(C)
                                if m =length(bd), n =length(ce) then
                                    if length(checking)=(length(bd)·length(ce)) then
                                        Multtable_{2,5} ← Null
                                        Multtable_{2,5} ← Null
                                        return Multtable
    print "No Contradictions Found"
```

We describe the algorithm with simpler notation for those not familiar with coding. We will refer to each vertex as x_i for some $i \in \{1, 2, \cdots, 5\}$. This algorithm first creates a 5×5 identity matrix to represent the relation matrix where the i, j entry is the relation containing (x_i, x_j). Since if $i = j$, $(x_i, x_j) \in 1$, there are 1s on the main diagonal. Next, the algorithm loops from relation 2 to relation 4 and assigns the relation to the pair (x_1, x_2). Then, it finds which products of relations contain the relation that contains (x_1, x_2). The algorithm then loops through all the possible products and assigns the first of the two relations to (x_1, x_3) and (x_3, x_2) accordingly. Within the overarching loop for the relation containing (x_1, x_2), the algorithm then repeats the same procedure with x_1, x_4, x_5. Then for all (x_i, x_j), it assigns the inverse of the relation that contains the pair to (x_j, x_i). The algorithm then determines the relation that contains (x_3, x_1) and the relation that contains (x_1, x_4) and takes the relations' product. The algorithm loops through all the non-identity relations, assigning the relation to the pair (x_3, x_4).

Then the algorithm finds the intersection between the product of relations containing (x_2, x_3), (x_3, x_4) and (x_2, x_1), (x_1, x_4). The set of the possible relations in the intersection is then looped, assigning the relation to (x_2, x_4). The algorithm uses the same process to loop through the possible relations in the intersection between the product of relations containing (x_3, x_4), (x_4, x_5) and (x_3, x_1), (x_1, x_5) and assigns the relation to (x_3, x_5). Lastly, the algorithm checks if the intersection between the product of relations containing (x_2, x_1), (x_1, x_5), and (x_2, x_3), (x_3, x_5), and (x_2, x_4), (x_4, x_5) contains any relations. If the intersection set is empty, then there is a contradiction.

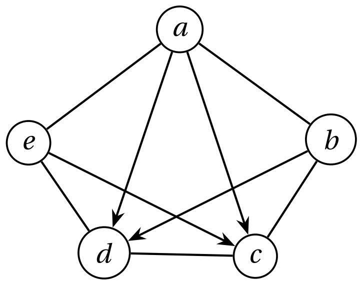
In essence, if the algorithm finds that a pair of vertices cannot be contained in any relation, we have a contradiction. We can then use this algorithm to find hypergroups that cannot be realized as infinite association schemes. We find nine hypergroups in this category. We present the proofs of each hypergroup next.

Hypergroup 9 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	p	$1, p, p^*, r$	p, r
p^*	p^*	$1, p, p^*, r$	p^*	p^*, r
r	r	p, r	p^*, r	$1, p, p^*$

Proposition 37. *Hypergroup 9 cannot be realized as an infinite association scheme.*

Proof. Let (X, S) be a infinite scheme realizing hypergroup 9. Let $a \in X$ be arbitrary and choose $c \in X$ such that $(a, c) \in p$. Since $p \in pr$, there exists $d \in X$ such that $(a, d) \in p$ and $(d, c) \in r$. Also, since $p \in rr$, there exist $b, e \in X$ such that $(a, b) \in r$, $(b, c) \in r$ and $(a, e) \in r$, $(e, d) \in r$. Notice $(b, c) \in r$, $(c, d) \in r$ and $(b, a) \in r$, $(a, d) \in p$. Since $rr \cap rp = \{p\}$, $(b, d) \in p$. Similarly, $(e, d) \in r$, $(d, c) \in r$ and $(e, a) \in r$, $(a, c) \in p$. Since $rr \cap rp = \{p\}$, $(e, c) \in p$. However, notice that $(e, a) \in r$, $(a, b) \in r$ and $(e, d) \in r$, $(d, b) \in p^*$, and $(e, c) \in p$, $(c, b) \in r$. Since $rr \cap rp^* \cap pr = \{\emptyset\}$, this contradicts the association scheme being a partition of X.



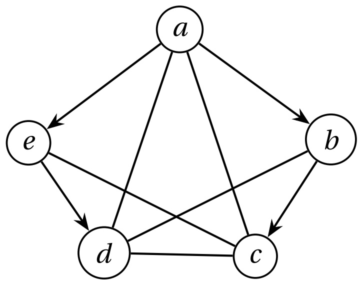
In the diagram above, directed edges represent p and solid edges represent r. Thus hypergroup 9 cannot be realized as an infinite scheme. $\square$

Hypergroup 12 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	p, r	$1, p, p^*$	p^*, r
p^*	p^*	$1, p, p^*$	p^*, r	p, r
r	r	p^*, r	p, r	$1, p, p^*, r$

Proposition 38. *Hypergroup 12 cannot be realized as an infinite association scheme.*

Proof. Let (X, S) be a infinite scheme realizing hypergroup 12. Let $a \in X$ be arbitrary and choose $c \in X$ such that $(a, c) \in r$. Since $r \in rr$, there exists $d \in X$ such that $(a, d) \in r$ and $(d, c) \in r$. Also, since $r \in pp$, there exist $b, e \in X$ such that $(a, b) \in p$, $(b, c) \in p$ and $(a, e) \in p$, $(e, d) \in p$. Notice $(b, c) \in p$, $(c, d) \in r$ and $(b, a) \in p^*$, $(a, d) \in r$. Since $pr \cap p^* r = \{r\}$, $(b, d) \in r$. Similarly, $(e, d) \in p$, $(d, c) \in r$ and $(e, a) \in p^*$, $(a, c) \in r$. Since $pr \cap p^* r = \{r\}$, $(e, c) \in r$. However, notice that $(e, a) \in p^*$, $(a, b) \in p$ and $(e, d) \in p$, $(d, b) \in r$, and $(e, c) \in r$, $(c, b) \in p^*$. Since $p^* p \cap pr \cap rp^* = \{\emptyset\}$, this contradicts the association scheme partitioning X.



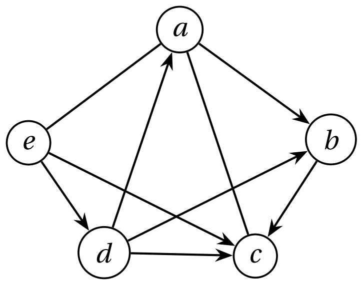
In the diagram above, directed edges represent p and solid edges represent r. Thus hypergroup 12 cannot be realized as an infinite scheme. $\square$

Hypergroup 13 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	p, r	$1, p, p^*$	p, p^*, r
p^*	p^*	$1, p, p^*, r$	p^*, r	p, r
r	r	p^*, r	p, p^*, r	$1, p, p^*$

Proposition 39. *Hypergroup 13 cannot be realized as an infinite association scheme.*

Proof. Let (X,S) be a infinite scheme realizing hypergroup 13. Let $a \in X$ be arbitrary and choose $c \in X$ such that $(a,c) \in r$. Since $r \in p^*p$, there exists $d \in X$ such that $(a,d) \in p^*$ and $(d,c) \in p$. Also, since $r \in pp$ and $p^* \in rp$, there exist $b,e \in X$ such that $(a,b) \in p$, $(b,c) \in p$ and $(a,e) \in r$, $(e,d) \in p$. Notice $(b,c) \in p$, $(c,d) \in p^*$ and $(b,a) \in p^*$, $(a,d) \in p^*$. Since $pp^* \cap p^*p^* = \{p^*\}$, $(b,d) \in p^*$. Similarly, $(e,d) \in p$, $(d,c) \in p$ and $(e,a) \in r$, $(a,c) \in r$. Since $pp \cap rr = \{p\}$, $(e,c) \in p$. However, notice that $(e,a) \in r$, $(a,b) \in p$ and $(e,d) \in p$, $(d,b) \in p$, and $(e,c) \in p$, $(c,b) \in p^*$. Since $rp \cap pp \cap pp^* = \{\emptyset\}$, this contradicts the association scheme partitioning X.



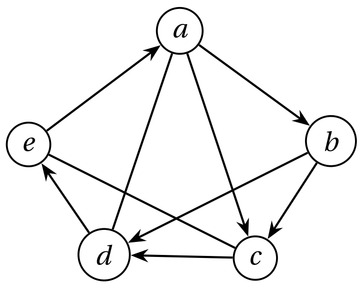
In the diagram above, directed edges represent p and solid edges represent r. Thus hypergroup 13 cannot be realized as an infinite scheme. □

Hypergroup 15 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	p,r	$1,p,p^*,r$	p,r
p^*	p^*	$1,p,p^*,r$	p^*,r	p,r
r	r	p,r	p,r	$1,r$

Proposition 40. *Hypergroup 15 cannot be realized as an infinite association scheme.*

Proof. Let (X,S) be a infinite scheme realizing hypergroup 15. Let $a \in X$ be arbitrary and choose $c \in X$ such that $(a,c) \in p$. Since $r \in rp^*$, there exists $d \in X$ such that $(a,d) \in r$ and $(d,c) \in p^*$. Also, since $p \in pp$ and $r \in p^*p^*$, there exist $b,e \in X$ such that $(a,b) \in p$, $(b,c) \in p$ and $(a,e) \in p^*$, $(e,d) \in p^*$. Notice $(b,c) \in p$, $(c,d) \in p$ and $(b,a) \in p^*$, $(a,d) \in r$. Since $pp \cap p^*r = \{p\}$, $(b,d) \in p$. Similarly, $(e,d) \in p^*$, $(d,c) \in p^*$ and $(e,a) \in p$, $(a,c) \in p$. Since $p^*p^* \cap pp = \{r\}$, $(e,c) \in r$. However, notice that $(e,a) \in p$, $(a,b) \in p$ and $(e,d) \in r$, $(d,b) \in p^*$, and $(e,c) \in p^*$, $(c,b) \in p^*$. Since $pp \cap rp^* \cap p^*p^* = \{\emptyset\}$, this contradicts the association scheme partitioning X.



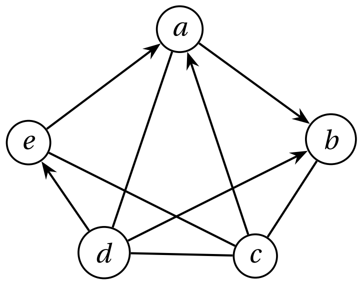
In the diagram above, directed edges represent p and solid edges represent r. Thus hypergroup 15 cannot be realized as an infinite scheme. □

Hypergroup 22 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	p^*, r	$1, r$	p, p^*, r
p^*	p^*	$1, r$	p, r	p, p^*, r
r	r	p, p^*, r	p, p^*, r	$1, p, p^*$

Proposition 41. *Hypergroup 22 cannot be realized as an infinite scheme.*

Proof. Let (X, S) be a infinite scheme realizing hypergroup 22. Let $a \in X$ be arbitrary and choose $c \in X$ such that $(a, c) \in p^*$. Since $p^* \in pr$, there exists $d \in X$ such that $(a, d) \in p$ and $(d, c) \in r$. Also, since $p^* \in pr$ and $r \in p^* p^*$, there exist $b, e \in X$ such that $(a, b) \in p$, $(b, c) \in r$ and $(a, e) \in p^*$, $(e, d) \in p^*$. Notice $(b, c) \in r$, $(c, d) \in r$ and $(b, a) \in p^*$, $(a, d) \in r$. Since $rr \cap p^* r = \{p^*\}$, $(b, d) \in p^*$. Similarly, $(e, d) \in p^*$, $(d, c) \in r$ and $(e, a) \in p$, $(a, c) \in p^*$. Since $p^* r \cap pp^* = \{r\}$, $(e, c) \in r$. However, notice that $(e, a) \in p$, $(a, b) \in p$ and $(e, d) \in p^*$, $(d, b) \in p$, and $(e, c) \in r$, $(c, b) \in r$. Since $pp \cap pp^* \cap rr = \{\emptyset\}$, this contradicts the association scheme partitioning X.



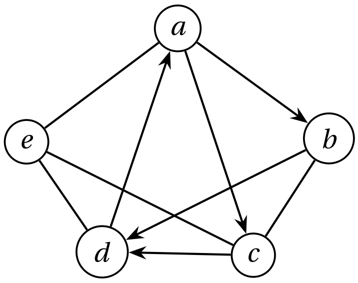
In the diagram above, directed edges represent p and solid edges represent r. Thus hypergroup 22 cannot be realized as an infinite scheme. □

Hypergroup 28 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	p, p^*	$1, p, p^*, r$	p, r
p^*	p^*	$1, p, p^*, r$	p, p^*	p^*, r
r	r	p, r	p^*, r	$1, p, p^*$

Proposition 42. *Hypergroup 28 cannot be realized by an infinite association scheme.*

Proof. Let (X, S) be a infinite scheme realizing hypergroup 28. Let $a \in X$ be arbitrary and choose $c \in X$ such that $(a, c) \in p$. Since $p \in p^* p^*$, there exists $d \in X$ such that $(a, d) \in p^*$ and $(d, c) \in p^*$. Also, since $p \in pr$ and $p^* \in rr$, there exist $b, e \in X$ such that $(a, b) \in p$, $(b, c) \in r$ and $(a, e) \in r$, $(e, d) \in r$. Notice $(b, c) \in r$, $(c, d) \in p$ and $(b, a) \in p^*$, $(a, d) \in p^*$. Since $rp \cap p^* p^* = \{p\}$, $(b, d) \in p$. Similarly, $(e, d) \in r$, $(d, c) \in p^*$ and $(e, a) \in r$, $(a, c) \in p$. Since $rp^* \cap rp = \{r\}$, $(e, c) \in r$. However, notice that $(e, a) \in r$, $(a, b) \in p$ and $(e, d) \in r$, $(d, b) \in p^*$, and $(e, c) \in r$, $(c, b) \in r$. Since $rp \cap rp^* \cap rr = \{\emptyset\}$, this contradicts the association scheme partitioning X.



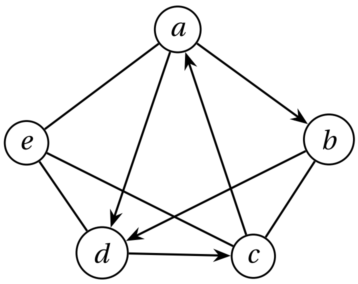
In the diagram above, directed edges represent p and solid edges represent r. Thus hypergroup 28 cannot be realized as an infinite scheme. $\square$

Hypergroup 31 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	p, p^*, r	$1, p, p^*$	p^*, r
p^*	p^*	$1, p, p^*$	p, p^*, r	p, r
r	r	p^*, r	p, r	$1, p, p^*$

Proposition 43. *Hypergroup 31 cannot be realized as an infinite scheme.*

Proof. Let (X, S) be a infinite scheme realizing hypergroup 31. Let $a \in X$ be arbitrary and choose $c \in X$ such that $(a, c) \in p^*$. Since $p^* \in pp$, there exists $d \in X$ such that $(a, d) \in p$ and $(d, c) \in p$. Also, since $p^* \in pr$ and $p \in rr$, there exist $b, e \in X$ such that $(a, b) \in p$, $(b, c) \in r$ and $(a, e) \in r$, $(e, d) \in r$. Notice $(b, c) \in r$, $(c, d) \in p^*$ and $(b, a) \in p^*$, $(a, d) \in p$. Since $rp^* \cap p^*p = \{p\}$, $(b, d) \in p$. Similarly, $(e, d) \in r$, $(d, c) \in p$ and $(e, a) \in r$, $(a, c) \in p^*$. Since $rp \cap rp^* = \{r\}$, $(e, c) \in r$. However, notice that $(e, a) \in r$, $(a, b) \in p$ and $(e, d) \in r$, $(d, b) \in p^*$, and $(e, c) \in r$, $(c, b) \in r$. Since $rp \cap rp^* \cap rr = \{\emptyset\}$, this contradicts the association scheme partitioning X.



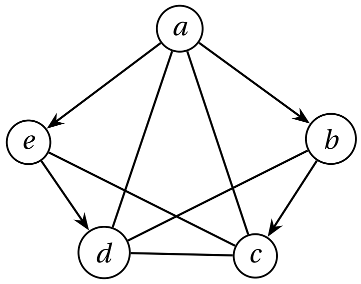
In the diagram above, directed edges represent p and solid edges represent r. Thus hypergroup 31 cannot be realized as an infinite scheme. $\square$

Hypergroup 32 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	p, p^*, r	$1, p, p^*$	p^*, r
p^*	p^*	$1, p, p^*$	p, p^*, r	p, r
r	r	p^*, r	p, r	$1, p, p^*, r$

Proposition 44. *Hypergroup 32 cannot be realized as an infinite association scheme.*

Proof. Let (X,S) be a infinite scheme realizing hypergroup 32. Let $a \in X$ be arbitrary and choose $c \in X$ such that $(a,c) \in r$. Since $r \in rr$, there exists $d \in X$ such that $(a,d) \in r$ and $(d,c) \in r$. Also, since $r \in pp$, there exist $b,e \in X$ such that $(a,b) \in p$, $(b,c) \in p$ and $(a,e) \in p$, $(e,d) \in p$. Notice $(b,c) \in p$, $(c,d) \in r$ and $(b,a) \in p^*$, $(a,d) \in r$. Since $pr \cap p^*r = \{r\}$, $(b,d) \in r$. Similarly, $(e,d) \in p$, $(d,c) \in r$ and $(e,a) \in p^*$, $(a,c) \in r$. Since $pr \cap p^*r = \{r\}$, $(e,c) \in r$. However, notice that $(e,a) \in p^*$, $(a,b) \in p$ and $(e,d) \in p$, $(d,b) \in r$, and $(e,c) \in r$, $(c,b) \in p^*$. Since $p^*p \cap pr \cap rp^* = \{\emptyset\}$, this contradicts the association scheme partitioning X.



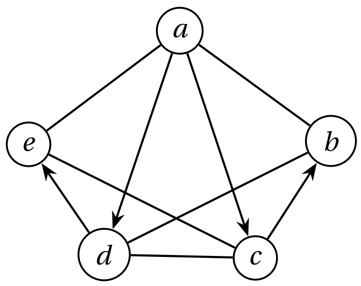
In the diagram above, directed edges represent p and solid edges represent r. Thus hypergroup 32 cannot be realized as an infinite scheme. □

Hypergroup 33 has the following hypermultiplication table

	1	p	p^*	r
1	1	p	p^*	r
p	p	p,p^*,r	$1,p,p^*$	p,p^*,r
p^*	p^*	$1,p,p^*,r$	p,p^*,r	p,r
r	r	p^*,r	p,p^*,r	$1,p,p^*$

Proposition 45. *Hypergroup 33 cannot be realized as an infinite association scheme.*

Proof. Let (X,S) be a infinite scheme realizing hypergroup 33. Let $a \in X$ be arbitrary and choose $c \in X$ such that $(a,c) \in p$. Since $p \in pr$, there exists $d \in X$ such that $(a,d) \in p$ and $(d,c) \in r$. Also, since $p \in rp^*$, there exist $b,e \in X$ such that $(a,b) \in r$, $(b,c) \in p^*$ and $(a,e) \in r$, $(e,d) \in p^*$. Notice $(b,c) \in p^*$, $(c,d) \in r$ and $(b,a) \in r$, $(a,d) \in p$. Since $p^*r \cap rp = \{r\}$, $(b,d) \in r$. Similarly, $(e,d) \in p^*$, $(d,c) \in r$ and $(e,a) \in r$, $(a,c) \in p$. Since $p^*r \cap rp = \{r\}$, $(e,c) \in r$. However, notice that $(e,a) \in r$, $(a,b) \in r$ and $(e,d) \in p^*$, $(d,b) \in r$, and $(e,c) \in r$, $(c,b) \in p$. Since $rr \cap p^*r \cap rp = \{\emptyset\}$, this contradicts the association scheme partitioning X.



In the diagram above, directed edges represent p and solid edges represent r. Thus hypergroup 33 cannot be realized as an infinite scheme. □

Each of the preceding proofs are almost identical in structure using a pentagon to show that a pair of vertices must be in a relation that is not consistent with the hypermultiplication table. We can also use higher degree polygons to compute a contradiction in much the same way. Notice, however, that increasing the degree of the polygon would increase the computational time exponentially. As a result, using such a method would require theory to reduce the number of possible configurations of the n-gon first and then a brute force method would be feasible. Notice, however, that these arguments do not use the assumption that the schemes realizing the hypergroups are infinite. Thus, each of the preceding hypergroups with proofs that they cannot be realized as infinite association schemes cannot be realized as finite association schemes either.

We would like to emphasize that not all proofs showing that a hypergroup is not realizable as an infinite scheme must follow this brute force method. One way to prove these claims about hypergroups more elegantly is to find repeating sequences of relations to show that two or more of such sequences force pairs of vertices to be in a relation that leads to a contradiction. While we do not present such a proof here, Professor C. French has used such a method for one of the hypergroups.

8 Concluding Remarks

Notice that over the last several sections we have asked all the questions we had sought to understand about the nonsymmetric hypergroups of rank 4: can the hypergroups be realized by finite association schemes? If not, can we prove that they cannot be realized as finite schemes? Can the hypergroups be realized by infinite schemes? If not, can prove that they cannot be realized as infinite schemes? We have that there are at least 11 hypergroups that can be realized as finite schemes, 20 that cannot be realized as finite schemes, 11 that can be realized as infinite schemes, and 10 that cannot be realized as infinite association schemes.

However, due to the lack of time, we were not able to answer each question about every hypergroup–there are still six nonsymmetric hypergroups of rank 4 that do not have all these answers. On the other hand, there are only six such hypergroups because our collaborator Bingyue He '18 has shown results for many of these hypergroups. While we do not present all the proofs and ideas that Bingyue developed, we use a table to organize all the information we have about the nonsymmetric hypergroups of order 4.

Index	Hyperproduct	Finite	Not Finite	Infinite	Not Infinite
1	$\begin{matrix} 4 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 1 \end{matrix}$	Thin			
2	$\begin{matrix} 4 & 1,4 & 2,3 \\ 1,4 & 4 & 2,3 \\ 2,3 & 2,3 & 1,4 \end{matrix}$		✗	✗	
3	$\begin{matrix} 4 & 1,4 & 2,3,4 \\ 1,4 & 4 & 2,3,4 \\ 2,3,4 & 2,3,4 & 1,2,3,4 \end{matrix}$		✗	✗	
4	$\begin{matrix} 2 & 1,2,3 & 4 \\ 1,2,3 & 3 & 4 \\ 4 & 4 & 1,2,3 \end{matrix}$		✗	✗	
5	$\begin{matrix} 2 & 1,2,3 & 4 \\ 1,2,3 & 3 & 4 \\ 4 & 4 & 1,2,3,4 \end{matrix}$		✗	✗	
6	$\begin{matrix} 2 & 1,2,3 & 2,4 \\ 1,2,3,4 & 3 & 4 \\ 4 & 3,4 & 1,2,3,4 \end{matrix}$		✗	✗	
7	$\begin{matrix} 2 & 1,2,3,4 & 2 \\ 1,2,3,4 & 3 & 3 \\ 2 & 3 & 1 \end{matrix}$		✗	✗	
8	$\begin{matrix} 2 & 1,2,3,4 & 2 \\ 1,2,3,4 & 3 & 3 \\ 2 & 3 & 1,4 \end{matrix}$		✗	✗	
9	$\begin{matrix} 2 & 1,2,3,4 & 2,4 \\ 1,2,3,4 & 3 & 3,4 \\ 2,4 & 3,4 & 1,2,3 \end{matrix}$		✗		✗
10	$\begin{matrix} 2 & 1,2,3,4 & 2,4 \\ 1,2,3,4 & 3 & 3,4 \\ 2,4 & 3,4 & 1,2,3,4 \end{matrix}$		✗	✗	

Index	Hyperproduct	Finite	Not Finite	Infinite	Not Infinite
11	$\begin{matrix} 2,4 & 1,2,3 & 3,4 \\ 1,2,3 & 3,4 & 2,4 \\ 3,4 & 2,4 & 1,2,3 \end{matrix}$		✗	✗	
12	$\begin{matrix} 2,4 & 1,2,3 & 3,4 \\ 1,2,3 & 3,4 & 2,4 \\ 3,4 & 2,4 & 1,2,3,4 \end{matrix}$		✗		✗
13	$\begin{matrix} 2,4 & 1,2,3 & 2,3,4 \\ 1,2,3,4 & 3,4 & 2,4 \\ 3,4 & 2,3,4 & 1,2,3 \end{matrix}$		✗		✗
14	$\begin{matrix} 2,4 & 1,2,3 & 2,3,4 \\ 1,2,3,4 & 3,4 & 2,4 \\ 3,4 & 2,3,4 & 1,2,3,4 \end{matrix}$		✗		✗
15	$\begin{matrix} 2,4 & 1,2,3,4 & 2,3 \\ 1,2,3,4 & 3,4 & 2,3 \\ 2,3 & 2,3 & 1,4 \end{matrix}$		✗		✗
16	$\begin{matrix} 2,4 & 1,2,3,4 & 2,3,4 \\ 1,2,3,4 & 3,4 & 2,3,4 \\ 2,3,4 & 2,3,4 & 1,2,3 \end{matrix}$				
17	$\begin{matrix} 2,4 & 1,2,3,4 & 2,3,4 \\ 1,2,3,4 & 3,4 & 2,3,4 \\ 2,3,4 & 2,3,4 & 1,2,3,4 \end{matrix}$				
18	$\begin{matrix} 3 & 1 & 4 \\ 1 & 2 & 4 \\ 4 & 4 & 1,2,3 \end{matrix}$	MH:6.4			
19	$\begin{matrix} 3 & 1 & 4 \\ 1 & 2 & 4 \\ 4 & 4 & 1,2,3,4 \end{matrix}$	MH:9.4,12.6,15.6, 18.6,21.5,24.8, 27.379,30.8,33.4			
20	$\begin{matrix} 3 & 1,4 & 2 \\ 1,4 & 2 & 3 \\ 2 & 3 & 1 \end{matrix}$	MH:6.6			
21	$\begin{matrix} 3 & 1,4 & 2 \\ 1,4 & 2 & 3 \\ 2 & 3 & 1,4 \end{matrix}$	MH:9.6,12.13,15.11, 18.14,21.10,24.23, 27.387,30.31,33.8			
22	$\begin{matrix} 3,4 & 1,4 & 2,3,4 \\ 1,4 & 2,4 & 2,3,4 \\ 2,3,4 & 2,3,4 & 1,2,3 \end{matrix}$		✗		✗
23	$\begin{matrix} 3,4 & 1,4 & 2,3,4 \\ 1,4 & 2,4 & 2,3,4 \\ 2,3,4 & 2,3,4 & 1,2,3,4 \end{matrix}$				
24	$\begin{matrix} 2,3 & 1,2,3 & 4 \\ 1,2,3 & 2,3 & 4 \\ 4 & 4 & 1,2,3 \end{matrix}$	MH:14.6,22.6, 30.25,30.26,38.7, 38.8,38.9			
25	$\begin{matrix} 2,3 & 1,2,3 & 4 \\ 1,2,3 & 2,3 & 4 \\ 4 & 4 & 1,2,3,4 \end{matrix}$	MH:21.8,28.14, 33.7			

Index	Hyperproduct	Finite	Not Finite	Infinite	Not Infinite
26	2,3 1,2,3,4 2 1,2,3,4 2,3 3 2 3 1	MH:14.5,22.5, 30.13,38.5,38.6			
27	2,3 1,2,3,4 2 1,2,3,4 2,3 3 2 3 1,4	MH:21.7,28.72, 33.6			
28	2,3 1,2,3,4 2,4 1,2,3,4 2,3 3,4 2,4 3,4 1,2,3		✗		✗
29	2,3 1,2,3,4 2,4 1,2,3,4 2,3 3,4 2,4 3,4 1,2,3,4				
30	2,3,4 1,2,3 3 1,2,3 2,3,4 2 3 2 1	MH:8.6,16.11, 24.14,32.14			
31	2,3,4 1,2,3 3,4 1,2,3 2,3,4 2,4 3,4 2,4 1,2,3		✗		✗
32	2,3,4 1,2,3 3,4 1,2,3 2,3,4 2,4 3,4 2,4 1,2,3,4		✗		✗
33	2,3,4 1,2,3 2,3,4 1,2,3,4 2,3,4 2,4 3,4 2,3,4 1,2,3		✗		✗
34	2,3,4 1,2,3 2,3,4 1,2,3,4 2,3,4 2,4 3,4 2,3,4 1,2,3,4		✗	✗	
35	2,3,4 1,2,3,4 2,3 1,2,3,4 2,3,4 2,3 2,3 2,3 1,4	MH:16.9			
36	2,3,4 1,2,3,4 2,3,4 1,2,3,4 2,3,4 2,3,4 2,3,4 2,3,4 1,2,3				
37	2,3,4 1,2,3,4 2,3,4 1,2,3,4 2,3,4 2,3,4 2,3,4 2,3,4 1,2,3,4			✗	

9 Acknowledgements

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