11 Oct. 2018

Review

- A real, $A = A^T \Rightarrow \lambda_i(A)$ real, $v^i \neq 0$ $Av^i = \lambda_i v^i$, $1 \le i \le n$

$\{\lambda_1, \dots, \lambda_n\}$ distinct $\Rightarrow \langle v^i, v^j \rangle = (v^i)^T v^j = 0 \quad i \neq j$.

HW6 treats the general case of repeated e-values.

Thm $A = A^T$ and real $\Rightarrow \exists\, Q$ orthogonal $[Q^T Q = I]$ such that $Q^T A Q = \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

$\therefore A = Q \Lambda Q^T$

- $x^T M x = x^T \underbrace{\left[\dfrac{M + M^T}{2}\right]}_{\text{"Symmetric Part of } M\text{"}} x$

- **Def.** $P = P^T$ is positive definite if $\forall x \in \mathbb{R}^n$, $x \neq 0 \Rightarrow x^T P x > 0$

HW#2 Prob 7: If $P = P^T$, then $\forall x \in \mathbb{R}^n$,

$$\lambda_{min}(P)\, x^T x \le x^T P x \le \lambda_{max}(P)\, x^T x$$

Thm P is pos. def $\iff \lambda_{min}(P) > 0$

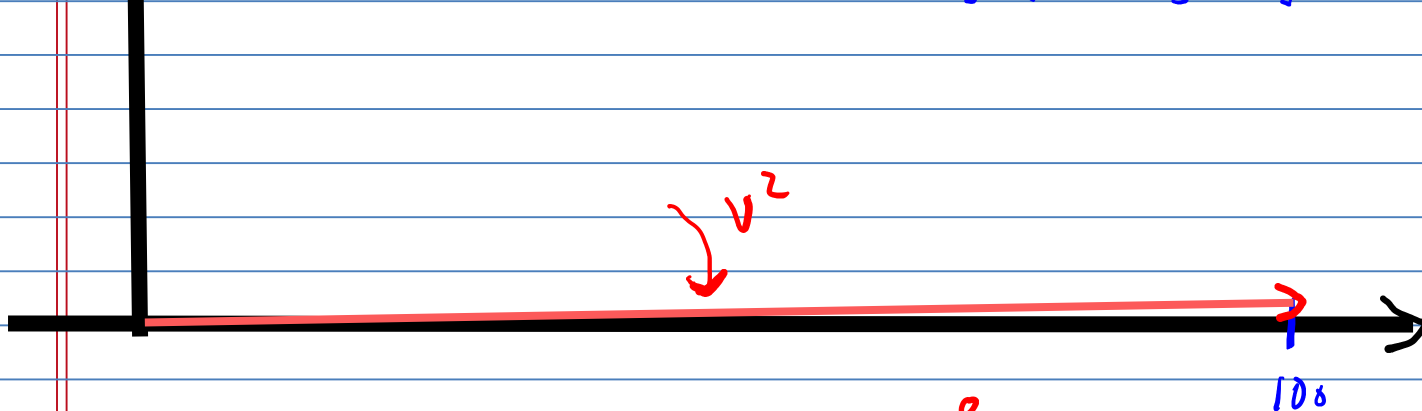
$$A = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix} \Rightarrow A^T A = \begin{bmatrix} 1 & 100 \\ 100 & 10{,}001 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -0.01 & -1 \\ -1 & 0.01 \end{bmatrix} \begin{bmatrix} 10{,}002 & 0 \\ 0 & -0.00009998 \end{bmatrix} \begin{bmatrix} -0.01 & -1 \\ -1 & 0.01 \end{bmatrix}$$

$$\frac{\lambda_{max}(A^T A)}{\lambda_{min}(A^T A)} \approx 10^8$$

is really $-\sqrt{(1-10^{-4})} = -0.999?$

$$v^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 100 \\ 1 \end{bmatrix}$$



angle of v^2 w.r.t. v^1 $\approx 0.57°$

Later: Check the "numerical" rank of a matrix and the "numerical linear independence" of sets of vectors.

Today

Symbol for pos. def.

Also see $\succ$

Important

Exercise: Show

$$P = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} > 0$$

Check $P = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ is not pos. def.

Definition: $P = P^\top$ is positive semidefinite if $x^\top P x \geq 0$ for all $x \neq 0$.

Theorem: P is positive semidefinite if and only if all eigenvalues of P are non-negative. (Notation: $P \geq 0$ or $P \succcurlyeq 0$.)

Definition: N is a square root of a symmetric matrix P if $N^\top N = P$.

Note: $N^\top N = \left(N^\top N\right)^\top \Rightarrow N^\top N$ is always symmetric.

Interesting

Theorem: $P \geq 0 \Leftrightarrow \exists N$ such that $N^\top N = P$.

Proof:

1. Suppose $N^\top N = P$, and let $x \in \mathbb{R}^n$.

$$x^\top P x = x^\top N^\top N x = (Nx)^\top (Nx) = \|Nx\|^2 \geq 0.$$

2. Now suppose $P \geq 0$. To show $\exists N$ such that $N^\top N = P$.

   Since P is symmetric, there exists an orthogonal matrix O such that

$$P = O^\top \Lambda O$$

   where $\Lambda = \text{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_n)$.
   Since $P \geq 0$, $\lambda_i \geq 0$ for all $i = 1, 2, \ldots, n$.
   Define $\Lambda^{1/2} := diag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n})$,

$$\Lambda = (\Lambda^{1/2})^\top \Lambda^{1/2} = \Lambda^{1/2}\Lambda^{1/2}.$$

   Let $N = \Lambda^{1/2} O$, then

$$N^\top N = O^\top \left(\Lambda^{1/2}\right)^\top \Lambda^{1/2} O = O^\top \Lambda O = P.$$

$$\therefore N^\top N = P. \; \square$$

Schur Complement Thm

- Means for checking if a matrix is positive definite
- It shows up when we study conditional Gaussian Random Vectors !!!

<u>Thm</u> Suppose $A = n \times n$, $B = n \times m$, and C is $m \times m$ and that

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \text{ is symmetric.}$$

$[A = A^T,\ C = C^T]$ Then TFAE:

a) $M > 0$

b) $A > 0$ and $\underbrace{C - B^T A^{-1} B}_{\text{Schur complement of } A \text{ in } M} > 0$

c) $C > 0$ and $\underbrace{A - B C^{-1} B^T}_{\text{Schur complement of } C \text{ in } M} > 0$

Proof Will show (a) $\Leftrightarrow$ (b) and skip (a) $\Leftrightarrow$ (c) because the proof is nearly identical.

a) $\Rightarrow$ b) $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0$. Hence, for all $\begin{bmatrix} x \\ y \end{bmatrix} \neq 0$ $\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} > 0.$

Claim 1 Let $x \neq 0$ be otherwise arbitrary and set $y = 0$. Then

$$0 < \begin{bmatrix} x \\ 0 \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}^T \begin{bmatrix} Ax \\ B^T x \end{bmatrix} =$$

$$= x^T A x \quad \therefore \quad A > 0.$$

Claim 2 $C - B^T A^{-1} B > 0$

Pf. We let $y \neq 0$ be otherwise arbitrary and we make a clever choice of x

$$Ax + By = 0$$

$$\therefore \; x = -A^{-1}By \qquad \text{(can do because } A > 0\text{)}$$

Because $y \neq 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -A^{-1}By \\ y \end{bmatrix} \neq 0$

$$\therefore\ 0 < \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 0 \\ B^T x + Cy \end{bmatrix}$$

$$= y^T B^T x + y^T C y$$

$$= y^T \left[-B^T A^{-1} B y + C y \right]$$

$$= y^T \left[C - B^T A^{-1} B \right] y$$

$$\therefore\ (C - B^T A^{-1} B) > 0$$

$\square$

(b) $\Rightarrow$ (a) $\quad A > 0$ and $(C - B^T A^{-1} B) > 0$

$$\Longrightarrow M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0.$$

Claim 3 $\quad (v^1 + v^2)^T M (v^1 + v^2) =$

$$= (v^1)^T M v^1 + 2 (v^1)^T M v^2 + (v^2)^T M v^2$$

$\boxed{\text{Recall } \langle x+y, x+y \rangle = \langle x, x \rangle + 2 \langle x, y \rangle + \langle y, y \rangle}$

Pf. Exercise.

Claim 4 Define $\tilde{x} = x + A^{-1}By$.

Then $\begin{bmatrix} \tilde{x} \\ y \end{bmatrix} \neq 0 \iff \begin{bmatrix} x \\ y \end{bmatrix} \neq 0$.

Pf. If $y \neq 0$, then the result is obvious.
If $y = 0$, then $\tilde{x} = x$ and the result is also obvious.

Claim 5

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \tilde{x}^T \underbrace{A}_{>0} \tilde{x} + y^T \underbrace{(C - B^T A^{-1} B)}_{>0} y$$

Pf. Multiply out LHS = RHS.
Too boring for a ROB 501 lecture.

$\square$

Examples

a) $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}_{2\times 2} > 0 \iff$ $a>0$ & $c - b(a)^{-1}b > 0$

$\iff a>0$ & $ac - b^2 > 0$

$\iff a>0$ & $\det(M) > 0$.

$\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} > 0$? $\qquad \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} > 0$?

b) $M = \left[\begin{array}{c:cc} 2 & 1 & 1 \\ \hdashline 1 & 2 & 1 \\ 1 & 1 & 3 \end{array}\right]$ $\quad A=2 \quad B=\begin{bmatrix} 1 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

$M>0$? We check $A=2>0$ ✓ and

we check $C - B^T A^{-1} B =$

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 5/2 \end{bmatrix} \succ 0 \quad \checkmark$$

$\therefore M \succ 0.$

Rob 501 Handout: Grizzle
Weighted Least Squares

Let M be an $n \times n$ positive definite matrix ($M \succ 0$) We revisit the over determined system of equations,

$$A\alpha = b,$$

where $A = n \times m, n \geq m, \text{r}ank(A) = m, \alpha \in \mathbb{R}^m, \text{and } b \in \mathbb{R}^n$.

We seek $\hat{\alpha}$ such that

$$\|A\hat{\alpha} - b\| = \min_{\alpha \in \mathbb{R}^m} \|A\alpha - b\|$$

where $\|x\| := \left(x^\top M x\right)^{1/2}$ and $M > 0$.

Solution: Define an appropriate inner product space $\mathcal{X} = \mathbb{R}^n, \mathcal{F} = \mathbb{R}, \langle x, y \rangle := x^\top M y$ and decompose A into its columns

$$A = \left[\; A_1 \;\middle|\; A_2 \;\middle|\; \cdots \;\middle|\; A_m \;\right]$$

We seek

$$\hat{x} := \underset{x \in \text{span}\{A_1, \ldots, A_m\}}{\text{argmin}} ||x - b||^2$$

Normal Equations:

$$\hat{x} = \hat{\alpha}_1 A_1 + \hat{\alpha}_2 A_2 + \cdots + \hat{\alpha}_m A_m$$

$$G^\top \hat{\alpha} = \beta, \text{ with } G = G^\top$$

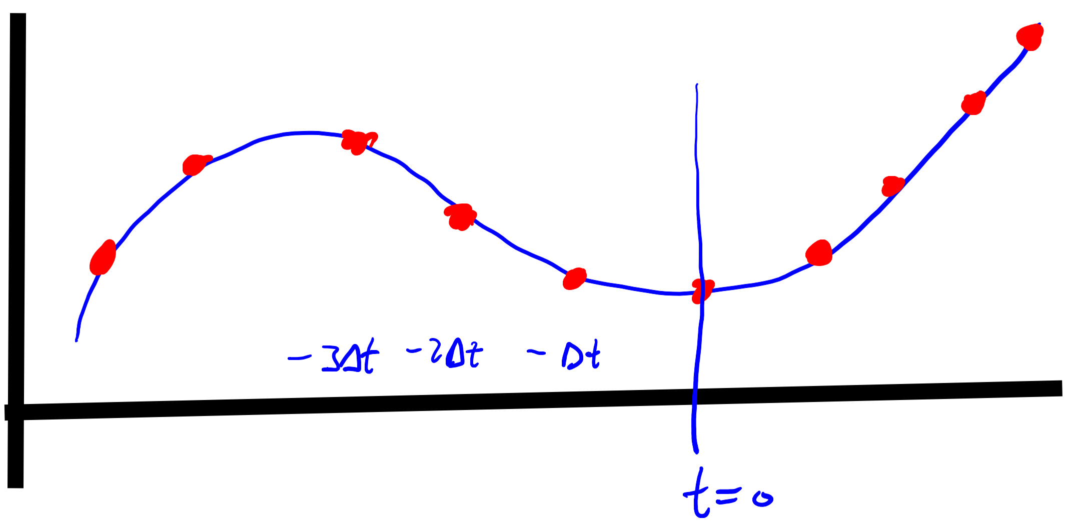
$$[G^\top]_{ij} = [G]_{ij} = \langle A_i, A_j \rangle = A_i^\top M A_j = [A^\top M A]_{ij}$$

$$\beta_i = \langle b, A_i \rangle = b^\top M A_i = A_i^\top M b = [A^\top M b]_i.$$

$$\boxed{\therefore \qquad A^\top M A\hat{\alpha} = A^\top M b}.$$

Because $\operatorname{rank}(A) = m$, its columns are linearly independent and thus the Gram matrix is invertible. Hence, we conclude that

$$\boxed{\hat{\alpha} = (A^\top M A)^{-1} A^\top M b}.$$



$$y = a_0 + a_1 t \quad , 4$$

$$Y = \begin{bmatrix} y_4 \\ y_3 \\ y_2 \\ y_1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 1 & -\Delta t \\ 1 & -2\Delta t \\ 1 & -3\Delta t \end{bmatrix} \quad \alpha = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$\hat{\alpha} = (A^\top A)^{-1} A Y$$

$$\dot{y}\Big|_{t=0} = a_1 = \alpha(2)$$

$$\dot{y} = \frac{y(t_k) - y(t_{k-1})}{\Delta t}$$