# ROB501 - HW06 KUAN-TING LEE

\# 50036944

## $\boxed{1.}$

Gram Schmidt process : $y_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $y_2 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$, $y_3 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$

$$V_1 = y_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$V_2 = y_2 - \frac{\langle V_1, y_2 \rangle}{\|V_1\|^2} V_1 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 1 \\ -1.5 \end{bmatrix}$$

$\langle V_1, y_2 \rangle = 4 - 1 = 3$

$\|V_1\|^2 = 1 + 4 + 1 = 6$

$$V_3 = y_3 - \frac{\langle V_1, y_3 \rangle}{\|V_1\|^2} V_1 - \frac{\langle V_2, y_3 \rangle}{\|V_2\|^2} V_2$$

$\langle V_1, y_3 \rangle = -2 - 4 + 3 = -3$

$\langle V_2, y_3 \rangle = -7 + 2 - 4.5 = -9.5$

$\|V_2\|^2 = 15.5$

$$= \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} - \frac{(-3)}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \frac{(-9.5)}{15.5}\begin{bmatrix} 3.5 \\ 1 \\ -1.5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix} + \frac{1}{62}\begin{bmatrix} 133 \\ 38 \\ -57 \end{bmatrix}$$

$$= 1/62 \begin{bmatrix} -124 + 31 + 133 \\ 124 - 62 + 38 \\ 186 + 31 - 57 \end{bmatrix}$$

$$= 1/62 \begin{bmatrix} 40 \\ 100 \\ 160 \end{bmatrix}$$

$$\Rightarrow V_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 3.5 \\ 1 \\ -1.5 \end{bmatrix}, \quad V_3 = 1/62 \begin{bmatrix} 40 \\ 100 \\ 160 \end{bmatrix}$$

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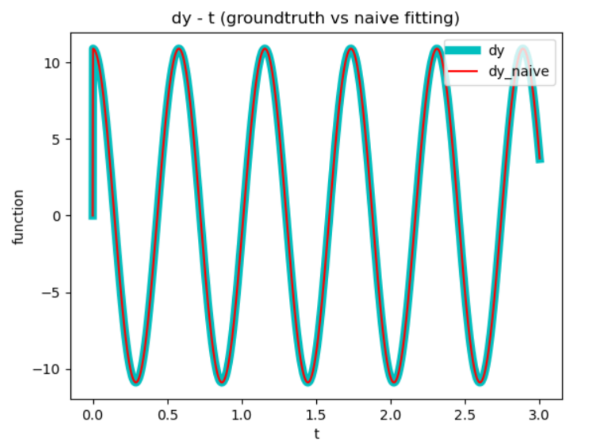
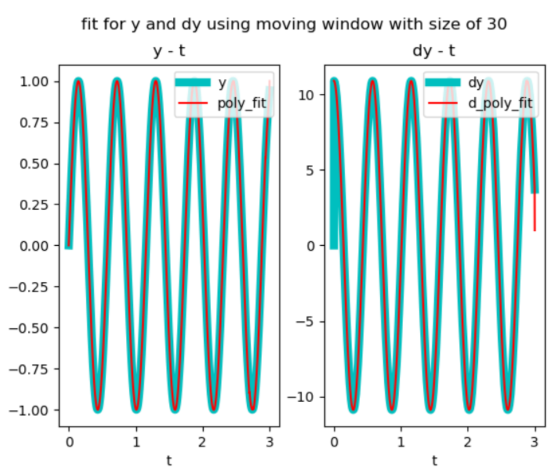
$\boxed{2}$

(a)




(b)

The basis for fitting is $\{1, t, t^2, t^3\}$ with window size 30

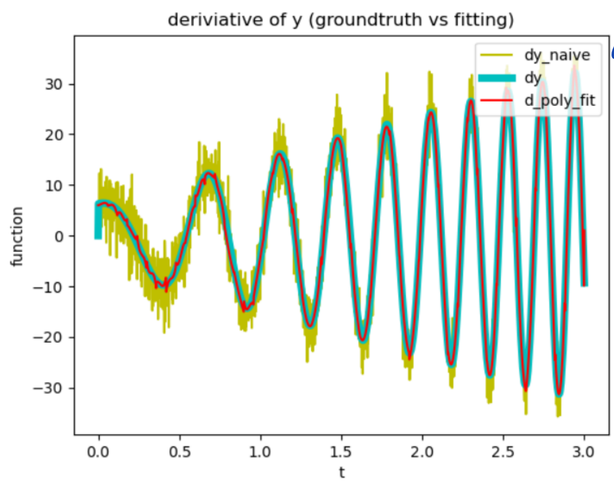



$\hat{y}_k(t)$

$\frac{d\hat{y}_k}{dt}(t)$

3.

(a)




naive estimate

estimate of $\frac{dy(t)}{dt}$

(b)

The basis for fitting is $\{1, t, t^2, t^3\}$ with window size 30

The Root Mean Square Error (RMSE) in this setting: 0.516

$\boxed{4.}$

Suppose $\hat{x}$ takes the form $\hat{x} = a_1 y^1 + a_2 y^2$, we need to solve $a_1, a_2$ using normal equation such that $\hat{x} = \underset{y \in M}{\operatorname{argmin}} \|x - y\|$.

$$G^T \alpha = \beta, \quad \text{where} \quad G^T = \begin{bmatrix} \langle y^1, y^1 \rangle, & \langle y^2, y^1 \rangle \\ \langle y^1, y^2 \rangle, & \langle y^2, y^2 \rangle \end{bmatrix}, \quad \beta = \begin{bmatrix} \langle x, y^1 \rangle \\ \langle x, y^2 \rangle \end{bmatrix}$$

$$\langle y^1, y^1 \rangle = \operatorname{trace}((y^1)^T y^1) = \operatorname{trace}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\right) = \operatorname{trace}\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} = 5$$

$$\langle y^2, y^1 \rangle = \langle y^1, y^2 \rangle = \operatorname{trace}((y^2)^T y^1) = \operatorname{trace}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\right) = \operatorname{trace}\left(\begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}\right)$$
$$= 3$$

$$\langle y^2, y^2 \rangle = \operatorname{trace}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \operatorname{trace}\left(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right) = 4$$

$$\Rightarrow G^T = \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix}$$

$$\langle x, y^1 \rangle = \operatorname{trace}(x^T y^1) = \operatorname{trace}\left(\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\right) = \operatorname{trace}\left(\begin{bmatrix} 4 & 0 \\ -1 & 0 \end{bmatrix}\right) = 4$$

$$\langle x, y^2 \rangle = \operatorname{trace}(x^T y^2) = \operatorname{trace}\left(\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \operatorname{trace}\left(\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\right) = 1$$

$$\Rightarrow \beta = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix} \alpha = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \Rightarrow \alpha = \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 13/11 \\ -7/11 \end{bmatrix}$$

$$\Rightarrow \hat{x} = 13/11 \cdot \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - 7/11 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= 1/11 \begin{bmatrix} 6 & -7 \\ 19 & -7 \end{bmatrix}$$

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$\boxed{5.}$

Suppose both $m_1 \in M$ and $m_2 \in M$ satisfy $\|x - m_i\| = d(x, M)$. The objective is to show that $m_1 = m_2$. Let $\gamma = d(x, M)$ and note that $\frac{m_1 + m_2}{2} \in M$.

$$\gamma = \inf_{y \in M} \|x - y\| \leq \left\| x - \frac{m_1 + m_2}{2} \right\| = \left\| \frac{x - m_1}{2} + \frac{x - m_2}{2} \right\|$$

$$\leq \frac{1}{2}\|x - m_1\| + \frac{1}{2}\|x - m_2\|$$

$$= \gamma/2 + \gamma/2$$

$$= \gamma$$

happens when $\frac{m_1 + m_2}{2} = m^*$

happens when $\frac{x - m_1}{2} = \alpha \left( \frac{x - m_2}{2} \right)$

$$\|x - m_1\| = \alpha \|x - m_2\|$$

$$\gamma = \alpha \gamma$$

$$\Rightarrow \alpha = 1$$

$$\Rightarrow x - m_1 = x - m_2$$

$$\Rightarrow m_1 = m_2$$

$\Rightarrow m^*$ is unique

Thus, $\exists\, m^* \in M$ s.t. $\|x - m^*\| = d(x, M) = \inf_{y \in M} \|x - y\|$, $m^*$ is unique. $\square$

$\boxed{6.}$

Let's prove (a) and (c)!

(a)

$$\|x+\alpha x\|_1 = \|(\alpha+1)x\|_1 = |(\alpha+1)x_1| + |(\alpha+1)x_2|$$
$$= (\alpha+1)|x_1| + (\alpha+1)|x_2|$$
$$= |x_1| + |x_2| + |\alpha x_1| + |\alpha x_2|$$
$$= \|x\|_1 + \|\alpha x\|_1 \quad . \square$$

(c)

$$\|x+\alpha x\|_\infty = \max\{|x_1+\alpha x_1|, |x_2+\alpha x_2|\}$$
$$= \max\{(\alpha+1)|x_1|, (\alpha+1)|x_2|\}$$
$$= \max\{|x_1|+|\alpha x_1|, |x_2|+|\alpha x_2|\}$$
$$= \max\{|x_1|, |x_2|\} + \max\{|\alpha x_1|, |\alpha x_2|\}$$
$$= \|x\|_\infty + \|\alpha x\|_\infty \quad . \square$$

$\boxed{7.}$

If $\mathrm{rank}(A) = n$, then by the Invertible Matrix Theorem, the only solution to $Ax = 0$ is the trivial solution $x = 0$. Hence, in this case, $\mathrm{nullspace}(A) = \{0\}$ so $\mathrm{nullity}(A) = 0$ and $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$.

Now suppose $\mathrm{rank}(A) = r < n$. In this case, there are $n-r > 0$ free variables in the solution to $Ax = 0$. Let $t_1, t_2 \ldots t_{n-r}$ denote these free variables (chosen as those variables not attached to a leading one in any row-echelon form of $A$), and let $x_1, x_2, \ldots, x_{n-r}$ denote the solutions obtained by sequentially setting each free variables to 1 and the remaining free variables to zero. Note that $\{x_1, x_2 \ldots x_{n-r}\}$ is linearly independent. Moreover, every solution to $Ax = 0$ is a linear combination of $x_1, x_2 \ldots x_{n-r}$:

$$x = t_1 x_1 + t_2 x_2 + \cdots t_{n-r} x_{n-r},$$

which shows that $\{x_1, x_2 \cdots x_{n-r}\}$ spans $\mathrm{nullspace}(A)$. Thus, $\{x_1, x_2 \ldots x_{n-r}\}$ is a basis for $\mathrm{nullspace}(A)$, and $\mathrm{nullity}(A) = n - r$

$\boxed{8.}$

```python
def matrixInverseLemma(A_inv, B, C, D):
    return A_inv - A_inv.dot(B).dot(np.linalg.inv(1 / C + D.dot(A_inv).dot(B))).dot(D).dot(A_inv)

def verify(A, B, C, D):
    return np.linalg.inv(A + 0.2 * B.dot(D))

A = np.array([[1, 0, 0, 0, 0], [0, 0.5, 0, 0, 0], [0, 0, 0.5, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 0.5]])
A_inv = np.linalg.inv(A)
B = np.array([[1, 0, 2, 0, 3]]).T
C = 0.2
D = B.T
print(matrixInverseLemma(A_inv, B, C, D))
print(verify(A, B, C, D))
```

The output is

$$\begin{bmatrix} 0.96875 & 0 & -0.125 & 0 & -0.1875 \\ 0 & 2 & 0 & 0 & 0 \\ -0.125 & 0 & 1.5 & 0 & -0.75 \\ 0 & 0 & 0 & 1 & 0 \\ -0.1875 & 0 & -0.75 & 0 & 0.875 \end{bmatrix}$$

$\boxed{9.}$

Suppose $\hat{x}$ takes the form $\hat{x} = \alpha_1 y^1 + \alpha_2 y^2 + \alpha_3 y^3$, we need to solve $\alpha_1, \alpha_2, \alpha_3$ using normal equation such that $\hat{x} = \arg\min_{y \in M} \|x - y\|$

We first define the inner products for $G$:

$y^1 = 1, \quad y^2 = t, \quad y^3 = \frac{1}{2}(3t^2 - 1), \quad x = e^t$

$\langle y^1, y^1 \rangle = \int_{-1}^{1} 1 \cdot 1 \cdot dt = 2, \quad \langle y^1, y^2 \rangle = \int_{-1}^{1} 1 \cdot t \, dt = 0, \quad \langle y^1, y^3 \rangle = \int_{-1}^{1} 1 \cdot \frac{1}{2}(3t^2 - 1)\, dt$

$= (t^3/2 - t/2)\Big|_{-1}^{1}$

$= 0$

$\langle y^2, y^2 \rangle = \int_{-1}^{1} t^2 \, dt = t^3/3 \Big|_{-1}^{1}$

$= 2/3$

$\langle y^2, y^3 \rangle = \int_{-1}^{1} \frac{1}{2}(3t^3 - t)\, dt$

$= 3/8\, t^4 - t^2/4 \Big|_{-1}^{1}$

$= 0$

$\langle y^3, y^3 \rangle = \int_{-1}^{1} \frac{1}{4}(9t^4 - 6t^2 + 1)\, dt$

$= \frac{9}{20} t^5 - \frac{1}{2} t^3 + \frac{t}{4} \Big|_{-1}^{1}$

$= 9/10 - 1 + 1/2$

$= 2/5$

$\langle x, y^1 \rangle = \int_{-1}^{1} e^t \cdot 1 \, dt = e^t \Big|_{-1}^{1} = e - 1/e$

$\langle x, y^2 \rangle = \int_{-1}^{1} e^t \cdot t \, dt = te^t - e^t \Big|_{-1}^{1}$

$= (e - e) - (-e^{-1} - e^{-1})$

$= 2/e$

$\langle x, y^3 \rangle = \int_{-1}^{1} e^t \cdot \frac{1}{2}(3t^2 - 1)\, dt$

$= \frac{3}{2} \int_{-1}^{1} t^2 e^t \, dt - \frac{1}{2} \int_{-1}^{1} e^t \, dt$

$= \frac{3}{2} \left( e^t (t^2 - 2t + 2) \right)\Big|_{-1}^{1} - \frac{1}{2} e^t \Big|_{-1}^{1}$

$= \frac{3}{2} \left[ e - 1/e \cdot 5 \right] - \frac{1}{2} (e - 1/e)$

$= e - 7 \cdot 1/e$

$$G^T \alpha = \beta, \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/5 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} e - 1/e \\ 2/e \\ e - 7/e \end{bmatrix}$$

Solve for $\alpha_1, \alpha_2, \alpha_3$, then you get $\hat{x}$

I discussed with Wan-Yi Yu # 74732586