ROB 501 Handout: Grizzle

Matrix Representation of a Linear Operator

Definition: Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be finite dimensional vector spaces. A function $L : \mathcal{X} \to \mathcal{Y}$ is a **linear operator** if for each x and z in $\mathcal{X}$, and α and β in $\mathcal{F}$,

$$L(\alpha x + \beta z) = \alpha L(x) + \beta L(z)$$

Definition: Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be finite dimensional vector spaces, and let $L : \mathcal{X} \to \mathcal{Y}$ be a linear operator. A **matrix representation of** L *with respect to the bases* $\{u\} = \{u^1, \cdots, u^m\}$ *for* $\mathcal{X}$ *and* $\{v\} = \{v^1, \cdots, v^n\}$ *for* $\mathcal{Y}$ is a matrix A with coefficients in $\mathcal{F}$ satisfying for every $x \in \mathcal{X}$,

$$[L(x)]_v = A[x]_u$$

Theorem: Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be finite dimensional vector spaces, and let $L : \mathcal{X} \to \mathcal{Y}$ be a linear operator. Then L has a matrix representation A, moreover,

$$A = [A_1 | A_2 | \cdots | A_m],$$

where A_i, the i-th column of A, is given by

$$A_i = \left[L\left(u^i\right)\right]_{\{v\}}$$

Proof: Let $x \in \mathcal{X}$, and write $x = \alpha_1 u^1 + \ldots + \alpha_m u^m$ so that

$$[x]_{\{u\}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}.$$

Then

$$\begin{aligned} L(x) &= L\left(\alpha_1 u^1 + \ldots + \alpha_m u^m\right) \\ &= \alpha_1 L\left(u^1\right) + \ldots + \alpha_m L\left(u^m\right) \end{aligned}$$

$$
\begin{aligned}
[L(x)]_{\{v\}} &= \left[\alpha_1 L\left(u^1\right) + \ldots + \alpha_m L\left(u^m\right)\right]_{\{v\}} \\
&= \alpha_1 \left[L\left(u^1\right)\right]_{\{v\}} + \ldots + \alpha_m \left[L\left(u^m\right)\right]_{\{v\}} \\
&= \left[\begin{array}{ccc} \left[L\left(u^1\right)\right]_{\{v\}} & \cdots & \left[L\left(u^m\right)\right]_{\{v\}} \end{array}\right] \left[\begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_m \end{array}\right] \\
&= A \left[\begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_m \end{array}\right] \\
&= A\,[x]_{\{u\}}
\end{aligned}
$$

End of Proof.

Key Point: The i-th column of A is given by $A_i = \left[\begin{array}{c} a_{1i} \\ \vdots \\ a_{ni} \end{array}\right] = \left[L(u^i)\right]_{\{v^1, \cdots, v^n\}}$

Useful Exercise: Let A be an $n \times n$ matrix with coefficients in a given field $\mathcal{F}$.

1. Let $e_1, \cdots, e_n$ denote the natural basis for $(\mathcal{F}^n, \mathcal{F})$. Show that $Ae_i = A_i$, where A_i is the i-th column of A.
2. Show that $L : \mathcal{F}^n \to \mathcal{F}^n$ by, for $x \in \mathcal{F}^n$, $L(x) = Ax$, is a linear operator.
3. Let $e_1, \cdots, e_n$ denote again the natural basis for $(\mathcal{F}^n, \mathcal{F})$. Find the matrix representation of L with respect to $e_1, \cdots, e_n$. Note: In this case, the bases $\{u\}$ and $\{v\}$ are the same and equal to the natural basis.

Remark: The above exercise can be extended to a non-square matrix, say $n \times m$. In that case, $L : \mathcal{F}^m \to \mathcal{F}^n$. If you take the natural bases on both $\mathcal{F}^m$ and $\mathcal{F}^n$, you will get the same result as part (3) of the exercise [namely, the matrix representation of L is A itself.]

A Worked Example of Matrix Representations

Let $(\mathcal{X}, \mathcal{F}) = \left(\mathbb{R}^2, \mathbb{R}\right)$, and define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L(e_1) = 3e_1 + 4e_2$, $L(e_2) = -e_1 + 6e_2$, where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the canonical basis elements.

1. What is the matrix representation of L with respect to $\{e_1, e_2\}$?
2. What is the matrix representation of L with respect to $\left\{v^1, v^2\right\}$ where $v^1 = e_1 + e_2$, $v^2 = 3e_1 - 4e_2$?

Solution:

1. Let A =matrix representation of L. Then the i^{th} column of $A = [L(e_i)]_{\{e_1,e_2\}}$.
$\therefore [L(e_1)]_{\{e_1,e_2\}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $[L(e_2)]_{\{e_1,e_2\}} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$
$\implies A = \begin{bmatrix} 3 & -1 \\ 4 & 6 \end{bmatrix}$

2. Let P be the change of coordinates from $\{e_1, e_2\}$ to $\left\{v^1, v^2\right\}$, and $\bar{P}$ be the change of coordinates from $\left\{v^1, v^2\right\}$ to $\{e_1, e_2\}$. Note that the i^{th} column of $\bar{P}$ is just the representation of v^i in $\{e_1, e_2\}$. That is,

$$\bar{P} = \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix}.$$

Recall that $\bar{P} = P^{-1}$, so

$$P = \left(\bar{P}\right)^{-1} = \frac{-1}{7} \begin{bmatrix} -4 & -3 \\ -1 & 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 4 & 3 \\ 1 & -1 \end{bmatrix}.$$

Therefore, if $\bar{A}$ is the representation of L in $\left\{v^1, v^2\right\}$, then

$$\bar{A} = PAP^{-1} = \frac{1}{7} \begin{bmatrix} 4 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -38 & 16 \\ -8 & 25 \end{bmatrix}.$$

Note: $\bar{P}$ was readily available, not P, as you may have guessed!!
3. Just to check, let's do thing the "long way":

$$
\begin{aligned}
L\left(v^1\right) &= L\left(e_1 + e_2\right) \\
&= L\left(e_1\right) + L\left(e_2\right) \\
&= \left(3e_1 + 4e_2\right) + \left(-e_1 + 6e_2\right) \\
&= 2e_1 + 10e_2 \\
L\left(v^2\right) &= L\left(3e_1 - 4e_2\right) \\
&= 3L\left(e_1\right) - 4L\left(e_2\right) \\
&= 3\left(3e_1 + 4e_2\right) - 4\left(-e_1 + 6e_2\right) \\
& 13e_1 - 12e_2
\end{aligned}
$$

$\left[L\left(v^1\right)\right]_{\{v^1,v^2\}}$ =? To find it, write

$$
\begin{bmatrix} 2 \\ 10 \end{bmatrix} = \bar{a}_{11} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{v^1} + \bar{a}_{21} \underbrace{\begin{bmatrix} 3 \\ -4 \end{bmatrix}}_{v^2} = \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} \bar{a}_{11} \\ \bar{a}_{12} \end{bmatrix}
$$

$$
\Longrightarrow \begin{bmatrix} \bar{a}_{11} \\ \bar{a}_{12} \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 38 \\ -18 \end{bmatrix}
$$

Similarly

$$
\begin{bmatrix} 13 \\ -12 \end{bmatrix} = \bar{a}_{12} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \bar{a}_{22} \begin{bmatrix} 3 \\ -4 \end{bmatrix}
$$

$$
\Longrightarrow \begin{bmatrix} \bar{a}_{12} \\ \bar{a}_{22} \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 16 \\ 25 \end{bmatrix}
$$

$$
\therefore \bar{A} = \frac{1}{7} \begin{bmatrix} 38 & 16 \\ -8 & 25 \end{bmatrix}
$$