Rob 501 Handout: Grizzle
Minimizing $||\cdot||_1$ and $||\cdot||_\infty$ Norms
Using Linear Programming

Linear Program: Minimize a scalar-valued linear function subject to linear equality and inequality constraints. For $x \in \mathbb{R}^n$, and $f \in \mathbb{R}^n$

$$\begin{aligned} \text{minimize} \quad & f^\top x \\ \text{subject to} \quad & A_{in}x \preceq b_{in} \\ & A_{eq}x = b_{eq} \end{aligned}$$

where $A_{in}x \preceq b_{in}$ means each row of $A_{in}x$ is less than or equal to the corresponding row of b_{in}.

Only restriction on A_{in} and A_{eq} is that the set

$$K = \{x \in \mathbb{R}^n \mid A_{in}x \preceq b_{in},\ A_{eq}x = b_{eq}\}$$

should be non-empty.

Remarks: Below is the MATLAB function call. Typically, the inequality constraints are not written with a subscript *in*. The next pages will show why I am doing this.

```
X = linprog(f,A,b) attempts to solve the linear programming problem:

      min f'*x    subject to:   A*x <= b
       x

X = linprog(f,A,b,Aeq,beq) solves the problem above while
satisfying the equality constraints Aeq*x = beq.
```

ℓ_1-**norm:** $||x||_1 = \sum_{i=1}^{n} |x_i|$.

Suppose that A is an $m \times n$ real matrix. Minimize $||Ax - b||_1$ is equivalent to the following linear program on $\mathbb{R}^{n+m}$

$$\begin{array}{rl} \text{minimize} & f^\top X \\ \text{subject to} & A_{in} X \preceq b_{in} \end{array}$$

with $X = \begin{bmatrix} x \\ s \end{bmatrix}$ ($s \in \mathbb{R}^m$ are called slack variables)

$$f := \begin{bmatrix} 0_{1\times n} & \mathbf{1}_{1\times m} \end{bmatrix}, \quad A_{in} := \begin{bmatrix} A & -I_{m\times m} \\ -A & -I_{m\times m} \end{bmatrix} \text{ and } b_{in} := \begin{bmatrix} b \\ -b \end{bmatrix}$$

If $\widehat{X} = [\widehat{x}^\top, \ \widehat{s}^\top]^\top$ is the solution of the linear programming problem, then $\widehat{x}$ solves the 1-norm optimization problem; that is

$$\widehat{x} = \arg \min_{x \in \mathbb{R}^n} ||Ax - b||_1$$

ℓ_∞**-norm:** $||x||_\infty = \max_{1\le i\le n} |x_i|$.

Suppose that A is an $m \times n$ real matrix. Minimize $||Ax - b||_\infty$ is equivalent to the following linear program on $\mathbb{R}^{n+1}$

$$\begin{aligned} \text{minimize} \quad & f^\top X \\ \text{subject to} \quad & A_{in} X \preceq b_{in} \end{aligned}$$

with $X = \begin{bmatrix} x \\ s \end{bmatrix}$ ($s \in \mathbb{R}$ is called a slack variable)

$$f := \begin{bmatrix} 0_{1\times n} & 1 \end{bmatrix}, \quad A_{in} := \begin{bmatrix} A & -\mathbf{1}_{m\times 1} \\ -A & -\mathbf{1}_{m\times 1} \end{bmatrix} \text{ and } b_{in} := \begin{bmatrix} b \\ -b \end{bmatrix}$$

If $\widehat{X} = [\widehat{x}^\top, \ \widehat{s}\ ]^\top$ solves the linear programming problem, then $\widehat{x}$ solves the max-norm optimization problem; that is

$$\widehat{x} = \arg \min_{x\in\mathbb{R}^n} ||Ax - b||_\infty$$

Remark: The following pages are my source for this material.
`https://www.princeton.edu/~chiangm/ele539l3a.pdf`

ELE539A: Optimization of Communication Systems
Lecture 3A: Linear Programming

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Lecture Outline

- Linear programming
- Norm minimization problems
- Dual linear programming
- Basic properties

Thanks: Stephen Boyd (some materials and graphs from Boyd and Vandenberghe)

Linear Programming

Minimize linear function over linear inequality and equality constraints:

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

Variables: $x \in \mathbf{R}^n$.

Standard form LP:

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& x \succeq 0
\end{array}
$$

Most well-known, widely-used and efficiently-solvable optimization

Appreciation-Application cycle starting for convex optimization

Transformation To Standard Form

Introduce slack variables s_i for inequality constraints:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx + s = h \\ & Ax = b \\ & s \succeq 0 \end{array}$$

Express x as difference between two nonnegative variables $x^+, x^- \succeq 0$: $x = x^+ - x^-$

$$\begin{array}{ll} \text{minimize} & c^T x^+ - x^T x^- \\ \text{subject to} & Gx^+ - Gx^- + s = h \\ & Ax^+ - Ax^- = b \\ & x^+, x^-, s \succeq 0 \end{array}$$

Now in LP standard form with variables x^+, x^-, s

Linear Fractional Programming

Minimize ratio of affine functions over polyhedron:

$$
\begin{array}{ll}
\text{minimize} & \frac{c^T x + d}{e^T x + f} \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

Domain of objective function: $\{x | e^T x + f > 0\}$

Not an LP. But if nonempty feasible set, transformation into an equivalent LP with variables y, z:

$$
\begin{array}{ll}
\text{minimize} & c^T y + dz \\
\text{subject to} & Gy - hz \preceq 0 \\
& Ay - bz = 0 \\
& e^T y + fz = 1 \\
& z \succeq 0
\end{array}
$$

Why: let $y = \frac{x}{e^T x + f}$ and $z = \frac{1}{e^T x + f}$

Norm Minimization Problems

- l_1 norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

Minimize $\|Ax - b\|_1$ is equivalent to this LP in $x \in \mathbf{R}^n, s \in \mathbf{R}^n$:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s \\ \text{subject to} & Ax - b \preceq s \\ & Ax - b \succeq -s \end{array}$$

- l_∞ norm: $\|x\|_\infty = \max_i \{|x_i|\}$

Minimize $\|Ax - b\|_\infty$ is equivalent to this LP in $x \in \mathbf{R}^n, t \in \mathbf{R}$:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \preceq t\mathbf{1} \\ & Ax - b \succeq -t\mathbf{1} \end{array}$$

Dual Linear Programming

1. Primal problem in standard form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

2. Write down Lagrangian using Lagrange multipliers λ, ν:

$$L(x, \lambda, \nu) = c^T x - \sum_{i=1}^{n} \lambda_i x_i + \nu^T (Ax - b) = -b^T \nu + (c + A^T \nu - \lambda)^T x$$

3. Find Lagrange dual function:

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = -b^T \nu + \inf_x [(c + A^T \nu - \lambda)^T x]$$

Since a linear function is bounded below only if it is identically zero, we have

$$g(\lambda, \nu) = \begin{cases} -b^T \nu & A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Dual Linear Programming

4. Write down Lagrange dual problem:

$$\begin{array}{ll} \text{maximize} & g(\lambda,\nu) = \begin{cases} -b^T\nu & A^T\nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases} \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

5. Make equality constraints explicit:

$$\begin{array}{ll} \text{maximize} & -b^T\nu \\ \text{subject to} & A^T\nu - \lambda + c = 0 \\ & \lambda \succeq 0 \end{array}$$

6. Simplify Lagrange dual problem:

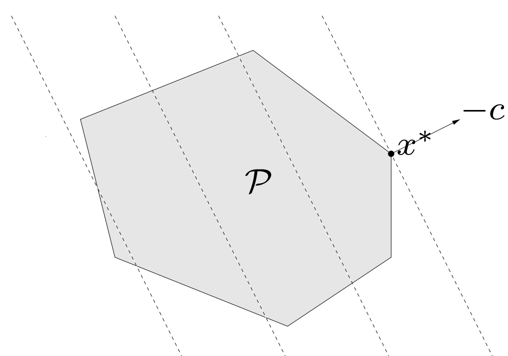
$$\begin{array}{ll} \text{maximize} & -b^T\nu \\ \text{subject to} & A^T\nu + c \succeq 0 \end{array}$$

which is an inequality constrained LP

Basic Properties

Definition: x in polyhedron P is an extreme point if there does not exist two other points $y, z \in P$ such that $x = \theta y + (1-\theta)z$ for some $\theta \in [0,1]$

Theorem: Assume that a LP in standard form is feasible and the optimal objective value is finite. There exists an optimal solution which is an extreme point



Algorithms

- Simplex Method
- Interior-point Method
- Ellipsoid Method
- Cutting-plane Method

Simplex method is very efficient in practice but specialized for LP: move from one vertex to another without enumerating all the vertices

We will cover interior point algorithms for general convex optimization later

Lecture Summary

- LP covers a wide range of interesting problems for communication systems
- Dual LP is LP
- There are very useful special structures in LP. But most of the important ones (computational efficiency, global optimality, Lagrange duality) can be generalized to convex optimization
- After another lecture on network flow LP, we will study the applications of nonlinear convex optimization, then nonlinear nonconvex optimization

Readings: Section. 4.3, 5.1-5.2 of Boyd and Vanderberghe