Minimum Variance Estimator

$$y = Cx + \varepsilon \qquad y \in \mathbb{R}^m, \; x \in \mathbb{R}^n$$

Stochastic Assumptions

means

$$E\{\varepsilon\} = 0, \quad E\{x\} = 0,$$

covariances

$$E\{\varepsilon\varepsilon^T\} = Q, \quad E\{xx^T\} = R, \quad E\{\varepsilon x^T\} = 0$$

(ε and x are uncorrelated)

$Q \geq 0$, $R \geq 0$, $\quad CRC^T + Q > 0$ (see why later)

Objective Minimum Variance

$$E\{\|\hat{x} - x\|^2\} = E \sum_{i=1}^{n} (\hat{x}_i - x_i)^2$$

$$= \sum_{i=1}^{n} E\{(\hat{x}_i - x_i)^2\}$$

n separate problems:

<u>Remark</u> $x = Ky$ is automatically unbiased; indeed

$$E\{x\} = E\{Ky\} = E\{K(Cx+\varepsilon)\}$$
$$= KC\,E\{x\} + K\,E\{\varepsilon\} = 0$$

As before, can reduce the problem to n deterministic problems in the rows k_i of K.

Here, we will instead formulate it is a minimum distance problem in a vector space of random variables

$\mathcal{F} = \mathbb{R}$

$\mathcal{X} = \text{span}\{x_1, .., x_n, \varepsilon_1, .., \varepsilon_n\}$

= Vector Space of the random variables $x_1, .., x_n, \varepsilon_1, .., \varepsilon_n$

$\langle z_1, z_2 \rangle := E\{z_1 z_2\}$ inner product

$\mathcal{M} = \text{span}\{y_1, .., y_m\} \subset \mathcal{X}$

$y_i = C_i x + \varepsilon_i = \sum_{j=1}^{n} c_{ij} x_j + \varepsilon_i$

$\hat{x}_i = \arg \min_{\hat{x}_i \in \mathcal{M}} \| x_i - m \| = d(x_i, \mathcal{M})$

<u>Exercise</u> $\{y_1, .., y_m\}$ linearly indep $\Leftrightarrow$ $CRC^T + Q > 0$ (see Gram Matrix below)

$\therefore \quad \hat{x}_i = \hat{\alpha}_1 y_1 + \hat{\alpha}_2 y_2 + \dots + \hat{\alpha}_m y_m$

where $\quad G^T \hat{\alpha} = \beta$

$G^T = G, \quad G_{ij} = \langle y_i, y_j \rangle \quad , \beta_j = \langle x_i, y_j \rangle$

$$
\begin{aligned}
\langle y_i, y_j \rangle &= E\{y_i y_j\} = \\
&= E\{(c_i x + e_i)(c_j x + e_j)\} \\
&= E\{(c_i x)(c_j x)\} + E\{e_i e_j\} \\
&= E\{c_i x (c_j x)^T\} + E\{e_i e_j\} \\
&= c_i E\{x x^T\} c_j^T + E\{e_i e_j\} \\
&= c_i R c_j^T + Q_{ij} \\
&= [C R C^T + Q]_{ij}
\end{aligned}
$$

MVE

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$$\beta_j = E\{x_i y_j\}$$

$$= E\{x_i (C_j x + \varepsilon_j)\}$$

$$= E\{x_i C_j x\} + \cancel{E\{x_i \varepsilon_j\}}^{\,0}$$

$$= E\{$$

$$= C_j \, E\{x_i x\} = C_j R_i$$

where $R = [R_1 | R_2 | \cdots R_n]$

$$\therefore \; G^T \hat{\alpha} = \beta$$

$$[C R C^T + Q]^{-1} \hat{\alpha} = C R_i$$

$$\hat{\alpha} = \hat{k}_i^T = [C R C^T + Q]^{-1} C R_i$$

$$\hat{K}^T = [C R C^T + Q]^{-1} C R$$

$$\boxed{\hat{K} = R C^T [C R C^T + Q]^{-1}}$$

$$\boxed{\hat{x} = \hat{K} y = RC^T[CRC^T + Q]^{-1}}$$

Exercise:

1. $E\{(\hat{x} - x)(\hat{x} - x)^T\} = R - RC^T[CRC^T + Q]^{-1} CR$

2. $\hat{x} = \hat{K} y = (C^T Q^{-1} C + R^{-1})^{-1} C^T Q^{-1} y$

Remark: $RC^T[CRC^T + Q^{-1}] CR$
has reduced the covariance of our estimate of x. It quantifies the "value" of the measurement $y = Cx + \varepsilon$. $\square$

BLUE vs MVE

BLUE $\quad \hat{x} = (C^T Q^{-1} C)^{-1} C^T Q^{-1} y$

MVE $\quad \hat{x} = R C^T [C R C^T + Q]^{-1} y$

$$= (C^T Q^{-1} C + R^{-1})^{-1} C^T Q^{-1} y$$

BLUE means $R^{-1} \equiv 0 \approx$ "$R \approx \infty$"

meaning, we have <u>no clue</u> what x is doing. (Infinite variance)

For BLUE to exist, need $\dim(y) \geq \dim(x)$.

For MVE, can have $\dim(y) < \dim(x)$

as long as $(C R C^T + Q) > 0$.