$\boxed{1.}$ # 50036744

(a) $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$. Let $x \in \mathbb{R}^2$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$(n=2)$

for $\|x\|_2 \le \|x\|_1$, $\|x\|_2^2 \le \|x\|_1^2$

$\Rightarrow x_1^2 + x_2^2 \le (|x_1| + |x_2|)^2 = x_1^2 + x_2^2 + 2|x_1||x_2|$

$\Rightarrow \|x\|_2 \le \|x\|_1$ #

for $\|x\|_1 \le \sqrt{n}\|x\|_2 \Rightarrow \|x\|_1^2 \le 2\|x\|_2^2$

$\Rightarrow (|x_1| + |x_2|)^2 = x_1^2 + x_2^2 + 2|x_1||x_2|$

$\because 2|x_1||x_2| \le x_1^2 + x_2^2$

$\Rightarrow (|x_1| + |x_2|)^2 \le 2(x_1^2 + x_2^2)$

$\Rightarrow \|x\|_1 \le \sqrt{n}\|x\|_2$ # $\square$

(b) $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, assume $|x_1| \ge |x_2|$

① $x_1^2 \le (x_1^2 + x_2^2)$

$\Rightarrow |x_1| \le \sqrt{x_1^2 + x_2^2}$

$\Rightarrow \|x\|_\infty \le \|x\|_2$ #

$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$

$\square$

② $|x_2| \le |x_1|$

$x_2^2 \le x_1^2$

$x_1^2 + x_2^2 \le 2x_1^2$

$\sqrt{x_1^2 + x_2^2} \le \sqrt{2}|x_1|$

$\|x\|_2 \le \sqrt{n}\|x\|_\infty$ #

(c) $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ assume $|x_1| \ge |x_2|$

① $|x_1| \le |x_1| + |x_2|$

$\Rightarrow \|x\|_\infty \le \|x\|_1$

② $|x_2| \le |x_1|$

$\Rightarrow |x_1| + |x_2| \le 2|x_1|$

$\Rightarrow \|x\|_1 \le n\|x\|_\infty$

$\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$

☑

$\boxed{2.}$

(a)

$\tilde{B}_{\frac{a}{k_2}}(x_0) \Rightarrow |||x - x_0||| < \frac{a}{k_2}$

$\because \frac{1}{k_2}||x - x_0|| \le |||x - x_0||| \Rightarrow ||x - x_0|| \le k_2 |||x - x_0|||$

$< \frac{a}{k_2} k_2 = a$

$\Rightarrow ||x - x_0|| < a \Rightarrow B_a(x_0) \quad \therefore \tilde{B}_{\frac{a}{k_2}}(x_0) \subset B_a(x_0) \quad — ①$

$B_a(x_0) \Rightarrow ||x - x_0|| < a, \quad \because k_1 |||x - x_0||| \le ||x - x_0||$

$\Rightarrow |||x - x_0||| \le \frac{1}{k_1} ||x - x_0|| < \frac{a}{k_1} \Rightarrow \tilde{B}_{\frac{a}{k_1}}(x_0)$

$\therefore B_a(x_0) \subset \tilde{B}_{\frac{a}{k_1}}(x_0) \quad — ②$

from ① and ② $\tilde{B}_{\frac{a}{k_2}}(x_0) < B_a(x_0) \subset \tilde{B}_{\frac{a}{k_1}}(x_0)$ $\square$

(b)

We know $\frac{1}{k_2}||x|| \le |||x||| \quad — ①$

$|||x||| \le \frac{1}{k_1}||x|| \quad — ②$

P is open in $(X, \mathbb{R}, ||\cdot||) \Leftrightarrow P$ is open in $(X, \mathbb{R}, |||\cdot|||)$

$P = P^0 = \{x \in X \mid d(x, \sim P) > 0\}$

$\Rightarrow \exists\, \varepsilon > 0, \forall y \in \sim P,$ s.t. $||x - y|| \ge \varepsilon$

by ①, $||x - y|| \ge \varepsilon \Rightarrow \frac{\varepsilon}{k_2} \le \frac{1}{k_2}||x - y|| \le |||x - y|||$

take $\varepsilon' = \frac{\varepsilon}{k_2} \Rightarrow |||x - y||| \ge \varepsilon'$

$\Rightarrow \exists\, \varepsilon' > 0,\ \forall y \in \sim P,\ s.t.\ |||x-y||| \geq \varepsilon'$

$\Rightarrow P$ is open in $(\mathcal{X}, \mathbb{R}, |||\cdot|||)$

$\Leftarrow)$ $P = P^\circ = \{ x \in \mathcal{X} \mid d(x, \sim P) > 0 \}$

$\Rightarrow \exists\, \varepsilon > 0,\ \forall y \in \sim P,\ s.t.\ |||x-y||| \geq \varepsilon$

by ② $|||x-y||| \geq \varepsilon$, $||x-y|| \geq k_1 |||x-y||| \geq k_1 \varepsilon$

take $\varepsilon' = k_1 \varepsilon \Rightarrow \exists\, \varepsilon',\ \forall y \in \sim P,\ s.t.\ ||x-y|| \geq \varepsilon'$

$\Rightarrow P$ is open in $(\mathcal{X}, \mathbb{R}, ||\cdot||)$ $\square$

(c) We need to prove (x_n) is Cauchy in $(\mathcal{X}, \mathbb{R}, ||\cdot||)$

$\Leftrightarrow$ (x_n) is Cauchy in $(\mathcal{X}, \mathbb{R}, |||\cdot|||)$

From definition $\frac{1}{k_2} ||x|| \leq |||x|||$ — ①

$|||x||| \leq \frac{1}{k_1} ||x||$ — ②

$\Rightarrow)$ $\forall \varepsilon > 0,\ \exists N(\varepsilon) < \infty,\ s.t.\ \forall n, m > N,\ ||x_m - x_n|| < \varepsilon$

from ②, $|||x_m - x_n||| \leq \frac{1}{k_1} ||x_m - x_n|| \leq \frac{\varepsilon}{k_1}$

take $\varepsilon' = \frac{\varepsilon}{k_1}$ $\because \varepsilon > 0,\ k_1 > 0 \Rightarrow \varepsilon' > 0$

$\Rightarrow \forall \varepsilon' > 0\ \exists N(\varepsilon') < \infty,\ s.t.\ \forall n, m > N,$

$|||x_m - x_n||| < \varepsilon'$

$\Leftarrow)$ $\forall \varepsilon > 0,\ \exists N(\varepsilon) < \infty,\ s.t.\ \forall n, m > N,\ |||x_m - x_n||| < \varepsilon$

from ①, $||x_m - x_n|| \leq k_2 |||x_m - x_n||| \leq k_2 \varepsilon$

take $\varepsilon' = k_2 \varepsilon$, $\because \varepsilon > 0,\ k_2 > 0 \Rightarrow \varepsilon' > 0$

$\Rightarrow \forall \varepsilon' > 0,\ \exists N(\varepsilon') < \infty,\ s.t.\ \forall n, m > N,\ ||x_m - x_n|| < \varepsilon'$

$\boxed{3.}$

- Def of continuous :

$\forall \varepsilon > 0, \exists \delta(\varepsilon, x_0) > 0 . \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon$

- Def of convergent sequence :

$\forall \varepsilon > 0, \exists N(\varepsilon) < \infty$ s.t. $\forall n \geq N, \|x - x_n\| < \varepsilon$

If X is a convergent sequence,

$\forall \varepsilon > 0, \exists N(\varepsilon) < \infty$ s.t. $\forall n \geq N, \|x - x_n\| < \varepsilon$

If $\lim_{n \to \infty} x_n = x_0$, and continuous at x_0

we can derive that $\forall \alpha > 0, \exists \varepsilon(\alpha, x_0) > 0$

s.t. $\|x - x_0\| < \varepsilon \Rightarrow \|f(x) - f(x_0)\| < \alpha$

Thus, if $\lim_{n \to \infty} x_n = x_0$, then $\lim_{n \to \infty} f(x_n) = f(x_0)$ $\square$

4.

Using Newton Raphson Algo.

$$h(x) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1^2 & x_1 x_2 \\ x_2 x_1 & x_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + x_1 + 2x_2 - x_1^2 - 2x_1 x_2 \\ 4 + 3x_1 + 4x_2 - x_1 x_2 - 2x_2^2 \end{bmatrix}$$

$$\frac{\partial h}{\partial x} = \begin{bmatrix} 1 - 2x_1 - 2x_2 & 2 - 2x_1 \\ 3 - x_2 & 4 - x_1 - 4x_2 \end{bmatrix}$$

```
initial guess =
 [[0]
 [0]]
h_x* =
 [[-2.22044605e-16]
 [ 5.20417043e-18]]
norm_h(x*) =
 2.2210558292432945e-16
x* =
 [[-1.2285874]
 [-0.0587784]]
d_norm =  1.7642626693499668e-12
steps =  8
```

```
initial guess =
 [[   10]
 [-1000]]
h_x* =
 [[1.94289029e-16]
 [4.55364912e-16]]
norm_h(x*) =
 4.950812361573011e-16
x* =
 [[-1.2285874]
 [-0.0587784]]
d_norm =  5.699415328275249e-16
steps =  16
```

```
initial guess =
 [[5000]
 [  20]]
h_x* =
 [[8.88178420e-16]
 [3.55271368e-15]]
norm_h(x*) =
 3.66205343881779e-15
x* =
 [[1.42953398]
 [2.77738978]]
d_norm =  4.2396471515301494e-16
steps =  16
```

With different initial guesses, I got different x^*

$\boxed{5.}$

(a) T, $X \sim \exp(0.001)$, $P(X < x \mid x = 365) = \int_0^{365} \lambda e^{-\lambda x} = -e^{-\lambda x}\Big|_0^{365}$

$= -e^{-365\lambda} + 1$

$= 0.3058 < 0.31$

(b) F, for unbiased estimator $E\{\hat{x}\} = x$

$E\{\hat{x}\} = E\{x + E\{k\varepsilon\}$

$= x + E\{k\varepsilon\}$, since ε is not zero mean,

the estimator is biased. *

(c) F, $\operatorname{cov}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \operatorname{cov}(x_1, x_1) & \operatorname{cov}(x_1, x_2) \\ \operatorname{cov}(x_2, x_1) & \operatorname{cov}(x_2, x_2) \end{bmatrix}$

since $\operatorname{cov}(x_1, x_2) = \operatorname{cov}(x_2, x_1) = 0$, x_1, x_2 are uncorrelated

But! It doesn't imply independence between x_1 and x_2