20 Sept. 2018

Summary

- A set B is a <u>basis</u> for $(\mathcal{X}, \mathcal{F})$ if
  1) B is linearly independent
  2) $\text{span}\{B\} = \mathcal{X}$

- $(\mathcal{X}, \mathcal{F})$ has <u>dimension</u> $n \geq 1$ if $\exists$ a linearly indep set with n elements and every set with $n+1$ elements is linearly dependent

- <u>$\dim(\mathcal{X}, \mathcal{F}) = \infty$</u> if $\forall n \geq 1$, $\exists$ a lin. indep. set with n elements. $\dim\{0\} = 0$.

- Suppose $\dim(\mathcal{X}, \mathcal{F}) = n \geq 1$ and $\{v^1, \ldots, v^n\}$ is a basis. For $x \in \mathcal{X}$

$$[x]_v = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathcal{F}^n \longleftrightarrow x = \alpha_1 v^1 + \cdots + \alpha_n v^n$$

Representation of x with respect to $v = \{v^1, \ldots, v^n\}$.

Thm (Change of Basis Matrix) $\dim(\mathcal{X}, \mathcal{F}) = n \geq 1$. Let $u = \{u^1, \ldots, u^n\}$ and $\bar{u} = \{\bar{u}^1, \ldots, \bar{u}^n\}$ be two bases. Then $\exists$ <u>$n \times n$</u> <u>invertible</u> matrix P such that

$$[x]_{\bar{u}} = P[x]_u \quad \text{and}$$

$$P = [P_1 | \cdots | P_n], \qquad P_i = [u^i]_{\bar{u}}.$$

Follows: $[x]_u = \bar{P}[x]_{\bar{u}}$, $\bar{P} = [\bar{P}_1, \ldots, \bar{P}_n]$,

$\bar{P}_i = [\bar{u}^i]_u$ **AND** $\bar{P} = P^{-1}$.

$\left\{ \therefore \text{ Compute either } P \text{ or } \bar{P} \text{ and get the other by matrix inversion.} \right\}$

Today: Example (Posted on Canvas)

Def. Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be two vector spaces (over the same field). $L: \mathcal{X} \to \mathcal{Y}$ is a <u>linear operator</u> if $\forall \alpha, \beta \in \mathcal{F}$, $\forall x, z \in \mathcal{X}$,

$$L(\alpha x + \beta z) = \alpha L(x) + \beta L(z).$$

<u>Equivalent:</u>

$$L(x+z) = L(x) + L(z) \quad \text{(additive)}$$

$$L(\alpha x) = \alpha L(x) \quad \text{(homogeneous)}$$

<u>Examples</u> $\mathcal{F}$ = field.

1. $A \in \mathcal{F}^{n \times m}$ ($n \times m$ matrix coeff in $\mathcal{F}$)
Then $L: \mathcal{F}^m \to \mathcal{F}^n$ by $\forall x \in \mathcal{F}^m$, $L(x) = Ax$ is a linear operator.

2. $\mathcal{X} = \mathbb{P}_3(t) = \{\text{poly's of degree} \leq 3\}$, $\mathcal{F} = \mathbb{R}$, $\mathcal{Y} = \mathbb{P}_3(t)$. Then $L: \mathcal{X} \to \mathcal{Y}$ by $\forall p \in \mathcal{X}$, $L(p) = \frac{d}{dt} p$ is a linear operator.

Def. Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be finite-dimensional vector spaces with bases $\{u^1, \dots, u^m\}$ for $\mathcal{X}$ and $\{v^1, \dots, v^n\}$ for $\mathcal{Y}$. Let $L : \mathcal{X} \to \mathcal{Y}$ be a linear operator. Then $\exists$ a matrix $A : \mathcal{F}^m \longrightarrow \mathcal{F}^n$ such that $\forall x \in \mathcal{X}$,

$$[L(x)]_{\{v\}} = A\,[x]_{\{u\}}.$$

A is called the matrix representation of L.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "L"] \arrow[d, "{[\cdot]_u}"'] & \mathcal{Y} \arrow[d, "{[\cdot]_v}"] \\ \mathcal{F}^m \arrow[r, "A"] & \mathcal{F}^n \end{tikzcd}$$

<u>Thm</u>. $(\mathcal{X},\mathcal{F})$, $(\mathcal{Y},\mathcal{F})$ finite dim vector spaces with bases $\{u^1,\dots,u^m\}$ for $\mathcal{X}$ and $\{v^1,\dots,v^n\}$ for $\mathcal{Y}$. Let $L:\mathcal{X}\to\mathcal{Y}$ be a linear operator. Then $\exists$ $A:\mathcal{F}^m\to\mathcal{F}^n$, the matrix representation of L, and $A=[A_1|A_2|\cdots|A_m]$ and

How to compute matrix rep. $\longrightarrow$ $A_i=[L(u^i)]_{\{v\}}$.

Example, then the proof!

$\mathcal{F}=\mathbb{R}$, $\mathcal{X}=\mathbb{P}_3(t)$, $\mathcal{Y}=\mathbb{P}_3(t)$.
Basis for $\mathcal{X}$ is $u=\{1,t,t^2,t^3\}$ and for $\mathcal{Y}$ is $v=\{1,t,t^2,t^3\}$.
$L:\mathcal{X}\to\mathcal{Y}$, $L(p)=\frac{d}{dt}p$.

Find the matrix rep. of L!

$A=[A_1|A_2|A_3|A_4]$, $A_i=[\frac{d}{dt}u^i]_{\{v\}}$.

$$A_1 = [L(1)]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_2 = [L(t)]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_3 = [L(t^2)]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$A_4 = [L(t^3)]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Does this make sense?

$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

and

$$[p(t)]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$A[p(t)]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix}$$

Does this correspond to differentiating the polynomial $p(t)$? We see that

$$\frac{d}{dt} p(t) = a_1 + 2a_2 t + 3a_3 t^2$$

$$[\frac{d}{dt} p(t)]_{\{1,t,t^2,t^3\}} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix}$$

and thus, yes indeed,

$$A[p(t)]_{\{1,t,t^2,t^3\}} = [\frac{d}{dt} p(t)]_{\{1,t,t^2,t^3\}}$$

.

Exercise: change basis to $\{1, t, \frac{t^2}{2!}, \frac{t^3}{3!}\}$. Find A.

Remark: $(\mathcal{X}, \mathcal{F})$ basis $\{u^1, \dots, u^m\}$, $(\mathcal{X}, \mathcal{F})$ with basis $\{\bar{u}^1, \dots, \bar{u}^m\}$. Define the indentity (linear) operator by $L(x) = x \ \forall x \in \mathcal{X}$ $(L : \mathcal{X} \to \mathcal{X})$. The matrix representatio of L is the change of basis matrix!!!

$$[L(u^i)]_{\{\bar{u}\}} = [u^i]_{\{\bar{u}\}}.$$

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "Id"] \arrow[d, "{[\ ]_u}"'] & \mathcal{X} \arrow[d, "{[\ ]_{\bar{u}}}"] \\
\mathcal{F}^m \arrow[r, "A"] & \mathcal{F}^m
\end{tikzcd}$$

Quick proof of matrix rep. theorem

$x \in \mathcal{X}$ $\quad x = \alpha_1 u^1 + \alpha_2 u^2 + \cdots + \alpha_m u^m$

$$[x]_u = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}$$

In the theorem

$$A_i = [L(u^i)]_v \qquad 1 \le i \le m$$

$$L(x) = L(\alpha_1 u^1 + \alpha_2 u^2 + \cdots + \alpha_m u^m)$$
$$= \alpha_1 L(u^1) + \alpha_2 L(u^2) + \cdots + \alpha_m L(u^m)$$

Now, compute representations in $\mathcal{Y}$ of both sides

$$[L(x)]_v = \alpha_1 [L(u^1)]_v + \cdots + \alpha_m [L(u^m)]_v$$

$$= \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_m A_m$$

$$= [A_1 | A_2 | \cdots | A_m] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_m \end{bmatrix}$$

$$[L(x)]_v = A [x]_u$$

• Eigenvalues and Eigenvectors

<u>Def</u> Let A be an $n \times n$ matrix with complex coefficients. [Real matrix is a special case]. $\lambda \in \mathbb{C}$ is an eigenvalue of A (e-value) if $\exists\, (v \in \mathbb{C}^n, v \neq 0)$ such that $Av = \lambda v$. The vector $v \in \mathbb{C}^n$ is called an eigenvector (e-vector) corresponding to λ.

<u>Fact</u> λ is an e-value $\Leftrightarrow \det(\lambda I - A) = 0$

<u>Example</u> $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\det(\lambda I - A) = \lambda^2 + 1 = 0$

$\Leftrightarrow \lambda_1 = j$ and $\lambda_2 = -j$. (A real, but e-values not necessarily real!) (Complex conjugate pairs).

Moreover

$v^1 = \begin{bmatrix} 1 \\ j \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -j \end{bmatrix}$ $\left\{ \begin{matrix} \text{complex} \\ \text{conjugate} \\ \text{pairs} \end{matrix} \right\}$

<u>Def.</u> $\Delta(\lambda) : \det(\lambda I - A)$ characteristic polynomial. $\det(\lambda I - A) = 0$ characteristic equation.

Fundamental Thm of Algebra says that

$$\Delta(\lambda) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2} \cdots (\lambda-\lambda_p)^{m_p}$$

where $\{\lambda_1, \dots, \lambda_p\}$ are distinct and $m_1 + m_2 + \dots + m_p = n$.

m_i is called the multiplicity of λ_i.

Theorem A be $n \times n$ with complex coeff. Suppose $m_1 = m_2 = \cdots = m_n = 1$ (e-values are distinct) [$\lambda_i \neq \lambda_j$ for $i \neq j$]. Then the corresponding e-vectors form a basis for $(\mathbb{C}^n, \mathbb{C})$. That is, $\{v^1, \ldots, v^n\}$ is linearly independent over $\mathbb{C}$.

Proof to be posted here!

(You are not responsible for the proof)

Remark: Restatement of the theorem: If $\{\lambda_1, \cdots, \lambda_n\}$ are distinct then $\{v^1, \cdots, v^n\}$ is a basis for $(\mathbb{C}^n, \mathbb{C})$.

Proof: We prove the contrapositive and show there is a repeated e-value ($\lambda_i = \lambda_j$ for some $i \neq j$).

$\{v^1, \cdots, v^n\}$ linearly dependent $\Rightarrow \exists\ \alpha_1, \cdots, \alpha_n \in \mathbb{C}$, not all zero, such that $\alpha_1 v^1 + \cdots + \alpha_n v^n = 0 (*)$.

Without loss of generality, we can suppose $\alpha_1 \neq 0$. (that is, we can always reorder of e-values so that the first coefficient is nonzero.)

Because v^i is an e-vector,

$$(A-\lambda_j I)v^i = Av^i - \lambda_j v^i = \lambda_i v^i - \lambda_j v^i = (\lambda_i - \lambda_j)v^i$$

<u>Side Note:</u> It is an easy exercise to show

$(A-\lambda_2 I)(A-\lambda_3 I)\cdots(A-\lambda_n I)v^i = (\lambda_i-\lambda_2)(\lambda_i-\lambda_3)\cdots(\lambda_i-\lambda_n)v^i, 2 \leq i \leq n$

Let $i = 1$

$$(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_n)v^1$$

Let $i = 2$

$$(\lambda_2-\lambda_2)(\lambda_2-\lambda_3)\cdots(\lambda_2-\lambda_n)v^2 = 0$$

Etc.

Combining the above with $(*)$, we obtain

$$0 = (A-\lambda_2 I)(A-\lambda_3 I)\cdots(A-\lambda_n I)(\alpha_1 v^1 + \cdots + \alpha_n v^n)$$

$$= \alpha_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_n)v^1$$

We know $\alpha_1 \neq 0$, as stated above, and $v^1 \neq 0$, by definition of e-vectors.

$$\therefore 0 = (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_n)$$

At least one the terms $(\lambda_1 - \lambda_k)$, $2 \leq k \leq n$, must be zero, and thus there is a repeated e-value $\lambda_1 = \lambda_k$ for some $2 \leq k \leq n$. $\square$

Def. Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be vector spaces. $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ is a <u>linear operator</u> if for all $x, z \in \mathcal{X}$, $\alpha, \beta \in \mathcal{F}$,

$$\mathcal{L}(\alpha x + \beta z) = \alpha\mathcal{L}(x) + \beta\mathcal{L}(z)$$

Equivalently,

$$\mathcal{L}(x+z) = \mathcal{L}(x) + \mathcal{L}(z)$$
$$\mathcal{L}(\alpha x) = \alpha\mathcal{L}(x)$$

Easy Facts on Representations:

1. Addition of vectors in $(\mathcal{X}, \mathcal{F}) \equiv$ Addition of the representations in $(\mathcal{F}^n, \mathcal{F})$.

$$[x+y]_v = [x]_v + [y]_v$$

2. Scalar multiplication in $(\mathcal{X}, \mathcal{F}) \equiv$ Scalar multiplication with the representations in $(\mathcal{F}^n, \mathcal{F})$.

$$[\alpha x]_v = \alpha [x]_v$$

3. Once a basis is chosen, any n-dimensional vector space $(\mathcal{X}, \mathcal{F})$ "looks like" $(\mathcal{F}^n, \mathcal{F})$.

Change of Basis Matrix: Let $\{u^1, \cdots, u^n\}$ and $\{\bar{u}^1, \cdots, \bar{u}^n\}$ be two bases for $(\mathcal{X}, \mathcal{F})$. Is there a relation between $[x]_u$ and $[x]_{\bar{u}}$?

Theorem: $\exists$ an invertible matrix P, with coefficients in $\mathcal{F}$, such that $\forall x \in (\mathcal{X}, \mathcal{F})$, $[x]_{\bar{u}} = P[x]_u$. Moreover, $P = [P_1|P_2|\cdots|P_n]$ with $P_i = [u^i]_{\bar{u}} \in \mathcal{F}^n$ where P_i is the i^{th} column of the matrix P and $[u^i]_{\bar{u}}$ is the representation of u^i with respect to $\bar{u}$.

<u>Proof:</u> Let $x = \alpha_1 u^1 + \cdots + \alpha_n u^n = \bar{\alpha}_1 \bar{u}^1 + \cdots + \bar{\alpha}_n \bar{u}^n$.

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = [x]_u$$

$$\bar{\alpha} = \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_n \end{bmatrix} = [x]_{\bar{u}}$$

$$\bar{\alpha} = [x]_{\bar{u}} = \left[\sum_{i=1}^{n} \alpha_i u^i \right]_{\bar{u}} = \sum_{i=1}^{n} \alpha_i [u^i]_{\bar{u}} = \sum_{i=1}^{n} \alpha_i P_i = P\alpha.$$

Therefore, $\bar{\alpha} = P\alpha = P[x]_u$.

Now we need to show that P is invertible:
Define $\bar{P} = [\bar{P}_1|\bar{P}_2|\cdots|\bar{P}_n]$ with $\bar{P}_i = [\bar{u}^i]_u$.
Do the same calculations and obtain $\alpha = \bar{P}\bar{\alpha}$.
Then, we can obtain that $\alpha = \bar{P}P\alpha$ and $\bar{\alpha} = P\bar{P}\bar{\alpha}$.
Therefore, $P\bar{P} = \bar{P}P = I$.
In conclusion, $\bar{P}$ is the inverse of P ($\bar{P} = P^{-1}$). $\square$

Example: $\mathcal{X}$=$\{2 \times 2$ matrices with real coefficients$\}$, $\mathcal{F} = \mathbb{R}$.

$$u = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
$$\bar{u} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

We have following relations:

$$\alpha = P\bar{\alpha}, P_i = [u^i]_{\bar{u}}, \quad \bar{\alpha} = \bar{P}\alpha, \bar{P}_i = [\bar{u}^i]_u$$
$$\bar{P}^{-1} = P, P^{-1} = \bar{P}$$

Typically, compute the easier of P or $\bar{P}$, and compute the other by inversion. For this example, we choose to compute $\bar{P}$

$$\bar{P}_1 = [\bar{u}^1]_u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{P}_2 = [\bar{u}^2]_u = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\bar{P}_3 = [\bar{u}^3]_u = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\bar{P}_4 = [\bar{u}^4]_u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $\bar{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $P = \bar{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .5 & .5 & 0 \\ 0 & .5 & -.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

What if we did it the other direction?

$$P_1 = [u^1]_{\bar{u}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_2 = [u^2]_{\bar{u}} = \begin{bmatrix} 0 \\ .5 \\ .5 \\ 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + .5 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_3 = [u^3]_{\bar{u}} = \begin{bmatrix} 0 \\ .5 \\ -.5 \\ 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - .5 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_4 = [u^4]_{\bar{u}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .5 & .5 & 0 \\ 0 & .5 & -.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\bar{P} = P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2+\alpha_3 \\ \alpha_2-\alpha_3 & \alpha_4 \end{bmatrix} \iff \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

Four equations in four unknowns.