ROB 501 - HW # 5

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$\boxed{1.}$

(a)

e-vectors v and e-values λ for a matrix A satisfy $(A - \lambda I)v = 0$

$$\det\left(\begin{bmatrix} 1-\lambda & 4 & 10 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = (1-\lambda)(2-\lambda)(3-\lambda)$$

$\Rightarrow$ e-values $\lambda = 1, 2, 3$

for $\lambda = 1$

$$\begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{aligned} v_1 + 4v_2 + 10v_3 &= v_1 \\ 2v_2 &= v_2 \\ 3v_3 &= v_3 \end{aligned} \Rightarrow \begin{aligned} v_1 &= 1 \\ v_2 &= 0 \\ v_3 &= 0 \end{aligned} \Rightarrow v^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = 2$

$$\begin{aligned} v_1 + 4v_2 + 10v_3 &= 2v_1 \\ 2v_2 &= 2v_2 \\ 3v_3 &= 2v_3 \end{aligned} \Rightarrow \begin{aligned} v_1 &= 4 \\ v_2 &= 1 \\ v_3 &= 0 \end{aligned} \Rightarrow v^2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = 3$

$$\begin{aligned} v_1 + 4v_2 + 10v_3 &= 3v_1 \\ 3v_2 &= 2v_2 \\ 3v_3 &= 3v_3 \end{aligned} \Rightarrow \begin{aligned} v_1 &= 5 \\ v_2 &= 0 \\ v_3 &= 1 \end{aligned} \Rightarrow v^3 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

e-vectors we got from MATLAB are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0.9701 \\ 0.2425 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0.9806 \\ 0 \\ 0.1961 \end{bmatrix}$

, which are unit vector solutions for the result I derived.

(b)

$$\det(A_4 - \lambda I) = (3-\lambda)(3-\lambda)(2-\lambda) \Rightarrow \lambda = 3, 3, 2$$

for $\lambda = 3$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 3 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{aligned} 3v_1 + v_2 &= 3v_1 \\ 3v_2 &= 3v_2 \\ 2v_3 &= 3v_3 \end{aligned} \Rightarrow \begin{aligned} v_1 &= 1 \\ v_2 &= 0 \\ v_3 &= 0 \end{aligned} \Rightarrow$$

geometric multiplicity is 1. This is the only e-vector for this e-value.

for $\lambda = 2$

$$\begin{aligned} 3v_1 + v_2 &= 2v_1 \\ 3v_2 &= 2v_2 \\ 2v_3 &= 2v_3 \end{aligned} \Rightarrow \begin{aligned} v_1 &= 0 \\ v_2 &= 0 \\ v_3 &= 1 \end{aligned}$$

e-values: 3, 3, 2

e-vectors: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\Rightarrow$ can't form basis for $\mathbb{R}^3$, since we need at least three vectors but only have two.

$\boxed{2.}$

$$
\begin{aligned}
\det(\lambda I - A) &= \det(\lambda I - PBP^{-1}) \\
&= \det(\lambda P I P^{-1} - PBP^{-1}) \\
&= \det(P \lambda I P^{-1} - PBP^{-1}) \\
&= \det(P(\lambda I - B)P^{-1}) \\
&= \det(P)\det(\lambda I - B)\det(P^{-1}) \\
&= \det(\lambda I - B)\det(PP^{-1}) \\
&= \det(\lambda I - B)
\end{aligned}
$$

$\Rightarrow$ Similar matrix A and B have the same characteristic equation. $\square$

$\boxed{3.}$

$$A_3 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

from Prob. 1 $\Rightarrow$ $\lambda = 1, 2, 3$, $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$

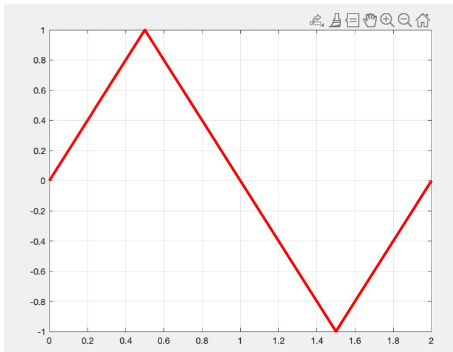
$$
\begin{aligned}
M &= [v_1 | v_2 | v_3], \quad AM = [Av_1 | Av_2 | Av_3] \\
&\qquad\qquad\qquad\quad = [\lambda_1 v_1 | \lambda_2 v_2 | \lambda_3 v_3] \\
&\qquad\qquad\qquad\quad = M \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}
\end{aligned}
$$

$\Rightarrow A = M \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} M^{-1}$

$\Rightarrow A$ is similar to a diagonal matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ ※

4.

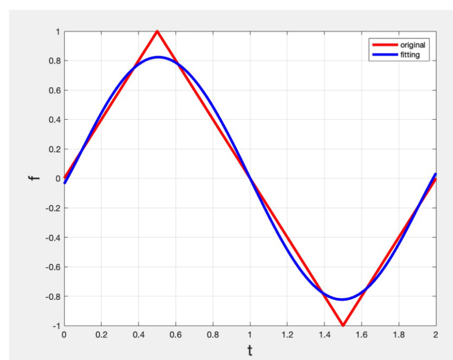
(a)



(b)

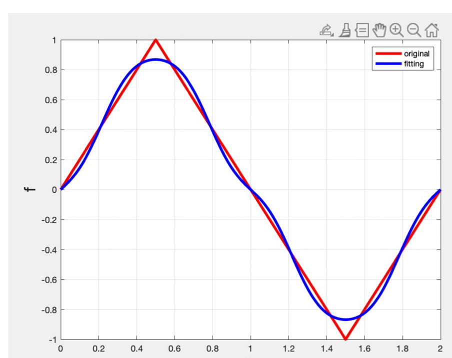
fit for the function is $c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5$

where
$$\begin{cases} c_0 = -0.0370 \\ c_1 = 2.2158 \\ c_2 = 2.8901 \\ c_3 = -11.2271 \\ c_4 = 7.6978 \\ c_5 = -1.5396 \end{cases}$$



(c) fit for the function is $\sum_{k=1}^{5} c_k \sin(k\pi t)$

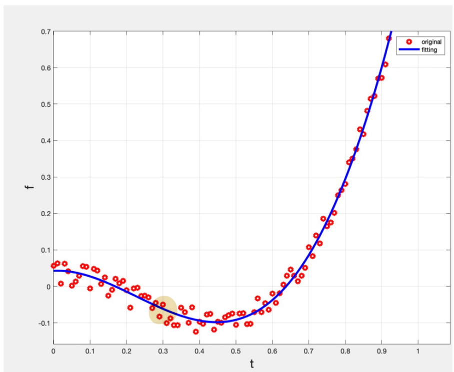
where
$$\begin{cases} c_1 = 0.8106 \\ c_2 = 0 \\ c_3 = -0.0901 \\ c_4 = 0 \\ c_5 = 0.0325 \end{cases}$$



$\boxed{5.}$

We use polynomial function to fit the data. Specifically, the basis of our fit is $\{1, t, t^2, t^3, t^4, t^5\}$

The fit is $c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5$, where



$$\begin{cases} c_0 = 0.0428 \\ c_1 = 0.0505 \\ c_2 = -2.4087 \\ c_3 = 3.6850 \\ c_4 = -0.2325 \\ c_5 = -0.1186 \end{cases}$$

$$f'(0.3) = c_1 + 2c_2 \cdot 0.3 + 3c_3 \cdot 0.3^2 + 4c_4 \cdot 0.3^3 + 5c_5 \cdot 0.3^4$$

$$= -0.4297 \quad \#$$

6.

- Definition in lecture: Let $(\mathcal{X}, \mathbb{C})$ be a vector space

  (a) $\forall x, y \in \mathcal{X}$, $\langle x, y\rangle = \overline{\langle y, x\rangle}$

  (b) $\forall x_1, x_2, y \in \mathcal{X}$, $\forall \alpha_1, \alpha_2 \in \mathbb{C}$
  $\langle \alpha_1 x_1 + \alpha_2 x_2, y\rangle = \alpha_1 \langle x_1, y\rangle + \alpha_2 \langle x_2, y\rangle$

  (c) $\forall x \in \mathcal{X}$, $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$

We want to show that on $(\mathbb{C}^n, \mathbb{C})$, the definition of $\langle x, y\rangle = x^T \bar{y}$ satisfies (a)(b)(c)

(a) $\langle x, y\rangle = x^T \bar{y} = \bar{y}^T x = \overline{(y^T \bar{x})} = \overline{\langle y, x\rangle}$. □

(b) $\langle \alpha_1 x_1 + \alpha_2 x_2, y\rangle = (a_1 x_1 + a_2 x_2)^T \bar{y} = (a_1 x_1)^T \bar{y} + (a_2 x_2)^T \bar{y}$
$= a_1 x_1^T \bar{y} + a_2 x_2^T \bar{y}$
$= a_1 \langle x_1, y\rangle + a_2 \langle x_2, y\rangle$. □

(c) $\langle x, x\rangle = x^T \bar{x}$, let $x = \begin{bmatrix} a_1 + b_1 i \\ \vdots \\ a_n + b_n i \end{bmatrix}$

$\langle x, x\rangle = \sum_{i=1}^{n} (a_i^2 + b_i^2) \geq 0$, and $\langle x, x\rangle = 0 \iff x = \mathbf{0}$. □

- Definition in 6.2.1, page 185, of Nagy

  (a) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

  (b) $\langle x, (ay + bz)\rangle = a\langle x, y\rangle + b\langle x, z\rangle$

  (c) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$

We want to show that on $(\mathbb{C}^n, \mathbb{C})$, the definition of $\langle x, y\rangle = \bar{x}^T y$ satisfies (a)(b)(c)

(a) $\langle x, y\rangle = \bar{x}^T y = y^T \bar{x} = \overline{(\bar{y}^T x)} = \overline{\langle y, x\rangle}$. □

(b) $\langle x, (ay+bz)\rangle = \bar{x}^T(ay+bz)$

$= \bar{x}^T ay + \bar{x}^T bz$

$= a\bar{x}^T y + b\bar{x}^T z$

$= a\langle x,y\rangle + b\langle x,z\rangle$. □

(c)

$\langle x, x\rangle = \bar{x}^T x$, let $x = \begin{bmatrix} a_1+b_1 i \\ \vdots \\ a_n+b_n i \end{bmatrix}$.

$\langle x, x\rangle = \sum_{i=1}^{n}(a_i^2+b_i^2) \geq 0$, and $\langle x,x\rangle = 0 \iff x = \mathbf{0}$. □

$\boxed{7.}$

We need to verify that $\langle p_0, p_3\rangle = 0$, $\langle p_1, p_2\rangle = 0$

$\langle p_0, p_3\rangle = \int_{-1}^{1} 1\cdot\frac{1}{2}(5x^3-3x)\,dx$

$= (5/8 \cdot x^4 - 3/2 \cdot x^2)\Big|_{-1}^{1}$

$= 0$ ※

$\langle p_1, p_2\rangle = \int_{-1}^{1} x\cdot\frac{1}{2}(3x^2-1)\,dx$

$= \int_{-1}^{1}(3/2 \cdot x^3 - x)\,dx$

$= (3/8\, x^4 - x^2/2)\Big|_{-1}^{1}$

$= 0$ ※

8.

(a)

The formula can be proven by checking that $(A+BCD)$ times its alleged inverse on the right side gives the identity matrix

$$(A+BCD)\left[A^{-1}-A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}\right]$$

$$= \left[I - B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}\right] + \left[BCDA^{-1} - BCDA^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}\right]$$

$$= \left[I + BCDA^{-1}\right] - \left[B(C^{-1}+DA^{-1}B)^{-1}DA^{-1} + BCDA^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}\right]$$

$$= I + BCDA^{-1} - (B + BCDA^{-1}B)(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$$

$$= I + BCDA^{-1} - BC(C^{-1}+DA^{-1}B)(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$$

$$= I + BCDA^{-1} - BCDA^{-1}$$

$$= I. \quad \square$$

(b)

$$(A+BCD)^{-1}$$

$$= A^{-1} - A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$$

$$= \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 1 & \\ & & & & 2 \end{bmatrix} - \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 1 & \\ & & & & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} \Bigg(5 + \underbrace{\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 1 & \\ & & & & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}}_{27} \Bigg)^{-1} \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 1 & \\ & & & & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 1 & \\ & & & & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ 6 \end{bmatrix} \cdot \frac{1}{32} \cdot \begin{bmatrix} 1 & 0 & 4 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \frac{1}{32} \begin{bmatrix} 1 & 0 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 16 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 24 & 0 & 36 \end{bmatrix}$$

#

$\boxed{9.}$

(a) To prove that $f(x)$ is a norm, we need to show the three norm properties.

① Positive definiteness

For all nonzero vectors x, and a positive definite matrix A, $f(x) = (x^TAx)^{1/2} > 0$

For $x = \mathbf{0}$, $f(x) = (x^TAx)^{1/2} = 0$ (trivial)

$\Rightarrow \forall x \in \mathcal{X}, f(x) \geq 0$ and $f(x) = 0 \Leftrightarrow \|x\| = 0$. $\square$

② Triangular inequality

$$\begin{aligned}\forall x, y \in \mathcal{X},\ f(x+y) &= [(x+y)^TA(x+y)]^{1/2} \\ &= [(x^T+y^T)A(x+y)]^{1/2} \\ &= (x^TAx + y^TAy + x^TAy + y^TAx)^{1/2} \\ &= (f(x)^2 + f(y)^2 + 2x^TAy)^{1/2} \\ &= (f(x)^2 + f(y)^2 + 2x^TA^{1/2}A^{1/2}y)^{1/2} \\ &= [f(x)^2 + f(y)^2 + 2(A^{1/2}x)^T(A^{1/2}y)]^{1/2} \\ &= [f(x)^2 + f(y)^2 + 2\langle A^{1/2}x, A^{1/2}y\rangle]^{1/2} \\ &\leq [f(x)^2 + f(y)^2 + 2(\langle A^{1/2}x, A^{1/2}x\rangle^{1/2} \cdot \langle A^{1/2}y, A^{1/2}y\rangle^{1/2})]^{1/2} \\ &= [f(x)^2 + f(y)^2 + 2((x^TA^{1/2}A^{1/2}x)^{1/2} \cdot (y^TA^{1/2}A^{1/2}y)^{1/2})]^{1/2} \\ &= \{f(x)^2 + f(y)^2 + 2((x^TAx)^{1/2} \cdot (y^TAy)^{1/2})\}^{1/2} \\ &= \{f(x)^2 + f(y)^2 + 2f(x)f(y)\}^{1/2} \\ &= [(f(x)+f(y))^2]^{1/2} \\ &= f(x) + f(y)\end{aligned}$$

A is symmetric positive definite, it admits a squared root

for $(\mathbb{R}^n, \mathbb{R})$, $\langle x, y\rangle = x^Ty$

Cauchy-Schwartz

$\Rightarrow \forall x, y \in \mathcal{X},\ f(x+y) \leq f(x) + f(y)$. $\square$

③

$\forall \alpha \in \mathcal{F}, \forall x \in \mathcal{X}$

$$\begin{aligned}f(\alpha x) = [(\alpha x)^TA(\alpha x)]^{1/2} &= (\alpha^2)^{1/2} \cdot (x^TAx)^{1/2} \\ &= |\alpha| \cdot (x^TAx).\ \square\end{aligned}$$

$\Rightarrow f(x)$ is a norm!

(b)

$$f_1(A) = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|_1 = 1}} \|Ax\|_1$$

We represent A by its column vectors $A = \left[a_1 \middle| a_2 \cdots \middle| a_n \right]$

$$f_1(A) = \max_{\|x\|_1 = 1} \|Ax\|_1$$

$$= \max_{\|x\|_1 = 1} \left\| (a_1 | \cdots | a_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_1$$

$$= \max_{\|x\|_1 = 1} \| a_1 x_1 + a_2 x_2 + \cdots a_n x_n \|_1$$

triangular inequality

$$\leq \max_{\|x\|_1 = 1} \|a_1 x_1\|_1 + \cdots + \|a_n x_n\|_1$$

$$= \max_{\|x\|_1 = 1} |x_1| \, \|a_1\|_1 + \cdots + |x_n| \, \|a_n\|_1$$

Let J be chosen so that $\max_{1 \leq j \leq n} \|a_j\|_1 = \|a_J\|_1$

$$\leq \max_{\|x\|_1 = 1} |x_1| \, \|a_J\|_1 + \cdots + |x_n| \, \|a_J\|_1$$

$$= \max_{\|x\|_1 = 1} (|x_1| + \cdots + |x_n|) \|a_J\|_1$$

$$= \|a_J\|_1$$

Also, $\|a_J\|_1 = \|Ae_J\|_1 \leq \max_{\|x\|_1 = 1} \|Ax\|_1$

Hence, $\|a_J\|_1 \leq \max_{\|x\|_1 = 1} \|Ax\|_1 \leq \|a_J\|_1$

which implies that $\max_{\|x\|_1 = 1} \|Ax\|_1 = \|a_J\|_1 = \max_{1 \leq j \leq n} \|a_j\|_1$ (maximum absolute column sum)

For $f_\infty(A)$

$$f_\infty(A) = \sup_{\substack{x\in\mathbb{R}^n \\ \|x\|_\infty=1}} \|Ax\|_\infty$$

$$= \max_{\|x\|_\infty=1} \left\| \begin{pmatrix} \hat{a}_1^T \\ \vdots \\ \hat{a}_n^T \end{pmatrix} x \right\|_\infty = \max_{\|x\|_\infty=1} \left\| \begin{pmatrix} \hat{a}_1^T x \\ \vdots \\ \hat{a}_n^T x \end{pmatrix} \right\|_\infty$$

$$= \max_{\|x\|_\infty=1} \left(\max_{1\le i\le n} |\hat{a}_i^T x| \right) = \max_{\|x\|_\infty=1} \max_{1\le i\le n} \left| \sum_{j=1}^n \hat{a}_{i,j} x_j \right| \le \max_{\|x\|_\infty=1} \max_{1\le i\le n} \sum_{j=1}^n |\hat{a}_{i,j} x_j|$$

$$= \max_{\|x\|_\infty=1} \max_{1\le i\le n} \sum_{j=1}^n |\hat{a}_{i,j}||x_j| \le \max_{\|x\|_\infty=1} \max_{1\le i\le n} \sum_{j=1}^n (|\hat{a}_{i,j}| \|x\|_\infty) = \max_{1\le i\le n} \sum_{j=1}^n |\hat{a}_{i,j}|$$

$$= \max_{1\le i\le n} \|\hat{a}_i\|_1$$

So that $\|A\|_\infty \le \max_{1\le i\le n} \|\hat{a}_i\|_1$

We also want to show that $\|A\|_\infty \ge \max_{1\le i\le n} \|\hat{a}_i\|_1$. Let k be such that $\max_{1\le i\le n} \|\hat{a}_i\|_1 = \|\hat{a}_k\|_1$

and pick $y = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$ so that $\hat{a}_k^T y = |a_{k,1}| + \cdots + |a_{k,n}| = \|\hat{a}_k\|_1$ (pick $|\psi_i|=1$ and $\psi_i a_{k,i} = |a_{k,i}|$)

Then
$$\max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{\|x\|_\infty=1} \left\| \begin{pmatrix} \hat{a}_1^T \\ \vdots \\ \hat{a}_n^T \end{pmatrix} x \right\|_\infty \ge \left\| \begin{pmatrix} \hat{a}_1^T \\ \vdots \\ \hat{a}_n^T \end{pmatrix} y \right\|_\infty$$

$$= \left\| \begin{pmatrix} \hat{a}_1^T y \\ \vdots \\ \hat{a}_n^T y \end{pmatrix} \right\|_\infty \ge |\hat{a}_k^T y| = \hat{a}_k^T y = \|\hat{a}_k\|_1$$

$$= \max_{1\le i\le n} \|\hat{a}_i\|_1$$

$$\max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{1\le i\le n} \|\hat{a}_i\|_1 = \max_{1\le i\le n} \sum_{j=1}^n |\hat{a}_{i,j}|$$

(max absolute row sum)

I discussed this HW with Wan-Yi Yu ※ 74732586