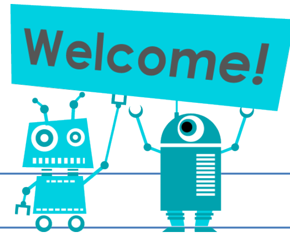
ROB 501

04 Sept 2018



Warnings about the course ==(It is assumed that you have read this statement and accept it):== We do lots of proofs and no realistic examples. This is a theory course on mathematical methods. If you are seeking practical knowledge about robots or mechanical systems, this is not your course. We cover linear algebra, and thus if you have had EECS 560 =

Introduction to Mathematical Arguments

Notation:

$\mathbb{N} = \{1, 2, 3, \cdots\}$ Natural numbers or counting numbers

$\mathbb{Z} = \mathcal{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ Integers or whole numbers

$\mathbb{Q} = \left\{ \frac{m}{q} | m, q \in \mathbb{Z}, q \neq 0, \text{no common factors (reduce all fractions)} \right\}$ Rational numbers

$\mathbb{R}$ = Real numbers

$\mathbb{C} = \{\alpha + j\beta \mid \alpha, \beta \in \mathbb{R}, j^2 = -1\}$ Complex numbers

$\forall$ means "for every", "for all", "for each".

$\exists$ means "for some", "there exist(s)", "there is/are", "for at least one".

$\in$ means "element of" as in "$x \in A$" (x is an element of the set A)

Every non-zero real number has a multiplicative inverse

$\forall (x \in \mathbb{R}, x \neq 0)$ $\exists\, y \in \mathbb{R}$ such that $xy = 1$

Obvious: the choice of y depends on x

Every real number x can be arbitrarily closely approximated by a rational number.

$\forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$, $\exists\, q \in \mathbb{Q}$ s.t. $|x-q| < 1/n$.

Ex $x = \pi$ $n = 10$ $q = 3.1$

$n = 100$ $q = 3.14$

q varies with n.

$\sim$ means "not". In books, and some of our handouts, you see $\neg$.

$p \Rightarrow q$ means "if p is true, then q is true.".

$p \iff q$ means "p is true if and only if q is true".

$p \iff q$ is logically equivalent to:

(a) $p \Rightarrow q$ and

(b) $q \Rightarrow p$.

* The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$ (logically equivalent).

* The converse of $p \Rightarrow q$ is $q \Rightarrow p$.

Relation: $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$

However, in general, $(p \Rightarrow q)$ DOES NOT IMPLY $(q \Rightarrow p)$, and vice-versa

$\square$ = Q.E.D. (Latin:"quod erat demonstrandum" = "thus it was demonstrated")

Review of Proof Techniques

Direct Proofs: We derive a result by applying the rules of logic to the given assumptions, definitions, and known theorems.

show = prove

HW 1: Direct Proofs

Example

Def. An integer n is even if $n = 2k$ for some integer k, and it is odd otherwise.

Very important remark: In a definition, "if" = "if and only if".

Proposition The sum of two odd integers is even.

Proof: [Direct]. Let a and b be two odd integers. Hence, $\exists$ two integers k_1 and k_2 such that

$a = 2k_1 + 1$ and

$b = 2k_2 + 1$.

$a + b = (2k_1+1) + (2k_2+1) = 2k_1 + 2k_2 + 2$

$= 2(k_1 + k_2 + 1)$

is even because $k_1 + k_2 + 1$ is an integer.

$\square$

Proof by Contrapositive To establish $p \Rightarrow q$ we show instead that $\sim q \Rightarrow \sim p$. (Logically Equivalent)

Proposition Let n be an integer. If n^2 is even, then n is even.

Proof: p: n^2 is even $\quad \sim p$: n^2 is odd

q: n is even $\quad \sim q$: n is odd.

To show: n odd $\Rightarrow n^2$ odd

n is odd $\Rightarrow \exists\, k \in \mathbb{Z}$ s.t $n = 2k+1$

$(n^2) = (2k+1)^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$ is odd

because $2k^2 + 2k$ is an integer.

$\square$

Proof by Exhaustion: Reduce the proof to a finite number of cases and then check every one of them.

Proofs by Induction

First Principle of Induction (Standard)

Let $P(n)$ be a statement about the counting numbers with the following properties:

a) Base case: $P(1)$ is true.

b) For $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ is true.

Then, $P(n)$ is true for all $n \geq 1$.

Claim [Formula for the sum of odd integers] For all $n \geq 1$,

$$1+3+5+\cdots+(2n-1) = n^2$$

$P(n)$: $1+3+5\cdots+(2n-1) = n^2$ $\quad n \geq 1$

Base Case: $P(1)$ holds because $1 = (1)^2$.

Induction Step: We assume $P(k)$ is true, namely $P(k): 1+3+5+\cdots+(2k-1) = k^2$ and we attempt to prove that $P(k+1)$ is also true:

$$P(k+1): 1+3+\cdots+(2k-1)+(2[k+1]-1) = (k+1)^2$$

We add $2(k+1)-1$ to both sides of the statement for $P(k)$

$$\begin{aligned} 1+3+\cdots+(2k-1)+(2[k+1]-1) &= k^2+2(k+1)-1 \\ &= k^2+2k+1 \\ &= (k+1)^2 \end{aligned}$$

Hence $P(k) \Rightarrow P(k+1)$ and hence $P(n)$ is true for all $n \geq 1$. $\square$

Question What if you want to prove something is true for $k_0 \geq 19$.

Induction $k \geq 19$, let $\tilde{P}(k) = P(k+18)$. Then do induction on $\tilde{P}(k)$.

Second Principle of Induction (Strong Induction): Let $P(n)$ be a statement about the natural numbers with the following properties:

a) Base case: $P(1)$ is true.

b) If $P(j)$ is true for all $1 \leq j \leq k$, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \geq 1$.

Facts Two induction methods are equivalent! But sometimes the second one is easier to apply.

$\wedge$ = and $\qquad p_1 \wedge p_2 = p_1$ and p_2

$\tilde{P}(k) = P(1) \wedge P(2) \wedge \cdots \wedge P(k)$

Do ordinary induction on $\tilde{P}(k)$

Example A natural number $n \geq 2$ is **composite** if $\exists\ a, b \in \mathbb{N}$ such that $n = ab$ and $2 \leq a, b \leq n-1$. Otherwise n is prime.

1 is neither prime nor composite.

Theorem [Fundamental Theorem of Arithmetic] Every natural number $n \geq 2$ can be written as a product of one or more primes.