KUAN-TING LEE # 50036744

HW 04 - ROB 501

1.

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_5 \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 2 & 1 & 3 \\ 2 & 0 & 8 & 1 & 3 \\ -1 & 0 & -4 & 1 & 0 \\ 3 & 2 & 8 & 1 & 6 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix}$$

$$= A \alpha$$

rank $(A) = 3$

dimension $= 3$ #

2.

$$\begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 2 \\ -3 \end{bmatrix}$$ #

$\boxed{3.}$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$u_1 s = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ u_2 s = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},\ u_3 s = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$[x]_u = P\,[x]_e,\quad P = [P_1 | P_2 | P_3]$$

$$P_i = [e^i]_u$$

In order to simplify the calculation, we choose to derive $\bar{P} = P^{-1}$ (setting reference to be standard basis)

$$\overline{P_i} = [u^i]_e$$

$$\overline{P_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \overline{P_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},\ \overline{P_3} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \bar{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\Rightarrow P = \bar{P}^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

※

4.

$[x]_R = P[x]_W$, $P = [P_1 | P_2]$

$P_i = [w^i]_R$

$P_1 = [w^1]_R = [X_W]_R = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$

$P_2 = [w^2]_R = [Y_W]_R = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$

$P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ ※

5. (a)

We must show that M is a linearly independent set and $\text{Span}(M) = \mathbb{R}^{2,2}$

$\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = 0$, only when $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

$$\begin{cases} \alpha_3 + \alpha_4 = 0 \\ \alpha_1 - \alpha_2 = 0 \\ \alpha_1 + \alpha_2 = 0 \\ \alpha_3 - \alpha_4 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \Rightarrow \text{linearly independent}$$

- To prove that $\text{span}(M) = \mathbb{R}^{2,2}$, we must show $\begin{cases} \text{span}(M) \subseteq \mathbb{R}^{2,2} \\ \mathbb{R}^{2,2} \subseteq \text{span}(M) \end{cases}$

(1) For the first inclusion, let $m \in \text{span}(M)$, there exists $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ such that $m = \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = \begin{bmatrix} \alpha_3 + \alpha_4 & \alpha_1 - \alpha_2 \\ \alpha_1 + \alpha_2 & \alpha_3 - \alpha_4 \end{bmatrix} \in \mathbb{R}^{2,2}$

$\Rightarrow \text{span}(M) \subseteq \mathbb{R}^{2,2}$ ($\because \alpha_3 + \alpha_4 \in \mathbb{R}$, $\alpha_1 - \alpha_2 \in \mathbb{R}$, $\alpha_1 + \alpha_2 \in \mathbb{R}$, $\alpha_3 - \alpha_4 \in \mathbb{R}$)

(2) For second inclusion, let $m \in \mathbb{R}^2$, there exist $a, b, c, d \in \mathbb{R}$, s.t. $m = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4$, where $\begin{cases} \alpha_1 = (b+c)/2 \in \mathbb{R} \\ \alpha_2 = (c-b)/2 \in \mathbb{R} \\ \alpha_3 = (a+d)/2 \in \mathbb{R} \\ \alpha_4 = (a-d)/2 \in \mathbb{R} \end{cases}$

$\Rightarrow \mathbb{R}^{2,2} \subseteq \text{span}(M)$

$\Rightarrow \text{span}(M) = \mathbb{R}^{2,2}$

$\Rightarrow M$ is a basis of $\mathbb{R}^{2,2}$. $\square$

(b)

$\alpha = \begin{bmatrix} 2.5 \\ 0.5 \\ 2.5 \\ -1.5 \end{bmatrix}$ ※

$\boxed{6.}$

(a)

let $\alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2 = r(x) = 2 + 3x - x^2$

$$\Rightarrow \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \#$$

(b)

let $\alpha_0 q_0 + \alpha_1 q_1 + \alpha_2 q_2 = r(x) = 2 + 3x - x^2$

$$\alpha_0 + \alpha_1 - \alpha_1 x + \alpha_2 x + \alpha_2 x^2 = 2 + 3x - x^2$$

$$\Rightarrow \begin{cases} \alpha_0 + \alpha_1 = 2 \\ \alpha_2 - \alpha_1 = 3 \\ \alpha_2 = -1 \end{cases} \Rightarrow \alpha_1 = -4, \ \alpha_0 = 6$$

$$\Rightarrow \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix} \quad \#$$

$\boxed{7.}$

(a)

$$L(\alpha M + \bar{\alpha} M) = \frac{1}{2}\left[(\alpha M + \bar{\alpha}\bar{M}) + (\alpha M + \bar{\alpha}\bar{M})^T \right]$$

for $A, B \in \mathbb{R}^{2,2}$

$(A+B)^T = A^T + B^T$

$$= \frac{1}{2}\left[\alpha M + (\alpha M)^T\right] + \frac{1}{2}\left[\bar{\alpha}\bar{M} + (\bar{\alpha}\bar{M})^T\right]$$

for $\alpha \in \mathbb{R}$, $M \in \mathbb{R}^{2,2}$

$(\alpha M)^T = \alpha M^T$

$$= \alpha \cdot \frac{1}{2}(M + M^T) + \bar{\alpha} \cdot \frac{1}{2}(\bar{M} + \bar{M}^T)$$

$$= \alpha L(M) + \bar{\alpha} L(\bar{M})$$

$\Rightarrow$ L is a linear operator. $\square$

(b)

$A = [A_1 | A_2 | A_3 | A_4]$

$A_1 = [L(\mathcal{E}^{11})]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $L(\mathcal{E}^{11}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$A_2 = [L(\mathcal{E}^{12})]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix}$, $L(\mathcal{E}^{12}) = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$

$A_3 = [L(\mathcal{E}^{21})]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix}$, $L(\mathcal{E}^{21}) = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$

$A_4 = [L(\mathcal{E}^{22})]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $L(\mathcal{E}^{22}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ ※

$\boxed{8.}$

(a)

For $(\mathbb{C}^n, \mathbb{C})$, natural basis $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$

$[L(x)]_{\mathcal{E}} = \hat{A} [x]_{\mathcal{E}}$, where $\hat{A} = [\hat{A}_1 | \hat{A}_2 | \cdots | \hat{A}_n]$

$$
\begin{aligned}
\hat{A}_i &= [L(\mathcal{E}^i)]_{\mathcal{E}} \\
&= [A \mathcal{E}^i]_{\mathcal{E}} \\
&= \left[A \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \right]_{\mathcal{E}} \\
&= [\text{ith column of } A]_{\mathcal{E}} \\
&= \text{ith column of } A
\end{aligned}
$$

$\Rightarrow \hat{A} = [\hat{A}_1 | \hat{A}_2 | \cdots | \hat{A}_n] = A$ ※

(b) e-values are distinct $\rightarrow$ e-vectors are linearly indep. and form a basis for $(\mathbb{C}^n, \mathbb{C})$

Let the set of all e-vectors $V = \{v^1, v^2, \cdots, v^n\}$, with corresponding e-values $\{\lambda^1, \lambda^2, \cdots, \lambda^n\}$

$[L(x)]_V = \hat{A} [x]_V$

$$
\begin{aligned}
\hat{A}_i &= [L(v^i)]_V \\
&= [A v^i]_V \\
&= [\lambda_i v^i]_V \\
&= \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix}
\end{aligned}
$$

$\Rightarrow \hat{A} = [\hat{A}_1 | \hat{A}_2 | \cdots | \hat{A}_n] = \begin{bmatrix} \lambda_1 & \cdots & & 0 \\ & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{bmatrix}$

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I discussed this HW with Wan-Yi Yu # 74732586