

ROB 501 Handout: Grizzle

Newton Raphson Algorithm

Let $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuously differentiable, and satisfy

$$\det \left(\frac{\partial h}{\partial x}(x) \right) \neq 0 \quad \forall x \in \mathbb{R}^n$$

Problem: For $y \in \mathbb{R}^n$ fixed, find a solution of $y = h(x)$; i.e, find $x^* \in \mathbb{R}^n$ s.t. $y = h(x^*)$. We note that this is equivalent to $h(x^*) - y = 0$. In other words, we are looking for a root of the equation $h(x) - y = 0$,

Approach: Find a convergent sequence $x_k \rightarrow x^*$ such that

$$\lim_{k \rightarrow \infty} h(x_k) - y = h(x^*) - y = 0$$

that is, $x^* = \lim_{k \rightarrow \infty} x_k$ is a root of $h(x) - y = 0$

Idea: Write $x_{k+1} = x_k + \Delta x_k$. We want

$$h(x_{k+1}) - y = h(x_k + \Delta x_k) - y \approx 0.$$

What should Δx_k look like ?

Apply Taylor's Theorem, to get

$$\begin{aligned} h(x_k) + \frac{\partial h}{\partial x}(x_k) \Delta x_k - y &\approx 0 \\ \therefore \frac{\partial h}{\partial x}(x_k) \Delta x_k &\approx -(h(x_k) - y) \\ \Delta x_k &\approx - \left(\frac{\partial h}{\partial x}(x_k) \right)^{-1} (h(x_k) - y) \end{aligned}$$

Recalling that $x_{k+1} = x_k + \Delta x_k$, we arrive at Newton's Algorithm, given by

$$x_{k+1} = x_k - \left(\frac{\partial h}{\partial x}(x_k) \right)^{-1} (h(x_k) - y)$$

In practice, the change in x_k given by $\Delta x_k = - \left(\frac{\partial h}{\partial x}(x_k) \right)^{-1} (h(x_k) - y)$ is often too large. Hence, one uses the so-called Damped Newton Algorithm

$$x_{k+1} = x_k - \epsilon \left(\frac{\partial h}{\partial x}(x_k) \right)^{-1} (h(x_k) - y)$$

where $\epsilon > 0$ provides step size control!

Remark: Looking ahead to our discussion of contraction mappings, a solution of $h(x) - y$ is a fixed point of

$$P(x) = x - \epsilon \left(\frac{\partial h}{\partial x}(x) \right)^{-1} (h(x) - y)$$

Indeed,

$$\begin{aligned} x^* &= P(x^*) \\ \Downarrow \\ x^* &= x^* - \epsilon \left(\frac{\partial h}{\partial x}(x^*) \right)^{-1} (h(x^*) - y) \\ \Downarrow \\ 0 &= -\epsilon \left(\frac{\partial h}{\partial x}(x^*) \right)^{-1} (h(x^*) - y) \\ \Downarrow \\ 0 &= (h(x^*) - y). \end{aligned}$$

It can be shown that P is a local contraction on an open ball around a solution of $h(x) - y = 0$.

Example Find the solution to the coupled NL equations

$$0 = h(x) = \begin{pmatrix} x_1 + 2x_2 - x_1(x_1 + 4x_2) - x_2(4x_1 + 10x_2) + 3 \\ 3x_1 + 4x_2 - x_1(x_1 + 4x_2) - x_2(4x_1 + 10x_2) + 4 \\ \sin(x_3)^7 + \frac{\cos(x_1)}{2} \\ x_4^3 - 2x_2^2 \sin(x_1) \end{pmatrix}$$

Initial Guess: $x_0 = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 10 \end{bmatrix}$

We do 16 iterations of Newton's Algorithm (a nonlinear root finding algorithm) and we obtain:

$$x^* = \begin{pmatrix} -2.25957308738366677539068499960 \\ 1.75957308738366677539068499960 \\ 189.50954100613333978330549312824 \\ -1.68458069860197189523093013800 \end{pmatrix}$$

And the error is:

$$h(x^*) = \begin{bmatrix} 3.6734198 \times 10^{-39} \\ 2.9387359 \times 10^{-39} \\ 1.2765134 \times 10^{-38} \\ -2.5915832 \times 10^{-32} \end{bmatrix}$$