27 Nov. 2018

Review $(X, \|\cdot\|)$ a given normed space

Def. A set of vectors indexed by the counting numbers is called a sequence. (x_n) or $\{x_n\}$

Def. A sequence (x_n) converges to a point $x \in X$ if, $\forall\, \varepsilon > 0$, $\exists\, N(\varepsilon) < \infty$ such that $\forall n \geq N$, $\|x_n - x\| < \varepsilon$. : $\lim_{n\to\infty} x_n = x$ or $x_n \longrightarrow x$

Example Find the solution to the coupled NL equations

$$0 = h(x) = \begin{pmatrix} x_1 + 2\,x_2 - x_1\,(x_1 + 4\,x_2) - x_2\,(4\,x_1 + 10\,x_2) + 3 \\ 3\,x_1 + 4\,x_2 - x_1\,(x_1 + 4\,x_2) - x_2\,(4\,x_1 + 10\,x_2) + 4 \\ \sin(x_3)^7 + \frac{\cos(x_1)}{2} \\ {x_4}^3 - 2\,{x_2}^2\,\sin(x_1) \end{pmatrix}$$

$$x_{k+1} = x_k - \left[\frac{\partial h}{\partial x}(x_k)\right]^{-1} h(x_k)$$ (see handout)

Initial Guess: $x_0 = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 10 \end{bmatrix}$

We do 16 iterations of Newton's Algorithm (a nonlinear root finding algorithm) and we obtain:

$$x^* = \begin{pmatrix} -2.25957308738366677539068499960 \\ 1.75957308738366677539068499960 \\ 189.509541006133339783305493128 24 \\ -1.68458069860197189523093013800 \end{pmatrix}$$

And the error is:

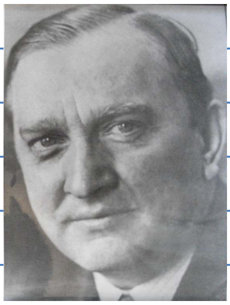
$$h\,\cancel{F}(x^*) = \begin{bmatrix} 3.6734198 \times 10^{-39} \\ 2.9387359 \times 10^{-39} \\ 1.2765134 \times 10^{-38} \\ -2.5915832 \times 10^{-32} \end{bmatrix}$$

Today

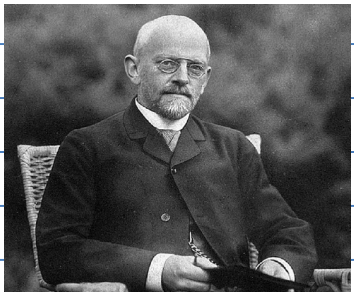
- Use sequences to characterize closed sets
- Learn how to prove convergence without knowing the limit point a priori.
- Cauchy sequences & Banach spaces

Cauchy
1789-1857

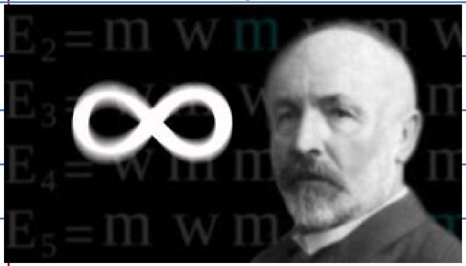
Stefan Banach
1892-1945



David Hilbert
1862 - 1943



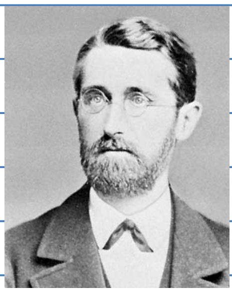
Georg Cantor
1845-1918



Proved there are different types of ∞!

Hermann Schwarz
1843-1921

Richard Dedekind
1831 - 1916



First careful construction of the real numbers

Prop. If $x_n \to x$ and $x_n \to y$, then $x = y$.
(Limits of sequences are unique)

<u>Pf.</u>

<u>Idea:</u> $$\|x-y\| = \|x - x_n + x_n - y\|$$
$$\leq \|x - x_n\| + \|x_n - y\| \xrightarrow[n\to\infty]{} 0.$$

Let $\varepsilon > 0$ be given. Because $x_n \to x$, $\exists$ $N(\varepsilon) < \infty$ such that $\forall n \geq N$, $\|x_n - x\| < \varepsilon/2$.

Because $x_n \to y$, $\exists$ $M(\varepsilon) < \infty$, s.t. $\forall m \geq M$, $\|x_m - y\| < \varepsilon/2$. Let $L = \max\{M(\varepsilon), N(\varepsilon)\} < \infty$, then, $\forall\, l \geq L$,

$$\|x-y\| \leq \|x - x_l\| + \|x_l - y\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore \|x-y\| = 0$ $\square$.

<u>Def.</u> Let $P \subset X$ be a subset and $x \in X$. Then x is a <u>limit point of P</u> if $\exists$ a sequence (x_n) satisfying

(a) $\forall n \geq 1$, $x_n \in P$

(b) $x_n \to x$.

Prop. x is a limit point of $P \iff x \in \overline{P}$

Proof.

If x is a limit point of P, $\exists$ a sequence (x_n) s.t. $\forall n \geq 1$, $x_n \in P$ and $x_n \to x$. i.e., $\forall \varepsilon > 0$, $\exists N(\varepsilon) < \infty$, s.t., $\forall n \geq N$, $\|x_n - x\| < \varepsilon$.

$\Rightarrow d(x, P) > 0 \Rightarrow x \in \overline{P}$.

Suppose $x \in \overline{P}$, then $d(x, P) = 0$. Hence, $\forall n < \infty$, $\exists$ $x_n \in P$ s.t. $\|x_n - x\| < 1/n$.

$\therefore$ $x_n \to x$ and $x_n \in P$ $\forall n$ $\therefore$ x is a limit point. $\square$

$\boxed{\text{Corollary } P \text{ is closed} \iff P \text{ contains its limit points.}}$

Drawback of the notion of a converging sequence is that you have to know an a priori guess for the limit in order to check $\|x_n - x\| \to 0$!!!

Def. A sequence (x_n) is Cauchy if, $\forall \varepsilon > 0$, $\exists N(\varepsilon) < \infty$, such that $\forall n, m \geq N$, $\|x_n - x_m\| < \varepsilon$.

Depends only on the elements of the sequence. You do not need to know = guess a candidate limit point!!!

Notation: $\|x_n - x_m\| \xrightarrow[n,m\to\infty]{} 0$

Prop. If $x_n \to x$, then (x_n) is Cauchy.

Pf.

Idea: $\|x_n - x_m\| = \| x_n - x + x - x_m \|$

$$\leq \|x_n - x\| + \|x - x_m\| \xrightarrow[n,m\to\infty]{} 0$$

Formal proof: Repeat the steps that limits of sequences are unique.

□

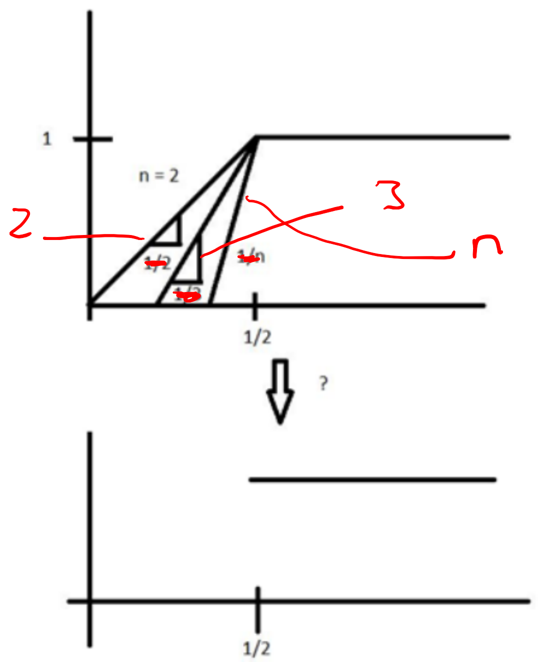
Question: Do all Cauchy sequences have limits???

Unfortunately, not all Cauchy sequences have limits.

Example: $X = \{f : [0,1] \to \mathbb{R} \mid \text{f continuous}\}$ and $\|f\|_1 = \int_0^1 |f(\tau)| d\tau$.

Define a sequence as follow

$$f_n(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ 1 + n(t - \frac{1}{2}) & \frac{1}{2} - \frac{1}{n} \le t \le \frac{1}{2} \\ 1 & t \ge \frac{1}{2} \end{cases}$$



$\|f_n - f_m\|_1 = \frac{1}{2} \left|\frac{1}{n} - \frac{1}{m}\right| \xrightarrow[n,\, m \to \infty]{} 0$, but there is no continuous f, such that $f_n \to f$.

Cauchy sequence in $(C[0,1], \|\cdot\|_1)$ that does not have a limit in $X = C[0,1]$.

Proposition Let $(X, \|\cdot\|)$ be a normed space. If X is finite dimensional, then every Cauchy sequence has a limit in X.

Def. A normed space is complete if every Cauchy sequence has a limit (in the given normed space). A complete normed space $(X, \|\cdot\|)$ is called a Banach space.

Def. Let $(X, \|\cdot\|)$ be a normed space. $S \subset X$ is complete if every Cauchy sequence constructed from elements of S has a limit that is also in S.

Proposition Let $(X, \|\cdot\|)$ be a normed space.

(a) If $S \subset X$ is complete, then S is closed.

(b) If $(X, \|\cdot\|)$ is complete and $S \subset X$ is closed, then S is complete.

(c) All finite-dimensional subspaces of X are complete.

Not on Final Exam = Completion of a Normed Space. Let $(X, \|\cdot\|_X)$ be a normed space. Then $(Y, \|\cdot\|_Y)$ is a completion of $(X, \|\cdot\|_X)$ if

(a) $(Y, \|\cdot\|_Y)$ is complete (Banach)

(b) $X \subset Y$, $\forall x \in X$, $\|x\|_Y = \|x\|_X$.

(c) $\overline{X} = Y$ (the closure of X in Y).

$Y = X + \{$limit points of Cauchy sequences in $X\}$.

Every normed space has a completion.

$(C[a,b], \|\cdot\|_1)$ is not complete.

$(C[a,b], \|\cdot\|_\infty)$ is complete.

Going back to $(C[a,b], \|\cdot\|_1)$, what is its completion? Its completion is called $L_1[a,b]$ and requires Lebesgue integration = measure theory.