HW01

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1.

(a)

$$\{AB\}_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$$

$$[Ab^1 | Ab^2 | \dots Ab^p]_{ij} = \{Ab^j\}_i = \sum_{k=1}^{m} A_{ik} \{b^j\}_k = \sum_{k=1}^{m} A_{ik} B_{kj} = \{AB\}_{ij} . \square$$

(b)

$$\{AB\}_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$$

$$\begin{bmatrix} a^1 B \\ a^2 B \\ \vdots \\ a^n B \end{bmatrix}_{ij} = \{a^i B\}_j = \sum_{k=1}^{m} \{a^i\}_k B_{kj} = \sum_{k=1}^{m} A_{ik} B_{kj} = \{AB\}_{ij} . \square$$

(c)

$$\{AB\}_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj} = \sum_{k=1}^{m} \{a^i\}_k \{b^j\}_k = a^i b^j . \square$$

2.

(a)

$$tr(A) = 1+5+9 = 15$$

(b)

$$tr(xx^T) = tr(x^T x) = tr(x_1^2 + x_2^2 + \dots x_n^2) = \sum_{i=1}^{n} x_i^2$$

(c)

$$K^T Q K = \begin{bmatrix} - k^{1T} - \\ - k^{2T} - \\ \vdots \\ - k^{mT} - \end{bmatrix} Q \begin{bmatrix} | & | & & | \\ k^1 & k^2 & \cdots & k^m \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} - k^{1T} Q - \\ - k^{2T} Q - \\ \vdots \\ - k^{mT} Q - \end{bmatrix} \begin{bmatrix} | & | & & | \\ k^1 & k^2 & \cdots & k^m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} k^{1T} Q k^1 & \cdots & & \vdots \\ \vdots & k^{2T} Q k^2 & & \vdots \\ & & \ddots & \\ \vdots & & \cdots & k^{mT} Q k^m \end{bmatrix}$$

$$\Rightarrow tr(K^T Q K) = \sum_{i=1}^{m} k^{iT} Q k^i$$

$\boxed{3.}$

(a)

To compute the e-vector and e-values:

$\quad$ e v $\quad$ e λ

$(M-\lambda I)\,v = 0$

For the case $v \neq 0$, $\det(M-\lambda I) = 0$

$\det(M-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 5 = 0 \Rightarrow \lambda = \underset{\sim\sim\sim}{\frac{5 \pm \sqrt{5}}{2}}$ (e-values)

To find the corresponding e-vectors:

(1) when $\lambda = \frac{5+\sqrt{5}}{2}$

$\Rightarrow \begin{bmatrix} -1/2 - \sqrt{5}/2 & 1 \\ 1 & 1/2 - \sqrt{5}/2 \end{bmatrix} v = 0$

$\Rightarrow \begin{bmatrix} -1/2 - \sqrt{5}/2 & 1 \\ -(-1/2 - \sqrt{5}/2)(1/2 - \sqrt{5}/2) + 1 & 0 \end{bmatrix} v = 0$

$\Rightarrow \begin{bmatrix} -1/2 - \sqrt{5}/2 & 1 \\ 0 & 0 \end{bmatrix} v = 0$

$\Rightarrow$ e-value $= \begin{bmatrix} 1 \\ 1/2 + \sqrt{5}/2 \end{bmatrix}$

(2) when $\lambda = \frac{5-\sqrt{5}}{2}$

$\Rightarrow \begin{bmatrix} -1/2 + \sqrt{5}/2 & 1 \\ 1 & 1/2 + \sqrt{5}/2 \end{bmatrix} v = 0$

$\Rightarrow \begin{bmatrix} -1/2 + \sqrt{5}/2 & 1 \\ -(1/2 + \sqrt{5}/2)(-1/2 + \sqrt{5}/2) + 1 & 0 \end{bmatrix} v = 0$

$\Rightarrow \begin{bmatrix} -1/2 + \sqrt{5}/2 & 1 \\ 0 & 0 \end{bmatrix} v = 0$

$\Rightarrow$ e-value $= \begin{bmatrix} 1 \\ 1/2 - \sqrt{5}/2 \end{bmatrix}$

$\Rightarrow \begin{cases} \lambda_1 = 5/2 + \sqrt{5}/2, & v_1 = \begin{bmatrix} 1 \\ 1/2 + \sqrt{5}/2 \end{bmatrix} \\ \lambda_2 = 5/2 - \sqrt{5}/2, & v_2 = \begin{bmatrix} 1 \\ 1/2 - \sqrt{5}/2 \end{bmatrix} \end{cases}$ ※

(b)

$$v^1 = \begin{bmatrix} 1 \\ 1/2 + \sqrt{5}/2 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 1 \\ 1/2 - \sqrt{5}/2 \end{bmatrix}$$

$$(v^1)^T v^2 = \begin{bmatrix} 1 & 1/2 + \sqrt{5}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 - \sqrt{5}/2 \end{bmatrix} = 1 + (1/2 + \sqrt{5}/2)(1/2 - \sqrt{5}/2) = 0$$

$\#$

(c)

A matrix A is symmetric $\Leftrightarrow$ $A = A^T$

$M = A^T A$, where A is any real $n \times m$ matrix

$M^T = (A^T A)^T = A^T (A^T)^T = A^T A = M$

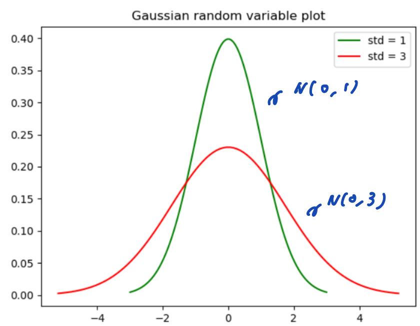
$\Rightarrow$ M is symmetric

(d)

- The inner product of e-vectors $(v^i)^T v^j$, where $i \neq j$, are always close to zero ($\sim 10^{-17}$)
- Sum of the e-values is the same as trace(M)
- product of the e-values is the same as det(M)

$\boxed{4.}$

(a)



(b) $P\{X \geq 4\} = 0.3446$

$P\{-2 \leq X \leq 4\} = 0.4436$

$P(X \in A)$, where $A = [-2, 4] \cup [8, \infty] = 0.5586$

(c) $X \sim N(2, 5^2)$ $\{ \mu_x = 2, \ \sigma_x = 5]$

$$Y = 2x + 4 \Rightarrow \begin{cases} \mu_y = 2\mu_x + 4 = 8 \\ \sigma_y = 2\sigma_x = 10 \end{cases}$$

$$\Rightarrow Y \sim N(8, 10^2)$$

$$f_Y(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}$$

$$= \frac{1}{10 \cdot \sqrt{2\pi}} e^{-\frac{(y-8)^2}{200}}$$

$\boxed{5.}$

(a)

$$k\int_0^1\int_0^2 (x^2+2xy+y^2)\,dy\,dx$$
$$= k\int_0^1 (x^2y+xy^2+y^3/3)\Big|_0^2\,dx$$
$$= k\int_0^1 2x^2+4x+8/3\,dx$$
$$= k\,(2x^3/3+2x^2+8/3\,x)\Big|_0^1$$
$$= k\,(16/3) = 1$$
$$\Rightarrow k = 3/16$$

(b)

$$f_X(x) = \int_0^2 f_{XY}(x,y)\,dy \cdot 3/16$$
$$= \int_0^2 (x^2+2xy+y^2)\,dy \cdot 3/16$$
$$= (x^2y+xy^2+y^3/3)\Big|_0^2 \cdot 3/16$$
$$= (2x^2+4x+8/3)\cdot 3/16$$

$$f_Y(y) = \int_0^1 f_{XY}(x,y)\,dx \cdot 3/16$$
$$= \int_0^1 (x^2+2xy+y^2)\,dx \cdot 3/16$$
$$= (x^3/3+x^2y+xy^2)\Big|_0^1 \cdot 3/16$$
$$= (1/3+y+y^2)\cdot 3/16$$

(c) $$f_{X|Y}(x|Y=y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$
$$= 3/16\cdot(x^2+y^2) \Big/ 3/16\cdot(1/3+y+y^2)$$
$$= (x^2+y^2)/(1/3+y+y^2)$$

$\boxed{6.}$

$$f(x_1,x_2) = x_1^2+x_2^2$$
$$g(x_1,x_2) = x_1+3x_2-4$$
$$L = x_1^2+x_2^2+\lambda(x_1+3x_2-4)$$
$$\nabla L = \left(\frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial x_2}, \frac{\partial L}{\partial \lambda}\right) = 0$$

$$\Rightarrow \begin{cases} 2x_1+\lambda = 0 \\ 2x_2+3\lambda = 0 \\ x_1+3x_2-4=0 \end{cases} \Rightarrow \begin{matrix} x_1 = -\lambda/2 \\ x_2 = -3/2\,\lambda \end{matrix} \Rightarrow -\lambda/2 - 9/2\,\lambda - 4 = 0$$

$$\Rightarrow -5\lambda - 4 = 0$$
$$\Rightarrow \lambda = -4/5$$
$$\Rightarrow \begin{cases} x_1 = 2/5 \\ x_2 = 6/5 \end{cases} .\ \min = x_1^2+x_2^2 = 8/5 \ \#$$

$\boxed{7.}$

covariance matrix $\Sigma = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} = \begin{bmatrix} 3 & \sqrt{5} \\ \sqrt{5} & 2 \end{bmatrix}$

$\Rightarrow \sigma_x = \sqrt{3}, \ \sigma_y = \sqrt{2}, \ \rho = \sqrt{5/6}$

We also know $\mu_x = 1, \ \mu_y = 2$

(a)

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} = \frac{1}{\sqrt{6\pi}} e^{-\frac{(x-1)^2}{6}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}} = \frac{1}{\sqrt{4\pi}} e^{-\frac{(y-2)^2}{4}}$$

(b)

$$f_{X|Y}(x|Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}}{\frac{1}{\sqrt{4\pi}} e^{-\frac{(y-2)^2}{4}}}$$

$$= \frac{\frac{1}{2\pi} e^{-3\left[\frac{(x-1)^2}{3} - 2\sqrt{\frac{5}{6}}\left(\frac{x-1}{\sqrt{3}}\right)\left(\frac{y-2}{\sqrt{2}}\right) + \frac{(y-2)^2}{2}\right]}}{\frac{1}{\sqrt{4\pi}} e^{-\frac{(y-2)^2}{4}}}$$

Another way to solve this question:

$$f_{X|Y}(x|y) = \frac{1}{\sigma_c\sqrt{2\pi}} e^{-\frac{(x-\mu_c)^2}{2\sigma_c^2}}, \text{ where } \begin{cases} \mu_c = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y) = 1 + \sqrt{\frac{5}{6}}\cdot\sqrt{\frac{3}{2}}\cdot(10-2) \\ \sigma_c = \sqrt{1-\rho^2}\,\sigma_x = \sqrt{\frac{1}{6}}\cdot\sqrt{3} = \sqrt{\frac{1}{2}} \end{cases}$$

(c)

$$f_{X|Y}(x|Y=y) = \frac{1}{\sqrt{2\pi}\cdot\left(\frac{1}{\sqrt{2}}\right)} e^{(\cdots)}$$

eg $\sigma_{X|Y}^2 = 1/2$

$\sigma_X^2 = 3$

$\Rightarrow$ variance of X given $Y = y$ is less than variance of X

(d)