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Review

(1) $\lambda \in \mathbb{C}$ is an evalue of $A = n\times n$ matrix $\Leftrightarrow$
$\exists (v \in \mathbb{C}^n,\ v \neq 0)$ such that $Av = \lambda v$

(2) Inner product on $(\mathbb{C}^n, \mathbb{C})$ is $\langle x, y\rangle := x^T \bar{y}$ where
$\bar{y}$ = Complex conjugate $(y = y_R + j\, y_{Im} \Leftrightarrow \bar{y} = y_R - j\, y_{Im})$

(3) $v \in \mathbb{C}^n$ is real $\Leftrightarrow v = \bar{v}$, $\lambda \in \mathbb{C}$ real $\Leftrightarrow \lambda = \bar{\lambda}$

(4) $\langle x, x\rangle = x^T \bar{x} = \sum_{i=1}^{n} x_i \cdot \bar{x}_i = \sum_{i=1}^{n} |x_i|^2 = \|x\|^2$

for $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

(5) Let $A = [A_1 | A_2 | \cdots | A_m]$, $B = [B_1 | B_2 | \cdots | B_m]$

$$\boxed{[B^T A]_{ij} = B_i^T A_j = \langle B_i, A_j \rangle}$$

$$B^T A = \begin{bmatrix} B_1^T \\ \hline B_2^T \\ \hline \vdots \\ \hline B_m^T \end{bmatrix} [A_1 | A_2 | \cdots | A_m]$$

(6) If $z_1, z_2 \in \mathbb{C}$, then $\overline{z_1 z_2} = (\bar{z}_1) \cdot (\bar{z}_2)$

7 One More Reminder: Claim If A is $n \times n$ and real, then its e-values and e-vectors occur in complex conjugate pairs.

Proof. Let $\lambda \in \mathbb{C}$ and $(v \in \mathbb{C}^n, v \neq 0)$ satisfy $Av = \lambda v$ To show: $A\bar{v} = \bar{\lambda}\bar{v}$.

$$\overline{Av} = \bar{A} \cdot \bar{v} = A \cdot \bar{v} \quad \text{(because } A \text{ is real)}$$

$$\overline{Av} = \overline{\lambda v} = \bar{\lambda}\bar{v} \quad \text{(because } Av = \lambda v\text{)}$$

$$\therefore A\bar{v} = \bar{\lambda}\bar{v}$$

$\square$

Symmetric Real Matrices

Def. An $n \times n$ matrix A is symmetric if $A = A^T$.

* Claim 1 If A is real and symmetric, $(A = \bar{A})$ and $(A = A^T)$, then $\forall\, x, y \in \mathbb{C}^n$,

$$\langle Ax, y \rangle = \langle x, Ay \rangle. \quad *$$

Proof. $\langle Ax, y\rangle = x^T A^T \bar{y} = x^T A \bar{y}$ $(A = A^T)$

$\langle x, Ay\rangle = x^T \overline{(Ay)} = x^T \bar{A} \bar{y} = x^T A \bar{y}$

$(A = \bar{A})$

$\square$

Claim 2 e-values of real symmetric matrices are real.

Pf. To show if $Av = \lambda v$, $v \neq 0$, then $\lambda = \bar{\lambda}$. Apply Claim 1 with $x = y = v$

$$\langle Av, v\rangle = \langle v, Av\rangle$$

$$\langle \lambda v, v\rangle = \langle v, \lambda v\rangle$$

$$\lambda \langle v, v\rangle = \bar{\lambda} \langle v, v\rangle$$

$$\lambda \|v\|^2 = \bar{\lambda} \|v\|^2$$

$$\therefore \lambda = \bar{\lambda} \quad (\|v\|^2 \neq 0)$$

$\square$

Remark: $(A - \lambda I)v = 0$ Can assume e-vectors are real !!!

Claim 3 e-vectors of symmetric real matrices are orthogonal

Pf. Assume $\lambda_1, \dots, \lambda_n$ distinct, here. The general case is treated in HW.

$\lambda_1 \neq \lambda_2$, $v^1, v^2 \neq 0$, $Av^1 = \lambda_1 v^1$, $Av^2 = \lambda_2 v^2$

To show: $\langle v^1, v^2 \rangle = 0.$

Apply Claim 1 with $x = v^1$ and $y = v^2$

$$\langle Av^1, v^2 \rangle = \langle v^1, Av^2 \rangle$$

$$\langle \lambda_1 v^1, v^2 \rangle = \langle v^1, \lambda_2 v^2 \rangle \qquad \text{(e-vectors)}$$

$$\lambda_1 \langle v^1, v^2 \rangle = \lambda_2 \langle v^1, v^2 \rangle \qquad (\lambda_2 \text{ real})$$

$$(\lambda_1 - \lambda_2) \langle v^1, v^2 \rangle = 0$$

$$\therefore \langle v^1, v^2 \rangle = 0 \qquad (\lambda_1 \neq \lambda_2).$$

$\square$

Def. An $n \times n$ real matrix Q is orthogonal if $Q^T Q = I$

$$Q = [Q_1 | Q_2 | \cdots | Q_n]$$

$$[Q^TQ]_{ij} = \langle Q_i, Q_j \rangle = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

If $Q^TQ = I$. Columns of Q are an orthonormal basis of $\mathbb{R}^n$.

$\|Q_i\|_2^2 = \langle Q_i, Q_i \rangle = 1$ for orthogonal matrix.

Claim 4 If A is real and symmetric, then $\exists$ an orthogonal matrix Q such that $Q^TAQ = \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

Pf. We assume $\lambda_1, \dots, \lambda_n$ distinct. See HW for general case. $\therefore$ $\exists$ $\{v^1, \dots, v^n\}$ linearly indep e-vectors that are also orthogonal. WLOG (Without Loss of Generality) we can assume $\|v^i\| = 1$, $1 \le i \le n$. Hence

$$Q = [v^1 | v^2 | \cdots | v^n]$$

is orthogonal.

$$AQ = A[v^1|v^2|\cdots|v^n] = [\lambda_1 v^1|\lambda_2 v^2|\cdots|\lambda_n v^n] =$$
$$= Q\Lambda$$

$\therefore$ $Q^{-1}AQ = \Lambda$, but $Q^{-1} = Q^T$, hence we are done.

$\square$.

$$A = Q\Lambda Q^T$$

Why orthogonal matrices are interesting.

$$\|Qx\|_2^2 = x^T \underbrace{Q^T Q}_{I} x = x^T x = \|x\|_2^2$$

"Obviously" numerically close to being dependent.

$$\begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 10^{-2} & 0 \end{bmatrix}}_{\text{small } \Delta} = \underbrace{\begin{bmatrix} 1 & 100 \\ 10^{-2} & 1 \end{bmatrix}}_{\text{singular}}$$

det = 1

Later will lead to SVD = Singular Value Decomposition = = Numerical method to check the rank of a matrix, etc.

Observation A^TA and AA^t are both symmetric!!!!

<u>Claim 5</u> e-values of A^TA and AA^T are non-negative. $(\lambda_i(A^TA) \geq 0, \lambda_i(AA^T) \geq 0)$

<u>Pf.</u> Let v be an e-vector of A^TA, $v \neq 0$.

$$A^TAv = \lambda v$$

$$\therefore \quad v^TA^TAv = \lambda v^Tv$$

$$\therefore \quad \|Av\|^2 = \lambda \|v\|^2$$

$\therefore \lambda \geq 0$ because $\|Av\| \geq 0$ and $\|v\| > 0$.

$\square$

Quadratic Forms

Building toward estimation problems where some measurements are "more certain" than others. Need unequal weights on our error terms!!

Let M be $n \times n$ and real. Let $x \in \mathbb{R}^n$

$$x^T M x = x^T \left[\underbrace{\frac{M+M^T}{2}}_{\text{symmetric}} + \underbrace{\frac{M-M^T}{2}}_{\text{skew symmetric}} \right] x$$

$$\left(\frac{M-M^T}{2}\right)^T = -\frac{(M-M^T)}{2}$$

$$= \underbrace{x^T \left[\frac{M+M^T}{2}\right] x}_{\text{scalar}} + \underbrace{\cancel{x^T \left[\frac{M-M^T}{2}\right] x}}_{\text{scalar}} \rightarrow 0$$

symmetric part of M.

When we form $x^T M x$ we always start with M symmetric.

Def. A real symmetric matrix P is positive definite if, $\forall x \in \mathbb{R}^n$, $x \neq 0$, $x^T P x > 0$.

Thm A symmetric matrix P is pos. def. if, and only if, all of its e-values are positive.

Pf. Exercise. $\square$

Remark: HW 02 you proved using Lagrange multipliers: $\forall\, x \in \mathbb{R}^n$, $\|x\|_2 = 1$, then

$$\lambda_{min} \leq x^T P x \leq \lambda_{max}$$