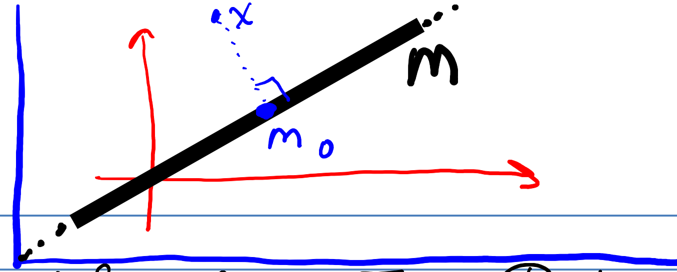
02 October 2018



Review $(X, \mathcal{F})$ a vector space, we defined an Inner Product Space for $\mathcal{F} = \mathbb{R}$ and $\mathcal{F} = \mathbb{C}$. For now, we focus on $\mathcal{F} = \mathbb{R}$. $\langle \cdot , \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ satisfies

1) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$

2) $\langle x, y \rangle = \langle y, x \rangle$

3) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

4) C. S. Inequality: $|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \cdot \langle y, y \rangle^{1/2}$

5) $x \perp S \iff \forall y \in S, \ \langle x, y \rangle = 0.$

6) $\|x\| := \langle x, x \rangle^{1/2}$ is a norm. (7) $x \perp y \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$

<u>Pre-projection Theorem</u> $(X, \mathbb{R}, \langle \cdot , \cdot \rangle)$ an inner product space, $M \subset X$ a subspace, $x \in X$ arbitrary:

(a') If $\exists \ m_0 \in M$ s.t. $\|x - m_0\| = d(x, M)$, then m_0 is unique $\left\{ d(x, M) = \inf_{m \in M} \|x - m\| \right\}$

(b') $m_0 \in M$ satisfies $\|x - m_0\| = d(x, M) \iff x - m_0 \perp M$ $\{x - m_0 = \text{the approximation error}\}$

Remark: As soon as we write $(\mathcal{X}, \mathcal{F}, \langle \cdot, \cdot \rangle)$, then the norm is always $\|x\|^2 := \langle x, x \rangle$; recall $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$.

Proof of the Pre-Projection Thm

<u>Claim 1</u> Let $m_0 \in \mathcal{M}$. If $\|x - m_0\| = d(x, \mathcal{M})$, then $x - m_0 \perp \mathcal{M}$.

<u>Pf.</u> Contrapositive: $x - m_0 \not\perp \mathcal{M} \Rightarrow \|x - m_0\| > d(x, \mathcal{M})$.

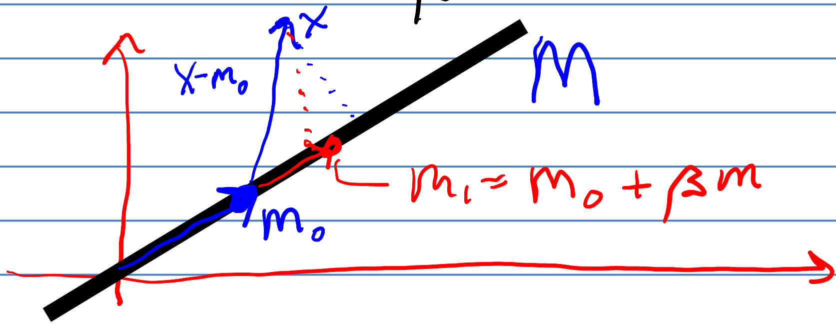
$x - m_0 \not\perp \mathcal{M} \Rightarrow \exists\, m \in \mathcal{M}$ such that $\langle x - m_0, m \rangle \neq 0$. Because

$$\left\langle x - m_0, \frac{m}{\|m\|} \right\rangle = \frac{1}{\|m\|} \langle x - m_0, m \rangle \neq 0,$$

thus we can suppose $\|m\| = 1$.

$$\beta = \langle x - m_0, m \rangle \neq 0$$

Define $m_1 = m_0 + \beta m$



$$\|x-m_1\|^2 = \|x-m_0-\beta m\|^2$$

$$= \langle x-m_0-\beta m, \ x-m_0-\beta m \rangle$$

$$= \langle x-m_0, x-m_0 \rangle - 2\beta \langle x-m_0, m \rangle + \langle -\beta m, -\beta m \rangle$$

$$= \|x-m_0\|^2 - 2\beta^2 + (-\beta)^2$$

$$= \|x-m_0\|^2 - \beta^2$$

Because $\beta \neq 0$,

$$\|x-m_1\|^2 < \|x-m_0\|^2 .$$

$\therefore$ m_0 is not a minimizer.

$$\left\{ \langle z_1+z_2, z_1+z_2 \rangle = \langle z_1, z_2 \rangle + 2\langle z_1, z_2 \rangle + \langle z_2, z_2 \rangle \right\}$$

<u>Claim 2</u> If $x-m_0 \perp M$, then $\|x-m_0\| = d(x, M)$ and m_0 is unique.

<u>Pf.</u> Assume $x-m_0 \perp M$ and let $m \in M$ be arbitrary.

$\|x-m\|^2 = \|x-m_0+m_0-m\|^2$

$m_0 \in M$, $m \in M$, Hence $m_0 - m \in M$.

Also, $x-m_0 \perp M \Rightarrow x-m_0 \perp m_0-m$

$\therefore \|x-m\|^2 = \|x-m_0\|^2 + \|m_0-m\|^2$ by Pythagora

$$d(x,M)^2 := \inf_{m\in M} \|x-m\|^2 = \inf_{m \in M}\left[\|x-m_0\|^2 + \|m_0-m\|^2\right]$$

$\therefore$ the minimizing vector is unique and equal to m_0 ! $\square$

Gram Schmidt Process

(Handout) Key Points

Initial data $\{y^1, \dots, y^n\}$ that is linearly indep.

Recursively compute $\{v^1, \dots, v^n\}$ such that

- $\forall\ 1 \le k \le n$, $\operatorname{span}\{v^1, \dots, v^k\} = \operatorname{span}\{y^1, \dots, y^k\}$
- $\forall\ i \ne j$, $1 \le i, j \le n$, $\langle v^i, v^j \rangle = 0$.

Back to Linear Algebra

Prop. Let $(\mathcal{X}, \mathcal{F})$ be a vector space of dimension n. If $1 \le k < n$, $\{v^1, .., v^k\}$ is linearly indep. Then, $\exists \{v^{k+1}, ..., v^n\}$ such that $\{v^1, .., v^k, v^{k+1}, ..., v^n\}$ is a basis. (Any set of linearly indep vectors can be completed to a basis).

Proof is by Induction on k. We establish a cute lemma that will allow you to do the proof by induction!

Lemma: $(\mathcal{X}, \mathcal{F})$ a vector, n-dim, $1 \le k < n$, $\{v^1, .., v^k\}$ linearly indep. Then $\exists\ v^{k+1} \in \mathcal{X}$ such that $\{v^1, .., v^{k+1}\}$ is linearly independent.

Pf. Contrapositive $p \Rightarrow q \iff$
$\sim q \Rightarrow \sim p$.

p: $k<n$, $\{v^1, .., v^k\}$ lin. indep.
q. $\exists\, v^{k+1}$ linearly indep of $\{v^1, .., v^k\}$.

$\sim q$: $\forall x \in X$, $x \in \operatorname{span}\{v^1, .., v^k\}$.
$\therefore X \subset \operatorname{span}\{v^1, .., v^k\}$.
$n = \dim\{X\} \le \dim\{v^1, .., v^k\} \le k$
$\therefore k \ge n \Rightarrow \sim p$.
$\square$

Back to Inner Product Spaces

$(\mathcal{X}, \mathcal{F} = \mathbb{R} \text{ or } \mathbb{C}, \langle \cdot, \cdot \rangle)$.

Def. Suppose $S \subset \mathcal{X}$ is a subset.
Then $S^\perp := \{x \in X \mid x \perp S\}$ is
orthogonal complement of S.

Exercise 1: $S^\perp$ is a subspace of X.
Exercise 2: $S^\perp = (\operatorname{span}\{S\})^\perp$.

Proposition Let $(\mathcal{X}, \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}, \langle \cdot, \cdot \rangle)$ be an inner product space and finite dimensional. Let $M \subset \mathcal{X}$ be a subspace. Then

$$X = M \oplus M^{\perp}.$$

Note $M \cap M^{\perp} = \{0\}$.

$$x \in M \ \& \ x \in M^{\perp} \Rightarrow \langle x, x \rangle = 0 \Leftrightarrow x = 0.$$