Easy Facts on Representations:

1. Addition of vectors in $(\mathcal{X}, \mathcal{F}) \equiv$ Addition of the representations in $(\mathcal{F}^n, \mathcal{F})$.

$$[x + y]_v = [x]_v + [y]_v$$

2. Scalar multiplication in $(\mathcal{X}, \mathcal{F}) \equiv$ Scalar multiplication with the representations in $(\mathcal{F}^n, \mathcal{F})$.

$$[\alpha x]_v = \alpha [x]_v$$

3. Once a basis is chosen, any n-dimensional vector space $(\mathcal{X}, \mathcal{F})$ "looks like" $(\mathcal{F}^n, \mathcal{F})$.

Change of Basis Matrix: Let $\{u^1, \cdots, u^n\}$ and $\{\bar{u}^1, \cdots, \bar{u}^n\}$ be two bases for $(\mathcal{X}, \mathcal{F})$. Is there a relation between $[x]_u$ and $[x]_{\bar{u}}$?

Theorem: $\exists$ an invertible matrix P, with coefficients in $\mathcal{F}$, such that $\forall x \in (\mathcal{X}, \mathcal{F})$, $[x]_{\bar{u}} = P[x]_u$. Moreover, $P = [P_1|P_2|\cdots|P_n]$ with $P_i = [u^i]_{\bar{u}} \in \mathcal{F}^n$ where P_i is the i^{th} column of the matrix P and $[u^i]_{\bar{u}}$ is the representation of u^i with respect to $\bar{u}$.

<u>Proof:</u> Let $x = \alpha_1 u^1 + \cdots + \alpha_n u^n = \bar{\alpha}_1 \bar{u}^1 + \cdots + \bar{\alpha}_n \bar{u}^n$.

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = [x]_u$$

$$\bar{\alpha} = \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_n \end{bmatrix} = [x]_{\bar{u}}$$

$$\bar{\alpha} = [x]_{\bar{u}} = \left[\sum_{i=1}^{n} \alpha_i u^i \right]_{\bar{u}} = \sum_{i=1}^{n} \alpha_i [u^i]_{\bar{u}} = \sum_{i=1}^{n} \alpha_i P_i = P\alpha.$$

Therefore, $\bar{\alpha} = P\alpha = P[x]_u$.

Now we need to show that P is invertible:
Define $\bar{P} = [\bar{P}_1|\bar{P}_2|\cdots|\bar{P}_n]$ with $\bar{P}_i = [\bar{u}^i]_u$.
Do the same calculations and obtain $\alpha = \bar{P}\bar{\alpha}$.
Then, we can obtain that $\alpha = \bar{P}P\alpha$ and $\bar{\alpha} = P\bar{P}\bar{\alpha}$.
Therefore, $P\bar{P} = \bar{P}P = I$.
In conclusion, $\bar{P}$ is the inverse of P ($\bar{P} = P^{-1}$). $\square$

Example: $\mathcal{X}$=$\{2 \times 2$ matrices with real coefficients$\}$, $\mathcal{F} = \mathbb{R}$.

$$u = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
$$\bar{u} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

We have following relations:

$$\bar{\alpha} = P\alpha, P_i = [u^i]_{\bar{u}}, \quad \alpha = \bar{P}\bar{\alpha}, \bar{P}_i = [\bar{u}^i]_u$$
$$\bar{P}^{-1} = P, P^{-1} = \bar{P}$$

Typically, compute the easier of P or $\bar{P}$, and compute the other by inversion. For this example, we choose to compute $\bar{P}$

$$\bar{P}_1 = [\bar{u}^1]_u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{P}_2 = [\bar{u}^2]_u = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\bar{P}_3 = [\bar{u}^3]_u = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\bar{P}_4 = [\bar{u}^4]_u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $\bar{P}= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $P = \bar{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .5 & .5 & 0 \\ 0 & .5 & -.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

What if we did it the other direction?

$$P_1 = [u^1]_{\bar{u}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_2 = [u^2]_{\bar{u}} = \begin{bmatrix} 0 \\ .5 \\ .5 \\ 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + .5 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_3 = [u^3]_{\bar{u}} = \begin{bmatrix} 0 \\ .5 \\ -.5 \\ 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - .5 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_4 = [u^4]_{\bar{u}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .5 & .5 & 0 \\ 0 & .5 & -.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\bar{P} = P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$