HW02 Kuan-Ting Lee # 50036?44

1.

(a)

$\neg(P \wedge Q) = (\neg P) \vee (\neg Q)$

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

the same

(b)

$\neg(P \vee Q) = (\neg P) \wedge (\neg Q)$

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

the same

2.

(a) There exists a integer n, $2n+1$ is even

(b) For all integer n, 2^n+1 is not prime

(c) $\forall v \in \mathbb{R}^n$, $v \neq 0$, $Av \neq \lambda v$

(d) $\exists \eta > 0$, $\forall \delta > 0$, $|x| \leq \delta \wedge |f(x)| > \eta |x|$

$\boxed{3.}$

- Use proof by contradiction

Let's assume $\sqrt{7}$ is rational, $\exists\, m, n \in \mathbb{N}$ s.t. $\begin{cases} R_1: m, n \text{ have no common factors} \\ R_2: \sqrt{7} = \frac{m}{n} \end{cases}$

$R_2 \Rightarrow 7 = m^2/n^2 \Rightarrow 7n^2 = m^2 \Rightarrow 7$ divides $m^2 \Rightarrow 7$ divides $m \Rightarrow \exists k \in \mathbb{N}$ s.t. $m = 7k$

$7n^2 = m^2 = (7k)^2 \Rightarrow n^2 = 7k^2 \Rightarrow 7$ divides $n^2 \Rightarrow 7$ divides n

$\therefore$ m, n have a common factor 7

$\therefore$ $\neg R_1$ is true

$\Rightarrow$ Contradiction $R_1 \wedge (\neg R_1)$ from assuming $\sqrt{7}$ is rational

$\Rightarrow$ $\sqrt{7}$ is irrational. $\square$

$\boxed{4.}$

- Use proof by contradiction

using $\neg(p \to q) \equiv p \wedge \neg q$

Assume $\det(A) = 0 \wedge A$ is invertible, $\exists\, B = A^{-1}$ s.t. $AB = BA = I$

$\det(AB) = \det(I) = 1 = \det(A)\det(B)$

$\Rightarrow \det(A) \neq 0$

$\Rightarrow$ Contradiction $(\det(A) = 0 \wedge \det(A) \neq 0)$

$\Rightarrow$ If $\det(A) = 0$, then A is not invertible. $\square$

$\boxed{5.}$

Let $P(n)$ be the statement, for $n \in \mathbb{N}, n \geq 1$, $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

Base case $(n=1)$:

$$P(1) = \sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{n}{n+1}$$

Inductive step:

We assume $P(i)$ is true:

$$P(i): \sum_{k=1}^{i} \frac{1}{k(k+1)} = \frac{i}{i+1}$$

and attempt to prove $P(i+1)$ is also true:

$$P(i+1) = \sum_{k=1}^{i+1} \frac{1}{k(k+1)} = P(i) + \frac{1}{(i+1)(i+2)} = \frac{i}{i+1} + \frac{1}{(i+1)(i+2)}$$

$$= \frac{i(i+2)+1}{(i+1)(i+2)}$$

$$= \frac{(i+1)(i+1)}{(i+1)(i+2)}$$

$$= \frac{i+1}{i+2}$$

That is, $P(k+1)$ also holds true, establishing the inductive step. $\square$

6.

(a)

Using proof by strong induction:

$P(n)$: for $n \in \mathbb{N}$, $n \geq 12$, there exists non-negative integers k_1 and k_2 such that

$$n = k_1 \cdot 4 + k_2 \cdot 5$$

Hypothesis: Suppose for some natural number $n \geq 12$, we have that for every $12 \leq k \leq n$ there exists non-negative integers k_1, k_2 such that $k = k_1 \cdot 4 + k_2 \cdot 5$

Base case:

$P(12)$ can be achieved by setting $k_1 = 3$, $k_2 = 0$

Inductive step:

Assume now that $12 \leq j \leq i$, $P(j)$ is true. Will show that $P(i+1)$ is true

If $i = 12$, $i+1 = 13 = 4 \cdot 2 + 5 \cdot 1 \Rightarrow P(i+1)$ is true

If $i = 13$, $i+1 = 14 = 4 \cdot 1 + 5 \cdot 2 \Rightarrow P(i+1)$ is true

If $i = 14$, $i+1 = 15 = 4 \cdot 0 + 5 \cdot 3 \Rightarrow P(i+1)$ is true

For $i \geq 15$, $i+1 = (i-3) + 4 = \underbrace{(4k_1 + 5k_2)}_{\exists k_1, k_2 \text{ that satisfy } P(i-3)} + 4 = 4(k_1+1) + 5k_2 \Rightarrow P(i+1)$ is true

. $\square$

The same statement is not true for $n \geq 8$.

Because $P(10)$ can not lead to $P(11)$, which means we can't find non-negative integers k_1, k_2 such that $11 = 4 \cdot k_1 + 5k_2$

(b)

We follow the same method as in (a)

Base case:

$P(6)$ can be achieved by setting $k_1 = 2$, $k_2 = 0$

Inductive step:

If $i = 6 \Rightarrow i+2 = 8 = 3\cdot 1 + 5\cdot 1 \Rightarrow P(i+2)$ is true

If $i = 8 \Rightarrow i+2 = 10 = 3\cdot 0 + 5\cdot 2 \Rightarrow P(i+2)$ is true

For $i \geq 10 \Rightarrow i+2 = (i-4)+6 = (3\cdot k_1 + 5\cdot k_2)+6 = 3(k_1+2)+5k_2$. $\square$

k_1, k_2 that satisfy $P(i-4)$

$\boxed{7.}$

$L(x,\lambda) = x^T M x - \lambda x^T x + \lambda = x^T(M - \lambda I)x + \lambda$

$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}(x^T M x - \lambda x^T x + \lambda) = 2(M - \lambda I)x = 0$

$\Rightarrow Mx = \lambda x$

Therefore, λ should be the eigenvalue of M, and x should be an eigenvector

Since M is an $n \times n$ real symmetric matrix, λ and x are both real.

Now, let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, be the eigenvalues of A, and $x_1, x_2 \cdots x_n$ be the corresponding eigenvectors, then we have

$A x_i = \lambda_i x_i$

$\Rightarrow x_i^T A x_i = x_i^T \lambda_i x_i = \lambda_i x_i^T x_i = \lambda_i$

Therefore, λ_1 is the maximum value when $x = x_1$

λ_n is the minimun value when $x = x_n$