04 October 2018

Review ① G.S. process: $\{y^1, \ldots, y^n\}$ linearly independent

$\to \{v^1, \ldots, v^n\}$ orthogonal and $\forall 1 \le k \le n$,

$\text{span}\{v^1, \ldots, v^k\} = \text{span}\{y^1, \ldots, y^k\}$ $\quad (\langle v^i, v^j \rangle = 0 \ i \neq j)$

② $(\mathcal{X}, \mathcal{F})$ n-dim vector space, $\{y^1, \ldots, y^k\}$ linearly indep.

$\Rightarrow \exists \{y^{k+1}, \ldots, y^n\}$ such that $\{y^1, \ldots, y^k, y^{k+1}, \ldots, y^n\}$ basis

③ (Pre-Projection Thm) $(\mathcal{X}, \mathbb{R}, \langle \cdot, \cdot \rangle)$ inner product space,

$M \subset \mathcal{X}$ subspace and $x \in \mathcal{X}$ given:

a) If $\exists \hat{x} \in M$ s.t. $\|x - \hat{x}\| = d(x, M) := \inf_{m \in M} \|x - m\|$,

then $\hat{x}$ is unique.

b) $\hat{x} \in M$ satisfies $\|x - \hat{x}\| = d(x, M) \Leftrightarrow x - \hat{x} \perp M$

$\Leftrightarrow x - \hat{x} \in M^\perp$ (using $\hat{x}$ to emphasise approx. of x]

④ $S \subset \mathcal{X}$ subset. $S^\perp := \{x \in \mathcal{X} \mid x \perp S\}$.

$(\text{span}\{S\})^\perp = S^\perp$ (exercise)

⑤ Proposition $(\mathcal{X}, \mathcal{F} = \mathbb{R} \text{ or } \mathbb{C}, \langle \cdot, \cdot \rangle)$ finite dimensional and $M \subset \mathcal{X}$ a subspace. Then $\mathcal{X} = M \oplus M^\perp$.

 Noted If $x \in M \cap M^\perp$ then $\langle x, x \rangle = 0 \Rightarrow x = 0$.

$\therefore M \cap M^\perp = \{0\}$.

Rest of Proof: Let $k = \dim(M)$ and $n = \dim(X)$. If $k = n$, then $M^\perp = \{0\}$ and we are done. Now assume $1 \le k < n$ and let $\{y^1, \dots, y^k\}$ be a basis for M. Complete this to a basis $\{y^1, \dots, y^k, y^{k+1}, \dots, y^n\}$ for X.

We apply G.S. to produce orthogonal vectors $\{v^1, \dots, v^n\}$ such that

$M = \operatorname{span}\{v^1, \dots, v^k\} = \operatorname{span}\{y^1, \dots, y^k\}$, and $\{v^{k+1}, \dots, v^n\} \perp \{v^1, \dots, v^k\}$.

$$X = \underbrace{\operatorname{span}\{v^1, \dots, v^k\}}_{M} + \underbrace{\operatorname{span}\{v^{k+1}, \dots, v^n\}}_{M^\perp}$$

$\therefore X = M + M^\perp$. Because $M \cap M^\perp = \{0\}$, $X = M \oplus M^\perp$. $\square$

Projection Thm Let $(X, \mathbb{R}, \langle \cdot, \cdot \rangle)$ be a finite-dim. inner product space, $M \subset X$ a subspace and $x \in X$. THEN

$\exists$ a unique $\hat{x} \in M$ such that
$\hat{x} = \underset{m \in M}{\arg\min} \|x - m\|$. Moreover, $\hat{x}$

is characterised by $x - \hat{x} \perp M$. $\square$

Pf. Only thing left to show is the existence of $\hat{x}$ such that $x - \hat{x} \perp M$. However, we know that X finite dimensional $\Longrightarrow X = M \oplus M^\perp$.

$\therefore \exists\, m \in M$ and $\tilde{m} \in M^\perp$ such that

$$x = m + \tilde{m}.$$

Then $x - m = \tilde{m} \in M^\perp$. Hence,

$$\hat{x} = m.$$

$\blacksquare$

Start Putting Stuff Together

Normal Equations
(know for ROB Qual Exam)

$(\mathcal{X}, \mathbb{R}, \langle \cdot , \cdot \rangle)$ finite dimensional

$M = \operatorname{span}\{y^1, \dots, y^k\}$, $\{y^1, \dots, y^k\}$ linearly independent. Given $x \in \mathcal{X}$, seek an explicit formula for $\hat{x}$.

$$\hat{x} = \arg\min_{m \in M} \|x - m\|$$

Know $\quad x - \hat{x} \perp M = \operatorname{span}\{y^1, \dots, y^k\}$

$$\Updownarrow \text{ exercise}$$

$$x - \hat{x} \perp y^i \quad 1 \le i \le k$$

$$\Updownarrow$$

$$\langle x - \hat{x}, y^i \rangle = 0 \quad 1 \le i \le k$$

$$\Updownarrow$$

$$\langle x, y^i \rangle - \langle \hat{x}, y^i \rangle = 0 \quad 1 \le i \le k$$

$$\Updownarrow$$

$$\langle \hat{x}, y^i \rangle = \langle x, y^i \rangle \quad 1 \le i \le k$$

$\hat{x} \in M \iff \hat{x} = \alpha_1 y^1 + \alpha_2 y^2 + \cdots + \alpha_k y^k$

$$\langle \alpha_1 y^1 + \alpha_2 y^2 + \cdots + \alpha_k y^k, y^i \rangle = \langle x, y^i \rangle$$

$1 \le i \le k$

$i=1$ $\quad \alpha_1 \langle y^1, y^1 \rangle + \alpha_2 \langle y^2, y^1 \rangle + \cdots + \alpha_k \langle y^k, y^1 \rangle = \langle x, y^1 \rangle$

$i=2$ $\quad \alpha_1 \langle y^1, y^2 \rangle + \alpha_2 \langle y^2, y^2 \rangle + \cdots + \alpha_k \langle y^k, y^2 \rangle = \langle x, y^2 \rangle$

$\vdots$

$i=k$ $\quad \alpha_1 \langle y^1, y^k \rangle + \alpha_2 \langle y^2, y^k \rangle + \cdots + \alpha_k \langle y^k, y^k \rangle = \langle x, y^k \rangle$

Write in Matrix Form

Because $\mathcal{F} = \mathbb{R}$, $\langle y^i, y^j \rangle = \langle y^j, y^i \rangle$

Define $[G(y^1, \ldots, y^k)]_{ij} := \langle y^i, y^j \rangle$

Note $G^T = G$

$$\alpha := \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}, \quad \beta := \begin{bmatrix} \langle x, y^1 \rangle \\ \langle x, y^2 \rangle \\ \vdots \\ \langle x, y^k \rangle \end{bmatrix}$$

$$\boxed{G^T \alpha = \beta}$$ Normal Equations

$$\boxed{\hat{x} = \alpha_1 y^1 + \cdots + \alpha_k y^k}$$

$g(g^1,\ldots,g^k) = \det G(g^1,\ldots,g^k)$ is called the Gram determinant and G is called the Gram Matrix.

Prop. $\{g^1,\ldots,g^k\}$ lin. indep $\Longleftrightarrow$ $g(g^1,\ldots,g^k) \neq 0$.

Pf. From Proj. Thm and uniqueness. Full drawn out, every gorey detail is given at the end of the lecture. Do not need to know it!

$\square$

Go back to $A\alpha = b$, with $A = n \times m$ real matrix, $b \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^m$. We tried to solve

$$\hat{\alpha} := \arg\min_{\alpha \in \mathbb{R}^m} \|A\alpha - b\|_2^2$$

"Magic" $\hat{\alpha} = (A^TA)^{-1}A^Tb$, $\text{rank}(A) = m$.

$$A = [A_1 | A_2 | \cdots | A_m]$$

$$A\alpha = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_m A_m$$

Real Problem Formulation

$\mathcal{X} = \mathbb{R}^n$, $\mathcal{F} = \mathbb{R}$, $\langle x, y \rangle = x^T y$

$M = \text{span}\{A_1, \ldots, A_m\}$ seek

$\hat{x} \in M$ such that $\|\hat{x} - b\|_2$ is minimized.

$$\hat{x} = \underset{m \in M}{\arg\min} \|b - m\|_2$$

x

By the normal equations

$$\hat{x} = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_m A_m$$

$$G^T \alpha = \beta$$

$$G_{ij} = \langle A_i, A_j \rangle, \quad \beta_i = \langle b, A_i \rangle$$

$$G_{ij} = \langle A_i, A_j \rangle = (A_i)^T A_j$$

$$[A^T A]_{ij} = \left[\begin{bmatrix} A_1^T \\ A_2 \\ \vdots \\ A_k^T \end{bmatrix} \begin{bmatrix} A_1 \mid A_2 \mid \cdots \mid A_k \end{bmatrix} \right]_{ij}$$

$$= A_i^T A_j = \langle A_i, A_j \rangle$$

$$\boxed{G^T = A^T A}$$

$$\beta_i = \langle b, A_i \rangle = b^T A_i = (b^T A_i)^T$$

$$= A_i^T b$$

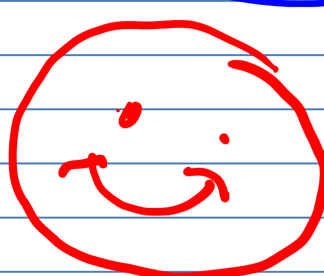
$$G^T \alpha = \beta$$

$$\Updownarrow$$

$$A^T A \alpha = A^T b$$

$$\therefore \boxed{\alpha = (A^T A)^{-1} A^T b}$$

!!!



Orthogonal Projection Operator

$(X, \mathbb{R}, \langle \cdot, \cdot \rangle)$ finite-dim. inner product space and $M \subset X$ a subspace. For $x \in X$ and $\hat{x} \in M$ we know TFAE:

a) $\hat{x} = \arg\min_{m \in M} \|x - m\|$

b) $x - \hat{x} \perp M$

c) $\exists\, \tilde{x} \in M^{\perp}$ such that $x = \hat{x} + \tilde{x}$.

Def. $P : X \to M$ by $P(x) = \hat{x}$, where $\hat{x}$ satisfies any one of (a), (b), or (c) is called **the orthogonal projection of x onto M.**

Exercises (a) $P : X \to M$ is a linear operator. Hint: use part (c) to show this.

(b) Let $\{v^1, \dots, v^n\}$ be an

orthonormal basis for M. Then,

$$P(x) = \sum_{i=1}^{k} \langle x, v_i \rangle v_i$$