ROB 501: Another Kalman Filter Derivation

We are given a linear system:

$$
\begin{aligned}
x_t &= Ax_{t-1} + Bu_t + \epsilon_t \\
z_t &= Cx_t + \delta_t
\end{aligned}
$$

where $\epsilon_t \sim \mathcal{N}(0, R_t)$, $\delta_t \sim \mathcal{N}(0, Q_t)$ and $x_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$. We assume x_0, δ and ϵ are uncorrelated. We seek to find an estimate for our state given our observation, $x_t \sim \mathcal{N}(\mu_{x|z}, \Sigma_{x|z})$.

Prediction Step

First, we update our distribution from our previous estimate. We use our equations for a linear combination of Gaussian random vectors (note that u_t is a known constant):

$$
\begin{aligned}
\bar{\mu}_t &= A\mu_{t-1} + Bu_t \\
\bar{\Sigma}_t &= A\Sigma_{t-1}A^\top + R_t
\end{aligned}
$$

Measurement Update

Next, we incorporate our measurement to estimate $x_t \sim \mathcal{N}(\mu_{x|z}, \Sigma_{x|z})$. We define a new random vector:

$$
\begin{bmatrix} x_t \\ z_t \end{bmatrix} = \begin{bmatrix} x_t \\ Cx_t + \delta_t \end{bmatrix} = \begin{bmatrix} I \\ C \end{bmatrix} x_t + \begin{bmatrix} 0 \\ \delta_t \end{bmatrix}
$$

Now we find the mean and covariance of our new random vector. Note that the new random vector is a linear function of x_t, and is jointly Gaussian distributed.

$$
E\left\{\begin{bmatrix} x_t \\ z_t \end{bmatrix}\right\} = \begin{bmatrix} I \\ C \end{bmatrix} E\{x_t\} + \begin{bmatrix} 0 \\ E\{\delta_t\} \end{bmatrix} = \begin{bmatrix} I \\ C \end{bmatrix} \bar{\mu}_t
$$

$$
\begin{aligned}
\operatorname{cov}\left(\begin{bmatrix} x_t \\ z_t \end{bmatrix}, \begin{bmatrix} x_t \\ z_t \end{bmatrix}\right) &= \begin{bmatrix} I \\ C \end{bmatrix} \bar{\Sigma}_t \begin{bmatrix} I & C^\top \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \\
&= \begin{bmatrix} \bar{\Sigma}_t & \bar{\Sigma}_t C^\top \\ C\bar{\Sigma}_t & C\bar{\Sigma}_t C^\top + Q \end{bmatrix}
\end{aligned}
$$

Recall from Gaussian random vectors that for a jointly Gaussian random vector $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$:

$$
\begin{aligned}
\mu_{1|2} &= \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2) \\
\Sigma_{1|2} &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}
\end{aligned}
$$

Applying this fact gives us:

$$
\begin{aligned}
\mu_{x|z} &= \bar{\mu}_t + \bar{\Sigma}_t C^\top (C\bar{\Sigma}_t C^\top + Q)^{-1}(z_t - C\bar{\mu}_t) \\
\Sigma_{x|z} &= \bar{\Sigma}_t - \bar{\Sigma}_t C^\top (C\bar{\Sigma}_t C^\top + Q)^{-1} C\bar{\Sigma}_t
\end{aligned}
$$

We can write this equation in terms of the Kalman gain, K_t:

$$
\begin{aligned}
K_t &= \bar{\Sigma}_t C^\top (C\bar{\Sigma}_t C^\top + Q)^{-1} \\
\mu_{x|z} &= \bar{\mu}_t + K_t(z_t - C\bar{\mu}_t) \\
\Sigma_{x|z} &= \bar{\Sigma}_t - K_t C\bar{\Sigma}_t
\end{aligned}
$$

$\square$