$\boxed{1.}$

$$f(x_1, x_2, x_3) = 3x_1 \left[2x_2 - (x_3)^3 \right] + (x_2)^{4/3}$$

(a)

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} &= 3\left[2x_2 - (x_3)^3\right] \\ \frac{\partial f}{\partial x_2} &= 6x_1 + 4/3\, x_2^{3} \\ \frac{\partial f}{\partial x_3} &= -9x_1 \cdot x_3^{2} \end{aligned}\right\} \Rightarrow \frac{\partial f(x)}{\partial x} = \left[3\left[2x_2 - x_3^3\right],\ 6x_1 + 4/3\, x_2^{3},\ -9x_1 \cdot x_3^2 \right]$$

$$\frac{\partial f(x)}{\partial x} \text{ at } x^* = [1\ 3\ -1]^T = \left[3(6+1),\ 6 + 4/3 \cdot 27,\ -9 \right]$$

$$= [21,\ 42,\ -9]$$

(b)

let $\delta_1 = \begin{bmatrix} 0.001 \\ 0 \\ 0 \end{bmatrix}$

$$\frac{\partial f(x^*)}{\partial x_1} = \frac{f(x^* + \delta_1) - f(x^* - \delta_1)}{2\delta_1} = \frac{48.0210 - 47.9790}{0.002} = 21.0000$$

let $\delta_2 = \begin{bmatrix} 0 \\ 0.001 \\ 0 \end{bmatrix}$

$$\frac{\partial f(x^*)}{\partial x_2} = \frac{f(x^* + \delta_2) - f(x^* - \delta_2)}{2\delta_2} = \frac{48.0420 - 47.9580}{0.002} = 42.0000$$

let $\delta_3 = \begin{bmatrix} 0 \\ 0 \\ 0.001 \end{bmatrix}$

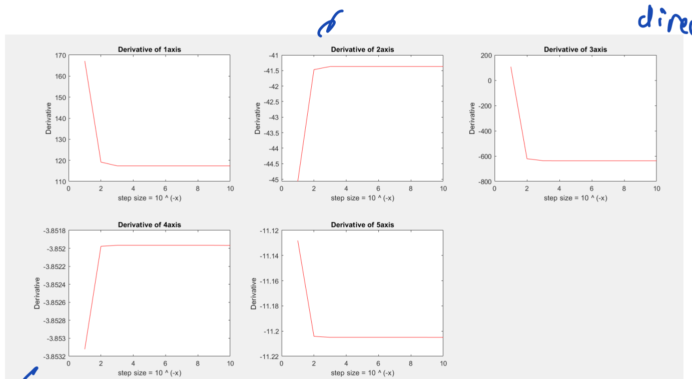
$$\frac{\partial f(x^*)}{\partial x_3} = \frac{f(x^* + \delta_3) - f(x^* - \delta_3)}{2\delta_3} = \frac{47.9910 - 48.0090}{0.002} = -9.0000$$

(c) Code in MATLAB

```matlab
% (c)
x0 = [1, 1, 1, 1, 1];
k = 1:10;
A = ones(1, 10);
B = A .* (10 .^ -k);
Derivative = [];
for axis = 1:5
    result = [];
    for i = 1: 10
        delta = B(i);
        x0(axis) = x0(axis) + delta;
        f_r = funcPartC(x0);
        x0(axis) = x0(axis) - 2 * delta;
        f_l = funcPartC(x0);
        d = (f_r - f_l) / (2 * delta);
        result = [result, d];
        x0(axis) = x0(axis) + delta;
    end
    Derivative = [Derivative, d];

    subplot(2, 3, axis)
    plot(result, 'r-')
    title("Derivative of " + axis + "axis")
    xlabel('step size = 10 \^ (-x)')
    ylabel('Derivative')
end

Derivative
```

step size from $10^{-1}, 10^{-2}, \ldots 10^{-10}$

Derivative w.r.t step size in each direction



Derivative converges when step size $< 10^{-4}$

$\Rightarrow$ Jacobian at $x^* = [1, 1, 1, 1, 1]$

$= [117.3523, -41.3685, -636.6021, -3.8520, -11.2049]$

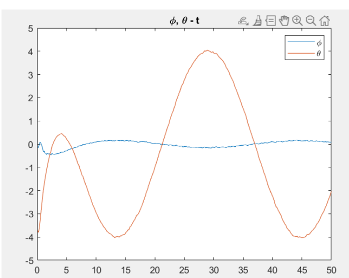
2.

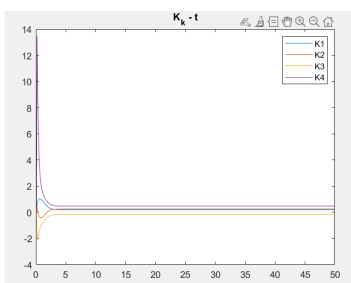
(a)

```
load SegwayData4KF.mat

phi = zeros(N, 1);
theta = zeros(N, 1);
phi_dot = zeros(N, 1);
theta_dot = zeros(N, 1);
K_1 = zeros(N, 1);
K_2 = zeros(N, 1);
K_3 = zeros(N, 1);
K_4 = zeros(N, 1);

x1 = x0;
P1 = P0;
tic
t=zeros(1,N);
for k =1:N
    uk = u(k);
    yk = y(k);

    K = P1 * C' / (C * P1 * C' + Q)
    x1 = A * x1 + B * uk + A * K * (yk - C * x1);
    P1 = A * (P1 - K * C * P1) * A' + G * R * G';

    x1_hat = x1;
    P1_hat = P1;

    phi(k)=[1 0 0 0] * x1_hat;
    theta(k)=[0 1 0 0] * x1_hat;
    phi_dot(k) = [0 0 1 0] * x1_hat;
    theta_dot(k) = [0 0 0 1] * x1_hat;
    K_1(k) = K(1);
    K_2(k) = K(2);
    K_3(k) = K(3);
    K_4(k) = K(4);

    t(k)=k*Ts;
    x1=x1_hat;
    P1 = P1_hat;
end
```

(b)

(c)



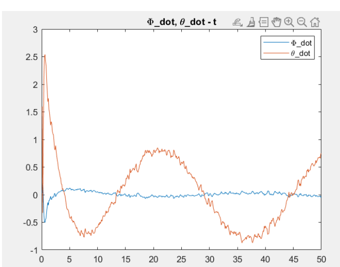



```
K =

    0.2113
    0.2559
   -0.1744
    0.4816
```

My implementation

```
Kss =

    0.2113
    0.2559
   -0.1744
    0.4816
```

Steady state



$\boxed{3.}$

Code in MATLAB

```
A = 1;
B = 0.1;
u = 10;
R = 16;
c = 3 * (10 ^ 8);
C = -2 / c;
Q = 10 ^ (-18);

z_1 = 2.2 * (10 ^ (-8));

X_0 = 1;
P_0 = 0.25;
X_hat = A * X_0 + B * u;

z_hat = 2 / c * (5 - X_hat);

P_hat = A * P_0 * A' + B * R * B';
K = P_hat * C' * inv(C * P_hat * C' + Q);
X_1_hat = X_hat + K * (z_1 - z_hat)
P_1_hat = P_hat -  K * C * P_hat
```

$x_1 \sim N(\mu_1, \Sigma_1) = N(1.7156, 0.0213)$

$\boxed{4.}$

$$\hat{x} = \underset{x^T x = 1}{\operatorname{argmin}} \; x^T A^T A x$$

let $\sigma = \min\limits_{x^T x = 1} x^T A^T A x$

$\Rightarrow x \sigma = \sigma x = A^T A x$

$\Rightarrow \sigma$ is the e-value, and $A^T A$ is the e-vector of x

$\Rightarrow$ Since we want the e-value σ to be minimized, we pick the corresponding e-vector to satisfy the condition.

$\because$ The smallest e-value is at the bottom right corner of Σ

$\Rightarrow$ the corresponding e-vector is the last column of V

($\because$ columns of V are e-vector of $A^T A$)

$\boxed{5.}$

$$\hat{A} = \begin{bmatrix} 4.0420 & 7.0450 & 3.0150 \\ 10.0426 & 17.0345 & 7.0245 \\ 16.0073 & 27.0037 & 11.0443 \end{bmatrix}$$

$$\Delta A = \begin{bmatrix} 0.0010 & -0.0010 & 0.0010 \\ -0.0024 & 0.0025 & -0.0025 \\ 0.0013 & -0.0013 & 0.0013 \end{bmatrix}$$

$$\|\Delta A\|_2 = \sqrt{\lambda_{max}(\Delta A^T \Delta A)} = 0.0051$$