

ROB 501 - HW 07

Kuan-Ting Lee (50036744)

1.

Use `numpy.linalg.eig(A)` in python. We can get approximate soln of its orthogonally diagonalized form.

$$A = O \Lambda O^T, \text{ where } \begin{cases} O \approx \begin{bmatrix} -0.500 & -0.707 & -0.500 \\ 0.707 & 0 & -0.707 \\ -0.500 & -0.707 & -0.500 \end{bmatrix} \\ \Lambda \approx \begin{bmatrix} 3.414 & 0 & 0 \\ 0 & 2.000 & 0 \\ 0 & 0 & 0.586 \end{bmatrix} \end{cases}$$

2.

$$(a) \quad A v^1 = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2 v^1$$

$$(b) \quad v^2 = \begin{bmatrix} 0.8165 \\ 0 \\ 0.5774 \end{bmatrix}, \quad v^3 = \begin{bmatrix} -0.5774 \\ 0 \\ 0.8165 \end{bmatrix}$$

$$[v^1 | v^2 | v^3] \cdot [v^1 | v^2 | v^3]^T = I_{3 \times 3} \Rightarrow \text{orthogonal}$$

$$(c) \quad V^T A V = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \text{ satisfies the form } \begin{bmatrix} 2 & 0_{1 \times 2} \\ 0_{2 \times 1} & A_2 \end{bmatrix}$$

with A_2 symmetric

$$(d) \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \text{ e-values} = 2, -1, \text{ e-vectors} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} U_2^T A_2 U_2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{diagonal} \end{aligned}$$

(e)

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, U^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$U U^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \Rightarrow \text{orthogonal}$$

(f)

$$O = V U = \begin{bmatrix} 0 & 0.8165 & -0.5774 \\ 1 & 0 & 0 \\ 0 & 0.5774 & 0.8165 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -0.5774 & 0.8165 \\ 1 & 0 & 0 \\ 0 & 0.8165 & 0.5774 \end{bmatrix}$$

$$O O^T = \begin{bmatrix} 0 & -0.5774 & 0.8165 \\ 1 & 0 & 0 \\ 0 & 0.8165 & 0.5774 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -0.5774 & 0 & 0.8165 \\ 0.8165 & 0 & 0.5774 \end{bmatrix}$$

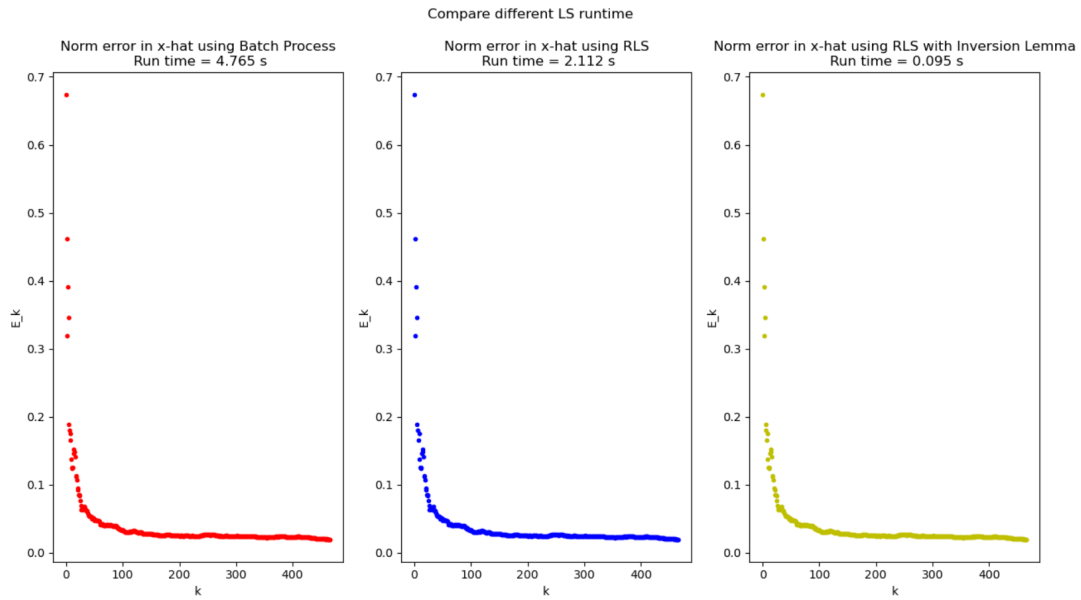
$$= I_{3 \times 3} \Rightarrow \text{orthogonal}$$

$$(g) O^T A O = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \text{diagonal!}$$

3.

(a) $n = 34$

(b) (c) (d)



<https://github.com/leekt0124/ROB501/tree/main/hw07>

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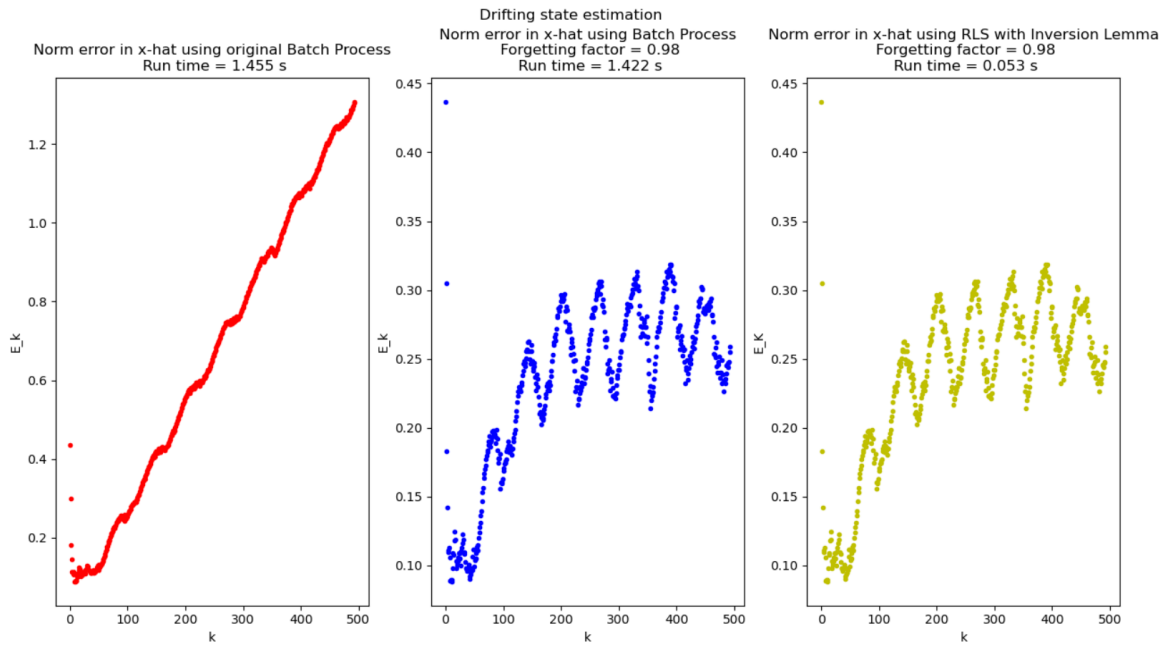
Code link

4.

(a) $n = 7$

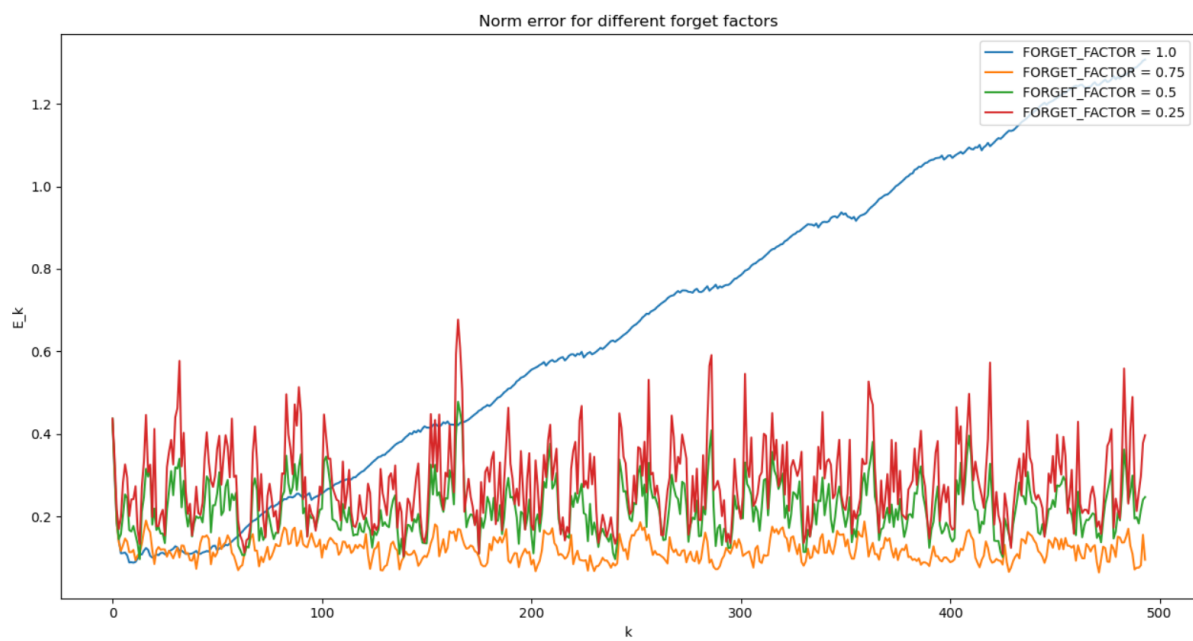
(b) (c) (d)

<https://github.com/leekt0124/ROB501/tree/main/hw07>



Code link

(bonus) just for fun !



5.

(a) eigenvalues of $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ are 0, 10 (all greater than or equal to 0)

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} = 0 \Lambda 0^T, \text{ where } \begin{cases} 0 \approx \begin{bmatrix} -0.9487 & -0.3162 \\ 0.3162 & -0.9487 \end{bmatrix} \\ \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix} \end{cases} \quad \Rightarrow \text{Positive Semidefinite}$$

$$P = 0 \Lambda 0^T = 0 \Lambda^{1/2} 0^T 0 \Lambda^{1/2} 0^T = N N^T$$

$$\Rightarrow N = 0 \Lambda^{1/2} 0^T = 0 \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{10} \end{bmatrix} 0^T \approx \begin{bmatrix} 0.3162 & 0.9487 \\ 0.9487 & 2.846 \end{bmatrix} \quad \times$$

(b) e-values of $\begin{bmatrix} 6 & 10 & 11 \\ 10 & 14 & 14 \\ 11 & 14 & 21 \end{bmatrix}$ are around 44.73, 0.18, 1.08 (all greater than 0)

\Rightarrow Positive Definite

$$\text{Conduct the same process as above, we get } \begin{cases} 0 \approx \begin{bmatrix} -0.3583 & -0.8745 & 0.3270 \\ -0.6407 & -0.0244 & -0.7674 \\ -0.6191 & -0.4845 & 0.5575 \end{bmatrix} \\ \Lambda = \begin{bmatrix} 44.73 & 0 & 0 \\ 0 & 0.18 & 0 \\ 0 & 0 & 1.08 \end{bmatrix} \end{cases}$$

$$\Rightarrow N = 0 \Lambda^{1/2} 0^T = \begin{bmatrix} 1.2093 & 1.2832 & 1.6325 \\ 1.2832 & 3.3589 & 2.4640 \\ 1.6325 & 2.4640 & 3.5019 \end{bmatrix} \quad \times$$

(c) e-values of $\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix}$ are 32.916, -2.916, 0 (not all greater than or equal to zero)

\Rightarrow Neither positive definite or positive semidefinite

6.

$$(a) \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

$$A = 1 > 0$$

$$C - B^T A^{-1} B = 8 - 3 \cdot \frac{1}{1} \cdot 3 = -1 < 0$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix} \text{ is not positive definite}$$

$$(b) \begin{bmatrix} 1 & 0 & 6 \\ 0 & 4 & 7 \\ 6 & 7 & 10 \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

$$A = 1 > 0$$

$$\begin{aligned} C - B^T A^{-1} B &= \begin{bmatrix} 4 & 7 \\ 7 & 10 \end{bmatrix} - \begin{bmatrix} 0 \\ 6 \end{bmatrix} \frac{1}{1} \begin{bmatrix} 0 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 7 \\ 7 & 10 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 36 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 7 \\ 7 & -26 \end{bmatrix} \end{aligned}$$

$$\text{Let } \begin{bmatrix} 4 & 7 \\ 7 & -26 \end{bmatrix} = \begin{bmatrix} a & b \\ b^T & c \end{bmatrix}$$

$$a = 4 > 0$$

$$c - b^T a^{-1} b = -26 - 7 \cdot \frac{1}{4} \cdot 7 < 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 6 \\ 0 & 4 & 7 \\ 6 & 7 & 10 \end{bmatrix} \text{ is not positive definite}$$

$$(c) \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 7 \\ 6 & 7 & a \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} > 0 \quad \left(\because \begin{cases} 1 > 0 \\ 5 - 2 \cdot \frac{1}{1} \cdot 2 = 1 > 0 \end{cases} \right)$$

$$\begin{aligned} C - B^T A^{-1} B &= a - [6 \ 7] \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 7 \end{bmatrix} \\ &= a - 61 \end{aligned}$$

$$C - B^T A^{-1} B > 0 \quad \text{when} \quad a - 61 > 0$$

\Rightarrow The range of a s.t. $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 7 \\ 6 & 7 & a \end{bmatrix}$ is p.d is $a > 61$ *

7.

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix} X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \text{underdetermined}$$

$$\begin{aligned} (a) \quad \hat{X} &= A^T (A A^T)^{-1} b \\ &= \begin{bmatrix} -0.0952 \\ 0.0416 \\ 0.4162 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} (b) \quad \hat{X} &= Q^{-1} A^T (A Q^{-1} A^T)^{-1} b, \quad Q = \begin{bmatrix} 5 & 1 & 9 \\ 1 & 2 & 1 \\ 9 & 1 & 17 \end{bmatrix} \\ &= \begin{bmatrix} -0.6497 \\ 0.3248 \\ 0.3376 \end{bmatrix} \end{aligned}$$

* Derivation of $\hat{X} = Q^{-1} A^T (A Q^{-1} A^T)^{-1} b$:

$$N(A)^\perp = \{ y \mid y^T Q x = 0, \forall x \in N(A) \}$$

$$N(A)^\perp = \{ y \mid (Q y)^T x = 0, \forall x \in N(A) \}$$

$$N(A)^\perp = \{ y \mid Q y = A^T \alpha, \alpha \in \mathbb{R}^m \}$$

$$N(A)^\perp = \{ y \mid y = Q^{-1} A^T \alpha, \alpha \in \mathbb{R}^m \}$$

\rightarrow choose $\hat{X} = Q^{-1} A^T \alpha$ ($\hat{X} \in N(A)^\perp$)

$$A \hat{X} = A Q^{-1} A^T \alpha$$

$$\alpha = (A Q^{-1} A^T)^{-1} A \hat{X} = (A Q^{-1} A^T)^{-1} b$$

$$\hat{X} = Q^{-1} A^T \alpha = Q^{-1} A^T (A Q^{-1} A^T)^{-1} b$$

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I discussed this assignment with Wan-ji (

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