30 October 2018

Review G. S. $\{y^1, \dots, y^n\}$ linearly independent.

$\exists \{v^1, \dots, v^n\}$ orthogonal such that, $\forall\, 1 \le k \le n$,

$\operatorname{span}\{v^1, \dots, v^k\} = \operatorname{span}\{y^1, \dots, y^k\}$.

$$v^1 = y^1$$

$$v^2 = y^2 - \frac{\langle y^2, v^1 \rangle v^1}{\langle v^1, v^1 \rangle}$$

$$v^3 = y^3 - \frac{\langle y^3, v^1 \rangle v^1}{\langle v^1, v^1 \rangle} - \frac{\langle y^3, v^2 \rangle}{\langle v^2, v^2 \rangle} v^2$$

etc.

- Let A be $m \times n$ real matrix. Then

$\operatorname{rank}(A) := \#$ lin. indep columns of A

$= \dim \operatorname{span}\{A_1, A_2, \dots, A_n\}$

where $A = [A_1 | A_2 | \cdots | A_n]$

Back Substitution

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 10 & ③ &\Rightarrow x_1 = -3/2 \\ x_2 - x_3 &= 2 & ② &\Rightarrow x_2 = 5 \\ 4x_3 &= 12 & ① &\Rightarrow x_3 = 3 \end{aligned}$$

$$\Updownarrow$$

$$\begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 12 \end{bmatrix}$$

Prop. Let A be an $m \times n$ matrix with linearly indep. columns. Then there exists an $m \times n$ matrix Q with orthonormal columns and an upper triangular matrix $R = n \times n$ such that $A = QR$.

Notes:

1) $Q^T \cdot Q = I_{n \times n}$
2) $[R]_{ij} = 0$ for $i > j$

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & & \cdots & r_{nn} \end{bmatrix}$$

3) Columns of A are linearly independent $\Longleftrightarrow$ R is invertible.

Utility? Why Important!?!

1) Suppose $Ax=b$ is over determined, columns of A are linearly independent. Write $A=QR$ and consider

$$A^TA\hat{x} = A^Tb$$

$$A^TA = R^TQ^TQR = R^TR$$

$$A^Tb = R^TQ^Tb$$

$$\therefore \cancel{R^T}R\hat{x} = \cancel{R^T}Q^Tb$$

rank $A=n \Leftrightarrow R$ invertible

$$\boxed{R\hat{x} = Q^Tb}$$

Solve for $\hat{x}$ by back substitution!!!

2) Suppose A is square and invertible. Write $A=QR$

$$A^{-1} = R^{-1}Q^{-1} = R^{-1}Q^T$$

MATLAB \ command!!!

3) Suppose $Ax=b$ is underdetermined with the rows of A linearly indep.

$\hat{x} = A^T(AA^T)^{-1}b$ is the "x" of smallest 2-norm satisfying $Ax=b$.

A^T has linearly indep columns

$A^T = QR$, $Q^TQ = I$, R upper triang.

$AA^T = R^TQ^TQR = R^TR$

$$\hat{x} = QR\left[R^TR\right]^{-1}b$$
$$= QRR^{-1}(R^T)^{-1}b$$
$$= Q(R^T)^{-1}b \qquad \square$$

How to compute the QR factorization?

$A = [A_1 | A_2 | \cdots | A_n]$, $A_i \in \mathbb{R}^m$

$\langle x,y \rangle = x^Ty$

G.S. with normalization

$\{A_1, A_2, \dots, A_n\} \longrightarrow \{v^1, v^2, \dots, v^n\}$

by

$$v^1 = \frac{A_1}{\|A_1\|}$$

$$v^2 = A_2 - \langle A_2, v^1\rangle v^1$$

$$v^2 = v^2 / \|v^2\|$$

$$\vdots$$

$$v^k = A_k - \langle A_k, v^1\rangle v^1 - \langle A_k, v^2\rangle v^2 \dots - \langle A_k, v^{k-1}\rangle v^{k-1}$$

$$v^k / \|v^k\|$$

$$\text{span}\{A_1, \dots, A_n\} = \text{span}\{v^1, \dots, v^n\}$$

$$Q = [v^1 | v^2 | \dots | v^n]$$

$$[Q^T Q]_{ij} = \langle v^i, v^j\rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

What About R?

$A_i \in \operatorname{span}\{\underbrace{v^1, v^2, \dots, v^i}_{\text{orthonormal}}\}$

$$A_i = \langle A_i, v^1\rangle v^1 + \langle A_i, v^2\rangle v^2 + \dots + \langle A_i, v^i\rangle v^i$$

$$[A_i]_{\{v\}} = \begin{bmatrix} r_{i1} \\ r_{i2} \\ \vdots \\ r_{ii} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \langle A_i, v^1\rangle \\ \langle A_i, v^2\rangle \\ \vdots \\ \langle A_i, v^i\rangle \\ 0 \\ \vdots \\ 0 \end{bmatrix} = R_i$$

$$QR_i = A_i \iff QR = A$$

$$\left[v^1 \mid v^2 \mid \dots \mid v^n\right] \begin{bmatrix} r_{i1} \\ r_{i2} \\ \vdots \\ r_{in} \end{bmatrix} = r_{i1}v^1 + r_{i2}v^2 + \dots + r_{in}v^n$$

for us $r_{ij} = 0$ for $j > i$.