08 November 2018

Review $(\Omega, \mathcal{F}, P)$ a probability space

$\mathcal{F}$ = Set of allowed events $\neq$ a field!

$X: \Omega \to \mathbb{R}$ (random variable)

$X: \Omega \to \mathbb{R}^p$ (random vector)

Density: $\forall x \in \mathbb{R}^p$,

$$P\left(\left\{\omega \in \Omega \,\middle|\, X(\omega) \leq \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}\right\}\right) = \int_0^{x_p} \cdots \int_0^{x_1} f_X(\bar{x}_1, \ldots \bar{x}_p)\, d\bar{x}_1 \cdots d\bar{x}_p$$

$$E\{g(X)\} = \int_{\mathbb{R}^p} g(x) f_X(x)\, dx$$

$\longleftarrow$ p-iterated integrals

mean $\mu := E\{X\}$

covariance $\mathrm{cov}(X) = \mathrm{cov}(X, X) := E\{\underbrace{(X-\mu)(X-\mu)^T}_{p \times p}\} \geq 0$

<u>Conditioning</u> $A, B \in \mathcal{F}$, $P(B) > 0$

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\text{"Joint"}}{\text{"Marginal"}}$$

$A|B$ is still an event and has probability $P(A|B)$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad \begin{aligned} X_1 &: \Omega \to \mathbb{R}^n \\ X_2 &: \Omega \to \mathbb{R}^m \end{aligned} \quad n+m=p$$

$$f_X(x) = f_{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}(x_1, x_2) = f_{X_1 X_2}(x_1, x_2)$$ joint density of X_1 and X_2

= density of $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$.

$f_{X_1}(x_1)$, $f_{X_2}(x_2)$ marginal densities

X_1, X_2 independent $\iff f_{X_1 X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$

In general, marginals are a pain to compute

$$f_{X_1}(x_1) = \int_{\mathbb{R}^m} f_{X_1 X_2}(x_1, x_2)\, dx_2$$

$$f_{X_2}(x_2) = \int_{\mathbb{R}^n} f_{X_1 X_2}(x_1, x_2)\, dx_1$$

Def. The conditional density of X_1 given $X_2 = x_2$ is

$$f_{X_1|X_2}(x_1, x_2) := \frac{f_{X_1 X_2}(x_1, x_2)}{f_{X_2}(x_2)}$$

Notation

$$X_1 \mid X_2 = x_2 \sim \frac{f_{X_1 X_2}(x_1, x_2)}{f_{X_2}(x_2)}$$

Shorthand

$$X_1 | X_2 \sim \frac{f_{X_1 X_2}}{f_{X_2}}$$

(We know we really have $X_2 = x_2$, x_2 is a value in $\mathbb{R}^m$)

$X_1 | X_2 = x_2$ is a random vector with density $f_{X_1|X_2}$.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$X \leftrightarrow x_k$

$Y \leftrightarrow y_k$

$Z \longleftrightarrow \{y_{k-1}, \cdots, y_1, y_0\}$

Note: $\begin{bmatrix} Y \\ Z \end{bmatrix} \longleftrightarrow \{y_k, y_{k-1}, \cdots, y_1, y_0\}$

Seek $X \Big| \begin{bmatrix} Y \\ Z \end{bmatrix} \longleftrightarrow x_k \Bigg| \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_0 \end{bmatrix}$ $\to \hat{x}_{k|k}$, $\to P_{k|k}$

From a previous step, we would

have $X|Z \to \hat{x}_{k|k-1}$, $\to P_{k|k-1}$ and

$Y|Z \to C\hat{x}_{k|k-1}$, $\to CP_{k|k-1}C^T$

$$(X|Z)\Big|(Y|Z) \sim \frac{f_{(X|Z)(Y|Z)}}{f_{Y|Z}} =$$

$$= \frac{f_{\begin{bmatrix} X \\ Y \end{bmatrix}|Z}}{f_{Y|Z}} = \frac{\dfrac{f_{XYZ}}{f_Z}}{\dfrac{f_{YZ}}{f_Z}} = \frac{f_{XYZ}}{f_{YZ}} \sim X \Big| \begin{bmatrix} Y \\ Z \end{bmatrix}$$

$\therefore$ If we have the joint density of $\cancel{X|Y}$ $X|Z$ and $\cancel{Z|Y}$ $Y|Z$, then we can compute $X \Big| \begin{bmatrix} Y \\ Z \end{bmatrix}$ by

$$X \Big| \begin{bmatrix} Y \\ Z \end{bmatrix} \sim (X|Z) \Big| (Y|Z)$$

Re-visit MVE for the special case of Normal Random Vectors. Our original solution is very general and does NOT require Normal Random Vectors.

$$y = Cx + \varepsilon$$

$x \in \mathbb{R}^n$, $x \sim N(\mu_x, \Sigma_{xx})$

$\varepsilon \in \mathbb{R}^m$, $\varepsilon \sim N(\mu_\varepsilon, \Sigma_{\varepsilon\varepsilon})$

$\Sigma_{x\varepsilon} = \Sigma_{\varepsilon x}^T = 0_{n \times m}$ (the unknown x and the measurement noise ε are uncorrelated)

Today

$$\mu_y = E\{Cx + \varepsilon\} = CE\{x\} + E\{\mu\}$$
$$= C\mu_x + \mu_\varepsilon$$

$$\Sigma_{yy} = E\{(y - \mu_y)(y - u_y)^T\}$$

$$(y - \mu_y) = C(x - \mu_x) + (\varepsilon - \mu_\varepsilon)$$

$$(y-\mu_y)(y-\mu_y)^T = [C(x-\mu_x) + (\varepsilon - \mu_\varepsilon)][C(x-\mu_x) + (\varepsilon-\mu_\varepsilon)]^T$$
$$= C(x-\mu_x)(x-\mu_x)^T C^T + (\varepsilon-\mu_\varepsilon)(\varepsilon-\mu_\varepsilon)^T +$$
$$+ C(x-\mu_x)(\varepsilon-\mu_\varepsilon)^T + (\varepsilon-\mu_\varepsilon)(x-\mu_x)^T C^T$$

$$\therefore \operatorname{cov}(y) = \boxed{\Sigma_{yy} = C\Sigma_{xx}C^T + \Sigma_{\varepsilon\varepsilon}} + 0 + 0.$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$E\{X\} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$$

$$\mathrm{cov}(X) = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}$$

$$\begin{aligned} \Sigma_{xy} &= E\{(x-\mu_x)(y-\mu_y)^T\} \\ &= E\{(x-\mu_x)[C(x-\mu_x) + (\varepsilon-\mu_\varepsilon)]^T\} \\ &= E\{(x-\mu_x)(x-\mu_x)^T C^T + (x-\mu_x)(\varepsilon-\mu_\varepsilon)^T\} \\ &= \Sigma_{xx} C^T + 0 \end{aligned}$$

$$\mathrm{cov}(X) = \mathrm{cov}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xx} C^T \\ C\Sigma_{xx} & C\Sigma_{xx}C^T + \Sigma_{\varepsilon\varepsilon} \end{bmatrix}$$

Next $x \mid y = \bar{y} = ?$

$$x \mid y = \bar{y} \sim N(\mu_{x|y}, \Sigma_{x|y})$$

$$\mu_{x|y} = \mu_x + \Sigma_{xx} C^T \left[C \Sigma_{xx} C^T + \Sigma_{\varepsilon\varepsilon} \right]^{-1} (\bar{y} - \mu_y)$$

$$\Sigma_{x|y} = \Sigma_{xx} - \Sigma_{xx} C^T \left[C \Sigma_{xx} C^T + \Sigma_{\varepsilon\varepsilon} \right]^{-1} C \Sigma_{xx}$$

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