Fields and Vector Spaces

Definition 2-1

A field consists of a set, denoted by $\mathscr{F}$, of elements called *scalars* and two operations called addition "+" and multiplication "·"; the two operations are defined over $\mathscr{F}$ such that they satisfy the following conditions:

1. To every pair of elements α and β in $\mathscr{F}$, there correspond an element $\alpha+\beta$ in $\mathscr{F}$ called the *sum* of α and β, and an element $\alpha \cdot \beta$ or $\alpha\beta$ in $\mathscr{F}$, called the *product* of α and β.
2. Addition and multiplication are respectively commutative: For any α, β in $\mathscr{F}$,
$$\alpha+\beta=\beta+\alpha \qquad \alpha\cdot\beta=\beta\cdot\alpha$$
3. Addition and multiplication are respectively associative: For any α, β, γ in $\mathscr{F}$,
$$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma) \qquad (\alpha\cdot\beta)\cdot\gamma=\alpha\cdot(\beta\cdot\gamma)$$
4. Multiplication is distributive with respect to addition: For any α, β, γ in $\mathscr{F}$,
$$\alpha\cdot(\beta+\gamma)=(\alpha\cdot\beta)+(\alpha\cdot\gamma)$$
5. $\mathscr{F}$ contains an element, denoted by 0, and an element, denoted by 1, such that $\alpha+0=\alpha$, $1\cdot\alpha=\alpha$ for every α in $\mathscr{F}$.
6. To every α in $\mathscr{F}$, there is an element β in $\mathscr{F}$ such that $\alpha+\beta=0$. The element β is called the *additive inverse*.
7. To every α in $\mathscr{F}$ which is not the element 0, there is an element γ in $\mathscr{F}$ such that $\alpha\cdot\gamma=1$. The element γ is called the *multiplicative inverse*. ∎

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Definition 2-2

A linear space over a field $\mathscr{F}$, denoted by $(\mathscr{X}, \mathscr{F})$, consists of a set, denoted by $\mathscr{X}$, of elements called *vectors*, a field $\mathscr{F}$, and two operations called *vector addition* and *scalar multiplication*. The two operations are defined over $\mathscr{X}$ and $\mathscr{F}$ such that they satisfy all the following conditions:

1. To every pair of vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathscr{X}$, there corresponds a vector $\mathbf{x}_1+\mathbf{x}_2$ in $\mathscr{X}$, called the sum of $\mathbf{x}_1$ and $\mathbf{x}_2$.
2. Addition is commutative: For any $\mathbf{x}_1, \mathbf{x}_2$ in $\mathscr{X}$, $\mathbf{x}_1+\mathbf{x}_2=\mathbf{x}_2+\mathbf{x}_1$.
3. Addition is associative: For any $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ in $\mathscr{X}$, $(\mathbf{x}_1+\mathbf{x}_2)+\mathbf{x}_3=\mathbf{x}_1+(\mathbf{x}_2+\mathbf{x}_3)$.
4. $\mathscr{X}$ contains a vector, denoted by $\mathbf{0}$, such that $\mathbf{0}+\mathbf{x}=\mathbf{x}$ for every $\mathbf{x}$ in $\mathscr{X}$. The vector $\mathbf{0}$ is called the zero vector or the origin.
5. To every $\mathbf{x}$ in $\mathscr{X}$, there is a vector $\bar{\mathbf{x}}$ in $\mathscr{X}$, such that $\mathbf{x}+\bar{\mathbf{x}}=\mathbf{0}$.
6. To every α in $\mathscr{F}$, and every $\mathbf{x}$ in $\mathscr{X}$, there corresponds a vector $\alpha\mathbf{x}$ in $\mathscr{X}$ called the *scalar product* of α and $\mathbf{x}$.
7. Scalar multiplication is associative: For any α, β in $\mathscr{F}$ and any $\mathbf{x}$ in $\mathscr{X}$, $\alpha(\beta\mathbf{x})=(\alpha\beta)\mathbf{x}$.
8. Scalar multiplication is distributive with respect to vector addition: For any α in $\mathscr{F}$ and any $\mathbf{x}_1, \mathbf{x}_2$ in $\mathscr{X}$, $\alpha(\mathbf{x}_1+\mathbf{x}_2)=\alpha\mathbf{x}_1+\alpha\mathbf{x}_2$.
9. Scalar multiplication is distributive with respect to scalar addition: For any α, β in $\mathscr{F}$ and any $\mathbf{x}$ in $\mathscr{X}$, $(\alpha+\beta)\mathbf{x}=\alpha\mathbf{x}+\beta\mathbf{x}$.
10. For any $\mathbf{x}$ in $\mathscr{X}$, $1\mathbf{x}=\mathbf{x}$, where 1 is the element 1 in $\mathscr{F}$. ∎

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