1.

Use numpy.linalg.eig (A) in python. We can get approximate Soln of its orthogonally diagonalized form.

$$A = O \Lambda O^T, \text{ where } \begin{cases} O \approx \begin{bmatrix} -0.500 & -0.707 & -0.500 \\ 0.707 & 0 & -0.707 \\ -0.500 & -0.707 & -0.500 \end{bmatrix} \\ \Lambda \approx \begin{bmatrix} 3.414 & 0 & 0 \\ 0 & 2.000 & 0 \\ 0 & 0 & 0.586 \end{bmatrix} \end{cases}$$

2.

(a) $$Av^1 = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2v^1$$

(b) $$v^2 = \begin{bmatrix} 0.8165 \\ 0 \\ 0.5774 \end{bmatrix}, \quad v^3 = \begin{bmatrix} -0.5774 \\ 0 \\ 0.8165 \end{bmatrix}$$

$$[v^1 | v^2 | v^3] \cdot [v^1 | v^2 | v^3]^T = I_{3\times3} \Rightarrow \text{orthogonal}$$

(c) $$V^T A V = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \text{ satisfies the form } \begin{bmatrix} 2 & 0_{1\times2} \\ 0_{2\times1} & A_2 \end{bmatrix}$$

with A_2 symmetric

(d) $$A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \text{ e-values} = 2, -1, \text{ e-vectors} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$U_2^T A_2 U_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{diagonal}$$

(e)

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad U^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$UU^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \Rightarrow \text{onthogonal}$$

(f)

$$O = VU = \begin{bmatrix} 0 & 0.8165 & -0.5774 \\ 2 & 0 & 0 \\ 0 & 0.5774 & 0.8165 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -0.5774 & 0.8165 \\ 2 & 0 & 0 \\ 0 & 0.8165 & 0.5774 \end{bmatrix}$$

$$OO^T = \begin{bmatrix} 0 & -0.5774 & 0.8165 \\ 2 & 0 & 0 \\ 0 & 0.8165 & 0.5774 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -0.5774 & 0 & 0.8165 \\ 0.8165 & 0 & 0.5774 \end{bmatrix}$$

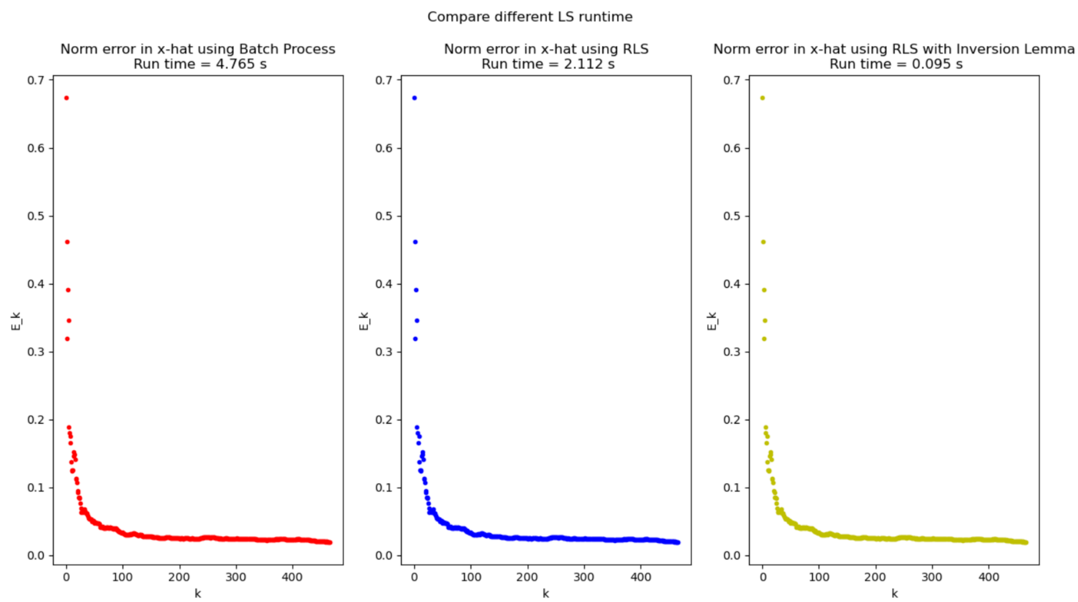
$$= I_{3\times 3} \Rightarrow \text{onthognal}$$

(g) $$O^T A O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \text{diagonal!}$$

3.

(a) $n = 34$

(b) (c) (d)



https://github.com/leekt0124/ROB501/tree/main/hw07

4

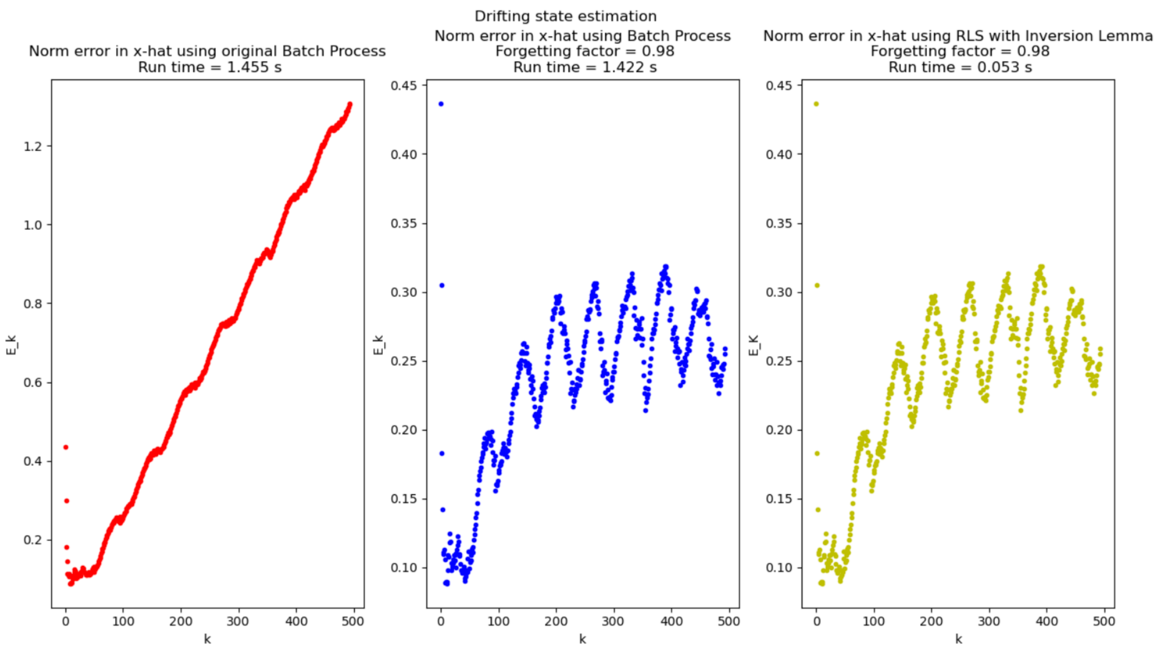
Code link

(4.)

(a) $n = 7$

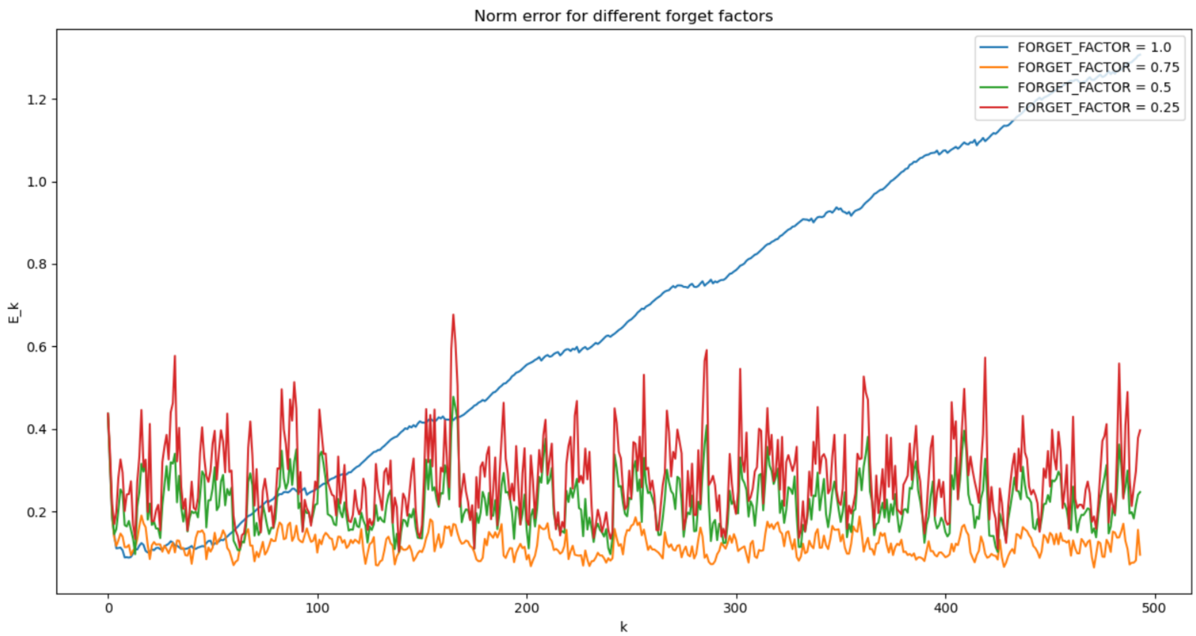
(b) (c) (d)

https://github.com/leekt0124/ROB501/tree/main/hw07



← Code link

(bonus) just for fun!



5.

(a) eigenvalues of $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ are $0, 10$ (all greater than or equal to 0) $\Downarrow$ Positive Semidefinite

$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} = Q \Lambda Q^T$, where $\begin{cases} Q \approx \begin{bmatrix} -0.9487 & -0.3162 \\ 0.3162 & -0.9487 \end{bmatrix} \\ \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix} \end{cases}$

$$P = Q \Lambda Q^T = Q \Lambda^{1/2} Q^T Q \Lambda^{1/2} Q^T = N N^T$$

$$\Rightarrow N = Q \Lambda^{1/2} Q^T = Q \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{10} \end{bmatrix} Q^T \approx \begin{bmatrix} 0.3162 & 0.9487 \\ 0.9487 & 2.846 \end{bmatrix}$$

(b) e-values of $\begin{bmatrix} 6 & 10 & 11 \\ 10 & 19 & 19 \\ 11 & 19 & 21 \end{bmatrix}$ are around $44.73, 0.18, 1.08$ (all greater than 0)

$\Rightarrow$ Positive Definite

Conduct the same process as above, we get $\begin{cases} Q \approx \begin{bmatrix} -0.3583 & -0.8745 & 0.3270 \\ -0.6407 & -0.0249 & -0.7674 \\ -0.6791 & -0.4845 & 0.5515 \end{bmatrix} \\ \Lambda = \begin{bmatrix} 44.73 & 0 & 0 \\ 0 & 0.18 & 0 \\ 0 & 0 & 1.08 \end{bmatrix} \end{cases}$

$$\Rightarrow N = Q \Lambda^{1/2} Q^T = \begin{bmatrix} 1.2993 & 1.2832 & 1.6325 \\ 1.2832 & 3.3589 & 2.4640 \\ 1.6325 & 2.4640 & 3.5019 \end{bmatrix}$$

(c) e-values of $\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix}$ are $32.916, -2.916, 0$ (not all greater than or equal to zero)

$\Rightarrow$ Neither positive definite or positive semidefinite

⑥

(a) $\begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$

$A = 1 > 0$

$C - B^T A^{-1} B = 8 - 3 \cdot 1/1 \cdot 3 = -1 < 0$

$\Rightarrow \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix}$ is not possitive definite

(b) $\begin{bmatrix} 1 & 0 & 6 \\ 0 & 4 & 7 \\ 6 & 7 & 10 \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$

$A = 1 > 0$

$C - B^T A^{-1} B = \begin{bmatrix} 4 & 7 \\ 7 & 10 \end{bmatrix} - \begin{bmatrix} 0 \\ 6 \end{bmatrix} 1/1 \begin{bmatrix} 0 & 6 \end{bmatrix}$

$= \begin{bmatrix} 4 & 7 \\ 7 & 10 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 36 \end{bmatrix}$

$= \begin{bmatrix} 4 & 7 \\ 7 & -26 \end{bmatrix}$

Let $\begin{bmatrix} 4 & 7 \\ 7 & -26 \end{bmatrix} = \begin{bmatrix} a & b \\ b^T & c \end{bmatrix}$

$a = 4 > 0$

$c - b^T a^{-1} b = -26 - 7 \cdot 1/4 \, 7 < 0$

$\Rightarrow \begin{bmatrix} 1 & 0 & 6 \\ 0 & 4 & 7 \\ 6 & 7 & 10 \end{bmatrix}$ is not positive definite

(c) $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 7 \\ 6 & 7 & a \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} > 0 \quad \left(\because \begin{cases} 1 > 0 \\ 5 - 2 \cdot 1/1 \cdot 2 = 1 > 0 \end{cases} \right)$$

$$C - B^T A^{-1} B = a - \begin{bmatrix} 6 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$= a - 61$$

$C - B^T A^{-1} B > 0$ when $a - 61 > 0$

$\Rightarrow$ The range of a s.t. $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 7 \\ 6 & 7 & a \end{bmatrix}$ is p.d is $a > 61$

7.

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \text{underdetermined}$$

(a) $\hat{x} = A^T (A A^T)^{-1} b$

$$= \begin{bmatrix} -0.0952 \\ 0.0476 \\ 0.4762 \end{bmatrix}$$

(b) $\hat{x} = Q^{-1} A^T (A Q^{-1} A^T)^{-1} b \quad , \quad Q = \begin{bmatrix} 5 & 1 & 9 \\ 1 & 2 & 1 \\ 9 & 1 & 17 \end{bmatrix}$

$$= \begin{bmatrix} -0.6497 \\ 0.3248 \\ 0.3376 \end{bmatrix}$$

※ Derivation of $\hat{x} = Q^{-1} A^T (A Q^{-1} A^T)^{-1} b$:

$N(A)^{\perp} = \{ y \mid y^T Q x = 0, \ \forall x \in N(A) \}$

$N(A)^{\perp} = \{ y \mid (Qy)^T x = 0, \ \forall x \in N(A) \}$

$N(A)^{\perp} = \{ y \mid Qy = A^T \alpha, \ \alpha \in \mathbb{R}^m \}$ ← row vector of A is normal to x

$N(A)^{\perp} = \{ y \mid y = Q^{-1} A^T \alpha, \ \alpha \in \mathbb{R}^m \}$

→ choose $\hat{x} = Q^{-1} A^T \alpha \quad (\hat{x} \in N(A)^{\perp})$

$A \hat{x} = A Q^{-1} A^T \alpha$

$\alpha = (A Q^{-1} A^T)^{-1} A \hat{x} = (A Q^{-1} A^T)^{-1} b$

$\hat{x} = Q^{-1} A^T \alpha = Q^{-1} A^T (A Q^{-1} A^T)^{-1} b$ ※

I discussed this assignment with Wan-Yi Yu

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