ROB 501 Handout: Grizzle

Inner Product Spaces

Definition: Let $(X, \mathbb{C})$ be a vector space. A function

$$< \cdot, \cdot >: X \times X \to \mathbb{C}$$

is an **inner product** if

(a) $< x, y > = \overline{< y, x >}$

(b) $< \alpha_1 x_1 + \alpha_2 x_2, y > = \alpha_1 < x_1, y > + \alpha_2 < x_2, y >$ (i.e., linear in the left argument)

(c) $< x, x > \geqslant 0$ for any $x \in X$, and $< x, x > = 0 \Leftrightarrow x = 0$.

In the case of a real vector space $(X, \mathbb{R})$, replace (a) with

(a$'$): $< x, y > = < y, x >$. It is easy to show that we then have linearity in both the left and right sides.

Examples:

(a) $(\mathbb{C}^n, \mathbb{C}) \quad < x, y > = x^\top \overline{y}$

(b) $(\mathbb{R}^n, \mathbb{R}) \quad < x, y > = x^\top y$

(c) The field is the real numbers $\mathbb{R}$ and the vector space is the set of $n \times m$ real matrices

$$X = \{A \mid A \text{ real } n \times m \text{ matrix}\},$$

with inner product

$$< A, B > = \mathrm{tr}\left(\mathrm{A}^\top \mathrm{B}\right)$$

(d) $X = C[a, b] =$ space of continuous real-valued functions on $[a, b]$

$$< f, g > = \int_a^b f(t) g(t) dt$$

Theorem: [Cauchy-Schwarz Inequality] Suppose that $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$. Let $(X, \mathcal{F}, < \cdot, \cdot >)$ be an **inner product space** (i.e. $(X, \mathcal{F})$ is a vector space and $< \cdot, \cdot >$ is an inner product on X). Then, for all $x, y \in X$,

$$|< x, y >| \leqslant < x, x >^{1/2} \cdot < y, y >^{1/2} .$$

Proof: If $y = 0$, the result is obviously true. Hence, assume $y \neq 0$. For all scalars λ we have that

$$0 \leq \quad < x - \lambda y, x - \lambda y > = < x, x > -\lambda < y, x > -\overline{\lambda} < x, y > +|\lambda|^2 < x, y >,$$

because the inner product of a vector with itself is a non-negative real number. For the particular choice $\lambda = \frac{<x,y>}{<y,y>}$, direct calculation shows

$$0 \leq \quad < x, x > - \frac{| < x, y > |^2}{< y, y >},$$

which gives

$$| < x, y > | \leq \sqrt{< x, x >< y, y >} = < x, x >^{1/2} \cdot < y, y >^{1/2} .$$

$\blacksquare$

Corollary: Let $(X, \mathcal{F}, < \cdot, \cdot >)$ be an inner product space. Then

$$\|x\| := < x, x >^{1/2}$$

is a norm on X.

Proof: The main thing to establish is the triangle inequality:

$$\|x+y\| \leqslant \|x\| + \|y\|.$$

This is equivalent to showing:

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2.$$

Brute force computation:

$$\begin{aligned}
\|x+y\|^2 &= < x+y, x+y > \\
&= < x, x+y > + < y, x+y > \\
&= \overline{< x+y, x >} + \overline{< x+y, y >} \\
&= \overline{< x, x > + < y, x >} + \overline{< x, y > + < y, y >} \\
&= < x, x > + < x, y > + \overline{< x, y >} + < y, y > \\
&= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\{< \mathrm{x}, \mathrm{y} >\}
\end{aligned}$$

where $\mathrm{Re}\{< \mathrm{x}, \mathrm{y} >\}$ denotes the real part of the complex number $< x, y >$. However, for any complex number α, $\mathrm{Re}\{\alpha\} \leq |\alpha|$, and thus we have

$$\begin{aligned}
\|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\{< \mathrm{x}, \mathrm{y} >\} \\
&\leqslant \|x\|^2 + \|y\|^2 + 2| < x, y > | \\
&\leqslant \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|,
\end{aligned}$$

where the last inequality is from the Cauchy-Schwarz Inequality. ■

Definition:

(a) Two vectors x and y are **orthogonal** if $< x, y >= 0$. **Notation:** $x \perp y$.

(b) A **set of vectors** S is **orthogonal** if

$$\forall x, y \in S, \quad x \neq y, \ < x, y >= 0 \ \ (\text{i.e. } \ x \perp y).$$

(c) If in addition $\|x\| = 1 \ \forall x \in S$, S is an **orthonormal set**.

How to Construct Orthonormal Sets ?

Gram-Schmidt Process: First orthogonalize, then normalize.

Let $\{y_i \mid i = 1, ..., n\}$ be a *linearly independent* set of vectors. Define a set of vectors $\{v_i \mid i = 1, ..., n\}$ recursively by
Step 1: Set

$$v_1 = y_1.$$

Then clearly Span $\{v_1\} = \text{Span}\{y_1\}$. From the linear independence of $\{y_1\}$, we see that $\{v_1\}$ is linearly independent and hence v_1 is nonzero.
Step 2: Set

$$v_2 = y_2 - a_{21}v_1,$$

and **choose** a_{21} so that $< v_2, v_1 >= 0$.

$$\begin{aligned} 0 =< v_2, v_1 > &=< y_2 - a_{21}v_1, v_1 > \\ &=< y_2, v_1 > -a_{21} < v_1, v_1 > \\ \therefore a_{21} &= \frac{< y_2, v_1 >}{\|v_1\|^2} \end{aligned}$$

where we could divide by $\|v_1\|^2$ because we know that $v_1 \neq 0$.

Claim Span $\{v_1, v_2\} = \text{Span}\{y_1, y_2\}$ and $v_1 \perp v_2$.

Proof of the claim: The orthogonality is by construction. Because we have Span $\{v_1\} = \text{Span}\{y_1\}$, in order to show Span $\{v_1, v_2\} = \text{Span}\{y_1, y_2\}$, it is enough to show that $v_2 \in \text{Span}\{y_1, y_2\}$ and $y_2 \in \text{Span}\{v_1, v_2\}$, which both follow from $v_2 = y_2 - a_{21}v_1$. Indeed,

$$y_2 = v_2 + a_{21}v_1$$

shows $y_2 \in \text{Span}\{v_1, v_2\}$, and

$$v_2 = y_2 - a_{21}v_1 = y_2 - a_{21}y_1$$

shows that $v_2 \in \text{Span}\{y_1, y_2\}$.

From the linear independence of $\{y_1, y_2\}$, we see that the set $\{v_1, v_2\}$ is linearly independent and hence v_1 and v_2 are nonzero. We could pass to the induction step now, but just to be extra clear, let's do one more:

Step 3: Write $v_3 = y_3 - a_{31}v_1 - a_{32}v_2$

$$
\begin{aligned}
0 =< v_3, v_1 > &=< y_3 - a_{31}v_1 - a_{32}v_2, v_1 > \\
&=< y_3, v_1, > -a_{31} < v_1, v_1 > -a_{32} \underbrace{< v_2, v_1 >}_{=0} \\
\therefore a_{31} &= \frac{< y_3, v_1 >}{\|v_1\|^2}
\end{aligned}
$$

$$
\begin{aligned}
0 =< v_3, v_2 > &=< y_3 - a_{31}v_1 - a_{32}v_2, v_2 > \\
&=< y_3, v_2 > -a_{31} \underbrace{< v_1, v_2 >}_{=0} -a_{32} < v_2, v_2 > \\
\therefore a_{32} &= \frac{< y_3, v_2 >}{\|v_2\|^2}
\end{aligned}
$$

Step k: In general, one obtains:

$$v_k = y_k - \sum_{j=1}^{k-1} \underbrace{\frac{< y_k, v_j >}{\|v_j\|^2}}_{a_{kj}} \cdot v_j.$$

By construction, for each $1 \leq k \leq n$, $\{v_1, \cdots, v_k\}$ is an **orthogonal** set.

Normalization: Define: $\tilde{v}_i = \dfrac{v_i}{\|v_i\|} \Rightarrow \{\tilde{v}_1, \cdots, \tilde{v}_k\}$ is **orthonormal**.

Remark: At each step, we need to prove that

$$\text{Span } \{v_1, \cdots, v_k\} = \text{Span}\{y_1, \cdots, y_k\}.$$

In this way we will have that the v_i are non-zero, and thus each step in the Gram-Schmidt process is well defined. First an example, then a proof.

Example: Construct a set of orthonormal vectors from

$$y_1^T = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \quad y_2^T = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}, \quad y_3^T = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$$

The vectors are easily checked to be linearly independent.

Orthogonalize: Let

$$\begin{aligned} v_1 &= y_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \\ v_2 &= y_2 - \frac{< y_2, v_1 >}{< v_1, v_1 >} v_1 = y_2, \quad \text{(because} \quad < y_2, v_1, >= 0) \end{aligned}$$

In this case y_1 and y_2 were already orthogonal, so there was nothing to do. Continuing,

$$\begin{aligned} v_3 &= y_3 - \frac{< y_3, v_1 >}{< v_1, v_1 >} v_1 - \frac{< y_3, v_2, >}{< v_2, v_2 >} v_2 && (1) \\ &= y_3 - \frac{2}{2} v_1 - \frac{4}{6} v_2 && (2) \\ &= \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}^T && (3) \end{aligned}$$

Normalize: Normalize v_i to get $\tilde{v}_i$:

$$\tilde{v}_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; \tilde{v}_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \; \tilde{v}_3 = \frac{v_1}{\|v_3\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

$\square$

Proposition: Let $\{y_1, \cdots, y_k\}$ be a linearly independent set and suppose the set $\{v_1, \cdots, v_{k-1}\}$ is orthogonal and

$$\text{span}\{v_1, \cdots, v_{k-1}\} = \text{span}\{y_1, \cdots, y_{k-1}\}.$$

Define

$$v_k = y_k - \sum_{j=1}^{k-1} \frac{< y_k, v_j >}{\|v_j\|^2} \cdot v_j \tag{4}$$

Then, $\{v_1, \cdots, v_k\}$ is orthogonal and

$$\text{span}\{v_1, \cdots, v_k\} = \text{span}\{y_1, \cdots, y_k\}$$

Proof: The orthogonality is by construction. We will show that $y_k \in \text{span}\{v_1, \cdots, v_k\}$ and $v_k \in \text{span}\{y_1, \cdots, y_k\}$.

From (4),

$$y_k = v_k + \sum_{j=1}^{k-1} \frac{< y_k, v_j >}{\|v_j\|^2} \cdot v_j \;\Rightarrow\; y_k \in \text{span}\{v_1, \cdots, v_k\}.$$

Left to show: $v_k \in \text{span}\{y_1, \cdots, y_k\}$. By hypothesis,

$$v_j \in \text{span}\{y_1, \cdots, y_{k-1}\} \text{ for all } 1 \leqslant j \leqslant k-1,$$

so

$$\sum_{j=1}^{k-1} \left(\frac{< v_j, y_k >}{\|v_j\|^2} \right) v_j \;\in\; \text{span}\{y_1, \cdots, y_{k-1}\} \;\subset\; \text{span}\{y_1, \cdots, y_k\}.$$

Clearly, $y_k \in \text{span}\{y_1, \cdots, y_k\}$.

$$\therefore v_k = y_k - \sum_{j=1}^{k-1} \left(\frac{< v_j, y_k >}{\|v_j\|^2} \right) v_j \;\in\; \text{span}\{y_1, \cdots, y_k\}$$

because $\text{span}\{y_1, \cdots, y_k\}$ is a subspace. ■

Definition: $x \perp y \Leftrightarrow < x, y >= 0; \quad x \perp S \Leftrightarrow \forall\, y \in S, \; < x, y >= 0.$

Claim: Suppose that $x \perp \{y_1, \cdots, y_k\}$ (i.e. $< x, y_i >= 0, 1 \leqslant i \leqslant k$). Then x is $\perp$ to span$\{y_1, \cdots, y_k\}$, i.e, $< w, x >= 0 \; \forall \; w \in$ span$\{y_1, \cdots, y_k\}$.

Proof of the claim: $< \sum_{i=1}^{k} \alpha_i y_i, x, >= \sum_{i=1}^{k} \alpha_i < y_i, x, >= 0$ ■

Claim: Let $\{v_1, \cdots, v_n\}$ be an **orthonormal basis** for a inner product space X. Then the representation of $x \in X$ with respect to $\{v_1, \cdots, v_n\}$ is

$$\boxed{x = \sum_{i=1}^{n} < x, v_i > v_i}$$

Proof of the claim: $\{v_1, \cdots, v_n\}$ a basis $\Rightarrow \exists \,!$ coeff. $\alpha_1, \cdots, \alpha_n$ such that

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

Now,

$$\begin{aligned} < x, v_i > &= < \sum_{k=1}^{n} \alpha_k v_k, v_i > \\ &= \sum_{k=1}^{n} \alpha_k \underbrace{< v_k, v_i >}_{0 \;\; k \neq i} \\ &= \alpha_i \end{aligned}$$

$$\therefore \; \boxed{\alpha_i = < x, v_i >}$$

■

Reciprocal Basis Vectors

Proposition: Let $(X, \mathbb{R}, < \cdot, \cdot >)$ be an n-dim. inner product space, and let $\{v_1, \cdots, v_n\}$ be a basis for X (*not* necessarily orthonormal). Show that for each $i = 1, 2, \cdots, n$, $\exists\ r_i \in X$ such that

$$< r_i, v_j >= \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \tag{5}$$

Solution: Suffices to prove this for $i = 1$. Apply the Gram-Schmidt process to the linearly independent set $\{v_2, v_3, \cdots, v_n, v_1\}$. Note that v_1 has been permuted to the end. (To compute r_3 for example, you would permute v_3 to the end and apply the same procedure.) The Gram-Schmidt process will produce n orthogonal vectors $\{w_2, w_3, \cdots, w_n, w_1\}$, such that

$$\text{span}\{w_2, w_3, \cdots, w_n\} = \text{span}\{v_2, v_3, \cdots, v_n\}$$

By construction, $< w_1, w_j >= 0$, $j = 2, \cdots, n$, which implies that

$$w_1 \perp \text{span}\{w_2, \cdots, w_n\}, \tag{6}$$

and thus

$$w_1 \perp \text{span}\{v_2, \cdots, v_n\}. \tag{7}$$

From the Gram-Schmidt Process,

$$w_1 = v_1 - \sum_{j=2}^{n} \frac{< w_j, v_1 >}{\|w_j\|^2} w_j.$$

Therefore, $v_1 = w_1 + \sum_{j=2}^{n} \frac{< w_j, v_1 >}{\|w_j\|^2} w_j$, and hence,

$$< w_1, v_1 >=< w_1, w_1 > \quad \text{by (6) and (7).}$$

$\therefore$ If we choose $r_1 = \frac{w_1}{\|w_1\|^2}$, we have

$$< r_1, v_j >= \begin{cases} 1 & j = 1 \\ 0 & j \neq 1 \end{cases} \quad \text{as desired.}$$

$\square$

Remarks: (a) $\{r_1, \cdots, r_n\}$ satisfying (5) is called a **reciprocal basis**. (b) In order to find r_k, simply rotate v_k to back as in $\{v_1, \cdots, v_{k-1}, v_{k+1}, \cdots, v_n, v_k\}$ and apply the Gram-Schmidt procedure as above.

Problem:[4] Let $(X, \mathbb{R}, < \cdot, \cdot >)$ be an n-dim. inner product space, let $\{v_1, \cdots, v_n\}$ be a basis for X, and let $\{r_1, \cdots, r_n\}$ be the corresponding reciprocal basis. Show that for all $x \in X$,

$$x = \sum_{i=1}^{n} < r_i, x > v_i \tag{8}$$

In other words, determining the representation of x with respect to $\{v_1, \cdots, v_n\}$ can be accomplished by computing inner products!

Solution: Because $\{v_1, \cdots, v_n\}$ is a basis, there exist unique $\alpha_i \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{n} \alpha_i v_i \tag{9}$$

Hence,

$$< r_i, x > = < r_i, \sum_{k=1}^{n} \alpha_k v_k > \tag{10}$$

$$= \sum_{k=1}^{n} \alpha_k < r_i, v_k > \tag{11}$$

$$= \alpha_i \tag{12}$$

because $< r_i, v_k >$ equals one if $i = k$ and zero otherwise. This proves (8).

Inner Products: Example Computation for $(\mathbb{R}^3, \mathbb{R})$

Given data:

$$< p, q >= p^T q = \sum_{i=1}^{3} p_i q_i$$

$$\{y_1, y_2, y_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Apply Gram-Schmidt to Produce an Orthogonal Basis:

$$v_1 = y_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\|v_1\|^2 = (v_1)^T v_1 = 2;$$

$$\begin{aligned} v_2 &= y_2 - \frac{< v_1, y_2 >}{\|v_1\|^2} v_1 \\ &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{3} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix} \end{aligned}$$

$$\|v_2\|^2 = 9\frac{1}{2} = \frac{19}{2};$$

$$\begin{aligned} v_3 &= y_3 - \frac{< v_1, y_3 >}{\|v_1\|^2} v_1 - \frac{< v_2, y_3 >}{\|v_2\|^2} v_2 \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{1} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \underbrace{\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{3\frac{1}{2}} \frac{1}{\frac{19}{2}} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{7}{38} \\ \frac{7}{38} \\ \frac{21}{19} \end{bmatrix} = \begin{bmatrix} -\frac{6}{19} \\ \frac{6}{19} \\ -\frac{2}{19} \end{bmatrix}. \end{aligned}$$

Normalize to obtain Orthonormal Basis:

$$\tilde{v}_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\tilde{v}_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} \frac{-1}{\sqrt{38}} \\ \frac{1}{\sqrt{38}} \\ 3\sqrt{\frac{2}{19}} \end{bmatrix}$$

$$\tilde{v}_3 = \frac{v_3}{\|v_3\|} = \frac{19}{\sqrt{76}} \begin{bmatrix} -\frac{6}{19} \\ \frac{6}{19} \\ -\frac{2}{19} \end{bmatrix}$$

Obtain Reciprocal Basis:

We seek a basis $\{r_1, r_2, r_3\}$ such that $< r_i, y_j >= \delta_{ij}$.
Step 1: r_1 is found by applying the Gram-Schmidt process to $\{y_2, y_3, y_1\}$.

$$v_1^1 := y_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$v_1^2 := y_3 - \frac{< v_1^1, y_3 >}{\|v_1^1\|^2} v_1^1 = \begin{bmatrix} -\frac{5}{14} \\ \frac{2}{7} \\ -\frac{1}{14} \end{bmatrix}$$

$$v_1^3 := y_1 - \frac{< v_1^1, y_1 >}{\|v_1^1\|^2} v_1^1 - \frac{< v_1^2, y_1 >}{\|v_1^2\|^2} v_1^2 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$

$$r_1 := \frac{v_1^3}{\|v_1^3\|^2} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Step 2: r_2 is found by applying the Gram-Schmidt process to $\{y_3, y_1, y_2\}$.

$$v_2^1 := y_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2^2 := y_1 - \frac{< v_1^1, y_1 >}{\|v_1^1\|^2} v_1^1 = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$v_2^3 := y_2 - \frac{< v_1^1, y_2 >}{\|v_1^1\|^2} v_1^1 - \frac{< v_1^2, y_1 >}{\|v_1^2\|^2} v_1^2 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$r_2 := \frac{v_2^3}{\|v_2^3\|^2} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Step 3: r_3 is found by applying the Gram-Schmidt process to $\{y_1, y_2, y_3\}$.

$$v_3^1 := y_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3^2 := y_2 - \frac{< v_1^1, y_2 >}{\|v_1^1\|^2} v_1^1 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}$$

$$v_3^3 := y_3 - \frac{< v_1^1, y_3 >}{\|v_1^1\|^2} v_1^1 - \frac{< v_1^2, y_3 >}{\|v_1^2\|^2} v_1^2 = \begin{bmatrix} -\frac{6}{19} \\ \frac{6}{19} \\ -\frac{2}{19} \end{bmatrix}$$

$$r_3 := \frac{v_3^3}{\|v_3^3\|^2} = \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Inner Products: Example Computation for $(C[0,1],\mathbb{R})$

Given data:

$$C[0,1]=\{f:[0,1]\to\mathbb{R}\mid f \ \ \text{continuous}\},\quad <f,g>=\int_0^1 f(\tau)g(\tau)d\tau$$

$$\{y_1,y_2,y_3\}=\{1,t,t^2\}$$

Apply Gram-Schmidt to Produce an Orthogonal Basis:

$$\begin{aligned}
v_1 &= y_1 = 1\\
\|v_1\|^2 &= \int_0^1 (1)^2 d\tau = 1;\\
\\
v_2 &= y_2 - \frac{<v_1,y_2>}{\|v_1\|^2}v_1\\
&= t - \underbrace{\int_0^1 1\cdot\tau d\tau}_{\frac{1}{2}}\cdot\frac{1}{1}\cdot 1 = t-\frac{1}{2}\\
\|v_2\|^2 &= \int_0^1(\tau-\frac{1}{2})^2 d\tau = \frac{1}{12};\\
\\
v_3 &= y_3 - \frac{<v_1,y_3>}{\|v_1\|^2}v_1 - \frac{<v_2,y_3>}{\|v_2\|^2}v_2\\
&= t^2 - \underbrace{\int_0^1 1\cdot\tau^2 d\tau}_{\frac{1}{3}}\cdot\frac{1}{1}\cdot 1 - \underbrace{\int_0^1(\tau-\frac{1}{2})\tau^2 d\tau}_{\frac{1}{12}}\left(\frac{1}{\frac{1}{12}}\right)\left(t-\frac{1}{2}\right)\\
&= t^2-\frac{1}{3}-\left(t-\frac{1}{2}\right)\\
&= t^2-t+\frac{1}{6}.
\end{aligned}$$

Doing Inner products on $C[a,b]$ in MATLAB

```
>> clear *
>> syms t % declare to be a symbolic variable

>> % INT(S,a,b) is the definite integral of S with respect to
   % its symbolic variable from a to b.  a and b are each
   % double or symbolic scalars.

>> y1=1+0*t % Otherwise MATLAB is too dumb to realize
            % that y1 is a trivial function of the symbolic
            % variable t
y1 = 1

>> y2=t;
>> y3=t^2;

% Start the G-S Procedure. Here we assume C[0,1], that is
% C[a,b], with [a,b]=[0,1]

>> v1=y1

v1=1

>> v2=y2-int(v1*y2,0,1)*v1/int(v1^2,0,1)

v2=t-1/2

>> v3=y3-int(v1*y3,0,1)*v1/int(v1^2,0,1)-
int(v2*y3,0,1)*v2/int(v2^2,0,1)

v3=t^2+1/6-t

% Next, normalize to length one

v1_tilde=v1/int(v1^2,0,1)^.5
```

```
v1_tilde=1

>> v2_tilde=v2/int(v2^2,0,1)^.5

v2_tilde=(t-1/2)*12^(1/2)

>> simplify(v2_tilde);

ans=(2*t-1)*3^(1/2)

>> v3_tilde=simplyfy(v3/int(v3^2,0,1)^.5)

v3_tilde=(6*t^2+1-6*t)*5^(1/2)
```