$\boxed{1.}$

For BLUE, $\hat{x} = (A^T Q^{-1} A)^{-1} A^T Q^{-1} y = Ky$, $\text{cov}(\hat{x}) = KQK^T$

(a) $\hat{x} = \begin{bmatrix} 0.6194 \\ 0.45915 \end{bmatrix}$, $\text{cov}(\hat{x}) = \begin{bmatrix} 4 & -2.75 \\ -2.75 & 2 \end{bmatrix}$

(b) $\hat{x} = \begin{bmatrix} -1.4303 \\ 1.8791 \end{bmatrix}$, $\text{cov}(\hat{x}) = \begin{bmatrix} 0.06992 & -0.02599 \\ -0.02599 & 0.11289 \end{bmatrix}$

(c) $\hat{x} = \begin{bmatrix} -1.2201 \\ 1.5368 \end{bmatrix}$, $\text{cov}(\hat{x}) = \begin{bmatrix} 0.048692 & 0.0053562 \\ 0.0053562 & 0.061839 \end{bmatrix}$

$\boxed{2.}$

jointly normal random variables (X, Y, Z) with

(a) mean $\mu = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, covariance $\Sigma = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} X \\ Y \end{bmatrix}$, $B = [Z]$. Then

$$\mu_A = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \mu_B = 1, \quad \Sigma_{AA} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}, \quad \Sigma_{BB} = 2$$

$$\Sigma_{AB} = \Sigma_{BA}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mu_{A|B} = \mu_A + \Sigma_{AB} \Sigma_{BB}^{-1} (B - \mu_B)$$

$$= \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} 1/2 \, (z - 1)$$

$$= \begin{bmatrix} -1 + 1/2\, z - 1/2 \\ z - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2\, z - 3/2 \\ z - 1 \end{bmatrix}$$

$$\Sigma_{A|B} = \Sigma_{AA} - \Sigma_{AB} \Sigma_{BB}^{-1} \Sigma_{BA}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} 1/2 \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 & 1 \\ 1 & 2 \end{bmatrix}$$

(b)

Let $A = [X]\Big|_{Z=z}$, $B = [Y]\Big|_{Z=z}$

Using results in (a)

$\mu_A = \frac{1}{2}z - \frac{3}{2}$, $\mu_B = z-1$

$\Sigma_{AA} = 3/2$, $\Sigma_{AB} = \Sigma_{BA}^T = 1$, $\Sigma_{BB} = 2$

$$\mu_{A|B} = \mu_A + \Sigma_{AB}\Sigma_{BB}^{-1}(B - \mu_B)$$

$$= (\tfrac{1}{2}z - \tfrac{3}{2}) + 1 \cdot \tfrac{1}{2} \cdot (y - z + 1)$$

$$= \tfrac{1}{2}z - \tfrac{3}{2} + \tfrac{1}{2}y - \tfrac{1}{2}z + \tfrac{1}{2}$$

$$= \tfrac{1}{2}y - 1$$

$$\Sigma_{A|B} = \Sigma_{AA} - \Sigma_{AB}\Sigma_{BB}^{-1}\Sigma_{BA}$$

$$= \tfrac{3}{2} - 1 \cdot \tfrac{1}{2} \cdot 1$$

$$= 1$$

(c)

Let $A = [X]$, $B = \begin{bmatrix} Y \\ Z \end{bmatrix}$

$$\mu_A = -1, \quad \mu_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Sigma_{AA} = 2, \quad \Sigma_{AB} = \Sigma_{BA}^T = [2 \ 1], \quad \Sigma_{BB} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\Sigma_{BB}^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

$$\mu_{A|B} = \mu_A + \Sigma_{AB} \Sigma_{BB}^{-1} (B - \mu_B)$$

$$= -1 + [2 \ 1] \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} y-0 \\ z-1 \end{bmatrix}$$

$$= -1 + [1/2 \ 0] \begin{bmatrix} y-0 \\ z-1 \end{bmatrix}$$

$$= -1 + 1/2\, y$$

$$\Sigma_{A|B} = \Sigma_{AA} - \Sigma_{AB} \Sigma_{BB}^{-1} \Sigma_{BA}$$

$$= 2 - [2 \ 1] \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= 2 - [1/2 \ 0] \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= 1$$

(d)

Results in (b) and (c) are the same.

$\boxed{3.}$ (a)

For Gram Matrix $[G]_{ij} = \langle y_i, y_j \rangle$

$$G_{k+1} = \begin{bmatrix} G_k & 0_{k\times 1} \\ 0_{1\times k} & \langle y_{k+1}, y_{k+1} \rangle \end{bmatrix} \quad (\because\ y_{k+1} \perp M_k \Rightarrow \forall\ 1 \le i \le k,\ y_{k+1} \perp y_i.)$$

$$G^T \alpha = \beta = G\alpha$$

$$\alpha = G^{-1}\beta$$

$$\alpha_k = G_k^{-1}\beta_k = G_k^{-1} \begin{bmatrix} \langle x, y_1 \rangle \\ \vdots \\ \langle x, y_k \rangle \end{bmatrix}$$

$$\alpha_{k+1} = G_{k+1}^{-1}\beta_{k+1} = G_{k+1}^{-1} \begin{bmatrix} \langle x, y_1 \rangle \\ \vdots \\ \langle x, y_k \rangle \\ \langle x, y_{k+1} \rangle \end{bmatrix}$$

$$G_{k+1}^{-1} = \begin{bmatrix} G_k & 0_{k\times 1} \\ 0_{1\times k} & \langle y_{k+1}, y_{k+1} \rangle \end{bmatrix}^{-1} = 1/\langle y_{k+1}, y_{k+1} \rangle \cdot \begin{bmatrix} G_k/\langle y_{k+1}, y_{k+1} \rangle & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= 1/\langle y_{k+1}, y_{k+1} \rangle \cdot \begin{bmatrix} G_k^{-1} \cdot \langle y_{k+1}, y_{k+1} \rangle & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} G_k^{-1} & 0_{k\times 1} \\ 0_{1\times k} & 1/\langle y_{k+1}, y_{k+1} \rangle \end{bmatrix}$$

$$\alpha_{k+1} = G_{k+1}^{-1}\beta_{k+1} = \begin{bmatrix} G_k^{-1} & 0_{k\times 1} \\ 0_{1\times k} & 1/\langle y_{k+1}, y_{k+1} \rangle \end{bmatrix} \begin{bmatrix} \langle x, y_1 \rangle \\ \vdots \\ \langle x, y_{k+1} \rangle \end{bmatrix}$$

$$= \begin{bmatrix} | \\ \alpha_k \\ | \\ \dfrac{\langle x, y_{k+1} \rangle}{\langle y_{k+1}, y_{k+1} \rangle} \end{bmatrix}$$

$$\Rightarrow \hat{x}_{k+1} = \hat{x}_k + \frac{\langle x, y_{k+1} \rangle}{\langle y_{k+1}, y_{k+1} \rangle} \cdot y_{k+1}$$

$$\Rightarrow \beta = \frac{\langle x, y_{k+1} \rangle}{\langle y_{k+1}, y_{k+1} \rangle} \quad ※$$

(b)

From projection theorem, $y_{k+1} - \hat{y}_{k+1|k}$ is orthogonal to M_k.

$$M_{k+1} = M_k \oplus \text{span}\{y_{k+1}\} = M_k \oplus \text{span}\{y_{k+1} - \hat{y}_{k+1|k}\}.$$

$$(\because \hat{y}_{k+1|k} \in M_k)$$

We can use the result in (a)

$$\Rightarrow \beta = \frac{\langle x, \Delta y \rangle}{\langle \Delta y, \Delta y \rangle}, \text{ where } \Delta y = y_{k+1} - \hat{y}_{k+1|k}$$

$\boxed{4.}$

For MVE, $\hat{x} = (A^T Q^{-1} A + P^{-1})^{-1} A^T Q^{-1} y$

$$\text{cov}(\hat{x}) = P - PA^T(APA^T + Q)^{-1} AP$$

(a)

$$\hat{x} = \begin{bmatrix} 0.3417 \\ 0.4271 \end{bmatrix}, \quad \text{Cov}(\hat{x}) = \begin{bmatrix} 0.2778 & -0.0278 \\ -0.0278 & 0.1528 \end{bmatrix}$$

(b) $$\hat{x} = \begin{bmatrix} 0.4503 \\ 0.4963 \end{bmatrix}, \quad \text{cov}(\hat{x}) = \begin{bmatrix} 0.1938 & -0.0813 \\ -0.0813 & 0.1188 \end{bmatrix}$$

(c) $$\hat{x} = \begin{bmatrix} -1.0134 \\ 1.2402 \end{bmatrix}, \quad \text{Cov}(\hat{x}) = \begin{bmatrix} 0.0545 & -0.0105 \\ -0.0105 & 0.0828 \end{bmatrix}$$

(d) $$\hat{x} = \begin{bmatrix} -1.0296 \\ 1.2667 \end{bmatrix}, \quad \text{Cov}(\hat{x}) = \begin{bmatrix} 0.0437 & 0.0072 \\ 0.0072 & 0.0538 \end{bmatrix}$$

5.

For Least squares: $\hat{x} = (A^T A)^{-1} A^T y$

For BLUE: $\hat{x} = (A^T Q^{-1} A)^{-1} A^T Q^{-1} y$

For MVE: $\hat{x} = (A^T Q^{-1} A + P^{-1})^{-1} A^T Q^{-1} y$

(a) $\hat{x} = \begin{bmatrix} -1.3169 \\ 1.4368 \end{bmatrix}$ (LS)

(b) $\hat{x} = \begin{bmatrix} -1.3169 \\ 1.4368 \end{bmatrix}$ (BLUE, assuming $Q = I$)

(c) $\hat{x} = \begin{bmatrix} -1.3163 \\ 1.4365 \end{bmatrix}$ (MVE, $Q = I$, $P = 100 I$)

$\hat{x} = \begin{bmatrix} -1.3168 \\ 1.4368 \end{bmatrix}$ (MVE, $Q = I$, $P = 10^6 I$)

(d)
- result from BLUE is the same as that from LS when $Q = I$
- result from MVE reduces to both BLUE and LS when P is a infinity identity matrix.

$\boxed{6.}$

$$\bar{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \bar{y} = C\bar{x} + \bar{\xi} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 5 \\ -6 \end{bmatrix}$$

$$\hat{x} = \bar{x} + PC^T(CPC^T + Q)^{-1}(y - \bar{y})$$

$$= \begin{bmatrix} -0.8836 \\ 1.0802 \end{bmatrix} \quad ※$$

I discussed with Wan-Ti Yu