$\boxed{1.}$

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- In this question, we use W to denote the subset.

(a)

Not a subspace. ∵ Not closed under multiplication by a constant, such as -1

(b)

Is a subspace

Given two arbitrary elements $u, v \in W$. $u = \begin{bmatrix} 0 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

$$au + bv = \begin{bmatrix} 0 \\ au_2 + bv_2 \\ \vdots \\ au_n + bv_n \end{bmatrix} \in W, \text{ for all } a, b \in \mathbb{R}$$

(c)

Not a subspace. ∵ Not closed under vector addition

$$\text{ex: } \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \notin W \quad (x_1 x_2 = 1 \cdot 1 = 1 \neq 0)$$

(d)

Is a subspace

Given two arbitrary elements $u, v \in W$. $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, where $\sum_{i=1}^{n} u_i = 0$, $\sum_{i=1}^{n} v_i = 0$

$$au + bv = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ \vdots \\ u_n + v_n \end{bmatrix} \in W, \text{ for all } a, b \in \mathbb{R}$$

$$\left(\sum_{i=1}^{n} (u_i + v_i) = \sum_{i=1}^{n} u_i + \sum_{i=1}^{n} v_i = 0 \right)$$

(e)

Not a subspace ∵ Not closed under vector addition

$$\text{ex: } \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin W \quad (x_1 + \cdots + x_n = 2 \neq 1)$$

(f)

Not a subspace ∵ Not closed under vector addition

ex: $Au = b$, $Av = b$

$\Rightarrow A(u+v) = 2b \neq b$

$\Rightarrow (u+v) \notin W$

$\boxed{2.}$

To prove this, we must show that

$$\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2) \quad \text{and} \quad \text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$$

For the first inclusion, let $v \in \text{span}(S_1 \cup S_2)$. This means that there exist $v_1, \dots v_n \in S_1 \cup S_2$ and $a_1, \dots a_n \in \mathbb{F}$ such that

$$v = a_1 v_1 + \cdots + a_n v_n$$

For each i, we have $v_i \in S_1$ or $v_i \in S_2$. By relabeling the indices, we can assume for some $1 \le k \le n$ that $v_1, \dots v_k \in S_1$ and $v_{k+1} \cdots v_n \in S_2$. Then

$$w_1 = a_1 v_1 + \cdots + a_k v_k \in \text{span}(S_1) \quad \text{and} \quad w_2 = a_{k+1} v_{k+1} + \cdots + a_n v_2 \in \text{span}(S_2)$$

so $v = w_1 + w_2 \in \text{span}(S_1) + \text{span}(S_2) \Rightarrow \text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$

For the second inclusion, let $v \in \text{span}(S_1) + \text{span}(S_2)$, so then $v = w_1 + w_2$ for $w_1 \in \text{span}(S_1)$, and $w_2 \in \text{span}(S_2)$. Then there exist $x_1, \dots x_n \in S_1$, $a_1, \dots a_n \in \mathbb{F}$ such that

$$w_1 = a_1 x_1 + \cdots + a_n x_n$$

and similarly, there exist $y_1 \cdots y_m \in S_2$ and $b_1, \dots b_m \in \mathbb{F}$ such that

$$w_2 = b_1 y_1 + \cdots + b_m y_m$$

This gives us

$$v = w_1 + w_2 = a_1 x_1 + \cdots + a_n x_n + b_1 y_1 + \cdots + b_m y_m$$

which is a linear combination of $x_1 \cdots x_n, y_1 \cdots y_m \in S_1 \cup S_2$. $\Rightarrow \text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 + S_2)$

$\therefore$ $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$ and $\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$

$\therefore$ $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$

3.

A finite set of vectors $x_1, \dots, x_k \in \mathcal{X}$ is **linearly dependent** if there exist scalars $\alpha_1, \dots, \alpha_k \in \mathcal{F}$, NOT ALL ZERO, such that $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_k x_k = 0$.

zero vector

(a)

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} = 0$$

$$\begin{bmatrix} \alpha_1 + 2\alpha_2 + \alpha_3 \\ 2\alpha_1 + \alpha_2 + 5\alpha_3 \\ 3\alpha_1 + 9\alpha_3 \end{bmatrix} = 0$$

There exist $\alpha_1 = -3,\ \alpha_2 = 1\ \ \alpha_3 = 1$ that satisfies the condition

$\Rightarrow$ linearly dependent

ex: $\begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(b)

dimension $= 3$, number of elements in the set $= 4$ $\Rightarrow$ linearly dependent

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} + \frac{1}{8} \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

(c)

$$\alpha_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

$$A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

$$\det(A) = 3 \cdot 0 - 1 \cdot 1 + 2 \cdot 0 = -1 \neq 0$$

$\Rightarrow$ linearly independent

$\boxed{4.}$

$$\left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \right\}$$

$$\alpha_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} \alpha_1 + 2\alpha_2 + 4\alpha_3 & 2\alpha_1 + \alpha_2 - \alpha_3 \\ 2\alpha_1 + \alpha_2 - \alpha_3 & \alpha_1 + \alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ 2\alpha_1 + \alpha_2 - \alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A\alpha = 0$$

$$\det(A) = 1 \cdot (2) - 2 \cdot 3 + 4 \cdot 1 = 0$$

$\Rightarrow$ Linearly dependent

$\boxed{5.}$

$S = \{u_1, u_2 \cdots u_n\}$, a finite set in a vector space X over $\mathbb{F}$

$Span(S) = \{u \in X : u = c_1 u_1 + \cdots + c_n u_n,\ c_1 \cdots c_n \in \mathbb{F},\ u_1, \dots u_n \in S\}$

- First, we need to prove if $S_1 \subset S_2$, then $Span(S_1) \subset Span(S_2)$.

  Let $v \in Span(S_1)$. Then there exist $v_1, \cdots v_n \in S_1$ and $a_1, \cdots a_n \in \mathbb{F}$ such that

  $$V = a_1 v_1 + \cdots + a_n v_n$$

  Since $S_1 \subset S_2$, we have $V_1, \dots V_n \in S_2$, so v is a linear combination of vectors in S_2. Hence $v \in Span(S_2) \Rightarrow Span(S_1) \subset Span(S_2)$. $\square$

- Second, we also know that a span of a subspace is the subspace itself.

If S is a subspace, S is a closed set under linear combination of elements in S. Since $span(S)$ is also the set of all linear combination of elements in S. We derive $span(S) = S$ if S is a subspace

- Since we know $S \subset Y \Rightarrow span(S) \subset span(Y)$

  Also we know Y is a subspace, therefore $span(Y) = Y$

  $$\Rightarrow Span\{S\} \subset Span\{Y\} = Y$$

  $$\Rightarrow span\{S\} \subset Y.$$

  $\square$

$\boxed{6.}$

(a) $\Rightarrow$ (b) :

let $Y = V + W$, for every $y \in Y$ there exist $v \in V$ and $w \in W$ such that $y = v + w$. Suppose there exists other vectors $\tilde{v} \in V$ and $\tilde{w} \in W$ such that $y = \tilde{v} + \tilde{w}$

Then, $0 = (v - \tilde{v}) + (w - \tilde{w}) \Leftrightarrow (v - \tilde{v}) = -(w - \tilde{w})$

Therefore $(v - \tilde{v}) \in W$ and so $(v - \tilde{v}) \in V \cap W$. Since $V \cap W = \{0\}$, we then conclude that $v = \tilde{v}$, which also says $w = \tilde{w}$. Then we obtain that for every $x \in V + W$, there exist unique $v \in V$ and $w \in W$ such that $x = v + w$

(b) $\Rightarrow$ (a)

If for every $y \in V + W$, there exist unique $v \in V$ and $w \in W$ such that $y = v + w$. Suppose that $y \in V \cap W$, then on the one hand, there exist $v \in V$ such that $y = v + 0$; on the other hand, there is $w \in W$ such that $y = 0 + w$. $v = 0$ and $w = 0$, so $V \cap W = \{0\}$

After proving that (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a), We prove that (a) and (b) are equivalent. $\square$

- I discussed with Wan-Yi Yu #94732586. In this assignment