# 13 Sept 2018

**Review:** $\mathcal{F}$ = Field $\longleftrightarrow$ 7 Axioms

Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$

Non-examples: $\mathbb{Z}$ (integers) (No multiplicative inverses) $\mathbb{I}$ (irrational numbers) (No zero = 0, No one = unit = 1)

$(V, \mathcal{F})$ or $(\mathcal{X}, \mathcal{F})$ Vector Space $\longleftrightarrow$ 10 Axioms

Examples: $(\mathbb{R}^2, \mathbb{R})$, $(\mathcal{F}^n, \mathcal{F})$, $(\mathcal{F}^{n \times m}, \mathcal{F})$

$(\mathcal{X}, \mathbb{R})$ where $\mathcal{X} = \{f : D \longrightarrow \mathbb{R}\}$ functions

$(\mathbb{C}, \mathbb{R})$ but $(\mathbb{R}, \mathbb{C})$ is not a vector space.

## Subspaces

Def. Let $(\mathcal{X}, \mathcal{F})$ be a vector space and $\mathcal{Y} \subset \mathcal{X}$. Then $\mathcal{Y}$ is a subspace of $\mathcal{X}$ [or of $(\mathcal{X}, \mathcal{F})$] if $(\mathcal{Y}, \mathcal{F})$ is a vector space when using the rules of vector addition and scalar times vector multiplication coming from $(\mathcal{X}, \mathcal{F})$.

**Remarks:** (a) In principle, must check all 10 axioms!

(b) 0 must be an element of $\mathcal{Y}$.

# TODAY

<u>Proposition</u> (TFAE) The Following Are Equivalent for a vector space $(\mathcal{X}, \mathcal{F})$ and $\mathcal{Y} \subset \mathcal{X}$ a subset:

a) $\mathcal{Y}$ is a subspace of $\mathcal{X}$

b) $\forall v^1, v^2 \in \mathcal{Y}$, $v^1 + v^2 \in \mathcal{Y}$ and $\forall \alpha \in \mathcal{F}$, $\forall v \in \mathcal{Y}$, $\alpha \cdot v \in \mathcal{Y}$.

c) $\forall \alpha \in \mathcal{F}$ and $\forall v^1, v^2 \in \mathcal{Y}$, $\alpha v^1 + v^2 \in \mathcal{Y}$

d) $\forall \alpha_1, \alpha_2 \in \mathcal{F}$, and $\forall v^1, v^2 \in \mathcal{Y}$,

$$\alpha_1 v^1 + \alpha_2 v^2 \in \mathcal{Y}.$$

$\square$

<u>Examples</u>

1) $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^2, \mathbb{R})$, $\mathcal{Y} = \left\{ \begin{bmatrix} \beta \\ 2\beta \end{bmatrix} \middle| \beta \in \mathbb{R} \right\}$.

$$\underbrace{\begin{bmatrix}\beta_1\\2\beta_1\end{bmatrix}}_{v^1} + \underbrace{\begin{bmatrix}\beta_2\\2\beta_2\end{bmatrix}}_{v^2} = \begin{bmatrix}\beta_1+\beta_2\\2(\beta_1+\beta_2)\end{bmatrix} \in \mathcal{Y}$$

$$\alpha \in \mathbb{R}, \quad \alpha\underbrace{\begin{bmatrix}\beta\\2\beta\end{bmatrix}}_{v} = \begin{bmatrix}\alpha\beta\\2(\alpha\beta)\end{bmatrix} \in \mathcal{Y}$$

$\therefore$ Subspace.

2) $\mathcal{F} = \mathbb{R}$, $\mathcal{X} = \{f : \mathbb{R} \to \mathbb{R}\}$

$\mathcal{Y} := P(t) := \{$polynomials in $t$ with real coefficients$\}$. Is a subspace by checking the Proposition.

$\tilde{\mathcal{Y}} := \{f : \mathbb{R} \to \mathbb{R},\ f$ differentiable and $\frac{d}{dt} f \equiv 0\}. \subset \mathcal{X}$

Also subspace.

**Non-example** $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^2, \mathbb{R})$

$$\mathcal{Y} = \left\{ \begin{bmatrix}\beta\\2\beta\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} \;\middle|\; \beta \in \mathbb{R} \right\}$$

$0 \notin \mathcal{Y}$. Not a subspace.

$0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in this case.

# Linear Combinations and Linear Independence

Let $(\mathcal{X}, \mathcal{F})$ be a vector space.

Def. A linear combination is any $\boxed{\text{finite}}$ sum of the form

$$\alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_n v^n$$

where $n \geq 1$, $\alpha_i \in \mathcal{F}$, $v^i \in \mathcal{X}$, $1 \leq i \leq n$. $\square$

Note $\sum_{i=1}^{\infty} \alpha_i v^i$ is not a linear combination.

Why? $\sum_{i=1}^{\infty} \alpha_i v^i \triangleq \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i v^i$

Without additional structure, we cannot define what it means to converge to something; i.e., has a limit. $\square$

**Mindset** Let $A$ be an $n\times n$ real matrix. Let $x\in\mathbb{R}^n$.

**Fact:** $A$ is invertible $\Leftrightarrow$ the only solution to $Ax=0$ is $x=0$. $\square$

$$A=[A_1|A_2|\cdots|A_n], \quad x=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$$

**Exercise:** $Ax = x_1A_1 + x_2A_2 + \cdots + x_nA_n$

**Def.** A finite set of vectors $\{v^1,\dots,v^k\}\subset \mathcal{X}$ is linearly dependent if there exist $\alpha_1,\dots,\alpha_k\in\mathcal{F}$ NOT ALL ZERO such that

$$\alpha_1v^1+\alpha_2v^2+\cdots+\alpha_kv^k=0.$$

Otherwise $\{v^1,\dots,v^k\}$ is linearly independent

[The only solution to
$$\alpha_1v^1+\alpha_2v^2+\cdots+\alpha_kv^k=0$$
is the trivial solution,
$$\alpha_1=\alpha_2=\cdots=\alpha_k=0$$
] $\square$

**Def.** An arbitrary set of vectors is linearly independent if every finite subset of vectors is linearly independent.

**Examples** $(\mathcal{V}, \mathcal{F}) = (\mathbb{R}^2, \mathbb{R})$

$v^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v^2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. Is the set $\{v^1, v^2\}$ linearly independent?

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be arbitrary.

$$\alpha_1 v^1 + \alpha_2 v^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} \alpha_1 + 4\alpha_2 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\alpha_1 = \alpha_2 = 0$ is the only solution



$\therefore$ Linearly independent.

$\mathcal{X} = \mathbb{R}^{2\times 3}$ $\quad \mathcal{F} = \mathbb{R}$

$$A_1 = \begin{bmatrix} 1 & 0 & 4 \\ 3 & -1 & 2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 4 & 1 & 0 \\ 6 & 0 & 6 \end{bmatrix}$$

Independent?

$$\alpha_1 A_1 + \alpha_2 A_2 \stackrel{?}{=} 0$$

$$\alpha_1 A_1 + \alpha_2 A_2 = \begin{bmatrix} \alpha_1 + 4\alpha_2 & \alpha_2 & 4\alpha_1 \\ 3\alpha_1 + 6\alpha_2 & -\alpha_1 & 2\alpha_1 + 6\alpha_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \iff \begin{aligned} \alpha_1 + 4\alpha_2 &= 0 \\ \alpha_2 &= 0 \\ 4\alpha_1 &= 0 \\ 3\alpha_1 + 6\alpha_2 &= 0 \\ -\alpha_1 &= 0 \\ 2\alpha_1 + 6\alpha_2 &= 0 \end{aligned}$$

$$\iff \alpha_1 = \alpha_2 = 0$$

$\therefore$ Linearly independent.

<u>Example</u> $(\mathbb{P}(t), \mathbb{R}) = (\mathcal{X}, \mathcal{F})$.

$$S = \{1, t, t^2, t^3, \ldots \} =$$

all monomials.

<u>Claim</u> $S$ is linearly independent.

<u>Pf.</u> (Lazy Induction) (Sketch of Proof)

Take an arbitrary, but finite, combination of elements of $S$

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k$$

To show: $p(t) = 0$ vector $\Longleftrightarrow a_0 = a_1 =$
$= \cdots = a_k = 0$.

**Important:** $p(t) = 0$ vector $\Longleftrightarrow \forall t_0 \in \mathbb{R}$, $p(t_0) = 0$.

<u>Observe</u> $p(0) = 0 \Longleftrightarrow a_0 = 0$.

<u>"The trick"</u> $p \equiv 0 \Rightarrow \frac{d}{dt} p \equiv 0$

$$\frac{d}{dt} p(t) = a_1 + 2a_2 t + \cdots + k a_k t^{k-1}$$

$\therefore 0 = \frac{d}{dt} p(t) \implies \dot{p}(0) = 0 \implies a_1 = 0$

Continue by induction. $\square$

Proof: **Correct Proof by Induction:** Let $k \geq 0$, and define the property $\mathcal{P}(k)$ by

$$\mathcal{P}(k) : \text{The set } \{1, t, \ldots, t^k\} \text{ is linearly independent}$$

**Base Case:** $\mathcal{P}(0)$ is true; that is, the set $\{1\}$ is linearly independent. (You can work this out at home).

**Induction Step:** For $k \geq 0$, we assume that $\mathcal{P}(k)$ is true and we must show that $\mathcal{P}(k+1)$ is true, that is,

$$\{1, t, \ldots, t^{k+1}\} \text{ is linearly independent}$$

Assume $p_{k+1}(t) := \alpha_0 + \alpha_1 t + \cdots + \alpha_{k+1} t^{k+1} = 0$, the zero polynomial, and hence, is zero for all t. Then,

$$0 = \frac{d^{k+1} p_{k+1}}{dt^{k+1}}|_{t=0} = (k+1)!\ a_{k+1}$$

and hence $a_{k+1} = 0$. It follows that

$$p_{k+1}(t) := \alpha_0 + \alpha_1 t + \cdots + \alpha_k t^k = 0.$$

By the induction step, this implies that

$$a_0 = 0, a_1 = 0, \ldots, a_k = 0,$$

and thus we are done.

$\square$

<u>Def.</u> Let $S \subset \mathcal{X}$, $(\mathcal{X}, \mathcal{F})$ a vector space. The **span of S** denoted

$\text{span}\{S\}$ or $\text{span}\, S$

is the set of all possible linear combinations of $S$. $\square$

$$\text{span}\{S\} = \{x \in \mathcal{X} \mid \exists\, k \geq 1,\ \alpha_1, \dots, \alpha_k \in \mathcal{F},\ v^1, \dots, v^k \in S \text{ such that } x = \alpha_1 v^1 + \cdots + \alpha_k v^k\}$$

Span$\{S\}$ is always a subspace of $\mathcal{X}$. $\{$closed under vector addition and scalar times vector multiplication$\}$.

Examples

$\mathcal{F} = \mathbb{R}$, $\mathcal{X} = \{f : \mathbb{R} \to \mathbb{R}\}$,

$S = \{1, t, t^2, \dots\}$.

$\text{span}\{S\} = \mathbb{P}(t)$

Question: Is $e^t \in \text{span}\{S\}$?

**No** only linear combinations (finite sums).

$\mathcal{F} = \mathbb{R}^{2\times 4}$, $A_1 = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 0 \end{bmatrix}$

$A_2 = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 1 & 1 & 1 & 5 \end{bmatrix}$

$\operatorname{span}\{A_1, A_2\} =$ exercise!

$\boxtimes$.