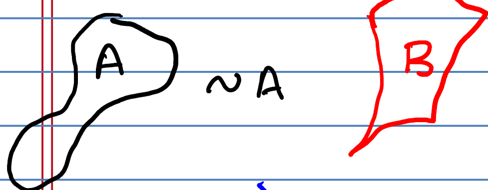
20 Nov. 2018

Review: $(X, \|\cdot\|)$ a normed space. We assume $\mathbb{F} = \mathbb{R}$

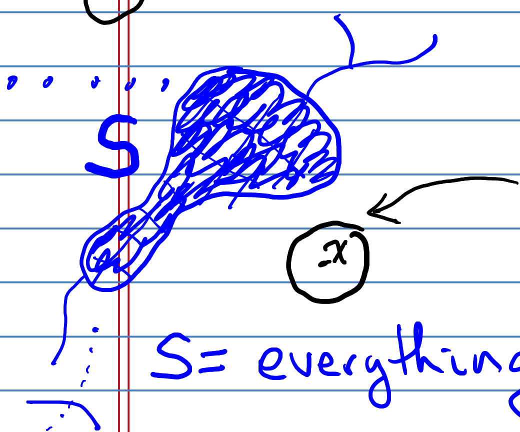
$d(x,y) := \|x-y\|$. $S \subset X$ subset, $d(x,S) := \inf_{y \in S} \|x-y\|$

Def. (Open Ball) $B_a(x_0) := \{x \in X \mid \|x - x_0\| < a\}$

Set complement of $A \subset X$ is $\sim A := \{x \in X \mid x \notin A\}$



$B \subset \sim A \Leftrightarrow B \cap A = \emptyset$



$B_\varepsilon(x) \cap S = \emptyset \Leftrightarrow B_\varepsilon(x) \subset \sim S$

S = everything in blue.

Important "Observations" $x \in X$, $S \subset X$

(a) $d(x,S) = 0 \Leftrightarrow \forall \varepsilon > 0, \exists y \in S, \|x-y\| < \varepsilon$

$\Leftrightarrow \forall \varepsilon > 0, B_\varepsilon(x) \cap S \neq \emptyset$

($\neq \emptyset \Leftrightarrow \exists y$ s.t. $y \in B_\varepsilon(x)$ and $y \in S$)

(b) $d(x,S) > 0 \Leftrightarrow \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap S = \emptyset$

$\Leftrightarrow \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset \sim S$

Today Open sets and Closed sets

<u>Def.</u> $P \subset X$ a subset.

(a) $p \in P$ is <u>an interior point of P</u> if $\exists\, \varepsilon > 0$, $B_\varepsilon(p) \subset P$.

(b) $\mathring{P} = \{p \in P \mid p \text{ is an interior point}\}$ is the <u>interior</u> of P

<u>Remark</u>

$$\mathring{P} = \{p \in P \mid \exists\, \varepsilon > 0,\ B_\varepsilon(p) \subset P\}$$
$$= \{p \in P \mid d(p, \sim P) > 0\}$$
$$= \{x \in X \mid d(x, \sim P) > 0\}$$

because, if $x \in \sim P$, then $d(x, \sim P) = 0$.

$$\boxed{\mathring{P} = \{x \in X \mid d(x, \sim P) > 0\}}$$

<u>Def.</u> P is open if $\mathring{P} = P$.

$\therefore$ P is open $\Leftrightarrow$ $P = \{x \in X \mid d(x, \sim P) > 0\}$

Examples ($X = \mathbb{R}$, $\|\cdot\| = |\cdot|$)

- Is $P = (0,1)$ open?

$x \in P \iff 0 < x < 1$

Let $\varepsilon = \min\{\frac{x}{2}, \frac{1-x}{2}\} > 0$

$\therefore B_\varepsilon(x) \subset P \implies x \in \mathring{P}$

$\therefore$ P is open.

- $\sim P = (-\infty, 0] \cup [1, \infty)$

$x \in P \iff 0 < x < 1$

$d(x, (-\infty, 0]) = x > 0$

$d(x, [1, \infty)) = 1 - x > 0$

$\therefore d(x, \sim P) = \min\{x, 1-x\} > 0$

$x \in P \iff d(x, \sim P) > 0$ $\therefore$ P is open.

Def. $P \subset X$ a subset.

(a) x is a closure point of P if $\forall \varepsilon > 0$, $\exists y \in P$, $\|x - y\| < \varepsilon$. (i.e., $d(x, P) = 0$)

(b) The closure of P, denoted $\overline{P}$, is

$$\overline{P} := \{x \in X \mid x \text{ is a closure point of } P\}$$
$$= \{x \in X \mid d(x, P) = 0\}.$$

Def. P is closed if $\overline{P} = P$.

Examples $(X = \mathbb{R}, \ \|\cdot\| = |\cdot|)$

- Check that $P = [0,1)$ is not closed.

Because $1 \notin P$, but $d(1, P) = 0$.

- $\overline{P} = [0,1]$

- $P = \mathbb{Q}$ = rational numbers.

$x = \sqrt{2} \notin P$, but $d(\sqrt{2}, P) = 0$

Fact: $\overline{P} = \mathbb{R}$

Theorem Let $(X, \|\cdot\|)$ be a normed space, and $P \subset X$ a subset. Then P is open if, and only if, $\sim P$ is closed.

- P open $\Leftrightarrow$ $\sim P$ closed
- P closed $\Leftrightarrow$ $\sim P$ open

Proof:

P open $\iff \mathring{P} = P$

$\iff P = \{ x \in X \mid d(x, \sim P) > 0 \}$

$\iff P = \{ x \in X \mid \exists\, \varepsilon > 0,\ B_\varepsilon(x) \cap \sim P = \phi \}$

$\iff P = \sim \{ x \in X \mid \forall\, \varepsilon > 0,\ B_\varepsilon(x) \cap \sim P \neq \phi \}$

P open $\iff \sim P = \{ x \in X \mid \forall\, \varepsilon > 0,\ B_\varepsilon(x) \cap \sim P \neq \phi \}$

$= \overline{\sim P}$

P open $\iff \sim P$ closed. $\square$

Can a set be both closed and open? Yes, such sets are (sometimes) called CLOPEN.

X = closed and open. as in $(X, \|\cdot\|)$

By convention ϕ is both closed and open.

X = entire normed space.

Exercises • Arbitrary unions of open sets are open

• Finite intersections of open sets are open.

• Arbitrary intersections of closed sets are closed.

• Finite unions of closed sets are closed.

• Example of a countably infinite intersection of open sets that is NOT open

$\forall n \geq 1$, define $a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$

$\therefore [-1, 1] \subset (-a_n, a_n) \quad \forall n \geq 1.$

We note that if $|x| > 1$, then $\exists k < \infty$ s.t. $\frac{k+1}{k} < |x| \Rightarrow x \notin (-a_k, a_k)$

$$\bigcap_{n=1}^{\infty} \underbrace{(-a_n, a_n)}_{\text{open}} = \underbrace{[-1, 1]}_{\text{not open}}$$

infinite # intersections

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$$

Def. **Boundary of a set** $\partial P = \overline{P} \cap \overline{(\sim P)}$

Exercise $\partial P = \overline{P} \setminus \mathring{P}$

$$= \{x \in \overline{P} \mid x \notin \mathring{P}\}$$

$$\partial X = \phi$$

$P = B_a(0) \quad a > 0$

$\partial P = \{x \in X \mid \|x\| = a\}$

$\overline{P} = \overline{B}_a(0)$

cannot just use $\partial P = \overline{P} \cap \sim P$

from a question in class.

Sequences

Example Find the solution to the coupled NL equations

$$0 = h(x) = \begin{pmatrix} x_1 + 2\,x_2 - x_1\,(x_1 + 4\,x_2) - x_2\,(4\,x_1 + 10\,x_2) + 3 \\ 3\,x_1 + 4\,x_2 - x_1\,(x_1 + 4\,x_2) - x_2\,(4\,x_1 + 10\,x_2) + 4 \\ \sin(x_3)^7 + \frac{\cos(x_1)}{2} \\ {x_4}^3 - 2\,{x_2}^2\,\sin(x_1) \end{pmatrix}$$

Initial Guess: $x_0 = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 10 \end{bmatrix}$ $\quad x_{k+1} = x_k - \left[\frac{\partial h}{\partial x}(x_k)\right]^{-1} h(x_k)$

We do 16 iterations of Newton's Algorithm (a nonlinear root finding algorithm) and we obtain:

$$x^* = \begin{pmatrix} -2.25957308738366677539068499960 \\ 1.75957308738366677539068499960 \\ 189.509541006133339783305493 12824 \\ -1.68458069860197189523093013800 \end{pmatrix}$$

And the error is:

$$F(x^*) = \begin{bmatrix} 3.6734198 \times 10^{-39} \\ 2.9387359 \times 10^{-39} \\ 1.2765134 \times 10^{-38} \\ -2.5915832 \times 10^{-32} \end{bmatrix}$$