

# Not on Final Exam

## CONTINUITY OF CONVEX FUNCTIONS IN NORMED SPACES

In this chapter, we consider continuity properties of real-valued convex functions defined on open convex sets in normed spaces. Recall that every infinite-dimensional normed space contains a discontinuous linear functional. Thus, in infinite-dimensional spaces, there exist discontinuous convex functions. One of the corollaries of the results of this chapter is that, in finite-dimensional spaces, this cannot happen.

Recall that  $B(x, r)$  and  $B^0(x, r)$  denote the open and closed ball of radius  $r$ , centered at  $x$ . A function  $f: E \rightarrow \mathbb{R}$  is  $L$ -Lipschitz if  $|f(x) - f(y)| \leq L\|x - y\|$  whenever  $x, y \in E$ .

**Proposition 0.1.** *Let  $X$  be a normed space,  $x_0 \in X$ ,  $r > 0$ ,  $\varepsilon \in (0, r)$ ,  $m, M \in \mathbb{R}$ . Let  $f: B^0(x_0, r) \rightarrow \mathbb{R}$  be a convex function.*

- (a) *If  $f(x) \leq m$  on  $B^0(x_0, r)$ , then  $|f(x)| \leq |m| + 2|f(x_0)|$  on  $B^0(x_0, r)$ .*
- (b) *If  $|f(x)| \leq M$  on  $B^0(x_0, r)$ , then  $f$  is  $(\frac{2M}{\varepsilon})$ -Lipschitz on  $B^0(x_0, r - \varepsilon)$ .*

*Proof.* By translation, we can suppose that  $x_0 = 0$ . Denote  $B = B^0(0, r)$  and  $C = B^0(0, r - \varepsilon)$ .

- (a) Since  $0 = \frac{1}{2}x + \frac{1}{2}(-x)$  ( $x \in B$ ), we have  $f(0) \leq \frac{1}{2}f(x) + \frac{1}{2}f(-x)$ . Consequently,  $f(x) \geq 2f(0) - f(-x) \geq 2f(0) - m$ , and hence

$$|f(x)| \leq \max\{m, m - 2f(0)\} \leq |m| + 2|f(0)| \quad (x \in B).$$

- (b) Consider two distinct points  $x, y \in C$ . The point

$$z = y + \frac{\varepsilon}{\|y - x\|}(y - x)$$

belongs to  $B$  and  $y \in (x, z)$ . An easy calculation shows that

$$y = \frac{\varepsilon}{\varepsilon + \|y - x\|}x + \frac{\|y - x\|}{\varepsilon + \|y - x\|}z \quad (\text{convex combination!}).$$

Use convexity of  $f$  and multiply by the common denominator to get

$$(\varepsilon + \|y - x\|)f(y) \leq \varepsilon f(x) + \|y - x\|f(z).$$

Then  $\varepsilon[f(y) - f(x)] \leq [f(z) - f(y)]\|y - x\| \leq 2M\|y - x\|$ . Thus

$$f(y) - f(x) \leq \frac{2M}{\varepsilon}\|y - x\|.$$

Interchanging the role of  $x$  and  $y$ , we obtain that  $f$  is  $(\frac{2M}{\varepsilon})$ -Lipschitz on  $C$ . □

**Observation 0.2.** *Let  $C$  be a convex set in a normed space  $X$ ,  $f: C \rightarrow \mathbb{R}$  a convex function,  $B := B^0(x_0, r) \subset C$ . Let  $x, y \in C$  be such that  $x = (1 - \lambda)x_0 + \lambda y$  with  $0 < \lambda < 1$ . If  $f \leq m$  on  $B$ , then*

$$f \leq \max\{m, f(y)\} \quad \text{on } \text{conv}[B \cup \{y\}]$$

(in particular, on  $B^0(x, (1 - \lambda)r)$ ).

*Proof.* Exercise. □

**Theorem 0.3.** *Let  $C$  be an open convex set in a normed space  $X$ , and  $f: C \rightarrow \mathbb{R}$  a convex function. The following assertions are equivalent:*

- (i)  $f$  is locally Lipschitz on  $C$ ;
- (ii)  $f$  is continuous on  $C$ ;
- (iii)  $f$  is continuous at some point of  $C$ ;
- (iv)  $f$  is locally bounded on  $C$ ;
- (v)  $f$  is upper bounded on a nonempty open subset of  $C$ .

*Proof.* The implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (v) and (ii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v) are obvious. It remains to show that (v) implies (i).

By (v), there exists an open ball  $B^0(x_0, r) \subset C$  on which  $f$  is upper bounded. Let  $x \in C$ . There exists  $y \in C$  such that  $x \in (x_0, y)$ . By Observation 0.2,  $f$  is upper bounded on some ball  $B^0(x, \varrho)$ . By Proposition 0.1,  $f$  is Lipschitz on  $B^0(x, \frac{\varrho}{2})$ .  $\square$

As an easy corollary, we obtain the following result on automatic continuity of convex functions in finite-dimensional spaces.

**Corollary 0.4.** *Each convex function on an open convex subset of  $\mathbb{R}^d$  is locally Lipschitz (hence continuous).*

*Proof.* Let  $C \subset \mathbb{R}^d$  be open and convex, and  $f: C \rightarrow \mathbb{R}$  a convex function. Fix  $x_0 \in C$ . There exist finitely many points  $c_1, \dots, c_n \in C_0$  such that  $x_0 \in U := \text{int}[\text{conv}\{c_1, \dots, c_n\}]$  (take, e.g., the vertices of a small  $d$ -dimensional cube centered at  $x_0$ ). By convexity,  $f \leq \max\{f(c_1), \dots, f(c_n)\}$  on  $U$ . By Theorem 0.3,  $f$  is locally Lipschitz on  $U$ .  $\square$

**Corollary 0.5.** *Let  $C$  be a finite-dimensional convex set in a normed space  $X$ . Then every convex function  $f: C \rightarrow \mathbb{R}$  is continuous on  $\text{ri}(C)$  (the relative interior of  $C$ ).*

*Proof.* Exercise. (Hint: use Corollary 0.4.)  $\square$

### Continuity of semicontinuous convex functions.

Let  $M$  be a topological space,  $x_0 \in M$ . Recall that a function  $f: M \rightarrow \overline{\mathbb{R}}$  is:

- *lower semicontinuous (l.s.c.)* at  $x_0$  if  $\forall t \in (-\infty, f(x_0)) \exists \delta > 0: f(x) > t$  whenever  $d(x, x_0) < \delta$ .
- *upper semicontinuous (u.s.c.)* at  $x_0$  if  $\forall t \in (f(x_0), +\infty) \exists \delta > 0: f(x) < t$  whenever  $d(x, x_0) < \delta$ .

Clearly,  $f$  is u.s.c. at  $x_0$  if and only if  $-f$  is l.s.c. at  $x_0$ .

**Observation 0.6.** *Let  $M, f$  be as above. Then the following assertions are equivalent:*

- (i)  $f$  is l.s.c.;
- (ii) for each  $t \in \mathbb{R}$ , the set  $\{f > t\}$  is open;
- (iii) for each  $t \in \mathbb{R}$ , the set  $\{f \leq t\}$  is closed;
- (iv) the epigraph  $\text{epi}(f) := \{(x, t) \in M \times \mathbb{R} : f(x) \leq t\}$  is closed (in  $M \times \mathbb{R}$ ).

*Proof.* Exercise.  $\square$

**Proposition 0.7.** *Let  $C$  be an open convex set in a normed space  $X$ ,  $f: C \rightarrow \mathbb{R}$  a convex function.*

- (a) *If  $f$  is u.s.c., then  $f$  is continuous on  $C$ .*

(b) If  $X$  is a Banach space and  $f$  is l.s.c., then  $f$  is continuous on  $C$ .

*Proof.* (a) Fix  $x_0 \in C$  and  $t > f(x_0)$ . Then the set  $\{x \in C : f(x) < t\}$  is a nonempty open subset of  $C$ , on which  $f$  is bounded above. Apply Theorem 0.3.

(b) If  $C = X$ , put  $F_n = \{x \in C : f(x) \leq n\}$ ; otherwise define

$$F_n = \{x \in C : f(x) \leq n, \text{dist}(x, X \setminus C) \geq \frac{1}{n}\}.$$

The sets  $F_n$  ( $n \in \mathbb{N}$ ) are closed in  $C$ ; but they are also closed in  $X$  since  $\overline{F_n} \subset C$ . By the Baire Category Theorem, there exists  $k \in \mathbb{N}$  such that  $F_k$  has a nonempty interior. This implies that  $f$  is upper bounded on a nonempty open set. Apply Theorem 0.3.  $\square$

**Families of convex functions.** Let  $\mathcal{F}$  be a family of functions on a set  $E$ . We say that  $\mathcal{F}$  is *pointwise bounded* if, for each  $x \in E$ , the set  $\mathcal{F}(x) = \{f(x) : f \in \mathcal{F}\}$  is bounded (in  $\mathbb{R}$ ).

**Theorem 0.8.** *Let  $C$  be an open convex set in a Banach space  $X$ . Let  $\mathcal{F}$  be a family of continuous convex functions on  $C$ . If  $\mathcal{F}$  is pointwise bounded, then  $\mathcal{F}$  is locally equi-Lipschitz and locally equi-bounded.*

*Proof.* The (real-valued!) function

$$g(x) = \sup_{f \in \mathcal{F}} f(x)$$

is easily seen to be l.s.c. (e.g., by Observation 0.6). By Proposition 0.7,  $g$  is continuous on  $C$ ; in particular,  $g$  is locally bounded. Hence  $\mathcal{F}$  is locally equi-bounded above on  $C$ . The rest follows from Proposition 0.1.  $\square$

**Theorem 0.9.** *Let  $C$  be an open convex set in a Banach space  $X$ . Let  $\{f_n\}$  be a sequence of continuous convex functions on  $C$  that converges pointwise on  $C$  to a (convex) function  $f : C \rightarrow \mathbb{R}$ . Then  $f$  is continuous and the convergence is uniform on compact sets.*

*Proof.* The sequence  $\{f_n\}$  is pointwise bounded, hence, by the previous theorem, locally equi-bounded and equi-Lipschitz. Consequently,  $f$  is locally bounded and hence continuous (Theorem 0.3). Moreover, on each compact set  $K \subset C$ , the restrictions  $f_n|_K$  ( $n \in \mathbb{N}$ ) are equi-bounded and equi-Lipschitz. An easy application of the Ascoli-Arzelà theorem (*exercise!*) gives that they converge uniformly on  $K$ .  $\square$

Let us recall the so-called *diagonal method*, a standard argument used in many areas of Mathematical Analysis. Given a sequence  $\sigma = (\sigma(1), \sigma(2), \dots)$  of elements of a set  $S$ , and  $m \in \mathbb{N}$ , we denote by  $\sigma|_{[m, \infty)}$  the sequence  $(\sigma(m), \sigma(m+1), \dots)$ .

**Lemma 0.10** (Diagonal method). *Let  $S$  be a set, and  $\sigma_n$  ( $n \in \mathbb{N}$ ) countably many sequences of elements of  $S$  such that  $\sigma_{n+1}$  is a subsequence of  $\sigma_n$  for each  $n \in \mathbb{N}$ . Then there exists a sequence  $\sigma_\infty$  of elements of  $S$ , such that, for each  $n \in \mathbb{N}$ ,*

$$\sigma_\infty|_{[n, \infty)} \text{ is a subsequence of } \sigma_n.$$

*Proof.* The “diagonal” sequence  $\sigma_\infty = (\sigma_1(1), \sigma_2(2), \dots)$  has the desired property.  $\square$

**Theorem 0.11.** *Let  $C$  be an open convex set in a separable Banach space  $X$ . Let  $\{f_n\}$  be a pointwise bounded sequence of continuous convex functions on  $C$ . Then there exists a subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  that converges pointwise and uniformly on compact sets to a continuous convex function on  $C$ .*

*Proof.* Fix a countable dense set  $D \subset C$ . Since  $\{f_n\}$  is pointwise bounded, an easy application of diagonal method produces a subsequence  $\{f_{n_k}\}$  that converges at each point of  $D$ . Let us show that this subsequence converges pointwise on  $C$ . Given  $x \in C$ , there exists an open ball  $U \subset C$ , centered in  $x$ , on which  $\{f_n\}$  is equi-Lipschitz with a certain Lipschitz constant  $L > 0$ . Choose  $d \in U$  such that  $\|d - x\| < \frac{1}{L}$ . Given  $\varepsilon > 0$ , there exists  $k_0 \in \mathbb{N}$  such that  $|f_{n_k}(d) - f_{n_j}(d)| < \varepsilon$  whenever  $k, j \geq k_0$ . Then, for such indices, we have  $|f_{n_k}(x) - f_{n_j}(x)| \leq |f_{n_k}(x) - f_{n_k}(d)| + |f_{n_k}(d) - f_{n_j}(d)| + |f_{n_j}(d) - f_{n_j}(x)| < L\|x - d\| + \varepsilon + L\|x - d\| < 3\varepsilon$ . It follows that  $\{f_{n_k}(x)\}$  is Cauchy and hence convergent. The rest follows from Theorem 0.9.  $\square$

**Direct applications to linear mappings and functionals.** Let us show that some significant results of Functional Analysis are easy consequences of the above theorems for families of convex functions. While usual proofs of the Banach-Steinhaus theorem (Corollary 0.12) go in a similar way as Theorem 0.8, the “subsequence theorem” Corollary 0.14 is usually proved using compactness of  $B_{X^*}$  (the dual unit ball) in the  $w^*$ -topology (Alaoglu’s theorem) and its metrizability.

Let  $X$  be a normed space, and  $x \in X$ . Recall the following well-known consequence of the Hahn-Banach theorem

$$\|x\| = \sup\{x^*(x) : x^* \in X^*, \|x^*\| = 1\}.$$

**Corollary 0.12** (Banach-Steinhaus Uniform Boundedness Principle). *Let  $X$  be a Banach space,  $Y$  a normed space. Let  $\mathcal{T}$  be a family of continuous linear mappings from  $X$  into  $Y$ . Suppose that  $\mathcal{T}$  is pointwise bounded, that is, for each  $x \in X$ , the set  $\{Tx : T \in \mathcal{T}\}$  is bounded in  $Y$ . Then the family  $\mathcal{T}$  is bounded in the normed space  $\mathcal{L}(X, Y)$  of all continuous linear mappings from  $X$  into  $Y$ .*

*Proof.* The family  $\{y^* \circ T : T \in \mathcal{T}, y^* \in Y^*, \|y^*\| = 1\}$  is a pointwise bounded family of continuous linear functionals on  $X$ ; indeed,  $|(y^* \circ T)(x)| \leq \|y^*\| \|Tx\| = \|Tx\|$  ( $x \in X$ ). By Theorem 0.8, there exist  $r > 0$  and  $M \in \mathbb{R}$  such that

$$y^*(Tx) = (y^* \circ T)(x) \leq M \quad (x \in B(0, r), \|y^*\| = 1, T \in \mathcal{T}).$$

Passing to supremum w.r.t.  $y^*$ , we obtain

$$\|Tx\| \leq M \quad (x \in B(0, r), T \in \mathcal{T}),$$

which easily implies that  $\|T\| \leq \frac{M}{r}$  for each  $T \in \mathcal{T}$ .  $\square$

**Corollary 0.13.** *Let  $X$  be a Banach space, and  $\{x_n^*\} \subset X^*$  a sequence that converge pointwise on  $X$  to a (linear) functional  $\ell : X \rightarrow \mathbb{R}$ . Then  $\{x_n^*\}$  is bounded,  $\ell \in X^*$ , and the convergence is uniform on compact sets.*

*Proof.* By Corollary 0.12,  $\{\|x_n^*\|\}$  is bounded, say, by a constant  $M$ . This easily implies that  $\|\ell\| \leq M$ . The last part follows by Theorem 0.9.  $\square$

Let us recall that a sequence  $\{x_n^*\} \subset X^*$  is  $w^*$ -convergent if there exists  $x^* \in X^*$  such that  $x_n^* \rightarrow x^*$  pointwise on  $X$ .

**Corollary 0.14.** *Let  $X$  be a separable Banach space. Then every pointwise bounded sequence  $\{x_n^*\} \subset X^*$  is bounded and admits a  $w^*$ -convergent subsequence.*

*Proof.* Apply Theorem 0.8 and Theorem 0.11. □

**Remarks on boundary points.** Let  $X$  be a normed space,  $C$  a convex set with  $\text{int}(C) \neq \emptyset$ , and  $f: C \rightarrow \mathbb{R}$  a convex function which is continuous on  $\text{int}(C)$ . The following example shows that, even for finite-dimensional  $X$ , the behaviour of  $f$  at boundary points of  $C$  can be very bad.

**Example 0.15.** Let  $|\cdot|_e$  denote the Euclidean norm on  $X = \mathbb{R}^d$ . Then each function  $f: C \rightarrow [0, +\infty)$  such that  $f|_{\text{int}(C)} = 0$  is convex on  $C$ .