Rob 501 Handout: Grizzle
Cauchy Sequence Example and Contraction Mapping Theorem

Suggested Exercise: Suppose A is a square invertible matrix and we want to solve $Ax = b$. You know a few ways to do this, such as inverting A or using QR-factorization. Here, I will let you investigate another method via Contraction Mappings! Recall in the following that we assume A is invertible.

- Let's first note that the solution to $A^\top Ax = A^\top b$ is the same as that of $Ax = b$.
- We recall that $A^\top A > 0$ hence its e-values are all positive.
- Find the range of $\alpha > 0$ such that $-1 < \lambda_{\max}(I - \alpha A^\top A) < 1$. **Hint**: For any square real matrix M, e-values of $I+M$ satisfy: $\lambda_i(I+M) = 1+\lambda_i(M)$.
- **Exercise:** Recall from the SVD Handout, $\sqrt{\lambda_{\max}(M^\top M)}$ is the *induced 2-norm* of the matrix M. Prove that if M is real and symmetric, then $\sqrt{\lambda_{\max}(M^\top M)} = |\lambda_{\max}(M)|$.
- Define $P(x) := x - \alpha\big(A^\top Ax - A^\top b\big)$, for an α you found above.
- Check that $x^* = P(x^*) \Leftrightarrow A^\top Ax^* - A^\top b = 0$
- Choose random A and b with A invertible. Choose a random initial condition x_0. Define
$$x_{k+1} = P(x_k)$$
and check that the resulting sequence approaches a solution to $Ax = b$.
- Choose different values of α and see what you get.
- **Remark:** $||P(x) - P(y)||_2 \leq |\lambda_{\max}(I - \alpha A^\top A)|||x - y||_2$. Hence, you will see in Thursday's lecture that you are building a Cauchy Sequence when you choose α such that $0 \leq |\lambda_{\max}(I - \alpha A^\top A)| < 1$.