Exam Number:________

ROB 501 Exam-II

You can pick any 36 hours between 4:30pm (ET) December 15, 2021 (Wednesday) and 11:59pm (ET) December 18, 2021 (Saturday) to solve this exam.

HONOR PLEDGE: Copy (NOW) and SIGN (**after the exam is completed**): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

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Kuan-Ting Lee

SIGNATURE
(Sign **after** the exam is completed)

Kuan-Ting, Lee

LAST NAME (PRINTED), *FIRST NAME*

RULES:

1. The exam is open book, open lecture handouts and slides, open recitation notes, open HW solutions, open internet (under the communication and usage restrictions mentioned below).
2. If you use MATLAB or any other scientific software to complete some parts of the exam. You are required to submit your script along with your solution in such case.
3. You are not allowed to communicate with anyone other than the Course instructor and the GSIs related to the exam during the entire period. If you have questions, you can post a private Piazza post for the instructors or email `necmiye@umich.edu` with GSIs on cc.
4. You are not allowed to use any online "course helper" sites like Chegg, Course Hero, and Slader, in any part of the exam. You are not allowed to post exam questions on the internet or discuss them online.
5. Please do not wait until the last minute to upload your solution to Gradescope and double-check to make sure you uploaded the correct pdf. If you run into problems with Gradescope, email your .pdf file as an attachment to Prof. Ozay as soon as practicable at `necmiye@umich.edu`.

SUBMISSION AND GRADING INSTRUCTIONS:

1. The maximum possible score is 80 (+3 bonus points). To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly.

2. You must submit your solutions in a single pdf. You will be asked to mark where each solution is.

3. **Honor Code:** The first page of your submitted pdf should include a hand-written and signed honor code (see the first page of this pdf). Without this, your exam will not be graded.

4. **For problems 1-5** Use this page to record your answers. We will NOT grade other pages and we do not care if you make a mistake when copying your answers to this page. Please be careful. If you are submitting handwritten (or word-processed) documents, make sure to make a similar table where you record all your True/False (and fill in the blanks for 1(a),(b),(c)) answers. There is no partial credit on these questions. You are welcome to leave some justification but we will not look at them.

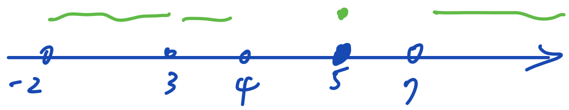
5. **For problems 6-7-8** Record your final answer in the box whenever one is provided. If you are submitting handwritten (or word-processed) documents, make sure to box or highlight the final result. However, you MUST show your work to get credit. In other words, a correct result with no reasoning or wrong reasoning could lead to no points.

Answers for Problem 1	
Problem 1(a)	$(-2,3) \cup (3,4) \cup (7,\infty)$
Problem 1(b)	$[-2,4] \cup [7,\infty)$
Problem 1(c)	$\{-2,3,4,5,7\}$
Problem 1(d)	True False

Answers for the True/False Part				
	(a)	(b)	(c)	(d)
Problem 2	T	T	F	T
Problem 3	T	F	T	F
Problem 4	F	F	F	F
Problem 5	T	F	T	T

Problems 1 - 5 (30 points: 5 × 6)

Instructions. For each problem, you should select True or False. **Make sure to record your answers on the second page. Only the second page will be graded!!!**



1. (Real Analysis I) Consider the normed space $(\mathbb{R}, \mathbb{R}, |\cdot|)$ (the set of reals equipped with the absolute value as the norm). Consider the set $S \subset \mathbb{R}$ given as $S = (-2, 3) \cup (3, 4) \cup \{5\} \cup [7, \infty)$. **For parts (a)-(b)-(c), provide the set being asked for (e.g., if you were asked about the interior of $\mathbb{R}$, the answer would be $\mathbb{R}$. Giving the definition of interior will not get you any points.). For part (d) Answer True or False. Record your answers on the second page.**

(a) The interior S° of S is $(-2,3) \cup (3,4) \cup (7,\infty)$.

(b) The set of all limit points of S is $[-2,4] \cup [7,\infty)$.

(c) The boundary ∂S of S is $\{-2, 3, 4, 5, 7\}$.

(d) Let ∂S be the boundary of S and let $f : \mathbb{R} \to \mathbb{R}$ be any continuous function. Then, there exists $y^* \in \partial S$ such that $f(y^*) \geq f(y)$ for all $y \in \partial S$. **True or False**.

let $f(y) = -(y-4)^2$

$y^* = 4$, $f(y^*) = 0 \geq f(y) \ \forall y \in \partial S$

2. (Real Analysis II) Consider finite-dimensional normed spaces $(\mathcal{X}, \mathbb{R}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \mathbb{R}, \|\cdot\|_{\mathcal{Y}})$. **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

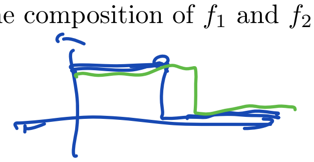
T **F** (a) Consider finite-dimensional normed spaces $(\mathcal{X}, \mathbb{R}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \mathbb{R}, \|\cdot\|_{\mathcal{Y}})$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a function that is continuous everywhere on $\mathcal{X}$. Let (x_n) be a Cauchy sequence with $x_n \in \mathcal{X}$ for all n. Define $y_n = f(x_n)$. Then, (y_n) is a Cauchy sequence in $\mathcal{Y}$.

complete   closed and bounded

T **F** (b) Consider a finite-dimensional normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$. If $S \subset \mathcal{X}$ is compact, then S is complete.

T **F** (c) Let $\mathcal{F} = \{f : [0,1] \to \mathbb{R} \mid \sup_{x\in[0,1]} |f(x)| < \infty\}$ be the set of real-valued bounded functions on $[0, 1]$. Define $f_k(x) = \begin{cases} 1 & x \in [0, 1 - \frac{1}{1+k}) \\ 0 & \text{otherwise} \end{cases}$. Consider the normed space $(\mathcal{F}, \mathbb{R}, ||\cdot||_\infty)$ with norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$. Then, (f_k) is a Cauchy sequence in this space. [1]

T **F** (d) Consider the functions $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ defined as $f_1(x) = 0.8x$ and $f_2(x) = -1.2x$, respectively. Define the composition of f_1 and f_2 as $p(x) = f_1(f_2(x))$. Then, p is a contraction mapping.



$p(x) = -0.96x$

$||f_n(x) - f_m(x)||_\infty$

$|| -0.96x + 0.96y || = 0.96\,|x - y|$

≤ 0.97

[1] The set $\mathcal{F}$ only includes bounded functions to make sure the given norm is well-defined for all elements of $\mathcal{F}$, hence, the normed space is well-defined. You do not really need to worry about boundedness. You just need to apply the definition of Cauchy sequence for the given norm.

3. (SVD, QR factorization) Let $A = U\Sigma V^\top$ be the SVD of a real square matrix A, with $\Sigma = \text{diag}(\sigma_1, 5, 3, \sigma_4)$, where $\sigma_1 \geq 5 \geq 3 \geq \sigma_4 \geq 0$. For $1 \leq i \leq 4$, let U_i and V_i denote the ith column of U and V respectively. The matrix norm used in this problem is $\|B\| = \sqrt{\lambda_{\max}(B^\top B)}$. **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T F (a) U_3 is an e-vector of $AA^\top$ corresponding to e-value 3.

T F (b) There exists an orthogonal matrix Q and an upper-triangular, invertible matrix R such that $A = QR$.

T F (c) There exists a matrix E with norm $\|E\| \leq 3$, rank$(A - E) = 3$.

T F (d) The rank of the matrix $M := U_1V_1^\top + U_2V_2^\top$ equals two.

8

A needs to have indep. col.

$$\begin{bmatrix} \sigma_1 & & & \\ & 5 & & \\ & & 3 & \\ & & & \sigma_4 \end{bmatrix}$$

$A - E = B$

4. (Estimators) Consider a problem with equation $y = Cx + \epsilon$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and both n and m are integers greater than one. You measure y but ϵ is unknown. **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T F (a) Suppose $\mathcal{E}\{x\} = 0$, $\mathcal{E}\{\epsilon\} = 0$, $\mathcal{E}\{x\epsilon^\top\} = 0$, $\mathcal{E}\{xx^\top\} = P = 10000I_{n\times n}$, and $\mathcal{E}\{\epsilon\epsilon^\top\} = Q = 0.01I_{m\times m}$. Also, assume $m > n$ and C is full rank. Then, a Minimum Variance Estimate (MVE) of x can be found and is approximately[2] equal to the ordinary least squares solution $\hat{x}$ that minimizes $\|y - Cx\|_2^2$.

T F (b) Suppose $\mathcal{E}\{x\} = 0$, $\mathcal{E}\{\epsilon\} = 0$, $\mathcal{E}\{x\epsilon^\top\} = 0$, $\mathcal{E}\{xx^\top\} = P \geq 0$, and $\mathcal{E}\{\epsilon\epsilon^\top\} = Q > 0$. Then the Minimum Variance Estimate (MVE) is given by $\hat{x} = PC^\top(CPC^\top + Q)^{-1}y$.

T F (c) Assume x is deterministic, $\mathcal{E}\{\epsilon\} = 0$, $\mathcal{E}\{\epsilon\epsilon^\top\} = Q > 0$, and columns of C are linearly independent. Then, $K = (C^\top QC)^{-1}C^\top Q^{-1}$ is well-defined and it satisfies $KC = I_{n\times n}$. BLUE ?

T F (d) Suppose x is deterministic, $\mathcal{E}\{\epsilon\} = 0$, $\mathcal{E}\{\epsilon\epsilon^\top\} = Q \geq 0$ and the columns of C are linearly independent. Then a Best Linear Unbiased Estimate (BLUE) of x can be determined.

[2]You can assume the values in the measurement y is in the order of 1 to 10 (i.e., small relative to P, large relative to Q).

5. (Probability) Consider three random variables X_1, X_2 and X_3 and the random vector $Z = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}^\top$. We are given the following information:

- X_1, X_2 and X_3 are jointly normally distributed (i.e., joint Gaussian random variables).
- $\mathcal{E}\{Z\} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^\top$.
- $\text{cov}(Z, Z) = \Sigma_Z = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 2 & 1 & 5 \end{bmatrix}$.

Circle True or False as appropriate for the following statements:

T F (a) X_1 and X_2 are both uncorrelated and independent. (For G : uncorrelated $\Leftrightarrow$ indep.)

~~T~~ F (b) There exists a random variable X_4 such that the covariance of the joint distribution of X_1 and X_3 conditioned on X_4 is $\text{cov}(Y, Y) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, where $Y = \begin{bmatrix} X_{1|X_4=x_4} \\ X_{3|X_4=x_4} \end{bmatrix}$

// $X_1 |_{X_2 = x_2}$ | $X_3 |_{X_2 = x_2}$

T F (c) The mean of X_1 given $X_2 = 0.5$ and $X_3 = 2$, $\mathcal{E}\{X_1 | X_2 = 0.5, X_3 = 2\}$, is equal to 2.

T ~~F~~ (d) The distribution of random variable X_1 conditioned on X_2 and X_3, or $X_{1|X_2=x_2,X_3=x_3}$, is the same as the distribution of $X_{1|X_3=x_3}$ conditioned on $X_{2|X_3=x_3}$.

$X_{2|x_3=x_3} = x_2$

(c)

$$\mu_{1|3} = \mu_1 + \Sigma_{13}\Sigma_{33}^{-1}(x_3 - \mu_3)$$
$$= 1 + 2 \cdot 1/5\,(2 - 0)$$
$$= 9/5$$

$$\mu_1|_x = \mu_1 + \Sigma_{1x}\Sigma_{xx}^{-1}(X_x - \mu_x)$$
$$= 1 + [0\ \ 2] \begin{bmatrix} 4 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0.5 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right)$$
$$= 1 + [0\ \ 2] \cdot \frac{1}{19} \begin{bmatrix} 5 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -1.5 \\ 2 \end{bmatrix}$$
$$= 1 + \frac{1}{19} [-2\ \ 8] \begin{bmatrix} -1.5 \\ 2 \end{bmatrix}$$
$$= 1 + \frac{1}{19} \cdot 19$$
$$= 2$$

(b)

$$\Sigma_{X|Z} = \Sigma_{XX} - \Sigma_{XZ}\Sigma_{ZZ}^{-1}\Sigma_{ZX}$$
$$= \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} [\cdot] [\cdot\ \ \cdot]$$
$$\Rightarrow \Sigma_{XZ}\Sigma_{ZZ}^{-1}\Sigma_{ZX} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$
$$\Rightarrow \Sigma_{XZ} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}, \quad \Sigma_{ZZ} = 1$$
$$\Rightarrow \Sigma = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 4 & 1 & 0 \\ 2 & 1 & 5 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 1 \end{bmatrix}$$

Partial Credit Section of the Exam

For the next problems, partial credit is awarded and you MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

"I do not know",

because answering "yes" or "no" would be unfair to everyone else. If you show the steps you followed in deriving your answer, you'll probably be fine. <u>If something was explicitly derived in lecture, handouts or homework, you do not have to re-derive it. You can state it as a known fact and then use it.</u> For example, we proved that real symmetric matrices have real e-values. So if you need this fact, simply state it and use it.

6. (15 points) (Place your answer in the box and show your work below.) Consider an estimation problem where the model is

$$y_0 = C_0 x + \epsilon_0$$

(a) **(10 points)** Determine the Best Linear Unbiased Estimate (BLUE) of $x \in \mathbb{R}^2$ when

$$y_0 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \quad C_0 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & -1 \end{bmatrix} \quad \mu_0 = \mathcal{E}\{\epsilon_0\} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad Q_0 = \mathcal{E}\{\epsilon_0 \epsilon_0^\top\} = \begin{bmatrix} 0.70 & -0.40 & -0.10 \\ -0.40 & 0.80 & 0.20 \\ -0.10 & 0.20 & 0.30 \end{bmatrix}.$$

$$\hat{x} = \begin{bmatrix} 1.1875 \\ 1.5625 \end{bmatrix}$$

(b) **(5 points)** Now assume you get an additional measurement x via:

$$y_1 = C_1 x + \epsilon_1,$$

where $C_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $y_1 = 4$, ϵ_1 is zero mean, independent from ϵ_0, and with $Q_1 = \mathcal{E}\{\epsilon_1^2\} = 0.01$. Use this information and your results from part (a) (possibly some extra information derived from part (a) too) to recursively determine the Best Linear Unbiased Estimate (BLUE) of $x \in \mathbb{R}^2$ for

$$\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} \epsilon_0 \\ \epsilon_1 \end{bmatrix}$$

Note: a batch solution will get you only 1 point.

$$\hat{x}_{new} = \begin{bmatrix} 1.0405 \\ 1.4834 \end{bmatrix}$$

Please show your work for question 6.

6. (a)

$$\hat{x} = \hat{K} y$$

$$= (C_0^T Q_0^{-1} C_0)^{-1} C_0^T Q_0^{-1} y_0$$

$$= \begin{bmatrix} 1.1875 \\ 1.5625 \end{bmatrix}$$

(b)

For recursive approach, $\hat{x}_1$ can be expressed as below:

$$\hat{x}_1 = \hat{x}_0 + M_1^{-1} C_1^T Q_1^{-1} (y_1 - C_1 \hat{x}_0), \quad \text{where } M_1 = M_0 + C_1^T Q_1^{-1} C_1$$

$$= \begin{bmatrix} 1.1875 \\ 1.5625 \end{bmatrix} + \begin{bmatrix} 106 & 198 \\ 198 & 422 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot 0.01^{-1} \cdot \left(4 - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1.1875 \\ 1.5625 \end{bmatrix}\right)$$

$$\approx \begin{bmatrix} 1.0605 \\ 1.4834 \end{bmatrix}$$

$$M_0 = C_0^T Q_0^{-1} C_0$$

$$= \begin{bmatrix} 6 & -2 \\ -2 & 22 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 6 & -2 \\ -2 & 22 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot 100 \cdot \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 \\ -2 & 22 \end{bmatrix} + \begin{bmatrix} 100 & 200 \\ 200 & 400 \end{bmatrix}$$

$$= \begin{bmatrix} 106 & 198 \\ 198 & 422 \end{bmatrix}$$

```python
# Q6
# (a)
y0 = np.array([[2], [4], [0]])
C0 = np.array([[1, 1], [0, 2], [1, -1]])
mu0 = np.array([[0], [0], [0]])
Q0 = np.array([[0.7, -0.4, -0.1], [-0.4, 0.8, 0.2], [-0.1, 0.2, 0.3]])

x_hat = np.linalg.inv(C0.T @ np.linalg.inv(Q0) @ C0) @  C0.T @ np.linalg.inv(Q0) @ y0
print(x_hat)

# # (b)
# Recursive method
x0 = np.array([[1.1875], [1.5625]])
C1 = np.array([[1, 2]])
y1 = 4
mu1 = 0
Q1 = np.array([[0.01]])
M0 = C0.T @ np.linalg.inv(Q0) @ C0
M1 = M0 + C1.T @ np.linalg.inv(Q1) @ C1
x1 =  x0 + np.linalg.inv(M1) @ C1.T @ np.linalg.inv(Q1) @ (y1 - C1 @ x0)
print("M0 = ", M0)
print("M1 = ", M1)
print("x1 = ", x1)
```

← Code for matrix computation

$$\begin{bmatrix} x_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ s_k \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} w_k$$

7. (20 points + 3 points) (Place your answers in the boxes and show your work below.) We are trying to estimate the position x_k and velocity s_k of a vehicle from sensor measurements. Let the state be $z_k = \begin{bmatrix} x_k & s_k \end{bmatrix}^\top$. Consider the vehicle's discrete-time dynamics

$$x_{k+1} = x_k + 0.1 s_k + 0.1 w_k \qquad P_{k+1} = A P_k A^T + R_k'$$
$$s_{k+1} = s_k + 0.2 w_k.$$

For the sensor measurements, we have two different options (we can only use one of these sensors but not both):

$$\text{Sensor 1}: \quad y_k^{(1)} = x_k + s_k + v_k^{(1)} := C^{(1)} z_k + v_k^{(1)} \qquad y_k^{(1)} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ s_k \end{bmatrix}$$

or,

$$\text{Sensor 2}: \quad y_k^{(2)} = 0.1 x_k + s_k + v_k^{(2)} := C^{(2)} z_k + v_k^{(2)} \qquad y_k^{(2)} = \begin{bmatrix} 0.1 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ s_k \end{bmatrix}$$

where w_k, $v_k^{(1)}$, $v_k^{(2)}$ are (scalar) zero mean white Gaussian (i.e. normal) noise processes, with constant covariances, satisfying all of the standard assumptions in our Kalman Filter handout. **Part (e) is on page 11.**

Data: At time $k = 2$ the Kalman filter has the data $\hat{z}_{2|2} := \mathcal{E}\{z_2 | y_0, y_1, y_2\} = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$ and $P_{2|2} := \mathcal{E}\{(z_2 - \hat{z}_{2|2})(z_2 - \hat{z}_{2|2})^T | y_0, y_1, y_2\} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$, and for all $k \geq 0$, $R = R_k = \text{cov}\{w_k\} = 2$, $Q^{(1)} = Q_k^{(1)} = \text{cov}\{v_k^{(1)}\} = 2$, and $Q^{(2)} = Q_k^{(2)} = \text{cov}\{v_k^{(2)}\} = 1$.

Given the information above:

(a) **(2 points)** What is the value of $\hat{z}_{3|2}$?

(b) **(6 points)** If sensor 1 is used, what is the Kalman gain $K_3^{(1)}$ at time $k = 3$? What is the corresponding $\hat{z}_{3|3}$ if $y_3 = 2.5$?

(c) **(6 points)** If sensor 2 is used, what is the Kalman gain $K_3^{(2)}$ at time $k = 3$? What is the corresponding $\hat{z}_{3|3}$ if $y_3 = 0.7$?

(d) **(2 points)** Given the measurements y_3 for the two different sensors from part (b) and (c), which sensor would give you a more accurate (less uncertain) estimate of the position x_3 at time $k = 3$? Explain why.

$\hat{z}_{3\|2} = \begin{bmatrix} 2.05 \\ 0.5 \end{bmatrix}$	$K_3^{(1)} = \begin{bmatrix} 0.3298 \\ 0.4920 \end{bmatrix}$ $\hat{z}_{3\|3} = \begin{bmatrix} 2.0335 \\ 0.4754 \end{bmatrix}$	$K_3^{(2)} = \begin{bmatrix} 0.3091 \\ 0.7836 \end{bmatrix}$ $\hat{z}_{3\|3} = \begin{bmatrix} 2.0485 \\ 0.4961 \end{bmatrix}$

Sensor choice and reason:

Sensor 1, since it $\| P_{3|3}^{(1)} \|$ is smaller than $\| P_{3|3}^{(2)} \|$

Please show your work for question 7 (Problem continues on the next page).

The model can be expressed as below

$$z_{k+1} = A z_k + e_k, \text{ where } z_k = \begin{bmatrix} x_k \\ s_k \end{bmatrix}, A = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, e_k = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} w_k$$

$$y_k^{(1)} = C^{(1)} z_k + v_k^{(1)}, \text{ where } C^{(1)} = [1 \quad 1]$$

$$y_k^{(2)} = C^{(2)} z_k + v_k^{(2)}, \text{ where } C^{(2)} = [0.1 \quad 1]$$

(a) $$\hat{z}_{3|2} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \hat{z}_{2|2}$$

$$= \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 2.05 \\ 0.5 \end{bmatrix}$$

(b) $$K_3^{(1)} = P_{3|2} C^{(1)T} \left(C^{(1)} P_{3|2} C^{(1)T} + Q^{(1)} \right)^{-1}$$

$$P_{3|2} = A P_{2|2} A^T + R_k', \text{ where } R_k' = \mathrm{cov}(e_k) = \mathrm{cov}\left(\begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} w_k \right) = \begin{bmatrix} 0.01 & 0.02 \\ 0.02 & 0.04 \end{bmatrix} \cdot 2$$

$$= \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix} + 2 \begin{bmatrix} 0.01 & 0.02 \\ 0.02 & 0.04 \end{bmatrix}$$

$$= \begin{bmatrix} 2.26 & 1.44 \\ 1.44 & 4.08 \end{bmatrix}$$

$$\Rightarrow K_3^{(1)} = \begin{bmatrix} 0.3298 \\ 0.4920 \end{bmatrix}$$

$$\hat{z}_{3|3} = \hat{z}_{3|2} + K_3^{(1)} \left(y_3^{(1)} - C^{(1)} \hat{z}_{3|2} \right)$$

$$= \begin{bmatrix} 2.0335 \\ 0.4754 \end{bmatrix}$$

(c)

$$K_3^{(2)} = P_{3|2} C^{(2)T} \left(C^{(2)} P_{3|2} C^{(2)T} + Q^{(2)} \right)^{-1}$$

$$\Rightarrow K_3^{(2)} = \begin{bmatrix} 0.3091 \\ 0.7836 \end{bmatrix}$$

$$\hat{z}_{3|3} = \hat{z}_{3|2} + K_3^{(2)} \left(y_3^{(2)} - C^{(2)} \hat{z}_{3|2} \right)$$

$$= \begin{bmatrix} 2.0485 \\ 0.4961 \end{bmatrix}$$

(d)

$$P_{3|3}^{(1)} = P_{3|2} - K_3^{(1)} C^{(1)} P_{3|2}$$

$$= \begin{bmatrix} 1.0399 & -0.3803 \\ -0.3803 & 1.3643 \end{bmatrix}$$

$$P_{3|3}^{(2)} = P_{3|2} - K_3^{(2)} C^{(2)} P_{3|2}$$

$$= \begin{bmatrix} 1.7451 & 0.1345 \\ 0.1345 & 0.7701 \end{bmatrix}$$

Here $\|\cdot\|$ is defined of 2-norm.

$$\| P_{3|3}^{(1)} \| = 1.7977, \quad \| P_{3|3}^{(2)} \| = 1.9170$$

$$\Rightarrow \| P_{3|3}^{(1)} \| < \| P_{3|3}^{(2)} \|$$

$\Rightarrow$ sensor 1 gives more accurate estimate ※

Code for Q7 (a)(b)(c)(d)

```python
# ================================================
# Q7
A = np.array([[1, 0.1], [0, 1]])
R = np.array([[0.02, 0.04], [0.04, 0.08]])
C1 = np.array([[1, 1]])
C2 = np.array([[0.1, 1]])

z2 = np.array([[2], [0.5]])
P2 = np.array([[2, 1], [1, 4]])
Q1 = 2
Q2 = 1

z3_2 = A @ z2
print(z3_2)
P3_2 = A @ P2 @ A.T + R
print("P3_2 = ", P3_2)

K3_1 = P3_2 @ C1.T @ np.linalg.inv(C1 @ P3_2 @ C1.T + Q1)
print("K3_1 = ", K3_1)
y3_1 = 2.5
z3_3_1 = z3_2 + K3_1 @ (y3_1 - C1 @ z3_2)
print("z3_3_1 = ", z3_3_1)


K3_2 = P3_2 @ C2.T @ np.linalg.inv(C2 @ P3_2 @ C2.T + Q2)
print("K3_2 = ", K3_2)
y3_2 = 0.7
z3_3_2 = z3_2 + K3_2 @ (y3_2 - C2 @ z3_2)
print("z3_3_2 = ", z3_3_2)

P3_3_1 = P3_2 - K3_1 @ C1 @ P3_2
print("P3_3_1 = ", P3_3_1)
print("norm of P3_3_1 = ", np.linalg.norm(P3_3_1))
# print((1.04 ** 2 + 2 * 0.3803 ** 2 + 1.3643 ** 2) ** 0.5)

P3_3_2 = P3_2 - K3_2 @ C2 @ P3_2
print("P3_3_2 = ", P3_3_2)
print("norm of P3_3_2 = ", np.linalg.norm(P3_3_2))
```

7. (e) **(4 points + 3 bonus points)** Consider a simplified version of the vehicle dynamics where we can control the velocity via the input u_k, that is, the position of the robot evolves according to:

$$x_{k+1} = x_k + 0.1u_k.$$

Assume at time $k = 2$, the position of the vehicle is $x_2 = 2$. Write a linear program (LP):

$$\begin{array}{ll} \min & c^\top X \\ \text{s.t.} & AX \leq b \\ & A_{eq}X = b_{eq} \end{array}$$

to compute the next two inputs $u := [u_2, u_3]^\top$ such that $\|u\|_\infty$ is minimized and the robot reaches the region $x_4 \in [-0.1, 0.1]$ at time $k - 4$. For a properly defined X, provide c, A, b, A_{eq}, b_{eq}. If you just formulate the LP and not implement it, you will get 4 points. You will get **3 bonus points** if you implement the LP in MATLAB (or similar software) and provide the optimizing u^* together with your implementation. You can get partial credit for this question even if you solve for u^* using a different method.

$\|u\|_\infty = \max(u_2, u_3)$

min $\max(u_2, u_3)$ can be formulated as min t s.t. $-t \leq u_2 \leq t$, $-t \leq u_3 \leq t$

$$\Rightarrow \min \|u\|_\infty \quad \text{s.t. } 2 + 0.1(u_2 + u_3) \leq 0.1, \quad 2 + 0.1(u_2 + u_3) \geq -0.1$$

$$\equiv \min t \quad \text{s.t. } \begin{array}{l} -u_2 - t \leq 0 \\ u_2 - t \leq 0 \\ -u_3 - t \leq 0 \\ u_3 - t \leq 0 \\ 0.1u_2 + 0.1u_3 \leq -1.9 \\ -0.1u_2 - 0.1u_3 \leq 2.1 \end{array}$$

$$\equiv \min c^T X \quad \text{s.t. } AX \leq b, \quad \text{where } X = \begin{bmatrix} t \\ u_2 \\ u_3 \end{bmatrix}$$

(No need for A_{eq}, b_{eq})

$$A = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0.1 & 0.1 \\ 0 & -0.1 & -0.1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1.9 \\ 2.1 \end{bmatrix}$$

$$c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

MATLAB implementation using linprog()

linear_prog.m

```
A = [-1 -1 0
    -1 1 0
    -1 0 -1
    -1 0 1
    0 0.1 0.1
    0 -0.1 -0.1]

b = [0
    0
    0
    0
    -1.9
    2.1]

f = [1 0 0]

X_optimized = linprog(f, A, b)

u_star = X_optimized(2:end)
```

Command Window

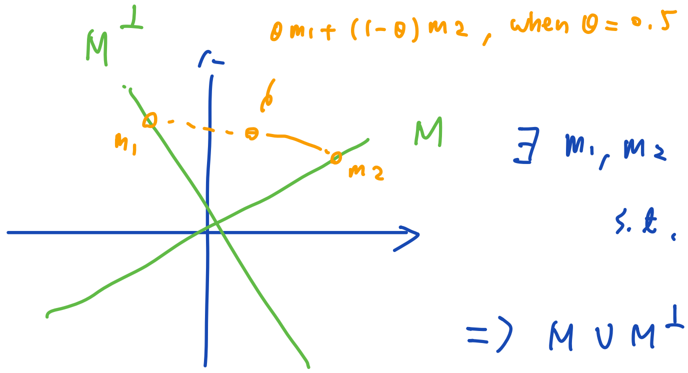
```
u_star =

   -9.5000
   -9.5000
```

$$\Rightarrow u^* = \begin{bmatrix} -9.5 \\ -9.5 \end{bmatrix}$$

8. (15 points) The following are three (3) short answer questions. You do not need to give a formal proof; only give a few short reasons/calculations why something is TRUE or FALSE. **Part (c) is on the next page.**

(a) **(5 Points)** Let $(\mathbb{R}^2, \mathbb{R}, <\cdot,\cdot>)$ be the two-dimensional real inner product space with the standard inner product. Let $M \subset \mathbb{R}^2$ be a one-dimensional subspace of $\mathbb{R}^2$. Then, $M \cup M^\perp$ is a convex set.

Circle **T** or **F**. Give a few short reasons/calculations why this is TRUE or FALSE:



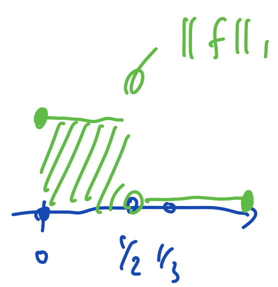
$\exists\ m_1, m_2 \in M \cup M^\perp,\ 0 \leq \theta \leq 1$

s.t. $\theta m_1 + (1-\theta) m_2 \notin M \cup M^\perp$

$\Rightarrow M \cup M^\perp$ is not a convex set

(b) **(5 Points)** Let $\mathcal{F} = \{f : [0,1] \to \mathbb{R} \mid \int_0^1 |f(x)|dx < \infty\}$ be the set of real-valued absolutely summable functions with domain $[0,1]$. Define $f_k(x) = \begin{cases} 1 & x \in [0, 1-\frac{1}{1+k}) \\ 0 & \text{otherwise} \end{cases}$. Consider the normed space $(\mathcal{F}, \mathbb{R}, ||\cdot||_1)$ with norm $||f||_1 = \int_0^1 |f(x)|dx$. Then, (f_k) is a Cauchy sequence in this space. [3]

Circle **T** or **F**. Give a few short reasons/calculations why this is TRUE or FALSE:



$||f_1||_1 = 1/2$

$||f_2||_1 = 2/3$

$\vdots$

$||f_k||_1 = k/(k+1)$

[3]The set $\mathcal{F}$ only includes absolutely summable functions to make sure the given norm is well-defined for all elements of $\mathcal{F}$.You do not need to worry about absolute summability to answer this question. You just need to apply the definition of Cauchy sequence for the given norm.

We define $f^*(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & x = 1 \end{cases}$

$$(f_n - f^*)(x) = \begin{cases} 0, & x \in [0, 1-\frac{1}{1+n}) \cup \{1\} \\ -1, & x \in [1-\frac{1}{1+n}, 1) \end{cases}$$

$$\| f_n - f^* \|_1 = 1 \cdot [1 - (1 - \frac{1}{1+n})] = \frac{1}{1+n}$$

$\Rightarrow \forall \varepsilon > 0$, we can pick $N = \frac{2-\varepsilon}{\varepsilon}$. $\forall n \geq N$, $\| f_n - f^* \|_1 = \frac{1}{1+n}$

$$\leq \frac{1}{1+N}$$

$$= \frac{1}{1 + 2-\varepsilon/\varepsilon}$$

$$= \frac{\varepsilon}{\varepsilon + 2 - \varepsilon}$$

$$= \varepsilon/2 < \varepsilon$$

$N(\varepsilon) = \frac{2-\varepsilon}{\varepsilon}$ as shown above

f_k converges to 1. $\because \forall \varepsilon > 0$, $\exists N(\varepsilon) < \infty$, $\forall n \geq N$, $\| f_n - f^* \|_1 < \varepsilon$

$\Rightarrow f_k \to f^* \Rightarrow f_k$ is a Cauchy sequence

(c) **(5 Points)** Consider the (real) inner product space $(\mathcal{X}, \mathbb{R}, <\cdot,\cdot>)$ where $\mathcal{X}$ is the set of 2×2 real matrices and the inner product of two matrices A and B is $<A,B>:=\mathrm{tr}(A^\top B)$. Let

$$Y_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } Y_2 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}.$$

Then, the unique solution to the optimization problem

$$\begin{aligned} A^* = \arg\min \quad & \sqrt{<A,A>} \\ \text{s.t.} \quad & <A,Y_1>=4 \\ & <A,Y_2>=2 \end{aligned}$$

is given by $A^* = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$.

Circle **T or F**. Give a few short reasons/calculations why this is TRUE or FALSE:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$<A, Y_1> = tr(A^T Y_1) = tr\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = 4$$

$$\Rightarrow a + b + d = 4$$

$$<A, Y_2> = tr(A^T Y_2) = tr\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}\right) = 2$$

$$\Rightarrow -a + c - d = 2$$

$$\sqrt{<A,A>} = \sqrt{tr(A^T A)} = \sqrt{a^2 + c^2 + b^2 + d^2}$$

since square root is a monotonous function, we can minimize $<A,A>$ instead.

In QP, we formulate our problem as below:

$$x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\min \ \frac{1}{2} x^T H x + f^T x \quad \text{such that} \quad A_{eq}\, x = b_{eq}$$

, where $H = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $A_{eq} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 \end{bmatrix}$, $b_{eq} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$$f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x^* = \begin{bmatrix} 0.4 \\ 3.2 \\ 2.8 \\ 0.4 \end{bmatrix}$$

$$\Rightarrow A^* = \begin{bmatrix} 0.4 & 3.2 \\ 2.8 & 0.4 \end{bmatrix}$$

```
H = eye(4) * 2
Aeq =[1 1 0 1; -1 0 1 -1]
beq = [4; 2]
f = zeros(4, 1)
options = optimoptions('quadprog','Display','iter');
[x,fval] = quadprog(H, f, [], [], Aeq, beq, [], [], [], options)
```

Command Window

```
x =

    0.4000
    3.2000
    2.8000
    0.4000
```

$$\Rightarrow A^* = \begin{bmatrix} 0.4 & 3.2 \\ 2.8 & 0.4 \end{bmatrix}$$