18 Sept 2018

Summary: Linear combinations are finite: $\sum_{i=1}^{k} \alpha_i v^i$

$(\mathcal{X}, \mathcal{F})$ vector space: $\{v^1, \dots, v^k\}$ linearly independent if the only solution to $\alpha_1 v^1 + \cdots + \alpha_k v^k = 0$ is the trivial solution $\alpha_1 = 0, \dots, \alpha_k = 0$. **Otherwise** the set is linearly dependent [$\exists\, \alpha_i \in \mathcal{F}$ NOT all zero such that $\alpha_1 v^1 + \cdots + \alpha_k v^k = 0$].

- $S \subset \mathcal{X}$ a (non-empty) subset. $\text{span}\{S\} = \{$all linear combinations$\}$
  span$\{S\}$ is always a subspace.
- If the zero vector is in a set, then the set is linearly dependent $0 \in S \Rightarrow S$ linearly dependent.

- For Thursday, study the definition of eigenvalues and eigenvectors

SEE PAGE 23 of NOTES TYPESET by STUDENTS!

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TODAY

Def. A set of vectors B is a **basis** for $(\mathcal{X}, \mathcal{F})$ if

a) B is linearly independent

b) $\mathcal{X} = \operatorname{span}\{B\}$

Examples

1. Natural Basis Vectors

$$\left.\begin{array}{l} (\mathcal{F}^n, \mathcal{F}) \\ (\mathbb{R}^n, \mathbb{R}) \\ (\mathbb{C}^n, \mathbb{C}) \\ (\mathbb{Q}^n, \mathbb{Q}) \end{array}\right\} \quad e^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \; e^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \; \dots, \; e^n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

2. $(\mathbb{C}^n, \mathbb{R})$ a basis is

$\{e^1, e^2, \dots, e^n, je^1, je^2, \dots, je^n\}$.

3. $(\mathbb{P}(t), \mathbb{R})$ a basis is

$\{1, t, t^2, t^3, \dots\}$ (set of monomials)

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Non-examples

- $(\mathbb{C}^n, \mathbb{R})$ then $\{e^1, \dots, e^n\}$ is not a basis because $\text{span}\{e^1, \dots, e^n\} \neq \mathbb{C}^n$. Indeed, $je^1 \notin \text{span}\{e^1, \dots, e^n\}$.

- $(\mathbb{C}^n, \mathbb{C})$ $\{e^1, \dots, e^n, je^1, \dots, je^n\}$ is not a basis because the set is linearly dependent.

$(\mathbb{C}^2, \mathbb{R})$, $x = \begin{bmatrix} 2+j \\ -4-6j \end{bmatrix} = 2e^1 - 4e^2 + je^1 - 6je^2$

$(\mathbb{C}^2, \mathbb{C})$ $x = (2+j)e^1 + (-4-6j)e^2$

Dimension **Def.** Let $n \geq 1$ be a finite integer. $(\mathcal{X}, \mathcal{F})$ has dimension n if

a) $\exists$ a set of n vectors in $\mathcal{X}$ that is linearly independent.

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b) Every set of n+1 vectors is linearly dependent. $\square$

<u>Def.</u> $(\mathcal{X}, \mathcal{F})$ is infinite dimensional if $\forall n \geq 1$, $\exists$ a set of n linearly independent vectors.

Examples

1. $\dim(\mathcal{F}^n, \mathcal{F}) = n$
2. $\dim(\mathbb{C}^n, \mathbb{R}) = 2n$
3. $\dim(\mathcal{P}(t), \mathbb{R}) = \infty$
4. $\dim(\mathbb{R}, \mathbb{Q}) = \infty$ (not on any exam)

Theorem: In an $n-$dimensional vector space <u>**ANY**</u> set of n linearly independent vectors is a basis.

Proof: Let $(\mathcal{X}, \mathcal{F})$ be $n-$dimensional and let $\{v^1, \ldots, v^n\}$ be a linearly independent set.

To Show: $\forall x \in \mathcal{X}$, $\exists \alpha_1, \ldots, \alpha_n \in \mathcal{F}$ such that $x = \alpha_1 v^1 + \ldots + \alpha_n v^n$

How: Because $(\mathcal{X}, \mathcal{F})$ is $n-$dimensional, $\{x, v^1, \ldots, v^n\}$ is linearly dependent. Otherwise, the $\dim \mathcal{X} > n$ which it isn't. Hence, $\exists \beta_0, \beta_1, \ldots, \beta_n \in \mathcal{F}$, NOT ALL ZERO, such that $\beta_0 x + \beta_1 v^1 + \ldots + \beta_n v^n = 0$.

Claim: $\beta_0 \neq 0$

Proof: Suppose that $\beta_0 = 0$. Then

1. At least one of $\beta_1, \ldots, \beta_n$ is non-zero
2. $\beta_1 v^1 + \ldots + \beta_n v^n = 0$

1 and 2 above, imply that $\{v^1, \ldots, v^n\}$ is linearly dependent, which is a contradiction. Hence $\beta_0 = 0$ cannot hold. Completing the proof, we write

$$\beta_0 x = -\beta_1 v^1 - \ldots - \beta_n v^n$$

$$x = \left(\frac{-\beta_1}{\beta_0}\right) v^1, \ldots - \left(\frac{-\beta_n}{\beta_0}\right) v^n$$

$$\therefore \alpha_1 = \frac{-\beta_1}{\beta_0}, \ldots, \alpha_n = \frac{-\beta_n}{\beta_0}$$

$\square$

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Proposition Let $(\mathcal{X},\mathcal{F})$ be a vector space and suppose that $B=\{b^1,b^2,\cdots\}$ is a basis for $(\mathcal{X},\mathcal{F})$. Let $x\in\mathcal{X}$ and suppose that

$$x=\alpha_1 b^1+\cdots+\alpha_k b^k$$

and

$$x=\bar{\alpha}_1 b^1+\cdots+\bar{\alpha}_k b^k$$

Then, $\alpha_i=\bar{\alpha}_i$ for all $1\le i\le k$.

Only one way to write x as a linear combination of the basis elements.

Proof:

$$\begin{aligned} 0=x-x &= \left(\alpha_1 b^1+\cdots+\alpha_k b^k\right)-\left(\bar{\alpha}_1 b^1+\cdots+\bar{\alpha}_k b^k\right) \\ &= (\alpha_1-\bar{\alpha}_1)b^1+\cdots+(\alpha_k-\bar{\alpha}_k)b^k \end{aligned}$$

Because $\left\{b^1,\cdots,b^k\right\}\subset B$ implies that $\left\{b^1,\cdots,b^k\right\}$ is linearly independent, we deduce that $\alpha_i-\bar{\alpha}_i=0$ for all $1\le i\le k$.

In other words, the "representation of a vector" x in terms of a given basis B is unique: if I compute one representation and you compute another, we both get the same answer.

Representations of Vectors

(or, how we can do most of our abstract linear algebra calculations in MATLAB)

Let $(\mathcal{X},\mathcal{F})$ be an n-dimensional vector space. Let $v=\{v^1,\ldots,v^n\}$

www.PrintablePaper.net and let $x\in\mathcal{X}$.

$$x \in \mathcal{X} \overset{\text{unique}}{\longleftrightarrow} \alpha_1 v^1 + \cdots + \alpha_n v^n$$

$$\overset{\text{unique}}{\longleftrightarrow} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathcal{F}^n$$

Def. $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$ is the representation of $x \in \mathcal{X}$ with respect to a basis $v = \{v^1, \dots, v^n\}$ if

$$x = \alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_n v^n$$ $\square$

- To emphasize the dependence on the basis, we will often write

$$[x]_v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \left(\Longleftrightarrow x = \alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_n v^n \right)$$

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Example: $\mathcal{F} = \mathbb{R}$, $\mathcal{X} = \{2 \times 2 \text{ matrices with real coefficients}\}$

$$\text{Basis 1: } v^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v^3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v^4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Basis 2: } w^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, w^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, w^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, w^4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix} = 5v^1 + 3v^2 + 1v^3 + 4v^4$$

Therefore, $[x]_v = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^4$.

$$x = \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix} = 5w^1 + 2w^2 + 1w^3 + 4w^4$$

* 4 simultaneous equations

Therefore, $[x]_w = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^4$.

$$\begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Remark: 1) Change basis, change the representation.

2) The order a basis is given

in matters. Nagy calls a basis $\{v^1, v^2, \ldots, v^n\}$ an **ordered basis.** In ROB 501, all bases we use are ordered. $\square$

Easy Facts:

1. Addition of vectors in $(\mathcal{X}, \mathcal{F}) \equiv$ Addition of the representations in $(\mathcal{F}^n, \mathcal{F})$.

$$[x + y]_v = [x]_v + [y]_v$$

2. Scalar multiplication in $(\mathcal{X}, \mathcal{F}) \equiv$ Scalar multiplication with the representations in $(\mathcal{F}^n, \mathcal{F})$.

$$[\alpha x]_v = \alpha [x]_v$$

3. Once a basis is chosen, any n-dimensional vector space $(\mathcal{X}, \mathcal{F})$ "looks like" $(\mathcal{F}^n, \mathcal{F})$.

$$(\mathcal{X}, \mathcal{F}) \xleftrightarrow{\text{basis}} (\mathcal{F}^n, \mathcal{F})$$

Change of Basis Matrix: Let $\{u^1, \cdots, u^n\}$ and $\{\bar{u}^1, \cdots, \bar{u}^n\}$ be two bases for $(\mathcal{X}, \mathcal{F})$. Is there a relation between $[x]_u$ and $[x]_{\bar{u}}$?

Change of Basis Matrix

Let $\{u^1, \ldots, u^n\}$ be a basis for $(\mathcal{X}, \mathcal{F})$.

Let $\{\bar{u}^1, \ldots, \bar{u}^n\}$ be another basis.

What is the relation between

$[x]_{\bar{u}}$ and $[x]_u$???

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Theorem $\exists$ an invertible $n\times n$ matrix P with coefficients in $\mathbb{F}$ such that, $\forall x \in \mathcal{X}$

$$[x]_{\bar{u}} = P\,[x]_u .$$

Moreover, $P = [P_1 | P_2 | \cdots | P_n]$ with

$$P_i = [u^i]_{\bar{u}}$$

Proof. Let $x \in X$, Hence,

$$x = \sum_{i=1}^{n} \alpha_i u^i = \sum_{i=1}^{n} \bar{\alpha}_i \bar{u}^i$$

$$\therefore\ [x]_u = \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\alpha} \qquad [x]_{\bar{u}} = \underbrace{\begin{bmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_n \end{bmatrix}}_{\bar{\alpha}}$$

$$\bar{\alpha} = [x]_{\bar{u}} = \left[\sum_{i=1}^{n} \alpha_i u^i\right]_{\bar{u}}$$

$$= \sum_{i=1}^{n} \alpha_i \underbrace{[u^i]_{\bar{u}}}_{P_i}$$

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$$= \sum_{i=1}^{n} \alpha_i P_i$$

$$= \left[P_1 \,|\, P_2 \,|\, \cdots \,|\, P_n \right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$= P\alpha$$

$$\boxed{\therefore \quad \bar{\alpha} = P\alpha, \qquad P_i = [u^i]_{\bar{u}}}$$

Invertibility of P?

By the same reasoning,

$\alpha = \bar{P}\bar{\alpha}$. Hence

$$\bar{\alpha} = P \cdot \alpha = P\bar{P}\bar{\alpha} \implies P\bar{P} = I.$$

$$\alpha = \bar{P}\bar{\alpha} = \bar{P}P\alpha \implies \bar{P}P = I.$$

$\therefore$ $\bar{P} = (P)^{-1}$ and hence P is invertible. $\square$

Do the following example in Thur.

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Example: $\mathcal{X}$={2×2 matrices with real coefficients}, $\mathcal{F}=\mathbb{R}$.

$$u=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$
$$\bar{u}=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\-1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$

We have following relations:

$\alpha=P\bar{\alpha}$, $P_i=[u^i]_{\bar{u}}$, $\bar{\alpha}=\bar{P}\alpha$, $\bar{P}_i=[\bar{u}^i]_u$. ($\bar{P}^{-1}=P$, $P^{-1}=\bar{P}$)

Typically, compute the easier of P or $\bar{P}$, and compute the other by inver

We choose to compute $\bar{P}$

$$\bar{P}_1=[\bar{u}^1]_u=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$\bar{P}_2=[\bar{u}^2]_u=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

$$\bar{P}_3=[\bar{u}^3]_u=\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

$$\bar{P}_4=[\bar{u}^4]_u=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Therefore, $\bar{P}=\begin{bmatrix}1&0&0&0\\0&1&1&0\\0&1&-1&0\\0&0&0&1\end{bmatrix}$ and $P=\bar{P}^{-1}$

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