06 Dec. 2018 Compact Sets

Review $(X, \|\cdot\|)$ given

- Let (x_n) be a sequence and $1 \le n_1 < n_2 < \cdots$ strictly increasing. Then (x_{n_i}) is a subsequence.
- A set $C \subset X$ is bounded if $\exists\, r < \infty$ s.t. $C \subset B_r(0)$.
- C is unbounded $\Leftrightarrow$ $\exists$ a sequence (x_n) s.t. $\forall\, n \ge 1$, $x_n \in C$ and $\|x_{n+1}\| \ge \|x_n\| + 1$. If (x_{n_i}) is a subsequence of (x_n) then (x_{n_i}) is not Cauchy. Indeed $\|x_{n_i} - x_{n_j}\| \ge |n_i - n_j|$
- $\boxed{\text{Handy inequality}}$ $\|x-y\| \ge \big|\,\|x\| - \|y\|\,\big|$
- $C \subset X$ is (segentially) compact if every sequence (x_n) with elements in C has a convergent subsequence with limit in C.
- **C compact $\Rightarrow$ C is closed & bounded.**

Converse is false in general.

Bolzano-Weierstrass Theorem

For a finite-dimensional normed space $(X, \|\cdot\|)$ and a subset $C \subset X$, TFAE

(a) C is compact

(b) C is closed and bounded.

- $(X, \|\cdot\|)$ and $(Y, |||\cdot|||)$ normed spaces.

$f: X \to Y$ is continuous at $x_0 \in X$ if

$\forall \varepsilon > 0$, $\exists\ \delta(x_0, \varepsilon) > 0$ s.t. $\|x - x_0\| < \delta \Rightarrow$
$|||f(x) - f(x_0)||| < \varepsilon$

$$\Longleftrightarrow \left[x \in B_\delta(x_0) \Rightarrow f(x) \in B_\varepsilon(f(x_0)) \right]$$

f is continuous at $x_0 \Longleftrightarrow \left[\forall (x_n) \text{ with } x_n \to x_0 \Rightarrow f(x_n) \to f(x_0) \right]$

"Sequences can be used to characterize continuity at a point."

Today

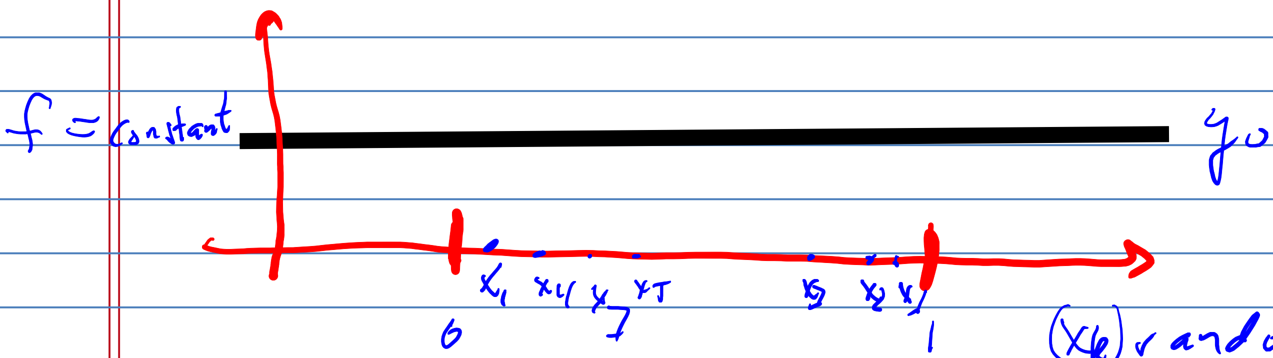
Aside

① $f : X \to Y$ continuous everywhere

② (x_n) in X, $y_n := f(x_n)$

③ $y_n \to y_0$ in $(Y, |||\cdot|||)$

④ $\exists$? x_0 s.t. $x_n \to x_0$ in $(X, ||\cdot||)$?



(x_k) randomly in $[0,1]$

$(y_n = f(x_n)) \longrightarrow y_0$

(x_n) does not converge.

Weierstrass Theorem If C is a compact subset of a normed space $(X, \|\cdot\|)$ and $f: C \to \mathbb{R}$ is continuous at each point of C, **THEN** f achieves its extreme values; i.e., $\exists$ $x^* \in C$ s.t. $f(x^*) = \sup_{x \in C} f(x)$

and $x_* \in C$ s.t. $f(x_*) = \inf_{x \in C} f(x)$.

One says that f achieves its max and min. $\square$

Claim: $f: C \to \mathbb{R}$ continuous and C compact $\Rightarrow f^* := \sup_{x \in C} f(x) < \infty$.

Pf. (Proof by Contradiction) $p \Rightarrow q \Leftrightarrow \sim(p \wedge \sim q)$

p: f is cont. and C is compact

q: $f^* < \infty$

Suppose $f^* = \infty$. Choose $x_1 \in X$ such that $f(x_1) \geq 1$. By induction, choose x_{n+1} such that $f(x_{n+1}) \geq f(x_n) + 1$.

$y_n := f(x_n)$ is a sequence in $\mathbb{R}$ and has no convergent subsequence.

However (x_n) is a sequence in C, which is compact. Hence, $\exists\, \tilde{x} \in C$ and a subsequence (x_{n_i}) s.t.

$$x_{n_i} \longrightarrow \tilde{x}.$$

But f is continuous, and thus

$$f(x_{n_i}) \xrightarrow[i \to \infty]{} f(\tilde{x})$$

(y_{n_i}) is a subsequence of (y_n) and $y_{n_i} \longrightarrow \tilde{y} := f(\tilde{x}) \in \mathbb{R}$.

This a contradiction!

Hence $(p \wedge \sim q)$ is false, and we have proved that $p \Rightarrow q$. $\square$

We return to the proof with the knowledge that $f^* := \sup_{x \in C} f(x) < \infty$.

$\therefore \forall n \geq 1, \ \exists \ x_n \in C, \text{ s.t. } |f(x_n) - f^*| < 1/n$

$\therefore \ y_n := f(x_n) \longrightarrow f^*$.

Invoke C is compact to choose a point $\tilde{x} \in C$ and a subsequence (x_{n_i}) of (x_n) such that $x_{n_i} \xrightarrow[i \to \infty]{} \tilde{x}$.

Question $f(\tilde{x}) = f^*$?

$(y_{n_i} := f(x_{n_i}))$ is a subsequence of (y_n), and $y_n \longrightarrow f^*$.

Hence $y_{n_i} \longrightarrow f^*$

$y_{n_i} \longrightarrow f(\tilde{x})$.

Limits are unique, hence $f^* = f(\tilde{x})$ and we denote now, $x^* = \tilde{x}$. $\square$

Done with Real Analysis

Learn More: Math 451

If you really want to learn about infinite dimensional normed spaces, EECS 600 or a math course on "functional analysis"