ROB530 - HW4

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1.

(1.1)

The main difference between a right and left Invariant EKF is in their measurement coordinates. For left invariant EKF, the measurement coordinate is world coordinate, while the measurement coordinate of right invariant EKF is in body frame.

(1.2)

The lie algebra of $SO(3)$ is $so(3) = \left\{ \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \in \mathbb{R}^3,\ \omega^\wedge = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times3} \right\}$

The lie algebra of $SE(3)$ is $se(3) = \left\{ \xi = \begin{bmatrix} \rho \\ \phi \end{bmatrix} \in \mathbb{R}^6,\ \rho \in \mathbb{R}^3,\ \phi \in so(3),\ \xi^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ 0^T & 0 \end{bmatrix} \in \mathbb{R}^{4\times4} \right\}$

The physical meaning behind the dimension of each Lie algebra is the number of degrees of freedom of the group transformation. Since Lie algebra is a vector space generated by differentiating the group transformation along chosen directions in the space, at the identity transformation.

(1.3)

For $R \in SO(3)$: $R(t+1) = R(t) \cdot \exp(\omega^\wedge \Delta t)$, where $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$ is the vector of angular velocity

For $T \in SE(3)$: $T(t+1) = T(t) \cdot \exp(\xi^\wedge \Delta t)$, where $\xi = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is the vector of angular velocity and translation velocity

(1.4)

$$\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & -RR^T p + p \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} I & -p+p \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

2.

In this question, we need to transform an error from $se(3)$ to $\mathbb{R}^6$. This transformation is a non-linear transform so we can apply unscented transform to parametrically deal with the nonlinearity.

To simplify the task, I used two functions to denote the transform:

$$\xi \in se(3) \xmapsto{f} \eta \in SE(3) \xmapsto{g} e \in \mathbb{R}^6$$

(1) f is an exponential map from $se(3)$ to $SE(3)$

An element of $se(3)$ is then represented as below

$$\xi = (u \ \ w)^T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^6$$

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 & u_1 \\ \omega_3 & 0 & -\omega_1 & u_2 \\ -\omega_2 & \omega_1 & 0 & u_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in se(3)$$

$$\eta = f(\xi) = \exp(\xi) = \begin{bmatrix} R & Vu \\ 0 & 1 \end{bmatrix},$$

where $u, w \in \mathbb{R}^3$

$\theta = \sqrt{\omega^T \omega}$

$A = \sin\theta / \theta$

$B = (1 - \cos\theta) / \theta^2$

$C = (1 - A) / \theta^2$

$R = I + A\,\omega_x + B\,\omega_x^2$

$V = I + B\,\omega_x + C\,\omega_x^2$

(2) g is a transform function from rotation to euler angle

η can be expressed as $\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$g(\eta) = e = \begin{bmatrix} roll \\ pitch \\ yaw \\ x \\ y \\ z \end{bmatrix} \in \mathbb{R}^6$$

, where $x = t_1$

$y = t_2$

$z = t_3$

$roll = \arctan(r_{32}/r_{33})$

$pitch = \arctan(-r_{31}/\sqrt{1-r_{31}^2})$

$yaw = \arctan(r_{21}/r_{11})$

(3) The whole problem :

$$\begin{tikzcd} \xi \in se(3) \arrow[r, mapsto, "f"] \arrow[rr, bend right, "h"'] & \eta \in SE(3) \arrow[r, mapsto, "g"] & e \in \mathbb{R}^6 \end{tikzcd}$$

I use function h to denote the entire transformation

$$\mu_e = \sum_{i=0}^{2n} w_i \cdot h(x_i)$$

$$\Sigma_e = \sum_{i=0}^{2n} w_i \, (h(x_i) - \mu_e)(h(x_i) - \mu_e)^T, \text{ where } x_i = \begin{cases} \mu_\xi, & i=0 \\ \mu_\xi + l_i, & i = 1 \ldots n \\ \mu_\xi - l_{i-n}, & i = n+1 \ldots 2n \end{cases}$$

$$w_i = \begin{cases} \frac{k}{n+k}, & i=0 \\ \frac{1}{2(n+k)}, & i = 1 \ldots 2n \end{cases}$$

l_i is the i-th column of $\sqrt{(n+k)}\,L$ and $\Sigma_\xi = LL^T$ can be decomposed using Cholesky decomposition, n is the dimension of the state, k is a user-defined parameter

3. A.

We need to derive the equations of RIEKF before proceeding to the coding part. Assume reading from gyroscope and accelerometer at time k is $w_k \in \mathbb{R}^3$, $a_k \in \mathbb{R}^3$ respectively

- $$\frac{d}{dt}\bar{X}_t = f_{u_t}(\bar{X}_t) = \bar{X}_t(\tilde{w}_t - w_t^g)^\wedge$$

$$\Rightarrow \bar{X}_{k+1} = \bar{X}_k \exp(w_k^\wedge \Delta t)$$

☆ Also, this process satisfies the group affine property:

$$f_{u_t}(X_1 X_2) = X_1 X_2 u_t^\wedge$$

$$f_{u_t}(X_1) X_2 + X_1 f_{u_t}(X_2) - X_1 f_{u_t}(I) X_2$$

$$= X_1 u_t^\wedge X_2 + X_1 X_2 u_t^\wedge - X_1 u_t^\wedge X_2 = X_1 X_2 u_t^\wedge$$

- $$\frac{d}{dt} P_t^r = A_t^r P_t^r + P_t^r A_t^{r\,T} + Ad_{\bar{X}_t} Q_t Ad_{\bar{X}_t}^T$$

(1) For prediction:

$$\bar{X}_{k+1} = \bar{X}_k \exp(w_k^\wedge \Delta t)$$

$$P_{k+1} = \Phi P_k \Phi^T + Ad_{X_k} Q Ad_{X_k}^T, \quad \text{where } Ad_{X_k} = X_k$$

$$\Phi = \exp(A\Delta t) = \exp(0) = I$$

※ A is zero because:

$$\frac{d}{dt}\eta = g(\eta) = f(\eta) - \eta f(I) = \eta w^\wedge - \eta w^\wedge = 0$$

$$\Rightarrow \frac{d}{dt}\xi = 0 \quad \text{and} \quad A = 0$$

(2) For correction:

$$\bar{X}_k^+ = \exp\left(L_k (\bar{X}_k Y_k - b) \right) \bar{X}_k$$

$$\bar{P}_k^+ = (I - L_k H) P_k (I - L_k H)^T + L_k \bar{N} L_k^T$$

where

$$H \xi = \xi^\wedge b, \quad \text{let } \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$H \xi = \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -b_2 \xi_3 + b_3 \xi_2 \\ b_1 \xi_3 - b_2 \xi_1 \\ -b_1 \xi_2 + b_2 \xi_1 \end{bmatrix}$$

$$H \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} -b_2 \xi_3 + b_3 \xi_2 \\ b_1 \xi_3 - b_3 \xi_1 \\ -b_1 \xi_2 + b_2 \xi_1 \end{bmatrix} \Rightarrow H = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{bmatrix}$$

in our case, $b = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \Rightarrow H = \begin{bmatrix} 0 & g & 0 \\ -g & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\bar{N}_k = \bar{X}_k \operatorname{cov}(V_k) \bar{X}_k^T$$

$$S = H P_k H^T + \bar{N}_k$$

$$L_k = P_k H^T S^{-1}$$

B.