1.

ROB 530 - HW1 Kuan-Ting Lee (leekt)

A.

$$\mathcal{E}[X+Y+Z] = \int_X\int_Y\int_Z (x+y+z)\, p(x,y,z)\, dz\,dy\,dx$$
$$= \int_X\int_Y\int_Z x\, p(x,y,z)\, dz\,dy\,dx + \int_X\int_Y\int_Z y\, p(x,y,z)\, dz\,dy\,dx$$
$$+ \int_X\int_Y\int_Z z\, p(x,y,z)\, dz\,dy\,dx$$
$$= \mathcal{E}[X] + \mathcal{E}[Y] + \mathcal{E}[Z]. \quad \square$$

B.

A covariance matrix is symmetric and positive semi-definite

& implies non-negative eigenvalues

(a) Valid

(b) Invalid, $\because$ one of its eigenvalues is negative

(c) Invalid, $\because$ It's not symmetric

(d) Invalid, $\because$ one of its eigenvalues is negative

(e) Valid

C.

(a)

$$\sum x_r x_r^T = 4 \cdot \mathcal{E}[x_r x_r^T] = 4\left(\mathrm{Cov}[x_r] + \mathcal{E}[x_r]\mathcal{E}[x_r]^T\right)$$
$$= 4 \cdot \left(\begin{bmatrix} 5 & 3 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \end{bmatrix} \right)$$
$$= 4 \cdot \left(\begin{bmatrix} 5 & 3 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \right)$$
$$= 4 \begin{bmatrix} 9 & 9 \\ 9 & 16 \end{bmatrix}$$
$$= \begin{bmatrix} 36 & 36 \\ 36 & 64 \end{bmatrix} \quad \star$$

(b) $\sum x_m x_m^T = 6 \cdot \mathcal{E}\{X_m X_m^T\} = 6 \cdot (\mathrm{Cov}[X_m] + \mathcal{E}[X_m]\mathcal{E}[X_m]^T)$

$$= 6 \cdot \left(\begin{bmatrix} 8 & 4 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} \begin{bmatrix} -2 & 2 \end{bmatrix} \right)$$

$$= 6 \cdot \left(\begin{bmatrix} 8 & 4 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \right)$$

$$= 6 \begin{bmatrix} 12 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 72 & 0 \\ 0 & 42 \end{bmatrix} \#$$

(c) $\mu_{m+r} = (4\mu_r + 6\mu_m)/10$

$$= \left(4\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 6\begin{bmatrix} -2 \\ 2 \end{bmatrix}\right) / 10$$

$$= \begin{bmatrix} -4 \\ 24 \end{bmatrix} / 10$$

$$= \begin{bmatrix} -0.4 \\ 2.4 \end{bmatrix} \#$$

(d) $\Sigma_{m+r} = \mathrm{Cov}(X_{m+r}) = \mathcal{E}[X_{m+r} X_{m+r}^T] - \mathcal{E}[X_{m+r}]\mathcal{E}[X_{m+r}]^T$

$$= \left(\sum x_r x_r^T + \sum x_m x_m^T\right)/10 - \mu_{m+r}\mu_{m+r}^T$$

$$= \left(\begin{bmatrix} 36 & 36 \\ 36 & 64 \end{bmatrix} + \begin{bmatrix} 72 & 0 \\ 0 & 42 \end{bmatrix} \right)/10 - \begin{bmatrix} -0.4 \\ 2.4 \end{bmatrix} \begin{bmatrix} -0.4 & 2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 108 & 36 \\ 36 & 106 \end{bmatrix} / 10 - \begin{bmatrix} 0.16 & -0.96 \\ -0.96 & 5.76 \end{bmatrix}$$

$$= \begin{bmatrix} 10.8 & 3.6 \\ 3.6 & 10.6 \end{bmatrix} - \begin{bmatrix} 0.16 & -0.96 \\ -0.96 & 5.76 \end{bmatrix}$$

$$= \begin{bmatrix} 10.64 & 4.56 \\ 4.56 & 4.84 \end{bmatrix} \#$$

D.

(a) If A and B are independent

$P(A,B) = P(A)P(B)$

For A^c and B^c, $P(A^c, B^c) = P(A^c) - P(A^c, B)$

$$= P(A^c) - [P(B) - P(A,B)]$$

$$= P(A^c) - P(B) + P(A)P(B)$$
$$= 1 - P(A) - P(B) + P(A)P(B)$$
$$= (1-P(A))(1-P(B))$$
$$= P(A^c)\,P(B^c)$$

$\Rightarrow$ A^c and B^c are also independent events. $\square$

(b)

Suppose we have a Gaussian random vector $x \sim N(\mu, \Sigma)$

There is an affine transform $y = Ax + b$, the mean and covariance after the transform will be:

$$\mathcal{E}[y] = \mathcal{E}[Ax+b] = A\,\mathcal{E}[x] + b = A\mu + b$$

$$Cov[y] = Cov[Ax+b] = \mathcal{E}\left[(y-\mu_y)(y-\mu_y)^T\right]$$
$$= \mathcal{E}\left\{(Ax+b-A\mu-b)(Ax+b-A\mu-b)^T\right\}$$
$$= \mathcal{E}\left[A(x-\mu)(x-\mu)^T A^T\right]$$
$$= A\,\mathcal{E}\left[(x-\mu)(x-\mu)^T\right]A^T$$
$$= A\Sigma A^T$$

Since the Gaussian distribution is completely determined by the first two moments. y is a Gaussian random vector $(2\pi)^{-1}|\Sigma_y|^{-\frac{1}{2}}\exp(-\frac{1}{2}(x-\mu_y)^T\Sigma^{-1}(x-\mu_y))$

, where $\mu_y = A\mu + b$, $\Sigma_y = A\Sigma A^T$

$\boxed{2.}$

A.

C: have cancer, P: tested positive for cancer

$P(C) = 0.01$
$P(\sim C) = 0.99$
$P(P|\sim C) = 0.2$
$P(\sim P|\sim C) = 0.8$
$P(\sim P|C) = 0.1$
$P(P|C) = 0.9$

(1)

We want to derive $P(C|P) = P(P|C)\,P(C) / P(P)$

$$= \frac{P(P|C)\,P(C)}{P(P|C)\,P(C) + P(P|\sim C)\,P(\sim C)}$$

$$= \frac{0.9 \cdot 0.01}{0.9 \cdot 0.01 + 0.2 \cdot 0.99}$$

$$= 0.009 / 0.009 + 0.198$$

$$= 0.009 / 0.207$$

$$\approx 4.35\%$$

(2)

Define accuracy as $P(C|P)$, which we would like to increase

Assume false positive is reduced to d times

$P(P|\sim C) = 0.2d$
$P(\sim P|\sim C) = 1 - 0.2d$
$P(\sim P|C) = 0.1$
$P(P|C) = 0.9$

$$\Rightarrow P(C|P) = \frac{P(P|C)\,P(C)}{P(P|C)\,P(C) + P(P|\sim C)\,P(\sim C)}$$

$$= \frac{0.9 \cdot 0.01}{0.9 \cdot 0.01 + 0.2d \cdot 0.99}$$

$$= \frac{0.009}{0.009 + 0.198d} = 4.39\%$$

(if $d = 0.99$)

Assume false negative is reduced to d times

$$p(p|\sim c) = 0.2$$
$$p(\sim p|\sim c) = 0.8$$
$$p(\sim p|c) = 0.1d$$
$$p(p|c) = 1-0.1d$$

$$\Rightarrow p(c|p) = \frac{p(p|c)\,p(c)}{p(p|c)\,p(c) + p(p|\sim c)\,p(\sim c)}$$
$$= \frac{(1-0.1d)\,0.01}{(1-0.1d)\,0.01 + 0.2\cdot 0.99}$$
$$= \frac{0.01 - 0.001d}{0.01 - 0.001d + 0.198}$$
$$= 4.35\%\ (\text{if } d = 0.99)$$

From the above calculation, we know that reducing false positive can increase accuracy more.

B.

We want to derive $P(x_{t+1} = \text{blank} \mid u_{t+1} = \text{paint}, z_{t+1} = \text{colored})$

To simplify the derivation, I use $bel(x_t)$ to denote $p(x_t \mid z_{1:t}, u_{1:t})$

$\overline{bel}(x_t)$ to denote $p(x_t \mid z_{1:t-1}, u_{1:t})$

Since we have no knowledge about the current state of the object,

$$bel(x_t = \text{blank}) = bel(x_t = \text{colored}) = 0.5$$

$$\overline{bel}(x_{t+1} = \text{blank}) = P(x_{t+1} = \text{blank} \mid u_{t+1} = \text{paint}, x_t = \text{blank})\, bel(x_t = \text{blank})$$
$$+ P(x_{t+1} = \text{blank} \mid u_{t+1} = \text{paint}, x_t = \text{colored})\, bel(x_t = \text{colored})$$
$$= 0.1\cdot 0.5 + 0\cdot 0.5$$
$$= 0.05$$

$$\overline{bel}(x_{t+1} = \text{colored}) = 1 - \overline{bel}(x_{t+1} = \text{blank}) = 1 - 0.05 = 0.95$$

$$bel(x_{t+1} = \text{blank}) = \eta\, P(z_{t+1} = \text{colored} \mid x_{t+1} = \text{blank})\, \overline{bel}(x_{t+1} = \text{blank})$$
$$= \eta \cdot 0.2 \cdot 0.05 = 0.01\eta$$

$$\text{bel}(X_{t+1} = \text{colored}) = \eta \, p(z_{t+1} = \text{colored} \mid x_{t+1} = \text{colored}) \, \overline{\text{bel}}(x_{t+1} = \text{colored})$$

$$= \eta \cdot 0.7 \cdot 0.95$$

$$= 0.665 \, \eta$$

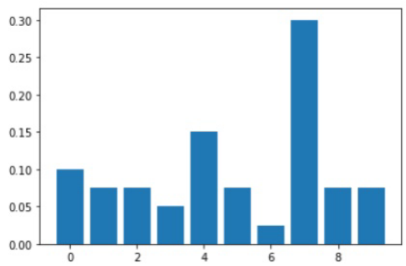
$$\therefore \; 0.01\,\eta + 0.665\,\eta = 1 \Rightarrow \eta = 1/0.675 \approx 1.48$$

$$\Rightarrow \text{bel}(X_{t+1} = \text{blank}) = 0.01 \cdot \eta = 0.0148 \approx 1.48\,\%$$

C.

Probability of robot position after execution

p



resulting belief :

$p(X=0) = 0.1$

$p(X=1) = 0.075$

$p(X=2) = 0.075$

$p(X=3) = 0.05$

$p(X=4) = 0.15$

$p(X=5) = 0.075$

$p(X=6) = 0.025$

$p(X=7) = 0.3$

$p(X=8) = 0.075$

$p(X=9) = 0.075$

$\boxed{3.}$

$P(A) = 0.3, \quad P(B) = 0.6, \quad P(C) = 0.1$

$P(G|A) = 0.7, \quad P(G|B) = 0.25, \quad P(G|C) = 0.05$

We want to get $P(A/G)$, $P(B/G)$, $P(C/G)$

$$\Rightarrow P(A|G) = \frac{P(G|A)\,P(A)}{P(G)} = \frac{0.7 \cdot 0.3}{P(G)} = \frac{0.21}{P(G)}$$

$$P(B|G) = \frac{P(G|B)\,P(B)}{P(G)} = \frac{0.25 \cdot 0.6}{P(G)} = \frac{0.15}{P(G)}$$

$$P(C|G) = \frac{P(G|C)\,P(C)}{P(G)} = \frac{0.05 \cdot 0.1}{P(G)} = \frac{0.005}{P(G)}$$

$$\because P(A|G) + P(B|G) + P(C|G) = 1 = \frac{0.365}{P(G)}$$

$$\Rightarrow P(G) = 0.365$$

$$\Rightarrow P(A|G) = 0.21/0.365 = 0.5753$$

$$P(B|G) = 0.15/0.365 = 0.4110$$

$$P(C|G) = 0.005/0.365 = 0.0137$$

#

$\boxed{4.}$

(1) F

(2) T

(3) F

(4) F

$\boxed{5.}$

(a) $\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}$ No Cholesky, $\because$ it has negative eigenvalue

(b) $\begin{bmatrix} 25 & -10 \\ -10 & 40 \end{bmatrix}$ has Cholesky $\begin{bmatrix} L_{00} & 0 \\ L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_{00} & L_{10} \\ 0 & L_{11} \end{bmatrix}$

$$= \begin{bmatrix} L_{00}^2 & L_{00}L_{10} \\ L_{10}L_{00} & L_{10}^2 + L_{11}^2 \end{bmatrix}$$

$$\Rightarrow L_{00} = 5,\ L_{10} = -2,\ L_{11} = 6$$

$$\Rightarrow L = \begin{bmatrix} 5 & 0 \\ -2 & 6 \end{bmatrix} \#$$

(c) $\begin{bmatrix} 3 & -6 \\ -4 & 6 \end{bmatrix}$ No Cholesky, $\because$ it's not symmetric

(d) $\begin{bmatrix} 9 & -3 \\ -3 & 5 \end{bmatrix}$ has Cholesky. $\begin{bmatrix} L_{00}^2 & L_{00}L_{10} \\ L_{10}L_{00} & L_{10}^2+L_{11}^2 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -3 & 5 \end{bmatrix}$

$\Rightarrow L_{00} = 3, \quad L_{10} = -1, \quad L_{11} = 2$

$\Rightarrow L = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$ ※

$\boxed{6.}$

$z \sim N(\mu_z, \sigma_z^2)$

$y = az + b,$

$\mu_y = \mathcal{E}[y] = \mathcal{E}[az+b] = a\,\mathcal{E}[z] + b = a\mu_z + b$

$$
\begin{aligned}
Cov(y) = Cov(az+b) &= \mathcal{E}[(y-\mu_y)^2] \\
&= \mathcal{E}[((az+b-(a\mu_z+b))^2] \\
&= \mathcal{E}[(a(z-\mu_z))^2] \\
&= a^2\,\mathcal{E}[(z-\mu_z)^2] \\
&= a^2\sigma_z^2
\end{aligned}
$$

$\Rightarrow y \sim N(\mu_y, \sigma_y^2) = N(a\mu_z+b, a^2\sigma_z^2)$

This is an exact model since we can express this in closed form.