Math 311

Numerical Methods

4.2: Richardson's Extrapolation

High Accuracy from Low Accuracy

S. K. Hyde

Burden and Faires, any ed.

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Introduction

Richardson's Extrapolation

Richardson's Extrapolation is a method to improve upon an approximation, $A(h)$, of an exact quantity, A.

It uses low-order formulas to generate results of high accuracy! It's almost magic! This method applies to any approx. for which the error structure is predictable

- For example, by Taylor's theorem, we can expand $f(x_0 + h)$ like this:

$$\boldsymbol{f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x)}{2}h^2 + \frac{f^{(3)}(x)}{3!}h^3 + \frac{f^{(4)}(\xi)}{4!}h^4} \quad (1)$$

- Let's solve (1) for $f'(x_0)$. We then will get the following

- Let's solve (1) for $f'(x_0)$. We then will get

$$\boxed{f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{f''(x)}{2}h - \frac{f^{(3)}(x)}{3!}h^2 - \frac{f^{(4)}(\xi)}{4!}h^3}$$

- Let's generalize the coefficients on the right to simply B, C, D, etc.

$$\begin{aligned} f'(x_0) &= \frac{f(x_0+h) - f(x_0)}{h} + Bh + Ch^2 + Dh^3 \\ A &= A(h) + E(h) \end{aligned} \tag{2}$$

- In this example, we are approximating $f'(x_0)$ with $\dfrac{f(x_0+h) - f(x_0)}{h}$.
- Note that $f(x_0+h)$ can be expanded in an infinite series, in which case, the more error structure for $A(h)$ in (2) would be

$$E(h) = Bh + Ch^2 + Dh^3 + Eh^4 + Fh^5 + Gh^6 + \cdots$$

- Sometimes only even terms (or odd terms) appear.

Error term – regular and irregular structures

- There are many different error structures.
- The ones that are predictable (constant spacing between powers of h) you can apply Richardson's Expratolation multiple times.
- However, the error terms must be known because each leads to a different Richardson's Extrapolation formula.
- This is the reason that there are two different formulas for Richardson's Extrapolation in our textbook. (Equation 4.13 and the last one on the same page).
- Note the different power in the denominator!

$$N_{j+1}(h) = N_j\left(\frac{h}{2}\right) + \frac{N_j(\frac{h}{2}) - N_j(h)}{2^j - 1} \quad \Bigg| \quad N_{j+1}(h) = N_j\left(\frac{h}{2}\right) + \frac{N_j(\frac{h}{2}) - N_j(h)}{4^j - 1}$$

- In general, the structure of $E(h)$ varies depending on what is being estimated. Here are several examples of possibilities of regular structure:

Regular Error Structure

$$E(h) = Bh + Ch^2 + Dh^3 + Eh^4 + \cdots \quad \text{(spacing of powers of } h \text{ is 1)} \tag{3}$$
$$E(h^2) = Bh^2 + Ch^4 + Dh^6 + Eh^8 + \cdots \quad \text{(spacing of powers of } h \text{ is 2)} \tag{4}$$
$$E(h^3) = Bh^3 + Ch^6 + Dh^9 + Eh^{12} + \cdots \quad \text{(spacing of powers of } h \text{ is 3)} \tag{5}$$
$$E(h^p) = Bh^p + Ch^{2p} + Dh^{3p} + Eh^{4p} + \cdots \quad \text{(spacing of powers of } h \text{ is } p) \tag{6}$$

- Richardson's Extrapolation works best when we have a regular difference that the powers of h follow (like multiples of 1, 2, or 3) as illustrated above.
- The error structure may not have a regular difference, however, it works with all the cases, as long as you know the powers of h of in the error term of what is being estimated, and apply the correct formula at the right time. We really need to know the entire sequence of error terms to make good use of this.
- The main cases are when we have spacing of 1 or 2 (for example, every term in a Taylor Series, or a Taylor Series with only even (or odd) terms.

Apply Richardson's with only one term in the Error

- To introduce Richardson's Extrapolation, we will illustrate one step of it.
- However, Richardson's real power is found by applying it several times, when the difference between powers of h in the error structure are constant.

Motivation for Richardson's Extrapolation

The goal of Richardson's Extrapolation is to eliminate the B coefficient in:

$$A = A(h) + Bh^p \tag{7}$$

- First, create both a fine (and coarse) approximation of A.
- Combine the finer and coarse approximations in such a way to eliminate the B term.
- What will be left is a formula with a better error term! (at least $O(h^{p+1})$)
- We will do it from the point of view of coarser approximations, which reduces the use of fractions (both ways yield equivalent answers).
- How to switch to the finer notation, follow equation (20).

- Let's start with only one term in the error structure.

Method 1: Substitution

$$A = A(h) + Bh^p \tag{8}$$

- A more coarse approximation to A from equation (8) with double the step size is

$$A = A(2h) + B(2h)^p$$
$$A = A(2h) + B \cdot 2^p h^p \tag{9}$$

- We can set both (8) and (9) equal to each other (since they are both A).

$$A(h) + Bh^p = A(2h) + B \cdot 2^p h^p$$

- Now solve this equation for B. Here we go!

$$\begin{aligned} A(h) - A(2h) &= B \cdot 2^p h^p - Bh^p \\ A(h) - A(2h) &= Bh^p(2^p - 1) \end{aligned} \quad \Longrightarrow \quad B = \frac{A(h) - A(2h)}{h^p(2^p - 1)}$$

- Thus, it follows that a better approximation to A is equation (8), with the B update:

$$A = A(h) + \frac{A(h) - A(2h)}{\cancel{h^p}(2^p - 1)}\cancel{h^p} \quad \Longrightarrow \quad A = A(h) + \frac{A(h) - A(2h)}{2^p - 1} \tag{10}$$

- Note that this eliminates the Bh^p term in the approximation equation. There will be an error term remaining, but this method doesn't allow us to find that term.
- To do so, we use elimination to find B. Here's an example of one step:

Method 2: Elimination

- Elimination is the main method used for several terms.
- Using (8) and (9), we can eliminate the $\boldsymbol{B}$ term.
- It follows that 2^p times (8) minus (9) yields:

$$\begin{array}{rcl} \mathbf{2}^p \boldsymbol{A} & = & \mathbf{2}^p \boldsymbol{A(h)} + \boldsymbol{B}\cdot\mathbf{2}^p \boldsymbol{h}^p \\ -\boldsymbol{A} & = & -\boldsymbol{A(2h)} - \boldsymbol{B}\cdot\mathbf{2}^p \boldsymbol{h}^p \\ \hline (\mathbf{2}^p - 1)\boldsymbol{A} & = & \mathbf{2}^p \boldsymbol{A(h)} - \boldsymbol{A(2h)} \end{array} \Rightarrow \boxed{\boldsymbol{A} = \frac{\mathbf{2}^p \boldsymbol{A(h)} - \boldsymbol{A(2h)}}{\mathbf{2}^p - 1}}$$

- Employing an add zero trick we can show this is equal to equation (10) from the previous section.

$$\boldsymbol{A} = \frac{(\mathbf{2}^p - 1 + 1)\boldsymbol{A(h)} - \boldsymbol{A(2h)}}{\mathbf{2}^p - 1} = \frac{(\mathbf{2}^p - 1)\boldsymbol{A(h)}}{\mathbf{2}^p - 1} + \frac{1\boldsymbol{A(h)} - \boldsymbol{A(2h)}}{\mathbf{2}^p - 1}$$

$$\boxed{\boldsymbol{A} = \boldsymbol{A(h)} + \frac{\boldsymbol{A(h)} - \boldsymbol{A(2h)}}{\mathbf{2}^p - 1}} \tag{11}$$

- Note that equation (11) is the preferred form as it shows that the approximation $\boldsymbol{A(h)}$ has been "improved" by the difference on the right.

- Now time for an example. Let's improve upon the First Order Derivative formula.

Improve the First Order Derivative formula

$$f'(x_0) = \frac{f(x_0 + \boldsymbol{h}) - f(x_0)}{h} - \frac{f''(\xi)}{2}\boldsymbol{h}$$
$$\boldsymbol{A} = \boldsymbol{A(h)} + E(\boldsymbol{h})$$

- It follows that a better approximation to $f'(x_0)$ is

$$\boldsymbol{A} = \boldsymbol{A(h)} + \frac{\boldsymbol{A(h)} - \boldsymbol{A(2h)}}{2^1 - 1}$$

$$\boldsymbol{A} = \frac{\boldsymbol{f(x_0 + h) - f(x_0)}}{\boldsymbol{h}} + \frac{1}{2^1 - 1}\left[\frac{\boldsymbol{f(x_0 + h) - f(x_0)}}{\boldsymbol{h}} - \frac{\boldsymbol{f(x_0 + 2h) - f(x_0)}}{\boldsymbol{2h}}\right]$$
$$= \frac{\boldsymbol{f(x_0 + h) - f(x_0)}}{\boldsymbol{h}} + \frac{\boldsymbol{f(x_0 + h) - f(x_0)}}{\boldsymbol{h}} - \frac{\boldsymbol{f(x_0 + 2h) - f(x_0)}}{\boldsymbol{2h}}$$
$$= \frac{1}{2h}\left[\boldsymbol{4f(x_0 + h) - 4f(x_0)} - \boldsymbol{f(x_0 + 2h) + f(x_0)}\right]$$

$$\boxed{f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + \boldsymbol{4f(x_0 + h)} - \boldsymbol{f(x_0 + 2h)}\right]} \qquad (12)$$

- Note that (12) is one of the three-point rules that we derived using Lagrange polynomials in the previous sections!
- Its error then was derived to be $\mathcal{O}(\boldsymbol{h}^2)$, which shows that the error has improved from $\mathcal{O}(\boldsymbol{h})$ to $\mathcal{O}(\boldsymbol{h}^2)$.

Improve the Central Second Order Derivative

- From Taylor series, it can be shown that

$$
\begin{aligned}
f'(x_0) &= \frac{f(x_0+h)-f(x_0-h)}{2h} \underbrace{-\frac{f^{(3)}(\xi)}{6}h^2}_{\mathcal{O}(h^2)} \\
A &= \qquad A(h) \qquad + E(h^2)
\end{aligned}
$$

- Note that we are not sure the structure that the error takes in this formula, but we do know that the first term in it is h^2.
- In section , we learn how to deal with a regular structure in a recursive manner. We'll talk about it later.
- To simplify the notation for this problem, let $y_k = f(x_0 + kh)$. Thus, the equation becomes

$$y'_0 = \frac{y_1 - y_{-1}}{2h} + \mathcal{O}(h^2)$$

- For this problem, $p = 2$.

- Equation (10) leads to

$$\begin{aligned}
\boldsymbol{A} &= \boldsymbol{A(h)} + \frac{\boldsymbol{A(h)} - \boldsymbol{A(2h)}}{2^2 - 1} = \frac{1}{3}\left[4\boldsymbol{A(h)} - \boldsymbol{A(2h)}\right] \\
&= \frac{1}{3}\left(4\left(\frac{\boldsymbol{y_1 - y_{-1}}}{\boldsymbol{2h}}\right) - \frac{\boldsymbol{y_2 - y_{-2}}}{\boldsymbol{4h}}\right) = \frac{1}{3}\left[\frac{1}{4\boldsymbol{h}}\left[\boldsymbol{8y_1} - \boldsymbol{8y_{-1}} - \boldsymbol{y_2} + \boldsymbol{y_{-2}}\right]\right] \\
&= \frac{1}{12\boldsymbol{h}}\left[y_{-2} - 8y_{-1} + 8y_1 - y_2\right]
\end{aligned}$$

- Note that this is the same as the central five point approximations to the first derivative we derived in the last section.
- In that section, the error of this formula is $\mathcal{O}\left(\boldsymbol{h}^4\right)$.
- Note for this one, we gained **TWO** orders of accuracy.
- This leads us to believe that the central difference first order approximation follows that error structure in equation (4) (but it might still be irregular).

Improve upon Trapezoid Rule

- Richardson's can also be apply to integration rules (which we will learn in the next section).
- We will learned that the Composite Trapezoid Rule can be written as (where $a = x_0$ and $b = x_n$, with a separation of $\boldsymbol{h} = \frac{b-a}{n}$) gives

$$\int_a^b f(x)dx = h\left[\frac{f(a)+f(b)}{2} + \sum_{k=1}^{n-1} f(x_0+kh)\right] + \mathcal{O}\left(\boldsymbol{h}^2\right)$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{A(h)} \quad \boldsymbol{+E(h^2)}$$

- Using a step size of $\boldsymbol{h}$ with 3 points gives the trapezoid rule:

$$\boldsymbol{A(h) = \frac{h}{2}f(x_0) + hf(x_1) + \frac{h}{2}f(x_2)}$$

- Using a more coarse approximation (only two points) with a step size of $2h$ gives

$$\boldsymbol{A(2h) = \frac{(2h)}{2}f(x_0) + \frac{(2h)}{2}f(x_2) = hf(x_0) + hf(x_2)}$$

- A better approximation to A using (10) ($p = 2$) gives

$$
\begin{aligned}
A &= A(h) + \frac{A(h) - A(2h)}{2^2 - 1} = \frac{4A(h) - A(2h)}{3} = \frac{1}{3}[4A(h) - A(2h)] \\
&= \frac{1}{3}\left(4\left[\frac{h}{2}f(x_0) + hf(x_1) + \frac{h}{2}f(x_2)\right] - (hf(x_0) + hf(x_2))\right) \\
&= \frac{h}{3}(2f(x_0) + 4f(x_1) + 2f(x_2) - f(x_0) - f(x_2))
\end{aligned}
$$

$$\boxed{A = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]}$$

- Note that this is Simpson's Rule!!
- We can apply this method to Simpson's Rule as well!

Improve upon Simpson's Rule

- Let's further apply the rule to the Composite Simpson's Rule.
- Note in this case, $\boldsymbol{n}$ must be an even integer.
- It can be written as (where $a = x_0$ and $b = x_n$, with a separation of $h = \frac{b-a}{n}$) gives

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + f(b) + 2\sum_{k=1}^{n/2-1} f(x_{2k}) + 4\sum_{k=1}^{n/2} f(x_{2k-1})\right] + \mathcal{O}\left(h^4\right)$$

$$\boldsymbol{A} \quad = \quad \boldsymbol{A(h)} \quad \boldsymbol{+\mathcal{O}\left(h^4\right)}$$

- Using a step size of $\boldsymbol{h}$ with only five points ($\boldsymbol{n = 4}$) gives Simpson's Rule rule:

$$\boldsymbol{A(h) = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]}$$

- Using a more coarse approximation (only three points) with a step size of $2h$ gives

$$\boldsymbol{A(2h) = \frac{(2h)}{3}\left[f(x_0) + 4f(x_2) + f(x_4)\right]}$$

- A better approximation to $\boldsymbol{A}$ using (10) ($\boldsymbol{p = 4}$) gives

- A better approximation to $\boldsymbol{A}$ using (10) ($\boldsymbol{p = 4}$) gives

$$\boldsymbol{A} = \boldsymbol{A(h)} + \frac{\boldsymbol{A(h)} - \boldsymbol{A(2h)}}{2^4 - 1} = \frac{2^4\boldsymbol{A(h)} - \boldsymbol{A(2h)}}{2^4 - 1} = \frac{1}{15}[16\boldsymbol{A(h)} - \boldsymbol{A(2h)}]$$

$$= \frac{1}{15}\left[16 \cdot \frac{\boldsymbol{h}}{\boldsymbol{3}}[\boldsymbol{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)}] - \frac{\boldsymbol{(2h)}}{\boldsymbol{3}}[\boldsymbol{f(x_0) + 4f(x_2) + f(x_4)}]\right]$$

$$= \frac{2h}{45}\Big(8[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] - [f(x_0) + 4f(x_2) + f(x_4)]\Big)$$

$$\boxed{\boldsymbol{A} = \frac{2h}{45}\big(7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\big)}$$

- Note that this is **Boole's Rule**!!

- Somehow, when we used Richardson's Approximation to improve Simpson's Rule, we skipped Simpson's three-eighth's Rule, and went directly to **Boole's Rule**!

- Sometimes this happens with Richardson's Extrapolation!

Apply Richardson's with regular error structure

- In this section, Richardson's will be applied to structures like (3), (4), (5), and more generally to (6).
- In this case, we can write the procedure recursively!
- The method will be applied three times, and then generalized in a table format.
- We will show it with an error structure of four terms:

$$E(h) = Bh^p + Ch^{2p} + Dh^{3p} + Eh^{4p}.$$

- Therefore, It follows that

$$A = A(h) + Bh^p + Ch^{2p} + Dh^{3p} + Eh^{4p} \tag{13}$$

- Remember, the method of Richardson's Extrapolation is to eliminate the B, C, D, E coefficients in two steps:
  1. first, get a finer (or more coarse) approximation of A.
  2. Then combine the two approximations in such a way to eliminate the B term.

- It will be derived from the point of view of coarser approximations rather than finer approximations. This is only done to reduce the use of fractions. It will also be shown how it is equivalent to the Burden and Faires version.

Step 1

- A more coarse approximation to $\boldsymbol{A}$ from equation (13) with double the step size is

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{A(2h)} + \boldsymbol{B}(2h)^{p} + \boldsymbol{C}(2h)^{2p} + \boldsymbol{D}(2h)^{3p} + \boldsymbol{E}(2h)^{4p} \\ \boldsymbol{A} &= \boldsymbol{A(2h)} + \boldsymbol{B}\cdot 2^{p}\boldsymbol{h}^{p} + \boldsymbol{C}\cdot 2^{2p}\boldsymbol{h}^{2p} + \boldsymbol{D}\cdot 2^{3p}\boldsymbol{h}^{3p} + \boldsymbol{E}\cdot 2^{4p}\boldsymbol{h}^{4p} \end{aligned} \tag{14}$$

- Using (13) and (14), we can eliminate the $\boldsymbol{B}$ term. It follows that 2^p times (13) minus (14) yields:

$$\begin{array}{rlllll} 2^{p}\boldsymbol{A} = & 2^{p}\boldsymbol{A(h)} & +\boldsymbol{B}\cdot 2^{p}\boldsymbol{h}^{p} + & \boldsymbol{C}\cdot 2^{p}\boldsymbol{h}^{2p} & + \quad \boldsymbol{D}\cdot 2^{p}\boldsymbol{h}^{3p} & + \quad \boldsymbol{E}\cdot 2^{p}\boldsymbol{h}^{4p} \\ -\boldsymbol{A} = & -\boldsymbol{A(2h)} & -\boldsymbol{B}\cdot 2^{p}\boldsymbol{h}^{p} - & \boldsymbol{C}\cdot 2^{2p}\boldsymbol{h}^{2p} & - \quad \boldsymbol{D}\cdot 2^{3p}\boldsymbol{h}^{3p} & - \quad \boldsymbol{E}\cdot 2^{4p}\boldsymbol{h}^{4p} \\ \hline (2^{p}-1)\boldsymbol{A} = & 2^{p}\boldsymbol{A(h)} - \boldsymbol{A(2h)} & & +\boldsymbol{C}(2^{p}-2^{2p})\boldsymbol{h}^{2p} & +\boldsymbol{D}(2^{p}-2^{3p})\boldsymbol{h}^{3p} & +\boldsymbol{E}(2^{p}-2^{4p})\boldsymbol{h}^{4p} \end{array}$$

$$\boldsymbol{A} = \frac{2^{p}\boldsymbol{A(h)} - \boldsymbol{A(2h)}}{2^{p}-1} - \boldsymbol{C}\frac{2^{p}-2^{2p}}{2^{p}-1}\boldsymbol{h}^{2p} - \boldsymbol{D}\frac{2^{p}-2^{3p}}{2^{p}-1}\boldsymbol{h}^{3p} - \boldsymbol{E}\frac{2^{p}-2^{4p}}{2^{p}-1}\boldsymbol{h}^{4p}$$

- Note that we have improved the approximation of $\boldsymbol{A}$ with this new formula from $\mathcal{O}(\boldsymbol{h}^p)$ to $\mathcal{O}\left(\boldsymbol{h}^{2p}\right)$:

$$\boldsymbol{A} = \frac{2^p \boldsymbol{A(h)} - \boldsymbol{A(2h)}}{2^p - 1} + \mathcal{O}\left(\boldsymbol{h}^{2p}\right) = \boldsymbol{A(h)} + \frac{\boldsymbol{A(h)} - \boldsymbol{A(2h)}}{2^p - 1} + \mathcal{O}\left(\boldsymbol{h}^{2p}\right)$$

- After staring at this for a little bit of time (and getting a headache), our epiphany is to simply rename the components with a subscript, then repeat the procedure:

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{A(h)} + \frac{\boldsymbol{A(h)} - \boldsymbol{A(2h)}}{2^p - 1} - \underbrace{\boldsymbol{C}\frac{2^p - 2^{2p}}{2^p - 1}}_{\boldsymbol{C}_1} \boldsymbol{h}^{2p} - \underbrace{\boldsymbol{D}\frac{2^p - 2^{3p}}{2^p - 1}}_{\boldsymbol{D}_1} \boldsymbol{h}^{3p} - \underbrace{\boldsymbol{E}\frac{2^p - 2^{4p}}{2^p - 1}}_{\boldsymbol{E}_1} \boldsymbol{h}^{4p} \\ \boldsymbol{A} &= \boldsymbol{A_2(h)} + \boldsymbol{C}_1 \boldsymbol{h}^{2p} + \boldsymbol{D}_1 \boldsymbol{h}^{3p} + \boldsymbol{E}_1 \boldsymbol{h}^{4p} \end{aligned} \tag{15}$$

- We can now employ the same method again to now eliminate the $\boldsymbol{C}_1$ term:

Step 2

- A more coarse approximation to $\boldsymbol{A}$ from equation (15) with double the step size is

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{A_2(2h)} + \boldsymbol{C}_1(2h)^{2p} + \boldsymbol{D}_1(2h)^{3p} + \boldsymbol{E}_1(2h)^{4p} \\ \boldsymbol{A} &= \boldsymbol{A_2(2h)} + \boldsymbol{C}_1 \cdot 2^{2p} \boldsymbol{h}^{2p} + \boldsymbol{D}_1 \cdot 2^{3p} \boldsymbol{h}^{3p} + \boldsymbol{E}_1 \cdot 2^{4p} \boldsymbol{h}^{4p} \end{aligned} \tag{16}$$

- Using (15) and (16), we can eliminate the C_1 term.
- It follows that 2^{2p} timefs (15) minus (16) yields:

$$\begin{array}{rlllll} 2^{2p}A = & 2^{2p}A_2(h) & +C_1\cdot 2^{2p}h^{2p} + & D_1\cdot 2^{2p}h^{3p} & + & E_1\cdot 2^{2p}h^{4p} \\ -A = & -A_2(2h) & -C_1\cdot 2^{2p}h^{2p} - & D_1\cdot 2^{3p}h^{3p} & - & E_1\cdot 2^{4p}h^{4p} \\ \hline (2^{2p}-1)A = & 2^{2p}A_2(h) - A_2(2h) & & +D_1(2^{2p}-2^{3p})h^{3p} & + & E_1(2^{2p}-2^{4p})h^{4p} \end{array}$$

$$A = \frac{2^{2p}A_2(h) - A_2(2h)}{2^{2p}-1} - D_1\frac{2^{2p}-2^{3p}}{2^{2p}-1}h^{3p} - E_1\frac{2^{2p}-2^{4p}}{2^{2p}-1}h^{4p}$$

- We have again improved the error in the approximation of A! from $\mathcal{O}\left(h^{2p}\right)$ to $\mathcal{O}\left(h^{3p}\right)$:

$$A = \frac{2^{2p}A_2(h) - A_2(2h)}{2^{2p}-1} + \mathcal{O}\left(h^{3p}\right) = A_2(h) + \frac{A_2(h) - A_2(2h)}{2^{2p}-1} + \mathcal{O}\left(h^{3p}\right)$$

- Let $A_3(h) = A_2(h) + \dfrac{A_2(h) - A_2(2h)}{2^{2p}-1}$. Renaming the coefficients again leads to the next step:

$$A = A_3(h) + D_2h^{3p} + E_2h^{4p} \tag{17}$$

Step 3

- Let's do it again!
- A more coarse approximation to $\boldsymbol{A}$ from equation (17) with double the step size is

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{A_3(2h)} + \ \boldsymbol{D}_2(2h)^{3p} + \ \boldsymbol{E}_2(2h)^{4p} \\ \boldsymbol{A} &= \boldsymbol{A_3(2h)} + \boldsymbol{D}_2 \cdot 2^{3p}\boldsymbol{h}^{3p} + \boldsymbol{E}_2 \cdot 2^{4p}\boldsymbol{h}^{4p} \end{aligned} \tag{18}$$

- Using (17) and (18), we can eliminate the $\boldsymbol{D}_2$ term.
- It follows that 2^{3p} times (17) minus (18) yields:

$$\begin{array}{rcllll} 2^{3p}\boldsymbol{A} &=& 2^{3p}\boldsymbol{A_3(h)} & + \ \boldsymbol{D}_1 \cdot 2^{3p}\boldsymbol{h}^{3p} & + & \boldsymbol{E}_1 \cdot 2^{3p}\boldsymbol{h}^{4p} \\ -\boldsymbol{A} &=& -\boldsymbol{A_3(2h)} & - \ \boldsymbol{D}_1 \cdot 2^{3p}\boldsymbol{h}^{3p} & - & \boldsymbol{E}_1 \cdot 2^{4p}\boldsymbol{h}^{4p} \\ \hline (2^{3p}-1)\boldsymbol{A} &=& 2^{3p}\boldsymbol{A_3(h)} - \boldsymbol{A_3(2h)} & & + & \boldsymbol{E}_1(2^{3p}-2^{4p})\boldsymbol{h}^{4p} \\ & & & & & \\ \boldsymbol{A} &=& \dfrac{2^{3p}\boldsymbol{A_3(h)} - \boldsymbol{A_3(2h)}}{2^{3p}-1} & & - & \boldsymbol{E}_1 \dfrac{2^{3p}-2^{4p}}{2^{3p}-1}\boldsymbol{h}^{4p} \end{array}$$

- We have improved the error in the approximation of $\boldsymbol{A}$ from $\mathcal{O}\left(\boldsymbol{h}^{3p}\right)$ to $\mathcal{O}\left(\boldsymbol{h}^{4p}\right)$:

$$
\begin{aligned}
A &= \frac{2^{3p}A_3(h) - A_3(2h)}{2^{3p}-1} + \mathcal{O}(h^{4p}) \\
A &= A_3(h) + \frac{A_3(h) - A_3(2h)}{2^{3p}-1} + \mathcal{O}(h^{4p}) \\
A &= A_4(h) + \mathcal{O}(h^{4p})
\end{aligned}
$$

- It follows that this process can be repeated many times (however, truncation error overtakes the advantages over time).
- We have the following formulas so far:

$$
\begin{aligned}
A_2(h) &= A(h) + \frac{A(h) - A(2h)}{2^p - 1} \\
A_3(h) &= A_2(h) + \frac{A_2(h) - A_2(2h)}{2^{2p} - 1} \\
A_4(h) &= A_3(h) + \frac{A_3(h) - A_3(2h)}{2^{3p} - 1}
\end{aligned}
$$

- If we equate $\boldsymbol{A_1(h) = A(h)}$, then for all integers $j \geqslant 1$ then we have the recursive equation

$$\boldsymbol{A} = \boldsymbol{A_{j+1}(h)} + \mathcal{O}\left(\boldsymbol{h}^{(j+1)p}\right),$$

$$\boldsymbol{A_{j+1}(h)} = \boldsymbol{A_j(h)} + \frac{\boldsymbol{A_j(h)} - \boldsymbol{A_j(2h)}}{\mathbf{2}^{jp} - 1} \tag{19}$$

- Last, it is typical to arrange the $\boldsymbol{h}$ values in decreasing fashion (finer) rather than increasing (coarser). Simply replace $\boldsymbol{h}$ by $\boldsymbol{h}/2$ on the right side only of (19):

$$\boxed{A_{j+1}(h) = A_j\left(\frac{h}{2}\right) + \frac{A_j\left(\frac{h}{2}\right) - A_j(h)}{\mathbf{2}^{jp} - 1} \tag{20}}$$

- The only difference bewteen this version and Burden Faires is that I'm using $\boldsymbol{A}$ instead of N.

Examples

Forward Difference Derivative Approximation

- In this section, the process will be applied to the derivative formula derived in (2).
- It follows that error structure in equation (3), so $\boldsymbol{p = 1}$.
- Given $\boldsymbol{f(x) = \dfrac{\sin x}{x}}$, we will find the derivative at the point $\dfrac{\pi}{4}$.
- Its exact value is

$$f'(\pi/4) = \frac{\frac{\pi}{4}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}}{\left(\frac{\pi}{4}\right)^2} = \frac{2\sqrt{2}(\pi - 4)}{\pi^2} = -0.246002020344$$

- It follows that using $\boldsymbol{h} = 0.1$, our approximation is

$$A_1(0.1) = \frac{f(\pi/4 + 0.1) - f(\pi/4)}{0.1} = -0.\underline{2}59446374241$$

- Next, when $\boldsymbol{h}$ is halved, we get

$$A_1(0.05) = \frac{f(\pi/4 + 0.05) - f(\pi/4)}{0.05} = -0.\underline{2}52787379972$$

- Using these, we can generate a new value with Richardson's that has $\mathcal{O}\left(\boldsymbol{h}^2\right)$ error:

$$\begin{aligned} A_2(0.1) &= A_1\left(\frac{0.1}{2}\right) + \frac{A_1\left(\frac{0.1}{2}\right) - A_1(0.1)}{\mathbf{2}^{1} - 1} \\ &= A_1(0.05) + \frac{A_1(0.05) - A_1(0.1)}{\mathbf{2}^{1} - 1} \\ &= -0.252787379972 + \frac{-0.252787379972 - (-0.259446374241)}{\mathbf{2}^{1} - 1} \\ &= -0.\underline{246}128385703 \end{aligned}$$

- The order we do the computations creates a new row as follows:

$$1 \quad A_1(\boldsymbol{h}) \Longrightarrow 2 \quad A_1\left(\frac{\boldsymbol{h}}{2}\right) \Longrightarrow 3 \quad A_2(\boldsymbol{h})$$

- These three calculations can be organized into a table, which we can expand to a larger size.

$\mathcal{O}(\boldsymbol{h})$	$\mathcal{O}\left(\boldsymbol{h}^2\right)$
$A_1(\boldsymbol{h})$	
$A_1(\frac{\boldsymbol{h}}{2})$	$A_2(\boldsymbol{h})$

$\mathcal{O}(\boldsymbol{h})$	$\mathcal{O}\left(\boldsymbol{h}^2\right)$
$A_1(0.1)$	
$A_1(0.05)$	$A_2(0.1)$

$\mathcal{O}(\boldsymbol{h})$	$\mathcal{O}\left(\boldsymbol{h}^2\right)$
1	
2	3

- Note that this value has three correct digits in the derivative!!
- Another row will generate another column of accuracy which gives another order of improvement. Thus,

$\mathcal{O}(h)$	$\mathcal{O}(h^2)$	$\mathcal{O}(h^3)$
$A_1(h)$		
$A_1(\frac{h}{2})$	$A_2(h)$	
$A_1(\frac{h}{4})$	$A_2(\frac{h}{2})$	$A_3(h)$

$\mathcal{O}(h)$	$\mathcal{O}(h^2)$	$\mathcal{O}(h^3)$
$A_1(0.1)$		
$A_1(0.05)$	$A_2(0.1)$	
$A_1(0.025)$	$A_2(0.05)$	$A_3(0.1)$

$\mathcal{O}(h)$	$\mathcal{O}(h^2)$	$\mathcal{O}(h^3)$
1		
2	3	
4	5	6

- Proceeding to 4, then 5 to finally 6 will produce an estimate that is $\mathcal{O}(h^3)$.
- So the fourth entry 4 is $A_1(0.025)$

$$A_1(0.025) = \frac{f(\pi/4 + 0.025) - f(\pi/4)}{0.025} = -0.\underline{2}49410195102$$

followed by the the fifth 5 entry as follows:

$$
\begin{aligned}
A_2(0.05) &= A_1\left(\frac{0.05}{2}\right) + \frac{A_1\left(\frac{0.05}{2}\right) - A_1(0.05)}{\mathbf{2}^{1} - 1} \\
&= A_1(0.025) + \frac{A_1(0.025) - (A_1(0.05))}{\mathbf{2}^{\mathbf{1}} - 1} \\
&= -0.249410195102 + \frac{-0.249410195102 - (-0.252787379972)}{\mathbf{2}^{\mathbf{1}} - 1} \\
&= -0.\underline{2460}33010233
\end{aligned}
$$

- Now, when computing the sixth 6 entry, the exponent on the $\mathbf{2}$ in the denominator increases to **2**:

$$
\begin{aligned}
A_3(0.1) &= A_2(0.05) + \frac{A_2(0.05) - A_2(0.1)}{\mathbf{2}^{\mathbf{2}} - 1} \\
&= -0.246033010233 + \frac{-0.246033010233 - (-0.246128385703)}{\mathbf{2}^{\mathbf{2}} - 1} \\
&= -0.\underline{24600}121841
\end{aligned}
$$

- This last entry is a $\mathcal{O}\left(\boldsymbol{h}^3\right)$ approximation.

- In this case, it can be seen that we have five digits of accuracy in the derivative.

- In summary, the table (with corresponding error rate) is:

$\mathcal{O}(\boldsymbol{h})$	$\mathcal{O}(\boldsymbol{h}^2)$	$\mathcal{O}(\boldsymbol{h}^3)$		$\mathcal{O}(\boldsymbol{h})$	$\mathcal{O}(\boldsymbol{h}^2)$	$\mathcal{O}(\boldsymbol{h}^3)$
$A_1(0.1)$				-0.259446374241		
$A_1(0.05)$	$A_2(0.1)$		$\Rightarrow$	-0.252787379972	-0.246128385703	
$A_1(0.025)$	$A_2(0.05)$	$A_3(0.1)$		-0.249410195102	-0.246033010233	$-0.\underline{24600}121841$

Central Difference Derivative Approximation

- In this section, the process will be applied to the derivative formula

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h}$$

with error structure described in equation (4), so $\boldsymbol{p = 2}$.

- Repeating the approximation from the last session, we will find the derivative of $\boldsymbol{f(x) = \frac{\sin x}{x}}$ at the point $\frac{\pi}{4}$. Its exact value is

$$f'\left(\frac{\pi}{4}\right) = \frac{2\sqrt{2}(\pi - 4)}{\pi^2} = -0.246002020344$$

The table we will fill out follows the error structure.

$\mathcal{O}(\boldsymbol{h}^2)$	$\mathcal{O}(\boldsymbol{h}^4)$	$\mathcal{O}(\boldsymbol{h}^6)$
$A_1(h)$		
$A_1(\frac{h}{2})$	$A_2(h)$	
$A_1(\frac{h}{4})$	$A_2(\frac{h}{2})$	$A_3(h)$

$\mathcal{O}(\boldsymbol{h}^2)$	$\mathcal{O}(\boldsymbol{h}^4)$	$\mathcal{O}(\boldsymbol{h}^6)$
$A_1(0.1)$		
$A_1(0.05)$	$A_2(0.1)$	
$A_1(0.025)$	$A_2(0.05)$	$A_3(0.1)$

$\mathcal{O}(\boldsymbol{h}^2)$	$\mathcal{O}(\boldsymbol{h}^4)$	$\mathcal{O}(\boldsymbol{h}^6)$
1		
2	3	
4	5	6

- For this example, we'll use $\boldsymbol{h} = 0.1$.
- It follows that the approximation for 1 is

$$A_1(0.1) = \frac{f(\frac{\pi}{4} + 0.1) - f(\frac{\pi}{4} - 0.1)}{2(0.1)} = -0.\underline{245}759076590$$

- Next, when $\boldsymbol{h}$ is halved, we get 2

$$A_1(0.05) = \frac{f(\frac{\pi}{4} + 0.05) - f(\frac{\pi}{4} - 0.05)}{2(0.05)} = -0.\underline{245}941268245$$

- Using these two values, we can generate a new value with $\mathcal{O}\big(\boldsymbol{h}^4\big)$ error (entry 3):

$$\begin{aligned} A_2(0.1) &= A_1\left(\frac{0.1}{2}\right) + \frac{A_1\left(\frac{0.1}{2}\right) - A_1(0.1)}{4^1 - 1} = A_1(0.05) + \frac{A_1(0.05) - A_1(0.1)}{4^1 - 1} \\ &= -0.245941268245 + \frac{-0.245941268245 - (-0.245759076590)}{\mathbf{4}^1 - 1} \\ &= -0.\underline{246001}998797 \end{aligned}$$

- So far this value has **six** correct digits in the derivative, which is better than the previous section.

- Completing the last row will give an approximation with $\mathcal{O}\left(\boldsymbol{h}^6\right)$ error. The fourth entry 4 is:

$$A_1(0.025) = \frac{f(\frac{\pi}{4} + 0.025) - f(\frac{\pi}{4} - 0.025)}{2(0.025)} = -0.245986831309$$

- The fifth entry 5 is calculated as follows:

$$\begin{aligned}
A_2(0.05) &= A_1\left(\frac{0.05}{2}\right) + \frac{A_1\left(\frac{0.05}{2}\right) - A_1(0.05)}{\mathbf{4}^1 - 1} \\
&= A_1(0.025) + \frac{A_1(0.025) - (A_1(0.05))}{\mathbf{4}^1 - 1} \\
&= -0.245986831309 + \frac{-0.245986831309 - (-0.245941268245)}{\mathbf{4}^1 - 1} \\
&= -0.\underline{24600201}8997
\end{aligned}$$

- This number has 8 digits correct!

- Finally, the sixth entry 6 is

$$\begin{aligned} A_3(0.1) &= A_2(0.05) + \frac{A_2(0.05) - A_2(0.1)}{4^2 - 1} \\ &= -0.246002018997 + \frac{-0.246002018997 - (-0.246001998797)}{\mathbf{4}^2 - 1} \\ &= -0.\underline{246002020344} \end{aligned}$$

- This last entry is a $\mathcal{O}\left(\boldsymbol{h}^6\right)$ approximation.
- In this case, we see that we have **twelve** digits of accuracy in the derivative.
- Such accuracy with just a few computations!
- This is amazing!
- We will do it again in the next section on Numerical Intgegration.
- Applying Richardson's to the Trapezoid Rule gives us **Romberg Integration**