Math 311

Practice Exam 3, Wi1963, (Chapter 4 in Burden and Faires, 5th)
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Show all your work to receive credit. All answers must be justified to get full credit.

Show Your Work
Show all work clearly and neatly. No work shown means no credit will be given. Use correct notation to get full credit. Reserve scratch paper work for scratch paper, which means only include necessary work on the exam. Erase all mistakes neatly. Keep it neat!

1. (15 pts) The following table lists the distance covered on a straight road by a car at various times. Using the table below and the three and five point formulas from the book (which you've memorized), estimate the **speed** and **acceleration** of the car at 5 seconds and at 10 seconds in **every way that is possible**.

Time (s)	Distance (m)
0	0
2	205
5	375
7	602
10	729
13	1030
16	1354

Acceleration first

$h=5 \quad f''(5) = \frac{1}{2h}(f(0) - 2f(5) + f(10)) = \frac{1}{10}(0 - 2(375) + 729) = \boxed{-2.1}$

$h=3 \quad f''(10) = \frac{1}{2h}(f(7) - 2f(10) + f(13)) = \frac{1}{6}(602 - 2(729) + 1030) = \frac{174}{6} = \boxed{29}$

No five pt formulas can be used. No spacing can use it.

$h=5 \quad f'(5) = \frac{1}{2h}(f(10) - f(0)) = \frac{1}{10}(729 - 0) = \boxed{72.9}$

$h=3 \quad f'(10) = \frac{1}{2h}(-3f(10) + 4f(13) - f(16)) = \frac{1}{6}(-3(729) + 4(1030) - 1354) = \frac{579}{6} = \boxed{96.5}$

$h=-5 \quad f'(10) = \frac{1}{2(-5)}(-3f(10) + 4f(5) - f(0)) = \frac{-1}{10}(-3(729) + 4(375) - 0) = \boxed{68.7}$

2. (10 pts) The forward-difference formula can be expressed as

$$f'(x_0) = \frac{1}{h}\left[f(x_0+h) - f(x_0)\right] - \frac{h}{2}f''(x_0) + O(h^2)$$

Use extrapolation to derive a formula for $f'(x_0)$ that is $O(h^2)$.

$$2\left[\, f'(x_0) = \frac{1}{h}\left[f(x_0+h) - f(x_0)\right] - \frac{h}{2}f''(x_0) + O(h^2) \right]$$

$$-\left[\, f'(x_0) = \frac{1}{2h}\left[f(x_0+2h) - f(x_0)\right] - \frac{2h}{2}f''(x_0) + O(h^2) \right]$$

$$2f'(x_0) - f'(x_0) = \frac{2}{h}\left[f(x_0+h) - f(x_0)\right] - hf''\cancel{(x_0)} + O(h^2)$$

$$- \frac{1}{2h}\left[f(x_0+2h) - f(x_0)\right] + \cancel{h}f''(x_0) + O(h^2)$$

$$f'(x_0) = \frac{4}{2h}f(x_0+h) - \frac{4}{2h}f(x_0) - \frac{1}{2h}f(x_0+2h) + \frac{1}{2h}f(x_0) + O(h^2)$$

$$= -\frac{3}{2h}f(x_0) + \frac{4}{2h}f(x_0+h) - \frac{1}{2h}f(x_0+2h) + O(h^2)$$

$$= \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right] + O(h^2)$$

3. (15 pts) Approximate $f'(x_0)$ for $f(x) = e^x\sqrt{x}$, where $x_0 = 2$ using Richardson's extrapolation, starting with $h = 0.1$ Use the approximation formula

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

to obtain the first column of the table. Show the work needed to find each answer for the table. Fill in the appropriate step size in the parentheses, and fill in the answer for each part. Compare your approximation to the exact value by finding the actual error (not the error bound).

$N_1(.1) =$ 13.842011293981		
$N_1(.05) =$ 13.444769189786	$N_2(.1) =$ 13.047527085592	
$N_1(.025) =$ 13.251661111546	$N_2(.05) =$ 13.058553033306	$N_3(.1) =$ 13.062228349211

$$N_1(.1) = \frac{f(2.1) - f(2)}{.1} = 13.842011293981$$

$$N_1(.05) = \frac{f(2.05) - f(2)}{.05} = 13.444769189786$$

$$N_1(.025) = \frac{f(2.025) - f(2)}{.025} = 13.251661111546$$

$$N_2(h) = N_1\left(\tfrac{h}{2}\right) + \frac{N_1\left(\frac{h}{2}\right) - N_1(h)}{1}$$ (2nd column)

$$N_3(h) = N_2\left(\tfrac{h}{2}\right) + \frac{N_2\left(\frac{h}{2}\right) - N_2(h)}{3}$$ (3rd column)

$$f'(x) = e^x\left(\frac{1}{2\sqrt{x}} + \sqrt{x}\,e^x\right) = e^x\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) = e^x\sqrt{x}\left(1 + \frac{1}{2x}\right)$$

$$f'(2) = e^2\sqrt{2}\left(1 + \frac{1}{4}\right) =$$

$$= 13.062129185304$$

error $= 9.91639 \times 10^{-5}$

4. (10 pts) Derive the approximating formula for $f''(x_0) = \frac{1}{h^2}[f(x_0 - h) - 2f(x_0) + f(x_0 + h)]$ as derived in class. (Hint: use Taylor polynomials)

Taylor series

$$f(x_0+h) = f(x_0) + \cancel{f'(x_0)h} + f''(x_0)\frac{h^2}{2} + f'''(x_0)\frac{h^3}{3!} + o(h^4)$$

$$f(x_0-h) = f(x_0) \cancel{- f'(x_0)h} + f''(x_0)\frac{h^2}{2} - f^{(3)}(x_0)\frac{h^3}{3!} + o(h^4)$$

$$f(x_0-h) + f(x_0+h) = 2f(x_0) + \cancel{2}f''(x_0)\frac{h^2}{\cancel{2}} + o(h^4)$$

$$f''(x_0)h^2 = f(x_0-h) - 2f(x_0) + f(x_0+h) + o(h^4)$$

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0-h) - 2f(x_0) + f(x_0+h)\right] + o(h^4)$$

$$\underbrace{a \qquad\qquad\qquad b}_{h}$$

5. (10 pts) Derive Trapezoid Rule for approximating $\int_a^b f(x)dx$ by using the Lagrange interpolating polynomial (ignore the error term).

$$f(x) = \frac{x-a}{b-a} y_b + \frac{x-b}{a-b} y_a = \frac{y_b}{h}(x-a) - \frac{y_a}{h}(x-b)$$

$$\int_a^b f(x)dx = \frac{y_b}{h}\int_a^b x-a\,dx - \frac{y_a}{h}\int_a^b x-b\,dx$$

$$= \frac{y_b}{h}\left.\frac{(x-a)^2}{2}\right]_a^b - \frac{y_a}{h}\left.\frac{(x-b)^2}{2}\right]_a^b$$

$$= \frac{y_b}{h}\frac{h^2}{2} - \frac{y_a}{h}\frac{h^2}{2}$$

$$= \frac{h}{2}(y_a + y_b)$$

1	1.25	1.5	1.75	2
x_0	x_1	x_2	x_3	x_4

$\int_1^2 \frac{1}{x}dx = \boxed{\ln 2}$

6. Approximate $\displaystyle\int_1^2 \frac{1}{x}dx$ using the following methods. Also, find the absolute value of the actual error for each of the problems. Indicate which one came the closest.

(a) (10 pts) Use Trapezoid Rule with $n = 4$.

$$T = \frac{h}{2}\left(f(1) + 2f(\underbrace{1.25}_{5/4}) + 2f(\underbrace{1.5}_{3/2}) + 2f(\underbrace{1.75}_{7/4}) + f(2)\right)$$

$$= \frac{.25}{2}\left(\frac{1}{1} + 2\left(\frac{4}{5}\right) + 2\left(\frac{2}{3}\right) + 2\left(\frac{4}{7}\right) + \frac{1}{2}\right)$$

$$= \boxed{\frac{1171}{1680}} = \boxed{.697023809524}$$

Perfect is ln 2.

So

$$\text{error} = \left|\frac{1171}{1680} - \ln 2\right| = 0.00387662896$$

(b) (10 pts) Use Simpson's Rule with $n = 4$.

$$S = \frac{h}{3}\left(f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2)\right)$$

$$= \frac{.25}{3}\left(1 + 4\left(\frac{4}{5}\right) + 2\left(\frac{2}{3}\right) + 4\left(\frac{4}{7}\right) + \frac{1}{2}\right)$$

$$= \frac{1747}{2520} = 0.69325396825 4$$

$$\text{error} = \left|\frac{1747}{2520} - \ln 2\right| = 0.0001067876 9$$

$1 \quad \frac{7}{6} \quad \frac{4}{3} \quad \frac{3}{2} \quad \frac{5}{3} \quad \frac{11}{6} \quad 2$

(c) (10 pts) Use Midpoint Rule with $n = 4$.

Two Methods

Book $= h = \frac{b-a}{4+2} = \frac{1}{6}$

$$M = 2h\left(f\left(\tfrac{7}{6}\right) + f\left(\tfrac{3}{2}\right) + f\left(\tfrac{11}{6}\right)\right)$$

$$\doteq \frac{478}{693} = \boxed{.689754689755}$$

$$\text{error} = \boxed{0.00339249080\,5}$$

Just use 4 rectangles!

$h = \frac{1}{4}$

$1 \quad \frac{9}{8} \quad \frac{5}{4} \quad \frac{11}{8} \quad \frac{6}{4} \quad \frac{13}{8} \quad \frac{7}{4} \quad \frac{15}{8} \quad 2$

$$M = h\left(f\left(\tfrac{9}{8}\right) + f\left(\tfrac{11}{8}\right) + f\left(\tfrac{13}{8}\right) + f\left(\tfrac{15}{8}\right)\right)$$

$$= \frac{1}{4}\left[\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right]$$

$$= \boxed{0.69121989122}$$

$$\text{error} = \boxed{0.00192728934005}$$

(d) (10 pts) Use Boole's Rule:

$$\int_{x_0}^{x_4} f(x)dx \approx \frac{2h}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right]$$

$$B = \boxed{0.69317460317\,5} \qquad \text{error} = \boxed{0.000027422614\,6578}$$

(e) (10 pts) Using Gaussian Quadrature with $n = 4$. The coefficients and roots are

x_i	−0.8611363116	−0.3399810436	0.3399810436	0.8611363116
c_i	0.3478548451	0.6521451549	0.6521451549	0.3478548451

$$G = \sum_{i=1}^{4} w_i \cdot f(t_i), \text{ where } t_i = \frac{(b-a)}{2} x_i + \frac{b+a}{2} = \frac{1}{2} x_i + \frac{3}{2}$$

Using program.

$$= \frac{1}{2}\sum_{i=1}^{4} c_i \cdot f\left(\tfrac{1}{2}x_i + \tfrac{3}{2}\right) = \boxed{0.693146417445}$$

$$\text{error} = \boxed{0.00000076311446\,2399}$$

Gaussian has the smallest error!

(f) (10 pts) Using Romberg Integration (Complete to $R_{3,3}$)

Using Computer

0.75

0.708333333333 0.694444444444

0.697023809524 0.693253968254 0.69317460317̄5

So $\boxed{0.6931746031\bar{75}}$

error $= \boxed{0.0000271422614657 8}$