LOG(M) PROJECT: LARGE SCALE GEOMETRY OF INTEGERS DRAFT 1

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1. Introduction

The study of groups through the investigation of the geometric structures that they create often leads to insightful and applicable results. In this study, we take an geometric group theoretic approach towards investigating the large scale structure of the set $\mathbb{Z}$ of integers. [2] Our approach will be to survey finitely generated groups of or related to integers and to develop various results regarding them, using the 'word-metric' as our primary investigative tool. In particular, we will develop algorithms to compute *word-metrics* of finitely generated groups of integers and for H(4), the Heisenberg group of 4x4 matrices[5], and work towards the open problem of the Rough Isometry of Integers. [4]

2. Basic Definitions/Approach

In order to approach this investigation, we must first establish some important definitions and results. We start with the definition of a group and of a finitely generated group.

Definition 2.1. Group, Group Structure [3]
Let G be a set and let $* : G \times G \to G$ be a binary operation on G. Suppose the following conditions hold:

(1) Associativity: $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$
(2) Identity: $\exists e \in G$ such that $a * e = a$ for all $a \in G$
(3) Inverse: For all $a \in G$, $\exists a^{-1} \in G$ such that $a * a^{-1} = e$

Then (G,*) is called a *group*, and G is said to have a *group structure*.

Definition 2.2. Subgroup, Finitely Generated Subset, Finitely Generated Group [3]
Let $(G, +)$ be a group (such that for all $z, x \in G$, $-z$ is the inverse of z and $x - z = x + (-z)$), and let S be a set such that $S \subset G$ and (S,+) forms a group. Then, S is called a *subgroup* of G.

Suppose $X \subset G$ is finite. Then, X is called a finitely generated set.

$< X >$:=$\{z \in G$: $z = \pm z_1 \pm z_2 \pm ... \pm z_n$, where $n \in \mathbb{N}, z_1, z_2, ..., z_n \in X\}$, and the subgroup $< X >$ is called the *finitely generated subgroup* of X.

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These two definitions give a sense of the primary objects of study in Geometric Group Theory, which are, indeed, finitely generated groups. Next, we define the 'word-metric', which gives us a concrete sense of *length* or *distance* as they pertain to groups.

Definition 2.3. Word Metric
Let $< X >$ be a finitely generated subgroup of G.
Define $|| :< X > \to \mathbb{N}_{\geq 0}$ such that for $x \in X$,
$|x| = \min\{n \in \mathbb{N} : x = \pm x_1 \pm x_2 \pm ... \pm x_n$, such that $x_1, x_2, ..., x_n \in < X >\}$.
Suppose $d : X \times X \to \mathbb{N}_{\geq 0}$ such that for $x, y \in X$, $d(x, y) = |-x + y|$. Then, d is called a *word-metric* over G.

To briefly justify this definition, we may, in essence, view each element a of $< X >$ as a *word* generated by X, and the number of elements in the shortest way to generate a as a string of products of elements in $< X >$ is the *word-metric* of a with respect to X. The next sections present results on our work on computing *word-metrics* on specific finitely generated groups.

3. Result 1: Computing the Word-Metric on $\mathbb{Z}$

In this section, we will present our algorithm to compute the *word-metric* on finitely generated subgroups of integers. Specifically, we focus on finitely generated subsets that generate $\mathbb{Z}$ with our group operation being addition, so our *word-metric* just computes the shortest way to add up the elements of our finitely generated subsets to get to the integer we want. We present a useful lemma:

Lemma 3.1. *Suppose X is a finitely generated subset of $\mathbb{Z}$ that generates $\mathbb{Z}$, with $|X| = k$. We define $M = \frac{x_1+x_2}{2}x_1 + \frac{x_2+x_3}{2}x_2 + ... + \frac{x_k+x_{k-1}}{2}x_{k-1} + x_k$, with x_i being the ith term when we order X. Then, for all $n > M$, $|n| = |n - x_k| + 1$, with $|n|$ being the word-metric $d(n, 0)$ over X.*

This lemma was proved using elementary number theory. What this tells us is that for a large n, (large enough so that $n > M$), we can compute the *word-metric* recursively. Thus, our algorithm is as follows:

Theorem 3.2. *Integer Word-Metric Algorithm*
Fix some ordered $X \subset \mathbb{Z}$. In order to find $d(x - y)$, the word-length between $x, y \in \mathbb{Z}$ with respect to $< X >$, we use the following steps:

1. *Ensure that $< X >= \mathbb{Z}$ by checking that $GCD(abs(x - y)) = 1$ (by Bezout's lemma [6])*
2. *Find $M = \frac{x_1+x_2}{2}x_1 + \frac{x_2+x_3}{2}x_2 + ... + \frac{x_k+x_{k-1}}{2}x_{k-1} + x_k$, $x_i :=$ i'th element in X*
3. *Compute the word-lengths of all $n \in \mathbb{Z}$ such that $0 \leq n \leq M$*
4. *Compute $d(x - y) = d(abs(x - y))$ recursively:*
   - *(a) if $d(abs(x - y)) > M$, $d(abs(x - y)) = d(abs(x - y - x_k)) + 1$*
   - *(b) else, we have already precomputed $d(x - y)$ in (3)*

The proof of this algorithm follows trivially from the lemma. Our team was able to translate this into code, and a python algorithm was developed.

4. Result 2: Computing the Word-Metric on $H(4)$

In this section, we will present our solution to computing the word-length of elements of the group $H(4)[\mathbb{Z}]$, the Heisenberg group of integer $4x4$ matrices. We first present a definition.

Definition 4.1. $H(4)[\mathbb{Z}]$, or the $4x4$ Heisenberg group

$H(4)[\mathbb{Z}]$ is the group of all matrices in the form $H = \begin{bmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

where $x_1, x_2, y_1, y_2, z \in \mathbb{Z}$.

$H(4)[\mathbb{Z}]$ is an important example of a *nilpotent group*. It is known that

$$X = \left\{ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

$:= \{X_1, X_2, Y_1, Y_2\}$ (suppose the generators are ordered in this way)

generates $H(4)[\mathbb{Z}]$ (this fact is trivial and may be simply computed), but an exact method or algorithm to find the minimum *word-length* has not yet been found. This is the problem that we are currently working on. We suspect that we can prove that the set generated by X_1, Y_2 is isomorphic to the Heisenberg group $H(3)[\mathbb{Z}]$, and that $< X_1, Y_2 >$ is also isomorphic to $H(3)[\mathbb{Z}]$. There is a known analytic formula to compute the word-length on $H(3)$[1], and we have deduced that $H(4)$ is isomorphic to $H(3) \times H(3)$. Thus, we simply extend the analytic formula, which breaks into cases quite trivially, to $H(4)$, which leads us to be able to solve the word-metric of $H(4)$ and, indeed, $H(n), n \in N$, using a bit of manipulation of the formula. We have written this argument into code, and testing shows the code to be very effective.

5. Open Problem: Rough Isometry

Definition 5.1. A **rough isometry** on $\mathbb{Z}$ is a map $f : \mathbb{Z} \to \mathbb{Z}$ such that there exists a constant $M > 0$ with the following properties:

- $|d(f(m), f(n)) - d(m, n)| \leq M$ for all $m, n \in \mathbb{Z}$,
- For all $m \in \mathbb{Z}$, there exists n such that $d(f(n), m) \leq M$.

The metric d is left-invariant:

$$d(l + x, l + y) = d(x, y) \quad \forall x, y, l \in \mathbb{Z}.$$

Additionally, we have the following property:

$$\lim_{n\to\infty} \frac{d(0,n)}{n} = 1.$$

Definition 5.2. We define an equivalence relation $\sim$ on $\mathbb{Z}$ as follows:

$$f \sim g \text{ if } \exists M > 0 \text{ such that } d(f(n), g(n)) \leq M \text{ for all } n \in \mathbb{Z}.$$

The open problem we aim to solve is as follows:

Open Question: For an arbitrary $(\mathbb{Z}, d)$, what is the rough isometry equivalence relation $\sim$? We suspect that all rough isometries are either equivilent to the identity or the negative identity.

We have made minor progress proving that some algebraic functions are indeed equivilent to the identity or negative identity but overall are still working on developing an approach. We are considering topological and number theoretic approaches as well.

References

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