UNIVERSITY OF
EXETER

MTHM005 - MATHEMATICAL SCIENCES PROJECT

Pricing Asian Options
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Abstract

Options are contracts that grant the right, but not the obligation to buy or sell an underlying asset at a set price on a specified date.

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Chapter 1: Introduction

This report studies a vital financial derivative in today's markets, namely options. The popularity of options in today's market can easily be seen by the exponential growth in their trading volume from when standardized, exchange-traded stock options were first listed in The Chicago Board Options Exchange in 1973 [Mar02, pp.51-52], shown in Figure 1.1.
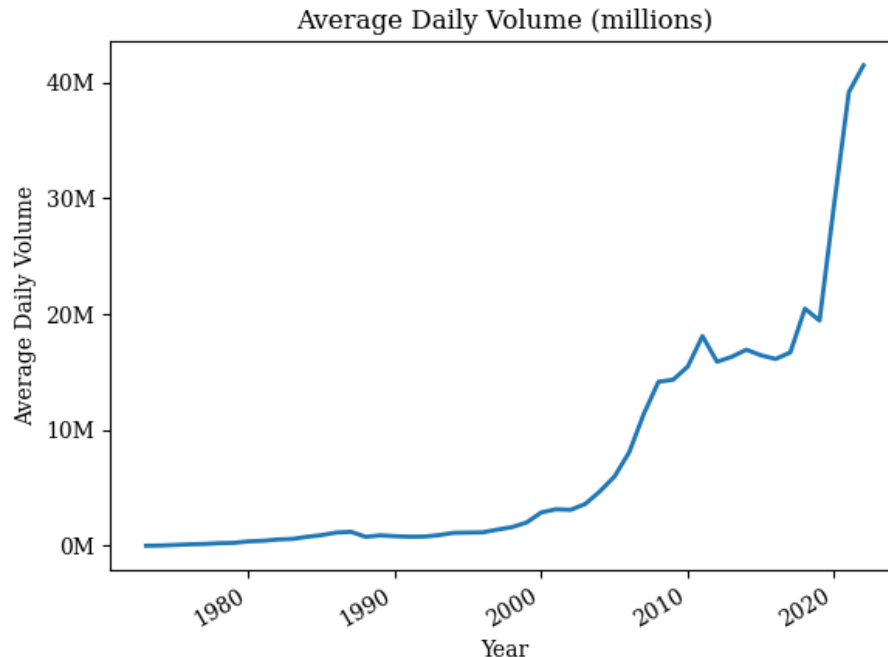

Figure 1.1: Time series plot of the average daily option and future contracts trading volume per annum. Data provided by the Options Clearing Corporation (OCC) [Opt]. Source code Listing B.1

1.1   A brief history of options

The option market is widely considered a venue for informed trading [Bla75, Li21, Hu14, CGM04], that is investors trading with superior knowledge of the probability distribution of share prices, through either access to private information or skillful processing of public information.

1.2   Standard options

A standard option is a contract between two parties which gives the holder the right to buy (or sell) an asset for an agreed upon (exercise) price prior to, or on a determined (expiry) time in the future; regardless of the current (spot) price. Since the holder of the contract is not obliged to exercise the contract at the expiry time, they would not hold any liability in the absence of a price to purchase the option. The problem then becomes what is the correct price to charge the holder of the option to balance this inequality of liability.

1.3   Asian options

Whilst standard options, namely European and American style involve using the spot price as the underlying value of the asset; this is not always the case with so-called exotic options. Exotic options differ in their payment structures, expiration dates, and/or strike prices. In the case of exotic fixed-strike price Asian options, the averaging price of the asset is used in place of the underlying asset value. This differs from fixed-price Asian options which instead use the averaging price of the asset to take place of the strike price. These are the two main variations of Asian style options but both of these can be varied further in how the averaging is calculated, for example: geometrically, arithmetically, average taken every day or average taken at the start of each month and so on. They can be varied further by having an expiry structure matching a European or American style option.

Bibliography

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[Hu14]   Jianfeng Hu. Does option trading convey stock price information? *Journal of Financial Economics*, 111(3):625–645, 2014.

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[Mar02]   J.W. Markham. *A Financial History of the United States Volume III (1970-2001) From the Age of Derivatives to the New Millennium*. A Financial History of the United States. M.E. Sharpe, 2002.

[Opt]   Options Clearing Corporation. Annual Volume and Open Interest Statistics. https://www.theocc.com/Market-Data/Market-Data-Reports/Volume-and-Open-Interest/Historical-Volume-Statistics. Accessed: 08-06-2022.

Appendix A: Matlab Files

All files can be found: https://github.com/leele2/Mathematics-in-Business-Project/tree/master/MATLAB%20Files

Listing A.1: ../../MATLAB Files/BinoAsian.m

```matlab
%function BinoAsian(S0,E,T,r,sigma,N,F)
%% Test
clear; tic; S0=50; E=40; T=1.0; r=0.1; sigma=0.3; N=500; F=@(S,A)max(A-E,0);
%% Function to evaluate European Call option by Binomial Method
%    Parameters:
%    S0 = initial share price
%    E = exercise price`
%    T = time to expiry
%    r = riskfree interest rate
%    sigma = volatility
%    N = Number of steps
%    F = Option Payoff (European Call in given)
%% Calculated parameters
dt   = T/N;                  %Timestep
u    = exp(sigma*sqrt(dt));  %Up price movement
d    = 1/u;                  %Down price movement
disf = exp(-r*dt);           %Discount factor over each timestep
p    = (1/disf - d)/(u-d);   %Risk-neutral probability
%% Initalizing Arrays and Functions
S    = zeros(N+1,N+1);   %Underlying Asset Price
S_k  = cell(N+1,N+1);    %S_max %% Cells are used so different sized vectors can be stored at each element in
          cell array
A    = zeros(N+1,N+1);   %Average of Underlying Asset Price
A_k  = cell(N+1,N+1);    %Representitive averages
C    = zeros(N+1,N+1);   %Price of Option
C_k  = cell(N+1,N+1);    %Option price for given representitive avg
%F    = @(S,A)max(A-E,0); %Option Payoff (European Call)
%% Functions to Calculate Maximum and Minimum Representative Averages
A_min = @(i,j) (1/(i+1)) * (sum(S0*d.^(0:i-j)) + sum(S0*d.^(0:(j-1)+i-(2*j))));
A_max = @(i,j) (1/(i+1)) * (sum(S0*u.^(0:j)) + sum(S0*u.^((0:(i-j-1))+(2*j)-i)));
%% Calculate Underlying asset price
for i = 0:N
    for j = 0:i
        S(i+1,j+1) = S0*u^(j)*d^(i-j);
    end
end
%% Calculating All S_max and Representitive Averages
for i = 0:N
    for j = 0:i %j indexes at j+1 due to matlab not allowing C_k{:,0}
        A_k{i+1,j+1} = zeros(j*(i-j)+1,1); %Create Vector to hold rep avgs
        S_k{i+1,j+1} = NaN(j*(i-j),1);     %Create Vector to hold S_max
        A_k{i+1,j+1}(1) = A_max(i,j);      %Assign A_max to first element in vector
        % Paths with only up (i = j) or down movements (j = 0) or i = 1 will only have one representative
                average
        if i < 1 || i == j || j == 0 || j*(i-j) == 1
            S_k{i+1,j+1}(1) = S0*(u^j)*d^(i-j);
            continue
        end
        %% Filling S_max Vector
        % Setting Paramaters
        n = j*(i-j);            %Size of vector
        rep = min([j;i-j]);     %Maximum element repetition in vector
        % Calculating unique elements in vector
        unique = 2*numel(1:rep-1) + (n - (2*sum(1:rep-1)))/rep;
        % First and last values
        S_k{i+1,j+1}(1)   = S0 * (u ^ j);
        S_k{i+1,j+1}(end) = S0 * (u ^ (j - unique + 1));
        %%The following if statement changes the final output%%
        if n ~= 2
            % Ascending/Descending repeated values
            k     = 2; %iterator
            count = 2; %Number of elements filled
            while k < rep
                S_k{i+1,j+1}(sum(1:k-1)+1:sum(1:k-1)+k) = repmat(S0*(u^(j-k+1)), k, 1);
                S_k{i+1,j+1}(n-sum(1:k-1)+1-k:n-sum(1:k-1)) = repmat(S0*(u^(j-unique+k)), k, 1);
                count = count + 2*k;
                k = k + 1;
            end
            % "Middle" repeated values
            for l = 0 : (n-count)/rep - 1
                S_k{i+1,j+1}(count/2+1+l*rep:end) = [repmat(S0*u^(j-k+1), rep, 1);
                    S_k{i+1,j+1}(count/2+1+l*rep+rep:end)];
                k = k + 1;
            end
        end
        %% Calculating Representitive Averages Using S_max
        A_k{i+1,j+1}(j*(i-j) + 1) = A_min(i,j); %Assign A_min to last element in vector
        for k = 2:j*(i-j)
```

3

```matlab
77              A_k{i+1,j+1}(k) = A_k{i+1,j+1}(k-1) - (1/(i+1))* ...
78                  (S_k{i+1,j+1}(k-1) - S_k{i+1,j+1}(k-1)*(d^2));
79          end
80      end
81  end
82  %% Pricing Option Value at Final Time (N)
83  for j = 0:N
84      C_k{N+1,j+1} = F(S(N+1,j+1),A_k{N+1,j+1});
85  end
86  %% Pricing Option
87  err = 1e-3;
88  for i = N-1:-1:0
89      for j = 0:i
90          C_k{i+1,j+1} = zeros(j*(i-j)+1,1);
91          for k = 1:j*(i-j)+1
92              %% Find K_u
93              Ku = ( (i+1)*A_k{i+1,j+1}(k) + u*S(i+1,j+1) )/(i+2);
94              [loc, ubound, lbound] = findInSorted(Ku,A_k{i+2,j+2},err);
95              % If found set accordingly
96              if loc > 0
97                  Cu = C_k{i+2,j+2}(loc);
98              else
99                  % If not found, interpolate between closest values
100                 Cu = C_k{i+2,j+2}(lbound)+(Ku-A_k{i+2,j+2}(lbound))*(...
101                     (C_k{i+2,j+2}(ubound)-C_k{i+2,j+2}(lbound))/...
102                     (A_k{i+2,j+2}(ubound)-A_k{i+2,j+2}(lbound)));
103             end
104             %% Find K_d
105             Kd = ( (i+1)*A_k{i+1,j+1}(k) + d*S(i+1,j+1) )/(i+2);
106             [loc, ubound, lbound] = findInSorted(Kd,A_k{i+2,j+1},err);
107             %If found set accordingly
108             if loc > 0
109                 Cd = C_k{i+2,j+1}(loc);
110             else
111                 %If not, interpolate between closest values
112                 Cd = C_k{i+2,j+1}(lbound)+(Kd-A_k{i+2,j+1}(lbound))*(...
113                     (C_k{i+2,j+1}(ubound)-C_k{i+2,j+1}(lbound))/...
114                     (A_k{i+2,j+1}(ubound)-A_k{i+2,j+1}(lbound)));
115             end
116             %% Calculate option value at previous node (i,j decreasing)
117             C_k{i+1,j+1}(k) = disf*(p*Cu + (1-p)*Cd);
118         end
119     end
120 end
121 disp(C_k{1,1})
122 toc;
```

Listing A.2: ../../MATLAB Files/Functions/findInSorted.m

```matlab
1   function [loc, A, B] = findinSorted(x,range,err)
2   %%%%%%%%%%%%%%%% inputs %%%%%%%%%%%%%%%%
3   % x = value to find                    %
4   % range = (sorted) vector to search    %
5   % err = tolerence on search            %
6   %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
7   %% Outputs
8   A   = 1;                %left boundary
9   B   = numel(range); %right boundary
10  loc = 0;                %initially set to 0 implying not found
11  %% Binary Search Algorithm
12  while B-A > 1
13      mid = floor((A+B)/2);
14      if x > range(mid)
15          B = mid;
16      else
17          A = mid;
18      end
19  end
20  %% Returning returned value if found (within tolerance)
21  if abs(range(A) - x) < err
22      loc = A;
23      return
24  elseif abs(range(B) - x) < err
25      loc = B;
26      return
27  end
```

Appendix B: Python Files

All files can be found:

Listing B.1: ../../Python Files/OptionVolume.py

```python
# -*- coding: utf-8 -*-
"""
Created on Wed Jun  8 18:05:55 2022
Scraping and plotting data from theocc.com
@author: leele2
"""
import pandas as pd
import matplotlib as mpl
import matplotlib.pyplot as plt
import matplotlib.dates as mdates
from datetime import datetime
from bs4 import BeautifulSoup
from os import listdir
from pathlib import Path

def string_to_int(string):
    if not string:
        return 0
    if string[0] == "$":
        string = string[1:]
    return int(string.replace(',', ''))

# Ensuring html file is on path
"""
html file taken from:
https://www.theocc.com/Market-Data/Market-Data-Reports/Volume-and-Open-Interest/Historical-Volume-Statistics
"""
html_file = [f for f in listdir('.') if f.endswith('.html')]
if len(html_file) != 1:
    raise ValueError('should be exactly one html file in the current directory')
# Importing html file
with open(html_file[0], 'r', encoding='utf-8') as file:
    soup = BeautifulSoup(file, 'lxml')
# Finding Table
table = soup.find_all("table")
# Selecting Data
table_data = table[0].find_all("tr")
# Finding Table Names
table_names = []
for names in table_data[0]:
    table_names.append(str(names.find("span").string))
del table_names[0]
del table_names[-1]
# Finding Headers
headers = []
for header in table_data[1]:
    headers.append(str(header.find("span").string))
# Populating Data
Data = {}
for rows in table_data[2:]:
    temp = {}
    for i, data in enumerate(rows):
        if i > 6 and i < 10 or i == 0:
            temp[headers[i]] = data.get_text()
    Date = datetime.strptime(temp["Date"], "%Y")
    del temp["Date"]
    Data[Date] = {k: string_to_int(v) for k, v in temp.items()}
# Converting data into pandas dataframes
Data = pd.DataFrame.from_dict(Data, orient="index")
# Plotting time-series
plt.rc('font', family='serif')
fig, ax1 = plt.subplots()
plt.gcf().set_size_inches(6, 4.5)
ax1.plot(Data['Options']/1e6, linewidth=2, zorder=1, label="Options")
# ax1.plot(Data[Data['Futures'] != 0]['Futures']/1e6, linewidth=2, zorder=1, label="Futures")
# ax1.legend()
ax1.set_title('Average Daily Volume (millions)')
ax1.set_xlabel('Year')
ax1.xaxis.set_minor_formatter(mdates.DateFormatter("%Y"))
ax1.set_ylabel('Average Daily Volume')
ax1.yaxis.set_major_formatter(mpl.ticker.StrMethodFormatter('{x:,.0f}M'))
fig.autofmt_xdate()
plt.tight_layout()
plt.show()
# Saving plot
cwd = str(Path(__file__).parent.parent.absolute())
fig.savefig(cwd + "\Latex_Files\Main\Chapters\C1\plots\OptionVolume.png", bbox_inches='tight')
```