Analytical Valuation of American-Style Asian Options

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This article derives the first analytical pricing formulas for American-style Asian options of the so-called floating strike type. Geometric as well as arithmetic averaging is considered. The setup is a standard Black-Scholes framework where the price of the underlying security evolves according to a geometric Brownian motion. A decomposition result that splits up the value of the floating strike American option into the price of an otherwise equivalent European option and an early exercise premium is first presented. This decomposition result is then manipulated further for the two separate types of averaging. With geometric averaging we derive an exact pricing formula, whereas with arithmetic averaging we develop an analytical approximation formula that proves to be very precise. Numerical examples are provided.
(*Asian Options; American Options; Analytical Valuation Formulas; Numerical Work; Change of Numeraire*)

1. Introduction

Asian options are path-dependent options whose payoffs are based on an average. As such, this type of option belongs to the broader class of "exotic options" that are mostly OTC-traded, but Asian options are nevertheless extremely common in currency- and commodity-related transactions, particularly in the oil industry (see e.g., Heenk et al. 1990, and Chance and Rich 1996).

The dependence of the average on the options' payoff can be specified in several ways. In some cases the "underlying asset" is an average, in which case the option is a *fixed strike* Asian option, whereas in other cases the exercise price of the contract is an average of prices of the underlying asset. These latter options are labeled *floating strike* Asian options.

Furthermore, the type of averaging may differ. First, the average may be based on discretely sampled prices (e.g., once a day) or it may—at least in principle—be based on continuously sampled price observations. Second, averaging may be on an arithmetic or geometric basis, and third, the weighting of the observations in the average may be equal or *flexible* (as proposed in a series of papers by Zhang, i.e., Zhang 1994, 1995a, 1995b).[1]

The final characteristic to be mentioned, and of particular concern in this article, is the distinction between Asian options of *American* and *European* types, i.e., options that may and may not be exercised prior to their maturity date.[2]

Because most of the previous literature on Asian

[1] The geometric average is not merely an academic abstraction. Geometric average-based contracts *do* in fact trade, and sometimes even in heavy volumes, on organized exchanges. For an example and a joyous story, see Ritter (1996).

[2] Based on the same logic that gave Bermudan (or Mid-Atlantic) options their name (these options are something in between European and American options), Jørgensen et al. (1997) suggest that American-style Asian options are collectively labeled "Hawaiian" options.

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options has been confined to the study of the European type of Asian options, the issue of valuation and optimal management of a possible early exercise feature implicit in the American-type option has not received much attention. The reason for the apparent lack of research in this area is perhaps a seeming conflict between an early exercise feature in an Asian option and one of the motivations for introducing Asian options in the first place. Supposedly (see, e.g., the discussions in Kemna and Vorst 1990, Bergman 1985, and Chance and Rich 1996), Asian options were introduced to reduce the concern of investors that standard options could be subject to price manipulation of the underlying asset just before the maturity date. Therefore, as argued in Kemna and Vorst (1990), letting Asian options be of American type would reintroduce the risk of price manipulation, since these options can be exercised right after averaging has begun. However, as our numerical results indicate, this "risk" may not be particularly significant. This is also supported by the fact that American-type Asian options *do* in fact trade (Zhang 1995b).

This article focuses exclusively on American-style Asian options of the floating strike type, and our methodology does not generalize to options of the fixed strike type. We consider equally weighted continuous averaging on an arithmetic as well as on a geometric basis and, as is common for the literature in this area, we work within the framework of Black and Scholes (1973), where the price variable evolves according to the geometric Brownian motion.

As indicated above, previous research into the valuation of Asian options of American type is limited, but *some* work—mostly of numerical nature—has been done. We are aware of the following articles that more than just touch on the issue. Hull and White (1993) and Ritchken et al. (1993) suggest extended bi- and trinomial methods tailored to Asian options of American type. These papers also provide numerical evidence that suggests that the value of the early exercise feature is significant. A PDE approach is taken in papers by Dewynne and Wilmott (1995), Andreasen (1996), and Zvan et al. (1998). In short, the two papers first mentioned concentrate on the problems associated with discrete sampling, whereas the last paper performs an analysis into efficient implementation of finite difference solutions to the PDE associated with Asian options. Finally, Gao et al. (1996) provide a general framework for the analytical valuation of American path-dependent options. The spirit of their analysis is very similar to ours, but they mention only briefly the possibility of applying their methodology to American-style Asian options, and the essential switch of numeraire asset that makes the derivation of analytical formulas possible, and which is the key to the contributions of the present section, is not explored in Gao et al. (1996).[3]

An outline of this article is as follows: In §2 our basic setup is presented. We define the derivative contracts to be valued and, by using the stock as the numeraire asset, we are able to reduce the complexity of the general problem and derive a general decomposition result that splits up the American option value into the value of an otherwise identical European option and an early exercise premium. This last part of the analysis pursues ideas from the analysis of standard American options in, for example, Kim (1990), Jacka (1991), Jamshidian (1992), and Carr et al. (1992).

Section 3 deals with the further manipulation of our general decomposition result in the specific cases of arithmetic and geometric averaging. With geometric averaging we arrive at an exact analytical pricing formula, which is very easy to implement. In the case of arithmetic averaging, we introduce an approximating process for the average process. This, together with the general decomposition result, allows us to derive an analytical approximation formula that is implemented in the same manner as the exact geometric formula, and that is accurate and fast. Section 4 contains plots of optimal exercise boundaries, numerical pricing results, and evidence on the accuracy of the suggested numerical procedures. The final section contains our conclusions and some suggestions for further work.

[3] Change of numeraire techniques have been applied to standard (i.e., European-type) Asian option pricing problems in Ingersoll (1987) and Rogers and Shi (1995); cf. also later. For a general treatment of change of numeraire techniques in option pricing the reader is referred to Geman et al. (1995).

2. The Basic Model

As our basic model we use the setup from Black and Scholes (1973), i.e., we consider a simple financial market in which all activity occurs on a filtered probability space $(\Omega, \Im, (\Im_t), Q)$ supporting Brownian motion on the finite time interval $[0, T]$. Here Q is the standard risk-neutral probability measure (see e.g., Harrison and Kreps 1979) in which the price, $S(\cdot)$, of the basic risky asset (the *stock*) of the economy evolves according to the stochastic differential equation

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q(t), \quad S(0) = S_0, \quad (1)$$

where r denotes the constant and positive risk-free rate of interest, σ is the constant volatility of stock returns, and $W^Q(\cdot)$ is a standard Brownian motion under Q. Equation (1) thus defines the well-known geometric Brownian motion, given by

$$S(t) = S_0 e^{(r-1/2\sigma^2)t+\sigma W^Q(t)}, \quad 0 \le t \le T.$$

The risk-free asset of the economy—the *money market account*—has dynamics

$$dB(t) = rB(t)dt, \quad B(0) = 1$$
$$\Updownarrow$$
$$B(t) = e^{rt}.$$

In addition to the above, we maintain the standard assumptions about continuous-time perfect markets, i.e., we assume that assets trade continuously and that there are no frictions, e.g., transactions costs or taxes of any kind.

2.1. Definition of the Derivative Contracts

The class of option contracts that we consider in this article is a subclass of the class of Asian options. More specifically, we concentrate on floating strike Asian options with continuously sampled averages, and most important, we focus on options of American type. The contracts are initiated at time zero and their payoffs upon exercise at time t are given as

$$\text{Payoff} = [\rho(S(t) - A(t))]^+,$$

where

$$\rho = \pm 1$$

and

$$A(t) = \begin{cases} \dfrac{1}{t}\displaystyle\int_0^t S(u)du \\ \quad \text{if the average is arithmetic} \\ \exp\left\{\dfrac{1}{t}\displaystyle\int_0^t \ln S(u)du\right\} \\ \quad \text{if the average is geometric.} \end{cases}$$

We refer to the cases $\rho = 1$ and $\rho = -1$ as the floating strike call and put option, respectively.

The European versions of these contracts have been considered in the literature. For example, Bouaziz et al. (1994) derive approximation formulas for the contracts based on the arithmetic average, and Boyle (1993) provides closed formulas for the options based on geometric averaging. These latter option price formulas could be considered as variants of the formulas, for the value of a European option to exchange one asset for another, derived by Margrabe (1978); cf. also later.

2.2. General Valuation

To proceed with the analytical valuation of American-type contracts, we first rely on a result from Karatzas' general treatment of the fair valuation of American-type contingent claims (see Karatzas 1988, 1989). Letting $V(t)$ denote the option value at time t and applying Karatzas' result to this particular problem, we have

$$V(t) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} E_t^Q\{e^{-r(\tau-t)}[\rho(S(\tau) - A(\tau))]^+\}, \quad (2)$$

where $\mathcal{T}_{t,T}$ denotes the class of $(\Im_t)$ stopping times taking values in $[t, T]$ and E_t^Q is the Q-expectation operator conditional on $\Im_t$.[4]

Our first concern is the further manipulation of this valuation equation. As a first pass a change of probability measure will prove helpful. If we define the Q-martingale

$$\xi(t) = e^{-rt}\frac{S(t)}{S(0)} = \exp\left\{-\frac{1}{2}\sigma^2 t + \sigma W^Q(t)\right\} \quad (3)$$

and a new (equivalent) measure, Q', by

[4] For a definition of the *essential supremum* see, for example, Hoffmann-Jørgensen (1994).

$$dQ' = \xi(T)dQ, \tag{4}$$

then by Girsanov's Theorem (see e.g., Duffie 1996 or Karatzas and Shreve 1988) the process

$$W^{Q'}(t) = W^{Q}(t) - \sigma t$$

is a standard Brownian motion under Q'. Consequently, under Q' the stock price evolves according to the following stochastic differential equation

$$dS(t) = (r + \sigma^2)S(t)dt + \sigma S(t)dW^{Q'}(t).$$

More important, a distinguishing feature of the measure Q' is that when discounted by (measured in units of) the stock price, all asset prices will be Q'-martingales.[5] This fact plays an essential role in our progress toward an analytical form of (2), as will be clarified by the following steps. Applying rules of conditional expectation, invoking the optional sampling theorem, and defining $x(t) \equiv (A(t)/S(t))$, we can write

$$\begin{aligned}
V(t) &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} E_t^{Q}\{e^{-r(\tau-t)}[\rho(S(\tau) - A(\tau))]^+\} \\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} E_t^{Q'}\left\{\frac{\xi(t)}{\xi(T)}\, e^{-r(\tau-t)}[\rho(S(\tau) - A(\tau))]^+\right\} \\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} E_t^{Q'}\left\{\frac{S(t)}{e^{rt}}\, e^{-r(\tau-t)} \right. \\
&\qquad \left. \times [\rho(S(\tau) - A(\tau))]^+ E_\tau^{Q'}\left\{\frac{e^{rT}}{S(T)}\right\}\right\} \\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} E_t^{Q'}\left\{\frac{S(t)}{e^{rt}}\, e^{-r(\tau-t)} \right. \\
&\qquad \left. \times [\rho(S(\tau) - A(\tau))]^+ \frac{e^{r\tau}}{S(\tau)}\right\} \\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} E_t^{Q'}\left\{S(t)\left[\rho\left(1 - \frac{A(\tau)}{S(\tau)}\right)\right]^+\right\} \\
&= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} S(t)E_t^{Q'}\{[\rho(1 - x(\tau))]^+\}.
\end{aligned} \tag{5}$$

Now, if $x(\cdot)$ is a Markov process on the filtration generated by $x(\cdot)$, the problem in (5) is potentially much easier to solve than (2). Specifically, it follows from Øksendal (1992) (Theorem 11.3) that the optimal stopping time for this problem is contained in the set

$$\mathcal{T}^{x}_{t,T} = \{\tau(\omega, u) \in \mathcal{T}_{t,T} | \tau \equiv f(x_u, u),\, f \text{ measurable}\}.$$

Below we therefore examine $x(\cdot)$ a little further. First, using Itô's Lemma we obtain a stochastic differential equation describing the evolution in $x(\cdot)$:

$$\begin{aligned}
dx(t) &= d\left(\frac{A(t)}{S(t)}\right) \\
&= \frac{1}{S(t)}\, dA(t) - \frac{A(t)}{S^2(t)}\, dS(t) + \frac{A(t)}{S^3(t)}\, (dS(t))^2 \\
&= x(t)\, \frac{dA(t)}{A(t)} - x(t)((r + \sigma^2)dt + \sigma dW^{Q'}(t)) \\
&\quad + x(t)\sigma^2 dt \\
&= x(t)\, \frac{dA(t)}{A(t)} - rx(t)dt - \sigma x(t)dW^{Q'}(t).
\end{aligned} \tag{6}$$

The exact form of the term $dA(t)/A(t)$ of course depends on the type of averaging. For the arithmetic average we have

$$\frac{dA(t)}{A(t)} = \frac{1}{t}\, (x^{-1}(t) - 1)dt, \tag{7}$$

while the choice of geometric averaging yields

$$\frac{dA(t)}{A(t)} = -\frac{1}{t}\ln x(t)dt. \tag{8}$$

Note that regardless of the type of averaging, the average process is a process of bounded variation. Using subscripts A and G on $x(\cdot)$ to denote arithmetic and geometric averaging, respectively, and using (7) and (8), we can define

$$\mu_A(x_A(t), t) \equiv \frac{1}{t}\, (x_A^{-1}(t) - 1) - r \tag{9}$$

and

[5] To see this, recall that from Harrison and Kreps (1979) all asset prices, $f(\cdot)$, denominated in units of the money market account are Q-martingales, i.e., $e^{-rt}f(t) = E_t^{Q}\{e^{-rT}f(T)\}$. Using this along with (3) and (4) we get that

$$f(t) = E_t^{Q'}\left\{\frac{\xi(t)}{\xi(T)}\, f(T)\right\} = E_t^{Q'}\left\{S(t)\, \frac{f(T)}{S(T)}\right\},$$

so $(f(\cdot)/S(\cdot))$ is clearly a Q'-martingale.

$$\mu_G(x_G(t), t) \equiv -\left[\frac{1}{t} \ln x_G(t) + r\right], \qquad (10)$$

and thus represent two specific forms of (6):

$$\frac{dx_A(t)}{x_A(t)} = \mu_A(x_A(t), t)dt - \sigma dW^{Q'}(t) \qquad (11)$$

and

$$\frac{dx_G(t)}{x_G(t)} = \mu_G(x_G(t), t)dt - \sigma dW^{Q'}(t). \qquad (12)$$

From (11) and (12) we conclude that the processes $x_A(\cdot)$ and $x_G(\cdot)$ are both Markov processes on the filtration generated by $x(\cdot)$. Therefore (5) becomes

$$V(t) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}^x_{t,T}} S(t)E^{Q'}_t\{[\rho(1 - x(\tau))]^+\}. \qquad (13)$$

In other words, this allows us to characterize the optimal stopping rule relating to Problem (5) as follows (see e.g., Karatzas 1989)

$$\tau^*_t = \inf\{s \in [t, T] | x(s) = x^*(s)\}. \qquad (14)$$

The economic interpretation of (14) is very important. It says that regardless of the type of averaging, the American option is optimally exercised the first time the process $x(\cdot)$ reaches a critical value, which may, of course, be a function of time. Because $x(\cdot)$ is defined as the average divided by the contemporaneous stock price, it means that knowledge of this fraction (and of the critical value) is enough to decide whether to exercise the floating strike Asian option at a premature point in time. In other words, we have a *one state-variable problem* on our hands. This key observation is closely related to Ingersoll's (1987, pp. 377–378) brief discussion of the European-type floating strike Asian option. Ingersoll observes that a formulation in terms of a single state variable is possible, and the solution is characterized in terms of a partial differential equation. However, by using S/A as the state variable, and hence a nontraded variable as the numeraire, martingale pricing techniques cannot be exploited. Moreover, Ingersoll does not consider American-style contracts.

Returning to our formulation, the function $x^*(\cdot)$ is continuous for $t \in [0, T[$ and defines a separation of the state space into a continuation region and a stopping (exercise) region, which we denote by $\mathcal{C}$ and $\mathcal{S}$, respectively. Thus, for the floating strike call for example, $\{x(t) \le x^*(t)\}$ defines $\mathcal{S}$ and $\{x(t) > x^*(t)\}$ defines $\mathcal{C}$.[6] Finally, observe that (13) could also be interpreted as a standard American option pricing problem with $x(\cdot)$ as an auxiliary asset with an appropriate dividend stream.

Denoting the stock price denominated value of the American-style floating strike Asian option by $\bar{V}(x(t), t)$, we proceed to state our general valuation theorem.

THEOREM 1. *The stock price denominated time t value of the floating strike Asian option is given by*

$$\bar{V}(x_i(t), t) = \bar{v}(x_i(t), t) + \bar{e}(x_i(t), t), \qquad (15)$$

where

$$\bar{v}(x_i(t), t) \equiv E^{Q'}_t\{[\rho(1 - x_i(T))]^+\} \qquad (16)$$

and

$$\bar{e}(x_i(t), t) \equiv E^{Q'}_t\left\{\int_t^T \rho\mu_i(x_i(u), u)x_i(u)1_{\mathcal{S}}(x_i(u))du\right\} \qquad (17)$$

and where $\rho = \pm 1$ *and* $i \in \{A, G\}$.[7]

In economic terms Theorem 1 is a decomposition result that states that the (S-denominated) value of the American-style floating strike Asian option can be decomposed into the (S-denominated) value of an otherwise identical European option given by (16), *and* a residual given by (17), which by definition represents the (S-denominated) value of the right to exercise the American option before the maturity date. For this reason (17) is labeled the *early exercise premium*. It

[6] More precisely for the case discussed we have, for example,
$\mathcal{C} = \{(t, x) | t \in [0, T] \wedge x(t) > x^*(t)\}$.

[7] Since by domination $\bar{V} \ge \bar{v}$, the integrand in (15) must be positive. This has implications for the exercise boundary relating to the floating strike call option, i.e., the case of $\rho = 1$: Note first that on $\mathcal{S}$ we clearly have $x(u) < 1 \ \forall \ u \in [t, T[$. Therefore we must have $\mu(x(u), u) \ge 0$. By checking (9) and (10) it is observed that in the case of geometric averaging this implies $x^*(u) \le e^{-ru}$, and with arithmetic averaging we must have $x^*(u) \le (1/(1 + ru))$. The reader can easily check that with $\rho = -1$ there are no additional restrictions imposed on x^*.

is also interesting to observe that in the S-numeraire economy the discount rate is zero, and that the European part of the original floating strike call (put) option is transformed and priced as one would normally price a standard unit exercise price put (call) option on $x(\cdot)$ in a zero interest rate economy, simply by calculating its expected payoff at maturity.

The economic interpretation of the mathematical representation of the early exercise premium in (17) is similar to the interpretation of the early exercise premium in the plain vanilla case (see e.g., Kim 1990, Jacka 1991, Jamshidian 1992, and Carr et al. 1992). There are some extra twists here, however, because we are working under Q' and because we are dealing with an option on a nonprice variable such as $x(\cdot)$. To understand (17), consider, for example, the case of $\rho = 1$ corresponding to the situation where we are considering a unit exercise price put option on $x(\cdot)$. Suppose further that the put option is in the exercise region but that exercise for some reason must be delayed for an instant, du. Now, as a consequence of the delay the option holder would on one hand lose interest on the exercise price, but on the other hand gain from the "dividend" received from the "underlying asset." In the present case the lost interest amounts to zero, and although our state variable, $x(\cdot)$, is not the price of a security, the negative of its drift is nevertheless exactly like the drain in an asset price created by a continuous dividend stream. Hence, a fair compensation for a (hypothetical) loss of the right to exercise prematurely between time t and T would be to provide the option holder with the instantaneous cash flow $\rho\mu_i(x_i(u), u)x_i(u)du$ everywhere in the exercise region. It is thus also meaningful to label this entity as the S-deflated *cost of carry* of the American option in the exercise region. The S-deflated value of such a conditional cash flow is established in (17) by appropriate integration and thus establishes the early exercise premium.[8] It should finally be noted that it is also the presence of a nonzero drift in $x(\cdot)$ that provides the incentive for early exercise in the case of the call on $x(\cdot)$ (the floating strike put).

Proof of Theorem 1. We prove the result by first considering separately price changes on the continuation and stopping regions.

On $\mathscr{C}$ the option is "alive," and by Itô's Lemma $\bar{V}$ solves the stochastic differential equation

$$
\begin{aligned}
d\bar{V} &= \frac{\partial \bar{V}}{\partial x}\,dx + \frac{1}{2}\frac{\partial^2\bar{V}}{\partial x^2}(dx)^2 + \frac{\partial \bar{V}}{\partial t}\,dt \\
&= \frac{\partial \bar{V}}{\partial x}(\mu(x(t), t)x(t)dt - \sigma x(t)dW^{Q'}(t)) \\
&\quad + \frac{1}{2}\sigma^2 x^2(t)\frac{\partial^2\bar{V}}{\partial x^2}\,dt + \frac{\partial \bar{V}}{\partial t}\,dt \\
&= \left[\mu(x(t), t)x(t)\frac{\partial \bar{V}}{\partial x} + \frac{1}{2}\sigma^2 x^2(t)\frac{\partial^2\bar{V}}{\partial x^2} + \frac{\partial \bar{V}}{\partial t}\right]dt \\
&\quad - \sigma x(t)\frac{\partial \bar{V}}{\partial x}\,dW^{Q'}(t) \\
&= -\sigma x(t)\frac{\partial \bar{V}}{\partial x}\,dW^{Q'}(t),
\end{aligned}
\tag{18}
$$

where functional arguments of $\bar{V}$ and subscripts on μ and x have been suppressed for ease of notation. Note that the last equality follows from the fact that $\bar{V}$ is a Q'-martingale.

On the stopping region, $\mathscr{S}$, the option value (denominated by the stock price) is simply the intrinsic value of the option, i.e.,

$$\bar{V}(x(t), t) = \rho(1 - x(t)), \tag{19}$$

so that on $\mathscr{S}$ we have

$$
\begin{aligned}
d\bar{V}(x(t), t) &= -\rho dx(t) \\
&= -\rho(\mu(x(t), t)x(t)dt - \sigma x(t)dW^{Q'}(t)).
\end{aligned}
$$

Hence, on the entire state space we have

[8] Using similar *economic* arguments but working under the risk-neutral measure Q the reader can easily establish the following alternative characterization of the early exercise premium:

$$E_t^Q\left\{\int_t^T e^{-r(u-t)}\rho\left(\frac{dA(u)}{du} - rA(u)\right)1_{\mathscr{S}}(A(u), S(u), u)du\right\}.$$

The skeptical *mathematician* can try to change the probability measure in the above expression from Q to Q' and thereby establish the equivalence of this expression to ($S(t)$ times) Expression (17) in Theorem 1.

$$d\bar{V}(x(t), t) = -\rho 1_{\mathcal{S}}(x(t))\mu(x(t), t)x(t)dt + dM^{Q'}(t), \tag{20}$$

where $M^{Q'}(t)$ is a Q'-martingale.

Finally, integrating (20) from t to T and taking expectation gives the desired result.[9] □

3. Specific Analytical Valuation

In this section we examine the result stated in Theorem 1 more closely for the two kinds of averaging that we are considering. In the case of geometric averaging we derive exact and easily implementable analytical formulas for American-style floating strike Asian options. In the case of arithmetic averaging, approximation formulas with characteristics in common with the geometric case are proposed.

3.1. The Case of Geometric Averaging

In the case where averaging is geometric, the distribution of $x_G(t)$ can be determined and the valuation expression from Theorem 1 can thus be further manipulated. The following lemma concerns the conditional distribution of the logarithm of the state variable, i.e., the geometric average divided by the stock price.

LEMMA 1. *For $u > t$*

$$\ln x_G(u)|\mathcal{F}_t \sim \mathcal{N}(\alpha_G(t, u), \beta_G^2(t, u)), \tag{21}$$

where

$$\alpha_G(t, u) = \frac{t}{u}\ln x_G(t) - \frac{u^2 - t^2}{2u}\left(r + \frac{1}{2}\sigma^2\right) \tag{22}$$

and

$$\beta_G^2(t, u) = \frac{\sigma^2}{3u^2}(u^3 - t^3). \tag{23}$$

The lemma can be proven by the help of Lemma A.1 *and Lemma* A.2 *in the Appendix.*

Lemma 1 allows us to manipulate (15) further in the geometric average case. The following two theorems deal with the European option price and the early exercise premium in turn.

THEOREM 2 (VALUATION OF THE EUROPEAN-TYPE FLOATING STRIKE ASIAN OPTION). *For the call and put, respectively, let $c_G(t)$ and $p_G(t)$ denote the time t values of the European-type floating strike Asian option expiring at time T and based on geometric averaging initiated at time* 0. *Theorem* 1 *and Lemma* 1 *imply the following results:*

$$c_G(t) = S(t)\left[\Phi\left(-\frac{\alpha_G(t, T)}{\beta_G(t, T)}\right) - e^{\alpha_G(t,T)+1/2\beta_G^2(t,T)}\Phi\left(-\frac{\alpha_G(t, T)}{\beta_G(t, T)} - \beta_G(t, T)\right)\right], \tag{24}$$

$$p_G(t) = S(t)\left[e^{\alpha_G(t,T)+1/2\beta_G^2(t,T)}\Phi\left(\frac{\alpha_G(t, T)}{\beta_G(t, T)} + \beta_G(t, T)\right) - \Phi\left(\frac{\alpha_G(t, T)}{\beta_G(t, T)}\right)\right], \tag{25}$$

where $\alpha_G(\cdot, \cdot)$ and $\beta_G(\cdot, \cdot)$ are defined in (22) *and* (23) *and $\Phi(\cdot)$ denotes the standard normal distribution function.*

The result is not new. It is implicit in Margrabe (1978), where the value of an option to exchange one asset for another is established, and the result can also be found in Boyle (1993). Naturally, the result is included here because it is an essential part of the entire American option price expression.

PROOF OF THEOREM 2. The result follows immediately from Theorem 1, Lemma 1, and Lemma A.3 in the Appendix. □

Theorem 3 below presents the specific result regarding the early exercise premium in the case of geometric averaging.

THEOREM 3 (THE EARLY EXERCISE PREMIUM). *For the call and put option, respectively, let $e_G^c(t)$ and $e_G^p(t)$*

[9] An alternative proof is the following: Observe that $\bar{V}$ satisfies the following inhomogeneous partial differential equation in the entire state space:

$$\mu(x(t), t)x(t)\frac{\partial \bar{V}}{\partial x} + \frac{1}{2}\sigma^2 x^2(t)\frac{\partial^2 \bar{V}}{\partial x^2} + \frac{\partial \bar{V}}{\partial t} + \rho 1_{\mathcal{S}}(x(t))\mu(x(t), t)x(t) = 0$$

with boundary condition

$$\bar{V}(x(T), T) = [\rho(1 - x(T))]^+.$$

This is recognized as a standard Cauchy-problem, for which a probabilistic representation of the solution is known from, e.g., Friedman (1975). This solution is of course identical to (15). Also note that since the above expression is a simple PDE in one state variable and time, it can be used as the basis for finite difference solutions of the problems studied here.

denote the early exercise premium at time t in relation to the American-style floating strike Asian option expiring at time T and based on geometric averaging initiated at time 0. Then we have

$$\frac{e_G^c(t)}{S(t)} = \int_t^T e^{\alpha_G(t,u)+1/2\beta_G^2(t,u)} \times \left[\frac{\beta_G(t,u)}{u}\phi(\gamma_G(t,u)) - \Phi(-\gamma_G(t,u))\left\{\frac{\alpha_G(t,u)+\beta_G^2(t,u)}{u}+r\right\}\right]du \quad (26)$$

and

$$\frac{e_G^p(t)}{S(t)} = \int_t^T e^{\alpha_G(t,u)+1/2\beta_G^2(t,u)} \times \left[\frac{\beta_G(t,u)}{u}\phi(\gamma_G(t,u)) + \Phi(\gamma_G(t,u))\left\{\frac{\alpha_G(t,u)+\beta_G^2(t,u)}{u}+r\right\}du, \quad (27)\right.$$

where $\alpha_G(t,u)$ and $\beta_G(t,u)$ are as defined in (22) *and* (23), *$\phi(\cdot)$ denotes the standard normal density function, and*

$$\gamma_G(t,u) = \frac{(\alpha_G(t,u)+\beta_G^2(t,u)) - \ln x_G^*(u)}{\beta_G(t,u)}.$$

Of course, $x_G^*(u)$ is specific to the choice of ρ, i.e., whether a call or a put is considered.

Proof. We prove (27) and leave the similar proof of (26) to the reader. From Theorem 1 and the choice $\rho = -1$ we have

$$\frac{e_G^p(t)}{S(t)} = -E_t^{Q'}\left\{\int_t^T \mu_G(x_G(u),u)x_G(u)1_{\{x_G(u)\ge x^*_G(u)\}}du\right\}.$$

Substituting for $\mu_G(x(u),u)$ (from (10)) and interchanging the order of integration we get

$$\frac{e_G^p(t)}{S(t)} = \int_t^T E_t^{Q'}\left\{\left[\frac{1}{u}\ln x_G(u)+r\right] \times x_G(u)1_{\{x_G(u)\ge x^*_G(u)\}}\right\}du.$$

In light of Lemma 1 the remaining part of the proof is simple algebraic manipulation, which is helped by some results listed in Lemma A.3 of the Appendix:

$$\begin{aligned}\frac{e_G^p(t)}{S(t)} &= \int_t^T \left(\frac{1}{u}E_t^{Q'}\{\ln x_G(u)x_G(u)1_{\{x_G(u)\ge x^*_G(u)\}}\} + rE_t^{Q'}\{x_G(u)1_{\{x_G(u)\ge x^*_G(u)\}}\}\right)du \\ &= \int_t^T \left(\frac{1}{u}e^{\alpha_G(t,u)+1/2\beta_G^2(t,u)}[\beta_G(t,u)\phi(\gamma_G(t,u)) + (\alpha_G(t,u)+\beta_G^2(t,u))\Phi(\gamma_G(t,u))] + re^{\alpha_G(t,u)+1/2\beta_G^2(t,u)}\Phi(\gamma_G(t,u))\right)du \\ &= \int_t^T e^{\alpha_G(t,u)+1/2\beta_G^2(t,u)} \times \left[\frac{\beta_G(t,u)}{u}\phi(\gamma_G(t,u)) + \Phi(\gamma_G(t,u))\times\left\{r+\frac{\alpha_G(t,u)+\beta_G^2(t,u)}{u}\right\}\right]du. \quad \square\end{aligned} \quad (28)$$

Theorems 2 and 3 provide an analytical characterization of the components that make up the American option price in the geometric average case. We defer the discussion of the implementation of the formula until we have dealt with the arithmetic average options.

3.2. The Case of Arithmetic Averaging

Contrary to the case of geometric averaging, the distribution for the arithmetic average is not known. This implies that no exact closed form solution is available for European-style Asian options based on the arithmetic average.[10] Because no analytical formula is available for European-style Asian options, it

[10] In Milevsky and Posner (1998) the authors provide closed-form approximation formulas for arithmetic Asian options based on the observation that the *infinite sum* of correlated lognormal variables has a well-known distribution—the Reciprocal Gamma.

does not come as a surprise that an analytical solution to the similar American-style problem is not feasible either. Specifically, note that the exact analytical evaluation of the expectation terms of our general valuation Formula (15) is not possible because the distribution of $x_A(\cdot)$ is unknown. However, drawing on some ideas from the literature on the European-style Asian options one can still utilize (15) as the basis for derivation of approximation formulas for the value of arithmetic average-based American-style floating strike Asian options. That is the goal of the present section. Our approach is inspired by the treatment of European-style average options by Levy (1992) and Turnbull and Wakemann (1991). Their idea is to approximate the unknown density of the state variable(s) by some known density so that the evaluation of the necessary integrals becomes feasible.

Levy demonstrates that for realistic parameter values "the distribution of an arithmetic average is well-approximated by the log normal distribution when the underlying price process follows the conventional assumption of a geometric diffusion," see Levy (1992, p. 475). In particular, for floating strike Asian options Levy (1992) suggests approximating the bivariate distribution of the log of the average and the log of the underlying state variable by a bivariate normal distribution. Now, our approach will differ from this methodology in two important ways.

First, because we are dealing with American-style options, we need not just *one* approximate distribution of the variables at the expiration date, but a whole family of approximating distributions since, as clarified by Lemma 1, all the conditional distributions of $x_A(u)|\Im_t$ for $0 \leq t \leq u \leq T$ must be characterized. This will be done by approximating at any time t the remainder of the stochastic process $(x_A(u))_{u \geq t}$ by a geometric Brownian motion $(\hat{x}_A(u, t))_{u \geq t}$ augmented with appropriate time-varying coefficients.

Second, as a consequence of our numeraire approach we propose to approximate the log of the average *divided* by the log of the contemporaneous stock price to a normal distribution instead of approximating the log of the average and the log of the contemporaneous stock price to a bivariate normal distribution. Here the former methodology is superior to the latter in the sense that weaker assumptions are necessary. Indeed, if the log of the average and the log of the contemporaneous stock price are bivariate normally distributed, then the log of the fraction between these two variables is also normally distributed, whereas the reverse is not in general true.

The procedure for identifying the appropriate approximating processes is one of matching moments of the true and the approximating processes. This is feasible since the necessary (conditional) true moments of the state variable can be characterized analytically. In its general version this methodology is known as Edgeworth expansion, and in the special case where only first and second moments are matched to a log normal distribution, the exercise is known as the Wilkinson approximation; see again Levy (1992). Note here that the first two moments are sufficient to fully characterize a lognormal distribution.

To be more specific, for $u > t$ consider $\ln \hat{x}_A(t, u)|\Im_t$, which is normal with mean and variance parameters $\alpha_A(t, u)$ and $\beta_A^2(t, u)$, respectively. The Wilkinson approximation then prescribes that

$$\alpha_A(t, u) = 2 \ln E_t^{Q'}\{x_A(u)\} - \tfrac{1}{2} \ln E_t^{Q'}\{x_A^2(u)\} \quad (29)$$

and

$$\beta_A^2(t, u) = \ln E_t^{Q'}\{x_A^2(u)\} - 2 \ln E_t^{Q'}\{x_A(u)\}. \quad (30)$$

As suggested above, the first and second conditional moments (under Q') of $x_A(u)$ can be calculated analytically. Straightforward but tedious calculations (some of which have been relegated to Lemma A.4 in the Appendix) lead to

$$E_t^{Q'}\{x_A(u)\} = \frac{t}{u} x_A(t) e^{-r(u-t)} + \frac{1}{ru} (1 - e^{-r(u-t)}) \quad (31)$$

and

$$E_t^{Q'}\{x_A^2(u)\} = \left(\frac{t}{u}\right)^2 x_A^2(t)e^{-(2r-\sigma^2)(u-t)} + \frac{2(r-\sigma^2) - (4r-2\sigma^2)e^{-r(u-t)} + 2re^{-(2r-\sigma^2)(u-t)}}{u^2 r(2r-\sigma^2)(r-\sigma^2)} + x_A(t)\frac{2te^{-r(u-t)}}{u^2(r-\sigma^2)}(1-e^{-(r-\sigma^2)(u-t)}). \quad (32)$$

This brings us to a position in which we can state our final theorem, which contains approximation formulas for the two components—the European option price and the early exercise premium—that make up the value of the American-style floating strike Asian options based on arithmetic averaging. The theorem is based on evaluation of the expectations entering the result in Theorem 1 substituting the approximating process $\hat{x}_A$ for x_A; cf. the above considerations.

Theorem 4. *Let $\hat{c}_A(t)$ denote the approximate time t value of a European-style floating strike Asian call option based on continuous arithmetic averaging initiated at time 0. Further, let $\hat{e}_A^c(t)$ denote the approximate early exercise premium at time t of the similar American-style option and use $\hat{p}_A(t)$ and $\hat{e}_A^p(t)$ as notation for the similar premium in relation to the otherwise identical put options. Then we have*

$$\hat{c}_A(t) = S(t)\left[\Phi\left(-\frac{\alpha_A(t,T)}{\beta_A(t,T)}\right) - e^{\alpha_A(t,T)+1/2\beta_A^2(t,T)}\Phi\left(-\frac{\alpha_A(t,T)}{\beta_A(t,T)} - \beta_A(t,T)\right)\right], \quad (33)$$

$$\hat{p}_A(t) = S(t)\left[e^{\alpha_A(t,T)+1/2\beta_A^2(t,T)}\Phi\left(\frac{\alpha_A(t,T)}{\beta_A(t,T)} + \beta_A(t,T)\right) - \Phi\left(\frac{\alpha_A(t,T)}{\beta_A(t,T)}\right)\right], \quad (34)$$

where $\alpha_A(\cdot,\cdot)$ and $\beta_A(\cdot,\cdot)$ are defined in (29) *and* (30) *(via* (31) *and* (32)). *The early exercise premiums are as follows*:

$$\hat{e}_A^c(t) = S(t)\int_t^T\left[\frac{1}{u}\Phi(\beta_A(t,u) - \gamma_A(t,u)) - \left(r+\frac{1}{u}\right)e^{\alpha_A(t,u)+1/2\beta_A^2(t,u)}\Phi(-\gamma_A(t,u))\right]du \quad (35)$$

and

$$\hat{e}_A^p(t) = S(t)\int_t^T\left[\left(r+\frac{1}{u}\right)e^{\alpha_A(t,u)+1/2\beta_A^2(t,u)}\Phi(\gamma_A(t,u)) - \frac{1}{u}\Phi(\gamma_A(t,u) - \beta_A(t,u))\right]du, \quad (36)$$

where

$$\gamma_A(t,u) = \frac{(\alpha_A(t,u) + \beta_A^2(t,u)) - \ln \hat{x}_A^*(u)}{\beta_A(t,u)}.$$

Proof. The proof of (33) and (34) follows immediately from Theorem 1, (29)–(32), and Lemma A.3 in the Appendix. Below we prove (35) and leave the similar proof of (36) to the reader.

From Theorem 1 we have:

$$\frac{\hat{e}_A^c(t)}{S(t)} = E_t^{Q'}\left\{\int_t^T \mu_A(\hat{x}_A(t,u), u)\hat{x}_A(t,u)1_{\{\hat{x}_A(t,u)\le\hat{x}_A^*(u)\}}\right\}du.$$

Substituting for $\mu_A(\hat{x}_A(t,u), u)$ using (9) and interchanging the order of integration we get

$$\frac{\hat{e}_A^c(t)}{S(t)} = \int_t^T E_t^{Q'}\left\{\left[\frac{1}{u}(\hat{x}_A^{-1}(t,u)-1) - r\right] \times \hat{x}_A(t,u)1_{\{\hat{x}_A(t,u)\le\hat{x}_A^*(u)\}}\right\}du$$

$$= \int_t^T\left(\frac{1}{u}E_t^{Q'}\{1_{\{\hat{x}_A(t,u)\le\hat{x}_A^*(u)\}}\} - \left(r+\frac{1}{u}\right)E_t^{Q'}\{\hat{x}_A(t,u)1_{\{\hat{x}_A(t,u)\le\hat{x}_A^*(u)\}}\}\right)du$$

$$= \int_t^T\left(\frac{1}{u}\Phi(\beta_A(t,u) - \gamma_A(t,u)) - \left(r+\frac{1}{u}\right)e^{\alpha_A(t,u)+1/2\beta_A^2(t,u)} \times \Phi(-\gamma_A(t,u))\right)du,$$

where—to obtain the last equality—we have again used Lemma A.3. in the Appendix. □

4. Numerical Results

In this section the numerical implementation of our specific valuation formulas is discussed and pricing results are provided.

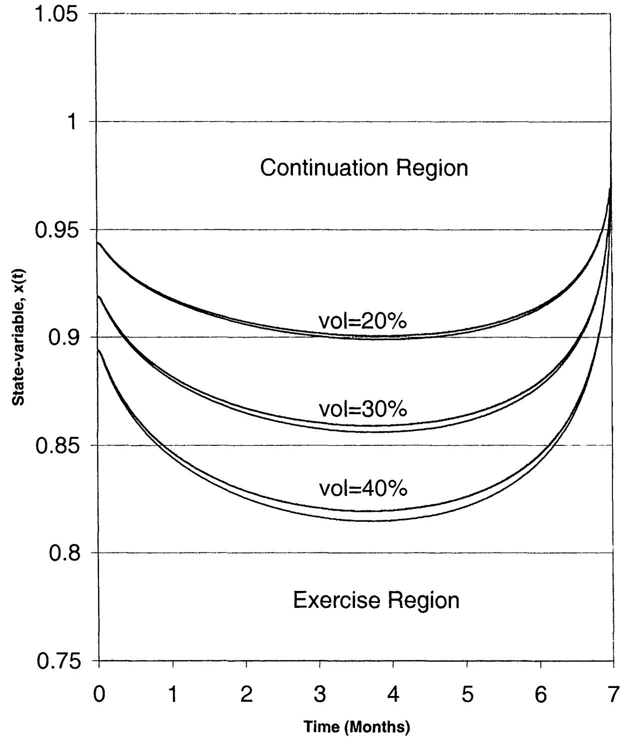
To implement the valuation formulas derived in §3, a numerical integration routine (a *Quadrature*) is needed to evaluate any of the $e(t)$ expressions in (26)–(27) and (35)–(36). The major problem here is that to calculate any of the early exercise premiums in principle, we need to know $x^*(u)$ for $u \in [t, T]$. Because at the outset of the analysis this optimal exercise boundary is unknown, it must be determined as part of the solution. For this purpose the boundary Condition (19), i.e.,

$$\bar{V}(x^*(t), t) = \rho(1 - x^*(t)), \tag{37}$$

is essential, and the analysis proceeds as follows: At the maturity date we obviously have $x^*(T) = 1$. To obtain intermediate values of x^*, a discretization scheme is used. The time interval $[0, T]$ is divided into n intervals of equal length Δt defining the time points $0 = t_0 < t_1 < \cdots < t_i < \cdots < t_n = T$, where $t_i - t_{i-1} = \Delta t = T/n$. The methodology is then to work "backwards" toward $t = 0$ and determine for each t_i the solution $x^*(t_i)$ to (37) using the numerical integration routine. It should be emphasized that the determination of $x^*(t_i)$ is contingent on $x^*(t_{i+1}), \ldots, x^*(t_n)$ having been determined already, and this also explains why the backwardation is necessary. When $t = 0$ is reached and $x^*(0)$ is determined, the early exercise premium, and hence the American option price, can be determined by one final numerical evaluation of the integral in question (see also Kim 1990 and Jørgensen 1997). The accuracy of the procedure will naturally increase in n, but at the expense of increased computation time; cf. also later.

Figures 1 and 2 show plots of selected optimal exercise boundaries in relation to some geometric as well as some arithmetic average-based options. The curves are obtained as output from the numerical integration procedures used to implement the earlier derived pricing formulas; cf. above. In each of the figures, time to maturity of the options is 7 months and the risk-free interest rate is fixed at 5%. For both the geometric and arithmetic options the optimal exercise boundary is plotted for each of the volatilities 20%, 30%, and 40%, yielding six curves per figure.

Figure 1 Optimal Exercise Boundaries, American-style Asian Call Options (T = 7 mths, r = 5%)



There are several interesting things to observe from these figures. First, the optimal exercise boundaries are nonmonotonic, contrary to the case of standard American options where these curves converge to horizontal lines as maturity is lengthened. In the present situation the distance from "at-the-money" to "critical value" (what one might call "critical moneyness") is relatively short at the beginning, larger in the middle, and smaller again toward the end of the life of the option.

It should be kept in mind, however, that the figure(s) show critical values of x, that is of A/S, as a function of time. If one feels uncomfortable with values measured in the S-numeraire, as an alternative one could illustrate the location of the continuation and stopping regions in an (S, A) diagram such as

Figure 2 **Optimal Exercise Boundaries, American-style Asian Put Options (T = 7 mths, r = 5%)**

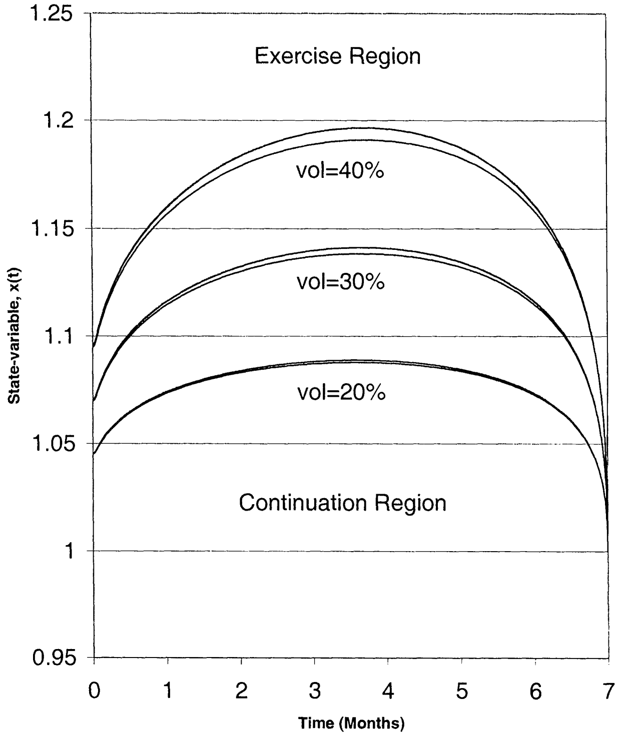


Figure 3. Here, for a single put option problem (r = 0.05, T = 7 months, and σ = 0.4) we have fixed t at three different times and plotted the straight line emitting from the origin, which separates the continuation and stopping regions at these specific dates. The slope of each of the lines is the critical value of the state variable, i.e., $x^*(t)$, in relation to the specific date.

Returning to Figures 1 and 2, note that for a given volatility the optimal exercise boundary for the geometric and arithmetic options are very close. (The curves relating to the arithmetic options are slightly above the curves relating to the geometric options in both figures.) Note also that increasing the volatility has the effect of enlarging the critical moneyness. This has the obvious implication that with a larger volatility the option holder would, ceteris paribus, wait longer before exercising the option relative to a low volatility situation.

Finally, note that at the inception of the averaging period at time 0 there is actually quite a distance from the starting point ($x(0)$ is equal to 1 by definition) to the critical value, $x^*(0)$. This confirms our statement from the Introduction that American-style Asian options are not trivial in the sense that they are particularly vulnerable to price manipulation and consequently have a too high probability of being exercised early because they always start out being "at-the-money."

Tables 1 and 2 show prices of a collection of floating strike Asian calls and puts, respectively. For a representative set of parameters the formulas from §3 were used (with n = 10,000 in the numerical integration routine; cf. above) to obtain prices of American- and European-style options based on the two types of averaging that we have studied.

For the case of arithmetic average options we supplement the values obtained from using our proposed approximation formula with values obtained from applying a finite difference scheme directly to the relevant partial differential equation (see Footnote 9). Using a sufficiently fine grid in the finite difference routine so that the prices converge makes an evaluation of the accuracy of our approximation approach possible. A comparison of the last four columns of Tables 1 and 2 reveals that the two numerical pricing procedures deliver results that are in remarkable ac-

Figure 3 **Exercise and Continuation Region in (S, A)-space American-style Asian Put Option (vol = 40%, r = 5%, T = 7 mths)**

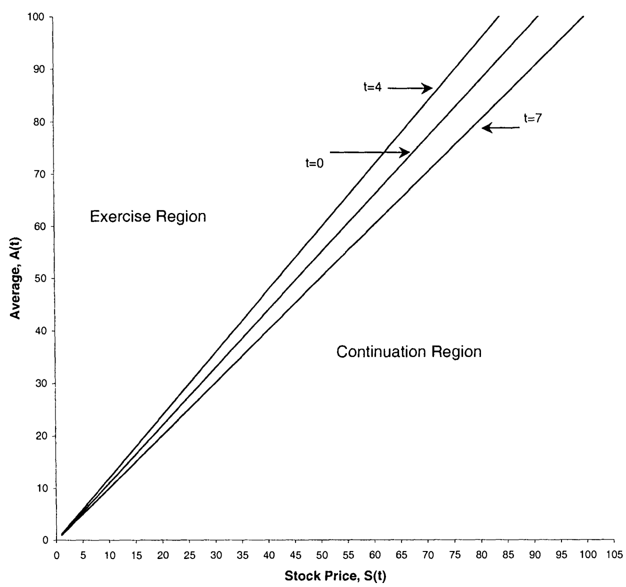


Table 1 Floating Strike Call Prices $S_0 = 100$

			Geometric Average (Exact Formula)		Arithmetic Average (Approximation Formula)		Arithmetic Average (Finite Difference Solution)	
r	T	σ	American	European	American	European	American	European
0.03	1/12	0.20	1.955	1.406	1.953	1.392	1.949	1.392
		0.30	2.909	2.088	2.905	2.056	2.895	2.056
		0.40	3.864	2.776	3.857	2.721	3.838	2.720
	4/12	0.20	4.007	2.967	3.998	2.907	3.980	2.907
		0.30	5.912	4.358	5.897	4.232	5.854	4.228
		0.40	7.818	5.774	7.799	5.560	7.718	5.548
	7/12	0.20	5.382	4.056	5.366	3.950	5.334	3.949
		0.30	7.898	5.917	7.874	5.697	7.796	5.688
		0.40	10.412	7.820	10.388	7.452	10.238	7.425
0.05	1/12	0.20	1.988	1.449	1.985	1.434	1.981	1.435
		0.30	2.941	2.130	2.936	2.098	2.926	2.097
		0.40	3.895	2.817	3.888	2.762	3.869	2.761
	4/12	0.20	4.138	3.143	4.127	3.079	4.110	3.079
		0.30	6.039	4.528	6.021	4.396	5.980	4.393
		0.40	7.942	5.941	7.918	5.719	7.839	5.709
	7/12	0.20	5.615	4.369	5.593	4.252	5.564	4.253
		0.30	8.119	6.217	8.088	5.982	8.014	5.975
		0.40	10.626	8.111	10.593	7.724	10.448	7.701
0.07	1/12	0.20	2.020	1.493	2.018	1.478	2.013	1.478
		0.30	2.973	2.173	2.968	2.140	2.958	2.140
		0.40	3.927	2.860	3.919	2.803	3.900	2.802
	4/12	0.20	4.275	3.325	4.261	3.257	4.245	3.258
		0.30	6.169	4.703	6.147	4.564	6.107	4.562
		0.40	8.067	6.110	8.040	5.880	7.963	5.872
	7/12	0.20	5.859	4.695	5.831	4.567	5.805	4.570
		0.30	8.348	6.525	8.309	6.274	8.238	6.271
		0.40	10.845	8.408	10.802	8.002	10.661	7.983

Number of Time Steps, n	Total Column CPU Time (Mean Relative Error)					
25	0.691 sec.		1.963 sec.		21.190 sec.	
	(2.048%)	(0.000%)	(2.821%)	(0.087%)	(3.547%)	(1.624%)
50	2.474 sec.		6.790 sec.		43.123 sec.	
	(1.075%)	(0.000%)	(1.793%)	(0.087%)	(0.624%)	(0.792%)
100	9.123 sec.		24.556 sec.		1 min. 27.035 sec.	
	(0.552%)	(0.000%)	(1.239%)	(0.087%)	(0.335%)	(0.374%)
200	34.600 sec.		1 min. 32.362 sec.		2 min. 54.952 sec.	
	(0.279%)	(0.000%)	(0.949%)	(0.087%)	(0.188%)	(0.159%)
400	2 min. 11.909 sec.		5 min. 50.404 sec.		5 min. 50.544 sec.	
	(0.138%)	(0.000%)	(0.800%)	(0.087%)	(0.113%)	(0.051%)

Table 2 Floating Strike Put Prices $S_0 = 100$

			Geometric Average (Exact Formula)		Arithmetic Average (Approximation Formula)		Arithmetic Average (Finite Difference Solution)	
r	T	σ	American	European	American	European	American	European
		0.20	1.858	1.254	1.861	1.267	1.864	1.267
	1/12	0.30	2.809	1.900	2.815	1.931	2.823	1.931
		0.40	3.759	2.540	3.771	2.596	3.784	2.595
		0.20	3.618	2.358	3.629	2.409	3.642	2.408
0.03	4/12	0.30	5.511	3.610	5.540	3.734	5.567	3.729
		0.40	7.398	4.834	7.457	5.062	7.501	5.050
		0.20	4.703	2.992	4.721	3.080	4.745	3.079
	7/12	0.30	7.197	4.613	7.250	4.827	7.296	4.818
		0.40	9.680	6.181	9.791	6.582	9.861	6.555
		0.20	1.827	1.214	1.830	1.226	1.833	1.226
	1/12	0.30	2.777	1.859	2.784	1.890	2.791	1.889
		0.40	3.727	2.498	3.739	2.554	3.752	2.552
		0.20	3.499	2.203	3.507	2.250	3.522	2.251
0.05	4/12	0.30	5.387	3.451	5.413	3.567	5.442	3.565
		0.40	7.272	4.671	7.326	4.890	7.372	4.880
		0.20	4.497	2.730	4.512	2.808	4.537	2.809
	7/12	0.30	6.982	4.339	7.028	4.537	7.077	4.531
		0.40	9.458	5.900	9.560	6.280	9.635	6.257
		0.20	1.797	1.174	1.799	1.187	1.803	1.187
	1/12	0.30	2.747	1.819	2.753	1.849	2.760	1.848
		0.40	3.696	2.458	3.707	2.512	3.720	2.511
		0.20	3.383	2.056	3.390	2.099	3.405	2.100
0.07	4/12	0.30	5.266	3.296	5.289	3.406	5.319	3.405
		0.40	7.147	4.512	7.198	4.722	7.246	4.714
		0.20	4.302	2.484	4.313	2.553	4.340	2.556
	7/12	0.30	6.774	4.076	6.814	4.260	6.866	4.257
		0.40	9.243	5.628	9.336	5.987	9.414	5.969

Number of Time Steps, n	Total Column CPU Time (Mean Relative Error)					
25	0.731 sec.		1.953 sec.		21.130 sec.	
	(3.104%)	(0.000%)	(2.472%)	(0.106%)	(4.453%)	(2.012%)
50	2.614 sec.		6.709 sec.		43.131 sec.	
	(1.636%)	(0.000%)	(1.060%)	(0.106%)	(0.979%)	(0.981%)
100	9.714 sec.		24.516 sec.		1 min. 26.995 sec.	
	(0.844%)	(0.000%)	(0.301%)	(0.106%)	(0.521%)	(0.465%)
200	36.793 sec.		1 min. 31.551 sec.		2 min. 55.012 sec.	
	(0.427%)	(0.000%)	(0.165%)	(0.106%)	(0.285%)	(0.198%)
400	2 min. 20.782 sec.		5 min. 46.288 sec.		5 min. 51.035 sec.	
	(0.212%)	(0.000%)	(0.303%)	(0.106%)	(0.163%)	(0.064%)

cordance. The biggest differences occur when the volatility and the time to maturity are both large.

We do not report finite difference pricing results for the geometric average-based options because these correspond exactly to the results obtained from the implementation of our exact analytical formulas when measured to three decimal places.

More generally, Tables 1 and 2 indicate that the option values increase in the volatility of the underlying stock returns and in the time to maturity, although this latter effect should not necessarily be observed in general for the European-type options. Naturally, the prices of the American-style options are always higher than those of the otherwise identical European options and, typically, significantly so. Furthermore, observe that for the calls, the geometric options are always more valuable than the otherwise identical arithmetic options, and vice versa for the puts. This should hold in general, because the geometric average is never bigger than an arithmetic average of the same numbers.

In the bottom parts of Tables 1 and 2 we provide some evidence of the speed, accuracy, and convergence properties of the numerical routines for various choices of the number of the associated time steps (in the finite difference algorithm we hold fixed the number of steps in the x-dimension at 10,000). The CPU times reported are the total computation times for the 27 pairs of prices in each column.[11] Accuracy is measured as *mean relative error*, i.e., as

$$MRE^n = \frac{1}{27} \sum_{i=1}^{27} \frac{|P_i^n - \bar{P}_i|}{\bar{P}_i} \cdot 100\%, \tag{38}$$

where P_i^n denotes the ith price of the table obtained from a numerical procedure using n time steps. $\bar{P}_i$ denotes the corresponding benchmark ("accurate") price obtained from the finite difference approach with $10{,}000 \times 10{,}000$ grid points (not reported).

The convergence properties of the numerical routines are seen to be quite good. For example, using 100 time steps in the quadrature, the set of geometric average-based American options is priced with a mean relative error below 1% in less than 10 seconds. The corresponding European options are priced without error independently of n by the closed Formulas (24) and (25). Note also that as n is doubled, the mean relative error is approximately cut in half and that the computation time approximately quadruples due to the structure of the quadrature described earlier.

As regards the arithmetic average-based Asian options, the implementation of the approximation formulas is not as fast as the implementation of the exact formulas in the geometric case. This is due to the rather complicated moment expressions that must be evaluated in this case. However, the procedure can still deliver quite accurate prices very quickly. Note that the European prices now contain an approximation error, but that this is again independent of n due to our use of the closed Formulas (33) and (34). It is finally emphasized that when implementing the approximation formulas for the arithmetic American options we should generally expect the mean relative error to converge—but not to zero—as n is increased.

For completeness we have also provided some details in relation to the speed and convergence properties of the finite difference procedure that was implemented to accurately price the arithmetic average-based options.

The results in Tables 1 and 2 suggest that the accuracy of our analytical approximation formulas for the options based on arithmetic averaging decreases in the volatility and in time to maturity. To understand the limitations of our approximation we therefore study a "difficult case" in Table 3. We fix volatility at a high value, 40%, and report the pricing accuracy of the analytical approximation as a function of time to maturity, letting T reach as much as 10 years. The analytical approximation formulas were implemented using 10,000 time steps in the quadrature, and the benchmark prices were again obtained from the finite difference algorithm with a $10{,}000 \times 10{,}000$ grid.

Table 3 confirms that our approximation formulas for arithmetic average-based options should be used with care when maturities and volatilities are large, especially if the options are calls. In the most extreme case where $T = 10$ years, our analytical approxima-

[11] The code was written in Delphi Pascal and implemented on a 450 MHz Pentium III-based PC. The CPU times were measured by means of Delphi Pascal's *DecodeTime* procedure.

Table 3 Prices of Arithmetic Average-Based American-Style Floating Strike Calls and Puts $S_0 = 100$, $\sigma = 0.40$, $r = 0.05$

		T					
		0.25	0.50	1.00	3.00	5.00	10.00
	Analytical Approximation	6.825	9.774	14.058	25.258	33.271	48.254
Calls	Accurate Finite Difference Benchmark	6.767	9.652	13.794	24.285	31.334	43.371
	Absolute Relative Error in %	0.860	1.269	1.912	4.005	6.181	11.258
	Analytical Approximation	6.380	8.888	12.301	20.162	25.055	33.106
Puts	Accurate Finite Difference Benchmark	6.416	8.954	12.413	20.351	25.213	32.891
	Absolute Relative Error in %	0.559	0.731	0.900	0.931	0.623	0.653

tion misprices the option by 11.25%. Interestingly, the mispricing is not nearly as severe for puts, for which the relative errors do not exceed 1% in our examples. The explanation for this difference is related to the fact that the expected lifetime of the American-style floating strike puts is generally shorter than that of the otherwise identical calls.

5. Conclusion

In this article we have derived analytical solutions for American-style Asian options based on floating strike prices. The derivation illustrated an important principle in that in some cases a change of numeraire can open up the possibility of reducing the dimensionality of the problem at hand and make the use of standard techniques from single state variable problems feasible.

Specifically, we defined the new state variable as the fraction of the average (geometric or arithmetic) and the stock price. The Markovian structure of this process, and the fact that boundary conditions could be stated in terms of this variable, allowed the optimal stopping problem to be characterized as a free boundary problem in terms of the fraction-process only. This led to a general decomposition result that characterized the value of the American option as the sum of the value of the otherwise identical European option and an early exercise premium.

Using the insights of previous research on the pricing of standard American options in the Black-Scholes setting (e.g., Kim 1990), we then gave an analytical solution to the option pricing problem depending on the distribution of the state variable. In the case of geometric averaging the process was shown to be lognormal, and on this basis the analytical formula was further simplified. In the case of arithmetic averaging, the distribution of the state variable is unknown. Inspired by the literature on European-style Asian options we suggested approximating the state variable by a lognormal process by matching of moments. This subsequently allowed for further manipulation of the analytical expressions for arithmetic average options.

Numerical examples illustrating the results were given. In particular these examples suggested that our approximation of the arithmetic average options is indeed accurate. Here the call options that have long time to maturity *and* that are written on very volatile assets displayed by far the most significant deviation from true values.

Future research could investigate the possibilities of applying the techniques proposed in this article to other similar American option pricing problems. For example, to the extent that the necessary distributions can be characterized our technique can be used to price American options with the strike price defined as a maximum or minimum of past prices.

Finally, we emphasize that our ability to derive analytical pricing formulas for the instruments considered here was crucially dependent on the assumption that the price variable followed a geometric Brownian motion. As pointed out by empiricists, this assumption is not always reasonable and it would therefore be interesting to explore possibilities for relaxing it, e.g., to allow for stochastic volatility and/or jumps in the underlying variable.[12]

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Appendix

Let the process $X(\cdot)$ evolve as a geometric Brownian motion, i.e.,

$$dX(t) = aX(t)dt + bX(t)dW(t), \qquad X(0) = X_0, \tag{A1}$$

where a and b are constant coefficients, X_0 is a constant, and $W(t)$ is a standard Wiener process. The solution to (A.1) is

$$X(T) = X(t) \cdot e^{(a-1/2b^2)(T-t)+b\int_t^T dW(u)}$$

for $0 \leq t \leq T$. It follows that

$$\ln X(T) = \ln X(t) + (a - \tfrac{1}{2}b^2)(T - t) + b\int_t^T dW(u), \tag{A2}$$

and hence that

$$\ln X(T) \sim \mathcal{N}(\ln X(t) + (a - \tfrac{1}{2}b^2)(T - t),\ b^2(T - t)). \tag{A3}$$

Lemma A1. *If we define*

$$Y(t) \equiv \exp\left\{\frac{1}{t}\int_0^t \ln(X(u))du\right\},$$

then

$$dY(t) = \frac{1}{t}Y(t)\left(\ln\frac{X(t)}{Y(t)}\right)dt$$

and

$$\ln Y(T) \sim \mathcal{N}\left(\frac{t}{T}\ln Y(t) + \frac{(T-t)}{T}\ln X(t) + \frac{(a - \frac{1}{2}b^2)}{2T}(T-t)^2,\ \frac{b^2(T-t)^3}{3T^2}\right). \tag{A4}$$

Proof of Lemma A1. To prove (A.4) we rewrite $\ln Y(T)$ using (A.2) and Fubini's Theorem

$$\begin{aligned}
\ln Y(T) &= \frac{1}{T}\int_0^T \ln X(u)du \\
&= \frac{1}{T}\int_0^t \ln X(u)du + \frac{1}{T}\int_t^T \ln X(u)du \\
&= \frac{t}{T}\ln Y(t) + \frac{1}{T}\int_t^T\left(\ln X(t) + \left(a - \frac{1}{2}b^2\right)(u-t) + b\int_t^u dW(s)\right)du \\
&= \frac{t}{T}\ln Y(t) + \frac{(T-t)}{T}\ln X(t) + \frac{(a - \frac{1}{2}b^2)}{T}\left|\frac{1}{2}u^2 - tu\right|_t^T + \frac{b}{T}\int_t^T\int_t^u dW(s)du \\
&= \frac{t}{T}\ln Y(t) + \frac{(T-t)}{T}\ln X(t) + \frac{(a - \frac{1}{2}b^2)}{2T}(T-t)^2 + \frac{b}{T}\int_t^T\int_s^T du dW(s) \\
&= \frac{t}{T}\ln Y(t) + \frac{(T-t)}{T}\ln X(t) + \frac{(a - \frac{1}{2}b^2)}{2T}(T-t)^2 + \frac{b}{T}\int_t^T (T-s)dW(s).
\end{aligned} \tag{A5}$$

The process $\ln Y(T)$ is thus clearly Gaussian with conditional mean and variance as follows:

$$E_t\{\ln Y(T)\} = \frac{t}{T}\ln Y(t) + \frac{(T-t)}{T}\ln X(t) + \frac{(a - \frac{1}{2}b^2)}{2T}(T-t)^2,$$

$$\begin{aligned}
\mathrm{Var}_t\{\ln Y(T)\} &= \frac{b^2}{T^2}\int_t^T (T-s)^2 ds \\
&= \frac{b^2}{T^2}\int_{T-t}^0 -u^2 du
\end{aligned}$$

$$= -\frac{b^2}{T^2}\left|\frac{1}{3}u^3\right|_{T-t}^{0}$$
$$= \frac{b^2(T-t)^3}{3T^2}. \quad \square$$

Lemma A2. *Defining*

$$Z(T) = \frac{Y(T)}{X(T)},$$

we have

$$\ln Z(T) = \frac{t}{T}\ln Z(t) - \frac{1}{2T}(T^2 - t^2)\left(a - \frac{1}{2}b^2\right) - \frac{b}{T}\int_t^T u\,dW(u),$$

and hence

$$\ln Z(T) \sim \mathcal{N}\left(\frac{t}{T}\ln Z(t) - \frac{1}{2T}(T^2 - t^2)\left(a - \frac{1}{2}b^2\right), \frac{b^2}{3T^2}(T^3 - t^3)\right).$$

Proof of Lemma A2.

$$\ln Z(T) = \ln Y(T) - \ln X(T)$$
$$= \frac{t}{T}\ln Y(t) + \frac{(T-t)}{T}\ln X(t) + \frac{(a - \frac{1}{2}b^2)}{2T}(T-t)^2 + \frac{b}{T}\int_t^T (T-u)\,dW(u) - \ln X(t) - (a - \tfrac{1}{2}b^2)(T-t) - b\int_t^T dW(u)$$
$$= \frac{t}{T}\ln Z(t) - \frac{(a - \frac{1}{2}b^2)}{2T}(T^2 - t^2) - \frac{b}{T}\int_t^T u\,dW(u). \quad \square \tag{A6}$$

Lemma A3. *Let*

$$v \equiv \ln V \sim \mathcal{N}(\alpha, \beta^2)$$

and define

$$\gamma \equiv \frac{(\alpha + \beta^2) - \ln K}{\beta}.$$

Assuming $K > 0$, and letting $\Phi(\cdot)$ and $\phi(\cdot)$ denote the standard normal cumulative distribution and density functions, respectively, we have:

1. $E\{V \cdot 1_{\{V \geq K\}}\} = e^{\alpha + 1/2\beta^2}\Phi(\gamma)$,
2. $E\{V \cdot 1_{\{V \leq K\}}\} = e^{\alpha + 1/2\beta^2}\Phi(-\gamma)$,
3. $E\{(V - K)^+\} = e^{\alpha + 1/2\beta^2}\Phi(\gamma) - K \cdot \Phi(\gamma - \beta)$,
4. $E\{(K - V)^+\} = K \cdot \Phi(\beta - \gamma) - e^{\alpha + 1/2\beta^2}\Phi(-\gamma)$,
5. $E\{V \cdot \ln V \cdot 1_{\{V \geq K\}}\} = e^{\alpha + 1/2\beta^2}(\beta\phi(\gamma) + (\alpha + \beta^2)\Phi(\gamma))$,
6. $E\{V \cdot \ln V \cdot 1_{\{V \leq K\}}\} = e^{\alpha + 1/2\beta^2}((\alpha + \beta^2)\Phi(-\gamma) - \beta\phi(\gamma))$.

Proof of Lemma A3.

1.

$$E\{V \cdot 1_{\{V \geq K\}}\} = \int_{\ln K}^{\infty} e^v f(v)\,dv$$
$$= \int_{\ln K}^{\infty} e^v \frac{1}{\sqrt{2\pi\beta^2}} e^{-1/(2\beta^2)(v-\alpha)^2}\,dv$$
$$= \int_{\ln K}^{\infty} \frac{1}{\sqrt{2\pi\beta^2}} e^{-1/(2\beta^2)(v^2 + \alpha^2 - 2\alpha v - 2\beta^2 v)}\,dv$$
$$= \int_{\ln K}^{\infty} \frac{1}{\sqrt{2\pi\beta^2}} e^{-1/(2\beta^2)(v - (\alpha + \beta^2))^2 - \alpha^2/(2\beta^2) + ((\alpha + \beta^2)^2)/(2\beta^2)}\,dv$$
$$= e^{1/(2\beta^2)[\alpha^2 + \beta^4 + 2\alpha\beta^2 - \alpha^2]} \int_{\ln K}^{\infty} \frac{1}{\sqrt{2\pi\beta^2}} e^{-1/(2\beta^2)(v - (\alpha + \beta^2))^2}\,dv$$
$$= e^{\alpha + 1/2\beta^2} \int_{1/\beta(\ln K - (\alpha + \beta^2))}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-1/2x^2}\,dx$$
$$= e^{\alpha + 1/2\beta^2}\Phi(\gamma). \tag{A7}$$

2.

$$E\{V \cdot 1_{\{V \leq K\}}\} = E\{V\} - E\{V \cdot 1_{\{V \geq K\}}\}$$
$$= e^{\alpha + 1/2\beta^2} - e^{\alpha + 1/2\beta^2}\Phi(\gamma)$$
$$= e^{\alpha + 1/2\beta^2}\Phi(-\gamma). \tag{A8}$$

3.

$$E\{(V - K)^+\} = E\{V \cdot 1_{\{V \geq K\}}\} - K \cdot E\{1_{\{V \geq K\}}\}$$
$$= e^{\alpha + 1/2\beta^2}\Phi(\gamma) - K \int_{\ln K}^{\infty} \frac{1}{\sqrt{2\pi\beta^2}} e^{-1/(2\beta^2)(v-\alpha)^2}\,dv$$
$$= e^{\alpha + 1/2\beta^2}\Phi(\gamma) - K \int_{(1/\beta)(\ln K - \alpha)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-1/2x^2}\,dx$$
$$= e^{\alpha + 1/2\beta^2}\Phi(\gamma) - K\Phi\left(\frac{\alpha - \ln K}{\beta}\right)$$
$$= e^{\alpha + 1/2\beta^2}\Phi(\gamma) - K\Phi(\gamma - \beta). \tag{A9}$$

4.

$$E\{(K-V)^+\} = K \cdot E\{1_{\{V\le K\}}\} - E\{V \cdot 1_{\{V\le K\}}\}$$
$$= K \cdot \Phi\left(\frac{\ln K - \alpha}{\beta}\right) - e^{\alpha+1/2\beta^2}\Phi(-\gamma)$$
$$= K \cdot \Phi(\beta-\gamma) - e^{\alpha+1/2\beta^2}\Phi(-\gamma). \quad \text{(A10)}$$

5.

$$E\{V \cdot \ln V \cdot 1_{\{V\ge K\}}\}$$
$$= \int_{\ln K}^{\infty} e^v \cdot v \cdot f(v)dv$$
$$= \int_{\ln K}^{\infty} e^v \cdot v \frac{1}{\sqrt{2\pi\beta^2}} e^{-1/(2\beta^2)(v-\alpha)^2}dv$$
$$= \int_{\ln K}^{\infty} v \frac{1}{\sqrt{2\pi\beta^2}} e^{-1/(2\beta^2)[v^2+\alpha^2-2\alpha v-2\beta^2 v]}dv$$
$$= \int_{\ln K}^{\infty} v \frac{1}{\sqrt{2\pi\beta^2}} e^{-1/(2\beta^2)[(v-(\alpha+\beta^2))^2-(\alpha+\beta^2)^2+\alpha^2]}dv$$
$$= e^{-1/(2\beta^2)[\alpha^2-\alpha^2-\beta^4-2\alpha\beta^2]}\int_{\ln K}^{\infty} v \frac{1}{\sqrt{2\pi\beta^2}} e^{-1/(2\beta^2)(v-(\alpha+\beta^2))^2}dv$$
$$= e^{\alpha+1/2\beta^2}\int_{1/\beta(\ln K-(\alpha+\beta^2))}^{\infty} (\beta x + (\alpha+\beta^2)) \frac{1}{\sqrt{2\pi}} e^{-1/2x^2}dx$$
$$= e^{\alpha+1/2\beta^2}\left(\beta \left| -\frac{1}{\sqrt{2\pi}} e^{-1/2x^2}\right|_{1/\beta(\ln K-(\alpha+\beta^2))}^{\infty} + (\alpha+\beta^2)\Phi(\gamma)\right)$$
$$= e^{\alpha+1/2\beta^2}(\beta\phi(\gamma) + (\alpha+\beta^2)\Phi(\gamma)). \quad \text{(A11)}$$

6.

$$E\{V \cdot \ln V \cdot 1_{\{V\le K\}}\}$$
$$= \cdots$$
$$= e^{\alpha+1/2\beta^2}\int_{-\infty}^{1/\beta(\ln K-(\alpha+\beta^2))} (\beta x + (\alpha+\beta^2)) \frac{1}{\sqrt{2\pi}} e^{-1/2x^2}dx$$
$$= e^{\alpha+1/2\beta^2}\left(\beta \left| -\frac{1}{\sqrt{2\pi}} e^{-1/2x^2}\right|_{-\infty}^{1/\beta(\ln K-(\alpha+\beta^2))} + (\alpha+\beta^2)\Phi(-\gamma)\right)$$
$$= e^{\alpha+1/2\beta^2}(-\beta\phi(\gamma) + (\alpha+\beta^2)\Phi(-\gamma)). \quad \text{(A12)}$$

Lemma A4. *Let*

$$dS(t) = (r+\sigma^2)S(t)dt + \sigma S(t)dW^{Q'}(t)$$

and define

$$x_A(t) = \frac{\frac{1}{t}\int_0^t S(u)du}{S(t)}.$$

Then we have

$$E_t^{Q'}\{x_A(u)\} = \frac{t}{u}\, x_A(t)e^{-r(u-t)} + \frac{1}{ru}(1-e^{-r(u-t)}) \quad \text{(A13)}$$

and

$$E_t^{Q'}\{x_A^2(u)\} = \left(\frac{t}{u}\right)^2 x_A^2(t)e^{-(2r-\sigma^2)(u-t)}$$
$$+ \frac{2(r-\sigma^2) - (4r-2\sigma^2)e^{-r(u-t)} + 2re^{-(2r-\sigma^2)(u-t)}}{u^2r(2r-\sigma^2)(r-\sigma^2)}$$
$$+ x_A(t)\frac{2te^{-r(u-t)}}{u^2(r-\sigma^2)}(1-e^{-(r-\sigma^2)(u-t)}). \quad \text{(A14)}$$

Proof of Lemma A4. First note that for $t < u$

$$E_t^{Q'}(x_A(u)) = E_t^{Q'}\left(\frac{1}{u}\int_0^u \frac{S_v}{S_u}dv\right)$$
$$= \frac{1}{u}\int_0^t \frac{S_v}{S_t}dv E_t^{Q'}\left(\frac{S_t}{S_u}\right) + \frac{1}{u}\int_t^u E_t^{Q'}\left(\frac{S_v}{S_u}\right)du,$$

and since for all $v \in [t, u]$

$$\frac{S_v}{S_u} = \exp\left(\left(r+\frac{1}{2}\sigma^2\right)(v-u) + \sigma^2(W_v^{Q'} - W_u^{Q'})\right),$$

we can apply the Laplace transformation of a normal distributed random variable to obtain

$$E_t^{Q'}\left(\frac{S_v}{S_t}\right) = \exp\left(\left(r+\frac{1}{2}\sigma^2\right)(v-t)\right)\exp\left(-\frac{1}{2}\sigma^2(v-t)\right)$$
$$= \exp(r(v-t)). \quad \text{(A15)}$$

Inserting above we can write

$$E_t^{Q'}(x_A(u)) = \frac{t}{u}\, x_A(t)\exp(-r(u-t)) + \frac{\exp(-ru)}{u}\int_t^u \exp(rv)dv$$
$$= \frac{t}{u}\, x_A(t)\exp(-r(u-t)) + \frac{\exp(-ru)}{ru}[\exp(rv)]_{v=t}^{v=u}$$
$$= \frac{t}{u}\, x_A(t)\exp(-r(u-t)) + \frac{1}{ru}(1-\exp(-r(u-t))),$$

which proves the first part of the Lemma. As for the second moment, we initially note that

$$E_t^{Q'}(x_A^2(u)) = E_t^{Q'}\left(\frac{1}{u^2}\left(\left(\int_0^u \frac{S_v}{S_u}dv\right)^2\right)\right)$$

$$= \frac{1}{u^2}\left\{\int_0^t\int_0^t \frac{S_z}{S_t}\frac{S_v}{S_t}dvdzE_t^{Q'}\left(\frac{S_t}{S_u}\frac{S_t}{S_u}\right)\right.$$

$$+\int_t^u\int_t^u E_t^{Q'}\left(\frac{S_z}{S_u}\frac{S_v}{S_u}\right)dvdz$$

$$\left.+\int_0^t \frac{S_z}{S_t}dzE_t^{Q'}\left(\frac{S_t}{S_u}\right)\int_t^u E_t^{Q'}\left(\frac{S_v}{S_u}\right)dv\right\}$$

$$= \frac{t^2}{u^2}x_A^2(t)E_t^{Q'}\left(\frac{S_t}{S_u}\frac{S_t}{S_u}\right)$$

$$+\frac{t}{u^2}x_A(t)E_t^{Q'}\left(\frac{S_t}{S_u}\right)\int_t^u E_t^{Q'}\left(\frac{S_v}{S_u}\right)dv$$

$$+\frac{1}{u^2}\int_t^u\int_t^u E_t^{Q'}\left(\frac{S_z}{S_u}\frac{S_v}{S_u}\right)dvdz. \quad \text{(A16)}$$

Now, as in the calculations on the first conditional moment of $x_A(u)$ we obtain

$$\frac{S_z}{S_u}\frac{S_v}{S_u} = \exp\left(\left(r+\frac{1}{2}\sigma^2\right)(z+v-2u)\right.$$

$$\left.+\sigma^2(W_z^{Q'}+W_v^{Q'}-2W_u^{Q'})\right).$$

Assuming that $z \geq v$ we can write

$$W_z^{Q'}+W_v^{Q'}-2W_u^{Q'} = -(2(W_u-W_z)+(W_z-W_v))$$

and therefore we obtain

$$E_t^{Q'}\left(\frac{S_z}{S_u}\frac{S_v}{S_u}\right) = \exp(r(z+v-2u)+\sigma^2(u-z)). \quad \text{(A17)}$$

By symmetry, note that for $z < v$ we will have

$$E_t^{Q'}\left(\frac{S_z}{S_u}\frac{S_v}{S_u}\right) = \exp(r(z+v-2u)+\sigma^2(u-v)). \quad \text{(A18)}$$

Finally, substitution of (A.15), (A.17), and (A.18) in (A.16) and performing calculations similar to those made for the first conditional moment gives us

$$E_t^{Q'}\{x_A^2(u)\} = \left(\frac{t}{u}\right)^2 x_A^2(t)e^{-(2r-\sigma^2)(u-t)}$$

$$+\frac{2(r-\sigma^2)-(4r-2\sigma^2)e^{-r(u-t)}+2re^{-(2r-\sigma^2)(u-t)}}{u^2r(2r-\sigma^2)(r-\sigma^2)}$$

$$+x_A(t)\frac{2te^{-r(u-t)}}{u^2(r-\sigma^2)}(1-e^{-(r-\sigma^2)(u-t)}). \quad \text{(A19)}$$

Observe that

$$\lim_{u\downarrow t} E_t^{Q'}(x_A(u)) = \lim_{u\downarrow t} E_t^{Q'}(x_A^2(u)) = 1,$$

implying that

$$\lim_{u\downarrow t} \mathrm{Var}_t^{Q'}(x_A(u)) = 0$$

as we expect. □

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