Arm Kinematics and Control Algorithm and Simulation Project Necessity: Arm Version 1.0

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This kinematic document is for Arm Version 1.0. The kinematic document includes math background on Forward Kinematics and Jacobian, and also control algorithm on Resolved Rates and Redundancy Resolution. Future development includes Task Priority Redundancy Resolution.
Nomenclature:

a. Bold text is for vector or matrix, unbold text is for scalar.

b. ${}^{i}A_j$: A of j defined in i.

c. Without specific notation, all the vectors and matrices are defined in the base frame/world frame. This means we omit superscript ^{0} in some occasions.

d. $[\mathbf{r}\times]$ is the cross product matrix of vector $\mathbf{r}$. `https://en.wikipedia.org/wiki/Cross_product`

1 Forward Kinematics

The forward kinematics takes URDF input exported from Solidworks and the current joint configuration and calculates the homogeneous transformation matrices. Figure 1 shows the arm home configuration with four important coordinate system. The joint information on i^{th} coordinate from URDF export contains the origin ${}^{i-1}\mathbf{d}_i$ of axis defined in the parent frame $i-1$. It also provides roll pitch yaw rotation with respect to the fixed frame: α, β, γ. And the URDF also defines the rotation axis of the active joint or translation axis for the prismatic joint: ${}^{i}\boldsymbol{\eta}_i$.

$$\begin{aligned} Translation &: {}^{i-1}\mathbf{d}_i \\ Rotation &: \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} \\ Axis &: {}^{i}\boldsymbol{\eta}_i \end{aligned} \tag{1}$$

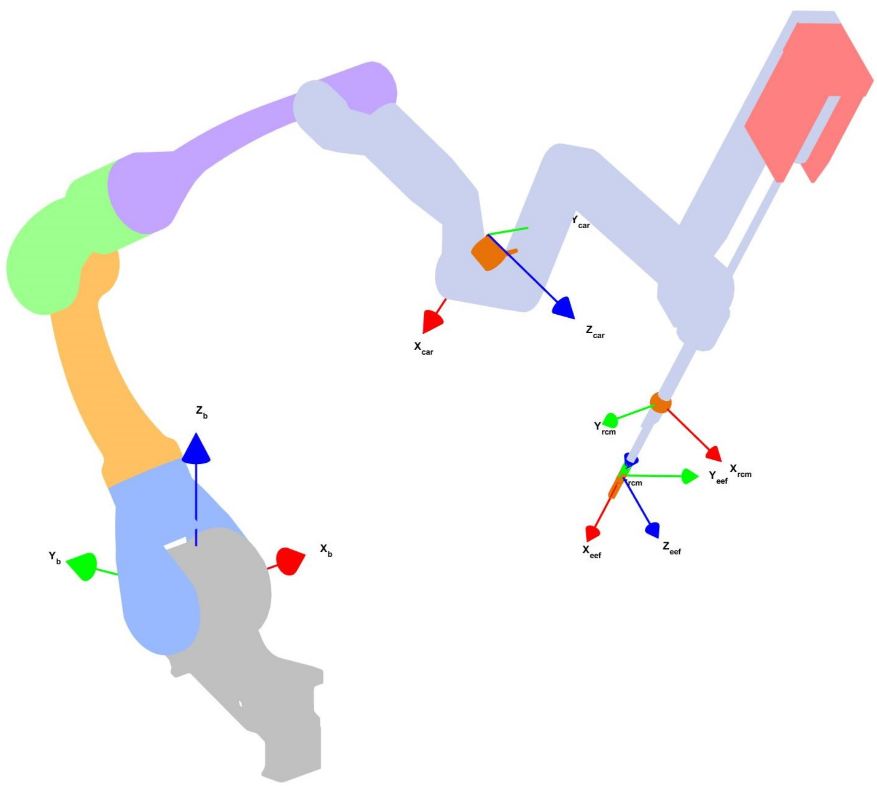


Figure 1: Arm home configuration: generated in Matlab simulation

The joint value is given in a format of 11×1 vector $\mathbf{q}$. Since we have the coupling on pitch arm of the spherical arm, the RCM joint value is defined as $\mathbf{q}_{rcm}$ with a transformation matrix $\mathbf{A} \in \mathbb{R}^{[13\times 11]}$

$$\begin{aligned} \mathbf{q}_{rcm} &= \mathbf{A}\mathbf{q} \\ \mathbf{A} &= \begin{bmatrix} \mathbf{I}_{6\times 6} & \mathbf{0}_{6\times 1} & \mathbf{0}_{6\times 4} \\ \mathbf{0}_{3\times 6} & \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} & \mathbf{0}_{3\times 4} \\ \mathbf{0}_{4\times 6} & \mathbf{0}_{4\times 1} & \mathbf{I}_{4\times 4} \end{bmatrix} \end{aligned} \tag{2}$$

According to fixed frame rotation matrix, the rotation matix ${}^{i-1}\mathbf{R}_i$ is given by:

$$\begin{aligned} Revolute: {}^{i-1}\mathbf{R}_i &= Rot(\hat{\mathbf{z}}, \gamma)Rot(\hat{\mathbf{y}}, \beta)Rot(\hat{\mathbf{x}}, \alpha)Rot({}^{i}\boldsymbol{\eta}_i, q_i) \\ Prismatic: {}^{i-1}\mathbf{R}_i &= Rot(\hat{\mathbf{z}}, \gamma)Rot(\hat{\mathbf{y}}, \beta)Rot(\hat{\mathbf{x}}, \alpha) \end{aligned} \tag{3}$$

The translation part is given as following:

$$\begin{aligned} Revolute: {}^{i-1}\mathbf{p}_i &= {}^{i-1}\mathbf{d}_i \\ Prismatic: {}^{i-1}\mathbf{p}_i &= {}^{i-1}\mathbf{d}_i + {}^{i-1}\mathbf{R}_i(q_i {}^{i}\boldsymbol{\eta}_i) \end{aligned} \tag{4}$$

The homogeneous transformation is then given by

$${}^{i-1}\mathbf{T}_i = \begin{bmatrix} {}^{i-1}\mathbf{R}_i & {}^{i-1}\mathbf{p}_i \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{5}$$

The homogeneous transformation with respect to the base frame is given by

$${}^{0}\mathbf{T}_i = {}^{0}\mathbf{T}_{i-1} {}^{i-1}\mathbf{T}_i \tag{6}$$

2 Jacobian

Most joints in our robotic arm is serial robot except for the spherical arm which has kinematic coupling. The Cartesian arm Jacobian $\mathbf{J}_{car5} \in \mathbb{R}^{[6\times 5]}$ from base to spherical roll joint origin $\mathbf{p}_{car}$ with 5 DoF joint input $(i = 1 \cdots 5)$

$$\mathbf{J}_{car5}(i) = \begin{bmatrix} \hat{\mathbf{z}}_i \times (\mathbf{p}_{car} - \mathbf{p}_i) \\ \hat{\mathbf{z}}_i \end{bmatrix} \tag{7}$$

Cartesian arm Jacobian from base to spherical roll joint origin $\mathbf{p}_{car}$ with 6 DoF joint input $(i = 1 \cdots 6)$. This Jacobian $\mathbf{J}_{car6} \in \mathbb{R}^{[6\times 6]}$ also include the spherical arm roll joint as an input, thus allowing us to perform full 6 DoF control on RCM position and orientation.

$$\mathbf{J}_{car6}(i) = \begin{bmatrix} \hat{\mathbf{z}}_i \times (\mathbf{p}_{car} - \mathbf{p}_i) \\ \hat{\mathbf{z}}_i \end{bmatrix} \tag{8}$$

With this $\mathbf{J}_{car5}$, we can obtain the twist $\mathbf{t}_{car}$.

$$\mathbf{t}_{car} = \mathbf{J}_{car5}\mathbf{q}_{1\cdots5} \tag{9}$$

For the spherical arm, because of the kinematic coupling on the pitch arm, we could not treat it as a simple serial robot. Here we present the geometric solution for Jacobian. First we look at the twist of the arm at the RCM point ($\mathbf{t}_{rcm}$) instantaneously. And we know for the pitch arm, the rotation axis $\hat{\mathbf{z}}_7$, $\hat{\mathbf{z}}_8$ and $\hat{\mathbf{z}}_9$ are always parallel to each other.

$$\begin{aligned}
\boldsymbol{\omega}_{rcm} &= q_6\hat{\mathbf{z}}_6 + q_7\hat{\mathbf{z}}_7 - q_7\hat{\mathbf{z}}_8 + q_7\hat{\mathbf{z}}_9 + q_9\hat{\mathbf{z}}_{11} \\
&= q_6\hat{\mathbf{z}}_6 + q_7\hat{\mathbf{z}}_7 + q_9\hat{\mathbf{z}}_{11} \\
\mathbf{v}_{rcm} &= q_8\hat{\mathbf{z}}_{10}
\end{aligned} \tag{10}$$

From the RCM point to the wrist joint origin, we can calculate the twist at the wrist $\mathbf{t}_{wr}$.

$$\begin{aligned}
\boldsymbol{\omega}_{wr} &= \boldsymbol{\omega}_{rcm} \\
&= \begin{bmatrix} \hat{\mathbf{z}}_6 & \hat{\mathbf{z}}_7 & \mathbf{0}_{3\times1} & \hat{\mathbf{z}}_{11} \end{bmatrix} \begin{bmatrix} q_6 \\ q_7 \\ q_8 \\ q_9 \end{bmatrix} \\
\mathbf{v}_{wr} &= q_8\hat{\mathbf{z}}_{10} + \boldsymbol{\omega}_{wr} \times (\mathbf{p}_{wr} - \mathbf{p}_{rcm}) \\
&= q_8\hat{\mathbf{z}}_{10} + \left[(\mathbf{p}_{rcm} - \mathbf{p}_{wr})\times\right] \boldsymbol{\omega}_{wr} \\
&= \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \hat{\mathbf{z}}_{10} & \mathbf{B}_3 \end{bmatrix} \begin{bmatrix} q_6 \\ q_7 \\ q_8 \\ q_9 \end{bmatrix} \\
\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \end{bmatrix} &= \left[(\mathbf{p}_{rcm} - \mathbf{p}_{wr})\times\right] \begin{bmatrix} \hat{\mathbf{z}}_6 & \hat{\mathbf{z}}_7 & \hat{\mathbf{z}}_{11} \end{bmatrix} \in \mathbb{R}^{[3\times3]}
\end{aligned} \tag{11}$$

In the current arm version, the end effector $\mathbf{p}_{eef}$ is set at the origin of the distal wrist axis.

$$\begin{aligned}
&\boldsymbol{\omega}_{eef} = \boldsymbol{\omega}_{wr} + q_{10}\hat{\mathbf{z}}_{12} + q_{11}\hat{\mathbf{z}}_{13} \\
&= \begin{bmatrix} \hat{\mathbf{z}}_6 & \hat{\mathbf{z}}_7 & \mathbf{0}_{3\times1} & \hat{\mathbf{z}}_{11} & \hat{\mathbf{z}}_{12} & \hat{\mathbf{z}}_{13} \end{bmatrix} \mathbf{q}_{6\cdots11} \\
&\mathbf{v}_{eef} = \mathbf{v}_{wr} + q_{10}\hat{\mathbf{z}}_{12} \times (\mathbf{p}_{eef} - \mathbf{p}_{12}) + q_{11}\hat{\mathbf{z}}_{13} \times (\mathbf{p}_{eef} - \mathbf{p}_{13}) \\
&= \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \hat{\mathbf{z}}_{10} & \mathbf{B}_3 & \hat{\mathbf{z}}_{12} \times (\mathbf{p}_{eef} - \mathbf{p}_{12}) & \hat{\mathbf{z}}_{13} \times (\mathbf{p}_{eef} - \mathbf{p}_{13}) \end{bmatrix} \mathbf{q}_{6\cdots11}
\end{aligned} \tag{12}$$

Thus, the 6DoF RCM Jacobian $\mathbf{J}_{rcm} \in \mathbb{R}^{[6\times6]}$ for the spherical arm plus the wrist joints is given here.

$$\mathbf{J}_{rcm} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \hat{\mathbf{z}}_{10} & \mathbf{B}_3 & \hat{\mathbf{z}}_{12} \times (\mathbf{p}_{eef} - \mathbf{p}_{12}) & \hat{\mathbf{z}}_{13} \times (\mathbf{p}_{eef} - \mathbf{p}_{13}) \\ \hat{\mathbf{z}}_6 & \hat{\mathbf{z}}_7 & \mathbf{0}_{3\times1} & \hat{\mathbf{z}}_{11} & \hat{\mathbf{z}}_{12} & \hat{\mathbf{z}}_{13} \end{bmatrix} \tag{13}$$

The overall arm Jacobian for all 11 DoF $\mathbf{J}_{all} \in \mathbb{R}^{[6\times11]}$ is given by putting two Jacobians side by side

$$\mathbf{J}_{all} = \begin{bmatrix} \mathbf{J}_{car5} & \mathbf{J}_{rcm} \end{bmatrix} \tag{14}$$

3 Resolved Rates

Resolved rates algorithm is for controlling the robot with given desired end-effector configuration. No matter the end-effector configuration is defined in any kind of format (homogeneous transformation, quarternion or roll pitch yaw). Here we use homegeneous transformation matrix to define target tracking.
The reference configuration of end effector is $\mathbf{T}_{ref}$.

$$\mathbf{T}_{ref} = \begin{bmatrix} \mathbf{R}_{ref} & \mathbf{p}_{ref} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{15}$$

The difference between current end effector configuration and the reference one.

$$\begin{aligned} \mathbf{p}_e &= \mathbf{p}_{ref} - \mathbf{p}_{eef} \\ \mathbf{R}_e &= \mathbf{R}_{ref}\mathbf{R}_{eef}^T \end{aligned} \tag{16}$$

Several variables defined for resolved rates algorithm.

a. Δt: discrete time step.

b. $\mathbf{v}_{max}$ and $\boldsymbol{\omega}_{max}$: maximum linear and angular velocity.

c. p_{eps} and θ_{eps}: Convergence criteria for translation and rotation. When error is within the criteria, we don't move the robot.

d. n_t and n_r: These two defines the cycles before approaching the target with maximum velocity. Within the cycle, the end-effector twist starts to slow down.

The end-effector linear velocity is given by following criteria.

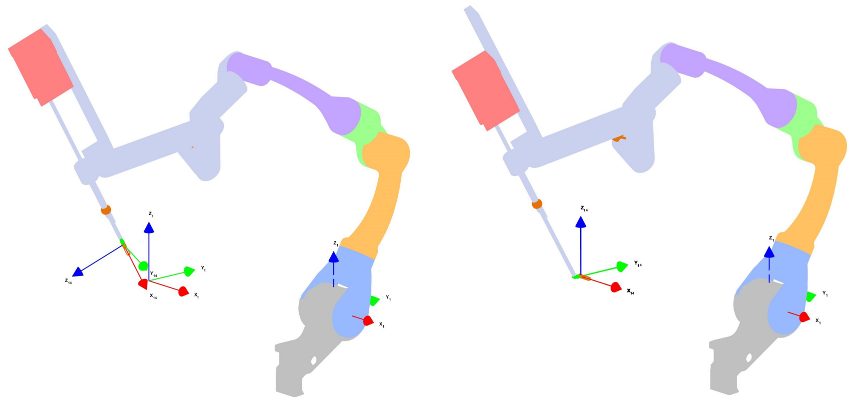
$$\mathbf{v}_{eef} = \begin{cases} \mathbf{0}_{3\times 1}, & |\mathbf{p}_e| < p_{eps} \\ \mathbf{p}_e/(n_t\Delta t), & p_{eps} \leqslant |\mathbf{p}_e| < n_t v_{max}\Delta t \\ v_{max}\mathbf{p}_e/|\mathbf{p}_e|, & |\mathbf{p}_e| \geqslant n_t v_{max}\Delta t \end{cases} \tag{17}$$

For rotation, we first need to convert rotation matrix to axis and angle of rotation.

$$\begin{aligned} \theta_e &= \arccos((tr(\mathbf{R}_e) - 1)/2) \\ \boldsymbol{\eta}_e &= \frac{1}{2}\begin{bmatrix} \mathbf{R}_e(3,2) - \mathbf{R}_e(2,3) \\ \mathbf{R}_e(1,3) - \mathbf{R}_e(3,1) \\ \mathbf{R}_e(2,1) - \mathbf{R}_e(1,2) \end{bmatrix} \end{aligned} \tag{18}$$

The angular velocity of end effertor is given as following.

$$\boldsymbol{\omega}_{eef} = \begin{cases} \mathbf{0}_{3\times 1}, & |\boldsymbol{\theta}_e| < \theta_{eps} \\ \theta_e\boldsymbol{\eta}_e/(n_r\Delta t), & \theta_{eps} \leqslant |\boldsymbol{\theta}_e| < n_r\omega_{max}\Delta t \\ \omega_{max}\boldsymbol{\eta}_e, & |\boldsymbol{\theta}_e| \geqslant n_r\omega_{max}\Delta t \end{cases} \tag{19}$$



(a) Before move (b) After move

So the end effector twist is given by

$$\mathbf{t}_{eef} = \begin{bmatrix} \mathbf{v}_{eef} \\ \boldsymbol{\omega}_{eef} \end{bmatrix} \tag{20}$$

Differenct control mode and different task will require using different Jacobian and select different end effector. We take the 6DoF spherical arm as an example here. The Jacobian is the 6DoF RCM Jacobian $\mathbf{J}_{rcm}$. And without redundancy in the system, we will use tseu-do inverse to solve the desired joint velocity.

$$\dot{\mathbf{q}}_{6\cdots 11} = \mathbf{J}_{rcm}^{+} \mathbf{t}_{eef} \tag{21}$$

The overall joint velocity

$$\dot{\mathbf{q}}_{1\cdots 11} = \begin{bmatrix} \mathbf{0}_{1\cdots 5} \\ \dot{\mathbf{q}}_{6\cdots 11} \end{bmatrix} \tag{22}$$

And the desired joint angle is given by

$$\mathbf{q}_{des} = \mathbf{q}_{cur} + \dot{\mathbf{q}}\Delta t \tag{23}$$

The simulation of targeting the robot end effector is shown in the figure.