THE DE GIORGI METHOD WITH APPLICATIONS TO FLUID DYNAMICS

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ABSTRACT. In 1957, De Giorgi presented a solution to Hilbert's 19th problem which concerns the regularity of variational solutions to elliptic problems. In this paper, we present a sketch of his proof, drawing attention to the geometric aspects of his method. We then discuss some examples of these techniques for problems in fluid dynamics, in particular partial regularity of solutions to the Navier-Stokes equations.

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1. HILBERT'S 19TH PROBLEM

Among the list of 23 problems proposed by Hilbert in the year 1900, the 19th problem asks if all minimizers of convex energy functionals of the form

$$\mathcal{E}(w) = \int_\Omega F(\nabla w)\,\mathrm{d}x \tag{1.1}$$

are smooth. A local minimizer refers to some function w such that

$$\mathcal{E}(w) \le \mathcal{E}(w+\varphi) \tag{1.2}$$

for all φ with compact support in Ω. We add the condition that F is uniformly convex, i.e.

$$\frac{1}{\Lambda} I \le D^2 F(p) \le \Lambda I \text{ for all } p \in \mathbb{R}^N \tag{1.3}$$

Date: October 2024.

with some fixed $\Lambda > 1$.

Hilbert devised this problem after noticing that certain classes of partial differential equations only admit analytic functions as solutions. In the year 1957, De Giorgi managed to resolve this problem [1]. Nash offered an independent solution [2] at around the same time. Subsequently, Moser offered a new formulation of the proof [3]. These methods are now acknowledged as De Giorgi-Nash-Moser techniques and have been applied to other degenerated parabolic or integral operators. In this paper, we present De Giorgi's proof in two main steps in Section 2. Sections 3 and 4 then adapt the first step to obtain some regularity results for the surface quasi-geostrophic equation and the Navier-Stokes equations respectively. The central result is the following.

Theorem 1.4. *Any local minimizer $w \in H^1(\Omega)$ of (1.1) lies in $C^\infty(\Omega)$.*

We now give an outline of the strategy behind De Giorgi's proof. First, any such minimizer w must satisfy an Euler-Lagrange equation.

Lemma 1.5. *Any $w \in H^1(\Omega)$ which is a local minimizer of (1.1) is a solution in the sense of distribution to*

$$\operatorname{div}(DF(\nabla w)) = 0, \tag{1.6}$$

where DF is the gradient of the functional $p \mapsto F(p)$.

Proof. Given any $\epsilon > 0$ and smooth function ϕ with compact support, we have

$$\int_\Omega F(\nabla w + \epsilon \nabla\phi)\,\mathrm{d}x \geq \int_\Omega F(\nabla w)\,\mathrm{d}x. \tag{1.7}$$

These integrals are finite since we have $\nabla w \in L^2$ and F is at most quadratic. By a Taylor expansion for F, we find that

$$\int_\Omega DF(\nabla w)\cdot\nabla\omega\,\mathrm{d}x \geq -\epsilon\Lambda \int_\Omega |\nabla\phi|^2\,\mathrm{d}x. \tag{1.8}$$

This inequality holds for any ϵ, which implies

$$\int_\Omega DF(\nabla w)\cdot\nabla\phi\,\mathrm{d}x \geq 0. \tag{1.9}$$

We also get the reverse inequality by replacing ϕ with $-\phi$. □

The Euler-Lagrange equation (1.6) can be written in non-divergence form as

$$D^2F(\nabla w) : D^2 w = 0. \tag{1.10}$$

Here, the strict convexity of F imposes a strict ellipticity property on w.

Second, we obtain a sufficient condition for the central result.

Lemma 1.11. *If $\nabla w \in C^\alpha$ for some α, then we have that $w \in C^\infty(\Omega)$.*

Proof. Assume that $\nabla w \in C^\alpha$. Then $D^2F(\nabla w)$ also lies in the space C^α. We obtain

$$A(x) : D^2 w = 0 \tag{1.12}$$

for some C^α elliptic matrix A. Schauder theory tells us that $\nabla w \in C^{1,\alpha}$ and $w \in C^{2,\alpha}$. When we differentiate (1.12), we have

$$D^2F(\nabla w) : D^2\partial_i w = -D^3F(\nabla w) \cdot \nabla\partial_i w : D^2 w. \tag{1.13}$$

Another application of Schauder theory on the linear problem

$$A(x) : D^2\partial_i w = f(x), \tag{1.14}$$

tells us that $w \in C^{3,\alpha}$. Repeated iteration eventually yields $w \in C^\infty(\Omega)$. □

Hence, our main goal to fill the gap between L^2 to C^α for ∇w. For every $1 \le i \le N$, we consider the derivative of (1.6) with respect to x_i. Denoting $u = \partial_i w$, we have

$$\operatorname{div}(F''(\nabla w)\nabla u) = 0, \text{ with } \frac{1}{\Lambda}I \le F''(\nabla w) \le \Lambda I \tag{1.15}$$

for every $x \in \Omega$ thanks to (1.3). By forgetting the dependence of $A(x) = F''(\nabla w)$ on w, we obtain

$$-\operatorname{div}(A(x)\nabla u) = 0, \text{ with } \frac{1}{\Lambda}I \le A(x) \le \Lambda I. \tag{1.16}$$

De Giorgi showed the following theorem.

Theorem 1.17. *Let $u \in H^1(\Omega)$ be a weak solution to (1.16). Then $u \in C^\alpha(\tilde{\Omega})$ for any $\tilde{\Omega} \subset\subset \Omega$, with*

$$||u||_{C^\alpha(\tilde{\Omega})} \le C||u||_{L^2(\Omega)}. \tag{1.18}$$

Here, the constant α depends only on Λ and N, while the constant C depends on Λ, N, Ω and $\tilde{\Omega}$.

In other words, we only need to check that $u \in H^1$ (or $\nabla w \in H^2$) and that u verifies (1.16) in the sense of distribution. Applying Theorem 1.17 to $u = \partial_i w$ gives us $\nabla w \in C^\alpha$, which yields our main result in the form of Theorem 1.4.

Definition 1.19. From this point forth, we denote by L any operator of the form $-\operatorname{div}(A(x)\nabla\cdot)$, where A is uniformly elliptic.

2. De Giorgi's solution

The proof of Theorem 1.17 can be split into two distinct steps. In the first step, we obtain an L^∞ bound from the L^2 norm of ∇u. The second step pushes the L^∞ bound to obtain C^α regularity.

Assume without loss of generality that $\Omega = B_1$ and $\tilde{\Omega} = B_{1/2}$. Otherwise, we apply a rescaling argument. Let $d > 0$ be the distance from $\tilde{\Omega}$ to Ω^c. For any $x_0 \in \tilde{\Omega}$, we define the function u_d on B_1 given by

$$u_d(y) := u(x_0 + dy). \tag{2.1}$$

Then u_d verifies (1.16) with the diffusion matrix $A_d(y) = A(x_0 + dy)$ which satisfies the same uniform elliptic estimates. Hence, we see that $u_d \in C^\alpha$ in $B_{1/2}$ and $u \in C^\alpha$ in $\tilde{\Omega}$ as desired.

2.1. **First step: Small energy to boundedness.** We consider a family of shrinking balls and decreasing truncated functions of the form

$$B_k := \{|x| \leq 1 + 2^{-k}\}, \tag{2.2}$$

$$u_k := (u - C_k)_+, \text{ where } C_k = 1 - 2^{-k}. \tag{2.3}$$

We then consider the energy along the level sets

$$U_k := \int_{B_k} |u_k|^2 \, \mathrm{d}x. \tag{2.4}$$

The goal is to derive a nonlinear estimate of the form

$$U_k \leq C^k U_{k-1}^\beta \text{ for some } \beta > 0. \tag{2.5}$$

Lemma 2.6. *Let $\{W_k\}_{k \geq 0} \subset \mathbb{R}$ be a sequence that satisfies a relation of the form*

$$0 \leq W_{k+1} \leq C^k W_k^\beta$$

for some $C, \beta > 1$. Then there exists some initial bound C_0^* such that $0 \leq W_0 \leq C_0^*$ gives us $\lim W_k = 0$.

Proof. The rate of decay of $\{W_k\}_k$ eventually dominates the geometric sequence $\{C_k\}_k$. □

Observe that we seek a non-linear estimate on a linear equation. We make use of the Sobolev inequality, energy inequality, as well as the nonlinear Tchebychev inequality with the truncations u_k. Here, the shrinking family of balls $(B_k)_k$ also controls the flux of energy in the energy inequality, i.e. the energy flowing through the boundary of B_k can be expressed in terms of the energy contained in B_{k-1}.

Lemma 2.7. *(Sobolev inequality) If v is supported in B_1, then*

$$||v||_{L^p(B_1)} \leq C||\nabla v||_{L^2(B_1)}, \tag{2.8}$$

where $p(d) = \frac{2d}{d-2}$.

Lemma 2.9. *(Energy inequality) If $u \geq 0$, $Lu \leq 0$ and $\varphi \in C_0^\infty(B_1)$, then*

$$\int_{B_1} (\nabla[\varphi u])^2 \, \mathrm{d}x \leq C||\nabla\varphi||_{L^\infty}^2 \int_{B_1 \cap supp \varphi} u^2 \, \mathrm{d}x. \tag{2.10}$$

Proof. Multiply $Lu = -\operatorname{div}(A(x)\nabla u)$ by $\varphi^2 u$. Since the first term is nonpositive and the second one nonnegative, we get

$$\int \nabla^T(\varphi^2 u) A \nabla u \, \mathrm{d}x \leq 0. \tag{2.11}$$

By Young's inequality, the estimate

$$\int \nabla^T(\varphi u) A u (\nabla\varphi) \, \mathrm{d}x \leq \epsilon \int \nabla^T(\varphi u) A \nabla(\varphi u) \, \mathrm{d}x + \frac{\Lambda}{\epsilon} \int |\nabla\varphi|^2 u^2 \, \mathrm{d}x. \tag{2.12}$$

enables us to transfer a φ from the left ∇ to the right ∇. □

Proposition 2.13. *There exists a constant $\delta = \delta(N, \Lambda)$ such that if $||u_+||_{L^2(B_1)}^2 \leq \delta$, then we have $\sup_{B_{1/2}} u_+ \leq 1$.*

Proof. Take a sequence of truncations $\varphi_k u_k$, where φ_k is a sequence of shrinking cut-off functions converging to $\chi_{B_{1/2}}$, i.e.

$$\varphi_k \equiv \begin{cases} 1 \text{ in } B_k, \\ 0 \text{ in } B_{k-1}^c, \end{cases} \qquad \text{where } |\nabla \varphi_k| \le C \cdot 2^k. \tag{2.14}$$

Since $u_{k+1} > 0$ implies $u_k > 2^{-(k+1)}$, we have

$$\{(\varphi_{k+1} u_{k+1}) > 0\} \subset \{(\varphi_k u_k) > 2^{-(k+1)}\}. \tag{2.15}$$

Hölder's inequality gives

$$U_{k+1} \le \int (\varphi_{k+1} u_{k+1})^2 \,\mathrm{d}x \tag{2.16}$$

$$\le \left(\int (\varphi_{k+1} u_{k+1})^p \,\mathrm{d}x \right)^{2/p} \cdot |\{\varphi_{k+1} u_{k+1} > 0\}|^{1/N} \quad (\text{with } p = \frac{2N}{N-2}) \tag{2.17}$$

$$\le C \left(\int (\nabla(\varphi_{k+1} u_{k+1}))^2 \,\mathrm{d}x \right) |\{\varphi_{k+1} u_{k+1} > 0\}|^{1/N}, \tag{2.18}$$

where the last inequality follows from the Sobolev inequality.

We control the two terms on (2.18) separately. For the first term, the energy inequality tells us that

$$\int |\nabla(\varphi_{k+1} u_{k+1})|^2 \,\mathrm{d}x \le C 2^{2k} \int_{\mathrm{supp}\varphi_{k+1}} u_{k+1}^2 \,\mathrm{d}x. \tag{2.19}$$

Since $\varphi_k \equiv 1$ on supp φ_{k+1} and $u_{k+1} \le u_k$, the above quantity is bounded by

$$C 2^{2k} \int (\varphi_k u_k)^2 \,\mathrm{d}x = C 2^{2k} U_k. \tag{2.20}$$

For the second term, (2.15) tells us that

$$|\{\varphi_{k+1} u_{k+1} > 0\}|^{1/N} \le |\{\varphi_k u_k > 2^{-k}\}|^{1/N} \le 2^{2k/N} \left(\int (\varphi_k u_k)^2 \right)^{1/N} \tag{2.21}$$

by Tchebychev's inequality. We combine our estimates for

$$U_{k+1} \le C 2^{4k} U_k^{1+\frac{1}{N}}. \tag{2.22}$$

The buildup of the exponent of U_k then forces U_k to go to zero. □

Applying Proposition 2.13 to $(\sqrt{\delta}/||u_+||_{L^2})u$ gives the following corollary.

Corollary 2.23. *If u is a solution of $Lu = 0$ in B_1, then*

$$||u_+||_{L^\infty(B_{1/2})} \le \frac{1}{\sqrt{\delta}} ||u_+||_{L^2(B_1)}. \tag{2.24}$$

2.2. **Second step: Boundedness to oscillation decay.** This second step is sometimes referred to as an oscillation lemma.

Definition 2.25. (oscillation) We have $\operatorname{osc}_D u := \sup_D u - \inf_D u$.

Proposition 2.26. *Let u be a solution of $Lu = 0$ in B_1, where A is uniformly elliptic. Then there exists $\lambda = \lambda(\Lambda, N) < 1$ such that*

$$\operatorname{osc}_{B_{1/2}} u \leq \lambda \operatorname{osc}_{B_1} u. \tag{2.27}$$

Proposition 2.26 implies C^α regularity of solutions by a rescaling argument similar to (2.1). Given any $x_0 \in B_{1/2}$, we introduce the functions

$$\overline{u}_1(y) := u(x_0 + y/2), \tag{2.28}$$

$$\overline{u}_n(y) := \overline{u}_{n-1}(y/2). \tag{2.29}$$

Then $\overline{u}_n$ constitute solutions corresponding to diffusion matrices $A_n(y) = A(x_0 + y/2^n)$. We can then recursively apply Proposition 2.26 for

$$\sup_{|x_0 - x| \leq 2^{-n}} |u(x_0) - u(x)| \leq 2||u||_{L^\infty(B_1)} \lambda^n, \tag{2.30}$$

where the right hand side does not depend on x_0. We see then that $u \in C^\alpha(B_{1/2})$ with $\alpha = -\log_2 \lambda$.

We claim that the following implies Proposition 2.26.

Proposition 2.31. *Let $v \leq 1, Lv = 0$ in B_2. Assume that $|B_1 \cap \{v \leq 0\}| \geq \mu > 0$. Then $\sup_{B_{1/2}} v \leq 1 - \lambda$, where λ depends only on μ, Λ, and N.*

Indeed, we consider the function

$$v(x) := \frac{2}{\operatorname{osc} u}\left(u(x) - \frac{\sup u + \inf u}{2}\right), \tag{2.32}$$

which immediately satisfies $-1 \leq v \leq 1$. Assume that $v < 0$ for more than half of in B_1. Then we can apply Proposition 2.31 which implies $\operatorname{osc}_{B_{1/2}} v \leq 2 - \lambda$ and $\operatorname{osc}_{B_{1/2}} u \leq (1 - \lambda/2) \operatorname{osc}_{B_2} u$. We get the same result by working with $(-v)$ when v is greater than 0 for at least half the space.

To prove Proposition 2.31, note that

$$|\{v \leq 0\}| \geq |B_1| - \frac{\delta}{4} \implies ||v_+||^2_{L^2(B_1)} \leq \frac{\delta}{4}, \tag{2.33}$$

which gives $u_+|_{B_{1/2}} \leq 1/2$ by Corollary 2.23. Our remaining goal is to show that $|\{v \leq 0\}| \geq |B_1|/2$ implies $|\{v \leq 0\}| \geq |B_1| - \frac{\delta}{4}$.

The main tool is the following inequality, which offers a quantitative version of the fact that a function with a jump discontinuity cannot be in H^1.

Lemma 2.34. *(Isoperimetric inequality) Consider w such that $\int_{B_1} |\nabla w_+|^2 \, dx \leq C_0$. Setting*

$$A = \{w \leq 0\} \cap B_1, C = \{w \geq 1\} \cap B_1, \text{ and } D = \{0 < w < 1\} \cap B_1, \tag{2.35}$$

we have

$$C_0|D| \geq C_1(|A||C|^{1-\frac{1}{n}})^2. \tag{2.36}$$

Proof. Consider $\overline{w} = \sup(0, \inf(w, 1))$. Note that $\nabla\overline{w} = \nabla w_+ \mathbf{1}_{\{0\leq 1\leq 1\}}$. For $x_0 \in C$, we can reconstruct $\overline{w}$ by integrating along any of the rays that go from x_0 to a point in A. We observe that

$$|A| \leq \int_D \frac{|\nabla\overline{w}(y)|\,\mathrm{d}y}{|x_0-y|^{n-1}}. \tag{2.37}$$

Integrating x_0 on C, we find that

$$|A||C| \leq \int_D |\nabla w_+(y)| \bigg(\int_C \frac{\mathrm{d}x_0}{|x_0-y|^{n-1}} \bigg)\,\mathrm{d}y, \tag{2.38}$$

where across all C

$$\int_C \frac{\mathrm{d}x_0}{|x_0-y|^{n-1}} \leq |C|^{1/n}. \tag{2.39}$$

Therefore, we obtain

$$|A||C| \leq |C|^{1/n} \bigg(\int_D |\nabla w_+|^2 \bigg)^{1/2} |D|^{1/2}, \tag{2.40}$$

which completes the proof. □

Proof of Proposition 2.31. We consider a sequence of truncations given by

$$w_k := 2^k[v-(1-2^{-k})] \tag{2.41}$$

with $w_k \leq 1$. From the energy inequality, we have

$$\int_{B_1} |\nabla(w_k)_+|^2\,\mathrm{d}x \leq C_0. \tag{2.42}$$

We also have $|\{w_k \leq 0\} \cap B_1| \geq \mu$.

As long as $\int_{B_1}(w_{k+1})_+^2\,\mathrm{d}x \geq \delta$, we apply Lemma 2.34 recursively on $2w_k$ for

$$|\{w_{k+1} \geq 0\} \cap B_1| = |\{2w_k \geq 1\} \cap B_1| \geq \int_{B_1} (w_{k+1})_+^2\,\mathrm{d}x \geq \delta. \tag{2.43}$$

Because there exists some constant α (independent of k) such that

$$|\{0 < w_k < 1/2\} \cap B_1| \geq \alpha. \tag{2.44}$$

we have

$$|\{w_k \leq 0\} \cap B_1| \geq |\{w_{k-1} \leq 0\} \cap B_1| + \alpha \geq \mu + k\alpha, \tag{2.45}$$

which must fail after a finite number of k.

For some k_0, we obtain

$$\int_{B_1} (w_{k_0+1})_+^2\,\mathrm{d}x \leq \delta. \tag{2.46}$$

Proposition 2.13 implies that $w_{k_0+1} \leq 1$ in $B_{1/2}$, before rescaling back to v. □

3. Global regularity for surface quasi-geostrophic equation

The surface quasi-geostrophic (SQG) equation models the temperature at the surface of the earth. Consider the potential of temperature θ on $\mathbb{R}^2$. We take

$$\theta_t + (v \cdot \nabla)\theta = (\Delta^{1/2})\theta, \quad t > 0, x \in \mathbb{R}^2, \tag{3.1}$$

$$(-v_2, v_1) = (R_1\theta, R_2\theta), \tag{3.2}$$

where R_j are the Riesz transforms of θ defined by

$$\widehat{R_j\theta} = \frac{i\xi_j}{|\xi|}\hat{\theta}. \tag{3.3}$$

In particular, we have $\operatorname{div} v = 0$. We claim the following.

Theorem 3.4. *For any initial value $\theta_0 \in L^2(\mathbb{R}^2)$, there exists a weak solution to the SQG equation which is smooth on $(0, \infty) \times \mathbb{R}^2$.*

The main strategy is similar to that for Hilbert's 19th problem: we first aim for C^α regularity, before potential theory and bootstrapping arguments give full regularity. However, we only present the part that corresponds to the first step of De Giorgi's proof, i.e. a global L^∞ estimate.

Theorem 3.5. *Let θ be a weak solution of*

$$\theta_t + v\nabla\theta = (\Delta^{1/2})\theta \text{ in } \mathbb{R}^N \times [0, \infty), \tag{3.6}$$

for some incompressible vector field v and initial data $\theta_0 \in L^2$. Then

$$||\theta(\cdot, t)||_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{t}||\theta(\cdot, 0)||_{L^2(\mathbb{R}^2)}. \tag{3.7}$$

3.1. **Harmonic extensions and energy inequality.** An harmonic extension allows us to replace a nonlocal description with a local one at the cost of an additional dimension. Given θ defined for $x \in \mathbb{R}^N$, we can extend θ to θ^* defined for $(x, y) \in (\mathbb{R}^{N+1})^+$ by combining it with the Poisson kernel:

$$P_y(x) = \frac{Cy}{(y^2 + |x|^2)^{\frac{N+1}{2}}} = y^{-N}P_1(x/y). \tag{3.8}$$

Then $\theta^*(x, y)$ satisfies $\Delta_{x,y}\theta^* = 0$ in $\mathbb{R}^N \times \mathbb{R}^+$. As such, we attain the energy inequality by multiplying the equation with a truncation of θ, given by $(\theta_\lambda) = (\theta - \lambda)_+$, before integrating in $\mathbb{R}^n \times [T_1, T_2]$. The transport term vanishes with

$$\frac{1}{2}\int [(\theta_\lambda)^2(y, T_2) - (\theta_\lambda)^2(y, T_1)]\, \mathrm{d}y + 0 = \int\int_{\mathbb{R}^n \times [T_1, T_2]} \theta_\lambda \Lambda^{1/2}\theta \, \mathrm{d}y \, \mathrm{d}t. \tag{3.9}$$

The last term corresponds to

$$\int_{T_1}^{T_2} \mathrm{d}t \bigg(\int_{\mathbb{R}^n} (\theta^*)_\lambda(y, 0, t) D_z(\theta^*)(y, 0, t) \, \mathrm{d}y \bigg) \tag{3.10}$$

$$= -\int_{T_1}^{T_2} \int\int_{\mathbb{R}^{n+1}_+} \nabla(\theta^*)_\lambda(y, z, t)\nabla\theta^*(y, z, t) \, \mathrm{d}y \, \mathrm{d}z \tag{3.11}$$

$$= -\int_{T_1}^{T_2} \mathrm{d}t \int\int_{\mathbb{R}^{n+1}_+} [\nabla\theta^*_\lambda]^2 \, \mathrm{d}y \, \mathrm{d}z. \tag{3.12}$$

Since $(\theta^*)_\lambda$ is an admissible extension of θ_λ, we have

$$||\theta^*_\lambda||_{H^1(\mathbb{R}^{n+1}_+)} \geq ||\theta_\lambda||_{H^{1/2}(\mathbb{R}^n)}. \tag{3.13}$$

Therefore, we obtain the following energy inequality

$$||\theta_\lambda(\cdot, T_2)||^2_{L^2} + \int_{T_1}^{T_2} ||\theta_\lambda||^2_{H^{1/2}}\, \mathrm{d}t \leq ||\theta_\lambda(T_1)||T^2_{L^2}. \tag{3.14}$$

3.2. **Recurrence Relation.** there is a layer at time $t = 0$, but no boundary in x. We need to "escape" from the layer $t = 0$ at a dyadic rate by considering cutoffs in time $T_k = -1 - 2^{-k}$. Take a sequence of increasing cutoffs $\lambda_k = 1 - 2^{-k}$ of θ, with $\theta_k = \theta_{\lambda_k}$.

With (3.14) and the Sobolev inequality, we have for any $s \leq T_k$,

$$\sup_{t \geq T_k} ||\theta_k(t)||^2_{L^2} + \int_{T_k}^{\infty} ||\theta_k(t)||^2_{L^p}\, \mathrm{d}t \leq ||\theta_k(s)||^2_{L^2}. \tag{3.15}$$

Here, we control the $L^\infty(L^2)$ and $L^2(L^p)$ norms of θ_k for some $p > 2$. By interpolation, there exists some $q > 2$ such that

$$||\theta_k||^2_{L^q(\mathbb{R}^N \times [T_k, \infty))} \leq ||\theta_k(S)||^2_{L^2} \tag{3.16}$$

for any $s < T_k$. Taking the mean value in s between T_{k-1} and T_k, we find

$$||\theta_k||^2_{L^q(\mathbb{R}^N \times [T_k, \infty))} \leq 2^k \int_{T_{k-1}}^{T_k} ||\theta_k(s)||_{L^2}\, \mathrm{d}s. \tag{3.17}$$

For $I_k = [T_k, \infty) \times \mathbb{R}^N$, we denote

$$U_k := \int \int_{I_k} (\theta_k)^2\, \mathrm{d}x\, \mathrm{d}t. \tag{3.18}$$

Applying Hölder with θ^2 and $\chi_{\{\theta_k > 0\}}$, we get

$$U_k \leq \left(\int \int_{I_k} \theta_k^q \right)^{2/q} |\{\theta_k > 0\} \cap I_k|^{1/\overline{q}} =: \alpha \cdot \beta, \tag{3.19}$$

where $\overline{q}$ is the conjugate exponent to $q/2$. In turn, we have $\alpha \leq 2^k U_{k-1}$. By going from k to $k-1$, we can estimate with Tchebychev's inequality that

$$\beta = |\{\theta_k > 0\} \cap I_k|^{1/\overline{q}} \leq \left(2^{2k} \int \int_{I_k} (\theta_{k-1})^2 \right)^{1/\overline{q}}. \tag{3.20}$$

We arrive at the recurrence relation

$$U_k \leq 2^{\overline{C}k} U_k^{1+1/(2\overline{q})}. \tag{3.21}$$

If $||\theta_0||_{L^2} \leq \delta_0$, then $||\theta(\cdot, t)||_{L^\infty} \leq 1$ for $t \geq 1$. Since the equation is linear in θ, applying this result to $\frac{\delta_0}{||\theta_0||_{L^2}} \theta(t_0 t, t_0 x)$ gives

$$||\theta(\cdot, t)||_{L^\infty} \leq \frac{||\theta_0||_{L^2}}{t_0^{N/2} \delta_0} \text{ for } t \geq t_0. \tag{3.22}$$

4. Partial regularity for Navier-Stokes equations

The Navier-Stokes equations in dimension $d = 3$ are given by

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla P - \Delta u = 0, \quad t \in (0, \infty), x \in \Omega, \tag{4.1}$$

$$\operatorname{div} u = 0, \tag{4.2}$$

where Ω is a regular subset of $\mathbb{R}^d$.

The initial boundary value problem is endowed with the conditions

$$u(0, \cdot) = u^0 \in L^2(\Omega),$$
$$u(t, x) = 0, \quad t \in (0, \infty), x \in \Omega.$$

We also rely on the notion of "suitable weak solutions" which verify the generalized energy inequality in the sense of distribution

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}(u\frac{|u|^2}{2}) + \operatorname{div}(uP) + |\nabla u|^2 - \Delta \frac{|u|^2}{2} \leq 0, \quad t \in (0, \infty), x \in \Omega. \tag{4.3}$$

In 1982, Caffarelli, Kohn and Nirenberg published the stunning result (see [10]) that such suitable weak solutions exist, and that the Hausdorff dimension of the singular set is bounded above by one. In this section, we present the first part of the proof, which corresponds to the following theorem.

Theorem 4.4. *Let $p > 1$. There exists a universal constant C^* such that any solution u of (4.1), (4.2), (4.3) in $[-1, 1] \times B(1)$ which satisfies*

$$\sup_{t \in [-1.1]} \left(\int_{B(1)} |u|^2 \,\mathrm{d}x \right) + \int_{-1}^{1} \int_{B(1)} |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}t + \left[\int_{-1}^{1} \left(\int_{B(1)} |P| \,\mathrm{d}x \right)^p \mathrm{d}t \right]^{2/p} \leq C^* \tag{4.5}$$

must be bounded by 1 on $[-1/2, 1] \times B(1/2)$. In particular, u is regular in this region.

4.1. Setup of truncations.

We discuss the preliminaries required to apply the De Giorgi method.

Definition 4.6. We take the following sets

$$B_k := B(\frac{1}{2}(1 + 2^{-3k})),$$
$$B_{k-1/3} := B(\frac{1}{2}(1 + 2 \cdot 2^{-3k})),$$
$$B_{k-2/3} := B(\frac{1}{2}(1 + 4 \cdot 2^{-3k})),$$

as well as cylinders of the form

$$T_k := \frac{1}{2}(-1 - 2^{-k}), \quad Q_k := [T_k, 1] \times B_k.$$

We will use $B(1)$ and B_0 interchangeably. We always have

$$[-\frac{1}{2}, 1] \times B(\frac{1}{2}) \subset \cdots \subset Q_{k+1} \subset Q_k \subset \cdots \subset [-1, 1] \times B(1).$$

Definition 4.7. We introduce the function

$$v_k := [|u| - (1 - 2^{-k})]_+.$$

The quantities v_k^2 act as level sets of energy since they vanish for $|u| < 1 - 2^{-k}$ and have order $|u|^2$ for $|u| >> 1 - 2^{-k}$.

Definition 4.8. We define the following quantities:

$$d_k^2 := \frac{(1-2^{-k})\mathbf{1}_{\{|u|\geq(1-2^{-k})\}}}{|u|}|\nabla|u||^2 + \frac{v_k}{|u|}|\nabla u|^2,$$

$$U_k := \sup_{t\in[T_k,1]}\left(\int_{B_k}|v_k(t,x)|^2\,\mathrm{d}x\right) + \int_{Q_k}|d_k(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

The quantity d_k is approximately equal to $|\nabla|u||$ when $|u|\simeq 1-2^{-k}$, and $|\nabla u|$ when $|u| >> 1-2^{-k}$. We also verify that U_0 corresponds to the first two terms on the left hand side of (4.5).

The main goal of this section is to prove the following proposition.

Proposition 4.9. *Let $p>1$. There exist universal constants $C_p,\beta_p>1$ such that for any solution to the NSE in $[1,1]\times B(1)$, the condition $U_0\leq 1$ implies*

$$U_k \leq C_p^k(1+||P||_{L^p(0,1;L^1(B_0))})U_{k-1}^{\beta}.$$

We claim that Proposition 4.9 implies Theorem 4.4 with the help of Lemma 2.6. For sufficiently small U_0, we see that $U_k\to 0$ as $k\to\infty$. Then $(u-1)_+=0$ on $B_{1/2}$ implies $u\leq 1$ on $B_{1/2}$.

4.2. **Pressure decomposition.** We rely on a decomposition of P_k (without proof) into a nonlocal term P_{k1} that exhibits good regularity, and a less regular local term P_{k2}.

Proposition 4.10. *Let (u,P) be a solution to the NSE in Q_{k-1}. Then $P|_{B_{k-2/3}}$ can be decomposed into two parts*

$$P|_{B_{k-2/3}} = P_{k1}|_{B_{k-2/3}} + P_{k2}|_{B_{k-2/3}},$$

where

$$\begin{aligned}&||\nabla P_{k1}||_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))} + ||P_{k1}||_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))}\\ &\leq C\cdot 2^{4dk}\Bigg(||P||_{L^p(T_{k-1},1;L^1(B_{k-1}))} + ||u||^2_{L^\infty(T_{k-1},1;L^2(B_{k-1}))}\Bigg).\end{aligned}$$

Moreover, P_{k2} is the solution on $[T_{k-1},1]\times\mathbb{R}^4$ to

$$-\Delta P_{k2} = \sum_{ij}\partial_i\partial_j[\phi_k u_j u_i].$$

We finish this subsection with a link between d_k and the gradient of v_k.

Lemma 4.11. *There exists a constant C such that for every k, and every $F\in L^\infty(T_k,1;L^2(B_k))$ and $\nabla F\in L^2(Q_k)$,*

$$||F||_{L^{2+4/d}(Q_k)} \leq C\big(||F||_{L^\infty(T_k,1;L^2(B_k))} + ||F||^{2/(d+2)}_{L^\infty(T_k,1;L^2(B_k))}||\nabla F||^{d/(d+2)}_{L^2(Q_k)}\big).$$

Proof. Sobolev embedding gives

$$||F||_{L^2(T_k,1;L^{2d/(d-2)}(B_k))} \leq C\big(||F||_{L^\infty(T_k,1;L^2(B_k))} + ||\nabla F||_{L^2(Q_k)}\big).$$

In order to satisfy

$$(1-\theta)\cdot\frac{1}{2} = \theta\cdot\frac{1}{2} + (1-\theta)\frac{d-2}{2d},$$

we take $\theta = 2/(d+2)$. Then we derive

$$\begin{aligned}||F||_{L^{2+4/d}(Q_k)} &\le ||F||^{\theta}_{L^\infty(T_k,1;L^2(B_k))}||F||^{1-\theta}_{L^2(T_k,1;L^{2d/(d-2)}(B_k))}\\ &\le C\big(||F||_{L^\infty(T_k,1;L^2(B_k))} + ||F||^{\theta}_{L^\infty(T_k,1;L^2(B_k))}||\nabla F||^{1-\theta}_{L^2(Q_k)}.\end{aligned}$$

from Hölder's inequality. □

We introduce cutoff functions $\phi_k \in C^\infty(\mathbb{R}^d)$ which verify

$$\phi_k(x) = 0 \text{ in } B_{k-2/3} \text{ and } \phi_k(x) = 0 \text{ in } B^c_{k-1}$$

with $|\nabla\phi_k| \le C2^{dk}$ and $|\nabla^2\phi_k| \le C2^{2dk}$.

Lemma 4.12. *We can split the function u into*

$$u = u\frac{v_k}{|u|} + u\bigg(1 - \frac{v_k}{|u|}\bigg), \text{ where } \bigg|u\bigg(1 - \frac{v_k}{|u|}\bigg)\bigg| \le 1 - 2^{-k}.$$

Moreover, the following gradients are bounded with respect to d_k

$$\frac{v_k}{|u|}|\nabla u|, \mathbf{1}_{\{|u|\ge 1-2^{-k}\}}|\nabla|u||, |\nabla v_k| \le d_k \text{ and } \bigg|\nabla\frac{uv_k}{|u|}\bigg| \le 3d_k.$$

Corollary 4.13. *We have $||v_{k-1}||_{L^{2+4/d}(Q_{k-1})} \le CU^{1/2}_{k-1}$.*

Proof. By Lemma 4.11, we have

$$\begin{aligned}&||v_{k-1}||_{L^{2+4/d}(Q_k)}\\ &\le C\big(||v_{k-1}||_{L^\infty(T_k,1;L^2(B_k))} + ||v_{k-1}||^{2/(d+2)}_{L^\infty(T_k,1;L^2(B_k))}||\nabla v_{k-1}||^{d/(d+2)}_{L^2(Q_k)}\big)\\ &\le C\big(||v_{k-1}||_{L^\infty(T_k,1;L^2(B_k))} + ||v_{k-1}||^{2/(d+2)}_{L^\infty(T_k,1;L^2(B_k))}||d_k||^{d/(d+2)}_{L^2(Q_k)}\big),\end{aligned}$$

which is bounded above by $CU^{1/2}_{k-1}$. □

4.3. **Proof of Proposition 4.9. Step 1: Evolution of v_k^2.**

Lemma 4.14. *Let u be a solution to NSE in $Q = (0,\infty)\times\Omega$. The level set energy function v_k verifies in the sense of the distribution*

$$\partial_t\frac{v_k^2}{2} + \operatorname{div}\bigg(u\frac{v_k^2}{2}\bigg) + d_k^2 - \Delta\frac{v_k^2}{2} + \operatorname{div}(uP) + (\frac{v_k}{|u|} - 1)u\cdot\nabla_x P \le 0. \tag{4.15}$$

Proof. Multiply (4.1) by $uv_k/|u|$. □

Step 2: Bound on U_k. We introduce cutoff functions $\eta_k \in C^\infty(\mathbb{R}^d)$ satisfying

$$\eta_k(x) = 1 \text{ in } B_k \text{ and } \eta_k(x) = 0 \text{ in } B^c_{k-1/3}$$

with $|\nabla\eta_k| \le C\cdot 2^{dk}$ and $|\nabla^2\eta_k| \le C\cdot 2^{2dk}$.

We multiply (4.15) by $\eta_k(x)$ and integrate on $[\sigma,\tau]\times\mathbb{R}^d$ for $T_{k-1}\le\sigma\le T_k\le t\le 1$ to find

$$\begin{aligned}
&\int \eta_k(x)\frac{|v_k(t,x)|^2}{2}\,\mathrm{d}x+\int_\sigma^t\int \eta_k(x)d_k^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\\
&\le \int \eta_k(x)\frac{|v_k(\sigma,x)|^2}{2}\,\mathrm{d}x+\int_\sigma^t\int \nabla\eta_k(x)u\frac{|v_k(s,x)|^2}{2}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_\sigma^t\int \Delta\eta_k(x)\frac{|v_k(s,x)|^2}{2}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\int_\sigma^t\int \eta_k(x)\Big\{\operatorname{div}(uP)+\Big(\frac{v_k}{|u|}-1\Big)u\nabla P\Big\}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We then integrate in σ between T_{k-1} and T_j and divide by their difference for

$$\begin{aligned}
&\sup_{t\in[T_k,1]}\Big(\int \eta_k(x)\frac{|v_k(t,x)|^2}{2}\,\mathrm{d}x+\int_{T_k}^t\int \eta_k(x)d_k^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\Big)\\
&\le 2^{k+1}\int_{T_{k-1}}^{T_k}\int \eta_k(x)\frac{|v_k(\sigma,x)|^2}{2}\,\mathrm{d}x\,\mathrm{d}\sigma+\int_{T_{k-1}}^1\Big|\int \nabla\eta_k(x)u\frac{|v_k(s,x)|^2}{2}\,\mathrm{d}x\Big|\,\mathrm{d}s\\
&\quad+\int_{T_{k-1}}^1\Big|\int \Delta\eta_k(x)\frac{|v_k(s,x)|^2}{2}\,\mathrm{d}x\Big|\,\mathrm{d}s\\
&\quad+\int_{T_{k-1}}^1\Big|\int \eta_k(x)\Big\{\operatorname{div}(uP)+\Big(\frac{v_k}{|u|}-1\Big)u\nabla P\Big\}\,\mathrm{d}x\Big|\,\mathrm{d}t.
\end{aligned}$$

Since $\eta_k\equiv 1$ on B_k, we have

$$\begin{aligned}
U_k&\le \sup_{t\in[T_k,1]}\Big(\int \eta_k(x)\frac{|v_k(t,x)|^2}{2}\,\mathrm{d}x\Big)+\int_{T_k}^1\int \eta_k(x)d_k^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\\
&\le 2\sup_{t\in[T_k,1]}\Big(\int \eta_k(x)\frac{|v_k(t,x)|^2}{2}\,\mathrm{d}x+\int_{T_k}^t\int \eta_k(x)d_k^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\Big).
\end{aligned}$$

We claim that

$$U_k\le C\cdot 2^{2dk}\int_{Q_{k-1}}|v_k(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s+C\cdot 2^{dk}\int_{Q_{k-1}}|v_k(s,x)|^3\,\mathrm{d}x\,\mathrm{d}s \tag{4.16}$$

$$+2\int_{T_{k-1}}^1\Big|\int \eta_k(x)\Big\{\operatorname{div}(uP)+\Big(\frac{v_k}{|u|}-1\Big)u\nabla P\Big\}\,\mathrm{d}x\Big|\,\mathrm{d}t. \tag{4.17}$$

Step 3: Raise of the power exponents. We want to bound the right hand side with some nonlinear power of U_{k-1} greater than 1.

Lemma 4.18. *There exists a constant C such that for all $k>1$ and $q>1$,*

$$||\mathbf{1}_{\{v_k>0\}}||_{L^q(Q_{k-1})}\le C2^{\frac{2k}{q}\cdot\frac{d+2}{d}}U_{k-1}^{\frac{1}{q}\cdot\frac{d+2}{d}},$$
$$||\mathbf{1}_{\{v_k>0\}}||_{L^\infty(T_{k-1},1;L^q(B_{k-1}))}\le C2^{\frac{2k}{q}}U_{k-1}^{\frac{1}{q}}.$$

Proof. First, $v_k > 0$ implies $v_{k-1} > 2^{-k}$. We apply Tchebychev's inequality and Corollary 4.13 to find

$$\begin{aligned}||\mathbf{1}_{\{v_k>0\}}||^q_{L^q(Q_{k-1})} &= \int_{Q_{k-1}} \mathbf{1}_{\{v_k>0\}}\,\mathrm{d}x\,\mathrm{d}t \le \int_{Q_{k-1}} \mathbf{1}_{\{v_{k-1}>2^{-k}\}}\,\mathrm{d}x\,\mathrm{d}t\\ &\le |\{v_{k-1} > 2^{-k}\} \cap Q_{k-1}|\\ &\le 2^{(2+4/d)k}\int_{Q_{k-1}} |v_{k-1}|^{2+4/d}\,\mathrm{d}x\,\mathrm{d}t\\ &= 2^{(2+4/d)k}||v_{k-1}||^{2+4/d}_{L^{2+4/d}(Q_{k-1})}\\ &\le 2^{(2+4/d)k}U^{1+2/d}_{k-1} = 2^{2k(d+2)/d}U^{(d+2)/d}_{k-1}.\end{aligned}$$

For the second statement, we observe that for every $t \in [T_{k-1}, 1]$,

$$\begin{aligned}||\mathbf{1}_{\{v_k(t,\cdot)>0\}}||^q_{L^q(B_{k-1})} &\le \int_{B_{k-1}} \mathbf{1}_{\{v_k(t,\cdot)>0\}}\,\mathrm{d}x \le \int_{B_{k-1}} \mathbf{1}_{\{v_{k-1}(t,\cdot)>2^{-k}\}}\,\mathrm{d}x\\ &\le |\{v_{k-1}(t,\cdot) > 2^{-k}\} \cap B_{k-1}|\\ &\le 2^{2k}\int_{B_{k-1}} |v_{k-1}(t,x)|^2\,\mathrm{d}x\\ &\le 2^{2k}\sup_{s\in[T_{k-1},1]}\int_{B_{k-1}} v^2_{k-1}(s,x)\,\mathrm{d}x \le 2^{2k}U_{k-1},\end{aligned}$$

before raising both sides to the $1/q$th power. □

We can now control (4.16) with

$$\begin{aligned}&C\cdot 2^{2dk}\int_{Q_{k-1}} |v_k(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s + C\cdot 2^{dk}\int_{Q_{k-1}} |v_k(s,x)|^3\,\mathrm{d}x\,\mathrm{d}s\\ &\le C2^{2dk}||v_k^2||_{L^{\frac{d+2}{d}}(Q_{k-1})}||\mathbf{1}_{\{v_k>0\}}||_{L^{\frac{d+2}{2}}(Q_{k-1})}\\ &\quad + C2^{dk}||v_k^3||_{L^{\frac{2(d+2)}{3d}}(Q_{k-1})}||\mathbf{1}_{\{v_k>0\}}||_{L^{\frac{2(d+2)}{4-d}}(Q_{k-1})}\\ &\le C2^{2dk}||v_{k-1}||^2_{L^{\frac{2(d+2)}{d}}(Q_{k-1})}||\mathbf{1}_{\{v_k>0\}}||_{L^{\frac{d+2}{2}}(Q_{k-1})}\\ &\quad + C2^{dk}||v_{k-1}||^3_{L^{\frac{2(d+2)}{d}}(Q_{k-1})}||\mathbf{1}_{\{v_k>0\}}||_{L^{\frac{2(d+2)}{4-d}}(Q_{k-1})}\\ &\le C2^{2dk}U_{k-1}\cdot C2^{\frac{4k}{d}}U^{\frac{2}{d}}_{k-1} + C2^{dk}U^{3/2}_{k-1}\cdot C2^{\frac{k(4-d)}{d}}U^{\frac{4-d}{2d}}_{k-1}\\ &\le C2^{2dk+\frac{4k}{d}}U^{\frac{d+2}{d}}_{k-1}.\end{aligned}$$

Then (4.17) is bounded by

$$\int_{T_{k-1}}^1 \left|\int_{B_{k-1/3}} \eta_k \frac{v_k u}{|u|}\nabla P_{k1}\,\mathrm{d}x\right|\mathrm{d}t \tag{4.19}$$

$$+\int_{T_{k-1}}^1 \left|\int_{B_{k-1/3}} \eta_k\{\mathrm{div}(uP_{k2}) + \left(\frac{v_k}{|u|} - 1\right)u\nabla P_{k2}\}\,\mathrm{d}x\right|\mathrm{d}t. \tag{4.20}$$

Step 4: Bound of the non-local pressure term (4.19) involving P_{k1}. Here, we apply the Hölder inequality on the quantities $v_k, \nabla P_{k1},$ and $\mathbf{1}_{\{v_k>0\}}$. We can bound $||\nabla P_{k1}||$ with Proposition 4.10 and use Lemma 4.18 on $||\mathbf{1}_{\{v_k>0\}}||$.

We then consider different values of p separately.

- For large values of $p \geq 2$ with $p' \leq 2$, we control (4.19) by

$$\begin{aligned}
&C||v_k||_{L^\infty(T_{k-1},1;L^2(B_{k-1}))}||\nabla P_{k1}||_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))}||\mathbf{1}_{\{v_k>0\}}||_{L^2(Q_{k-1})}\\
&\leq CU_{k-1}^{1/2}||\nabla P_{k1}||_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))}2^{\frac{k(d+2)}{d}}U_{k-1}^{\frac{d+2}{2d}}\\
&\leq C2^{\frac{k(d+2)}{d}}||\nabla P_{k1}||_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))}U_{k-1}^{\frac{d+1}{d}}.
\end{aligned}$$

- For $p < 2$, we get instead the weaker bound of

$$\begin{aligned}
&C||v_k||_{L^\infty(L^2)}||\nabla P_{k1}||_{L^p(L^\infty)}||\mathbf{1}_{\{v_k>0\}}||_{L^{p'}(L^{p'})}||\mathbf{1}_{\{v_k>0\}}||_{L^\infty(L^{\frac{2p}{2-p}})}\\
&\leq CU_{k-1}^{1/2}\left(||P||_{L^p(L^1)}+||u||^2_{L^\infty(L^2)}\right)2^{\frac{2k}{p'}\cdot\frac{d+2}{d}}U_{k-1}^{\frac{1}{p'}\cdot\frac{d+2}{d}}\,2^{\frac{2k}{2p/(2-p)}}U_{k-1}^{\frac{1}{2p/(2-p)}}\\
&\leq C2^{\frac{dkp+4pk-4k}{dp}}\left(||P||_{L^p(L^1)}+||u||^2_{L^\infty(L^2)}\right)U_{k-1}^{\frac{dp+2p-2}{dp}}.
\end{aligned}$$

  We check that $\frac{dp+2p-2}{dp} > 1$ for any $p > 1$.

Thus, for any $p > 1$ and $U_0 \leq 1$, there exists $\alpha_p > 0$ and $\beta_p > 1$ such that (4.19) is bounded by

$$C2^{k\alpha_p}U_{k-1}^{\beta_p}(||P||_{L^p(L^1)}+||u||^2_{L^\infty(L^2)}) \leq C2^{k\alpha_p}U_{k-1}^{\beta_p}(||P||_{L^p(L^1)}+1). \tag{4.21}$$

Step 5: Bound of the local pressure term (4.20) involving P_{k2}.

We split the term involving P_{k2} into three $P_{k2} = P_{k21} + P_{k22} + P_{k23}$, where $P_{k21}, P_{k22}, P_{k23}$ are defined by

$$\begin{aligned}
-\Delta P_{k21} &= \sum_{ij}\partial_i\partial_j\left\{\phi_k u_j\left(1-\frac{v_k}{|u|}\right)u_i\left(1-\frac{v_k}{|u|}\right)\right\}\\
-\Delta P_{k22} &= \sum_{ij}\partial_i\partial_j\left\{2\phi_k u_j\left(1-\frac{v_k}{|u|}\right)u_i\frac{v_k}{|u|}\right)\right\}\\
-\Delta P_{k23} &= \sum_{ij}\partial_i\partial_j\left\{\phi_k u_j\frac{v_k}{|u|}u_i\frac{v_k}{|u|}.\right\}
\end{aligned}$$

For all $1 < q < \infty$, the Riesz Theorem gives $||P_{k21}||_{L^q(Q_{k-1})} \leq C_q$. We also have

$$\operatorname{div}(uP_{k21}) + u\left(\frac{v_k}{|u|}-1\right)\nabla P_{k21} = \operatorname{div}\left(v_k\frac{u}{|u|}P_{k21}\right) - P_{k21}\operatorname{div}\left(\frac{uv_k}{|u|}\right). \tag{4.22}$$

For arbitrary $q < \infty$ with $q > \max\{2, \frac{d+2}{2}\}$, we check that

$$\begin{aligned}
&\int_{T_{k-1}}^1\left|\int\eta_k(x)\left\{\operatorname{div}(uP_{k21})+\left(\frac{v_k}{|u|}-1\right)u\nabla P_{k21}\right\}\mathrm{d}x\right|\mathrm{d}t\\
&\leq 2^{dk}C_q||v_k||_{L^{\frac{2(d+2)}{d}}}||P_{k21}||_{L^q}||\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}||_{L^{\frac{2q(d+2)}{dq+4q-2d-4}}}\\
&\quad + C_q||P_{k21}||_{L^q}||d_k||_{L^2}||\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}||_{L^{2q/(q-2)}}\\
&\leq C_q2^{k\alpha_q}(U_{k-1}^{\frac{(d+2)(q-1)}{dq}}+U_{k-1}^{\frac{dq+q-d-2}{dq}}).
\end{aligned}$$

We now turn to the terms involving P_{k22} and P_{k23}. We apply the Riesz Theorem and Corollary 4.13 again for

$$\begin{aligned}||P_{k22}||_{L^{\frac{2(d+2)}{d}}} &\le C\sum_{ij}||u_j(1-v_k/|u|)||_{L^\infty}||v_ku_i/|u|||_{L^{\frac{2(d+2)}{d}}}\\ &\le C||v_k||_{L^{\frac{2(d+2)}{d}}} \le CU_{k-1}^{1/2},\\ ||P_{k23}||_{L^{\frac{d+2}{d}}} &\le C\sum_{ij}||u_kv_k/|u|||_{L^{\frac{2(d+2)}{d}}}||v_ku_i/|u|||_{L^{\frac{2(d+2)}{d}}}\\ &\le C||v_k||^2_{L^{\frac{2(d+2)}{d}}} \le CU_{k-1}.\end{aligned}$$

We need to control their gradients too.

Lemma 4.23. *We can decompose ∇P_{k22} and ∇P_{k23} into*

$$\nabla P_{k22} = G_{221} + G_{222} + G_{223}, \text{ and } \nabla P_{k23} = G_{231} + G_{232},$$

where

$$\begin{aligned}||G_{221}||_{L^{\frac{2(d+2)}{d}}(Q_{k-1/3})} &\le C2^{dk}||v_k||_{L^{\frac{2(d+2)}{d}}(Q_{k-1})},\\ ||G_{222}||_{L^2(Q_{k-1/3})} &\le C||d_k||_{L^2(Q_{k-1})},\\ ||G_{223}||_{L^{\frac{d+2}{d+1}}(Q_{k-1/3})} &\le C||v_k||_{L^{\frac{2(d+2)}{d}}(Q_{k-1})}||d_k||_{L^2(Q_{k-1})},\\ ||G_{231}||_{L^{\frac{d+2}{d}}(Q_{k-1/3})} &\le C2^{dk}||v_k||^2_{L^{\frac{2(d+2)}{d}}(Q_{k-1})},\\ ||G_{232}||_{L^{\frac{d+2}{d+1}}(Q_{k-1/3})} &\le C||v_k||_{L^{\frac{2(d+2)}{d}}(Q_{k-1})}||d_k||_{L^2(Q_{k-1})}.\end{aligned}$$

We can now bound (4.20) by

$$\begin{aligned}&\int_{T_{k-1}}^1\left|\int\eta_k(x)\left\{\operatorname{div}(u(P_{k22}+P_{k23})) + \left(\frac{v_k}{|u|}-1\right)u\nabla(P_{k22}+P_{k23})\right\}\mathrm{d}x\right|\mathrm{d}t\\ &\le \int_{T_{k-1}}^1\int_{B_{k-1/3}}|\nabla\eta_k||u||P_{k22}+P_{k23}| + \eta_k(|\nabla P_{k22}|+|\nabla P_{k23}|)\,\mathrm{d}x\,\mathrm{d}t\\ &\le C2^{dk}\int_{T_{k-1}}^1\int_{B_{k-1/3}}(1+v_k)(|P_{k22}|+|P_{k23}|)(|\nabla P_{k22}|+|\nabla P_{k23}|)\,\mathrm{d}x\,\mathrm{d}t\\ &\le C2^{dk}\Bigg(||\mathbf{1}_{\{|u|\ge 1-2^{-k}\}}||_{L^{\frac{2(d+2)}{d+4}}(B_{k-1})}||P_{k22}||_{L^{\frac{2(d+2)}{d}}(B_{k-1/3})}\\ &\quad + ||\mathbf{1}_{\{|u|\ge 1-2^{-k}\}}||_{L^{\frac{d+2}{2}}(B_{k-1})}||P_{k23}||_{L^{\frac{d+2}{d}}(B_{k-1/3})}\Bigg)\\ &\le C2^{\alpha_{1k}}U_{k-1}^{\frac{d+2}{d}} + C2^{\alpha_{2k}}U_{k-1}^{\frac{d+1}{d}}.\end{aligned}$$

Step 6: Conclusion. From (4.16) and (4.17), for every $p>1$, there exists $C_p>1$ and $\beta_p>1$ such that for any solution to the NSE in $[-1,1]\times B(1)$, if $U_0\le 1$ then

$$U_k \le C_p^k(1+||P||_{L^p(0,1;L^1(B_0))})U_{k-1}^{\beta_p}. \tag{4.24}$$

Acknowledgments

I would like to thank my mentor, Dr. Jincheng Yang, for her invaluable guidance over this REU. Her generosity in sharing her insights and expertise helped me to appreciate many subtleties about the Navier-Stokes equations. I especially admire her willingness to toggle between high-level strategies and low-level technical details to address the source of my confusion. With her patient answers, I left every meeting inspired with a newfound sense of direction. I am also grateful to Minh Pham and Zichen Lu for sharing their ideas and working on problems together. Thank you to the REU lecturers and Professor Peter May for organizing this program. Last but not least, I must thank my family and friends for their love and support.

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