Singular Integral Operators

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1 Harmonic Analysis Facts

Theorem 1.1. *If T is a bounded linear transformation from $L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, then T commutes with translations iff there exists a measure in $\mathcal{B}(\mathbb{R}^n)$ such that $T(f) = f * \mu$.*

Theorem 1.2. *If T is a bounded linear transformation from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, then T commutes with translations if T is a multiplier operator, i.e. there exists bounded measurable $m(y)$ such that $(\hat{Tf})(y) = \hat{f}(y)m(y)$.*

2 Calderon-Zygmund Theory

Theorem 2.1. *Suppose a kernel $K(x)$ satisfies*

$$|K(x)| \leq \frac{B}{|x|^n} \quad |x| > 0, \tag{2.1}$$

$$\int_{R_1<|x|<R_2} K(x)\mathrm{d}x = 0 \quad 0 < R_1 < R_2 < \infty \tag{2.2}$$

and

$$\int_{|x|\geq 2|y|} |K(x-y) - K(x)|\mathrm{d}x \leq B. \tag{2.3}$$

Then for $f \in L^1 \cap L^p(\mathbb{R}^n), \epsilon > 0$ we define

$$T_\epsilon(f)(x) = \int_{|y|\geq\epsilon} f(x-y)K(y)\mathrm{d}y.$$

Then have the following bound

$$\|T_\epsilon f\|_p \leq A_p \|f\|_p .$$

Moreover, $\lim_{\epsilon\to 0} T_\epsilon(f) = T(f)$ exists in L^p norm.

The strategy is to prove that T_ϵ is a weak (1,1) and strong (2,2) operator, with bounds independent of epsilon. Then apply Marcinkiewicz interpolation, and then check that bounds are preserved as $\epsilon \to 0$.

Remark 2.2. The condition that $K \in C^1(\mathbb{R}^n \setminus \{0\})$ and $|\nabla K(x)| \leq B/|x|^{n+1}$ implies (2.3).

2.1 L^2 to L^2 boundedness

Lemma 2.3. *Take K as above and define $K_\epsilon(x) = K(x)\mathbf{1}_{|x|\geq\epsilon}(x)$, which lies in L^2 We have the estimates*

$$\sup_y |\hat{K}_\epsilon(y)| \leq CB$$

independently of ϵ.

Proof. It suffices to consider $\epsilon = 1$. $K_1(x)$ satisfies the same conditions as $K(x)$. The Fourier transform is given by

$$\begin{aligned}\hat{K}_1(y) &= \lim_{R\to\infty}\int_{|x|\leq R} e^{2\pi i x\cdot y} K_1(x)\mathrm{d}x = I_1 + I_2 \\ &= \int_{|x|\leq\frac{1}{|y|}} e^{2\pi i x\cdot y} K_1(x)\mathrm{d}x + \lim_{R\to\infty}\int_{\frac{1}{|y|}\leq|x|\leq R} e^{2\pi i x\cdot y} K_1(x)\mathrm{d}x.\end{aligned}$$

Observe via the cancellation condition that

$$\begin{aligned}I_1 &= \int_{|x|\leq\frac{1}{|y|}} (e^{2\pi i x\cdot y} - 1) K_1(x)\mathrm{d}x \\ |I_1| &\leq C|y|\int_{|x|\leq\frac{1}{|y|}} |x||K_1(x)|\mathrm{d}x \leq C|y|\int_{|x|\leq\frac{1}{|y|}} \frac{B|x|}{|x|^n}\mathrm{d}x,\end{aligned}$$

so we switch to polar coordinates to see that the above is bounded by CB.

For I_2, we define $z = \frac{1}{2}\frac{y}{|y|^2}$. Then $e^{2\pi i y\cdot z} = -1$, with $|z| = \frac{1}{2|y|}$. By a change of variables, we have

$$\int_{\mathbb{R}^n} K_1(x)e^{2\pi i x\cdot\xi}\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^n}[K_1(x) - K_1(x-z)]e^{2\pi i x\cdot\xi}\mathrm{d}x$$

Some rearrangement tells us that

$$\begin{aligned}I_2 &= I_3 + I_4 \\ &= \frac{1}{2}\lim_{R\to\infty}\int_{\frac{1}{|y|}\leq|x|\leq R}[K_1(x) - K_1(x-z)]e^{2\pi i\xi\cdot x}\mathrm{d}x - \frac{1}{2}\int_{|x|\leq\frac{1}{|y|}}[K_1(x) + K_1(x-z)]e^{2\pi i x\cdot\xi}\mathrm{d}x.\end{aligned}$$

Then for I_3 observe that $|z| \leq \frac{1}{2}|x|$, so apply (2.3). By a change of variables, we rewrite I_4 as

$$\int_{|x|\leq\frac{1}{|y|}} K_1(x)e^{2\pi i x\cdot\xi}\mathrm{d}x - \int_{|x+z|\leq\frac{1}{|y|}} K_1(x)e^{2\pi i x\cdot\xi}\mathrm{d}x.$$

The integral is bounded in some spherical shell $\frac{1}{2|y|} \leq |x| \leq \frac{2}{|y|}$, so we can apply (2.1) with changes to spherical coordinates.

If T corresponds to kernel K then $\tau_{\epsilon^{-1}}T\tau_\epsilon$ corresponds to kernel $\epsilon^{-n}K(\epsilon^{-1}(x)$. So let $K' = \epsilon^n K(\epsilon x)$. We can check that K' satisfies the conditions of the lemma with K'_1 satisfies $|\hat{K}'_1(y)| \leq CB$. But the Fourier transform of $K_\epsilon(x) = \epsilon^{-n}K'_1(\epsilon^{-1}x)$ is $\hat{K}'_1(\epsilon y)$. $\square$

If the kernel has bounded Fourier transform, Plancherel gives

$$\|Tf\|_2 \leq B \|f\|_2$$

2.2 Weak L^1 to L^1 boundedness

Let $\alpha > 0$ be fixed. Our goal is to find a constant C such that

$$m\{Tf > \alpha\} \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(x)| \mathrm{d}x.$$

First apply the Calderon-Zygmund decomposition, we can write $\mathbb{R}^n = F \cup \Omega$. Then define

$$g(x) := \begin{cases} f(x) & x \in F, \\ \fint_{Q_j} f(x) \mathrm{d}x & x \in Q_j^0. \end{cases}$$

Then take $b(x) = f(x) - g(x)$. Observe that $b(x)$ satisfies $b(x) = 0$ for $x \in F$ as well as

$$\fint_{Q_j} b(x) \mathrm{d}x = 0.$$

The triangle inequality tells us we can bound Tg and Tb separately since

$$m\{|Tf| > \alpha\} \leq m\{|Tg| > \frac{\alpha}{2}\} + m\{|Tb| > \frac{\alpha}{2}\}.$$

2.2.1 Estimate for Tg

We have

$$\begin{aligned} \|g\|_2^2 &= \int_F |g|^2 + \int_\Omega |g|^2 \\ &\leq \int_F \alpha |f| + m(\Omega)(2^n \alpha)^2 \\ &\leq \alpha \|f\|_1 + \frac{A}{\alpha} \|f\|_1 \cdot C^2 \alpha^2 \leq \frac{C}{\alpha} \|f\|_1 . \end{aligned}$$

But from the $L^2 \to L^2$ bound, we can apply Tchebychev for

$$m\{|Tg| > \frac{\alpha}{2}\} \leq \frac{C^2}{\alpha^2} \|g\|_2^2 .$$

2.2.2 Estimate for Tb

Here, we want the estimate for the integral of Tb over the good set F.

We can write $b_j(x) = b(x)\mathbf{1}_{Q_j}(x)$. We now expand the cubes each by $2\sqrt{n}$ times, denoted Q_j^*. Take F^* as the complement of the union of the expanded cubes. For $x \in F^*$, we can check that $|x - y^j| \geq 2|y - y^j|$ for all $y \in Q_j$, where y^j is the center of the cube. Then

$$\begin{aligned}
Tb_j(x) &= \int_{Q_j} K(x-y)b_j(y)\mathrm{d}y \\
&= \int_{Q_j} [K(x-y) - K(x-y^j)]b_j(y)\mathrm{d}y, \\
|Tb(x)| &\leq \sum_j \int_{Q_j} |K(x-y) - K(x-y^j)||b(y)|\mathrm{d}y, \\
\int_{F^*} |Tb(x)|\mathrm{d}x &\leq \int_{F^*} \sum_j \int_{Q_j} |K(x-y) - K(x-y^j)||b(y)|\mathrm{d}y\mathrm{d}x \\
&\leq \sum_j \int_{Q_j} |b(y)| \int_{F^*} |K(x-y) - K(x-y^j)|\mathrm{d}x\mathrm{d}y \quad \text{(Fubini's)}, \\
&\leq \sum_j \int_{Q_j} |b(y)| \int_{|x'| \geq 2|y'|} |K(x'-y') - K(x')|\mathrm{d}x'\mathrm{d}y \\
&\leq B \sum_j \int_{Q_j} |b(y)|\mathrm{d}y \leq C \left\|f\right\|_1.
\end{aligned}$$

We also recall that $m(\Omega) \leq \frac{C}{\alpha} \left\|f\right\|_1$.

2.3 Duality

How do we do this approximation if K does not lie in L^2? Answer: We can sidestep this because K_ϵ lies in L^2.

Recall that if $\psi \in L^1_{\text{loc}}$ and

$$\sup \left\{ \left| \int \psi\varphi \mathrm{d}x \right| : \varphi \in C_c, \left\|\varphi\right\|_q \leq 1 \right\} = A < \infty,$$

then $\left\|\psi\right\|_{L^p} = A$. We can write the double integral as

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x-y)f(y)\varphi(x)\mathrm{d}x\mathrm{d}y = \int f(y) \left(\int K(x-y)\varphi(y)\mathrm{d}x \right) \mathrm{d}y.$$

Replacing with the kernel $K(-x)$, we know that $\int K(x-y)\varphi(y)\mathrm{d}x$ belongs to L^q for $1 < q < 2$. Then apply Hölder's.

2.4 From T_ϵ to T

Consider $f \in C_c^1(\mathbb{R}^n)$. Then by the cancellation condition

$$T_\epsilon(f_1)(x) = \int_{|y|\geq 1} K(y) f_1(x-y)\mathrm{d}y + \int_{1\geq |y|\geq \epsilon} K(y)[f_1(x-y) - f_1(x)]\mathrm{d}y.$$

The first integral lies in L^p, second integral converges uniformly to 0. We can write arbitrary $f \in L^p$ as $f = f_1 + f_2$, where $f_1 \in C_c^1$ and $\|f_2\|_p$ is small.

3 SIOs which commute with dilations

If $\tau_{\epsilon^{-1}} T \tau_\epsilon = T$, then we are back to the requirement $K(\epsilon x) = \epsilon^{-n} K(x)$, where K is homogeneous of degree $-n$. Then

$$K(x) = \frac{\Omega(x)}{|x|^n},$$

where Ω is homogeneous of degree 0.

Remark 3.1. It suffices to consider Ω which satisfies the following smoothness and cancellation conditions

$$\sup\{|\Omega(x) - \Omega(x')| : |x - x'| \leq \delta, |x| = |x'| = 1\} = \omega(\delta) \implies \int_0^1 \frac{\omega(\delta)}{\delta}\mathrm{d}\delta < \infty, \tag{3.1}$$

$$\int_{S^{n-1}} \Omega(\sigma)\mathrm{d}\sigma = 0. \tag{3.2}$$

Theorem 3.2. *Let Ω homogeneous of degree 0 satisfying the above two properties. Let*

$$T_\epsilon(f)(x) = \int_{|y|>\epsilon} \frac{\Omega(y)}{|y|^n} f(x-y)\mathrm{d}y.$$

Then T_ϵ is bounded from L^p to L^p. The limit in the L^p norm exists (call it T), and T satisfies the same bounds.

In addition to convergence in L^p norm, we can also get convergence almost everywhere, with the help of the maximal function.

Theorem 3.3. *Take Ω as above. For $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$, the limit $\lim_{\epsilon\to 0} T_\epsilon(f)(x)$ exists for almost everywhere. Define the associated maximal function*

$$T^*(f)(x) = \sup_{\epsilon>0} |T_\epsilon(f)(x)|.$$

The result follows from the fact that T^* is of weak type $(1,1)$ and strong type (p,p).

Proof. Proof of (p,p) is deferred to latter chapter.

Again given $\alpha > 0$, we will consider a splitting of a function $f = g + b$. Take $x \in (Q_j^*)^c$. Suppose that there exists some $y \in Q_j$ such that $|x-y| = \epsilon$. Then there exists γ_n and γ_n' such that for every $y' \in Q_j$,

$$\gamma_n' \epsilon \leq |x-y| \leq \gamma_n \epsilon.$$

We claim that if $x \in F^*$,

$$\sup_{\epsilon>0} |T_\epsilon(b(x))| \leq \sum_j \int_{Q_j} |K(x-y) - K(x-y^j)||b(y)|\mathrm{d}y + C \sup_{r\to 0} \fint_{B(x,r)} |b(y)|\mathrm{d}y.$$

By definition,

$$T_\epsilon b(x) = \sum_j \int_{Q_j} K_\epsilon(x-y) b(y) \mathrm{d}y.$$

There are three different kinds of Q_j:

1. For all $y \in Q_j$, we have $|x-y| < \epsilon$. This term vanishes.
2. For all $y \in Q_j$, we have $|x-y| > \epsilon$. This appears as the first term.
3. There exists $y \in Q_j$ such that $|x-y| = \epsilon$. Use the above bounds, and estimate over $B(x, \gamma_n \epsilon)$.

Now define

$$\Lambda(f)(x) = \left| \limsup_{\epsilon\to 0} T_\epsilon(f)(x) - \liminf_{\epsilon\to 0} T_\epsilon(f)(x) \right| \leq 2(T^* f)(x).$$

□

Hypothesis 3.4. *The Calderón-Zygmund theorem gives us bounds on certain singular integral operators. Can we in fact describe them as Fourier multipliers?*

Proposition 3.5. *Suppose Ω is homogeneous of degree 0, and suppose that Ω satisfies the following cancellation and smoothness conditions*

$$\sup\{|\Omega(x)-\Omega(x')| : |x-x'| \leq \delta, |x| = |x'| = 1\} = \omega(\delta) \implies \int_0^1 \frac{\omega(\delta)}{\delta} \mathrm{d}\delta < \infty, \tag{3.3}$$

$$\int_{S^{n-1}} \Omega(\sigma) \mathrm{d}\sigma = 0. \tag{3.4}$$

Let Tf be the convolution of $\Omega(x)/|x|^n$ with $f(x)$, defined in the principal value sense.

If $f \in L^2(\mathbb{R}^n)$, then the Fourier transforms of f and Tf are related by $\widehat{Tf} = m\hat{f}$, where m is a homogeneous function of degree 0. Explicitly, we have

$$m(x) = \int_{\mathbb{S}^{n-1}} \left[\frac{\pi i}{2} \operatorname{sgn}(x \cdot y) + \log\left(\frac{1}{|x \cdot y|}\right)\right] \Omega(y) \mathrm{d}\sigma(y). \tag{3.5}$$

Proof. Since T is bounded and commutes with translations, we can write T as a multiplier operator. Moreover, such an operator commutes with dilations, so the multiplier is homogeneous of degree 0. How do we express the multiplier in terms of the kernel? Take

$$K_{\epsilon,\eta}(x) = \begin{cases} \frac{\Omega(x)}{|x|^n} & \epsilon \leq |x| \leq \eta, \\ 0 & \text{otherwise.} \end{cases}$$

We can check immediately that $K_\epsilon, \eta \in L^1$ (though not necessarily uniformly). Moreover, if $f \in L^2(\mathbb{R}^n)$, then $\widehat{K_{\epsilon,\eta} * f} = \hat{K}_{\epsilon,\eta}\hat{f}$.

We claim the following

1. $\sup_{\epsilon,\eta} |\widehat{K_{\epsilon,\eta}(y)}| \leq A$.
2. If $x \neq 0$, then the limit as $\epsilon \to 0, \eta \to \infty$ of $\hat{K}_\epsilon, \eta(x) = m(x)$.

To this end, we write $x = Rx', y = ry'$ in polar coordinates. Consider the following integral

$$I_{\epsilon,\eta}(x', y') := \int_\epsilon^\eta \frac{1}{r} \left[\exp(2\pi i Rr x' \cdot y') - \cos(2\pi Rr)\right] \mathrm{d}r.$$

With some calculus, we get

$$\begin{aligned} \operatorname{Im}(I_{\epsilon,\eta}(x', y')) &= \int_\epsilon^\eta \frac{\sin[2\pi Rr(x' \cdot y')]}{r} \mathrm{d}r \to \frac{\pi}{2} \operatorname{sgn}(x' \cdot y'), \\ \operatorname{Re}(I_{\epsilon,\eta}(x', y')) &= \int_\epsilon^\eta \frac{\cos[2\pi Rr(x' \cdot y')] - \cos(2\pi Rr)}{r} \mathrm{d}r \to \cos 0 \log \frac{1}{|x' \cdot y'|}, \end{aligned}$$

since

$$\int_0^\infty \frac{h(\lambda r) - h(\mu r)}{r} \mathrm{d}r = h(0) \log \frac{\mu}{\lambda}.$$

Combining the real and imaginary parts, we have

$$I(x', y') \to \log \frac{1}{|x' \cdot y'|} + i\frac{\pi}{2}\mathrm{sgn}(x' \cdot y').$$

Rewriting in spherical coordinates, we can express the Fourier transform of the kernel as

$$\begin{aligned} \hat{K}_{\epsilon,\eta}(x) &= \int_{\mathbb{S}^{n-1}} \int_\epsilon^\eta e^{2\pi i R r x' \cdot y'} \Omega(y') \frac{\mathrm{d}r}{r} \mathrm{d}\sigma(y') \\ &= \int_{\mathbb{S}^{n-1}} I_{\epsilon,\eta}(x', y') \Omega(y') \mathrm{d}\sigma(y') \end{aligned}$$

with the help of the cancellation property. But we can just take the norm of the real and imaginary parts of I_ϵ, η for the uniform bound

$$|\hat{K}_{\epsilon,\eta}(x)| \leq A \int_{\mathbb{S}^{n-1}} [1 + \log \frac{1}{|x' \cdot y'|}] |\Omega(y')| \mathrm{d}\sigma(y').$$

Then apply DCT.

To show convergence to T, first take ϵ fixed and take $\eta \to \infty$. Then consider $\epsilon \to 0$. $\square$

Remark 3.6. No boundednesson L^1, L^∞, take Hilbert transform of characteristic function of interval (a, b).

4 Hilbert and Riesz transforms

Definition 4.1. The Hilbert transform is given by

$$Hf(x) := \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{f(x - y)}{y} \mathrm{d}y.$$

It turns out that the Hilbert transform satisfies the properties of the singular integral operators for which we proved the Calderón-Zygmund theorems. Here, we have

$$K(x) = \frac{1}{\pi x}, \Omega(x) = \frac{1}{\pi}\mathrm{sgn}(x) = \frac{1}{\pi}\frac{x}{|x|.}$$

Then $\widehat{Hf} = m\hat{f}$, with $m(x) = i\mathrm{sgn}(x)$. Then $H^2 = -I$.

Proposition 4.2. *Suppose T is a bounded operator on $L^2(\mathbb{R}^1)$ which*

1. *commutes with translations,*
2. *commutes with positive dilations,*
3. *anticommutes with reflection.*

Then T is a multiple of the Hilbert transform.

Proof. Since T commutes with translations, we can write $\widehat{Tf} = m\hat{f}$. To simplify notation, we write $\mathcal{F}f = \hat{f}$. Then $\mathcal{F}T = m\mathcal{F}$.

We recall the effect of a Fourier transform on dilation, given by

$$\begin{aligned}(\mathcal{F}\tau_\delta f)(y) &= \int e^{2\pi ixy} f(\delta x)\mathrm{d}x \\ &= |\delta|^{-1} \int e^{2\pi ixy/\delta} f(x)\mathrm{d}x = |\delta|^{-1}\tau_{\delta^{-1}}\mathcal{F}.\end{aligned}$$

Now our remaining assumptions imply that $T\tau_\delta = \mathrm{sgn}(\delta)\tau_\delta T$. We have

$$\begin{aligned}\tau_\delta m &= \tau_\delta(\mathcal{F}T\mathcal{F}^{-1}) = |\delta|^{-1}\mathcal{F}\tau_{\delta^{-1}}T\mathcal{F}^{-1} \\ &= \delta^{-1}\mathcal{F}T\tau_{\delta^{-1}}\mathcal{F}^{-1} \\ &= \mathrm{sgn}(\delta)\mathcal{F}T\mathcal{F}^{-1}\tau_\delta = \mathrm{sgn}(\delta)m\tau_\delta.\end{aligned}$$

Specifically, we have

$$m(\delta y)\hat{f}(\delta y) = \mathrm{sgn}(\delta)m(y)\hat{f}(\delta y).$$

Then $m(y)$ is a constant multiple of $\mathrm{sgn}(y)$. □

Remark 4.3. If T is a bounded linear operator on $L^2(\mathbb{R}^1)$ that commutes with translations and all dilations, then its Fourier multiplier is a constant. Then T is a constant multiple of the identity.

Let ρ denote a rotation. We define $\rho(f)(x) = f(\rho^{-1}x)$. We can verify $\mathcal{F}\rho = \rho\mathcal{F}$.

Lemma 4.4. *Let $m(x) = (m_1(x), m_2(x), \ldots, m_n(x))$ be an n-tuple of functions on $\mathbb{R}^n$. Suppose that*

1. *m is homogeneous of degree 0,*
2. *m transforms like a vector, i.e.*

$$\rho(m)(x) = m(\rho^{-1}x) = \rho(m(x)), \quad m_j(\rho^{-1}x) = \sum_k \rho_{jk}m_k(x). \tag{4.1}$$

Here we take the induced action of ρ on a function.

Then the function m takes the form

$$m(x) = c\frac{x}{|x|}, \quad m_j(x) = c\frac{x_j}{|x|}.$$

Proof. It suffices to consider x on the unit sphere. Let $\{e_i : 1 \leq i \leq n\}$ denote the standard basis. Set $c = m_1(e_1)$.

Let ρ be a rotation that fixes e_1. For $2 \leq j \leq n$, we have $\rho_{j1} = 0$ which means

$$m_j(e_1) = \sum_{k=2}^{n} \rho_{jk} m_k(e_1).$$

So the $n-1$ dimensional vector $(m_2(e_1), m_3(e_1), \ldots, m_n(e_1))$ is left fixed by all the rotations on this $n-1$ dimensional space orthogonal to e_1. So $m_2(e_1) = m_3(e_1) = \cdots = m_n(e_1) = 0$. We obtain

$$m_j(\rho^{-1}e_1) = \rho_{j1} m_1(e_1) = c\rho_{j1}.$$

If $\rho^{-1}e_1 = x$, then $\rho_{j1} = x_j$. Then $m_j(x) = cx_j$. $\square$

Definition 4.5. Let $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$. We define

$$R_j(f)(x) = \lim_{\epsilon \to 0} c_n \int_{|y| \geq \epsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\mathrm{d}y, \quad c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}$$

The associated kernel is

$$K_j(x) = c_n \frac{x_j}{|x|^{n+1}}, \quad \Omega_j(x) = c_n \frac{x_j}{|x|}.$$

Observe that the mapping from kernel Ω to multipliers m commutes with rotations. Moreover, the kernels satisfy the transformation law (4.1), which means that the multipliers. also satisfy the transformation law.

But the m_j's are homogeneous of degree 0, so the lemma shows that $m_j(x) = cx_j/|x|$. Notice that

$$c = \int_{\mathbb{S}^{n-1}} \left(\frac{\pi i}{2}\mathrm{sgn}(y_1) + \log|\frac{1}{y_1}|\right) \cdot c_n \frac{y_j}{|y|} \mathrm{d}\sigma(y) = i$$

In this case we just need to check that

$$\frac{2\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}} = \int_{\mathbb{S}^{n-1}} |\cos\theta| \mathrm{d}\sigma(y). \tag{4.2}$$

In other words, we have

$$\widehat{R_j f}(y) = i\frac{y_j}{|y|}\hat{f}(y).$$

The Riesz operators obey

$$\rho^{-1}R_j\rho f = \sum_k \rho_{jk}R_k f.$$

If we denote $\hat{R}_j = m_j$, then

$$\rho(m_j\rho^{-1}(f)) = \sum_k \rho_{jk}m_k f \iff m_j(\rho^{-1}x) = \sum_k \rho_{jk}m_k(x).$$

Proposition 4.6. *Let $T = (T_1, T_2, \ldots, T_n)$ be an n-tuple of bounded transformations on $L^2(\mathbb{R}^n)$. Suppose*

1. *Each T_j commutes with translation*
2. *Each T_j commutes with dilations*
3. *For every rotation ρ, $\rho^{-1}T_j\rho f = \sum_k \rho_{jk}T_k f$.*

Then the T_j's are a constant multiple of the Riesz transforms.

4.1 Applications of Riesz transforms

Proposition 4.7. *Suppose $f \in C_C^2$. Then for $1 < p < \infty$.*

$$\left\| \frac{\partial^2 f}{\partial x_j \partial x_k} \right\|_p \leq A_p \left\| \Delta f \right\|_p . \tag{4.3}$$

Proof. We want to apply the identity

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = -R_j R_k \Delta f.$$

Recall the action of the Fourier transform of derivatives

$$\hat{f}(y) = \int e^{2\pi i x\cdot y} f(x) \mathrm{d}x \implies \widehat{\frac{\partial f}{\partial x_j}}(y) = -2\pi i y_j \hat{f}(y).$$

Then

$$\begin{aligned} \widehat{\frac{\partial^2 f}{\partial x_j \partial x_k}}(y) &= -4\pi^2 y_j y_k \hat{f}(y) \\ &= -\frac{iy_j}{|y|}\frac{iy_k}{|y|}(-4\pi|y|^2)\hat{f}(y) = -\widehat{R_j R_k \Delta} f(y). \end{aligned}$$

□

Proposition 4.8. *Suppose $f \in C^1(\mathbb{R}^2)$ with compact support. Then for $1 < p < \infty$,*

$$\left\|\frac{\partial f}{\partial x_1}\right\|_p + \left\|\frac{\partial f}{\partial x_2}\right\|_p \le A_p \left\|\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2}\right\|_p .$$

The identity used is

$$\frac{\partial f}{\partial x_j} = -R_j(R_1 - iR_2)\left(\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2}\right).$$

5 Poisson integrals

Recall the Dirichlet problem for the Laplace equation: We restrict ourself to $\mathbb{R}^{n+1}_+$. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, we want to find a harmonic function $u(x, y)$ on $\mathbb{R}^{n+1}_+$ whose boundary values on $\mathbb{R}^n$ are $f(x)$.

Here is a solution with L^2 theory. Let $f \in L^2$. Consider

$$u(x, y) = \int \hat{f}(t)e^{-2\pi it\cdot x}e^{-2\pi|t|y}\mathrm{d}t.$$

This integral converges absolutely, and can be differentiated. Then check

$$\Delta u = \frac{\partial^2 u}{\partial y^2} + \sum_{k=1}^{n}\frac{\partial^2 u}{\partial x_k^2} = 0,$$

because $e^{2\pi it\cdot x}e^{-2\pi|t|y}$ satisfies this property. We have L^2 convergence of $u(x, y)$ to $f(x)$ as $y \to 0$.

Definition 5.1. Define the Poisson kernel $P_y(x)$ by

$$P_y(x) = \int e^{-2\pi it\cdot x}e^{-2\pi|t|y}\mathrm{d}t.$$

Then we can write $u(x, y)$ as a convolution in the x variable called the Poisson integral

$$u(x, y) = (P_y * f)(x) = \int P_y(t)f(x - t)\mathrm{d}t$$

Proposition 5.2. *The explicit expression of the Poisson kernel is*

$$P_y(x) = \frac{c_n y}{(|x|^2 + y^2)^{\frac{n+1}{2}}},$$

with c_n as in the Riesz transform.

We can describe the boundary behavior of Poisson integrals as follows.

Theorem 5.3. *Let $f \in L^p(\mathbb{R}^n), 1 \le p \le \infty$. Let $u(x,y)$ be its associated Poisson integral. Then*

1. $\sup_{y>0} |u(x,y)| \le Mf(x)$, *where Mf is the maximal function.*
2. $\lim_{y\to 0} u(x,y) = f(x)$ *for almost every x.*
3. *If $p < \infty$, then $u(x,y)$ converges to $f(x)$ in the $L^p(\mathbb{R}^n)$ norm as $y \to 0$.*

Theorem 5.4. *[Approximations to the identity] Let $\varphi \in L^1(\mathbb{R}^n)$, and set $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(\epsilon^{-1}x)$. Suppose that the least decreasing radial majorant of φ is integral, i.e.*

$$\psi(x) = \sup_{|y|\ge|x|} |\varphi(y)| \textit{ satisfies } \int \psi(x)\mathrm{d}x = A < \infty.$$

With the same A,

1. $\sup_{\epsilon>0} |(f * \varphi_\epsilon)(x)| \le AMf(x)$, *for $f \in L^p(\mathbb{R}^n), 1 \le p \le \infty$.*
2. *If φ has integral one, then $\lim_{\epsilon\to 0}(f * \varphi_\epsilon)(x) = f(x)$ almost everwyhere.*
3. *If $p < \infty$, then $\|f * \varphi_\epsilon - f\|_p \to 0$ as $\epsilon \to 0$.*

Proof. To prove (3), it suffices to take φ integrable. Recall the L^p norm is continuous with respect to translation. If f_1 is continuous with compact support, we in fact have uniform convergence of $f_1(x-y)$ to $f_1(x)$. Otherwise write $f = f_1 + f_2$, where $\|f_2\|_p \le \delta$. Now

$$\begin{aligned}\|f(x-y) - f(x)\|_p &\le \|f_1(x-y) - f_1(x)\|_p + \|f_2(x-y) - f_2(x)\|_p\,, \\ \|f_2(x-y) - f(x)\|_p &\le 2\delta,\end{aligned}$$

so $\|f(x-y) - f(x)\|_p \to 0$. We can write with Fubini's

$$\begin{aligned} f * \varphi_\epsilon - f &= \int [f(x-y) - f(x)]\varphi_\epsilon(y)\mathrm{d}y \\ \|f * \varphi_\epsilon - f\|_p &\le \int \|f(x-y) - f(x)\|_{L^p_x} |\varphi_\epsilon(y)|\mathrm{d}y \\ &= \int \|f(x-\epsilon y) - f(x)\|_{L^p_x} |\varphi(y)|\mathrm{d}y \end{aligned}$$

which converges to zero by DCT.

To prove (1), write $\psi(r) = \psi(x)$, since ψ is radial. We claim that $r^n\psi(r) \to 0$ as $r \to 0, \infty$. Indeed, we can write

$$\int_{\frac{r}{2}\le|x|\le r} \psi(x)\mathrm{d}x \ge \psi(r) \int_{\frac{r}{2}\le|x|\le r} \mathrm{d}x = \psi(r)cr^n,$$

then we apply the fact that $\psi \in L^1, \psi(r)$ is decreasing (and absolute continuity of the integral).

We now want to show that

$$(f * \psi_\epsilon)(x) \le A(Mf)(x).$$

By translation and dilation invariance, it suffices to show that

$$(f * \psi)(0) \le A(Mf)(0).$$

Write

$$\lambda(r) = \int_{\mathbb{S}^{n-1}} f(rx)\mathrm{d}\sigma(x),$$
$$\Lambda(r) = \int_{|x|\le r} f(x)\mathrm{d}x = \int_0^r \lambda(t)^{n-1}\mathrm{d}t,$$

by polar coordinates. Then

$$\begin{aligned}(f * \psi)(0) = \int f(x)\psi(x)\mathrm{d}x = \int_0^\infty \lambda(r)\psi(r)r^{n-1}\mathrm{d}r \\ = \lim_{\epsilon\to 0, N\to\infty} \int_\epsilon^N \lambda(r)\psi(r)r^{n-1}\mathrm{d}r = -\lim_{\epsilon\to 0, N\to\infty}\int_\epsilon^N \Lambda(r)\mathrm{d}\psi(r).\end{aligned}$$

We have an error of the form $\Lambda(N)\psi(N) - \Lambda(\epsilon)\psi(\epsilon)$, but we can check that

$$\Lambda(r) \le |B(1)|r^n Mf(0).$$

Therefore

$$f * \psi(0) \le VMf(0)\int_0^\infty r^n \mathrm{d}(-\psi(r)).$$

To prove (2), if $f \in L^p, 1 \le p < \infty$, proof is analogous to Lebesgue Differentiation Thm. Now take $p = \infty$. Given any ball B, we want to show that

$$\lim_{\epsilon\to 0}(f * \varphi_\epsilon)(x) = f(x)$$

for almost every $x \in B$¿ Let B_1 be a different ball that strictly contains B and δ be the distance from B to the complement of B.

Take $f_1(x) = f(x)\mathbf{1}_B(x), f(x) = f_1(x) + f_2(x)$. We have $f_1 \in L^1$. For $x \in B$,

$$\begin{aligned}|(f_2 * \varphi_\epsilon)(x)| = \left|\int f_2(x-y)\varphi_\epsilon(y)\mathrm{d}y\right| \le \int_{|y|\ge\delta>0} |f_2(x-y)|\varphi_\epsilon(y)\mathrm{d}y \\ \le \|f\|_\infty \int_{|y|\ge\frac{\delta}{\epsilon}} |\varphi(y)|\mathrm{d}y \to 0\end{aligned}$$

as $\epsilon \to 0$. $\square$

5.1 Conjugate harmonic functions

There is an interesting relation between Riesz transform and the theory of harmonic functions.

Theorem 5.5. *Let $f, f_i \in L^2(\mathbb{R}^n)$, with their respective Poisson integrals*

$$u_0(x,y) = P_y * f, u_i(x,y) = P_y * f_i.$$

Then $f_j = R_j(f)$ iff the following generalized Cauchy-Riemann equations hold

$$\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0, \quad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}.$$

Proof. ($\implies$) Since $f_j = R_j(f)$, we know that $\hat{f}_j(t) = \frac{it_j}{|t|}\hat{f}(t)$. Then the formula for the Poisson integral is

$$u_j(x,y) = \int \hat{f}(t)\frac{it_j}{|t|}e^{-2\pi i t\cdot x}e^{-2\pi|t|y}\mathrm{d}t.$$

Then just differentiate under the integral sign.

($\impliedby$) Consider the formula for the Poisson integral. We have

$$-2\pi i t_j \hat{f}_0(t)e^{-2\pi|t|y} = -2\pi|t|\hat{f}_j(t)e^{-2\pi|t|y},$$

so $\hat{f}_j(t) = \frac{it_j}{|t|}\hat{f}_0(t)$ and $f - J = R_j(f)$. $\square$

5.2 L^p bounds on maximal singular operator

Lemma 5.6. *If* $T^*f(x) = \sup_{\epsilon>0} |T_\epsilon(f)|(x)$, *then*

$$\|T^*f\|_p \leq A_p \|f\|_p, \quad 1 < p < \infty.$$

Proof. We already proved existence of $T(f)$ as limit as $\epsilon \to 0$ in the L^p norm. We want to show that

$$T^*(f)(x) \leq M(Tf)(x) + CM(f)(x).$$

Let φ be a smooth non-negative function on $\mathbb{R}^n$, supported in unit ball, with integral equal to one, radial and decreasing in $|x|$. Consider

$$K_\epsilon(x) = \begin{cases} \frac{\Omega(x)}{|x|^n} & |x| \geq \epsilon, \\ 0 & |x| < \epsilon. \end{cases}$$

Then we define $\Phi = \varphi * K - K_1$.

We claim that the smallest decreasing radial majorant of Φ is integrable.

- If $|x| < 1$, then $\Phi = \varphi * K$ and we can write

  $$\Phi = \int K(y)[\varphi(x-y) - \varphi(x)]\mathrm{d}y$$

  which is bounded due to smoothness of φ.

- If $1 \leq |x| \leq 2$, then $\Phi(x) = K * \varphi - K(x)$ is also bounded.

- When $|x| \geq 2$,

  $$\Phi(x) = \int_{\mathbb{R}^n} K(x-y)\varphi(y)\mathrm{d}y - K(x) = \int_{|y|\leq 1} [K(x-y) - K(x)]\varphi(y)\mathrm{d}y$$

  so

  $$|\Phi(x)| \leq C' \frac{\omega(c/|x|)}{|x|^n}.$$

Obseve too that

$$\varphi_\epsilon * K - K_\epsilon = \Phi_\epsilon.$$

We claim now that for any $f \in L^p(\mathbb{R}^n)$, we have

$$(\varphi_\epsilon * K) * f(x) = T(f) * \varphi_\epsilon(x)$$

We conclude $T_\epsilon(f) = (Tf) * \varphi_\epsilon - f * \Phi_\epsilon$, so we can apply Theorem 5.4. □

6 Higher Riesz transforms and spherical harmonics

Definition 6.1. $\mathcal{H}_k$ is the linear space of homogeneous polynomials of degree k, also known as the solid spherical harmonic of degree k. This space has inner prorduct

$$(P, Q) = \int_{\mathbb{S}^{n-1}} P(x)\overline{Q(x)}\mathrm{d}\sigma(x).$$

Proposition 6.2. *The space $\{\mathcal{H}_k\}_{k=0}^{\infty}$ are orthogonal.*

Proof. If $P \in \mathcal{H}_k, Q \in \mathcal{H}_j$ then

$$\begin{aligned}(k-j)\int_{\mathbb{S}^{n-1}} P\overline{Q}\mathrm{d}\sigma(x) &= \int_{\mathbb{S}^{n-1}} \left(\overline{Q}\frac{\partial P}{\partial \nu} - P\frac{\partial \overline{Q}}{\partial \nu}\right)\mathrm{d}\sigma(x) \\ &= \int_{B_1} (\overline{Q}\Delta P - P\Delta Q)\mathrm{d}x = 0,\end{aligned}$$

because P and Q are both harmonic. □

Proposition 6.3. *Suppose P is homogeneous of degree k. Then*

$$P = P_1 + |x|^2 P_2,$$

where P_1 is homogeneous of degree k, harmonic and P_2 is homogeneous of degree $k-2$.

Proof. Iterate the previous proposition for

$$P(x) = P_1(x) + |x|^2 P_2(x) + |x|^4 P_3(x) + \cdots =$$

□

Proposition 6.4. *Let H_k denote the linear space of restrictions of $\mathcal{H}_k$ to the unit sphere, also known as the surface spherical harmonics of degree k. Then in the sense of Hilbert spaces,*

$$L^2(\mathbb{S}^{n-1}) = \sum_{k=0}^{\infty} H_k.$$

Proposition 6.5. *Let f be written as*

$$f(x) = \sum_{k=0}^{\infty} Y_k(x).$$

Then f is smooth on $\mathbb{S}^{n-1}$ if and only if

$$\int_{\mathbb{S}^{n-1}} |Y_k(x)|^2 \mathrm{d}\sigma(x) = O(k^{-N}). \tag{6.1}$$

Theorem 6.6. *Suppose $P_k(x)$ is a homogeneous polynomial of degree k. Then*

$$\mathcal{F}(P_k(x)e^{-\pi|x|^2}) = i^k P_k(x)e^{-\pi|x|^2}.$$

Proof. We want to show that

$$\int P_k(x)\exp\left(-\pi|x|^2 + 2\pi i x\cdot y\right)\mathrm{d}x = i^k P_k(y)e^{-\pi|y|^2}. \tag{6.2}$$

The Fourier transform of a Gaussian is a Gaussian, so

$$\int \exp[-\pi|x|^2 + 2\pi i x\cdot y]\mathrm{d}x = \exp(-\pi|y|^2).$$

Now apply the operator $P_k(\partial_y)$ to both sides for

$$\int P_k(2\pi i x)\exp[-\pi|x|^2 + 2\pi i x\cdot y]\mathrm{d}x = P_k(-2\pi y)\exp(-\pi|y|^2).$$

□

Theorem 6.7. *Let $P_k(x)$ be a homogeneous harmonic polynomial of degree k. Then the multiplier corresponding to the convolution operator with the kernel $P_k(x)/|x|^{k+n}$ is*

$$\gamma_k \frac{P_k(x)}{|x|^k}, \text{ where } \gamma_k = i^k \pi^{n/2}\frac{\Gamma(k/2)}{\Gamma((k+n)/2)}.$$

Lemma 6.8. *For all $k \in \mathbb{N}, 0 < \alpha < n$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\int \frac{P_k(x)}{|x|^{k+n-\alpha}}\hat{\varphi}(x)\mathrm{d}x = \gamma_{k,\alpha}\int \frac{P_k(x)}{|x|^{k+\alpha}}\varphi(x)\mathrm{d}x.$$

Proof of lemma. □

Proof. We claim that

$$\lim_{\alpha\to 0^+} \frac{P_k(x)}{|x|^{k+n-\alpha}}\hat{\varphi}(x)\mathrm{d}x = \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon}\frac{P_k(x)}{|x|^{k+n}}\hat{\varphi}(x)\mathrm{d}x. \tag{6.3}$$

The left hand side is

$$\int \frac{P_k(x)}{|x|^{k+n-\alpha}}\hat{\varphi}(x)\mathrm{d}x = \int_{|x|\leq 1}\frac{P_k(x)}{|x|^{k+n-\alpha}}[\hat{\varphi}(x)-\hat{\varphi}(0)]\mathrm{d}x + \int_{|x|\geq 1}\frac{P_k(x)}{|x|^{k+n-\alpha}}\hat{\varphi}(x)\mathrm{d}x.$$

Then pass to the limit as $\alpha \to 0$.

$$\lim_{\alpha\to 0}\int_{|x|\leq 1}\frac{P_k(x)}{|x|^{k+n-\alpha}}[\hat{\varphi}(x)-\hat{\varphi}(0)]\mathrm{d}x = \lim_{\epsilon\leq|x|\leq 1}\frac{P_k(x)}{|x|^{k+n}}\hat{\varphi}(x)\mathrm{d}x$$

In any case let f be sufficiently smooth with compact support. Set $f(x-y)=\hat{\varphi}(y)$, so $\varphi(y)=\hat{\hat{f}}(y)e^{-2\pi i x\cdot y}$, so we get

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{|y|\geq\epsilon}\frac{P_k(y)}{|y|^{k+n}}f(x-y)\mathrm{d}y &= \gamma_k\int\frac{P_k(y)}{|y|^k}\hat{f}(y)e^{-2\pi i x\cdot y}\mathrm{d}y\\ &= \int m(y)\hat{f}(y)e^{-2\pi i x\cdot y}\mathrm{d}y\end{aligned}$$

with the help of the lemma. With the definition of the multiplier, we arrive at

$$m(y)=\gamma_k\frac{P_k(y)}{|y|^k}.$$

□

Theorem 6.9. *The classes of transformation defined by*

$$T(f)=c\cdot f+\lim_{\epsilon\to 0}\int_{|y|\geq\epsilon}\frac{\Omega(y)}{|y|^n}f(x-y)\mathrm{d}y \qquad \textit{where } \Omega \textit{ is smooth} \tag{6.4}$$

$$\widehat{Tf}(y)=m(y)\hat{f}(y) \qquad \textit{where } m \textit{ is smooth} \tag{6.5}$$

are identical.

Proof. ($\Longrightarrow$) Suppose that T takes the first form. We already showed that

$$m(x)=c+\int_{\mathbb{S}^{n-1}}\Gamma(x\cdot y)\Omega(y)\mathrm{d}\sigma(y).$$

We can express m in terms of the spherical harmonics

$$\Omega(y)=\sum_{k=1}^{\infty}Y_k(y),\quad m(x)=\sum_{k=0}^{\infty}\tilde{Y}_k(X).$$

The previous theorem tells us that the ratios of spherical harmonics are explicit constants

$$\tilde{Y}_k(x)=\gamma_k Y_k(x).$$

For $N\neq M$, we get

$$\sup_{x\in\mathbb{S}^{n-1}}|m_M(x)-m_N(x)|\leq\sup_x\|\Gamma(x\cdot y)\|_{L^2_y(\mathbb{S}^{n-1})}\|\Omega_M-\Omega_N\|_{L^2_y(\mathbb{S}^{n-1})}\to 0$$

as $N,M\to\infty$. First term in the product is bounded because

$$\Gamma(t)=\frac{\pi i}{2}\mathrm{sgn}(t)+\log\frac{1}{|t|}$$

implies that

$$\sup_x \|\Gamma(x \cdot y)\|_{L^2_y(\mathbb{S}^{n-1})} \leq c_1 + c_2 \int_0^\pi |\log|\cos\theta||^2 (\sin\theta)^{n-2} \mathrm{d}\theta < \infty.$$

So the sequence is Cauchy.

The smoothness of Ω allows us to meet condition of (6.1).

($\Longleftarrow$) Suppose $m(x)$ is smooth on the unit sphere and set its spherical harmonics as above. Take

$$c = \tilde{Y}_0 \text{ and } Y_k(x) = \frac{1}{\gamma_k} \tilde{Y}_k(x).$$

Then Ω is infinitely differentiable. $\square$

7 The Littlewood-Paley g-function

Definition 7.1. Let $f \in L^p(\mathbb{R}^n)$. We write $u(x,y)$ for its Poisson integral

$$u(x,y) = \int_{\mathbb{R}^n} P_y(t) f(x-t) \mathrm{d}t.$$

Then we define $g(f)$ by

$$g(f)(x) = \left(\int_0^\infty |\nabla u(x,y)|^2 y \mathrm{d}y \right)^{1/2}.$$

Theorem 7.2. *Suppose $f \in L^p(\mathbb{R}^n), 1 < p < \infty$. Then $g(f)(x) \in L^p(\mathbb{R}^n)$ with*

$$A'_p \|f\|_p \leq \|g(f)\|_p \leq A_p \|f\|_p .$$

Proof of $p = 2$ case. We have

$$\|g(f)\|_2^2 = \iint y |\nabla u(x,y)|^2 \mathrm{d}x \mathrm{d}y.$$

The formula for the Poisson integral

$$u(x,y) = \int \hat{f}(t) e^{-2\pi i t \cdot x} e^{-2\pi t|y|} \mathrm{d}t.$$

We get

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \int -2\pi |t| \hat{f}(t) e^{-2\pi i t \cdot x - 2\pi |t| y} \mathrm{d}t, \\
\frac{\partial u}{\partial x} &= \int -2\pi i t_j \hat{f}(t) e^{-2\pi i t \cdot x - 2\pi |t| y} \mathrm{d}t.
\end{aligned}$$

Together, we get

$$\int |\nabla u(x,y)|^2 \mathrm{d}x = \int 8\pi^2 |t|^2 |\hat{f}(t)|^2 e^{-4\pi |t| y} \mathrm{d}t.$$

Applying Fubini's, we have

$$\begin{aligned}
\|g(f)\|_2^2 &= \int \left(\int |\nabla u(x,y)|^2 \mathrm{d}x \right) y \mathrm{d}y \\
&= \int \left(\int 8\pi^2 |t|^2 |\hat{f}(t)|^2 e^{-4\pi |t| y} \mathrm{d}t \right) y \mathrm{d}y \\
&= \int \left(\int 8\pi^2 |t|^2 y e^{-4\pi |t| y} \mathrm{d}y \right) |\hat{f}(t)|^2 \mathrm{d}t, \quad \text{(the constant is } \Gamma(2))
\end{aligned}$$

which implies

$$\|g(f)\|_2 = 2^{-1/2} \|f\|_2 .$$

□

Remark 7.3. If we introduce

$$g_1(f)(x) = \left(\int_0^\infty \left|\frac{\partial u}{\partial y}\right|^2 y\mathrm{d}y\right)^{1/2},$$
$$g_x(f)(x) = \left(\int_0^\infty |\nabla_x u(x,y)|^2 y\mathrm{d}y\right)^{1/2},$$

then $g^2 = g_1^2 + g_x^2$ and we actually showed that

$$\|g_1(f)\|_2 = \|g_x(f)\|_2 = \frac{1}{2}\|f\|_2 .$$

Proof when $p \neq 2$. (second inequality) When $p \neq 2$, we consider the Hilbert spaces $\mathcal{H}_1 = \mathbb{R}$ and

$$\mathcal{H}_2^0 = \left\{ f : |f|^2 = \int_0^\infty |f(y)|^2 y\mathrm{d}y < \infty \right\},$$
$$\mathcal{H}_2 = \bigoplus_{i=1}^{n+1} \mathcal{H}_2^0.$$

Recall the definition and explicit expression of the Poisson kernel

$$P_y(x) = \int_{\mathbb{R}^n} e^{-2\pi i t\cdot x} e^{-2\pi|t|y}\mathrm{d}t = \frac{c_n y}{(|x|^2+y^2)^{\frac{n+1}{2}}}$$

We define

$$K_\epsilon(x) = \left(\frac{\partial P_{y+\epsilon}(x)}{\partial y}, \frac{\partial P_{y+\epsilon}(x)}{\partial x_1} \ldots, \frac{\partial P_{y+\epsilon}(x)}{\partial x_k}\right).$$

For each x, we have $K_\epsilon(x) \in \mathcal{H}^2$ from the explicit formula for the Poisson kernel. In particular, we have

$$\left|\frac{\partial P_y}{\partial y}\right|, \left|\frac{\partial P_y}{\partial x}\right| \leq \frac{A}{(|x|^2+y^2)^{\frac{n+1}{2}}}.$$

Then for fixed x,

$$\|K_\epsilon(x)\|_{\mathcal{H}_2,y}^2 \leq A^2(n+1)\int_0^\infty \frac{y\mathrm{d}y}{(|x|^2+(y+\epsilon)^2)^{n+1}} \leq A|x|^{-2n},$$

which means $\|K_\epsilon(x)\|_{\mathcal{H}_2,y} \in L^2(\mathbb{R}^n)$.

A similar estimate yields

$$\left\|\frac{\partial K_\epsilon(x)}{\partial x_j}\right\|_{\mathcal{H}_2,y}^2 \leq A\int_0^\infty \frac{y\mathrm{d}y}{(|x|^2+(y+\epsilon))^{n+2}} \leq \frac{A}{|x|^{2n+2}}$$

We consider the operator T_ϵ defined by

$$T_\epsilon(f)(x) = \int K_\epsilon(t) f(x-t)\mathrm{d}t.$$

Observe that

$$|T_\epsilon(f)(x)| = \left(\int_0^\infty |\nabla u(x, y+\epsilon)|^2 y \mathrm{d}y\right)^{1/2} \le g(f)(x).$$

Then $\|T_\epsilon f(x)\|_2 \le 2^{-1/2} \|f\|_2$, which means $|\hat{K}_\epsilon(x)| \le 2^{-1/2}$.

Then apply the Calderón-Zygmund theorem for Hilbert spaces.

(first inequality) Applying polarization to the identity

$$\|g_1(f)\|_2 = \frac{1}{2} \|f\|_2,$$

we have

$$\begin{aligned}
\int f_1 \overline{f_2} \mathrm{d}x &= 4 \int_{\mathbb{R}^n} \int_0^\infty y \frac{\partial u_1}{\partial y} \overline{\frac{\partial u_2}{\partial y}} \mathrm{d}y \mathrm{d}x \\
&\le 4 \int g_1(f_1) g_1(f_2) \mathrm{d}x \\
&\le 4 \|g_1(f_1)\|_p \|g_1(f_2)\|_q \\
&\le 4A_q \|g_1(f_1)\|_p
\end{aligned}$$

with $\|f_2\|_q \le 1$.

□

7.1 Positive function

Lemma 7.4. *Suppose u is harmonic and strictly positive. Then*

$$\Delta(u)^p = p(p-1)u^{p-2}|\nabla u|^2.$$

Lemma 7.5. *Suppose $F(x,y)$ is continuous in $\overline{\mathbb{R}}_+^{n+1}$, of class C^2 in $\mathbb{R}_+^{n+1}$, and suitably small at infinity. Then*

$$\int_{\mathbb{R}_+^{n+1}} y \Delta F(x,y) \mathrm{d}x \mathrm{d}y = \int_{\mathbb{R}^n} F(x,0) \mathrm{d}x.$$

Proof. Green's theorem asserts

$$\int_D (y \Delta F(x,y)) \mathrm{d}x \mathrm{d}y = \int_{\partial D} \left(y \frac{\partial F}{\partial \nu} - F \frac{\partial y}{\partial \nu} \right) \mathrm{d}\sigma,$$

where $D = B_r \cap \mathbb{R}^{n+1}_+$. We observe that the spherical part of the boundary of D vanishes as $r \to \infty$ under suitable decay conditions for F, namely

$$|F| \le \frac{C}{(|x|+y)^{n+\epsilon}}, \quad |\nabla F| \le \frac{C}{(|x|+y)^{n+1+\epsilon}}.$$

$\square$

Definition 7.6. We define the positive function g^*_λ as

$$(g^*_\lambda(f)(x))^2 = \int_0^\infty \int_{\mathbb{R}^n} \left(\frac{y}{|t|+y}\right)^{\lambda n} |\nabla u(x-t,y)|^2 y^{1-n} \mathrm{d}t\mathrm{d}y.$$

Definition 7.7. Let Γ be a fixed proper cone in $\mathbb{R}^{n+1}_+$ which has a vertex at the origin and contains $(0,1)$. We may take

$$\Gamma = \{(t,y) \in \mathbb{R}^{n+1}_+ : |t| < y, y > 0\}$$

and let $\Gamma(x)$ denote the translated cone. We define the positive function $S(f)(x)$ by

$$\begin{aligned}
[S(f)(x)]^2 &= \int_{\Gamma(x)} |\nabla u(t,y)|^2 y^{1-n} \mathrm{d}y\mathrm{d}t \\
&= \int_{\Gamma} |\nabla u(x-t),y)|^2 y^{1-n} \mathrm{d}y\mathrm{d}t
\end{aligned}$$

Proposition 7.8. *We assert that*

$$g(f)(x) \le CS(f)(x) \le C_\lambda g^*_\lambda(f)(x).$$

Proof. For the second inequality, check that within the cone we have

$$|t| < y \implies |t| + y < 2y \implies \frac{y}{|t|+y} > \frac{1}{2}.$$

For the first inequality we just want to show that $g(f)(0) \le CS(f)(0)$. Let B_y be the ball in $\mathbb{R}^{n+1}_+$ centered at $(0,y)$ and tangent to the boundary of the cone Γ (in some sense this is the maximal ball that is still contained in the upper half plane). The radius of B_y is proportional to y. The partial derivatives of u are also harmonic functions, and obey the mean value property

$$\begin{aligned}
\frac{\partial u}{\partial y}(0,y) &= \frac{1}{|B_y|}\int_{B_y} \frac{\partial u}{\partial y}(x,s)\mathrm{d}x\mathrm{d}s, \\
\implies \left|\frac{\partial u}{\partial y}(0,y)\right|^2 &= \frac{1}{|B_y|}\int_{B_y} \left|\frac{\partial u}{\partial y}(x,s)\right|^2 \mathrm{d}x\mathrm{d}s,
\end{aligned}$$

by Jensen's inequality. Now multiply by y and integrate with respect to y for

$$\int_0^\infty y\left|\frac{\partial u(0,y)}{\partial y}\right|^2 \mathrm{d}y \le C\int_0^\infty y^{-n}\int_{B_y}\left|\frac{\partial u}{\partial y}(x,s)\right|^2 \mathrm{d}x\mathrm{d}s\mathrm{d}y$$
$$\le C\int_\Gamma y^{1-n}\left|\frac{\partial u}{\partial y}(x,y)\right|^2 \mathrm{d}x\mathrm{d}y,$$

because $(x,s) \in B_y$ implies that y is comparable to s. Now repeat for the remaining partial derivatives. □

Theorem 7.9. *Let λ be a parameter which is greater than 1. Suppose $f \in L^p(\mathbb{R}^n)$. Then*

1. *For every $x \in \mathbb{R}^n, g(f)(x) \le C_\lambda g^*_\lambda(f)(x)$.*
2. *If $1 < p < \infty$, and $p > 2/\lambda$, then*

$$\|g^*_\lambda(f)\|_p \le A_{p,\lambda}\|f\|_p\,.$$

Definition 7.10. Let $\mu \ge 1$, and write

$$M_\mu(f)(x) = \left(\sup_{r>0}\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)|^\mu \mathrm{d}y\right)^{1/\mu.}$$

Check that for $p > \mu$, we have

$$\|M_\mu(f)\|_p \le A_{p,\mu}\|f\|_p\,.$$

Lemma 7.11. *Let $f \in L^p(\mathbb{R}^n)$ for $p \ge \mu \ge 11$. If u is the Poisson integral of f, then*

$$|u(x-t,y)| \le A\left(1+\frac{|t|}{y}\right)^n M(f)(x), \tag{7.1}$$

$$|u(x-t,y)| \le A_\mu\left(1+\frac{|t|}{y}\right)^{n/\mu} M_\mu(f)(x). \tag{7.2}$$

Proof of lemma. The inequality is unchanged under dilation $(x,t,y) \mapsto (\delta x, \delta t, \delta y)$, so we only need to consider $y = 1$.

We have

$$|u(x-t,1)| = f(x) * P_1(x-t), \quad P_1(x) = \frac{c_n}{(1+|x|^2)^{\frac{n+1}{2}}}.$$

Theorem 5.4 tells us that

$$|u(x-t,1)| \le A_t(Mf)(x), \quad A_t = \int Q_t(x)\mathrm{d}x,$$

where $Q_t(x)$ is the smallest decreasing radial majorant of $P_1(x-t)$, given by

$$Q_t(x) = c_n \cdot \sup_{|x'|\geq|x|} \left(\frac{1}{(1+|x'-t|^2)^{(n+1)/2}} \right).$$

We have the following estimates

$$\begin{aligned} Q_t(x) &\leq c_n & |x| &\leq 2|t|, \\ Q_t(x) &\leq A'(1+|x|^2)^{-\frac{n+1}{2}} & |x| &\geq 2|t|, \end{aligned}$$

so $A_t \leq A(1+|t|^n)$ gives us (7.1).

To raise to the μth power, observe that

$$\begin{aligned} u(x-t,y) &= \int P_y(s) f(x-t-s)\mathrm{d}s, \\ |u(x-t,y)|^\mu &\leq \int P_y(s)|f(x-t-s)|^\mu \mathrm{d}s = U(x-t,y), \end{aligned}$$

where U is the Poisson integral of $|f|^\mu$. So we can apply (7.1) to U for

$$|u(x-t,y)| \leq A^{1/\mu}(1+|t|/y)^{n/\mu}(M(|f|^\mu))(x))^{1/\mu}.$$

□

Proof. ($p \geq 2$ case) Let ψ be positive function on $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} (g_\lambda^*(f))(x))^2 \psi(x)\mathrm{d}x \leq A_\lambda \int_{\mathbb{R}^n} (g(f)(x))^2 (M\psi)(x)\mathrm{d}x.$$

The left hand side equals

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_0^\infty\int_{\mathbb{R}^n} \left(\frac{y}{|t|+y}\right)^{\lambda n} |\nabla u(x-t,y)|^2 y^{1-n}\psi(x)\mathrm{d}t\mathrm{d}y\mathrm{d}x \\ &= \int_0^\infty \int_{\mathbb{R}^n} \left[y^{-n} \int_{\mathbb{R}^n} \left(\frac{y}{|t-x|+y}\right)^{\lambda n} \psi(x)\mathrm{d}x \right] |\nabla u(t,y)|^2 y \mathrm{d}t\mathrm{d}y, \end{aligned}$$

so we can apply Theorem 5.4 with $\epsilon := y$.

In the case that $p > 2$, set $1/q + 2/p = 1$, and take the supremum on the left hand side over all $\|\psi\|_q \leq 1$. Then apply Hölder duality, inequalities for the g-function and the boundedness of the maximal function operator.

For $p < 2$, we need the restriction that $p > 2/\lambda$. We can find μ close to p with $1 \leq \mu < p$ such that

$$\lambda' = \lambda - \frac{2-p}{\mu} > 1,$$

Then apply the lemma for

$$|u(x-t,y)|\left(\frac{y}{y+|t|}\right)^{n/\mu}\leq AM_{\mu}(f)(x).$$

We have

$$\begin{aligned}(g_{\lambda}^{*}(f)(x))^{2}&=\frac{1}{p(p-1)}\int_{\mathbb{R}_{+}^{n+1}}y^{1-n}\left(\frac{y}{y+|t|}\right)^{\lambda n}u^{2-p}|\Delta u^{p}|\mathrm{d}t\mathrm{d}y\\&\leq A^{2-p}(M_{\mu}(f)(x))^{2-p}I^{*}(x),\end{aligned}$$

where

$$I^{*}(x)=\int_{\mathbb{R}_{+}^{n+1}}y^{1-n}\left(\frac{y}{y+|t|}\right)^{\lambda' n}\Delta u^{p}(x-t,y)\mathrm{d}t\mathrm{d}y.$$

We check that

$$\begin{aligned}\int_{\mathbb{R}^{n}}I^{*}(x)\mathrm{d}x&=\int_{\mathbb{R}_{+}^{n+1}}\int_{x\in\mathbb{R}^{n}}y^{1-n}\left(\frac{y}{y+|t-x|}\right)^{\lambda' n}\Delta u^{p}(t,y)\mathrm{d}x\mathrm{d}t\mathrm{d}y\\&=C_{\lambda'}\int_{\mathbb{R}_{+}^{n+1}}y\Delta u^{p}(t,y)\mathrm{d}t\mathrm{d}y=C_{\lambda'}\int_{\mathbb{R}^{n}}u^{p}(t,0)\mathrm{d}t\leq C_{\lambda'}\left\|f\right\|_{p}^{p}\end{aligned}$$

by a change of variables $x\mapsto xy$. $\square$

8 Multipliers

Recall that if m is a bounded measurable function on $\mathbb{R}^n$, we can the define the operator T on $L^2 \cap L^p$ given by

$$(T_m f)^\wedge(x) = m(x)\hat{f}(x).$$

Definition 8.1. We say that m is a multiplier for L^p if $f \in L^2 \cap L^p \implies T_m f \in L^p$, with

$$\|T_m(f)\|_{L^p} \leq A \|f\|_p .$$

Let $\mathcal{M}_p$ denote the set of multipliers for L^p.

Recall from earlier that $\mathcal{M}_2$ is precisely the set of all bounded measurable functions, whose norm equals the L^∞ norm. Moreover, $\mathcal{M}_1$ is the set of Fourier transforms of Borel measures on $\mathbb{R}^n$, whose norm equals the norm on $\mathcal{B}(\mathbb{R}^n)$.

The theory of singular integral operators also tells us that if m is homogeneous of degree zero and smooth on the unit sphere, then $m \in \mathcal{M}_p$ for $1 < p < \infty$.

Lemma 8.2. *We have* $\overline{\check{m}}(x) = \check{\overline{m}}(x)$.

We have the following form of dual symmetry.

Proposition 8.3. *If p, p' are Hölder conjugates, then $\mathcal{M}_p = \mathcal{M}_{p'}$.*

Proof. Let σ be the involution defined by $\sigma(f)(x) = \overline{f(-x)}$. Check that $\sigma^{-1} T_m \sigma = T_{\overline{m}}$, so T_m and $T_{\overline{m}}$ have the same $\mathcal{M}_p$ norms.

Suppose $m \in \mathcal{M}_p$. By Plancherel and polarization ad the first and last types, we have

$$\begin{aligned}
\int T_m f \overline{g} &= \int m(x)\hat{f}(x)\overline{\hat{g}(x)}\mathrm{d}x \\
&= \int \hat{f}(x)\overline{\overline{m(x)}\hat{g}(x)}\mathrm{d}x \\
&= \int f(x)\overline{T_{\overline{m}} g(x)}\mathrm{d}x.
\end{aligned}$$

Then

$$\left| \int T_m f \overline{g} \right| \leq \|f\|_{p'} \|T_{\overline{m}} g\|_p \leq C \|f\|_{p'}$$

We take the supremum over all g such that $\|g\|_p \leq 1$. Therefore $m \in \mathcal{M}_{p'}$ $\square$

Theorem 8.4. *Suppose that $m \in C^k(\mathbb{R}^n \setminus \{0\})$, where $k > n/2$. Assume that for all $|\alpha| \leq k$, we have*

$$\left| \left(\frac{\partial}{\partial x}\right)^{\alpha} m(x) \right| \leq B|x|^{-|\alpha|}.$$

Then $m \in \mathcal{M}_p$ for all $1 < p < \infty$.

Lemma 8.5. *Let $f \in L^2(\mathbb{R}^n)$ and $F(x) = (T_m f)(x)$. Then*

$$g_1(F, x) \leq B_\lambda g_\lambda^*(f, x), \text{ where } \lambda = \frac{2k}{n}.$$

Proof. (why lemma implies theorem) Our assumption implies that $\lambda > 1$. Then for $p \geq 2$, we have

$$\|g_\lambda^*(f, x)\|_p \leq A_{\lambda,p} \|f\|_p.$$

We also have

$$\|F\|_p \leq A_p \|g_1(F, x)\|_p.$$

Then the lemma tells us that $m \in \mathcal{M}_p$ for $2 \leq p < \infty$. □

Proof of lemma. Let $u(x, y)$ be the Poisson integral of f and $U(x, y)$ be the Poisson integral of F. Let $\wedge$ denote the Fourier transform with respect to x. Then

$$\begin{aligned} u(x, y) &= \int \hat{f}(t) e^{-2\pi i t \cdot x} e^{-2\pi |t| y} \mathrm{d}y, \\ \hat{U}(x, y) &= e^{-2\pi |x| y} \hat{F}(x) \\ &= e^{-2\pi |x| y} m(x) \hat{f}(x). \end{aligned}$$

Analogously, we may define

$$\begin{aligned} M(x, y) &= \int e^{-2\pi i x \cdot t} e^{-2\pi |t| y} m(t) \mathrm{d}t, \\ \hat{M}(x, y) &= e^{-2\pi |x| y} m(x). \end{aligned}$$

Therefore

$$\begin{aligned} \hat{U}(x, y_1 + y_2) &= e^{-2\pi |x| (y_1 + y_2)} m(x) \hat{f}(x) \\ &= e^{-2\pi |x| y_1} m(x) e^{-2\pi |x| y_2} \hat{f}(x) \\ &= \hat{M}(x, y_1) \hat{u}(x, y_2). \end{aligned}$$

Differentiate k times with respect to y_1 and once with respect to y_2, set $y_1 = y_2 = y/2$. Then

$$U^{(k+1)}(x, y) = \int M^{(k)}(t, \frac{y}{2}) u^{(1)}(x - t, \frac{y}{2}) \mathrm{d}t. \tag{8.1}$$

We claim that M satisfies

$$|M^{(k)}(t,y)| \leq \frac{B'}{y^{n+k}}, \tag{8.2}$$

$$\int |t|^{2k}|M^{(k)}(t,y)|^2 \mathrm{d}t \leq \frac{B'}{y^n}. \tag{8.3}$$

Why? Since m is bounded, we have

$$|M^{(k)}(x,y)| \leq B \int |t|^k e^{-2\pi|t|} y \mathrm{d}y = \int_0^\infty r^k e^{-2\pi r y} r^{n-1} \mathrm{d}r = B'' y^{-n-k}.$$

We can in fact show that

$$\int |t^\alpha M^{(k)}(t,y)| \mathrm{d}t \leq \frac{B'}{y^n}.$$

Since a derivative of a Fourier Transform acts in terms of multiplication, Plancherel asserts

$$\begin{aligned}
\left\|t^\alpha M^{(k)}(t,y)\right\|_{L_t^2}^2 &= \left\|(2\pi)^{2k}\left(\frac{\partial}{\partial x}\right)^\alpha (|x|^k m(x) e^{-2\pi|x|y})\right\|_{L_x^2}^2 \\
&\leq \sum_{j=0}^{k} y^{2j} \int |x|^{2(k-j)} e^{-4\pi|x|y} \mathrm{d}x
\end{aligned}$$

because

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha (|x|^k m(x))\right| \leq B'|x|^{k-|\alpha|}.$$

But

$$y^{2r} \int |x|^{2r} e^{-4\pi|x|y} \mathrm{d}x \leq C y^{-n}$$

Then returning to (8.1),

$$\begin{aligned}
|U^{(k+1)}(x,y)|^2 &= \int_{|t|<\frac{y}{2}} |M^{(k)}(t,\frac{y}{2}) u^{(1)}(x-t,\frac{y}{2})|^2 \mathrm{d}t + \int_{|t|\geq\frac{y}{2}} |M^{(k)}(t,\frac{y}{2}) u^{(1)}(x-t,\frac{y}{2})|^2 \mathrm{d}t \\
&\leq \frac{A}{y^{n+2k}} \int_{|t|<\frac{y}{2}} |u^{(1)}(x-t,\frac{y}{2})|^2 \mathrm{d}t + \frac{A}{y^n} \int_{|t|\geq\frac{y}{2}} \frac{|u^{(1)}(x-t,\frac{y}{2})|^2}{|t|^{2k}} \mathrm{d}t \\
&= I_1(y) + I_2(y).
\end{aligned}$$

Now

$$(g_{k+1}(F,x))^2 = \int_0^\infty |U^{(k+1)}(x,y)|^2 y^{2k+1} \mathrm{d}y \leq \int_0^\infty (I_1(y)+I_2(y)) y^{2k+1} \mathrm{d}y.$$

We can control both terms

$$\begin{aligned}
\int_0^\infty I_1(y)y^{2k+1}\mathrm{d}y &\leq B\int_{|t|\leq \frac{y}{2}} |u^{(1)}(x-t,\frac{y}{2})|^2 y^{1-n}\mathrm{d}t\mathrm{d}y \\
&\leq B'\int_\Gamma |\nabla u(x-t),y)|^2 y^{1-n}\mathrm{d}t\mathrm{d}y \\
&= B'(S(f,x))^2 \leq B_\lambda g_\lambda^*(f,x)^2, \\
\int_0^\infty I_1(y)y^{2k+1}\mathrm{d}y &\leq \int_{|t|\geq \frac{y}{2}} y^{-n+2k+1}|t|^{2k}|\nabla u(x-t),y)|^2\mathrm{d}t\mathrm{d}y \\
&\leq B_\lambda g_\lambda^*(f,x)^2.
\end{aligned}$$

Now

$$g_1(F,x) \lesssim_\lambda g_{k+1}(F,x) \lesssim_\lambda g_\lambda^*(f,x).$$

□

8.1 Partial sum operators

Definition 8.6. Let ρ be a rectangle in $\mathbb{R}^n$, by which we mean that the sides are parallel to the axes. We define the associated "partial sum operator" by

$$S_\rho(f)^\wedge = \mathbf{1}_\rho \cdot \hat{f}, \quad f \in L^2 \cap L^p(\mathbb{R}^n).$$

Theorem 8.7. *For $1 < p < \infty, f \in L^2 \cap L^p$, we have*

$$\|S_\rho(f)\|_p \leq A_p \|f\|_p,$$

where the constant A_p is independent of the rectangle ρ.

Definition 8.8. Let $\mathcal{R} = \{\rho_j\}_{j=1}^\infty$ be a sequence of rectangles. We define the operator $S_\mathcal{R} : L^2(\mathbb{R}^n, \ell^2) \to L^2(\mathbb{R}^n, \ell^2)$, given by

$$S_\mathcal{R}((f_1, f_2, \dots)) = (S_{\rho_1}(f_1), \dots, S_{\rho_j}(f_j), \dots).$$

We get the following generalized theorem

Theorem 8.9. *Let $f \in L^2(\mathbb{R}^n, \ell^2) \cap L^p(\mathbb{R}^n, \ell^2)$. Then for $1 < p < \infty$,*

$$\|S_\mathcal{R}(f)\|_p \leq A_p \|f\|_p,$$

where A_p is independent of the family of rectangles $\mathcal{R}$.

Proof. Stage 1: Take $n = 1$, with the rectangles as semi-infinite rectangles. Recall that the Hilbert transform has multiplier $i\mathrm{sgn}(x)$. Then

$$S_{(-\infty,0)} = \frac{I + iH}{2}.$$

Lemma 8.10. *Let* $f(x) = (f_1(x), \dots, f_j(x), \dots) \in L^2(\mathbb{R}, \ell^2), L^p(\mathbb{R}^n \ell^)$. *Then for* $1 < p < \infty$

$$\left\|\tilde{H}f\right\|_p \leq A_p \|f\|_p$$

Stage 2: Here $n = 1$, and rectangles are intervals $(-\infty, a_1), (-\infty, a_2), \dots, (-\infty, a_j), \dots$. Now we want to shift the multiplier for the Hilbert operator

$$\begin{aligned}(f(x)e^{-2\pi i x \cdot a})^\wedge &= \hat{f}(x+a),\\ H(e^{-2\pi i x \cdot a} f)^\wedge &= i\mathrm{sgn}\hat{f}(x+a),\\ [e^{2\pi i x \cdot a} H(e^{-2\pi i x \cdot a f})]^\wedge &= i\mathrm{sgn}(x-a)\hat{f}(x).\end{aligned}$$

Then

$$(S_{(-\infty,a_j)} f_j)(x) = \frac{f_j + i e^{2\pi i x \cdot a_j} H(e^{-2\pi i x \cdot a_j} f_j)}{2}(x),$$

So we may write

$$S_{\mathcal{R}} f = \frac{f + i e^{2\pi i x \cdot a} \tilde{H}(e^{-2\pi i x \cdot a} f)}{2}$$

Stage 3: We move to general n, and take the rectangles as half spaces $x_1 < a_j$. Let $S^{(1)}_{(-\infty,a_j)}$ denote the operator defined on $L^2(\mathbb{R}^n)$, which acts only on the x_1 variable. We claim that

$$S_{\rho_j} = S^{(1)}_{(-\infty,a_j)},$$

because separable functions are dense in L^2. Then apply previous stage.

Final stage: Every general bounded rectangle is the intersection of $2n$ half spaces, each half-space having its boundary hyperplane perpendicular to the axes. Then take some limit argument to unbounded rectangle.

□

8.2 Dyadic decomposition

Let Δ denote the dyadic decomposition of $\mathbb{R}^n$. In the sense of L^2 convergence, we expect

$$\sum_{\rho \in \Delta} s_\rho = \mathrm{id}.$$

Because the blocks are mutually orthogonal, we actually have the stronger statement

$$\sum_{\rho \in \Delta} \|S_\rho f\|_2^2 = \|f\|_2^2 . \tag{8.4}$$

Theorem 8.11. *Suppose $f \in L^p(\mathbb{R}^n), 1 < p < \infty$. Then $(\sum_{\rho\in\Delta} |S_\rho f(x)|^2)^{1/2} \in L^p(\mathbb{R}^n)$ and in fact*

$$\left\| (\sum_{\rho\in\Delta} |S_\rho f(x)|^2)^{1/2} \right\|_p \quad \textit{and} \quad \|f\|_p$$

are comparable.

Example 8.12. As a detour, we consider the Rademacher functions, defined on $(0,1)$ with

$$r_0(t) = \begin{cases} 1 & 0 \le t \le 1/2, \\ -1 & 1/2 < t \le 1, \end{cases}$$

and we extend r_0 outside the unit interval by periodicity, and define $r_m(t) = r_0(2^m t)$. The sequence of Rademacher functions are orthonormal.

If $a_m \in \ell^2$ and $F(t) = \sum_{m=0}^\infty a_m r_m(t)$, then $F(t) \in L^p([0,1])$ for all $p < \infty$, and we get

$$A_p \|F\|_p \le \|F\|_2 = \|(a_m)\|_{\ell^2} \le B_p \|F\|_p .$$

An n-dimensional analog holds as well.

Proof. The $(\ge)$ direction of the inequality does not require any new machinery if we assume the other direction $(\le)$. Applying polarization to (8.4), we have

$$\int f\overline{g}\mathrm{d}x = \sum_{\rho\in\Delta} S_\rho(f)\overline{S_\rho(g)}\mathrm{d}x.$$

We then apply the Cauchy-Schwarz inequality to the sum and Hölder's inequality to the integral for

$$\begin{aligned} \left| \int f\overline{g}\mathrm{d}x \right| &\le \int \left(\sum_\rho |S_\rho f|^2 \right)^{\frac{1}{2}} \left(\sum_\rho |S_\rho g|^2 \right)^{\frac{1}{2}} \mathrm{d}x \\ &\le \left\| \left(\sum_\rho |S_\rho f|^2 \right)^{\frac{1}{2}} \right\|_p \left\| \left(\sum_\rho |S_\rho g|^2 \right)^{\frac{1}{2}} \right\|_{p'} \end{aligned}$$

Take the supremum over all g such that $\|g\|_{p'} \le 1$. The right hand side is controlled by the other direction of the inequality.

Our goal is to show that for $1 < p < \infty$, we have

$$\left\| \left(\sum_\rho |S_\rho f|^2 \right)^{\frac{1}{2}} \right\|_p \le A_p \|f\|_p .$$

Take first the case $n = 1$. Let Δ_1 be a family of dyadic intervals in $\mathbb{R}$. We define $\varphi \in C^1(\mathbb{R})$ to be mollified bump function such that

$$\varphi(x) = \begin{cases} 1 & 1 \leq x \leq 2, \\ 0 & x \leq 1/2 \text{ or } x \geq 4. \end{cases}$$

The associated multiplier operator to an interval I of the form $[2^k, 2^{k+1}]$ is given by

$$(\tilde{S}_I f)^\wedge(x) = \varphi(2^{-k}x)\hat{f}(x) = \varphi_I(x)\hat{f}(x).$$

Observe too that $S_I \tilde{S}_I = S_I$.

We consider the multiplier transformation given by

$$\tilde{T}_t = \sum_{m=0}^{\infty} r_m(t)\tilde{S}_{I_m},$$

i.e. the multiplier associated with $\tilde{T}_t$ is

$$m_t(x) = \sum_m r_m(t)\varphi_{I_m}(x).$$

For fixed x, there can be at most three nonzero terms in the sum. Then after absorbing some constant, we have the uniform bounds

$$|m_t(x)| \leq B, \quad \left|\frac{\mathrm{d}m_t}{\mathrm{d}x}(x)\right| \leq \frac{B}{|x|.}$$

By the multiplier theorem, we get

$$\begin{aligned} \left\|\tilde{T}_t f\right\|_p &\leq A_p \left\|f\right\|_p, \\ \implies \left(\int_0^1 \left\|\tilde{T}_t(f)\right\|_p^p \mathrm{d}t\right)^{1/p} &\leq A_p \left\|f\right\|_p. \end{aligned}$$

However

$$\begin{aligned} \int_0^1 \left\|\tilde{T}_t(f)\right\|_p^p \mathrm{d}t &= \int_{\mathbb{R}^n}\int_0^1 |\sum_m r_m(t)(\tilde{S}_{I_m}f)(x)|^p \mathrm{d}t\mathrm{d}x \\ &\geq A_p \int \left(\sum_m |\tilde{S}_{I_m}f(x)|^2\right)^{p/2} \mathrm{d}x, \end{aligned}$$

by the property of Rademacher functions. The apply theorem about partial sums.

Write $T_t = \sum_m r_m(t) S_{I_m}$. We claim that

$$\|T_t(f)\|_p \leq A_p \|f\|_p, \quad 1 < p < \infty,$$

because

$$B_p \left\|T_t^N f\right\|_p \leq \left\| \left(\sum^N |S_{I_m} f|^2 \right)^{\frac{1}{2}} \right\|_p \leq C - P \|f\|_p .$$

For the n-dimensional case, define $T_{t_1}^{(1)}$ as the operator T_{t_1} acting only on the x_1 variable, so

$$\int_{\mathbb{R}} |T_{t_1}^{(1)} f(x_1, x_2, \ldots, x_n)|^p \mathrm{d}x_1 \leq A_p^p \int_{\mathbb{R}^1} |f(x_1, \ldots, x_n)|^p \mathrm{d}x_1$$

for almost every fixed $x_2, x_3, \ldots, x_n$. Then integrate with respect to $x_2, \ldots, x_n$ for

$$\left\|T_{t_1}^{(1)} f\right\|_p \leq A_p \|f\|_p .$$

Iterating yields

$$\|T_t(f)\|_p \leq A_p^n \|f\|_p, \text{ where } T_t = \sum_{\rho_m \in \Delta} r_m(t) S_{\rho_m}.$$

Now raise to the pth power and integrate with respect to t, making use of properties of Rademacher functions. □

8.3 Marcinkiewicz multiplier theorem

Theorem 8.13. *Let m be a bounded function on $\mathbb{R}^n$ which is of bounded variation on every finite interval not containing the origin. Suppose also*

1. $|m(x)| \leq B$,
2. *for each $0 < k \leq n$*

   $$\sup_{x_{k+1}, \ldots, x_n} \int_\rho \left| \frac{\partial^k m}{\partial x_1 \partial x_2 \ldots \partial x_k} \right| \mathrm{d}x_1 \ldots \mathrm{d}x_k \leq B$$

   as ρ ranges over the dyadic rectangles of $\mathbb{R}^k$.
3. *The condition analogous to (b) is valid for every one of the $n!$ permutations of the variables $x_1, x_2, \ldots, x_n$.*

Then $m \in \mathcal{M}_p$ for $1 < p < \infty$.

Proof. We only consider the case $n = 2$. Let $f \in L^2 \cap L^p(\mathbb{R}^n)$ and write $F = T_m f$.

Let Δ denote the dyadic rectangles, for each $\rho \in \Delta$, write $f_\rho = S_\rho$, $F_\rho = S_\rho F$, and thus $F_\rho = T_m f_\rho$. By Theorem 8.11 it suffices to show that

$$\left\| \left(\sum_{\rho \in \Delta} |F_\rho|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left(\sum_{\rho \in \Delta} |f_\rho|^2 \right)^{1/2} \right\|_p$$

Assume ρ takes the form $[2^k, 2^{k+1}] \times [2^\ell, 2^{\ell+1}]$. The fundamental theorem of calculus tells us that

$$\begin{aligned} m(x_1, x_2) = \int_{2^k}^{x_1} \int_{2^\ell}^{x_2} \frac{\partial^2 m(t_1, t_2)}{\partial t_1 \partial t_2} \mathrm{d}t_1 \mathrm{d}t_2 + \int_{2^k}^{x_1} \frac{\partial}{\partial t_1} m(t_1, 2^\ell) \mathrm{d}t_1 \\ + \int_{2^\ell}^{x_2} \frac{\partial}{\partial t_2} m(2^k, t_2) \mathrm{d}t_2 + m(2^k, 2^\ell). \end{aligned}$$

Let S_t denote the multiplier corresponding to $(2^k, t_1) \times (2^\ell, t_2)$. Let $S^1_{t_1}$ be the multiplier corresponding to $(2^k, t_1) \times \mathbb{R}$ and $S^2_{t_2}$ be the multiplier corresponding to $\mathbb{R} \times (2^\ell, t_2)$. Then $S_t = S^1_{t_1} \cdot S^2_{t_2}$, so we have

$$\begin{aligned} S_\rho T_m = \int_{2^\ell}^{2^{\ell+1}} \int_{2^k}^{2k+1} S_t \frac{\partial^2 m}{\partial t_1 \partial t_2} \mathrm{d}t_1 \mathrm{d}t_2 + \int_{2^k}^{2^{k+1}} S^1_{t_1} \frac{\partial}{\partial t_1} m(t_1, 2^\ell) \mathrm{d}t_1 \\ + \int_{2^\ell}^{2^{\ell+1}} \cdots + m(2^k, 2^\ell) S_\rho. \end{aligned}$$

Now use the fact that $S_\rho T_m f = F_\rho$ for

$$\begin{aligned} |F_\rho|^2 \lesssim \iint_\rho |S_t(f_\rho)|^2 c + \int_{I_1} |S^1_{t_1}(f_\rho)|^2 \left| \frac{\partial^2 m(t_1, 2^\ell)}{\partial t_1} \right| \mathrm{d}t_1 \\ + + \int_{I_2} |S^2_{t_2}(f_\rho)|^2 \left| \frac{\partial^2 m(2^k, t_2)}{\partial t_1} \right| \mathrm{d}t_1 + |f_\rho|^2 \end{aligned}$$

Then we can apply Theorem 8.9 with the measure

$$\mathrm{d}\gamma = \int_{I_1} |S^1_{t_1}(f_\rho)|^2 \left| \frac{\partial^2 m(t_1, 2^\ell)}{\partial t_1} \right| \mathrm{d}t_1 \mathrm{d}t_2.$$

$\square$