The Prime Number Theorem via Complex Analysis

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1 The gamma function

We define the gamma function on $(0, \infty) \subset \mathbb{R}$ by

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt. \tag{1.1}$$

Proposition 1.1. *The improper integral in (1.1) converges for $s > 0$.*

Proof. We split the improper integral into two parts

$$\Gamma(s) = \lim_{a \to 0^+} \int_a^1 e^{-t} t^{s-1} dt + \lim_{b \to \infty} \int_1^b e^{-t} t^{s-1} dt. \tag{1.2}$$

For the first term, we know that e^{-t} is bounded on $[0, 1]$, so it suffices to consider the behavior of t^{s-1}. Since we also have $s > 0$, the primitive of t^{s-1} is given by $\frac{t^s}{s}$, which is defined for positive s at both $t = 1$ and $t = 0$.

For the second term, we apply the fact that $e^{-t} t^{s-1} \lesssim e^{-\frac{t}{2}}$ because exponential growth exceeds polynomial growth. But the integral $\int_1^\infty e^{-\frac{t}{2}} dt$ also converges. Since $e^{-t} t^{s-1}$ is bounded below by 0 and bounded above by $e^{-\frac{t}{2}}$ for sufficiently large t, we conclude that its integral also converges. ☐

We now extend the domain of Γ to the right half of the complex plane.

Proposition 1.2. *The gamma function extends to an analytic function in the half plane* $\mathrm{Re}(s) > 0$*, where the same integral formula (1.1) holds.*

Proof. Given any $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 0$, we denote $\sigma := \mathrm{Re}(s)$. We know there exist some $\delta, M \in \mathbb{R}$ such that $0 < \delta < \sigma < M$. For $\varepsilon > 0$, we write

$$I_\varepsilon(s) = \int_\varepsilon^{1/\varepsilon} e^{-t} t^{s-1} dt. \tag{1.3}$$

It thus suffices to show that as $\varepsilon \to 0$, we have $I_\varepsilon(s) \to \Gamma(s)$. We split the difference

$$|\Gamma(s) - I_\varepsilon(s)| \le \left| \int_0^\varepsilon e^{-t} t^{s-1} dt \right| + \left| \int_{1/\varepsilon}^\infty e^{-t} t^{s-1} dt \right| \tag{1.4}$$

and consider both terms separately again.

For the first term, we observe that

$$\begin{aligned}
\left|\int_0^{\varepsilon} e^{-t}t^{s-1}dt\right| &\leq \int_0^{\varepsilon} |e^{-t}t^{s-1}|dt \\
&\leq \int_0^{\varepsilon} |t^{s-1}|dt \quad (\text{because } |e^{-t}| \leq 1 \text{ for } 0 \leq t \leq 1) \\
&\leq \left[\frac{|t|^{\sigma}}{\sigma}\right]_0^{\varepsilon} = \frac{\varepsilon^{\sigma}}{\sigma} < \frac{\varepsilon^{\delta}}{\delta} \quad (\text{for } 0 < \varepsilon < 1),
\end{aligned}$$

which approaches zero as $\varepsilon \to 0$. For the second term, we already know that $t^{M-1} = O(e^{-t})$ and $e^{-t}t^{M-1} = O(e^{-\frac{t}{2}})$ for sufficiently large t because exponential growth exceeds polynomial growth. Then we have

$$\begin{aligned}
\left|\int_{1/\varepsilon}^{\infty} e^{-t}t^{s-1}dt\right| &\leq \int_{1/\varepsilon}^{\infty} |e^{-t}t^{s-1}|dt \leq \int_{1/\varepsilon}^{\infty} e^{-t}t^{M-1}dt \\
&\leq C\int_{1/\varepsilon}^{\infty} e^{-\frac{t}{2}}dt = -2C\left[e^{-\frac{t}{2}}\right]_{1/\varepsilon}^{\infty} \to 0
\end{aligned}$$

as ε approaches zero and $1/\varepsilon$ approaches infinity. Therefore we conclude that $|\Gamma(s) - I_{\varepsilon}(s)| \to 0$ as desired. ☐

In fact, we can even extend the gamma function Γ to a meromorphic function on the complex plane $\mathbb{C}$. We introduce the following lemma which reveals a surprising connection between the Γ function and the factorial function on natural numbers.

Lemma 1.3. *If Γ is defined at both $s+1, s \in \mathbb{C}$, then we have $\Gamma(s+1) = s\Gamma(s)$. In particular, we have $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.*

Proof. We apply the integration by parts formula for

$$\int_0^{\infty} \frac{d}{dt}(e^{-t}t^{s})dt = -\int_0^{\infty} e^{-t}t^{s}dt + s\int_0^{\infty} e^{-t}t^{s-1}dt = -\Gamma(s+1) + s\Gamma(s). \tag{1.5}$$

We observe that the left hand side converges to 0 by the Fundamental Theorem of Calculus because $e^{-t}t^{s} \to 0$ as $t \to 0$ or $t \to \infty$. For $n \in \mathbb{N}$, we verify the base case where

$$\Gamma(1) = \int_0^{\infty} e^{-t}dt = \left[-e^{-t}\right]_0^{\infty} = 1, \tag{1.6}$$

and apply (1.5) repeatedly to close the induction. ☐

We are now ready to prove the analytic continuation of Γ to the complex plane.

Theorem 1.4. *We can find a meromorphic continuation of the gamma function Γ to the complex plane with simple poles at $s = 0, -1, -2, \ldots$. The residue of Γ at $-n$ is $(-1)^n/n!$.*

Proof. We show that Γ has a meromorphic continuation to the half-plane $\mathrm{Re}(s) > -m$ for all $m \in \mathbb{N}$. For $m = 1$, we define via a change of variables

$$\Gamma(s) := \frac{\Gamma(s+1)}{s}, \tag{1.7}$$

which agrees with Lemma 1.3. We compute the residue at the simple pole of $s = 0$,

$$\mathrm{res}_0 \Gamma = \lim_{s \to 0} \Gamma(s+1) = \Gamma(1) = 1.$$

We now define an analogous formula for arbitrary m by

$$\Gamma(s) := \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\dots(s+1)(s)} \tag{1.8}$$

$$= \frac{\Gamma(s+m)}{(s-(-m+1))(s-(-m+2))\dots(s-(-1))(s)}, \tag{1.9}$$

which likewise agrees with Lemma 1.3. We verify that this new Γ is well-defined because a meromorphic continuation is unique. For $-n \in \{-m+1, -m+2, \dots, 0\}$, we see that

$$\begin{aligned} \mathrm{res}_{-n}\Gamma &= \frac{\Gamma(m-n)}{(-n-(-m+1))(-n-(m+2))\dots 1 \cdot (-1)(-2)\cdots(-n)} \\ &= \frac{\Gamma(m-n)}{(m-n-1)(m-n-2)\cdots 1 \cdot (-1)(-2)\cdots(-n)} \\ &= \frac{(m-n-1)!}{(m-n-1)!(-1)^n n!} = \frac{(-1)^n}{n!} \end{aligned}$$

which gives us our desired residue. ☐

We next consider the symmetry of Γ about the line $\mathrm{Re}(s) = \frac{1}{2}$.

Theorem 1.5. *For all $s \in \mathbb{C}$, we have*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}. \tag{1.10}$$

In particular, we obtain $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

This identity is remarkable because $\Gamma(s)$ has poles at $s = 0, -1, -2, \dots$, while $\Gamma(1-s)$ has poles at $s = 1, 2, \dots$. The location of these poles aligns with fact that $\frac{\pi}{\pi s}$ has poles at the integers.

Proof. It suffices to prove this identity for real s between 0 and 1 because it follows for the rest of the complex plane by analytic continuation. We first note that for $0 < a < 1$, we have

$$\int_0^\infty \frac{v^{a-1}}{1+v} dv = \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin \pi a}, \tag{1.11}$$

where the first equality follows from the change of variables $v = e^x$. The second equality follows from a contour integral along a rectangle with vertices at $\{-R, R, R+2\pi i, -R+2\pi i\}$. We can also rewrite

$$\Gamma(1-s) = \int_0^\infty e^{-u} u^{-s} du = t \int_0^\infty e^{-vt} (vt)^{-s} dv, \tag{1.12}$$

where we apply the change of variables $u = vt$ for some positive t. Then we see that

$$\begin{aligned}
\Gamma(1-s)\Gamma(s) &= \int_0^\infty e^{-t} t^{s-1} \cdot t \int_0^\infty e^{-vt} (vt)^{-s} dv dt \\
&= \int_0^\infty \int_0^\infty e^{-t(1+v)} v^{-s} dt dv \quad \text{(by Fubini's Theorem)} \\
&= \int_0^\infty v^{-s} \left[\frac{e^{-t(1+v)}}{-(1+v)} \right]_0^\infty dv \\
&= \int_0^\infty \frac{v^{-s}}{1+v} dv \\
&= \frac{\pi}{\sin \pi(1-s)} \quad \text{(from (1.11))} \\
&= \frac{\pi}{\sin \pi s}
\end{aligned}$$

because the sine function is symmetric about the line $\frac{\pi}{2}$. □

We proceed to consider the surprisingly simple properties of the reciprocal of the gamma function.

Theorem 1.6. $1/\Gamma(s)$ *is an entire function with simple zeroes at* $s = 0, -1, -2, \ldots$ *and vanishes nowhere else.*

Proof. By Theorem 1.5, we may write

$$\frac{1}{\Gamma(s)} = \Gamma(1-s) \frac{\sin \pi s}{\pi}. \tag{1.13}$$

The simple poles of $\Gamma(1-s)$ at $s = 1, 2, 3, \ldots$ thus cancel out the positive simple zeroes of $\sin \pi s$ at the natural numbers. We deduce that the product has simple zeroes only at $s = 0, -1, -2, -3, \ldots$. □

2 The Poisson summation formula

We take a brief detour into the one-dimensional theory of the Fourier transform to derive the Poisson summation formula from contour integration. We will use this theorem subsequently to obtain the functional equation of the Riemann zeta function.

If f is a function on $\mathbb{R}$ that satisfies appropriate regularity and decay conditions, then its Fourier transform is defined by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi} dx, \xi \in \mathbb{R}. \tag{2.1}$$

For each $a > 0$ we denote by $\mathfrak{F}_a$ the class of all functions f that satisfy the following two conditions

(i) The function f is holomorphic in the horizontal strip

$$S_a = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < a\}. \tag{2.2}$$

(ii) There exists a constant $A > 0$ such that

$$|f(x+iy)| \leq \frac{A}{1+x^2} \text{ for all } x \in \mathbb{R} \text{ and } |y| < a. \tag{2.3}$$

We denote by $\mathfrak{F}$ the class of all functions that belong to $\mathfrak{F}_a$ for some a.

Theorem 2.1. *(Poisson summation formula) If $f \in \mathfrak{F}$, then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n). \tag{2.4}$$

Proof. Suppose that $f \in \mathfrak{F}_a$ and take some b such that $0 < b < a$. The function $f(z)/(e^{2\pi i z} - 1)$ has simple poles at the integers n, with residues $f(n)/2\pi i$. We consider the rectangular contour γ_N which passes through the vertices

$$-N - \frac{1}{2} + ib, -N - \frac{1}{2} - ib, N + \frac{1}{2} - ib, N + \frac{1}{2} + ib.$$

We can apply the residue formula for

$$\sum_{|n|\leq N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi i z} - 1} dz.$$

Because f has moderate decrease, we know that the left hand side converges to $\sum_{n\in\mathbb{Z}} f(z)$ and that the integral along the vertical segments vanishes as N tends to infinity. We get

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} \frac{f(z)}{e^{2\pi i z} - 1} dz - \int_{L_2} \frac{f(z)}{e^{2\pi i z} - 1} dz,$$

where L_1 and L_2 are the real line shifted down and up by b respectively. Since we have $|e^{2\pi iz}| > 1$ on L_1 and $|e^{2\pi iz}| < 1$ on L_2, we find that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} f(n) &= \int_{L_1} f(z)\left(e^{2\pi iz}\sum_{n=0}^{\infty} e^{-2\pi inz}\right)dz + \int_{L_2} f(z)\left(\sum_{n=0}^{\infty} e^{2\pi inz}\right)dz \\
&= \sum_{n=0}^{\infty}\int_{L_1} f(z)e^{-2\pi i(n+1)z}dz + \sum_{n=0}^{\infty}\int_{L_2} f(z)e^{2\pi inz}dz \\
&= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-2\pi i(n+1)x}dx + \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} f(x)e^{2\pi inx}dx \quad \text{(omit vertical components)} \\
&= \sum_{n=0}^{\infty}\hat{f}(n+1) + \sum_{n=0}^{\infty}\hat{f}(-n) = \sum_{n\in\mathbb{Z}}\hat{f}(n),
\end{aligned}$$

which gives us the Poisson summation formula. ☐

3 The zeta function

Similarly to the case of the gamma function, we first define the zeta function on $(1,\infty) \subset \mathbb{R}$ by the convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{3.1}$$

We begin with a simple extension of ζ to a half-plane in $\mathbb{C}$.

Proposition 3.1. *The series defining $\zeta(s)$ converges for $\mathrm{Re}(s) > 1$ and the function ζ is holomorphic in this half-plane.*

Proof. If $s = \sigma + it$ where $\sigma, t \in \mathbb{R}$, then we have

$$|n^{-s}| = |e^{-s\log n}| = e^{-\sigma\log n} = n^{-\sigma}.$$

Then if $\sigma > 1 + \delta > 1$, the series defining ζ is uniformly bounded (across different σ) by $\sum_{n=1}^{\infty} 1/n^{1+\delta}$, which converges. Therefore, the series $\sum 1/n^s$ converges uniformly (across different s) on every half-plane $\mathrm{Re}(s) > 1 + \delta > 1$. ☐

We now introduce the theta function.

Definition 3.2. The theta function is defined for real $t > 0$ by

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}. \tag{3.2}$$

Lemma 3.3. *The theta function obeys the functional equation*

$$\vartheta(t) = t^{-1/2}\vartheta(1/t). \tag{3.3}$$

Proof. A contour integral tells us that the function $e^{-\pi x^2}$ is its own Fourier transform

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx = e^{-\pi\xi^2}.$$

For a fixed value of $t > 0$, the change of variables $x \mapsto t^{1/2}x$ shows that the Fourier transform of the function $f(x) = e^{-\pi t x^2}$ is $\hat{f}(\xi) = t^{-1/2}e^{-\pi\xi^2/t}$. We apply Theorem 2.1 for

$$\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = \sum_{n=-\infty}^{\infty} t^{-1/2} e^{-\pi n^2/t},$$

which corresponds exactly with $\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$. □

Lemma 3.4. *The growth and decay of ϑ satisfy*

$$\vartheta(t) \le Ct^{-1/2} \qquad \text{as } t \to 0, \text{ and} \tag{3.4}$$

$$|\vartheta(t) - 1| \le Ce^{-\pi t} \qquad \text{for some } C > 0, \text{ and all } t \ge 1. \tag{3.5}$$

Proof. The bounds for small t follow from the functional equation, while the bounds for large t follows from the fact that $t \ge 1$ implies $\sum_{n\ge1} e^{-\pi n^2 t} \le \sum_{n\ge1} e^{-\pi n t} \le Ce^{-\pi t}$. □

We then introduce an important relation among ζ, Γ, and ϑ.

Proposition 3.5. *If* $\mathrm{Re}(s) > 1$, *then we have*

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^{\infty} u^{s/2-1}(\vartheta(u) - 1)du. \tag{3.6}$$

Proof. We begin with the following observation for $n \ge 1$,

$$\int_0^{\infty} e^{-\pi n^2 u} u^{s/2-1} du = \int_0^{\infty} \frac{e^{-t}(t/\pi n^2)^{s/2-1}}{\pi n^2} dt = \pi^{-s/2} n^{-s}\Gamma(s/2), \tag{3.7}$$

which follows from a change of variables $u = t/\pi n^2$ on the left hand side. We also note that we have

$$\frac{\vartheta(u) - 1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 u}. \tag{3.8}$$

Our estimates for ϑ from Lemma 3.4 tell us that it is uniformly bounded so we can interchange the infinite sum with the integral for

$$\begin{aligned}
\frac{1}{2}\int_0^{\infty} u^{s/2-1}(\vartheta(u) - 1)du &= \sum_{n=1}^{\infty}\int_0^{\infty} u^{s/2-1} e^{-\pi n^2 u} du \quad \text{(from (3.8))} \\
&= \pi^{-s/2}\Gamma(s/2)\sum_{n=1}^{\infty} n^{-s} \quad \text{(from (3.7))} \\
&= \pi^{-s/2}\Gamma(s/2)\zeta(s),
\end{aligned}$$

which gives us our desired identity. □

We also consider a modification of the ζ function called the xi function which appears more symmetric.

Definition 3.6. The xi function is defined for $\mathrm{Re}(s) > 1$ by $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Theorem 3.7. *The function ξ is holomorphic for $\mathrm{Re}(s) > 1$ and has an analytic continuation to all of $\mathbb{C}$ as a meromorphic function with simple poles at $s = 0$ and $s = 1$. Moreover, we have $\xi(s) = \xi(1-s)$ for all $s \in \mathbb{C}$.*

Proof. Let $\psi(u) = \frac{\vartheta(u)-1}{2}$. The functional equation for the theta function in Lemma 3.3 implies

$$\psi(u) = u^{-1/2}\psi(1/u) + \frac{1}{2u^{1/2}} - \frac{1}{2}. \tag{3.9}$$

By Theorem 3.5 for $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned}
\xi(s) &= \pi^{-s/2}\Gamma(s/2)\zeta(s) \\
&= \int_0^\infty u^{s/2-1}\psi(u)du \\
&= \int_0^1 u^{s/2-1}\psi(u)du + \int_1^\infty u^{s/2-1}\psi(y)du \\
&= \int_0^1 u^{s/2-1}\left[u^{-1/2}\psi(1/u) + \frac{1}{2u^{1/2}} - \frac{1}{2}\right]du + \int_1^\infty u^{s/2-1}\psi(u)du \text{ (from (3.9))} \\
&= \frac{1}{2}\int_0^1 (u^{s/2-3/2} - u^{s/2-1})du + \int_\infty^1 v^{-s/2+3/2}\psi(v)(-v^{-2})dv + \int_1^\infty u^{s/2-1}\psi(u)du \\
&= \frac{1}{2}\left[\frac{u^{(s-1)/2}}{(s-1)/2}\right]_0^1 - \frac{1}{2}\left[\frac{u^{s/2}}{s/2}\right]_0^1 + \int_1^\infty v^{-s/2-1/2}\psi(v)dv + \int_1^\infty u^{s/2-1}\psi(u)du \\
&= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (u^{-s/2-1/2} + u^{s/2-1})\psi(u)du.
\end{aligned}$$

Since the function ψ has exponential decay at infinity, we obtain an entire function that has analytic continuation to all of $\mathbb{C}$ with simple poles at $s = 0$ and $s = 1$. We see its symmetry because we can replace s by $1 - s$. $\square$

We finally arrive at the analytic continuation and functional equation for the zeta function.

Theorem 3.8. *The zeta function has a meromorphic continuation into the entire complex plane, whose only singularity is a simple pole at $s = 1$.*

Proof. We take $\zeta(s) = \pi^{s/2}\frac{\xi(s)}{\Gamma(s/2)}$. From Theorem 1.6, we know that $1/\Gamma(s/2)$ is entire with simple zeros at $0, -2, -4, \ldots$. The simple pole of ξ at the origin is cancelled by the corresponding zero of $1/\Gamma(s/2)$. $\square$

We end this section on the zeta function with growth estimates of $\zeta(s)$ near the line $\mathrm{Re}(s) = 1$.

Lemma 3.9. *There exists a sequence of entire functions $\{\delta_n(s)\}_{n=1}^{\infty}$ which satisfy the estimate $|\delta_n(s)| \leq |s|/n^{\sigma+1}$, where $s = \sigma + i$, and such that we have*

$$\sum_{1\leq n<N} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{1\leq n<N} \delta_n(s). \tag{3.10}$$

for integers $N \geq 1$.

Proof. We set $\delta_n(s) = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx$. The mean-value theorem applied to $f(x) = x^{-s}$ yields

$$\left|\frac{1}{n^s} - \frac{1}{x^s}\right| \leq \frac{|s|}{n^{\sigma+1}}$$

whenever $n \leq x \leq n+1$. Therefore we have $|\delta_n(s)| \leq |s|/n^{\sigma+1}$. □

Corollary 3.10. *For* $\mathrm{Re}(s) > 0$*, we have*

$$\zeta(s) - \frac{1}{s-1} = H(s), \tag{3.11}$$

where $H(s) = \sum_{n=1}^{\infty} \delta_n(s)$ is holomorphic in the half-plane $\mathrm{Re}(s) > 0$.

Proof. We already have uniform convergence for $\mathrm{Re}(s) > 1$ by our estimate for $|\delta_n(s)|$. In fact, the convergence holds in any half-plane $\mathrm{Re}(s) \geq \delta$ when $\delta > 0$. □

We conclude this section by showing that the growth of $\zeta(s)$ near the line $\mathrm{Re}(s) = 1$ is "mild".

Proposition 3.11. *Suppose $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. Then for each $\sigma_0, 0 \leq \sigma_0 \leq 1$, and every $\varepsilon > 0$, there exists a constant c_ε so that*

(i) $|\zeta(s)| \leq c_\varepsilon |t|^{1-\sigma_0+\varepsilon}$ if $\sigma_0 \leq \sigma$ and $|t| \geq 1$.

(ii) $|\zeta'(s)| \leq c_\varepsilon |t|^{\varepsilon}$, if $1 \leq \sigma$ and $|t| \geq 1$.

Proof. We combine the estimates $|\delta_n(s)| \leq |s|/n^{\sigma+1}$ and $|\delta_n(s)| \leq 2/n^{\sigma}$ for

$$|\delta_n(s)| \leq \left(\frac{|s|}{n^{\sigma_0+1}}\right)^{\delta} \left(\frac{2}{n^{\sigma_0}}\right)^{1-\delta} \leq \frac{2|s|^{\delta}}{n^{\sigma_0+\delta}},$$

as long as δ is non-negative. Now we choose $\delta = 1 - \sigma_0 + \varepsilon$. By Corollary 3.10, we find that for $\sigma = \mathrm{Re}(s) \geq \sigma_0$,

$$|\zeta(s)| \leq \left|\frac{1}{s-1}\right| + 2|s|^{1-\sigma_0+\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}},$$

which gives us conclusion (i). By the Cauchy integral formula,

$$\zeta'(s) = \frac{1}{2\pi r} \int_0^{2\pi} \zeta(s + re^{i\theta}) e^{i\theta} d\theta,$$

where the integration is taken over a circle of radius r centered at the point s. Now we choose $r = \varepsilon$ and observe that the circle lies in the half-plane $\mathrm{Re}(s) \geq 1 - \varepsilon$, which gives us conclusion (ii). □

4 Zeros of the zeta function

We introduce informally Euler's identity, which states that for $\mathrm{Re}(s) > 1$ the zeta function can be expressed as an infinite product

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}. \tag{4.1}$$

The identity for $\mathrm{Re}(s) > 1$ from the case for $s > 1$. We observe that $1/(1 - p^{-s})$ can be written as a convergent geometric power series

$$1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}} + \ldots,$$

and taking formally the product of these series over all the primes p yields the desired result. We mention a proposition regarding infinite products.

Proposition 4.1. *If $\sum |a_n| < \infty$, then the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges. Moreover, the product converges to 0 if and only if one of its factors is 0.*

Proof. Since $\sum |a_n|$ converges, we know that $|a_n| < 1/2$ for sufficiently large n. We may then define

$$\log(1 + a_n) = -\sum_{r=1}^{\infty} (-1)^r \frac{a_n^r}{r}$$

and rewrite the partial products as

$$\prod_{n=1}^{N}(1 + a_n) = \prod_{n=1}^{N} e^{\log(1+a_n)} = e^{B_N},$$

where $B_N = \sum_{n=1}^{N} \log(1 + a_n)$. The power series expansion tells us that $|\log(1 + a_n)| \leq 2|a_n| < 1$, so B_N converges as $N \to \infty$. If $1 + a_n \neq 0$ for all n, then the product converges to a non-zero limit which is expressed as e^B. $\square$

Proposition 4.1 tells us that $\zeta(s)$ does not vanish where $\mathrm{Re}(s) > 1$. We wish to obtain further information about the location of the zeros of ζ.

Theorem 4.2. *The only zeros of ζ outside the strip $0 \leq \mathrm{Re}(s) \leq 1$ are at the negative even integers.*

Proof. We may write the fundamental relation $\xi(s) = \xi(1 - s)$ in the form

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1 - s)/2)\zeta(1 - s) \implies \zeta(s) = \pi^{s-1/2}\frac{\Gamma((1 - s)/2)}{\Gamma(s/2)}\zeta(1 - s).$$

We see that $\zeta(1 - s)$ has no zeros because $\mathrm{Re}(1 - s) > 1, \Gamma((1 - s)/2)$ is zero free, and $1/\Gamma(s/2)$ has zeros as $s = -2, -4, -6, \ldots$. $\square$

For the proof of the prime number theorem, we will require the property that $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}$. We proceed to a sequence of lemmas.

Lemma 4.3. *If* $\mathrm{Re}(s) > 1$, *then*

$$\log \zeta(s) = \sum_{p,m} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s} \tag{4.2}$$

for some $c_n \geq 0$.

Proof. We prove the identity $s > 1$, which extends to $\mathrm{Re}(s) > 1$ by analytic continuation because ζ has no zeros in the half plane $\mathrm{Re}(s) > 1$. Taking the logarithm of the Euler product formula and the power series for the logarithm, we have

$$\log\left(\frac{1}{1-x}\right) = \sum_{m=1}^{\infty} \frac{x^m}{m},$$

which holds for $0 \leq x < 1$, we find that

$$\log \zeta(s) = \log \prod_p \frac{1}{1-p^{-s}} = \sum_p \log \frac{1}{1-p^{-s}} = \sum_{p,m} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s},$$

where $c_n = 1/m$ if $n = p^m$ and $c_n = 0$. $\square$

Lemma 4.4. *If* $\theta \in \mathbb{R}$, *then* $3 + 4\cos\theta + \cos 2\theta \geq 0$.

Proof. The double angle formula tells us $3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2$. $\square$

Corollary 4.5. *If* $\sigma > 1$ *and* t *is real, then*

$$\log |\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 0.$$

Proof. Let $s = \sigma + it$ and note that

$$\mathrm{Re}(n^{-s}) = \mathrm{Re}(e^{-(\sigma+it)\log n}) = e^{-\sigma \log n}\cos(t\log n) = n^{-\sigma}\cos(t\log n).$$

Therefore, we have from $\log|z| = \mathrm{Re}\log z$,

$$\begin{aligned} \log |\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| &= 3\log|\zeta\sigma| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \\ &= \sum c_n n^{-\sigma}(3 + 4\cos t\log n + \cos 2t\log n) \geq 0, \end{aligned}$$

where the positivity follows from Lemma 4. $\square$

Theorem 4.6. *The zeta function has no zeros on the line* $\mathrm{Re}(s) = 1$.

Proof. Assume for the sake of contradiction that $\zeta(1+it_0)=0$ for some $t_) \neq 0$. Since ζ is holomorphic at $1+it_0$, it must vanish at least to order 1, hence

$$|\zeta(\sigma+it_0)|^4 = O(\sigma-1)^4 \text{ as } \sigma \to 1.$$

Since $s=1$ is a simple pole for $\zeta(s)$, we have

$$|\zeta(\sigma)|^3 = O(\sigma-1)^{-3} \text{ as } \sigma \to 1.$$

Since ζ is holomorphic as the points $\sigma+2it_0$, the quantity $|\zeta(\sigma+2it_0)|$ remains bounded as $\sigma \to 1$. Putting these facts together yields

$$|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \to 0,$$

which contradicts Corollary 4.5. $\square$

We end this section by understanding the growth of $1/\zeta$.

Proposition 4.7. *For every $\varepsilon > 0$, we have $1/|\zeta(s)| \leq c_\varepsilon |t|^\varepsilon$ when $s=\sigma+it, \sigma \geq 1$, and $|t| \geq 1$.*

Proof. Corollary 4.5 already tells us that $|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 1$, whenever $\sigma \geq 1$. Proposition 3.11 also gives us for all $\sigma \geq 1$ and $|t| \geq 1$,

$$|\zeta^4(\sigma+it)| \geq c|\zeta^{-3}(\sigma)||t|^{-\varepsilon} \geq c'(\sigma-1)^3|t|^{-\varepsilon}.$$

This implies

$$|\zeta(\sigma+it)| \geq c'(\sigma-1)^{3/4}|t|^{-\varepsilon/4}.$$

There are now two possible cases:

- If $\sigma - 1 \geq A|t|^{-5\varepsilon}$, then we already have $|\zeta(\sigma+it)| \geq A'|t|^{-4\varepsilon}$.
- If $\sigma - 1 < A|t|^{-5\varepsilon}$, then we select $\sigma' > \sigma$ with $\sigma'-1 = A|t|^{-5\varepsilon}$. The triangle inequality implies
$$|\zeta(\sigma+it)| \geq |\zeta(\sigma'+it)| - |\zeta(\sigma'+it)-\zeta(\sigma+it)|,$$
and an application of the mean value theorem gives
$$|\zeta(\sigma'+it)-\zeta(\sigma+it)| \leq c''|\sigma'-\sigma||t|^\varepsilon \leq c''|\sigma'-1||t|^\varepsilon$$

These observations show that

$$|\zeta(\sigma+it)| \geq c'(\sigma'-1)^{3/4}|t|^{-\varepsilon/4} - c''(\sigma'-1)|t|^\varepsilon.$$

We now pick $A = (c'/(2c''))^4$, and recall that $\sigma'-1 = A|t|^{-5\varepsilon}$. This gives precisely

$$c'(\sigma'-1)^{3/4}|t|^{-\varepsilon/4} = 2c''(\sigma'-1)|t|^\varepsilon,$$

so we arrive at

$$\zeta(\sigma+it)| \geq A''|t|^{-4\varepsilon}.$$

The proof of the proposition is complete once we replace 4ε by ε. $\square$

5 Reduction to the functions ψ and ψ_1

In his study of primes, Tchebychev introduced an auxiliary function whose behavior is to a large extent equivalent to the asymptotic distribution of primes, but which is easier to manipulate than $\pi(x)$. We define the following two functions

$$\psi(x) := \sum_{p^m \leq x} \log p, \tag{5.1}$$

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

We see immediately that

$$\psi(x) = \sum_{n=1}^{x} \Lambda(n) = \sum_{p \leq x} \lfloor \frac{\log x}{\log p} \rfloor \log p.$$

We can reduce the prime number theorem to a corresponding asymptotic statement about Tchebychev's psi-function ψ.

Proposition 5.1. *If $\psi(x) \sim x$ as $x \to \infty$, then $\pi(x) \sim x/\log x$ as $x \to \infty$.*

Proof. We first prove that $1 \leq \liminf_{x\to\infty} \pi(x)\frac{\log x}{x}$. We have the estimates

$$\psi(x) = \sum_{p \leq x} \lfloor \frac{\log x}{\log p} \rfloor \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x.$$

We next prove that $\limsup_{x\to\infty} \pi(x)\frac{\log x}{x} \leq 1$. Given $0 < \alpha < 1$, we note that

$$\psi(x) \geq \sum_{p \leq x} \log p \geq \sum_{x^\alpha < p \leq x} \log p \geq (\pi(x) - \pi(x^\alpha)) \log x^\alpha,$$

which gives $\psi(x) + \alpha\pi(x^\alpha) \log x \geq \alpha\pi(x) \log x$. When we divide both sides by x, we find that

$$1 \geq \alpha \limsup_{x\to\infty} \pi(x) \frac{\log x}{x}.$$

This inequality completes the proof since $\alpha < 1$ was arbitrary. □

We can simplify this further this a close cousin of the ψ function. We deefine the function ψ_1 by

$$\psi_1(x) = \int_1^x \psi(u) du.$$

We next show that the prime number theorem follows from the asymptotics of ψ_1.

Proposition 5.2. *If $\psi_1(x) \sim x^2/2$ as $x \to \infty$, then $\psi(x) \sim x$ as $x \to \infty$.*

Proof. If $\alpha < 1 < \beta$, then we apply the fact that ψ is increasing for

$$\frac{1}{(1-\alpha)x}\int_{\alpha x}^{x}\psi(u)du \leq \psi(x) \leq \frac{1}{(\beta-1)x}\int_{x}^{\beta x}\psi(u)du.$$

As a consequence, we find that

$$\psi(x) \leq \frac{1}{(\beta-1)x}[\psi_1(\beta x) - \psi_1(x)],$$

and therefore

$$\frac{\psi(x)}{x} \leq \frac{1}{(\beta-1)}\left[\frac{\psi_1(\beta x)}{(\beta x)^2}\beta^2 - \frac{\psi_1(x)}{x^2}\right].$$

In turn this implies

$$\limsup_{x\to\infty}\frac{\psi(x)}{x} \leq \frac{1}{\beta-1}\left[\frac{1}{2}\beta^2 - \frac{1}{2}\right] = \frac{1}{2}(\beta+1).$$

Since this result is true for all $\beta > 1$, we have proved that $\limsup_{x\to\infty}\psi(x)/x \leq 1$. A similar argument with $\alpha < 1$, then shows that $\liminf_{x\to\infty}\psi(x)/x \geq 1$. □

We now relate ψ_1 to ζ. By differentiating the expression in (4.2), we have

$$-\frac{\zeta'(s)}{\zeta(s)} = -\sum_{m,p}(\log p)p^{-ms} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}.$$

We record this formula for $\mathrm{Re}(s) > 1$ as

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}. \tag{5.3}$$

Lemma 5.3. *If $c > 0$, then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^s}{s(s+1)}ds = \begin{cases} 0 & \text{if } 0 < a \leq 1, \\ 1 - 1/a & \text{if } 1 \leq a. \end{cases}$$

Proof. We verify that the integral converges since $|a^s| = a^c$. We consider two cases separately.

($1 \leq a$) We write $a = e^{\beta}$ with $\beta = \log a \geq 0$. Let

$$f(s) = \frac{a^s}{s(s+1)} = \frac{e^{s\beta}}{s(s+1)}.$$

We obtain $\mathrm{res}_{s=0}f = 1$ and $res_{s=-1}f = -1/a$. For $T > 0$, we consider the positively oriented contour $\Gamma(T)$ with a vertical segment $S(T)$ from $c - iT$ to $c + iT$ and of the half-circle $C(T)$

centered at c of radius T lying to the left of the vertical segment. For sufficiently large T, we apply the residue formula for

$$\frac{1}{2\pi i}\int_{\Gamma(T)} f(s)ds = 1 - 1/a.$$

It thus suffices to show that the integral over the half-circle $C(T)$ goes to 0 as T tends to infinity. Note that if $s = \sigma + it \in C(T)$, then for all large T we have

$$|s(s+1)| \geq (1/2)T^2$$

and since $\sigma \leq c$ we also have the estimate $|e^{\beta s}| \leq e^{\beta c}$. Therefore

$$\left|\int_{C(T)} f(s)ds\right| \leq \frac{C}{T^2}2\pi T \to 0 \text{ as } T \to \infty.$$

For $0 \leq a \leq 1$, we consider an analogous contour but with the half-circle lying to the right of the line $\mathrm{Re}(s) = c$, noting that there are no poles in the interior of this contour. $\square$

Proposition 5.4. *For all $c > 1$*

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds. \tag{5.4}$$

Proof. We observe that $\psi(u) = \sum_{n=1}^{\infty}\Lambda(n)f_n(u)$, where $f_n(u) = 1$ if $n \leq u$ and $f_n(u) = 0$ otherwise. Therefore, we have

$$\begin{aligned}\psi_1(x) = \int_0^x \psi(u)du &= \sum_{n=1}^{\infty}\int_0^x \Lambda(n)f_n(u)du \\ &= \sum_{n\leq x}\Lambda(n)\int_n^x du = \sum_{n\leq x}\Lambda(n)(x-n).\end{aligned}$$

We now apply (5.3) and Lemma 5.3 with $a = x/n$ for

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds &= x\sum_{n=1}^{\infty}\Lambda(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(x/n)^s}{s(s+1)}ds \\ &= x\sum_{n\leq x}\Lambda(n)\left(1-\frac{n}{x}\right) = \psi_1(x),\end{aligned}$$

as was to be shown. $\square$

We dedicate our final theorem toward the asymptotic $\psi_1(x) \sim x^2/2$.

Theorem 5.5. *As $x \to \infty$, we have $\psi_1(x) \sim x^2/2$.*

Proof. The key ingredients in the argument are

1. the connection between ψ_1 and ζ from Proposition 5.4,
2. the non-vanishing of the zeta function on $\mathrm{Re}(s) = 1$ from Theorem 4.6,
3. the estimates for ζ near that line from Propositions 3.11 and 4.7.

Let $c > 1$ and assume x is also fixed fgor the moment with $x \geq 2$. Let $F(s)$ denote the integrand

$$F(s) = \frac{x^{s+1}}{s(s+1)}\Big(- \frac{\zeta'(s)}{\zeta(s)}\Big).$$

First we deform the vertical line from $c - i\infty$ to $c + i\infty$ to the path $\gamma(T)$. The usual and familiar arguments using Cauchy's theorem allow us to see that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)ds = \frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds. \tag{5.5}$$

Indeed, we know on the basis of Propositions 3.11 and 4.7 that $|\zeta'(s)/\zeta(s)| \leq A|t|^{\eta}$ for any fixed $\eta > 0$, whenever $s = \sigma + it, \sigma \geq 1$ and $|t| \geq 1$. Thus $|F(s)| \leq A'|t|^{-2+\eta}$ in the two infinite rectangles bounded by the line $(c - i\infty, c + i\infty)$ and $\gamma(T)$.

Next, we pass from the contour $\gamma(T)$ to the contour $\gamma(T, \delta)$. For fixed T, we choose $\delta > 0$ small enough so that ζ has no zeros in the box

$$\{s = \sigma + it, 1 - \delta \leq \sigma \leq 1, |t| \leq T\}.$$

Such a choice can be made since ζ does not vanish on the line $\sigma = 1$. We decompose the contour $\gamma(T, \delta)$ as follows

$$\begin{aligned}
\gamma_1 &: (-\infty, -T] \to \mathbb{C}, \gamma_1(t) = 1 + it,\\
\gamma_2^- &: [-\delta, 0] \to \mathbb{C}, \gamma_2(t) = (1 + t) - iT,\\
\gamma_3 &: [-T, T] \to \mathbb{C}, \gamma_3(t) = 1 - \delta + it,\\
\gamma_4 &: [0, \delta] \to \mathbb{C}, \gamma_4(t) = (1 - \delta + t) + iT,\\
\gamma_5 &: [T, \infty) \to \mathbb{C}, \gamma_5(t) = 1 + it.
\end{aligned}$$

Now $F(s)$ has a simple pole at $s = 1$. In fact, by Corollary 3.10, we know that $\zeta(s) = 1/(s-1) + H(s)$, where $H(s)$ is regular near $s = 1$. Hence $-\zeta'(s)/\zeta(s) = 1/(s-1) + h(s)$, where $h(s)$ is holomorphic near $s = 1$, and so the residue $F(s)$ at $s = 1$ equals $x^2/2$. As a result

$$\psi_1(x) = \frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds = \frac{x^2}{2} + \frac{1}{2\pi i}\int_{\gamma(T,\delta)} F(s)ds.$$

We now decompose the contour $\gamma(T, \delta)$ as $\gamma_1 + \gamma_2 + \cdots + \gamma_5$ and estimate each of the integrals $\int_{\gamma_j} F(s)ds$.

(γ_1, γ_5) For sufficiently large T, we claim that $|\int_{\gamma_1} F(s)ds| \leq \frac{\varepsilon}{2}x^2$. We first note that for $s \in \gamma_1$ we have $|x^{1+s}| = x^{1+\sigma} = x^2$. By Propositions 3.11 and 4.7, we know that $|\zeta'(s)/\zeta(s)| \leq A|t|^{1/2}$, so

$$\left| \int_{\gamma_1} F(s)ds \right| \leq Cx^2 \int_T^\infty \frac{|t|^{1/2}}{t^2} dt.$$

Since the integral converges, we can make the right-hand side smaller than $\varepsilon x^2/2$ upon taking T sufficiently large. The argument over γ_5 is the same.

(γ_3) Having now fixed T, we choose δ appropriately small. On γ_3 note that

$$|x^{1+s}| = x^{1+1-\delta} = x^{2-\delta},$$

from which we conclude that there exists a constant C_T such that

$$\left| \int_{\gamma_3} F(s)ds \right| \leq C_T x^{2-\delta}.$$

(γ_2, γ_4) Finally, on the small horizontal segment γ_2 (and similarly on γ_4), we can estimate the integral as follows

$$\left| \int_{\gamma_2} F(s)ds \right| \leq C_T' \int_{1-\delta}^{\delta} x^{1+\sigma} d\sigma \leq C_T' \frac{x^2}{\log x}.$$

We conclude that there exist constants C_T and C_T' such that

$$\left| \psi_1(x) - \frac{x^2}{2} \right| \leq \varepsilon x^2 + C_T x^{2-\delta} + C_T' \frac{x^2}{\log x}.$$

Dividing by $x^2/2$, we see that

$$\left| \frac{2\psi_1(x)}{x^2} - 1 \right| \leq 2\varepsilon + 2C_T x^{-\delta} + 2C_T' \frac{1}{\log x}.$$

We observe that for all large x we have

$$\left| \frac{2\psi_1(x)}{x^2} - 1 \right| \leq 4\varepsilon.$$

This conclude the proof that $\psi_1(x) \sim x^2/2$, which gives us the prime number theorem. $\square$