THE REGULARITY AND RIGIDITY OF THE BURGERS' EQUATION

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Abstract. We introduce several regularity and rigidity results for genuinely nonlinear scalar conservation laws, in particular the Burgers' equation. We first prove that entropy solutions to the Burgers' equation are of bounded variation for all $t > 0$. For generalized solutions of scalar conservation laws, we obtain a regularity result with Littlewood-Paley decomposition and velocity averaging techniques. Finally, we obtain two additional properties of solutions to the Burgers' equation: an Onsager-type rigidity result, and a dispersion result using the Hopf-Lax formula.

Contents

1. Introduction

We study entropy solutions to scalar conservation laws

$$u_t + \nabla \cdot f(u) = 0 \text{ in } (0, \infty) \times \mathbb{R}^d. \tag{1.1}$$

Here $f : \mathbb{R} \to \mathbb{R}^d$ is a given function, known as the flux. We use the standard notation $a(v) = f'(v)$. In this paper, we draw special attention to the most important scalar conservation law, Burgers' equation with $d = 1$

$$u_t + uu_x = 0, \tag{1.2}$$

for which we take $f(v) = \frac{1}{2}v^2$ to be the flux.

We will first motivate the scalar conservation law (1.1) by deriving the 1D equation from a traffic flow problem. Consider a very long highway. The position of the cars along the highway is parametrized by a spatial variable $x \in \mathbb{R}$. Suppose the cars are all driving to the right. We take the spatial density of cars at position x and time t to be $u(t,x)$. The number of cars between points a and b is approximately given by the integral

$$\int_a^b u(t,x)\,\mathrm{d}x.$$

We make the assumption that the average velocity of the cars depends on the density, as given by function $g(u)$. We can express the change in the number of cars between points a and b as the flux through point a minus the flux through point b. As the flux through a point is given by the density times the velocity, we obtain the following integral equation

$$\partial_t \int_a^b u(t,x)\,\mathrm{d}x = g(u(t,a))u(t,a) - g(u(t,b))u(t,b).$$

If we let $f(u) \coloneqq ug(u)$, we obtain the simpler identity

$$\partial_t \int_a^b u(t,x)\,\mathrm{d}x = f(u(t,a)) - f(u(t,b)) = -\int_a^b \partial_x f(u(t,x))\,\mathrm{d}x.$$

As the identity holds for all intervals $[a,b]$, we conclude

$$u_t + \partial_x f(u) = 0.$$

This is the 1D case of the scalar conservation law equation (1.1).

Higher-dimensional scalar conservation law equations appear in physics. For example, the continuity equation

$$\frac{\partial u}{\partial t} + \nabla \cdot (uV) = 0$$

is a scalar conservation law with flux $f(u,x) = uV(x)$. The continuity equation appears in electromagnetism and fluid mechanics like in the Euler systems of equations. The equation becomes linear, and therefore a transport equation, if the velocity field is divergence free. For constant coefficient transport equations, the initial data is transported along characteristic curves of slope $-1/V$, so there is no gain in regularity for solutions to linear conservation laws. By contrast, for solutions to nonlinear conservation laws, there can be a gain in regularity, and in fact a more nonlinear flux yields a greater gain in regularity.

In general, classical solutions to scalar conservation laws (1.1) do not exist beyond a small time interval and weak solutions are not unique (see [11]), so we will consider entropy solutions to scalar conservation laws instead. Entropy solutions are weak solutions with an added time-irreversible entropy condition. Informally, the entropy condition ensures no information is created or destroyed at shocks, allowing us to recover uniqueness.

When defining entropy solutions, it is more convenient to consider the time-independent scalar conservation law

$$a(u) \cdot \nabla u = 0. \tag{1.3}$$

There is no loss of generality when compared with (1.1), as we may take $\tilde{a} : \mathbb{R} \to \mathbb{R}^{d+1}$ given by $\tilde{a}(t, x_1, \ldots, x_d) = (1, a(t,x))$. Then a solution to (1.3) with flux $\tilde{a}$ is a solution to (1.1) with flux a.

Now we recount the standard formal definition of entropy solutions and subsolutions for (1.3). There are other equivalent definitions, but this definition is the most useful one for proving the theorems in this paper.

Definition 1.4. For every convex function $\eta : \mathbb{R} \to \mathbb{R}$, let $q : \mathbb{R} \to \mathbb{R}^d$ be a function such that

$$q_i'(v) = \eta'(v) a_i(v)$$

for $i = 1, \dots, d$. We say that (η, q) forms an *entropy-entropy flux pair.*

Definition 1.5. Let $u : \mathbb{R}^d \to \mathbb{R}$ belong to $L^\infty(\mathbb{R}^d)$. We say that u is an *entropy solution* to $a(u) \cdot \nabla u = 0$ if for every convex function η, and any nonnegative $\varphi \in C_c^\infty$, we have

$$\int_{\mathbb{R}^d} q(u) \cdot \nabla \varphi \, \mathrm{d}x \geq 0. \tag{1.6}$$

When (1.6) holds only for η convex and non-decreasing, we say u is an *entropy subsolution.*

Taking $\eta = \pm 1$, we see that every entropy solution is a weak solution, and entropy solutions to (1.1) satisfy both existence and uniqueness in $L^\infty([0, \infty) \times \mathbb{R}^d) \cap C([0, \infty), L^1_{\text{loc}}(\mathbb{R}^d))$ for initial data $u(0, x) = u_0 \in L^\infty(\mathbb{R}^d)$ (see [7]). From this point onward until we reach Section 6, we will assume $u_0 \in L^\infty$.

In this work, we will obtain several regularity and rigidity results for scalar conservation laws, in particular Burgers' equation. The first result will tell us that entropy solutions to Burgers' are of bounded variation. The space of bounded variation functions can be characterized in a number of equivalent manners. Some of the most useful definitions are given below.

Definition 1.7. A function $f : \mathbb{R} \to \mathbb{R}$ lies in $BV(\mathbb{R})$ if any of the following equivalent definitions hold.

(1) The function f can be written as $f = f_1 - f_2$ where f_1, f_2 are monotone increasing functions.
(2) Every distributional derivative of u is a measure with finite total variation.
(3) The norm $\|f\|_{BV} < \infty$, where the BV norm is given by

$$\|f\|_{BV} = \|f\|_{L^1} + \sup_{h \in \mathbb{R}} \frac{\|f - f(\cdot - h)\|}{|h|}.$$

However, solutions to Burgers' can develop shocks (discontinuities) in finite time. For example, consider initial data

$$u_0(x) = \begin{cases} 1 & \text{if } x < 0, \\ 1 - x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x > 1. \end{cases}$$

The method of characteristics tells us that the position x satisfies the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u(t, x(t)),$$

if and only if the function u satisfies the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t, x) = u_t(t, x) + \frac{\mathrm{d}x}{\mathrm{d}t} u_x(t, x) = u_t + u u_x = 0,$$

i.e. the characteristics are lines of slope u_0. While u_0 is continuous, the solution u to the Burgers' equation with initial data u_0 develops a shock at $t = 1$, as the characteristic curve $x = t$ emanating from $(0, 0)$ intersects the characteristic curve $x = 1$ emanating fron $(0, 1)$.

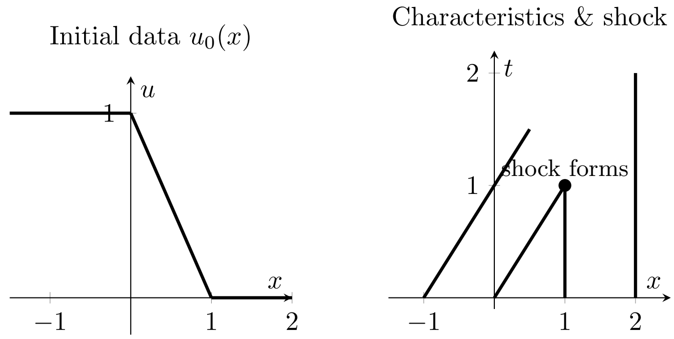

FIGURE 1. Burgers equation $u_t + \frac{1}{2}(u^2)_x = 0$. Left: initial data $u_0(x)$. Right: characteristics in (x, t), meeting at $(1, 1)$.

Thus, solutions to Burgers' cannot be any better than BV for all $t > 0$ in the sense of Sobolev spaces, as any better Sobolev spaces contains $W^{1,1}$, the space of absolutely continuous functions.

However, any solution to Burgers' with bounded initial data will become bounded variation for positive time.

Proposition 1.8. *Let $u : \mathbb{R} \to \mathbb{R}$ be an entropy solution to*

$$\begin{cases} u_t + uu_x = 0 & t > 0, \\ u(0, \cdot) = u_0 \in L^\infty(\mathbb{R}). \end{cases}$$

Then $u \in BV(\mathbb{R})$ for $t > 0$ and furthermore $u \in W^{s,p}(\mathbb{R})$ for all $s < 1/p$ and $0 < s < 1$.[1]

Thus, there is a meaningful sense that BV is the most natural space in which to study entropy solutions to the Burgers' equation.

We can generalize this sort of regularity result to a more general class of conservation laws and a more general notion of solution. To do so, we introduce the kinetic formulation of a scalar conservation law. We use the entropy-flux pairs (η, q) given in Definition 1.4 of entropy solutions. First we define the kinetic function.

Definition 1.9. Consider a function $u(x)$. The kinetic function $\chi(x, v)$ is given by

$$\chi(x, v) := \begin{cases} 1 & \text{if } 0 \le v < u(x), \\ -1 & \text{if } u(x) \le v < 0, \\ 0 & \text{otherwise.} \end{cases}$$

We then have the following equivalence between the kinetic formulation and the standard definition of entropy solutions (the proof is given in Section 2).

Proposition 1.10. *A function $u : \mathbb{R}^d \to \mathbb{R}$ is an entropy solution of $a(u) \cdot \nabla u = 0$ for $x \in \Omega$ if and only if there exists a nonnegative Borel measure m in $\Omega \times \mathbb{R}$ such that*

$$a(v) \cdot \nabla \chi = \partial_v m, \tag{1.11}$$

We call m the entropy dissipation measure. Moreover, the following relations hold

[1] See page 7 for the definition of fractional Sobolev spaces

(1) $u = \int \chi \, \mathrm{d}v$.
(2) *For any entropy pair* (η, q), $\mu = -\nabla \cdot q(u)$ *is equal to the integral of* m *against* $\eta''(v)$ *with respect to* v, *i.e. for any test function* $\varphi(x)$,
$$-\int \nabla \cdot q(u)\varphi \, \mathrm{d}x \, \mathrm{d}t = \iint \eta''(v)\varphi(x) \, \mathrm{d}m.$$

The kinetic formulation of scalar conservation laws then motivates a notion of generalized solutions.

Definition 1.12. We say u is a generalized solution to (1.3) if and only if $\partial_t \chi + a(v)\chi_x = \partial_v m$ for some signed measure m.

Generalized solutions of scalar conservation laws can be useful in the theory of PDEs. For example, the kinetic fomulation of the isentropic Euler equation relies on a more general class of solutions to the Burgers' equation than entropy solutions.

Using velocity averaging techniques, we can prove a similar result to Proposition 1.8 for generalized solutions to genuinely nonlinear scalar conservation laws. Informally, a scalar conservation law as in (1.3) is considered genuinely nonlinear if the function a does not concentrate on any hyperplane (the formal definition is given at the beginning of Section 3). Heuristically, we can understand the usefulness of genuine nonlinearity as follows. If we take a Fourier transform of the kinetic equation (1.11), we obtain
$$a(v) \cdot i\xi\hat{\chi}(\xi, v) = \partial_v(\hat{m}(\xi, v)),$$
With $\hat{\xi} = \xi/|\xi|$, genuine nonlinearity gives us that $|a(v) \cdot \hat{\xi}| > \delta$ except on an exceptional set of small measure, precisely $O(\delta^\alpha)$ for fixed α. Accordingly, we can write
$$|a(v) \cdot \delta|\hat{\chi}(\xi, v) = |\partial_v(\hat{m}(\xi, v))| > c|\xi|\hat{\chi}(\xi, v)\mathbf{1}_{G(v,\xi)} + O(\delta|\hat{\chi}(\xi, v)|)\mathbf{1}_{B(v,\xi)}$$
where G is the good set on which the symbol is elliptic and B is the remaining bad set. The bad term is controlled quite tightly by the genuine nonlinearity condition, so we almost already have a Sobolev bound. By averaging, letting
$$\hat{u}(\xi) = \int \hat{\chi}(\xi, v) \, \mathrm{d}v,$$
we blur out the bad set. As ellipticity of the symbol implies Sobolev space regularity (see for example the fractional Laplacian, in which case $(-\Delta)^s u \in L^2$ implies $u \in H^{2s}$), we obtain the following Sobolev space regularity result for u.

Theorem 1.13. *Let* $u : \mathbb{R}^d \to \mathbb{R}$ *be a generalized solution to*
$$a(u) \cdot \nabla u = 0 \text{ in } B_2.$$
Then $u \in W^{s,r}(B_1)$ *for all* $s < \theta$ *and* $1/r = (1+\theta)/2$, *where* $\theta = \alpha/(2+\alpha)$ *and* α *is the order of genuine nonlinearity. Furthermore, an inequality holds*
$$\|u\|_{W^{s,r}(B_1)} \leq C \|u\|_{L^1(B_2)}^{(1+\theta)/2}$$
for a constant C *that depends on dimension* d, *the order of genuine nonlinearity* α, *and* $\|u\|_{L^\infty}$.

This theorem is in some sense optimal in regularity for generalized solutions, but is suboptimal in integrability for generalized solutions, and is not optimal for entropy solutions to equations such as Burgers. As Burgers' equation has order of genuine nonlinearity $\alpha = 1$, Theorem 1.13 yields the following corollary.

Corollary 1.14. *Let $u : \mathbb{R} \to \mathbb{R}$ be a generalized solution to*

$$u_t + uu_x = 0 \text{ in } B_2.$$

Then $u \in W^{s,3/2}(\mathbb{R})$ for all $s < 1/3$.

This corollary motivates the Onsager-type rigidity result we will prove for Burgers, which tells us that if u has regularity past a certain threshold ($u \in W^{1/3,3}$), then u becomes Lipschitz continuous.

Theorem 1.15. *Let $\Omega = I \times J$ with $I, J \subset \mathbb{R}$ two intervals and $u \in L^4(\Omega)$ be a distributional solution of (1.2) which belongs to the space $L^3(I, W^{1/3,3}(J))$, namely*

$$\int_I \int_{J\times J} \frac{|u(t,x) - u(t,\xi)|^3}{|x-\xi|^2} \, \mathrm{d}x \, \mathrm{d}\xi \, \mathrm{d}t < \infty.$$

Then u is locally Lipschitz.

For both regularity and rigidity, $s = 1/3$ is the critical order of regularity. More specifically, we have that $u \in W^{s,3}$ for all $s < 1/3$, and if $u \in W^{1/3,3}$ then u is Lipschitz, which is slightly stronger than $u \in BV$ for entropy solutions. Thus generalized solutions either have $1/3$ derivative in L^3 or a full derivative in L^3.

The criticality of the exponent $1/3$ is also inspired by the Onsager conjecture for the Euler equation: for $\alpha > 1/3$, any weak solution to the Euler equations $u \in C^0_t C^{0,\alpha}$ obeys a form of energy conservation (see [1] for the full statement and proof).

Our final result concerns the dispersive nature of the Burgers' equation. We will use the relation of the Hamilton-Jacobi equation to the Burgers' equation and the Hopf-Lax representation formula to show that entropy solutions to Burgers' are essentially bounded by the L^1 norm of the initial data.

Theorem 1.16. *Let $u : \mathbb{R} \to \mathbb{R}$ be an entropy solution to (1.2). Then for all $t > 0$, we have $\|u(t,\cdot)\|_\infty \le Ct^{-1/2} \|u_0\|_1$ for some absolute constant C.*

This theorem allows us to extend existence and uniqueness of entropy solutions to settings in which the initial data is only L^1, not L^∞. Note that these last two theorems both rely upon an important connection between scalar conservation laws and the Hamilton-Jacobi equation. Consider the one-dimensional Hamilton-Jacobi equation

$$h_t + f(h_x) = 0 \tag{1.17}$$

In one dimension, solutions to conservation law equations are spatial derivatives of solutions to the Hamilton-Jacobi equation in the following sense: let $u_0 \in L^\infty$ and define

$$h_0(x) := \int_0^x u_0(y) \, \mathrm{d}y.$$

If h solves the Hamilton-Jacobi equation (1.17) with initial condition h_0, we can differentiate with respect to the spatial variable x to get that $u = h_x$ solves the corresponding conservation law equation (1.1) with initial condition u_0.

2. The Burgers' Equation and Bounded Variation Functions

In this section, we prove that entropy solutions to the Burgers' equation (1.2) are of bounded variation for all $t > 0$. This is the optimal regularity result for the Burgers' equation, and relates nicely to a regularity result in terms of fractional Sobolev spaces. The proof of the proposition relies on the vanishing viscosity method. Let u_ε solve

$$\begin{cases} \partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon = \varepsilon \partial_{xx} u_\varepsilon & t > 0, \\ u_\varepsilon(0, \cdot) = u_0 \in L^\infty(\mathbb{R}). \end{cases} \tag{2.1}$$

We call this equation a viscous conservation law. Solutions to (2.1) always exist, because (2.1) is an inhomogenous heat equation (see [11],[7]). Furthermore if u is the unique entropy solution to (1.2), then $u_\varepsilon \to u$ in L^1 as $\varepsilon \to 0$ (see [7]).

We also require a comparison principle for a particular form of inhomogenous heat equations, given below. This is a classical result, but the proof is brief and given here.

Lemma 2.2. *Consider the equation*

$$\begin{cases} u_t + u^2 + f(x,t) \cdot \nabla u = \varepsilon \Delta u & t > 0 \\ u(0, x) = u_0 \in L^\infty \end{cases} \tag{2.3}$$

for some smooth function $f : \mathbb{R}^{d+1} \to \mathbb{R}$. If both v and w solve (2.3) and for some $t_0 > 0$, $v_0 \leq w_0$ for all $x \in \mathbb{R}^d$ and $t < t_0$, then $v \leq w$ for all x and t.

Proof. Let $g(t) := \min_x [w(t,x) - v(t,x)]$. At the minimum, $\nabla w - \nabla v = 0$ and $\Delta w - \Delta v \geq 0$. Thus, we have that

$$\begin{aligned} g'(t) &= w_t(t,x) - v_t(t,x) \\ &= v^2 - w^2 + f(x,t) \cdot (\nabla v - \nabla u) + \varepsilon(\Delta u - \Delta v) \\ &\geq -g(t)(v + w) \end{aligned}$$

If we restrict to an arbitrary time interval $[0, T]$, v and w are bounded, so we conclude

$$\frac{g'(t)}{g(t)} \geq -C(T)$$

We then solve the differential inequality. We obtain

$$\min_x [w(t,x) - v(t,x)] = g(t) \geq e^{-C(T)t} > 0$$

Thus $v(t,x) \leq w(t,x)$ for all x and t. □

Now we can prove the main proposition.

Proposition 2.4. *Let u be an entropy solution to*

$$\begin{cases} u_t + u u_x = 0 & t > 0, \\ u(0, \cdot) = u_0 \in L^\infty(\mathbb{R}). \end{cases}$$

Then $u \in BV$ for $t > 0$.

Proof. Let u_ε solve the viscous conservation law (2.1). Take $v_\epsilon = \partial_x u_\epsilon$. Differentiating (2.1) with respect to x, we obtain

$$\partial_t v_\epsilon + v_\epsilon^2 + u_\epsilon \partial_x v_\epsilon = \epsilon \partial_{xx} v_\epsilon.$$

Observe that the function $w(t,x) = \frac{1}{t}$ solves the same equation with initial data $w(0,x) = +\infty$. However, $u_0 \in L^\infty$, so v is bounded near zero, and there exists some t_0 close to zero such that $v(t_0,x) \leq w(t_0)$. We also have that $v(t,x) \leq 1/t$ for all $t < t_0$. Therefore, we may restart the equation at t_0 and apply the comparison principle given above to see that $v \leq w$ for all $t > t_0$.

The above estimate implies that u_ϵ is uniformly across ϵ in BV, since we can write

$$u = \left(u - \frac{x}{t}\right) + \frac{x}{t}$$

as the difference of two monotonically increasing functions. Thus, as the space of bounded variation functions is closed in L^1, we may conclude that $u \in BV$ for $t > 0$ as $u_\varepsilon \to u$ in L^1. □

Remark 2.5. Using the dispersion result from Section 6, one can show that Proposition 2.4 holds for any initial data $u_0 \in L^1$.

We can rewrite this regularity result in terms of fractional Sobolev spaces. This step is important, because it motivates the sort of regularity result that we will prove more generally for generalized solutions to genuinely nonlinear scalar conservation laws in Section 4. We recall the definition of a fractional Sobolev space.

Definition 2.6. Given $s \in (0,1)$ and $1 \leq p < \infty$, we define the fractional Sobelev semi-norm

$$[u]_{W^{s,p}} = \iint \frac{|u(x) - u(y)|^p}{|x-y|^{1+sp}} \,\mathrm{d}x\,\mathrm{d}y.$$

We define the fractional Sobolev norm as $\|u\|_{W^{s,p}} = \|u\|_p + [u]_{W^{s,p}}$. We say that $u \in W^{s,p}$ if and only if $\|u\|_{W^{s,p}} < \infty$.

We then have the following inclusions of fractional Sobolev spaces into BV given below (for a proof of these inclusions, see Appendix A).

Proposition 2.7. *We have that $BV \subset W^{s,p}$ for all $p \in [1,\infty)$ and $s < 1/p$.*

We thus obtain the following corollary.

Corollary 2.8. *Let u be an entropy solution to*

$$\begin{cases} u_t + uu_x = 0 & t > 0, \\ u(0,\cdot) = u_0 \in L^\infty(\mathbb{R}). \end{cases}$$

Then $u \in W^{s,p}$ for all $s < 1/p$.

3. The Kinetic Formulation

We prove the equivalence of the kinetic formulation to the standard definition of the entropy solutions for the time-independent scalar conservation law (1.3). The kinetic formulation expresses the scalar conservation law as a kinetic equation, which is useful for proving regularity results.

Proposition 3.1. *A function $u : \mathbb{R}^d \to \mathbb{R}$ is an entropy solution of $a(u) \cdot \nabla u = 0$ for $x \in \Omega$ if and only if there exists a nonnegative Borel measure m in $\Omega \times \mathbb{R}$ such that*

$$a(v) \cdot \nabla \chi = \partial_v m, \tag{3.2}$$

We call m the entropy dissipation measure. Moreover, the following relations hold

(1) $u = \int \chi \, \mathrm{d}v$.
(2) *For any entropy pair* (η, q), $\mu = -\nabla \cdot q(u)$ *is equal to the integral of* m *against* $\eta''(v)$ *with respect to* v, *i.e. for any test function* $\varphi(x)$,

$$-\int \nabla \cdot q(u)\varphi \, \mathrm{d}x \, \mathrm{d}t = \iint \eta''(v)\varphi(x) \, \mathrm{d}m.$$

Proof. Fix a flux $a(v)$. For any convex η, let (η, q) be an entropy pair. Now let the bilinear functional F be given by

$$F(\eta, \varphi) := \int q(u) \cdot \nabla\varphi \, \mathrm{d}x$$

If u is an entropy solution $a(u) \cdot \nabla u$, then $F(\eta, \varphi) \geq 0$ for all convex η and $\varphi \in C_c^\infty$. Thus $F(\eta, \cdot)$ is a positive linear functional on C_c^∞, so by the Riesz representation theorem, there exists some positive Borel measure m_η such that

$$F(\eta, \varphi) = \int \varphi \, \mathrm{d}m_\eta.$$

Using integration by parts, we can rewrite

$$\begin{aligned} F(\eta, \varphi) &= \int q(u) \cdot \nabla\varphi \, \mathrm{d}x = \iint q'(v)\chi \cdot \nabla\varphi \, \mathrm{d}v \, \mathrm{d}x \\ &= \iint \eta'(v)a(v)\chi \cdot \nabla\varphi \, \mathrm{d}v \, \mathrm{d}x = -\iint \varphi\eta'(v)a(v) \cdot \nabla\chi \, \mathrm{d}v \, \mathrm{d}x \end{aligned}$$

and

$$F(\eta, \varphi) = \int \varphi \, \mathrm{d}m_\eta = -\int \varphi v \partial_v m_\eta \, \mathrm{d}v \, \mathrm{d}x.$$

If we take $\eta(v) = \frac{1}{2}v^2$, and let $m = m_\eta$, we have that for all $\varphi \in C_c^\infty$,

$$-\iint \varphi v a(v) \cdot \nabla\chi \, \mathrm{d}v \, \mathrm{d}x = -\iint \varphi v \partial_v m \, \mathrm{d}v \, \mathrm{d}x$$

Thus, $a(v) \cdot \nabla\chi = \partial_v m$ for some nonnegative Borel measure m, so the kinetic formulation holds. Conversely, if $a(v) \cdot \nabla\chi = \partial_v m$ for nonnegative Borel m, we may conclude that exists a nonnegative Borel measure $m_\eta := \eta''(v)m$ such that

$$F(\eta, \varphi) = \int \varphi \, \mathrm{d}m_\eta$$

so $F(\eta, \varphi)$ is a nonnegative functional, and therefore u is an entropy solution to $a(u) \cdot \nabla u = 0$.

We will now prove (1) and (2). Identity (1) is immediate. If $u \geq 0$, then

$$\int \chi \, \mathrm{d}v = \int_0^u 1 \, \mathrm{d}v = u.$$

If $u \leq 0$, then

$$\int \chi \, \mathrm{d}v = \int_u^0 -1 \, \mathrm{d}v = u.$$

Identity (2) is also immediate, as by definition of m,

$$\int q(u) \cdot \nabla\varphi \, \mathrm{d}x = \int \eta''(u)\varphi \, \mathrm{d}m$$

so after integration by parts

$$-\int \nabla \cdot q(u)\varphi \, \mathrm{d}x = \iint \eta''(u)\varphi \, \mathrm{d}m.$$

□

As an example, we compute the entropy dissipation measure for the Riemann problem for Burgers' equation, which concerns piecewise constant initial data

$$u_0(x) = \begin{cases} u_L & \text{if } x \le 0, \\ u_R & \text{if } x > 0. \end{cases} \tag{3.3}$$

The entropy solution to the Riemann problem for $u_R \le u_L$ is the traveling shock

$$u(x,t) = \begin{cases} u_L & \text{if } x \le ct, \\ u_R & \text{if } x > ct. \end{cases}, \quad c = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{1}{2}(u_L + u_R) \tag{3.4}$$

where c is given by the Rankine-Hugoniot condition (see [11]).

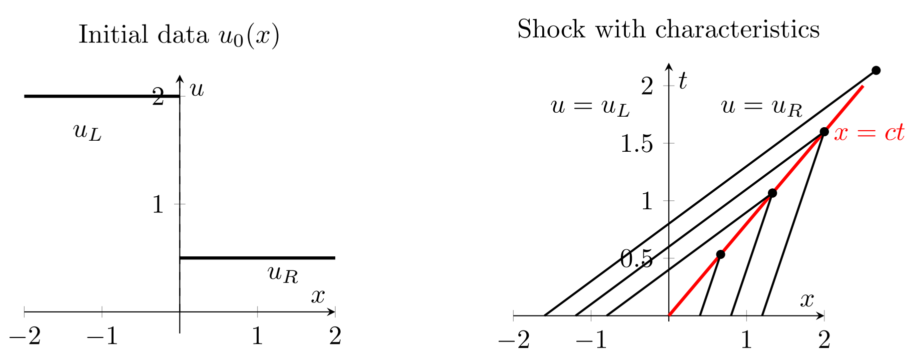

FIGURE 2. Riemann problem with $u_R < u_L$. Left: initial discontinuous data. Right: characteristics from both sides intersecting with the shock $x = ct$.

To calculate the entropy dissipation measure, take $\eta(v) = \frac{1}{2}v^2$. If we consider time as a variable, for Burgers' equation $a(v) = (1, v)$. Thus $q'(v) = a(v)\eta'(v) = (v, v^2)$. Integrating against a test function, we have

$$\begin{aligned}
\iint \varphi \, \mathrm{d}m &= -\int \nabla \cdot q(u)\varphi \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_{\{x=ct\}} (q(u_L) - q(u_R)) \cdot \nu\varphi \, \mathrm{d}\sigma \quad \text{(by the divergence theorem)} \\
&= \int \begin{pmatrix} u_L^2/2 - u_R^2/2 \\ u_L^3/3 - u_R^3/3 \end{pmatrix} \cdot \frac{1}{\sqrt{c^2+1}} \begin{pmatrix} -c \\ 1 \end{pmatrix} \varphi(t, ct) \cdot \sqrt{1+c^2} \, \mathrm{d}t \text{ (CVF)} \\
&= \int \frac{1}{12}(u_L - u_R)^3 \varphi(t, ct) \, \mathrm{d}t.
\end{aligned}$$

Here we can explicitly compute the measure

$$m = \frac{(u_L - u_R)^3}{12}\delta(x - ct), \tag{3.5}$$

so $m \geq 0$ implies that an entropy solution to the Riemann problem satisfies $u_L \geq u_R$ at a shock. For 1D conservation laws, the contribution to the entropy at a shock of a BV weak solution (in the *time-independent* formulation) is given by

$$-c(\eta(u_R) - \eta(u_L)) + q(u_r) - q(u_L) \text{ (see [3, Section 8.5] for details.)}$$

Conversely, the entropy solution to the Riemann problem for $u_R > u_L$ is the rarefaction wave

$$u(t,x) = \begin{cases} u_L & \text{if } x \leq u_L t \\ u_L + \frac{x - u_L t}{(u_R - u_L)t} & \text{if } u_L t < x \leq u_R t \\ u_R & \text{if } x > u_R t \end{cases}. \tag{3.6}$$

For Lipchitz solutions such as the rarefaction wave when $u_R \leq u_L$, we have a more general theorem that the entropy dissipation measure vanishes.

Theorem 3.7. *Let u be an entropy solution to the conservation law $a(u) \cdot \nabla u = 0$. If $u \in C^{1/2+\varepsilon}$ for any $\varepsilon > 0$, then the kinetic entropy dissipation measure vanishes.*

The theorem admits straightforward measure-theoretic proof from the following bounding lemma.

Lemma 3.8. *Let $u : B_R \to \mathbb{R}$ be an entropy solution to the conservation law $a(u) \cdot \nabla u = 0$. For $r < R$*

$$m(B_r \times \mathbb{R}) \lesssim \frac{(\max |a|)}{R - r} \int_{B_R} |u|^2 \, \mathrm{d}x.$$

Proof. We take $\phi \in C_c^\infty(B_R)$ with $0 \leq \phi \leq 1$ such that $\phi \equiv 1$ on B_r and $|\nabla \phi| \leq \frac{C}{R-r}$. We then multiply $a(v) \cdot \nabla \chi(x,v) = \partial_v m$ by $-v\phi(x)$ and integrate for

$$\begin{aligned} m(B_r \times \mathbb{R}) &\leq \int_{B_R \times \mathbb{R}} \phi \, m(\mathrm{d}x, \mathrm{d}v) = \int_{B_R \times \mathbb{R}} -(\partial_v m)\phi v \quad \text{(IBP in } v\text{)} \\ &= -\int_{B_R \times \mathbb{R}} a(v) \cdot \nabla_x \chi(x,v) \phi v \, \mathrm{d}x \, \mathrm{d}v \\ &= \int_{B_R \times \mathbb{R}} a(v) \cdot \chi(x,v) v \nabla \phi \, \mathrm{d}x \, \mathrm{d}v \quad \text{(IBP in } x\text{)} \\ &\leq (\max |a|) \cdot \frac{C}{R - r} \int_{B_R} \left(\int_{\mathbb{R}} \chi(x,v) v \, \mathrm{d}v \right) \mathrm{d}x \\ &= \frac{C(\max |a|)}{R - r} \int_{B_R} |u|^2 \, \mathrm{d}x. \end{aligned}$$

□

Proof of Theorem 3.7. We will begin by bounding from above the measure of a ball of radius r using the bound from Lemma 3.8. As $u \in C^{1/2+\varepsilon}$, there exists $K \in \mathbb{R}$ such that $|u(x) - u(y)| \leq K|x - y|^{1/2+\varepsilon}$. We thus have that

$$m(B_r \times \mathbb{R}) \lesssim \frac{(\max |a|)}{R - r} \int_{B_R} |u|^2 dx \lesssim \frac{(\max |a|)}{R - r} \int_{B_R} (R^{1/2+\varepsilon})^2 dx \lesssim \frac{R^{1+2\varepsilon} R^d}{R - r}$$

Now we take $R = 2r > r$. We have that

$$m(B_r \times \mathbb{R}) \lesssim \frac{R^{1+2\varepsilon} R^d}{R - r} = \frac{(2r)^{1+2\varepsilon}(2r)^d}{r} \sim r^{d+2\varepsilon}$$

Take any set $A \subset \mathbb{R}^n$ of finite Lebesgue measure. Now fix some r. By the Besicovitch covering theorem, there exists some $c(d)$ collections of disjoint balls such that $A \subseteq \bigcup_{k=1}^{c(d)} \bigsqcup_{i=1}^{N(k)} B_r(x_{k,i})$ where $c(d)$ is the Besicovitch constant in dimension d, and $N(k)$, possibly infinite, is the number of balls in collection k. For any given collection, we have that

$$N(k) r^d = \sum_{i=1}^{N(k)} \mu(B_r(x_{k,i})) = \mu\left(\bigcup_{i=1}^{N(k)} B_r(x_{k,i})\right) \leq \mu(A)$$

Thus $N(k) \leq \mu(A)/r^d$. Summing over the k's, we get

$$m(A \times \mathbb{R}) \leq m\left(\bigcup_{k=1}^{c(d)} \bigsqcup_{i=1}^{N(k)} B_r(x_{k,i}) \times \mathbb{R}\right) \leq \sum_{k=1}^{c(d)} \sum_{i=1}^{N(k)} m(B_r(x_{k,i}) \times \mathbb{R})$$

$$\lesssim c(d) \frac{\mu(A)}{r^d} r^{d+2\varepsilon} \lesssim r^{2\varepsilon}$$

However, r was arbitrary, so we have that

$$m(A \times \mathbb{R}) \lesssim \lim_{r \to 0} r^{2\varepsilon} = 0$$

Then as A was arbitrary and $\mathbb{R}^d$ is σ-finite, we may conclude that $m(A \times \mathbb{R}) = 0$, so $m = 0$. □

Thus, for all Lipchitz solutions to Burgers, including the rarefaction wave, the entropy dissipation measure vanishes.

Remark 3.9. In fact, if u is a continuous entropy solution to a scalar conservation law, then $m \equiv 0$ (see [10] for proof).

4. Velocity Averaging and Regularity Results

We will apply velocity averaging techniques to the kinetic formulation given in Section 3 to prove a regularity for generalized solutions to genuinely nonlinear conservation laws. As discussed in the introduction, in order to obtain $W^{s,p}$ estimates for the averaged kinetic solution $u = \bar{\chi}$, it is necessary to ensure that $|a(v) \cdot \xi| > \delta$ except on a set of measure $O(\delta^\alpha)$. This motivates the genuinely nonlinearity condition. We recall the formal definition below.

Definition 4.1. We say the equation $a(u) \cdot \nabla u = 0$ is *genuinely nonlinear* with order α if the function a is C^1 and there exists an $\alpha \in (0, 1]$ and $C > 0$ so that for every $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, and any $\delta > 0$, we have

$$|\{v \in \mathcal{I} : |a(v) \cdot \xi| < \delta\}| \leq C\delta^\alpha.$$

Here $\mathcal{I}$ is a closed interval containing the image of u.

In light of the kinetic formulation, we will consider results for the generic transport equation

$$a(v) \cdot \nabla_x f = \partial_v g \tag{4.2}$$

where f and g are compactly supported functions functions or measures. We also define the averaged function $\bar{f}$ as usual

$$\bar{f} = \int f \, \mathrm{d}v.$$

Next, we recount the Littlewood-Paley decomposition from harmonic analysis, which we use to relate the regularity of f to the decay of its Fourier transform $\hat{f}$.

Definition 4.3. Let $\psi_0 : \mathbb{R}^d \to \mathbb{R}$ be given by $\psi_0 = \mathbf{1}_{B_1}$. For $j \in \mathbb{N}$, define $\psi_j(\xi) = \psi_0(2^{-j}\xi) - \psi_0(2^{-j+1}\xi)$. Then for any $f : \mathbb{R}^d \to \mathbb{R}$, we define its *Littlewood-Paley blocks* to be the functions $\Delta_j f$ such that

$$(\Delta_j f)^\wedge = \psi_j(\xi)\hat{f}(\xi)$$

We begin by proving some straightforward lemmata.

Lemma 4.4. *Given $f \in L^2(\mathbb{R}^d)$, we have that*

(1) $f = \sum_{j=0}^\infty (\Delta_j f)$ (with convergence in L^2).
(2) $||f||_{L^2}^2 = \sum_{j=0}^\infty ||\Delta_j f||_{L^2}^2$
(3) If f and g satisfy (4.2) and $f_j = \Delta_j^x f$ and $g_j = \Delta_j^x g$, then

$$a(v) \cdot \nabla_x f_j = \partial_v g_j$$

(4) For any $p \in [1, \infty]$, we have that $||\Delta_j f||_{L^p} \leq C||f||_{L^p}$ with a constant C independent of j.

Proof. By definition of the Littlewood-Paley blocks, we have that

$$\left(\sum_{j=0}^\infty (\Delta_j f) \right)^\wedge = \hat{f}(\xi) \sum_{j=0}^\infty \psi_j(\xi) = \hat{f}(\xi) \lim_{n \to \infty} \psi_0(2^{-n}\xi) = \hat{f}(\xi).$$

By Plancherel's identity,

$$\sum_{j=0}^\infty \|\Delta_j f\|_{L^2}^2 = \sum_{j=0}^\infty \|(\Delta_j f)^\wedge\|_{L^2}^2 = \sum_{j=0}^\infty \|\psi_j \hat{f}\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2 = \|f\|_{L^2}.$$

For any j, we may convolve with $\check{\psi}_j$ on each side to obtain $a(v) \cdot \nabla_x f_j = \partial_v g_j$. By Young's inequality, we have that

$$\|\Delta_j f\|_{L^p} = \|\check{\psi}_j * f\|_{L^p} \leq \|\check{\psi}_j\|_{L^1} \|f\|_{L^p} \leq 2\|\check{\psi}_0\|_{L^1}\|f\|_{L^p}$$

□

We now decompose solutions to (4.2). We first write f in terms of its Littlewood-Paley blocks and then we decompose each Littlewood-Paley block as

$$f_j = \Delta_j f = \sum_{k=0}^\infty f_{jk}$$

where

$$\hat{f}_{jk}(\xi, v) = \psi_k \left(\frac{a(v) \cdot \xi}{\delta_j} \right) \psi_j(\xi) \hat{f}(\xi, v).$$

The advantage of this decomposition is in the support of each f_{jk} we have a lower bound for $|a(v) \cdot \xi|$, allowing us to estimate that norm of the f_{jk} in terms of the g_j. In this spirit, we will use Fourier multiplier lemmata to bound $\bar{f}_{j0}$ and $\bar{f}_{jk}$ terms for all values of j. For the remainder term $\bar{f}_{j0}$, we will need to use the genuine nonlinearity condition. First we prove the following Fourier multiplier lemma.

Lemma 4.5. *Let T_k be the operator whose Fourier multiplier is $\psi_k\left(\frac{a(v)\cdot\xi}{\delta}\right)$. Then T_k is bounded from L^q to L^q uniformly with respect to k, v, δ. Moreover, the norm of T_0 is $\left\|\check{\psi}_0\right\|_{L^1}$ and the norm of T_k is $\left\|\check{\psi}_1\right\|_{L^1}$ for any $k \geq 1$.*

Proof. First observe that for $k \geq 1$,

$$\left\|\check{\psi}_k\right\|_{L^1} = \left\|\check{\psi}_1\right\|_{L^1}.$$

By the inversion formula, we can write

$$\widehat{T_k f}(\xi, v) = \psi_k\left(\frac{a(v)\cdot\xi}{\delta}\right)\hat{f}(\xi, v) = \left(\int_{\mathbb{R}} e^{-2\pi i t\frac{a(v)\cdot\xi}{\delta}}\check{\psi}_k(t)\,\mathrm{d}t\right)\hat{f}(\xi, v).$$

Then by applying Fubini's theorem,

$$\begin{aligned} T_k f(x, v) &= \int_{\mathbb{R}} e^{2\pi i x\cdot\xi}\int_{\mathbb{R}} e^{-2\pi i t\frac{a(v)\cdot\xi}{\delta}}\check{\psi}_k(t)\,\mathrm{d}t\hat{f}(\xi, v)\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}}\check{\psi}_k(t) f\left(x - \frac{a(v)}{\delta}t\right)\mathrm{d}t. \end{aligned}$$

Now we apply Young's inequality to obtain

$$\begin{aligned} \|T_k f\|_{L^q} &\leq \left(\frac{\delta}{a(v)}\right)^d\left\|\check{\psi}_k\left(\frac{\delta}{a(v)}t\right)\right\|_{L^1}\|f\|_{L^q} \\ &\leq \left(\frac{\delta}{a(v)}\right)^d\left|B_{a(v)/\delta}\right|\|f\|_{L^q} = C(d)\|f\|_{L^q} \end{aligned}$$

so T_k is uniformly bounded with respect to k, v, δ. □

We then use this lemma to prove a set of upper bounds for $\bar{f}_{j0}$ using the genuine nonlinearity condition.

Proposition 4.6. *Assume that a is genuinely nonlinear. For any $j \geq 1$, we have the estimates*

(1) $\left\|\bar{f}_{j0}\right\|_{L^2} \leq C\min(1, \delta_j^{\alpha/2}2^{-\alpha j/2})\|f_j\|_{L^2}$
(2) $\left\|\bar{f}_{j0}\right\|_{L^1} \leq C\|f_j\|_{L^1}$
(3) $\left\|\bar{f}_{j0}\right\|_{L^p} \leq C\min(1, \delta_j^{\alpha}2^{-\alpha j})\|f_j\|_{L^p}$

Proof. Begin with item (1). By Plancherel, we have that

$$\begin{aligned} \left\|\bar{f}_{j0}(x)\right\|_{L^2_x} &= \left\|\int\bar{f}_{j0}(x, v)\,\mathrm{d}v\right\|_{L^2_x} \\ &= \left\|\int\psi_0\left(\frac{a(v)\cdot\xi}{\delta_j}\right)\psi_j(\xi)\hat{f}(\xi, v)\,\mathrm{d}v\right\|_{L^2_\xi}. \end{aligned}$$

We will then apply Cauchy-Schwartz in v. We want to bound the support of the above integrand in v. For any $\xi \in \mathbb{R}^d$, write $\xi = |\xi|e$, where e is a unit vector. Let $\mathcal{I}$ be the interval on which f takes its values. Then

$$\begin{aligned} \operatorname{supp} &\subset \{v \in \mathcal{I} : |a(v)\cdot\xi| < \delta_j \text{ and } |\xi| \leq 2^j\} \\ &\subset \{v \in \mathcal{I} : |a(v)\cdot|\xi|e| < \delta_j\} \text{ and } |\xi| \leq 2^j\} \\ \implies |\operatorname{supp}| &\leq C\frac{\delta_j^\alpha}{2^{\alpha j}} \end{aligned}$$

by the genuine nonlinearity condition. Then, by Cauchy-Schwartz, we conclude

$$\left\|\bar{f}_{j0}(x)\right\|_{L^2_x} \leq \left(C\frac{\delta_j^\alpha}{2^{\alpha j}}\right)^{1/2} \left\|\psi_j(\xi)\hat{f}(\xi,v)\right\|_{L^2_v L^2_\xi} = \left(C\frac{\delta_j^\alpha}{2^{\alpha j}}\right)^{1/2} \|f_j\|_{L^2_x L^2_v}.$$

The last inequality follows by definition of f_j and Plancherel.

Now consider item (2). We bound using Young's inequality and Jensen's inequality

$$\begin{aligned}\left\|\bar{f}_{j0}(x,v)\right\|_{L^1_x} &\leq \left\|\check{\psi}_0\right\|_{L^1} \left\|\check{\psi}_j(x) * \bar{f}(x,v)\right\|_{L^1_x} \\ &\leq \left\|\check{\psi}_0\right\|_{L^1} \|f_j(x,v)\|_{L^1_x L^1_v}.\end{aligned}$$

Finally, for item (3), we use Riesz-Thorin interpolation. □

Now we prove another Fourier multiplier lemma.

Lemma 4.7. *For any a in a bounded set, the operators with Fourier multipliers*

$$\left(\frac{a'(v)\cdot\xi}{2^j}\right)\psi_j(\xi) \text{ and } \tilde{\psi}_1\left(\frac{a(v)\cdot\xi}{2^k\delta}\right),$$

are bounded from L^p to L^p, for any $p \in [1,\infty]$, with their norms bounded independently of k, j and δ.

Proof. We can write the first Fourier multiplier operator as

$$T_k f = \left[\left(\frac{a'(v)\cdot\xi}{2^j}\right)\psi_j(\xi)\right]^\vee * f = \left[\frac{2^{jd}}{i}\left(a'(v)\right)\cdot(\check{\psi}_1)'(2^j x)\right] * f.$$

Then apply Young's inequality. After a change of variables, the L^1 norm is bounded by $\left\|a'(v)\cdot(\check{\psi}_1)'\right\|_{L^1}$, which is finite because $\psi_1 \in C^\infty$ is smooth, and $\check{\psi}_1 \in \mathcal{S}$.

For the second multiplier operator, we have

$$\left\|\left[\tilde{\psi}_1\left(\frac{a(v)\cdot\xi}{2^k\delta}\right)^\vee(x)\right]\right\|_{L^1_x} \leq \left\|\tilde{\psi}_1(a(v)\cdot\xi)^\vee(x)\right\|_{L^1_x} < \infty,$$

again since $\tilde{\psi}_1$ is a Schwartz function. □

Now we combine these results to bound the $\bar{f}_{jk}$.

Proposition 4.8. *For any $q \in [1,\infty]$,*

$$\left\|\sum_{k=1}^\infty \bar{f}_{jk}\right\|_{L^q} \leq C\frac{2^j}{\delta_j^2}\|g\|_{L^q}.$$

Here, as before, $\bar{f}_j = \int f_j \,\mathrm{d}v$.

Proof. Let j be fixed. Let k be given. We can write

$$\begin{aligned}(\bar{f}_{jk})^\wedge(\xi) &= i\int \hat{g}(\xi,v)\cdot\psi_j(\xi)\left(\frac{a'(v)\cdot\xi}{2^j}\right)\cdot\tilde{\psi}_1'\left(\frac{a(v)\cdot\xi}{2^k\delta_j}\right)\cdot\frac{2^{j-k}}{2^k\delta_j^2}\,\mathrm{d}v \\ &= i\frac{2^{j-k}}{\delta_j^2}\int \hat{g}(\xi,v)\cdot\psi_j(\xi)\left(\frac{a'(v)\cdot\xi}{2^j}\right)\cdot\tilde{\psi}_1'\left(\frac{a(v)\cdot\xi}{2^k\delta_j}\right)\mathrm{d}v.\end{aligned}$$

Then by Young's inequality and Lemma 4.7 (while replacing $\tilde{\psi}_1$ with $\tilde{\psi}_1'$)

$$\left\|\bar{f}_{jk}\right\|_{L^q} \leq \frac{C2^{j-k}}{\delta_j^2} \|g\|_{L^q}.$$

Then we sum over the geometric series. □

Now we will use a result concerning the interpolation of L^p spaces, a particular case of the K-method of real interpolation which can be found in [8, Chapter 26].

Proposition 4.9. *Let $\varphi : M \to \mathbb{R}$ be any measurable function from a measure space M. Let $1 \leq p < q \leq \infty$. Assume that*

$$\inf_{\varphi_0+\varphi_1=\varphi} (\|\varphi_0\|_{L^p} + t\|\varphi_1\|_{L^q}) \leq C_0 t^\theta, \tag{4.10}$$

then

$$\|\varphi\|_{L^{r,\infty}} := \sup\{\lambda |\{x : \varphi(x) > \lambda\}|^{1/r} : \lambda > 0\} \lesssim C_0,$$

for r given by $\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p}$.

We combine the bounds on $\bar{f}_{jk}$ from Propositions 4.6, 4.8 with the interpolation result from Proposition 4.9 in order to bound $\bar{f}_j$. The critical step in the proof involves picking the proper choice of δ_j in terms of the other parameters.

Proposition 4.11. *We have that for $p \in [1,2]$ and $q \in [1,\infty]$,*

$$\left\|\bar{f}_j\right\|_{L^{r,\infty}} \leq 2^{-j\theta} \|f\|_{L^p}^{1-\theta} \|g\|_{L^q}^\theta$$

where

$$\theta = \frac{\alpha}{2p'+\alpha} \text{ and } \frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p}.$$

Proof. As above, we decompose f_j as

$$f_j = \sum_{k=0}^\infty \bar{f}_{jk}$$

We then take $t := \|f\|_{L^p} / \|g\|_{L^q}$. We have that

$$\inf_{\varphi_0+\varphi_1 \leq \bar{f}_j} (\|\varphi_0\|_{L^p} + t\|\varphi_1\|_{L^q}) \leq \left\|\bar{f}_{j0}\right\|_{L^p} + \frac{\|f\|_{L^p}}{\|g\|_{L^q}} \left\|\sum_{k=1}^\infty \bar{f}_{jk}\right\|_{L^q}$$

Now we apply the bounds from Propositions 4.6 and 4.8

$$\begin{aligned}\inf_{\varphi_0+\varphi_1 \leq \bar{f}_j} (\|\varphi_0\|_{L^p} + t\|\varphi_1\|_{L^q}) &\leq \left\|\bar{f}_{j0}\right\|_{L^p} + \frac{\|f\|_{L^p}}{\|g\|_{L^q}} \left\|\sum_{k=1}^\infty \bar{f}_{jk}\right\|_{L^q} \\ &\leq C_1(\delta_j^\alpha 2^{-\alpha j})^{1/p'} \|f_j\|_{L^p} + C_2 \frac{2^j}{\delta_j^2} \|f\|_{L^q} \\ &\leq \frac{2^{j\theta}}{C_3}\left(C_1(\delta_j^\alpha 2^{-\alpha j})^{1/p'} + C_2\frac{2^j}{\delta_j^2}\right) C_3 2^{-j\theta} \|f\|_{L^p}\end{aligned}$$

Now pick C_3 sufficiently large such that the equation $C_1 x^{\alpha/p'+2} - C_3 x^2 + C_2 \leq 0$ has solutions, and then take

$$\delta_j := 2^{\frac{j(p'+\alpha)}{2p'+\alpha}} \beta$$

where β satisfies $C_1\beta^{\alpha/p'+2} - C_3\beta^2 + C_2 \le 0$. We then have that

$$\frac{2^{j\theta}}{C_3}\left(C_1(\delta_j^\alpha 2^{-\alpha j})^{1/p'} + C_2\frac{2^j}{\delta_j^2}\right) \le 1$$

so we obtain

$$\inf_{\varphi_0+\varphi_1\le\bar{f}_j}(\|\varphi_0\|_{L^p} + t\,\|\varphi_1\|_{L^q}) \le C_3 2^{-j\theta}\,\|f\|_{L^p} = \left(\frac{\|f\|_{L^p}}{\|g\|_{L^q}}\right)^\theta (C_3 2^{-j\theta}\,\|f\|_{L_p}^{1-\theta}\,\|g\|_{L^q}^\theta)$$

Thus we apply Proposition 4.9 to conclude that

$$\left\|\bar{f}_j\right\|_{L^{r,\infty}} \le C_3 2^{-j\theta}\,\|f\|_{L_p}^{1-\theta}\,\|g\|_{L^q}^\theta$$

□

Now we will prove a characterization of fractional Sobolev spaces which when combined with Proposition 4.11 will yield our desired regularity result. However, first we must recall the relation between Besov spaces and fractional Sobolev spaces. We recount the definition of the Besov norm.

Definition 4.12. Given $s \in (0,1)$ and $1 \le p < \infty$, we define the Besov norm

$$\|f\|_{B^s_{p,p}} := \|f\|_p + \left(\int\sum_{j=0}^\infty 2^{jsp}|\Delta_j f(x)|^p\,\mathrm{d}x\right).$$

We then have the following lemma (see Appendix B for proof).

Lemma 4.13. *The $B^s_{p,p}$ Besov norm and the $W^{s,p}$ fractional Sobolev norm are comparable.*

With this lemma in hand, we can prove the following useful characterization of fractional Sobolev spaces.

Proposition 4.14. *Given any function $\varphi : \mathbb{R}^d \to \mathbb{R}$, assume that*

$$\|\Delta_j\varphi\|_{L^{p,\infty}} \le C2^{-j\theta}.$$

for all $j \in \mathbb{N}$. Then $\varphi \in W^{s,r}(B_1)$ whenever $s \in (0,\theta]$ and $r \in (1,p]$ and either $s < \theta$ or $r < p$.

Proof. Assume that $\|\Delta_j\varphi\|_{L^{p,\infty}} \le C2^{-j\theta}$ for all $j \in \mathbb{N}$. Note that this assumption implies that $\varphi \in L^{p,\infty}$. In the following arguments we always consider μ restricted to B_1. We will show that both $\|\varphi\|_r < \infty$ and $[\varphi]_{W^{s,r}} < \infty$ for $s < \theta$ and $r < p$. First, we have that

$$\begin{aligned}\|\varphi\|_r = \int_{B_1}|\varphi|^r &= r\int_0^M \lambda^{r-1}\mu\{|\varphi| > \lambda\}\,\mathrm{d}\lambda + r\int_M^\infty \lambda^{r-1}\mu\{|\varphi| > \lambda\}\,\mathrm{d}\lambda\\ &\le r\mu(B_1)\int_0^M \lambda^{r-1}d\lambda + r\int_M^\infty \lambda^{r-1}\frac{\|\varphi\|_{p,\infty}^p}{\lambda^p}d\lambda\\ &\lesssim \mu(B_1)M^r + r\int_M^\infty \lambda^{r-p-1}\,\mathrm{d}\lambda\\ &\le \mu(B_1)M^r + \frac{r}{r-p}M^{r-p} < \infty\end{aligned}$$

as $r < p$. We recall the definition of the fractional Besov norm and use Lemma 4.13 to apply a similar decomposition as above for the fractional Sobolev seminorm.

$$\begin{aligned}
(\|\varphi\|_{W^{s,r}} - \|\varphi\|_r)^r &\lesssim (\|\varphi\|_{B^s_{r,r}} - \|\varphi\|_r)^r \\
&= \int_{\mathbb{R}^d} \sum_{j=0}^{\infty} 2^{jsr} |\Delta_j \varphi(x)|^r \,\mathrm{d}x = \sum_{j=0}^{\infty} 2^{jsr} \int_{\mathbb{R}^d} |\Delta_j \varphi(x)|^r \,\mathrm{d}x \\
&\leq \sum_{j=0}^{\infty} 2^{jsr} \left(\int_0^{M_j} \lambda^{r-1} \mu(B_1) \,\mathrm{d}\lambda + \int_{M_j}^{\infty} \lambda^{r-1} \frac{\|\Delta_j \varphi\|_{p,\infty}^p}{\lambda^p} \,\mathrm{d}\lambda \right) \\
&\lesssim \sum_{j=0}^{\infty} 2^{jsr} (M_j^r + 2^{-jp\theta} M_j^{r-p}) \\
&\leq \sum_{j=0}^{\infty} (2^{jr(s-\theta)} + 2^{-j\theta r}) = \frac{1}{1 - 2^{jr(s-\theta)}} + \frac{1}{1 - 2^{-j\theta r}} < \infty
\end{aligned}$$

after picking $M_j = 2^{-j\theta}$. Thus, we conclude that $\varphi \in W^{s,p}(B_1)$. $\square$

Now, having proved Propositions 4.11 and 4.14, we can finally prove our main result of this section, a regularity result in terms of fractional Sobolev spaces for generalized solutions to genuinely nonlinear conservation laws. Velocity averaging does not yield an optimal regularity result because it is a linear argument and neglects that we have $m \geq 0$.

Theorem 4.15. *Let $u : B_2 \to \mathbb{R}$ be a generalized solution of a genuinely nonlinear scalar conservation law*

$$a(u) \cdot \nabla u = 0 \text{ in } B_2.$$

Then $u \in W^{s,r}(B_1)$ for all $s < \theta$ and $1/r = (1+\theta)/2$, where $\theta = \alpha/(2+\alpha)$.

Proof. By definition of generalized solutions, we have that

$$a(v) \cdot \nabla \chi = \partial_v m$$

where

$$\chi(x, v) \coloneqq \begin{cases} 1 & \text{if } 0 \leq v \leq u(x) \\ -1 & \text{if } u(x) \leq v < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Now we have that

$$u = \bar{\chi} = \int \chi \,\mathrm{d}v.$$

Let $\varphi_\varepsilon = \eta_\varepsilon * \chi_{B^2}$. We have that

$$\iiint \varphi \,\mathrm{d}m = -\iint \nabla \cdot q(u) \varphi \,\mathrm{d}x \leq \iint q'(u) \cdot \nabla u \varphi \,\mathrm{d}x \leq \|\nabla u\|_{L^\infty} \|q''\|_{L^\infty(B_2)} \|u\|_{L^1}$$

with $\eta(v) = \frac{1}{2} v^2$. By Fatou's lemma, we obtain

$$\|m\|_{L^1} \leq \liminf_{\varepsilon \to 0} \iiint \varphi_\varepsilon \,\mathrm{d}m \leq \|\nabla u\|_{L^\infty} \|q''\|_{L^\infty(B_2)} \|u\|_{L^1}$$

We also have that $\|\chi\|_{L^\infty} = 1$. Now by Proposition 4.11, we have that

$$\|u_j\|_r = \|\bar{\chi}_j\|_r \leq 2^{-j\theta} \|\chi\|_{L^p}^{1-\theta} \|m\|_{L^q}^{\theta}.$$

for some $p \in [1,2]$ and $q \in [1,\infty]$. Thus, if we take $p = \infty$ and $q = 1$, we obtain

$$\|u_j\|_r \leq 2^{-j\theta} \|\chi\|_{L^2}^{1-\theta} \|m\|_{L^1}^{\theta} \leq C2^{-j\theta}$$

where C depends on dimension, the genuine nonlinearity parameters, and $\|u\|_{L^\infty}$ with $\theta = \alpha/(2+\alpha)$ and $1/r = (1+\theta)/2$. We then have that

$$\|u_j\|_r \leq C2^{-j\theta}$$

so we apply Proposition 4.14 to conclude that $u \in W^{s,r}(B_1)$. □

In the special case where the scalar conservation law is the Burgers' equation, which has order of genuine nonlinearity $\alpha = 1$, we have the following corollary, which will help motivate the rigidity result in the following section.

Corollary 4.16. *Let u be a generalized solution to*

$$u_t + uu_x = 0 \text{ in } B_2.$$

Then $u \in W^{s,3/2}$ for all $s < 1/3$.

5. An Onsager-type Rigidity Result for the Burgers' Equation

The next theorem gives an Onsager-type rigidity result for the Burgers' equation to complement the regularity result proved above. We will show that if a distributional solution of (1.2) satisfies $u \in L^4 \cap L^3(I, W^{1/3,3})$, then u is Lipschitz continuous. We will first, however, need a few propositions. To begin, our goal is to characterize when distributional solutions to the Burgers' equation become entropy solutions (see [5, Theorem 2.4]).

In order to prove this characterization, we need to recall the relation between entropy solutions to scalar conservation laws and viscosity solutions to Hamilton-Jacobi equation (1.17). We recall the following definition of a viscosity solution

Definition 5.1. We say a continuous function $h : [0,T] \times \Omega \to \mathbb{R}$ is a *viscosity subsolution* of (1.17) for every C^1 function φ, $r > 0$ and point $(t_0, x_0) \in (0,T] \times \Omega$ such that

$$\varphi(t_0,x_0) = h(t_0,x_0)$$
$$\varphi(t,x) \geq h(t,x) \text{ for all } t \in (t_0 - r, t_0] \times B_r(x_0)$$

then $\varphi_t + f(\varphi_x) \leq 0$.

Similarly, we say a continuous function $h : [0,T] \times \Omega \to \mathbb{R}$ is a *viscosity supersolution* of (1.17) for every C^1 function φ, $r > 0$ and point $(t_0, x_0) \in (0,T] \times \Omega$ such that

$$\varphi(t_0,x_0) = h(t_0,x_0)$$
$$\varphi(t,x) \leq h(t,x) \text{ for all } t \in (t_0 - r, t_0] \times B_r(x_0)$$

then $\varphi_t + f(\varphi_x) \geq 0$.

We say a continuous function $h : [0,T] \times \Omega \to \mathbb{R}$ is a *viscosity solution* of (1.17) if it is both a subsolution and a supersolution.

A classical result (see [5]) relates entropy solutions to (1.1) and viscosity solutions to (1.17).

Lemma 5.2. *The function h is a viscosity solution to (1.17) if and only if the function $u = h_x$ is entropy solution to (1.1).*

Now to prove our main result, we will need the following technical lemmas.

Definition 5.3. Let μ be a probability measure on $\mathbb{R}$ and fix open set Ω. For every vector valued function $F \in L^1(\mathbb{R}, \mu)$, we set the average $\langle F \rangle = \int_\Omega F(v)\,\mathrm{d}\mu(v)$. Now let $f, \eta \in W^{1,\infty}_{\mathrm{loc}}$ and $q(v) \coloneqq \int_0^v f'(\tau)\eta'(\tau)\,\mathrm{d}\tau$. If μ is compactly supported, we define the bilinear form

$$\begin{aligned} B(f,\eta) &\coloneqq \langle (\eta, q)\cdot(-f, v)\rangle - \langle(\eta,q)\rangle\cdot\langle(-f,v)\rangle \\ &= \langle vq\rangle - \langle \eta f\rangle + \langle\eta\rangle\langle f\rangle - \langle v\rangle\langle q\rangle. \end{aligned}$$

Lemma 5.4. *If μ has compact support and f and η are convex, then $B(f,\eta) \geq 0$.*

Proof. First we will prove the lemma for η of the form $\eta(v) = |v-k|$. In this case, we have that

$$q(v) = \int_0^v \mathrm{sgn}(\tau - k) f'(\tau)\,\mathrm{d}\tau = \mathrm{sgn}(v-k)(f(v) - f(k)).$$

We then obtain

$$\begin{aligned} B(f,\eta) &= \langle (u-k)q\rangle - \langle \eta(f(u)-f(k))\rangle + \langle f(u)-f(k)\rangle\langle\eta\rangle - \langle u-k\rangle\langle q\rangle \\ &= \langle f(u)-f(k)\rangle\langle|u-k\rangle - \langle u-k\rangle\langle \mathrm{sgn}(u-k)(f(u)-f(k))\rangle \\ &= \langle f(u)-f(k)-f'(k)(u-k)\rangle\langle|u-k|\rangle \\ &- \langle u-k\rangle\langle \mathrm{sgn}(u-k)(f(u)-f(k)-f'(k)(u-k))\rangle. \end{aligned}$$

The last expression is non-negative, as by the convexity of f,

$$f(u) - f(k) - f'(k)(u-k) \geq 0.$$

Now if η is smooth and η'' has compact support, we can then write

$$\eta(u) = \int_{-\infty}^{+\infty} \frac{1}{2}\eta''(k)|u-k|\,\mathrm{d}k + C.$$

As B is bilinear, we then have that

$$B(f,\eta) = \int_{-\infty}^{\infty} \frac{1}{2}\eta''(k) B(f, |u-k|)\,\mathrm{d}k \geq 0$$

as above. Now we can bound B as a function of η in $L^1(\mathbb{R},\mu)$. First we bound q

$$\begin{aligned} \|q\|_{L^1(\mu)} &= \int \left| \int_0^v f'(\tau)\eta'(\tau)\,\mathrm{d}\tau \right| \mathrm{d}\mu(v) \\ &\leq \|f'\|_{L^\infty(\mu)} \int \left| \int_0^v \eta'(\tau)\,\mathrm{d}\tau\right| \mathrm{d}\mu(v) \\ &= \|f'\|_{L^\infty(\mu)} \int |\eta(v) - \eta(0)|\,\mathrm{d}\mu(v) \\ &\leq 2\,\|f'\|_{L^\infty(\mu)}\,(\mathrm{supp}\mu)\,\|\eta\|_{L^\infty(\mu)} \end{aligned}$$

Then we bound B

$$\begin{aligned} |B(f,\eta)| &\leq \|v\|_{L^\infty(\mu)}\|q\|_{L^1(\mu)} + \|f\|_{L^1(\mu)}\|\eta\|_{L^\infty(\mu)} \\ &\quad + \|\eta\|_{L^1(\mu)}\|f\|_{L^1(\mu)} + \|v\|_{L^1(\mu)}\|q\|_{L^1(\mu)} \\ &\leq C\,\|\eta\|_{L^\infty(\mu)} \end{aligned}$$

As C^∞ is dense in $W^{1,\infty}_{\mathrm{loc}}$ and $B(f,\eta) \geq 0$ for smooth convex compactly-supported η, we conclude that $B(f,\eta) \geq 0$ for all convex compactly-supported $\eta \in W^{1,\infty}_{\mathrm{loc}}$.

Moreover, as μ is compactly supported, we may assume without loss of generality that η has compact support, so the lemma holds for all convex $\eta \in W^{1,\infty}_{\mathrm{loc}}$. □

Lemma 5.5. *If μ has compact support and $f'', \eta'' \geq 2c$, then*

$$3B(f,\eta) \geq c^2 \langle (u - \langle u \rangle)^4 \rangle.$$

Proof. Jensen's inequality yields $3B(u^2, u^2) \geq \langle (u - \langle u \rangle)^4 \rangle$. But we also have

$$B(f,\eta) = B(f(u) - u^2, \eta(u) - u^2) + B(f(u) - u^2, u^2) + B(u^2, \eta - u^2) + B(u^2, u^2)$$

by applying bilinearity. □

Lemma 5.6. *Let u be a distributional solution to (1.2) and consider a function h such that $h_t = -\frac{u^2}{2}$ and $h_x = u$. Now let ζ be a smooth function with $\zeta(0,0) = 0$ and such that $h - \zeta$ has a minimum in $(0,0)$. Then $[\zeta_t(0,0) + \frac{1}{2}\zeta_x^2(0,0)] = 0$.*

Note that h satisfies the desired identity. The crux of the lemma lies in careful estimates of the average difference between ζ and h on some small neighborhood about $(0,0)$, where $\zeta_t + \frac{1}{2}\zeta_x^2$ is approximately bounded by the average oscillation of u given by $\langle (u - \langle u \rangle)^4 \rangle$.

To simplify notation, we will write $g(\varepsilon,\delta) \lesssim h(\varepsilon,\delta)$ whenever there exist constants $C_1, C_2 > 0$ such that $g(\varepsilon,\delta) \leq C_1 h(\varepsilon,\delta)$ for all $|\varepsilon|, |\delta| \leq C_2$.

Proof. As h and ζ are only defined up to their derivatives, without loss of generality take $h(0,0) = \zeta(0,0) = 0$. For $0 < \varepsilon \leq 1$, set $\zeta_\varepsilon = \zeta - \epsilon|(t,x)|$, so that $h - \zeta_\varepsilon$ has a strict minimum at 0. For any $\delta > 0$, we define the neighborhood

$$\Omega_{\varepsilon,\delta} \coloneqq \{(t,x) : [h - \zeta_\varepsilon](t,x) < \delta\}$$

with the associated average

$$\langle f \rangle_{\varepsilon,\delta} \coloneqq \frac{1}{|\Omega_{\varepsilon,\delta}|} \int_{\Omega_{\varepsilon,\delta}} f(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

In fact, we have the inclusion $\Omega_{\varepsilon,\delta} \subseteq B_{\delta/\epsilon}$: any $(t_0,x_0) \in \Omega_{\varepsilon,\delta}$ satisfies

$$\begin{aligned} \delta > [h - \zeta_\varepsilon](t_0,x_0) &= [h - \zeta](t_0,x_0) + \varepsilon|(t_0,x_0)| \\ &\geq [h - \zeta](0,0) + \varepsilon|(t_0,x_0)| = \varepsilon|(t_0,x_0)|. \end{aligned}$$

As our first step, we will show

$$\langle (\zeta_{\varepsilon t}, \zeta_{\varepsilon x}) \rangle_{\epsilon,\delta} = \langle (-\frac{u^2}{2}, u) \rangle_{\varepsilon,\delta}. \tag{5.7}$$

By definition of h,

$$\langle (-\frac{u^2}{2}, u) \rangle_{\varepsilon,\delta} - \langle (\zeta_{\varepsilon t}, \zeta_{\varepsilon x}) \rangle = \langle (h - \zeta_\varepsilon)_t, (h - \zeta_\varepsilon)_x \rangle.$$

By the fundamental theorem of calculus, we have

$$\langle (h - \zeta_\varepsilon)_t \rangle_{\varepsilon,\delta} = \frac{1}{|\Omega_{\varepsilon,\delta}|} \int_{\mathbb{R}^2} (\min\{h - \zeta_\varepsilon - \delta, 0\})_t \,\mathrm{d}t\,\mathrm{d}x = 0$$

$$\langle (h - \zeta_\varepsilon)_x \rangle_{\varepsilon,\delta} = \frac{1}{|\Omega_{\varepsilon,\delta}|} \int_{\mathbb{R}^2} (\min\{h - \zeta_\varepsilon - \delta, 0\})_x \,\mathrm{d}t\,\mathrm{d}x = 0$$

as $\min\{h-\zeta_\varepsilon-\delta,0\}$ is compactly supported on $\Omega_{\varepsilon,\delta}\subseteq B_{\varepsilon,\delta}$. As ζ is smooth, $\langle\zeta_{\varepsilon t}\rangle$ and $\langle\zeta_{\varepsilon x}\rangle$ are bounded, so from (5.7) we deduce

$$\langle u^2\rangle_{\varepsilon,\delta}\lesssim\langle\zeta_{\varepsilon t}\rangle\lesssim 1 \text{ and } \langle u\rangle_{\varepsilon,\delta}\lesssim\langle\zeta_{\varepsilon x}\rangle\lesssim 1. \tag{5.8}$$

Jensen's inequality gives that $\langle|u|\rangle_{\varepsilon,\delta}\lesssim 1$, so we likewise conclude that

$$\langle|(h_t,h_x)|\rangle_{\varepsilon,\delta}\lesssim 1.$$

Therefore, we may ascertain the stronger bounds on the absolute values that

$$\langle|((h-\zeta_\varepsilon)_t,(h-\zeta_\varepsilon)_x)|\rangle_{\varepsilon,\delta}\le\langle|(h_t,h_x)|\rangle+\langle|(\zeta_{\varepsilon t},\zeta_{\varepsilon x})|\rangle\lesssim 1. \tag{5.9}$$

As our second step, we will show that

$$\delta^2\lesssim|\Omega_{\varepsilon,\delta}|. \tag{5.10}$$

We rewrite (5.9) to obtain

$$\int_{\mathbb{R}^2}|(\min\{h-\zeta_\varepsilon-\delta,0\})_t,(\min\{h-\zeta_\varepsilon-\delta,0\})_x)|\,\mathrm{d}t\,\mathrm{d}x\lesssim|\Omega_{\varepsilon,\delta}|.$$

We then apply the Sobolev inequality (see [6]) for

$$\left[\int_{\mathbb{R}^2}(\min\{h-\zeta_\varepsilon-\delta,0\})^2\,\mathrm{d}t\,\mathrm{d}x\right]^{1/2}\lesssim|\Omega_{\varepsilon,\delta}|.$$

Cauchy-Schwartz and the above bound then gives us

$$\begin{aligned}&-\int_{\mathbb{R}^2}\min\{h-\zeta_\varepsilon-\delta,0\}\,\mathrm{d}t\,\mathrm{d}x\\&\quad\le\left[\int_{\mathbb{R}^2}|\chi_{\Omega_{\varepsilon,\delta}}|^2\,\mathrm{d}x\,\mathrm{d}t\right]^2\left[\int_{\mathbb{R}^2}(\min\{h-\zeta_\varepsilon-\delta,0\})^2\,\mathrm{d}t\,\mathrm{d}x\right]^{1/2}\lesssim|\Omega_{\varepsilon,\delta}|^{3/2}.\end{aligned}$$

We then introduce a function

$$I(\delta):=-\int_{\mathbb{R}^2}\min\{h-\zeta_\varepsilon-\delta,0\}=\int_0^\delta|\Omega_{\varepsilon,s}|\,\mathrm{d}s=0,$$

which allows us to construct a differential inequality

$$I(\delta)\lesssim|\Omega_{\varepsilon,\delta}|^{3/2}\,\mathrm{d}x\,\mathrm{d}t=\left[\frac{\mathrm{d}}{\mathrm{d}\delta}I(\delta)\right]^{3/2}. \tag{5.11}$$

As $I(\delta)>0$ for $\delta>0$, we can solve the differential inequality (5.11) by separation of variables, yielding $\delta^3\lesssim I(\delta)$. Then, as $|\Omega_{\varepsilon,\delta}|$ is a non-decreasing function of δ, we conclude

$$\delta|\Omega_{\varepsilon,\delta}|\ge\int_0^\delta|\Omega_{\varepsilon,\delta}|\,\mathrm{d}s\gtrsim\delta^3.$$

As our third step, we will show the following upper bound

$$\langle(-\frac{u^2}{2},u)\cdot(\frac{u^2}{2},\frac{u^3}{3})\rangle-\langle(\zeta_{\varepsilon t},\zeta_{\varepsilon x})\cdot(\frac{u^2}{2},\frac{u^3}{3})\rangle\lesssim\frac{\mu(B_{\delta/\varepsilon}(0,0))}{\delta}. \tag{5.12}$$

The bound follows from the following set of computations

$$
\begin{aligned}
&\langle(-\frac{u^2}{2},u)\cdot(\frac{u^2}{2},\frac{u^3}{3})\rangle_{\varepsilon,\delta}-\langle(\zeta_{\varepsilon t},\zeta_{\varepsilon x})\cdot(\frac{u^2}{2},\frac{u^3}{3})\rangle_{\varepsilon,\delta}\\
&=\langle((h-\zeta_\varepsilon)_t,(h-\zeta_\varepsilon)_x)\cdot(\frac{u^2}{2},\frac{u^3}{3})\rangle_{\varepsilon,\delta}\\
&=\frac{1}{|\Omega_{\varepsilon,\delta}|}\int_{\mathbb{R}^2}((\min\{h-\zeta_\varepsilon-\delta,0\})_t,(\min\{h-\zeta_\varepsilon-\delta,0\})_x)\cdot(\frac{u^2}{2},\frac{u^3}{3})\\
&=\frac{1}{|\Omega_{\varepsilon,\delta}|}\int_{\mathbb{R}^2}(-\min\{h-\zeta_\varepsilon-\delta,0\})\cdot((\frac{u^2}{2})_t,(\frac{u^3}{3})_x)\\
&\le\frac{\delta}{|\Omega_{\varepsilon,\delta}|}\mu(\Omega_{\varepsilon,\delta})\le\frac{\delta}{|\Omega_{\varepsilon,\delta}|}\mu(B_{\delta/\varepsilon}(0,0))\overset{(5.10)}{\lesssim}\frac{\mu(B_{\delta/\varepsilon}(0,0))}{\delta}.
\end{aligned}
$$

As our fourth step, we show that $\langle u^4\rangle_{\varepsilon,\delta}\lesssim\varepsilon^{-1}$ and $\langle|u^3|\rangle_{\varepsilon,\delta}\lesssim\varepsilon^{-1/2}$. We have

$$
\begin{aligned}
\langle u^4\rangle_{\varepsilon,\delta}&\lesssim\langle(-\frac{u^2}{2},u)\cdot(\frac{u^2}{2},\frac{u^3}{3})\rangle_{\varepsilon,\delta}\\
&\lesssim\langle(\zeta_{\varepsilon t},\zeta_{\varepsilon x})\cdot(\frac{u^2}{2},\frac{u^3}{3})\rangle_{\varepsilon,\delta}+\frac{\mu(B_{\delta/\varepsilon}(0,0))}{\delta}\\
&\lesssim\sup_{\Omega_{\varepsilon,\delta}}|(\zeta_{\varepsilon t},\zeta_{\varepsilon x})|\langle|(\frac{u^2}{2},\frac{u^3}{3})|\rangle_{\varepsilon,\delta}+\frac{1}{\varepsilon}\left[\frac{\varepsilon}{\delta}\mu(B_{\delta/\varepsilon}(0,0)\right]\\
&\lesssim\langle u^2\rangle_{\varepsilon,\delta}+\langle|u^3|\rangle_{\varepsilon,\delta}+\varepsilon^{-1}\overset{(5.8)}{\lesssim}1+[\langle u^4\rangle_{\varepsilon,\delta}]^{1/2}[\langle u^2\rangle_{\varepsilon,\delta}]^{1/2}+\varepsilon^{-1}.
\end{aligned}
$$

We obtain $\langle u^4\rangle_{\varepsilon,\delta}\lesssim\varepsilon^{-1}$ from Young's inequality. Then by Hölder's inequality

$$\langle|u|^3\rangle_{\varepsilon,\delta}\le[\langle u^4\rangle_{\varepsilon,\delta}]^{1/2}[\langle u^2\rangle_{\varepsilon,\delta}]^{1/2}\lesssim\varepsilon^{-1/2}.$$

As our fifth step, we show that

$$|\langle(\zeta_{\varepsilon t},\zeta_{\varepsilon x})\cdot(\frac{u^2}{2},\frac{u^3}{3})\rangle-\langle(\zeta_{\varepsilon t},\zeta_{\varepsilon x})\rangle\cdot\langle(\frac{u^2}{2},\frac{u^3}{3})\rangle|\lesssim\varepsilon^{1/2}+\frac{\delta}{\varepsilon^{3/2}}. \tag{5.13}$$

We have that

$$
\begin{aligned}
&\left|\left\langle(\zeta_{\varepsilon t},\zeta_{\varepsilon x})\cdot(\frac{u^2}{2},\frac{u^3}{3})\right\rangle_{\varepsilon,\delta}-\langle(\zeta_{\varepsilon t},\zeta_{\varepsilon x})\rangle_{\varepsilon,\delta}\cdot\left\langle(\frac{u^2}{2},\frac{u^3}{3})\right\rangle_{\varepsilon,\delta}\right|\\
&\le\sup_{\Omega_{\varepsilon,\delta}}\left|(\zeta_{\varepsilon t},\zeta_{\varepsilon x})-\langle(\zeta_{\varepsilon t},_{\varepsilon x})\rangle_{\varepsilon,\delta}\right|\left\langle\left|(\frac{u^2}{2},\frac{u^3}{3})\right|\right\rangle_{\varepsilon,\delta}\\
&\le\left[\operatorname{osc}_{\Omega_{\varepsilon,\delta}}(\zeta_{\varepsilon t},\zeta_{\varepsilon x})+2\varepsilon\right]\left(\left\langle u^2\right\rangle_{\varepsilon,\delta}+\left\langle u^3\right\rangle_{\varepsilon,\delta}\right)\\
&\overset{(5.10),(5.12)}{\lesssim}\left(\frac{\delta}{\varepsilon}+\varepsilon\right)(1+\varepsilon^{-1/2})\lesssim\frac{\delta}{\varepsilon^{3/2}}+\varepsilon^{1/2}
\end{aligned}
$$

As our final step, we combine (5.7), (5.12), (5.13) to obtain

$$
\begin{aligned}
&\left\langle(-\frac{u^2}{2},u)\cdot(\frac{u^2}{2},\frac{u^3}{3})\right\rangle_{\varepsilon,\delta}-\left\langle(-\frac{u^2}{2},u)\right\rangle_{\varepsilon,\delta}\cdot\left\langle(\frac{u^2}{2},\frac{u^3}{3})\right\rangle_{\varepsilon,\delta}\\
&\quad\lesssim\frac{\delta}{\varepsilon^{3/2}}+\varepsilon^{1/2}+\frac{\mu(B_{\delta/\varepsilon}(0,0))}{\delta}.
\end{aligned}
$$

We have arrived at bilinear form $B(\eta, f)$ with $f(u) = \eta(u) = u^2/2$. Hence Lemma 5.5 allows us to conclude that

$$\left\langle u - \langle u \rangle_{\varepsilon,\delta}^4 \right\rangle_{\varepsilon,\delta} \lesssim \frac{\delta}{\varepsilon^{3/2}} + \varepsilon^{1/2} + \frac{\mu(B_{\delta/\varepsilon}(0,0))}{\delta}. \tag{5.14}$$

Thus, finally, we may obtain

$$\begin{aligned}
\left|\zeta_t(0,0) + \frac{1}{2}\zeta_x^2(0,0)\right| &\lesssim \left|\langle \zeta_{\varepsilon t}\rangle_{\varepsilon,\delta} + \frac{1}{2}\left[\langle \zeta_{\varepsilon x}\rangle_{\varepsilon,\delta}\right]^2\right| + \varepsilon + \frac{\delta}{\varepsilon} \\
&\overset{(5.7)}{=} \left|-\left\langle \frac{u^2}{2}\right\rangle_{\varepsilon,\delta} + \frac{1}{2}\left[\langle u\rangle_{\varepsilon,\delta}\right]^2\right| + \varepsilon + \frac{\delta}{\varepsilon} \\
&= \frac{1}{2}\left\langle (u - \langle u\rangle_{\varepsilon,\delta})^2\right\rangle_{\varepsilon,\delta} + \varepsilon + \frac{\delta}{\varepsilon} \\
&\lesssim \left[\left\langle (u - \langle u\rangle_{\varepsilon,\delta})^4\right\rangle_{\varepsilon,\delta}\right]^{1/2} + \varepsilon + \frac{\delta}{\varepsilon} \\
&\overset{(5.14)}{\lesssim} \left(\frac{\delta}{\varepsilon^{3/2}} + \varepsilon^{1/2} + \frac{\mu(B_{\delta/\varepsilon}(0,0))}{\delta}\right)^{1/2} + \varepsilon + \frac{\delta}{\varepsilon}.
\end{aligned}$$

Letting $\delta \to 0$ and then $\varepsilon \to 0$, we obtain $\zeta_t(0,0) + \frac{1}{2}\zeta_x^2(0,0) = 0$. □

Proposition 5.15. *Let $\Omega \subset \mathbb{R}^2$ be open, and assume that the function $u \in L^4_{\mathrm{loc}}(\Omega)$ is distributional solution to (1.2) such that*

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_x \leq \mu \text{ in } \mathcal{D}'(\Omega) \tag{5.16}$$

for some non-negative Radon measure μ with

$$\lim_{r\downarrow 0} \frac{\mu(B_r(t,x))}{r} = 0 \text{ for every } (t,x) \in \Omega. \tag{5.17}$$

Then u is an entropy solution to (1.2).

Proof. Assume that u satisfies the conditions above, including (5.16). We want to show that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_t, L^2_{\mathrm{loc}}(\mathbb{R}_x))$. Pick $(t_0, x_0) \in \Omega$. As $u \in L^4_{\mathrm{loc}}$, there exists some $r > 0$ such that $\|u\|_{L^4(B_r(t_0)\times B_r(x_0))} < \infty$. By (5.17), we can also pick r sufficiently small such that $\mu(B_{r\sqrt{2}}(t_0, x_0)) < 1$. Furthermore, as

$$\|u\|^4_{L^4(B_r(t_0)\times B_r(x_0))} = \int_{B_r(x_0)} \|u(\cdot,x)\|^4_{L^4(B_r(t_0))}\,\mathrm{d}x$$

we can pick s such that $\|u(\cdot, x+s)\|_{L^4(B_r(t_0))} < \infty$ and $\|u(\cdot, x-s)\|_{L^4(B_r(t_0))} < \infty$. Now pick $a \in B_r(t_0)$. Integrating (5.16) twice, we have that

$$\int_{t_0}^a \int_{B_s(x_0)} \varphi \left(\frac{u^2}{2}\right)_t \mathrm{d}x\,\mathrm{d}t + \int_{t_0}^a \int_{B_s(x_0)} \varphi \left(\frac{u^3}{3}\right)_x \mathrm{d}x\,\mathrm{d}t \leq \int_{t_0}^a \int_{B_s(x_0)} \varphi\,\mathrm{d}\mu$$

for all test functions $\varphi \in C_c^\infty$. Integrating by parts, we obtain that

$$\frac{1}{2}\left(\|u(a,\cdot)\|^2_{L^2(B_s(x_0))} - \|u(t_0,\cdot)\|^2_{L^2(B_s(x_0))}\right) - \int_{t_0}^a \int_{B_s(x_0)} \varphi_t \frac{u^2}{2}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\frac{1}{3}\left(\|u(\cdot,x_0+s)\|^3_{L^3([a,t_0])} - \|u(\cdot,x_0-s)\|^3_{L^3([a,t_0])}\right) - \int_{t_0}^a \int_{B_s(x_0)} \varphi_x \frac{u^3}{3}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \int_{t_0}^a \int_{B_s(x_0)} \varphi\,\mathrm{d}\mu.$$

Now take φ such that $\varphi = 1$ on $B_{r\sqrt{2}}(x_0,t_0) \supset B_r(x_0)\times B_r(t_0)$, $\varphi = 0$ on $B_{2r}(x_0,t_0)$, and $\|\varphi_x\|_{L^\infty} + \|\varphi_t\|_{L^\infty} < \infty$. We may bound as follows

$$\frac{1}{2}\left|\|u(a,\cdot)\|^2_{L^2(B_s(x_0))} - \|u(t_0,\cdot)\|^2_{L^2(B_s(x_0))}\right|$$
$$\leq \frac{|a-t_0|}{2}\|\varphi_t\|_{L^\infty}\|u\|^2_{L^2(B_r(t_0)\times B_s(x_0))}$$
$$+\frac{1}{3}\left(\left|\|u(\cdot,x_0+s)\|^3_{L^3(B_r(t_0)\times B_s(x_0))}\right| + \left|\|u(\cdot,x_0-s)\|^3_{L^3(B_r(t_0)\times B_s(x_0))}\right|\right)$$
$$+\frac{|a-t_0|}{3}\|\varphi_x\|_{L^\infty}\|u\|^3_{L^3(B_r(t_0)\times B_s(x_0))} + \mu(B_{r\sqrt{2}}(x_0,t_0)) < \infty.$$

Thus, there exists some constant C such that

$$\left|\|u(a,\cdot)\|^2_{L^2(B_s(x_0))} - \|u(t_0,\cdot)\|^2_{L^2(B_s(x_0))}\right| \leq C < \infty.$$

Note that $u \in L^4_{\mathrm{loc}}$ implies that for t_0 almost everywhere, $u(t_0,\cdot) \in L^4_{\mathrm{loc}} \subset L^2_{\mathrm{loc}}$. Moreover, $a \in B_r(t_0)$ was arbitrary, so we may ascertain that

$$\|u\|_{L^\infty(B_r(t_0))L^2(B_s(x_0))} < \infty \implies u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_t, L^2_{\mathrm{loc}}(\mathbb{R}_x)).$$

We consider a function h such that $h_t = -\frac{u^2}{2}$ and $h_x = u$. Fix $t_0 \in \Omega_t$. For any $b > a \in \Omega_x$, by Cauchy-Schwartz, we have that

$$|h(t_0,b) - h(t_0,a)| = \left|\int_a^b h_x(t_0,x)\,\mathrm{d}x\right|$$
$$\leq \|u(t_0,\cdot)\|_{L^2([a,b])}|b-a|^{1/2}.$$

Then we have$[h(t,\cdot)]_{C^{0,1/2}(\bar{U})} \leq \|u(t,\cdot)\|_{L^2(\bar{U})}$, so as $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_t, L^2_{\mathrm{loc}}(\mathbb{R}_x))$, we may conclude that $h \in L^\infty_{\mathrm{loc}}(\mathbb{R}_t, C^{0,1/2}_{\mathrm{loc}}(\mathbb{R}_x))$. We also have $h \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}_t, L^1_{\mathrm{loc}}(\mathbb{R}_x))$ as for arbitrary $s < r \in \Omega_t$ and $b > a \in \Omega_x$, we have that

$$\|h(s,\cdot) - h(r,\cdot)\|_{L^1} = \int_a^b |h(s,x) - h(r,x)|\,\mathrm{d}x \leq \frac{1}{2}\int_a^b \int_r^s |u(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x$$
$$\leq \|u\|_{L^\infty_t([r,s])L^2_x([a,b])}|s-r|$$

Thus, we conclude that $[h]_{C^{0,1}_t([r,s])L^1_x([b,a])} \leq \|u\|_{L^\infty_t([r,s])L^2_x([a,b])}$, so as we have that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_t, L^2_{\mathrm{loc}}(\mathbb{R}_x))$, then $h \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}_t, L^1_{\mathrm{loc}}(\mathbb{R}_x))$. Thus we have

$$h \in L^\infty_{\mathrm{loc}}(\mathbb{R}_t, L^2_{\mathrm{loc}}(\mathbb{R}_x)), \tag{5.18}$$

$$h \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}_t, L^1_{\mathrm{loc}}(\mathbb{R}_x)). \tag{5.19}$$

We will interpolate between (5.18) and (5.19) to show that $h \in L^\infty_{\mathrm{loc}}(\mathbb{R}_x, C^{0,1/3}_{\mathrm{loc}}(\mathbb{R}_t))$. Let ρ_ε denote the standard mollifier. We take $\rho \geq 0$ such that $\operatorname{supp}\rho \subset (-1,1)$ and

$\int \rho \, \mathrm{d}x = 1$ and then let $\rho_\varepsilon := \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right)$. Let $*$ denote convolution in x. For $s, r \in \Omega_t$, we have that

$$\begin{aligned}
&|h(s,x) - h(r,x)| \\
&\leq |h(s,x) - (\rho_\varepsilon * h)(s,x)| + |(\rho_\varepsilon * h)(s,x) - (\rho_\varepsilon * h)(r,s)| \\
&\qquad + |(\rho_\varepsilon * h)(r,x) - h(r,x)| \\
&\leq \varepsilon^{1/2} \sup_{y, |x-y| < \varepsilon} \frac{|h(s,x) - h(s,y)|}{|x-y|^{1/2}} + \frac{\|\rho\|_{L^\infty}}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |h(s,y) - h(r,y)| \, \mathrm{d}y \\
&\qquad + \varepsilon^{1/2} \sup_{y, |x-y| < \varepsilon} \frac{|h(r,x) - h(r,y)|}{|x-y|^{1/2}} \\
&\leq C\left(\varepsilon^{1/2} + \frac{|s-r|}{\varepsilon}\right)
\end{aligned}$$

after applying (5.18) and (5.19). Then, after picking $\varepsilon = |s-r|^{2/3}$, we obtain $h \in L^\infty_{\mathrm{loc}}(\mathbb{R}_x, C^{0,1/3}_{\mathrm{loc}}(\mathbb{R}_t))$.

Now we will show that h is a viscosity solution to Hamilton-Jacobi equation

$$h_t + \frac{h_x^2}{2} = 0 \tag{5.20}$$

We begin by showing that h is a viscosity subsolution. By construction of h

$$\rho_t = -\frac{u^2}{2} = -\frac{h_x^2}{2} \text{ a.e. in } \Omega.$$

Let ρ_ε be the standard mollifier. That is, we take $\rho \in C_c^\infty(\mathbb{R}^2)$ nonnegative with $\int \rho \, \mathrm{d}t \, \mathrm{d}x = 1$ and then set $\rho_\varepsilon(t,x) := \frac{1}{\varepsilon^2}\rho\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$. By Jensen's inequality,

$$0 = \left(h_t + \frac{h_x^2}{2}\right) * \rho_\varepsilon \geq h_t * \rho_\varepsilon + \frac{(h_t * \rho_\varepsilon)^2}{2} = (h * \rho_\varepsilon)_t + \frac{(h * \rho_\varepsilon)_t^2}{2}.$$

We therefore conclude that $h * \rho_\varepsilon$ is a classical subsolution to (5.20) and thus also a viscosity subsolution (see [2]). As h is continuous $h * \rho_\varepsilon \to h$ uniformly as $\varepsilon \downarrow 0$. Thus h is a viscosity subsolution (see [2]).

The fact that h is viscosity supersolution to (5.20) follows immediately from Lemma 5.6, so h is viscosity solution to (5.20). By Lemma 5.2, we conclude that u is an entropy solution to Burgers' equation (1.2). $\square$

Proposition 5.21. *Assume that $\mathscr{C}$, the set of points of continuity of a distributional solution u to Burgers' equation* (1.2)*, has nonempty interior $\mathscr{C}^\circ$. Then u is locally Lipschitz on $\mathscr{C}^\circ$.*

Proof. Take $(t_0, x_0) \in \mathscr{C}^\circ$ and pick $r > 0$ such that $B_r(t_0, x_0) \subseteq \mathscr{C}$. Then pick $(t, x) \in B_r(t_0, x_0)$. The unique characteristic curves through (t_0, x_0) and (t, x) are straight lines with slopes $u(t_0, x_0)$ and $u(t, x)$ respectively. As $B_r(t, x) \subseteq \mathscr{C}$, these characteristic curves must not intersect within $B_r(t, x)$, as intersections of characteristic curves produce discontinuous shocks in the solution. A trigonometric estimate yields

$$|u(t,x) - u(t_0, x_0)| \leq \frac{c}{r}|(t,x) - (t_0, x_0)|$$

Thus, as $(t_0, x_0) \in \mathscr{C}$ was arbitrary, u is locally Lipschitz on $\mathscr{C}^\circ$. $\square$

Proposition 5.22. *Let $\Omega = I \times J$ with $I, J \subset \mathbb{R}$ two intervals and $u \in L^4(\Omega)$ be a distributional solution of (1.2) which belongs to the space $L^3(I, W^{1/3,3}(J))$. Then u satisfies the following identity in the weak sense*

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_s = 0.$$

Proof. We consider the standard mollifier $\rho \in C_c^\infty((-1,1))$ with $\rho_\epsilon(x) = \frac{1}{\epsilon}\rho(\frac{s}{\epsilon})$, and denote $u_\epsilon := u * \rho_\epsilon$, where we take mollification in only the spatial variable s.

Given a smooth test function $\zeta \in C_c^\infty(\Omega)$, we want to show that

$$\lim_{\epsilon\to 0}\int_\Omega \zeta\left(\left(\frac{u_\epsilon^2}{2}\right)_t + \left(\frac{u_\epsilon^3}{3}\right)_s\right)\mathrm{d}t\,\mathrm{d}x = 0.$$

Without loss of generality, we only consider ϵ sufficiently small such that $\Omega_\epsilon \subset \operatorname{supp}(\zeta)$. Then integrating on the support of ζ, we notice that u_ϵ is now smooth, so we can take classical derivatives in the above expression. We also have from the Burgers' equation that

$$(u_\epsilon)_t + \left(\frac{u^2 * \rho_\epsilon}{2}\right)_s = 0. \tag{5.23}$$

Then we can estimate with the chain rule

$$\begin{aligned}\int_\Omega \zeta\left(\left(\frac{u_\epsilon^2}{2}\right)_t + \left(\frac{u_\epsilon^3}{3}\right)_s\right)\mathrm{d}t\,\mathrm{d}x &= \int \zeta u_\epsilon\left((u_\epsilon)_t + \left(\frac{u_\epsilon^2}{2}\right)_s\right)\mathrm{d}t\,\mathrm{d}x\\ &= \frac{1}{2}\int \zeta u_\epsilon\left(u_\epsilon^2 - u^2 * \rho_\epsilon\right)_s \mathrm{d}t\,\mathrm{d}x \quad \text{(by (5.23))}\\ &= -\frac{1}{2}\int (\zeta u_\epsilon)_s\left(u_\epsilon^2 - u^2 * \rho_\epsilon\right)\mathrm{d}t\,\mathrm{d}x \quad \text{(IBP)},\end{aligned}$$

which we can split into two terms by the product rule (up to a constant of $-1/2$)

$$\int \zeta_s u_\epsilon(u_\epsilon^2 - u^2 * \rho_\epsilon)\,\mathrm{d}t\,\mathrm{d}x + \int \zeta(u_\epsilon)_s(u_\epsilon^2 - u^2 * \rho_\epsilon)\,\mathrm{d}t\,\mathrm{d}x.$$

The first term vanishes because u_ϵ is uniformly bounded in L^3 and $(u_\epsilon^2 - u^2 * \rho_\epsilon)$ converges to zero in $L^{3/2}$. Here we use the fact that $u \in L^4(\Omega) \subset L^3(\Omega)$.

For the second term, we check that

$$\begin{aligned}(u_\epsilon)_s(t,x) &= \int_{-\epsilon}^{\epsilon} \rho_\epsilon'(\xi)v(t,x-\xi)\,\mathrm{d}\xi\\ &= \frac{1}{\epsilon^2}\int_{-\epsilon}^{\epsilon}\rho'\left(\frac{\xi}{\epsilon}\right)(v(t,x-\xi) - v(t,x))\,\mathrm{d}\xi,\\ |(u_\epsilon)_s(t,x)| &\le \frac{C}{\epsilon^2}\left(\int_{-\epsilon}^{\epsilon}|v(t,x-\xi)-v(t,x)|^3\,\mathrm{d}\xi\right)^{1/3}\left(\int_{-1}^{1}\epsilon\,|\rho'(\xi)|^{3/2}\,\mathrm{d}\xi\right)^{2/3}\\ &\le \frac{C}{\epsilon^{4/3}}\left(\int_{-\epsilon}^{\epsilon}|v(t,x-\xi)-v(t,x)|^3\,\mathrm{d}\xi\right)^{1/3}\\ &\le \frac{C}{\epsilon}\left(\fint_{-\epsilon}^{\epsilon}|v(t,x-\xi)-v(t,x)|^3\,\mathrm{d}\xi\right)^{1/3}.\end{aligned}$$

We can also apply a doubling variables trick similar to the computation of the integral of a Gaussian for

$$
\begin{aligned}
((u_\epsilon)^2 - (u^2)_\epsilon)(t,x) &= -\frac{1}{2} \iint \rho_\epsilon(\xi)\rho_\epsilon(\xi')(v(t,x-\xi) - v(t,x-\xi'))^2 \,\mathrm{d}\xi \,\mathrm{d}\xi', \\
|(u_\epsilon)^2 - (u^2)_\epsilon|(t,x) &\le \frac{1}{2\epsilon^2} \iint \rho\left(\frac{\xi}{\epsilon}\right)\rho\left(\frac{\xi'}{\epsilon}\right)(v(t,x-\xi) - v(t,x-\xi'))^2 \,\mathrm{d}\xi \,\mathrm{d}\xi' \\
&\le \frac{1}{2\epsilon^2} \iint_{(-\epsilon,\epsilon)^2} (v(t,x-\xi) - v(t,x-\xi'))^2 \,\mathrm{d}\xi \,\mathrm{d}\xi' \\
\text{(Young's)} &\le \frac{C}{\epsilon} \int_{-\epsilon}^{\epsilon} (v(t,x-\xi) - v(t,x))^2 \,\mathrm{d}\xi \\
&\le C \left(\fint_{-\epsilon}^{\epsilon} |v(t,x-\xi) - v(t,x)|^3 \,\mathrm{d}\xi \right)^{2/3}
\end{aligned}
$$

by Jensen's inequality. Together we have

$$
\begin{aligned}
\left| \int_\Omega \zeta(u_\epsilon)_s(u_\epsilon^2 - u^2 * \rho_\epsilon) \,\mathrm{d}t \,\mathrm{d}x \right| &\le \frac{C}{\epsilon} \int_I \int_J \fint_{-\epsilon}^{\epsilon} |v(t,x-\xi) - v(t,x)|^3 \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t \\
&= \frac{C}{\epsilon^2} \int_I \int_J \int_{-\epsilon}^{\epsilon} |v(t,x-\xi) - v(t,x)|^3 \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t \\
&\le C \int_I \int_J \int_{s-\epsilon}^{s+\epsilon} \frac{|v(t,\xi) - v(t,x)|^3}{|x-\xi|^2} \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}
$$

by a change of variables. Therefore, we can apply the absolute continuity of the Lebesgue integral (5.25) to conclude that the above quantity vanishes as $\epsilon \to 0$. $\square$

Theorem 5.24. *Let $\Omega = I \times J$ with $I, J \subset \mathbb{R}$ two intervals and $u \in L^4(\Omega)$ be a distributional solution of (1.2) which belongs to the space $L^3(I, W^{1/3,3}(J))$, namely*

$$
\int_I \int_{J\times J} \frac{|u(t,x) - u(t,\xi)|^3}{|x-\xi|^2} \,\mathrm{d}x \,\mathrm{d}\xi \,\mathrm{d}t < \infty. \tag{5.25}
$$

Then u is locally Lipschitz.

Proof. Since Proposition 5.22 holds, u is an entropy solution of the Burgers' equation by Proposition 5.15. Then by Proposition 2.4, the function u lies in BV.

Moreover, the equality in Proposition 2 implies that

$$
\eta(u)_t + q(u)_s = 0 \text{ with } \eta(u) = \frac{v^2}{2}, \text{ and } q(u) = \frac{u^3}{3}
$$

where the choice of (η, q) forms an entropy-entropy flux pair. Then the associated entropy dissipation measure vanishes. In particular, the entropy contribution at each shock must vanish, so we recall the result of the explicit computation for Burgers' (3.5)

$$
-\frac{(u_R - u_L)^3}{12} = 0 \implies u_R = u_L.
$$

Therefore, u has no nontrivial shocks. [3, Theorem 11.3.2] tells us that u is everywhere continuous, so u is locally Lipschitz by Proposition 5.21. $\square$

6. The Dispersive Nature of the Burgers' Equation

Up until this section, every result we have proved concerning entropy solutions to Burgers' has assumed L^∞ intial data. In this section, we will prove a dispersion result, allowing us to extend these results to L^1 initial data by approximation.

The Hopf-Lax formula tells us that the solution of the Hamilton-Jacobi equation $u_t + f(\nabla u) = 0$ is given by the minimization problem

$$u(x,t) = \min_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + u_0(y) \right\}, \tag{6.1}$$

where $L = f^*$ is the convex conjugate (or Legendre transform) of f.

Now recall that in one dimension, we expect solutions to conservation law equations to be spatial derivatives of solutions to the Hamilton-Jacobi equation. As such, we expect from (6.1) that

$$u(x,t) = \frac{\partial}{\partial x}\left[\min_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + h(y) \right\} \right]$$

is a weak solution of the conservation law equation. Note here that in general w is differentiable only almost everywhere, so u is only defined at most everywhere.

Theorem 6.2. *(Lax-Oleinik formula) Assume $f : \mathbb{R} \to \mathbb{R}$ is smooth, uniformly convex, and $u_0 \in L^\infty$. Take h to be a spatial antiderivative of u_0, i.e. such that $h_x = u_0$. Then for all $t > 0$, for almost every $x \in \mathbb{R}$, there exists a unique point $y(x,t)$ such that the minimum of*

$$y \mapsto \left\{ tL\left(\frac{x-y}{t}\right) + h(y) \right\}$$

is achieved at $y(x,t)$. Moreover, we can write $G := (f')^{-1}$ because f' is strictly increasing and surjective, with

$$u(x,t) = \frac{\partial}{\partial x}\left[\min_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + h(y) \right\} \right] = G\left(\frac{x-y(x,t)}{t}\right).$$

Then the above u is the unique entropy solution of the scalar conservation law with flux f.

Theorem 6.3. *Let u be an entropy solution to a scalar conservation law in one spatial dimension with initial data u_0. Then for $t > 0$, the function u satisfies*

$$\|u(t,\cdot)\|_{L^\infty} \le \frac{C}{t^{1/2}} \|u_0\|_{L^1},$$

where C depends on the flux f.

Proof. We want to carry out estimates around the point $\sigma := f'(0)$ which lies in the domain of G and L. By definition $G(\sigma) = 0$, which means

$$L(\sigma) = \sigma G(\sigma) - f(G(\sigma)) = 0, L'(\sigma) = 0.$$

By the uniform convexity of L, there exists some constant C such that

$$\begin{aligned} tL\left(\frac{x-y}{t}\right) &= tL\left(\frac{x-y-\sigma t}{t}+\sigma\right) \\ &\geq t\left[L(\sigma)+L'(\sigma)\left(\frac{x-y-\sigma t}{t}\right)+\frac{1}{C}\left(\frac{x-y-\sigma t}{t}\right)^2\right] \\ &= \frac{1}{C}\left(\frac{|x-y-\sigma t|^2}{t}\right). \end{aligned}$$

We can rearrange and specify the minimizing point $y = y(x,t)$ for

$$\begin{aligned} \frac{|x-y-\sigma t|^2}{t} &\leq CtL\left(\frac{x-y}{t}\right) \\ &\leq C\left(tL\left(\frac{x-y}{t}\right)+h(y)+\|u_0\|_{L^1}\right) \\ &\leq C\left(tL\left(\frac{x-(x-\sigma t)}{t}\right)+h(x-\sigma t)+\|u_0\|_{L^1}\right) \\ &\leq C(2\|u_0\|_{L^1}), \\ \implies \left|\frac{x-y(x,t)}{t}-\sigma\right| &\leq \frac{C}{t^{1/2}}\|u_0\|_{L^1}. \end{aligned}$$

We conclude with the representation of u in terms of G

$$\begin{aligned} |u(x,t)| &= \left|G\left(\frac{x-y(x,t)}{t}-\sigma+\sigma\right)-G(\sigma)\right| \\ &\leq \operatorname{Lip}(G)\left|\frac{x-y(x,t)}{t}-\sigma\right|, \end{aligned}$$

where we use the fact that $G(\sigma)=0$. □

Appendix A. Fractional Sobolev Spaces

We prove some inclusions of fractional Sobolev spaces of the form $W^{s,p} \subset BV$, for use in Section 2.

Proposition A.1. *We have that $BV \subset W^{s,p}$ for all $p \in [1,\infty)$ and $s < 1/p$.*

Proof. We will prove the cases where $p=1$ and $p \geq 2$ and then we can interpolate. For the $p=1$ case, we first change variables to obtain

$$\|u\|_{W^{s,1}} = \|u\|_1 + \iint \frac{|u(x)-u(y)|}{|x-y|^{1+s}}\,\mathrm{d}x\,\mathrm{d}y.$$

We can then split the domain of the integral where $\{|h| \leq 1\}$ and $\{|h| > 1\}$, and we bound each integral

$$\begin{aligned} &\iint_{|h|>1} \frac{|u(x+h)-u(x)|}{|h|^{1+s}}\,\mathrm{d}x\,\mathrm{d}h \leq 2\|u\|_1 \int_{|h|>1} \frac{1}{|h|^{1+s}}\,\mathrm{d}h < \infty, \\ &\iint_{|h|\leq 1} \frac{|u(x+h)-u(x)|}{|h|^{1+s}}\,\mathrm{d}x\,\mathrm{d}h \leq \int_{|h|\leq 1} \frac{|h|\,\|u'\|_{\mathcal{M}}}{|h|^{1+s}} \leq \|u'\|_{\mathcal{M}} \int_{-1}^{1} \frac{1}{|h|^s}\,\mathrm{d}h < \infty. \end{aligned}$$

The $p \geq 2$ case is analogous, but we apply the following property of the Fourier transform

$$[u(\cdot + h) - u(\cdot)]^\wedge(\xi) = \hat{u}(\xi)(e^{2\pi i h\xi} - 1).$$

The Hausdorff-Young inequality yields

$$\int |u(x+h) - u(x)|\,\mathrm{d}x \leq \left(\int |\hat{u}(\xi)|^{p'} |e^{2\pi i \xi h} - 1|^{p'}\,\mathrm{d}\xi\right)^{1/p'}$$

since $1 \leq p' \leq 2$. Moreover, if m is the measure represented by u', we can write the Fourier transform as

$$\hat{u}(\xi) = \int e^{-2\pi i x\xi} u(x)\,\mathrm{d}x = \frac{1}{2\pi i|\xi|}\int e^{-2\pi i x\xi}\,\mathrm{d}m(x) \lesssim \frac{C\,\|m\|_{\mathcal{M}}}{|\xi|},$$

if $u \in BV$. □

Appendix B. Besov and Sobolev Norms

We prove that the $B^s_{p,p}$ Besov norm and the $W^{s,p}$ fractional Sobolev norms are comparable. We first prove a collection of bounding lemmata.

Lemma B.1. *The Besov norm is bounded by*

$$\|f\|^p_{B^s_{p,p}} \lesssim \|f\|^p_p + \left\|\check{\psi}_1\right\|_1^{p-1} \int_{\mathbb{R}^d} \sum_{j=1}^{\infty} 2^{jsp} \int_{\mathbb{R}^d} |\check{\psi}_j(x-y)||f(x) - f(y)|^p\,\mathrm{d}x\,\mathrm{d}y.$$

Proof. For $j = 0$, we apply Young's inequality for

$$\int_{\mathbb{R}^d} |\Delta_0 f(x)|^p\,\mathrm{d}x = \left\|f * \check{\psi}_0\right\|^p_{L^p} \leq \|f\|^p_{L^p}\left\|\check{\psi}_0\right\|^1_{L^p}.$$

For $j \geq 1$, we apply Hölder's inequality for

$$\begin{aligned} |\Delta_j f(x)| &\leq \int |\check{\psi}_j(x-y) f(y)|\,\mathrm{d}y \\ &\leq \left\|\check{\psi}_j(x-y)^{\frac{p-1}{p}}\right\|_{L^{p'}_y} \left\||\check{\psi}_j(x-y)|^{\frac{1}{p}}(f(y) - f(x))\right\|_{L^p_y} \end{aligned}$$

The $\check{\psi}_j$'s have the same L^1 norm since $\left\|\check{\psi}_j\right\|_{L^1} = \left\|2^{jd}\check{\psi}_1(2^j\cdot)\right\|_{L^1}$. □

Lemma B.2. *The following kernel satisfies an estimate of the form*

$$K(x-y) = \sum_{j=1}^{\infty} 2^{jsp}|\check{\psi}_j(x-y)| \lesssim |x-y|^{-d-sp}.$$

Proof. Since $\check{\psi}_1 \in \mathcal{S}$, we know that there exists R, C such that

$$|\check{\psi}_1(x)| \leq C \text{ for } x \in B_R, \quad |\check{\psi}_1(x)| \leq \frac{C}{|x|^{2(sp+d)}} \text{ for } x \notin B_R.$$

We can also rewrite $\check{\psi}_j(x-y) = 2^{jd}\check{\psi}_1(2^j(x-y))$, and

$$|2^j(x-y)| \leq R \iff j \leq \log_2 \frac{R}{|x-y|} =: M$$

To obtain the estimate, we split the sum based on the threshold M. □

Lemma B.3. *We have the following Bernstein-type estimate*

$$\|\Delta_j f - \tau_h \Delta_j f\|_p \lesssim \min(1, |h|)\,\|\nabla \Delta_j f\|_p \lesssim \min(1, 2^j|h|)\,\|\Delta_j f\|_p.$$

Proof. We focus on the version with the $2^j|h|$ factor on the right hand side: the first inequality follows frm an application of the fundamental theorem of calculus along the direction of h. The second inequality relies on $\Delta_j f = \check{\varphi}_{2^j} * \Delta_j f$ since $\Delta_j f$ is supported in B_{2^j} in frequency space. Differentiating yields $\nabla \Delta_j f = \nabla \check{\varphi}_{2^j} * \Delta_j f$.

We get the alternate inequality with $\|\Delta_j f - \tau_h \Delta_j\|_p \leq 2\|\Delta_j f\|_p$. ☐

Lemma B.4. *Define the increasing function* $\omega_f(r) = \sup_{|h|\leq r} \|f - \tau_h f\|_p$. *Then*

$$\|f\|_{W^{s,p}}^p \lesssim \sum_{\ell=-\infty}^{\infty} 2^{\ell sp} \omega_f(2^{-\ell})^p.$$

Proof. Changing the integral in the seminorm $[f]_{W^{s,p}}$ to polar coordinates yields

$$\left(\int \frac{1}{|h|^{d+sp}} \|f - \tau_h f\|_p^p \, \mathrm{d}h\right)^{1/p} \leq |S^{d-1}|^{1/p} \left(\int_0^\infty \frac{1}{r^{sp+1}} \omega_f(r)^p \, \mathrm{d}r\right)^{1/p}.$$

☐

Lemma B.5. *We have that*

$$(\|f\|_{W^{s,p}} - \|f\|_p)^p \lesssim \sum_{\ell=-\infty}^{\infty} \sum_{j=0}^{\infty} \left| 2^{(\ell-j)s} \min(1, 2^{j-\ell}) 2^{js} \|\Delta_j f\|_p \right|^p.$$

Proof. We combine the previous two lemmas, applying also the Littlewood-Paley decomposition $\|f\|_p \leq \sum_{j=0}^\infty \|\Delta_j f\|_p$. ☐

Proposition B.6. *The* $B^s_{p,p}$ *Besov norm and the* $W^{s,p}$ *fractional Sobolev norm are comparable, i.e.* $\|f\|_{B^s_{p,p}} \lesssim \|f\|_{W^{s,p}} \lesssim \|f\|_{B^s_{p,p}}$.

Proof. The first inequality follows from Lemma B.2. For the second inequality, Lemma B.5 tells us that

$$(\|f\|_{W^{s,p}} - \|f\|_p)^p \lesssim \sum_{\ell=-\infty}^{\infty} \sum_{j=0}^{\infty} \left| 2^{(\ell-j)s} \min(1, 2^{j-\ell}) 2^{js} \|\Delta_j f\|_p \right|^p = \|\alpha * \beta\|_p$$

where α and β are sequences defined by

$$\alpha_n := \begin{cases} 2^{ns} \|\Delta_j f\|_p & n \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$
$$\beta_n := 2^{ns} \min(1, 2^{-n}).$$

We conclude by invoking Young's convolution inequality for sequences. ☐

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