SUBQUADRATIC CAFFARELLI–KOHN–NIRENBERG EPSILON REGULARITY THEOREM

MICHAEL LEE AND JINCHENG YANG

Abstract. This expository note presents an existing interior regularity criterion for suitable weak solutions of the 3D incompressible Navier-Stokes equations, attributed to the work of Gustafson-Kang-Tsai [GKT07]. We show that at an interior point of the domain, control of the scaled $L^{5/3}$ norm of the velocity gradient suffices for regularity. We reformulate the proof in terms of a direct modification of the presentation by Vasseur [Vas07].

Contents

1. Introduction

We consider the incompressible Navier-Stokes equations in dimension 3, given by

$$\begin{aligned} &\partial_t u + (u \cdot \nabla) u + \nabla P = 0, \quad t \in (0, \infty), x \in \Omega, \\ &\mathrm{div} u = 0. \end{aligned} \tag{1}$$

where Ω is a regular subset of $\mathbb{R}^3$. Here, the vector field u represents the velocity and the scalar P represents the pressure. We take initial data in L^2 with Dirichlet boundary conditions.

Leray and Hopf proved the existence of weak solutions, which satisfy (1) in terms of distribution. However, the question of regularity remains open. The first series of papers studying partial regularity are attributed to Scheffer. Subsequently, Caffarelli-Kohn-Nirenberg proved the stunning result [CKN82] that for suitable weak solutions, the set of singular points has Hausdorff dimension at most one.

Date: November 9, 2025.
2020 *Mathematics Subject Classification.* 76D05, 35Q30.
Key words and phrases. Navier–Stokes equation, partial regularity.
Acknowledgement. The second author was partially supported by the US National Science Foundation under Grant No. DMS-1926686.

Further results by Scheffer show that this result is optimal, i.e. it possible to construct suitable weak solutions that exhibit a singular set with Hausdorff dimension arbitrarily close to one.

The existence of suitable weak solutions is also attributed to Caffarelli-Kohn-Nirenberg. They are defined as follows.

Definition 1.1. The suitable weak solutions satisfy the generalized energy inequality in the sense of distribution, which is given by

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\frac{|u|^2}{2}\right) + \operatorname{div}(uP) + |\nabla u|^2 - \Delta \frac{|u|^2}{2} \le 0 \tag{2}$$

for $t \in (0, \infty), x \in \Omega$.

This paper concerns the following regularity criterion, whose $p = 2$ case offers the key to the partial regularity result. It tells us that at an interior point (t_0, x_0), the smallness of scaled L^p norm of the velocity gradient yields boundedness, and therefore regularity of the solution.

Theorem 1.2. *Let u be a suitable weak solution. For every $p \ge \frac{5}{3}$, there exists a universal constant $\varepsilon_p > 0$, such that if near $(t_0, x_0) \in (0, \infty) \times \Omega$ the following holds*

$$\limsup_{r\to 0} \frac{1}{r^{5-2p}} \int_{Q_r(t_0,x_0)} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t < \varepsilon_p, \tag{3}$$

then (t_0, x_0) is a regular point.

Remark 1.3. A smaller choice of p represents a weaker assumption and stronger result, because we work on a bounded domain and can apply Hölder's inequality.

This existing result can be found in Theorem 1.1(ii) of [GKT07], which shows that if we satisfy (3) for $p = \frac{5}{3}$, then we have the same result for $p = 2$ and can therefore appeal to [CKN82]. We present the proof in terms of the linear iteration in Section 5 of [Vas07] and incorporate the interpolation of the cubic velocity term applied by Gustafon-Kang-Tsai in Lemmas 3.2, 3.3.

In addition to this result on velocity gradient criteria, the paper by Gustafson-Kang-Tsai presents a wide array of alternative sufficient conditions on velocity, velocity gradient, as well as vorticity gradient.

2. Notation and Preliminaries

In this section, we introduce the notations which will be used throughout the article. We also list auxiliary results related to our constructions.

C denotes a universal large constant, which may change from line to line.

For $r > 0$, let B_r represent a ball in $\mathbb{R}^3$ of radius r centered at the origin, and denote $Q_r = (-r^2, r^2) \times B_r$. We define the rescaled solutions (u_r, P_r) by

$$u_r(t, x) = ru(r^2t, rx), \qquad\qquad P_r(t, x) = r^2P(r^2t, rx).$$

If (u, P) is a solution to the Navier–Stokes equation in Q_r, then (u_r, P_r) is a solution to the Navier–Stokes equation in Q_1.

We fix a smooth radial cutoff function $\eta \in C_c^\infty(B_1)$ satisfying $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_{\frac{1}{2}}$, and $\int_{B_1} \eta \,\mathrm{d}x = 1$. Take $\eta_\lambda(x) = \eta(\frac{x}{\lambda})$. We define the weighted average velocity and pressure by

$$\tilde{u}(t) = \int_{B_1} u(t, x)\eta(x) \,\mathrm{d}x, \qquad\qquad \tilde{P}(t) = \int_{B_1} P(t, x)\eta(x) \,\mathrm{d}x.$$

$\tilde{u}_r$ and $\tilde{P}_r$ denotes the weighted average of u_r and P_r. For instance, $\tilde{u}_r$ means first rescale, then average:

$$\tilde{u}_r(t) = \int_{B_1} u_r(t,x)\eta(x)\,\mathrm{d}x = \int_{B_1} ru(r^2t, rx)\eta(x)\,\mathrm{d}x.$$

We will also use the heat kernel

$$\psi_\lambda(t,x) = \frac{1}{(2\lambda^2 - t)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4(2\lambda^2 - t)}}.$$

This function ψ_λ solves the backwards heat equation in $(-\infty, 2\lambda^2) \times \mathbb{R}^3$

$$\partial_t \psi_\lambda + \Delta \psi_\lambda = 0, \qquad\qquad \psi_\lambda(2\lambda^2, x) = \delta_0(x)$$

where δ_0 is the Dirac delta at the origin. It is easy to verify that ψ_λ has the following properties:

$$|\psi_\lambda(-1,x)| \le 1 \qquad \text{for } x \in \mathbb{R}^3, \tag{4}$$

$$|\psi_\lambda(t,x)| \ge \frac{1}{C\lambda^3} \qquad \text{for } x \in B_\lambda, -\lambda^2 \le t \le \lambda^2, \tag{5}$$

$$|\psi_\lambda| + |\nabla \psi_\lambda| \le C \qquad \text{for } x \in \mathbb{R}^3 \setminus B_{\frac{1}{2}}, -1 \le t \le \lambda^2, \tag{6}$$

$$|\psi_\lambda| \le \frac{C}{\lambda^3}, |\nabla \psi_\lambda| \le \frac{C}{\lambda^4} \qquad \text{for } x \in B_1, -1 \le t \le \lambda^2. \tag{7}$$

For $1 \le p, q \le \infty$, we write $L^p_t L^q_x(Q_r)$ to denote $L^p(-r^2, r^2; L^q(B_r))$.

3. Main result

Proposition 3.1. *There exists $\lambda \in (0, \frac{1}{4})$ and $C_\lambda > 1$, such that any suitable weak solution (u, P) in Q_1 verifies*

$$\begin{aligned}
&\|u_\lambda\|^2_{L^\infty_t L^2_x(Q_1)} + \|\nabla u_\lambda\|^2_{L^2(Q_1)} + \frac{1}{\lambda^2}\left(\|u_\lambda\|^3_{L^3(Q_1)} + \|P_\lambda\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}\right) \\
&\quad \le \frac{1}{4}\left(\|u\|^2_{L^\infty_t L^2_x(Q_1)} + \frac{1}{\lambda^2}\left(\|u\|^3_{L^3(Q_1)} + \|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}\right)\right) \\
&\qquad + \left(\|u\|^2_{L^\infty_t L^2_x(Q_1)} + \|\nabla u\|^2_{L^2_{t,x}(Q_1)}\right)\|\nabla u\|_{L^{\frac{5}{3}}_{t,x}(Q_1)}.
\end{aligned} \tag{8}$$

Proof. We will first control the kinetic energy and energy dissipation terms using the local energy inequality, then split the cubic term into the average and oscillation. Control of the pressure is achieved thanks to a local/nonlocal decomposition.

3.1. Control of the quadratic velocity terms.

Recall η and ψ_λ introduced in Section 2. We multiply (2) by $\eta(x)\psi_\lambda(t,x)$ and integrate on $[-1,s] \times \mathbb{R}^3$ for some

$s \in [-1, \lambda^2]$ to find

$$\int_{\mathbb{R}^3} \psi_\lambda(s,x)\eta(x)\frac{|u(s,x)|^2}{2}\,\mathrm{d}x + \int_{-1}^{\lambda^2}\int_{\mathbb{R}^3} \psi_\lambda(s,x)\eta(x)|\nabla u(s,x)|^2\,\mathrm{d}x\,\mathrm{d}t \tag{9}$$

$$\leq \int_{\mathbb{R}^3} \psi_\lambda(-1,x)\eta(x)\frac{|u(-1,x)|^2}{2}\,\mathrm{d}x \tag{10}$$

$$+ \int_{-1}^{\lambda^2} \left| \int_{\mathbb{R}^3} \nabla(\eta\psi_\lambda)\cdot u\left(\frac{|u|^2}{2} - \frac{|\tilde{u}|^2}{2}\right)\mathrm{d}x \right| \mathrm{d}t \tag{11}$$

$$+ \int_{-1}^{\lambda^2} \left| \int_{\mathbb{R}^3} (\psi_\lambda \Delta\eta + 2\nabla\eta\cdot\nabla\psi_\lambda)\frac{|u|^2(t,x)}{2}\,\mathrm{d}x \right| \mathrm{d}t \tag{12}$$

$$+ \int_{-1}^{\lambda^2} \left| \int_{\mathbb{R}^3} \nabla(\eta\psi_\lambda)\cdot u(P - \tilde{P})\,\mathrm{d}x \right| \mathrm{d}t, \tag{13}$$

where we use the fact that $\operatorname{div} u = 0$ and $\tilde{u}, \tilde{P}$ are independent of x.

We first show that the left-hand side (9) controls the energy of u_λ thanks to (5). For $-\lambda^2 \leq s \leq \lambda^2$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \psi_\lambda(s,x)\eta(x)\frac{|u(s,x)|^2}{2}\,\mathrm{d}x &\geq \frac{1}{C\lambda^3}\int_{B_\lambda} \frac{|u(s,x)|^2}{2}\,\mathrm{d}x \\
&= \frac{1}{C\lambda^2}\left\| u_\lambda\left(\frac{s}{\lambda^2}\right)\right\|_{L^2(B_1)}^2, \\
\int_{-1}^{\lambda^2}\int_{\mathbb{R}^3} \psi_\lambda(s,x)\eta(x)|\nabla u(s,x)|^2\,\mathrm{d}x\,\mathrm{d}t &\geq \frac{1}{C\lambda^3}\int_{-\lambda^2}^{\lambda^2}\int_{B_\lambda} |\nabla u(s,x)|^2\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{C\lambda^2}\|\nabla u_\lambda\|_{L^2(Q_1)}^2.
\end{aligned}$$

Taking the supremum for $s \in [-\lambda^2, \lambda^2]$ yields

$$\begin{aligned}
&\|u_\lambda\|_{L_t^\infty L_x^2(Q_1)}^2 + \|\nabla u_\lambda\|_{L^2(Q_1)}^2 \\
&\quad \leq C\lambda^2\left(\int_{\mathbb{R}^3} \psi_\lambda(s,x)\eta(x)\frac{|u(s,x)|^2}{2}\,\mathrm{d}x + \int_{-1}^{s}\int_{\mathbb{R}^3} \psi_\lambda(s,x)\eta(x)|\nabla u(s,x)|^2\,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned}$$

Next, we bound the four terms (10)-(13) on the right hand side. (10) can be controlled using bound (4):

$$\int_{\mathbb{R}^3} \psi_\lambda(-1,x)\eta(x)\frac{|u(-1,x)|^2}{2}\,\mathrm{d}x \leq \frac{1}{2}\|u\|_{L_t^\infty L_x^2(Q_1)}^2.$$

For (11), we apply (7) for

$$\begin{aligned}
&\int_{-1}^{\lambda^2} \left| \int_{\mathbb{R}^3} \nabla(\eta\psi_\lambda)\cdot u\left(\frac{|u|^2}{2} - \frac{|\tilde{u}|^2}{2}\right)\mathrm{d}x \right| \mathrm{d}t \\
&\qquad \leq \frac{C}{\lambda^4}\int_{-1}^{\lambda^2}\int_{B_1} |u||u-\tilde{u}||u+\tilde{u}|\,\mathrm{d}x\,\mathrm{d}t \leq \frac{C}{\lambda^4}\|u\|_{L^3(Q_1)}^3.
\end{aligned}$$

For (12), we obtain with (6)

$$\int_{-1}^{\lambda^2}\int_{\mathbb{R}^3} \left(|\nabla\eta||\nabla\psi_\lambda| + \psi_\lambda|\Delta\eta|\right)\frac{|u|^2}{2}\,\mathrm{d}x\,\mathrm{d}t \leq C\|u\|_{L_t^\infty L_x^2(Q_1)}^2.$$

The last term (13) is bounded by

$$\int_{-1}^{\lambda^2}\left|\int_{\mathbb{R}^3}\nabla(\eta\psi_\lambda)\cdot u(P-\tilde{P})\,\mathrm{d}x\right|\mathrm{d}t \leq \frac{C}{\lambda^4}\|u\|_{L^3(Q_1)}\|P\|_{L^{\frac{3}{2}}(Q_1)} \\ \leq \frac{C}{\lambda^4}\left(\|u\|^3_{L^3(Q_1)}+\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}}\right).$$

When combining the above estimates for (10)-(13), we always take the smallest power of λ. We conclude the bounds for the kinetic energy and energy dissipation of u_λ by the right-hand side of (8):

$$\|u_\lambda\|^2_{L_t^\infty L_x^2(Q_1)}+\|\nabla u_\lambda\|^2_{L^2(Q_1)} \\ \leq C\left(\lambda^2\|u\|^2_{L_t^\infty L_x^2(Q_1)}+\frac{1}{\lambda^2}\left(\|u\|^3_{L^3(Q_1)}+\|P\|^{\frac{3}{2}}_{L^3(Q_1)}\right)\right).$$

3.2. **Control of the cubic velocity term.** Here, we apply an argument from [GKT07, Lemmas 3.2, 3.3].

Lemma 3.2. *Suppose* $\lambda\in(0,\frac{1}{4})$. *Then*

$$\int_{-1}^{\lambda^2}\int_{B_\lambda}|u|^3\,\mathrm{d}x\,\mathrm{d}t \leq C\left(\lambda^3\|u\|^3_{L^3(Q_1)}+\left(\|u\|^2_{L_t^\infty L_x^2(Q_1)}+\|\nabla u\|^2_{L^2(Q_1)}\right)\|\nabla u\|_{L^{\frac{5}{3}}(Q_1)}\right).$$

Proof. We split $|u|\leq|\tilde{u}|+|u-\tilde{u}|$. We consider the scaling of the average

$$\int_{B_\lambda}|\tilde{u}(\cdot)|^3\,\mathrm{d}x \leq \int_{B_\lambda}\left(\frac{1}{|B_1|}\int_{B_1}|u(\cdot,x')|\,\mathrm{d}x'\right)^3\mathrm{d}x \leq C\lambda^3\left(\int_{B_1}|u(\cdot,x)|\,\mathrm{d}x\right)^3,$$

so we integrate in time to get

$$\int_{-1}^{\lambda^2}\int_{B_\lambda}|\tilde{u}|^3\,\mathrm{d}x\,\mathrm{d}t \leq C\lambda^3\|u\|_{L^3(Q_1)}.$$

For the velocity oscillation, we can interpolate for

$$\|u-\tilde{u}\|^3_{L^3_{t,x}} \leq \|u-\tilde{u}\|^{\frac{6}{5}}_{L_t^\infty L_x^2}\|u-\tilde{u}\|^{\frac{4}{5}}_{L_t^2L_x^6}\|u-\tilde{u}\|_{L_t^{\frac{5}{3}}L_x^{\frac{15}{4}}} \\ \leq \|u\|^{\frac{6}{5}}_{L_t^\infty L_x^2}\|\nabla u\|^{\frac{4}{5}}_{L^2_{t,x}}\|\nabla u\|_{L^{\frac{5}{3}}_{t,x}},$$

whichis bounded above by $C\left(\|u\|^2_{L_t^\infty L_x^2}+\|\nabla u\|^2_{L^2_{t,x}}\right)\|\nabla u\|_{L^{\frac{5}{3}}_{t,x}}$. □

We apply Lemma 3.2 to get

$$\|u_\lambda\|^3_{L^3(Q_1)} = \int_{-1}^{1}\int_{B_1}\left|\lambda u(\lambda^2t,\lambda x)\right|^3\mathrm{d}x\,\mathrm{d}t \\ \leq \frac{C}{\lambda^2}\int_{-\lambda^2}^{\lambda^2}\int_{B_\lambda}\left(|\tilde{u}|^3+|u-\tilde{u}|^3\right)\mathrm{d}x\,\mathrm{d}t \\ \leq \frac{C}{\lambda^2}\left(\lambda^3\|u\|^3_{L^3(Q_1)}+\left(\|u\|^2_{L_t^\infty L_x^2}+\|\nabla u\|^2_{L^2_{t,x}}\right)\|\nabla u\|_{L^{\frac{5}{3}}_{t,x}}\right) \\ \leq C\lambda\|u\|^3_{L^3(Q_1)}+\frac{C}{\lambda^2}\left(\|u\|^2_{L_t^\infty L_x^2}+\|\nabla u\|^2_{L^2_{t,x}}\right)\|\nabla u\|_{L^{\frac{5}{3}}_{t,x}}.$$

3.3. **Control of the pressure term.** The pressure is handled with a decomposition of the following form. Recall that taking the divergence of the Navier–Stokes equation yields

$$\operatorname{div}\operatorname{div}(u\otimes u)+\Delta P=0.$$

As u is divergence-free, we can subtract $\tilde{u}$ from it and have

$$\operatorname{div}\operatorname{div}((u-\tilde{u})\otimes(u-\tilde{u}))+\Delta P=0.$$

We define the local pressure by computing the Newtonian potential of the localized advection divergence

$$P^{\mathrm{loc}}=(-\Delta)^{-1}\operatorname{div}\operatorname{div}\left(\eta(u-\tilde{u})\otimes(u-\tilde{u})\right),$$

and the non-local pressure $P^{\mathrm{har}}=P-P^{\mathrm{loc}}$. The notation is due to the fact that P^{har} is harmonic in the set $\{\eta=1\}$:

$$\Delta P^{\mathrm{har}}=\operatorname{div}\operatorname{div}((1-\eta)(u-\tilde{u})\otimes(u-\tilde{u})).$$

For the local pressure term P^{loc}, recall that η is supported in B_1. We apply the Riesz Theorem for the singular integral operator $(-\Delta)^{-1}\operatorname{div}\operatorname{div}$ and conclude

$$\begin{aligned}\left\|P^{\mathrm{loc}}\right\|_{L^{\frac{3}{2}}(Q_1)}&\le\left\|\eta(u-\tilde{u})\otimes(u-\tilde{u})\right\|_{L^{\frac{3}{2}}(Q_1)}\\&\le C\left\|u-\tilde{u}\right\|^2_{L^3(Q_1)}.\end{aligned}$$

As for the non-local term P^{har}, we apply the mean value property of harmonic functions for

$$\begin{aligned}\int_{B_\lambda}|P^{\mathrm{har}}(x)|^{\frac{3}{2}}\,\mathrm{d}x&=\int_{B_\lambda}\left|\frac{1}{|B_{\frac{1}{8}}|}\int_{B_{\frac{1}{8}}}|P^{\mathrm{har}}(x+y)|\,\mathrm{d}y\right|^{\frac{3}{2}}\mathrm{d}x\\&\le\int_{B_\lambda}\frac{1}{|B_{\frac{1}{8}}|}\int_{B_{\frac{1}{8}}}|P^{\mathrm{har}}(x+y)|^{\frac{3}{2}}\,\mathrm{d}y\,\mathrm{d}x\\&\le\frac{1}{|B_{\frac{1}{8}}|}\int_{B_\lambda}\int_{B_{\frac{3}{8}}}|P^{\mathrm{har}}(z)|^{\frac{3}{2}}\,\mathrm{d}z\,\mathrm{d}x\\&\le(8\lambda)^3\int_{B_{\frac{3}{8}}}|P^{\mathrm{har}}(z)|^{\frac{3}{2}}\,\mathrm{d}z.\end{aligned}$$

Then we apply the triangle inequality at the larger scale for

$$\left\|P^{\mathrm{har}}\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_\lambda)}\le C\lambda^3\left\|P^{\mathrm{har}}\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}\le C\lambda^3\left(\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}+\left\|P^{\mathrm{loc}}\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}\right).$$

Together with the estimate on P^{loc} we conclude

$$\begin{aligned}\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_\lambda)}&\le C\lambda^3\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}+C\left\|P^{\mathrm{loc}}\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}\\&\le C\lambda^3\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}+C\left\|u-\tilde{u}\right\|^3_{L^3(Q_1)}\\&\le C\lambda^3\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)}+C\left(\|u\|^2_{L^\infty_tL^2_x}+\|\nabla u\|^2_{L^2_{t,x}}\right)\|\nabla u\|_{L^{\frac{5}{3}}_{t,x}}.\end{aligned}$$

The pressure term scales in the following way

$$\begin{aligned}\|P_\lambda\|_{L^{\frac{3}{2}}(Q_1)}^{\frac{3}{2}} &= \frac{1}{\lambda^2}\|P\|_{L^{\frac{3}{2}}(Q_\lambda)}^{\frac{3}{2}} \\ &\leq C\lambda\|P\|_{L^{\frac{3}{2}}(Q_1)}^{\frac{3}{2}} + \frac{C}{\lambda^2}\left(\|u\|_{L_t^\infty L_x^2}^2 + \|\nabla u\|_{L_{t,x}^2}^2\right)\|\nabla u\|_{L_{t,x}^{\frac{5}{3}}}.\end{aligned}$$

Together with the control of the velocity terms, we have shown that

$$\begin{aligned}&\|u_\lambda\|_{L_t^\infty L_x^2(Q_1)}^2 + \|\nabla u_\lambda\|_{L^2(Q_1)}^2 + \frac{1}{\lambda^2}\left(\|u_\lambda\|_{L^3(Q_1)}^3 + \|P_\lambda\|_{L^{\frac{3}{2}}(Q_1)}^{\frac{3}{2}}\right) \\ &\leq C\lambda\left(\|u\|_{L_t^\infty L_x^2(Q_1)}^2 + \frac{1}{\lambda^2}\left(\|u\|_{L^3(Q_1)}^3 + \|P\|_{L^{\frac{3}{2}}(Q_1)}^{\frac{3}{2}}\right)\right) \\ &\quad + \frac{C}{\lambda^4}\left(\|u\|_{L_t^\infty L_x^2(Q_1)}^2 + \|\nabla u\|_{L_{t,x}^2(Q_1)}^2\right)\|\nabla u\|_{L_{t,x}^{\frac{5}{3}}(Q_1)}.\end{aligned} \tag{14}$$

The proof is completed by choosing λ small such that $C\lambda \leq \frac{1}{4}$. □

Corollary 3.3. *For every $\varepsilon_2 > 0$, there exists $\varepsilon_{\frac{5}{3}} > 0$, such that if*

$$\limsup_{r\to 0}\|\nabla u_r\|_{L^{\frac{5}{3}}(Q_1)} < \varepsilon_{\frac{5}{3}},$$

then

$$\limsup_{r\to 0}\|\nabla u_r\|_{L^2(Q_1)} < \varepsilon_2.$$

Proof. Define

$$U(r) := \|u_r\|_{L_t^\infty L_x^2(Q_1)}^2 + \|\nabla u_r\|_{L^2(Q_1)}^2 + \frac{1}{\lambda^2}\left(\|u_r\|_{L^3(Q_1)}^3 + \|P_r\|_{L^{\frac{3}{2}}(Q_1)}^{\frac{3}{2}}\right).$$

Then by applying Proposition 3.1 to u_r, we can find some $\lambda \in (0, \frac{1}{4})$ and obtain the recursive relation

$$U(\lambda r) \leq \frac{1}{4}U(r) + C_\lambda\left(\|u_r\|_{L_t^\infty L_x^2}^2 + \|\nabla u_r\|_{L_{t,x}^2}^2\right)\|\nabla u_r\|_{L_{t,x}^{\frac{5}{3}}}.$$

We drop the dependence of C_λ in λ because λ has been fixed.

By our assumption, there exists $r_0 > 0$ such that for all $r < r_0$ we have

$$\|\nabla u_r\|_{L^{\frac{9}{5}}(Q_1)} \leq \varepsilon_{\frac{5}{3}},$$

where the value of $\varepsilon_{\frac{5}{3}}$ will be determined later. Therefore, for all $r \leq r_0$ we have

$$U(\lambda r) \leq \frac{1}{4}U(r) + CU(r)\varepsilon_{\frac{5}{3}} \leq \frac{1}{2}U(r)$$

provided $\varepsilon_{\frac{5}{3}}$ is sufficiently small. Thus we conclude that

$$\limsup_{r\to 0}\|\nabla u_r\|_{L^2(Q_1)}^2 \leq \limsup_{r\to 0}U(r) = 0.$$

□

References

[CKN82] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the navier-stokes equations. *Communications on Pure and Applied Mathematics*, 35(6):771–831, 1982.

[GKT07] Stephen Gustafson, Kyungkeun Kang, and Tai-Peng Tsai. Interior regularity criteria for suitable weak solutions of the navier-stokes equations. *Communications in Mathematical Physics*, 273(1):161–176, March 2007.

[Vas07] Alexis F. Vasseur. A new proof of partial regularity of solutions to navier-stokes equations. *Nonlinear Differential Equations and Applications NoDEA*, 14(5):753–785, Dec 2007.

Department of Mathematics, University of Chicago, 5734 S University Ave, Chicago, IL 60637, USA

Email address: leemichaeljh@uchicago.edu

School of Mathematics, Institute for Advanced Study, 1 Einstein Dr, Princeton, NJ 08540, USA

Email address: jcyang@ias.edu