MATRIX FACTORIZATIONS

- A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices.
- Whereas matrix multiplication involves a *synthesis* of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an *analysis* of data.

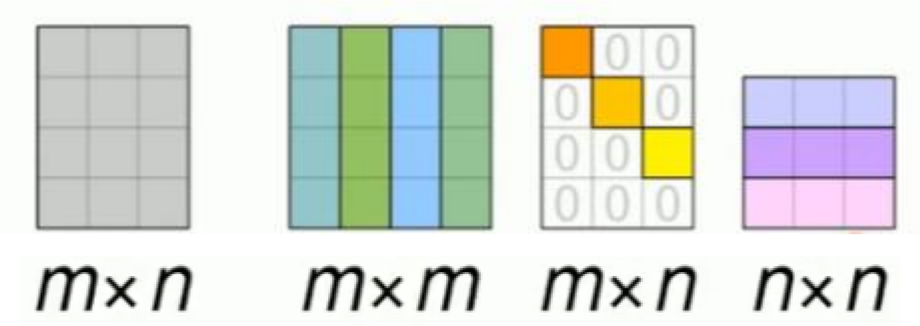
Examples? Diagonalize matrix $A = PDP^{-1}$

Eigen Decomposition.

THE SINGULAR VALUE DECOMPOSITION

- **Theorem: The Singular Value Decomposition** Let A be an $m \times n$ matrix with rank r. Then there exists an $m \times n$ diagonal matrix Σ whose diagonal entries are non-negative (the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$), and there exist an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that

$$A = U\Sigma V^T$$



THE SINGULAR VALUE DECOMPOSITION

- The columns of U in such a decomposition are called **left singular vectors** of A, and the columns of V are called **right singular vectors** of A.
- The diagonal entries of Σ are called the **singular values** of A

$$A = U\Sigma V^T$$

THE SINGULAR VALUE DECOMPOSITION

- **Proof** Let λ_i and v_i be as in Theorem , so that $\{Av_1, . . . , Av_r\}$ is an orthogonal basis for Col A.
- Normalize each Av_i to obtain an orthonormal basis $\{u_1, . . . , u_r\}$, where

$$u_i = \frac{1}{||Av_i||} Av_i = \frac{1}{\sigma_1} Av_i$$

(*)

- And

$$Av_i = \sigma_i u_i \qquad (1 \le i \le r)$$

- Now extend $\{u_1, . . . , u_r\}$ to an orthonormal basis $\{u_1, . . . , u_m\}$ of $\mathbb{R}^m$, and let

$$U = [u_1 \ \ u_2 \ \dots \ um] \ \text{ and } V = [v_1 \ \ v_2 \ \dots \ vm]$$

- By construction, U and V are orthogonal matrices.

THE SINGULAR VALUE DECOMPOSITION

- Also, from (*),

  $AV = [Ax_1 \;\; \ldots \;\; Avr \;\;\; 0 \;\; \ldots \;\; 0] = [\sigma_1 u_1 \ldots \sigma_r u_r \;\; 0 \ldots 0]$

- Let D be the diagonal matrix with diagonal entries $\sigma_1, \ldots, \sigma_r$, and let Σ be as follow. Then

$$U\Sigma = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_m\,]\left[\begin{array}{cccc|c} \sigma_1 & & & 0 & \\ & \sigma_2 & & & 0 \\ & & \ddots & & \\ 0 & & & \sigma_r & \\ \hline & & 0 & & 0 \end{array}\right]$$

$$= [\,\sigma_1\mathbf{u}_1 \quad \cdots \quad \sigma_r\mathbf{u}_r \quad \mathbf{0} \quad \cdots \quad \mathbf{0}\,]$$

$$= AV$$

- Since V is an orthogonal matrix, $U\Sigma V^T = AVVT = A$.

THE SINGULAR VALUE DECOMPOSITION

- **Example** Construct a singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution A construction can be divided into three steps.

- ***Step 1. Find an orthogonal diagonalization of A^TA***. That is, find the eigenvalues of A^TA and a corresponding orthonormal set of eigenvectors
- ***Step 2. Set up V and Σ***. Arrange the eigenvalues of A^TA in decreasing order.
- ***Step 3. Construct U***. When A has rank r, the first r columns of U are the normalized vectors obtained from $Av_1, \ldots, Av_r$.

THE SINGULAR VALUE DECOMPOSITION

- **Example** Construct a singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - ***Step 1. Find an orthogonal diagonalization of $A^T A$.***

$$A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{det(A^T A - \lambda I) = 0}$$

The eigenvalues are $\lambda = 2$, $\lambda = 1$, and $\lambda = 0$

THE SINGULAR VALUE DECOMPOSITION

- **Example** Construct a singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ***Step 1. Find an orthogonal diagonalization of A^TA.***

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{det(A^TA-\lambda I)=0}$$

  - Basis for $\lambda = 2$: $\mathrm{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

  - Basis for $\lambda = 1$: $\mathrm{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and Basis for $\lambda = 0$: $\mathrm{v}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

THE SINGULAR VALUE DECOMPOSITION

- ***Step 2*. *Set up V and Σ*.** Arrange the eigenvalues of A^TA in **decreasing order**. The corresponding unit eigenvectors, v_1, v_2, and v_3, are the right singular vectors of *A*.

$$V = [v_1 \ \ v_2 \ \ v_3] = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

- The square roots of the eigenvalues are the singular values:

$$\sigma_1 = \sqrt{2}, \quad \sigma_2 = 1, \quad \sigma_3 = 0$$

THE SINGULAR VALUE DECOMPOSITION

- The matrix Σ is

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- ***Step 3*. *Construct U*.** When A has rank r, the first r columns of U are the normalized vectors obtained from $Av_1, \ldots, Av_r$.

$$A = U\Sigma V^T \qquad AV = U\Sigma$$

THE SINGULAR VALUE DECOMPOSITION

- Thus

$$u_1 = \frac{1}{\sigma_1} A v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$u_2 = \frac{1}{\sigma_2} A v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Note that $\{u_1, u_2\}$ is already a basis for $\mathbb{R}^2$. Thus no additional vectors are needed for U, and $U = [u_1 \ \ u_2]$. The singular value decomposition of A is

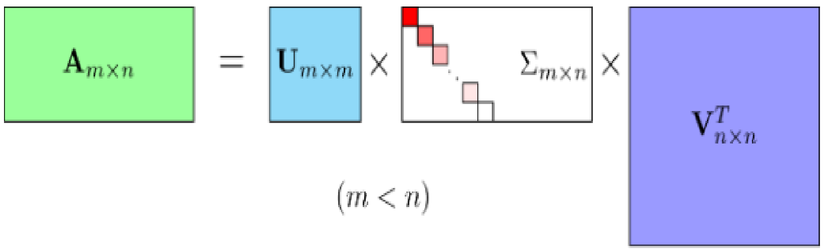
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

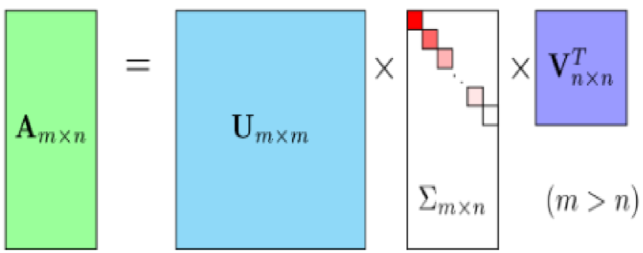
THE SINGULAR VALUE DECOMPOSITION

- **Example** Construct a singular value decomposition of

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

The Singular Value Decomposition

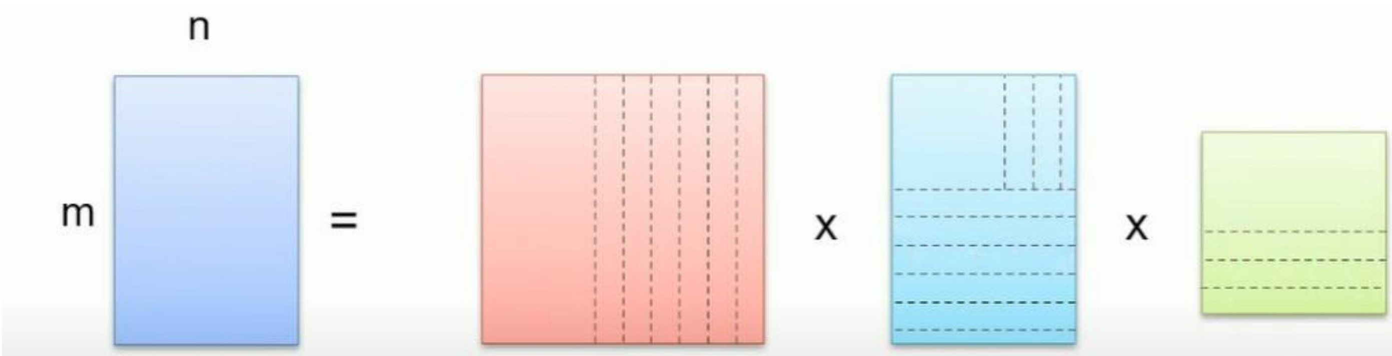


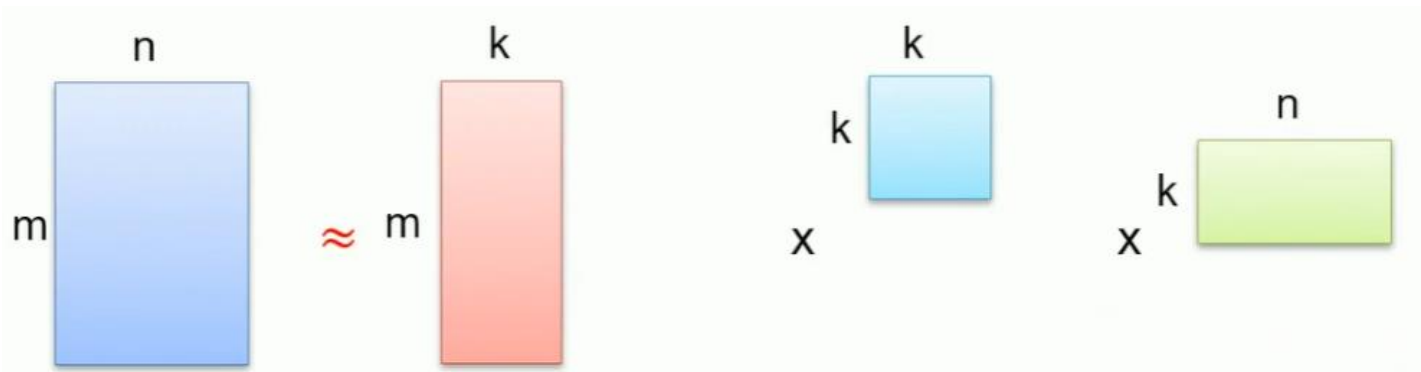


$$\mathbf{A} \approx \mathbf{A}_k = \mathbf{U}_k\Sigma_k(\mathbf{V}_k)^T = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^2 + \cdots + \sigma_k\mathbf{u}_k\mathbf{v}k^T$$

$$||\mathbf{A} - \mathbf{A}_k||_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$$

Image Compression





$$nk + k + km$$

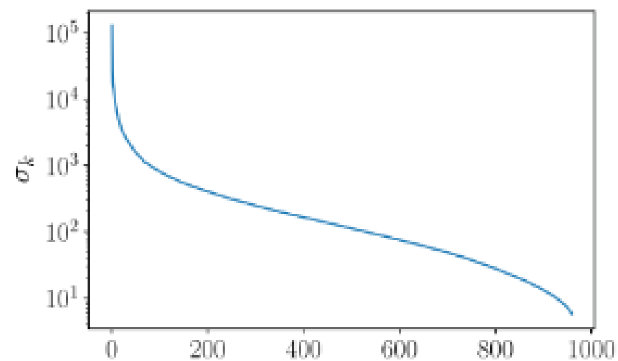
$$nm$$

$$\frac{nk+k+km}{nm}$$

The Singular Value Decomposition

$$\mathbf{A} \approx \mathbf{A}_k = \mathbf{U}_k \Sigma_k (\mathbf{V}_k)^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^2 + \cdots + \sigma_k \mathbf{u}_k \mathbf{v} k^T$$

$$||\mathbf{A} - \mathbf{A}_k||_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$$



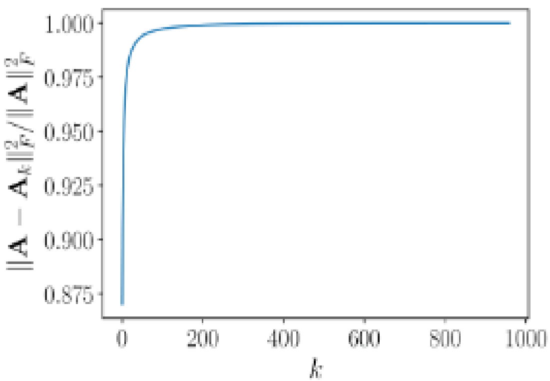
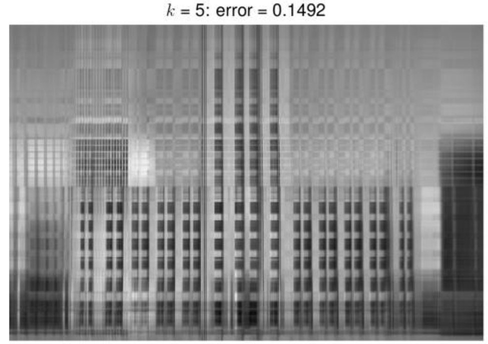
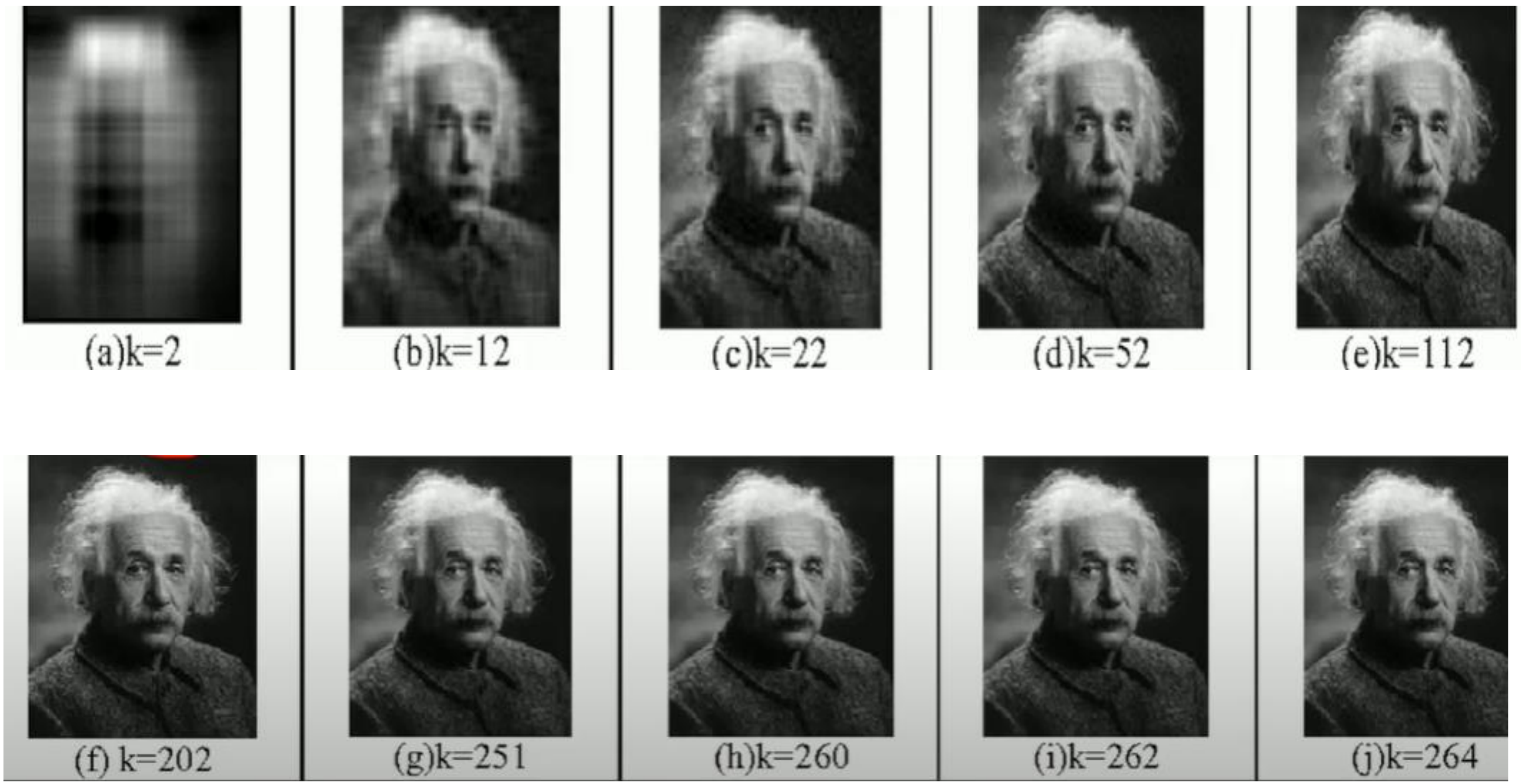


Image Compression

960×1440.

k = 5: error = 0.1492




(a)k=2
(b)k=12
(c)k=22
(d)k=52
(e)k=112
(f) k=202
(g)k=251
(h)k=260
(i)k=262
(j)k=264

QR FACTORIZATION OF MATRICES

- **Theorem: The QR Factorization**
- If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for Col$A$ and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

- **Proof** The columns of A form a basis $\{x_1, \ldots, x_n\}$ for Col A. Construct an orthonormal basis $\{u_1, \ldots, u_n\}$ for W = Col A with property (1) in Theorem. This basis may be constructed by the Gram-Schmidt process or some other means.

QR FACTORIZATION OF MATRICES

- Let

$$Q = [\mathbf{u}_1 \;\; \mathbf{u}_2 \;\; \ldots \;\; \mathbf{u}_n]$$

- For $k = 1, \ldots, k$, $\mathbf{x}_k$ is in Span{$x_1, \ldots, x_k$} = Span{$u_1, \ldots, u_k$}. So there are constants, $r_{1k}, \ldots, r_{kk}$, such that

$$x_k = r_{1k}\mathbf{u}_1 + \cdots + r_{kk}\mathbf{u}_k + 0 \cdot \mathbf{u}_{k+1} + \cdots + 0 \cdot \mathbf{u}_n$$

- We may assume that $r_{kk} \geq 0$. This shows that x_k is a linear combination of the columns of Q using as weights the entries in the vector

QR FACTORIZATION OF MATRICES

$$\mathrm{r}_k = \begin{bmatrix} r_{1k} \\ \cdot \\ \cdot \\ \cdot \\ r_{kk} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

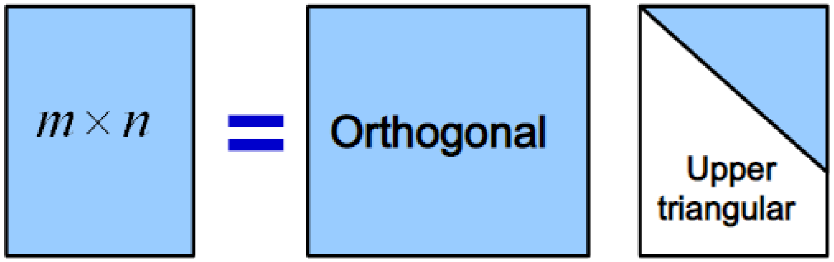
- That is, $\mathrm{x}_k = Q\mathrm{r}_k$ for $k = 1, \ldots, n$. Let $R = [\mathrm{r}_1 \ \ldots \ \mathrm{r}_n]$. Then

$$A = [\mathrm{x}_1 \ \ldots \ \mathrm{x}_n] = [Q\mathrm{r}_1 \ \ldots \ Q\mathrm{r}_n] = QR$$

- The fact that R is invertible follows easily from the fact that the columns of A are linearly independent. Since R is clearly upper triangular, its nonnegative diagonal entries must be positive.

QR FACTORIZATION OF MATRICES

$$A = QR$$



To find QR decomposition:

1.) Q: Use Gram-Schmidt to find orthonormal basis for column space of A

2.) Let $R = Q^T A$

QR FACTORIZATION OF MATRICES

$$A = QR$$

$$Q^{-1}A = Q^{-1}QR$$

$$Q^{-1}A = R$$

Q has orthonormal columns:

Thus $Q^{-1} = Q^T$

Thus $R = Q^{-1}A = Q^TA$

THE GRAM-SCHMIDT PROCESS

- **The Gram-Schmidt Process**
- Given a basis $\{x_1, \ldots, x_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$\vdots$$

$$v_p = xp - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \ldots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}$$

- Then $\{v_1, \ldots, v_p\}$ is an orthogonal basis for W. In addition

$$\text{Span}\{v_1, \ldots, v_k\} = \text{Span}\{x_1, \ldots, x_k\} \quad \text{for } 1 \le k \le p \qquad (1)$$

QR FACTORIZATION OF MATRICES

- **Example:** Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
  - **Solution** The columns of A are the vectors x_1, x_2, and x_3. An orthogonal basis for Col A = Span$\{x_1, x_2, x_3\}$ was found in that example:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

QR FACTORIZATION OF MATRICES

- To simplify the arithmetic that follows, scale $\mathbf{v}_3$ by letting $\mathbf{v}_3 = 3\mathbf{v}_3$. Then normalize the three vectors to obtain $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, and use these vectors as the columns of Q:

$$Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}.$$

- By construction, the first k columns of Q are an orthonormal basis of Span$\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$.

QR FACTORIZATION OF MATRICES

- From the proof of Theorem, $A = QR$ for some R. To find R, observe that $Q^TQ = I$, because the columns of Q are orthonormal. Hence

$$Q^T A = QT(QR) = IR = R$$

- and

$$R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 3/\sqrt{12} \end{bmatrix}$$

Find the QR decomposition of

$$A = \begin{bmatrix} 1 & -1 & 6 \\ 1 & 2 & 0 \\ 1 & 3 & -1 \\ 1 & 4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{10}} \\ \frac{1}{2} & 0 & \frac{-2}{\sqrt{10}} \\ \frac{1}{2} & \frac{1}{\sqrt{14}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{2} & \frac{2}{\sqrt{14}} & \frac{2}{\sqrt{10}} \end{bmatrix} = Q$$

$$\begin{bmatrix} 1 & -3 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-3}{\sqrt{14}} & 0 & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{10}} & \frac{-2}{\sqrt{10}} & \frac{-1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 1 & -1 & 6 \\ 1 & 2 & 0 \\ 1 & 3 & -1 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & \frac{14}{\sqrt{14}} & \frac{-2}{\sqrt{14}} \\ 0 & 0 & \frac{5}{\sqrt{10}} \end{bmatrix}$$

Matrix A:

$$\begin{pmatrix} 1.0 & -1.0 & 4.0 \\ 1.0 & 4.0 & -2.0 \\ 1.0 & 4.0 & 2.0 \\ 1.0 & -1.0 & 0.0 \end{pmatrix}$$

Matrix with orthonormal basis:

$$\begin{pmatrix} -0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & -0.5 \\ -0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & 0.5 \end{pmatrix}$$

Upper diagonal matrix:

$$\begin{pmatrix} -2.0 & -3.0 & -2.0 \\ 0.0 & -5.0 & 2.0 \\ 0.0 & 0.0 & -4.0 \\ 0.0 & -0.0 & -0.0 \end{pmatrix}$$

Find the QR decomposition of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

1.) Use Gram-Schmidt to find orthogonal basis for column space of A

$$\text{col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$$

$$= \text{span}\left\{\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}\right\}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$$

QR Decomposition-Based Algorithm for Background Subtraction

Figure 1. Sample frames of a traffic movie[1].

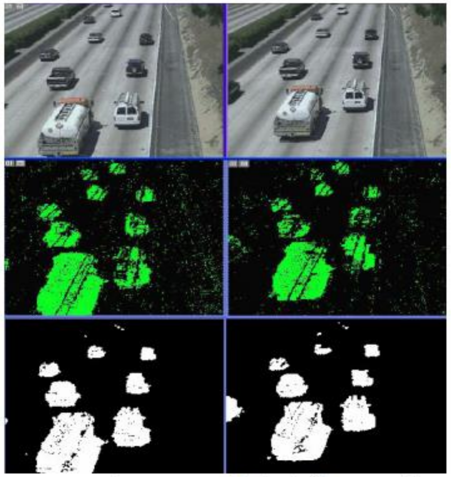


Figure 5. From top to bottom: original frames, foreground object detection using GMM (taken from [11]) and the proposed approach.

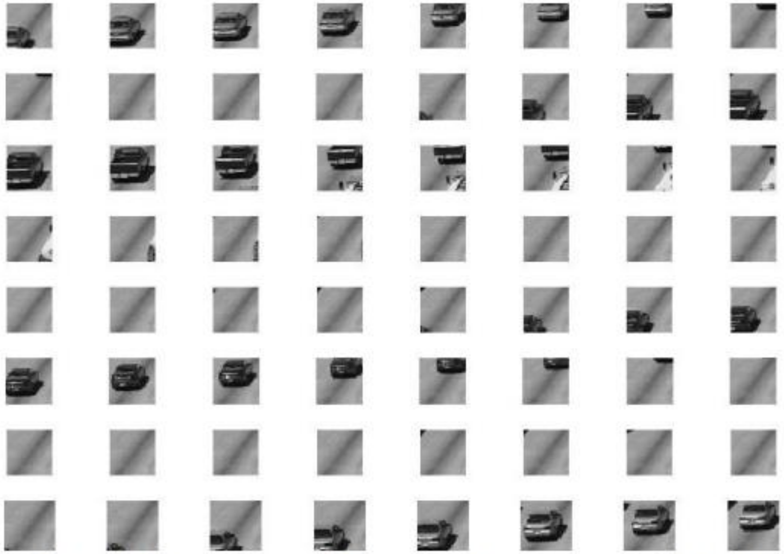


Figure 2. 64 Consequence frames of an instance block in its original order.

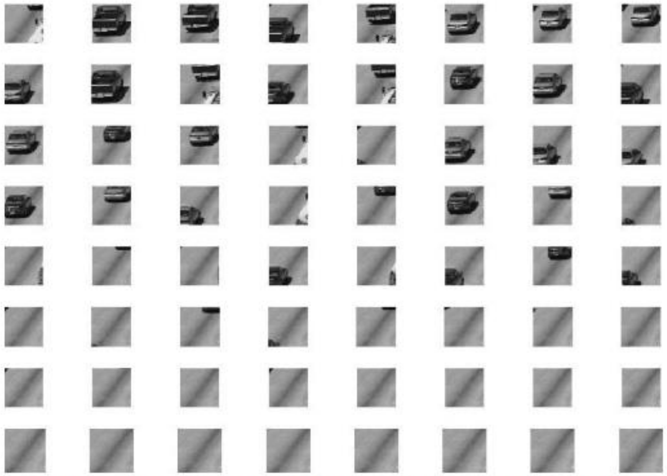


Figure 3. Rearrangement of blocks in Figure 2 based on their R-values' order. A can be seen, background frames has been shifted to the end.