Landmark-based Registration

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Aalto-universitetet
Aalto University

Medical Image Analysis

Koen Van Leemput

Fall 2024

Image registration

Combine information contained in different scans

Images need to be spatially aligned!

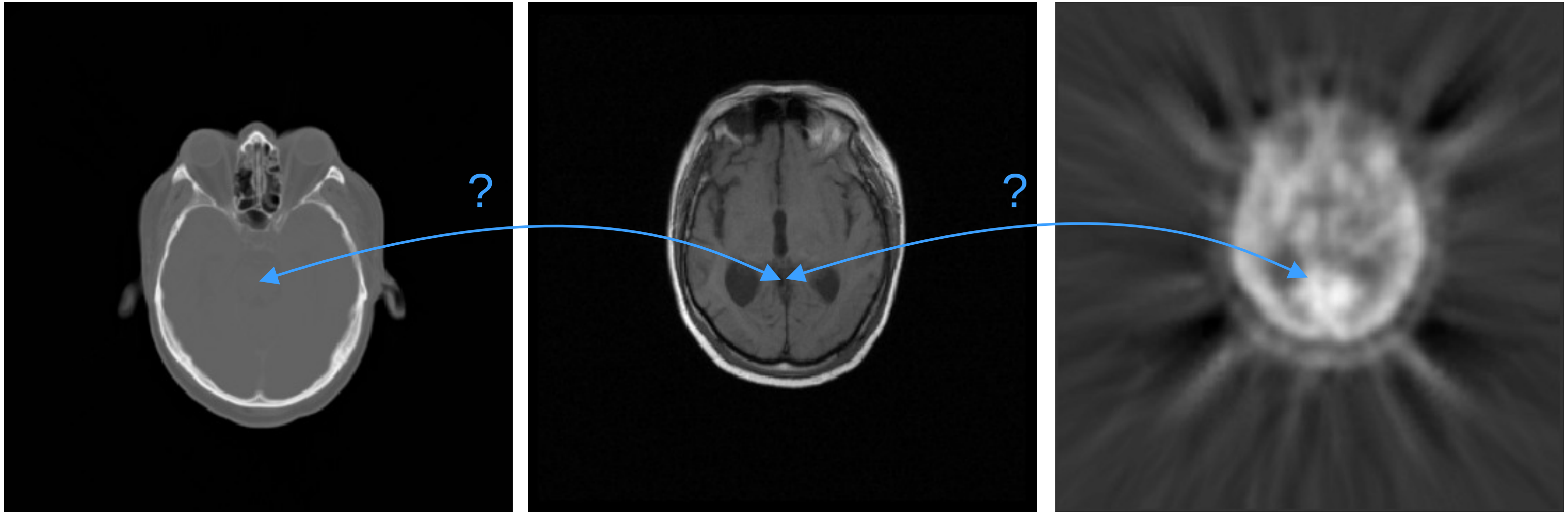
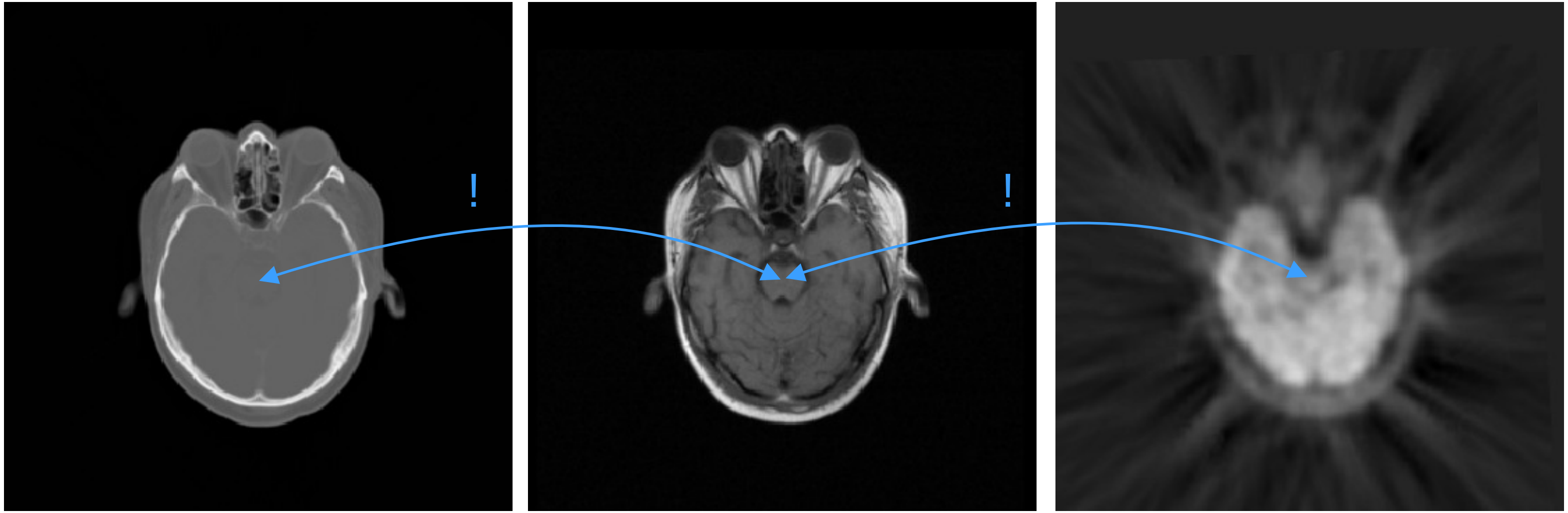


Image registration

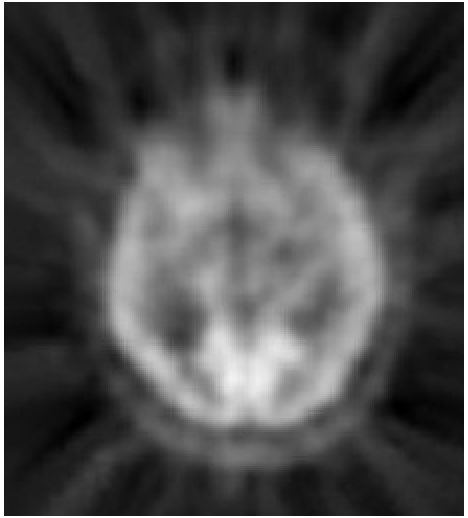
Combine information contained in different scans

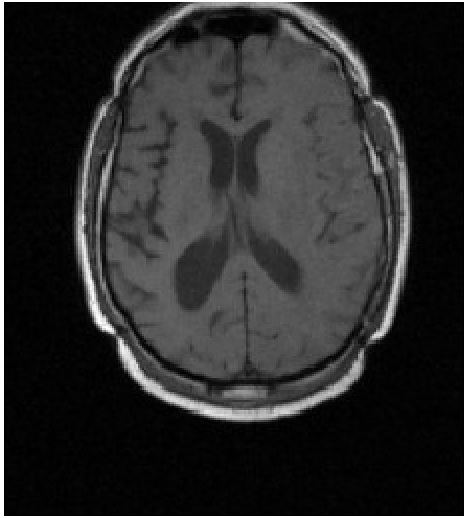
Images need to be spatially aligned!



Example: PET/MR

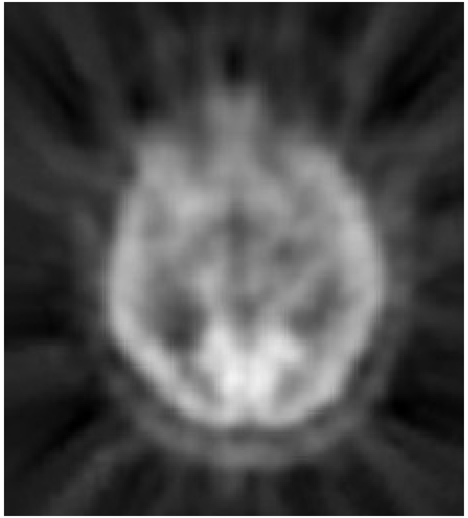
Before registration...

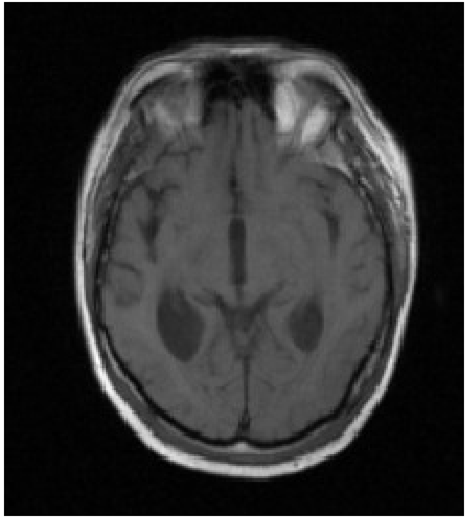


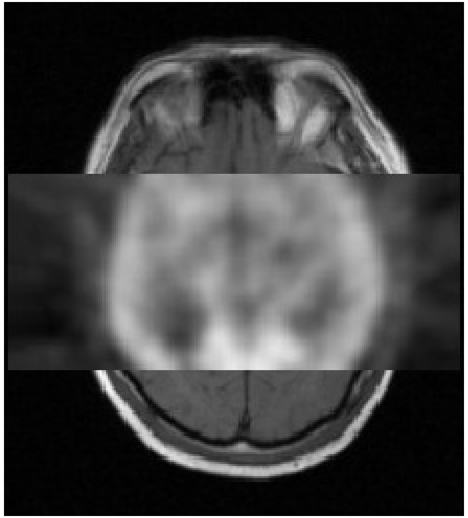


Example: PET/MR

… after registration

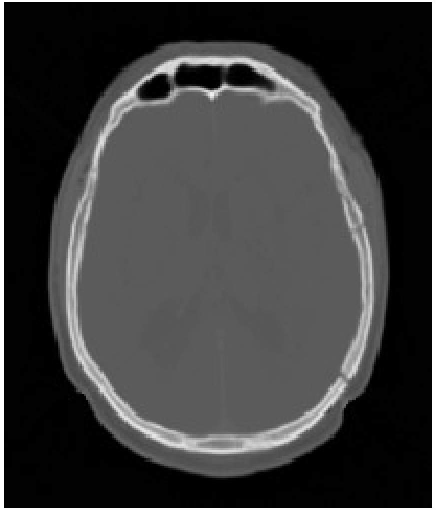


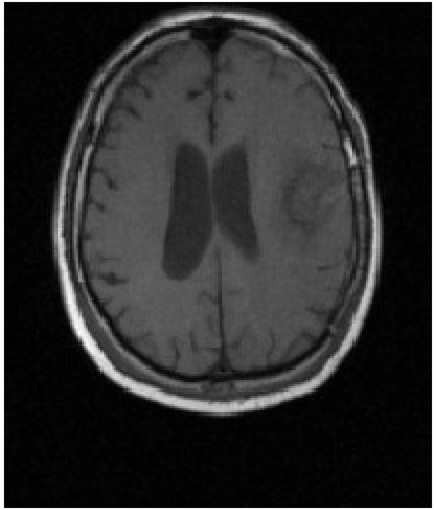




Example: CT/MR

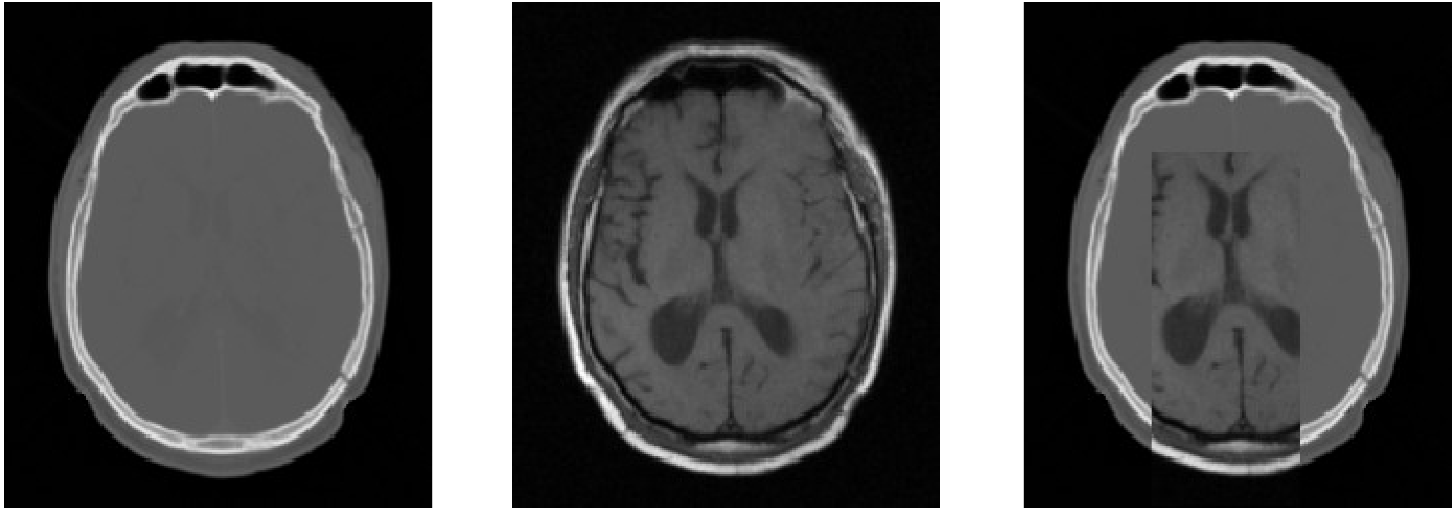
Before registration…





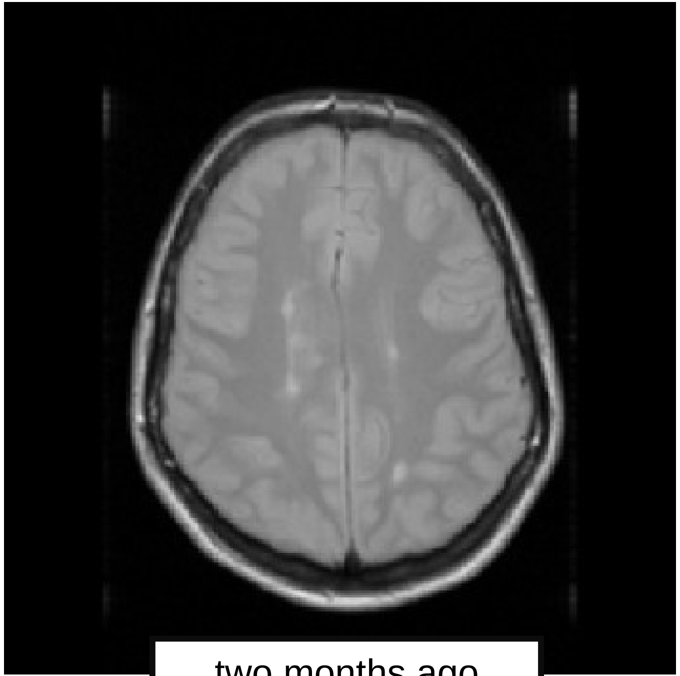
Example: CT/MR

… after registration



Example: longitudinal scans

Patient with multiple sclerosis (MS)



two months ago

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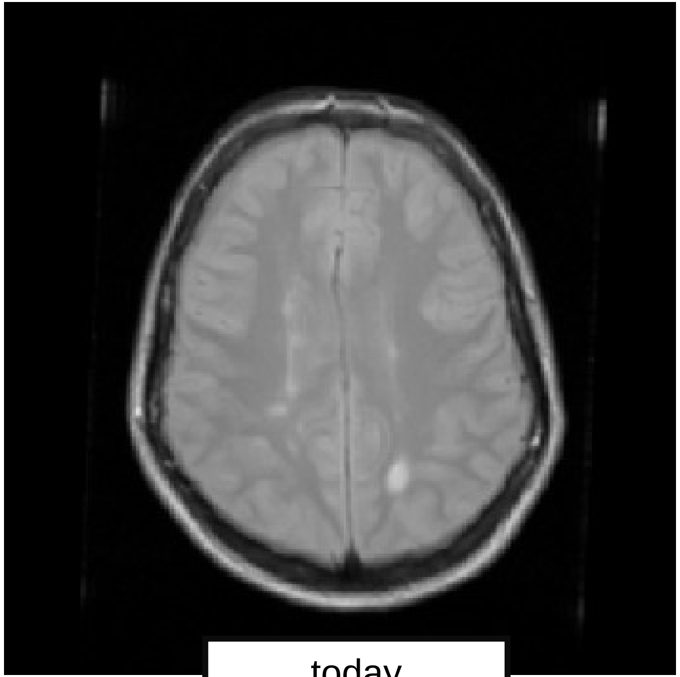
Example: longitudinal scans

Patient with multiple sclerosis (MS)



today

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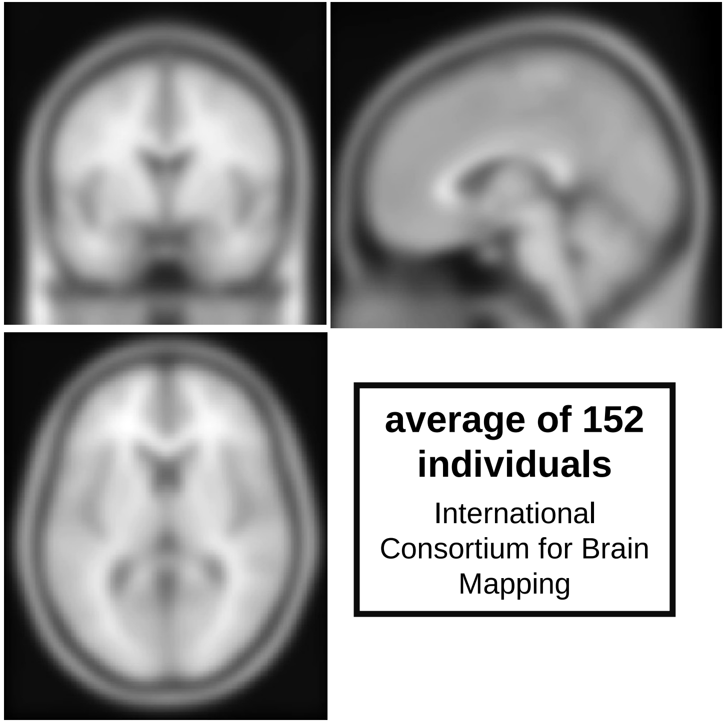
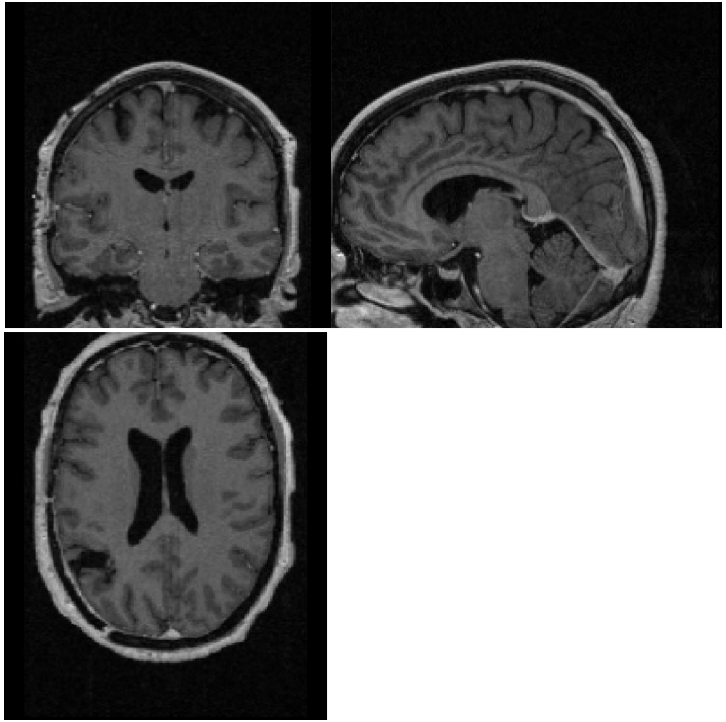
Example: population studies

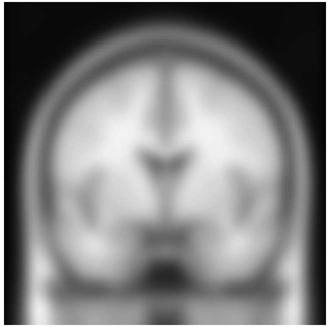
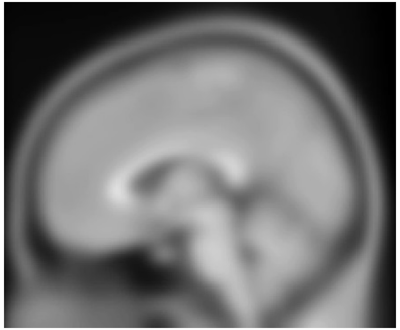
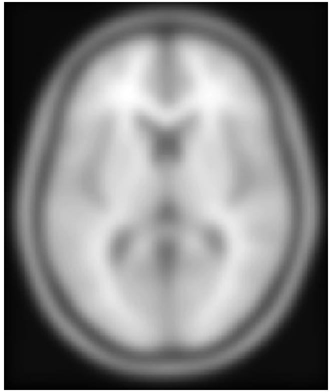


average of 152 individuals

International Consortium for Brain Mapping



Example: population studies

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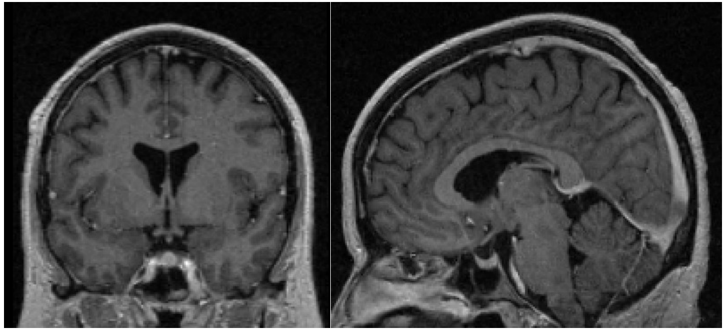
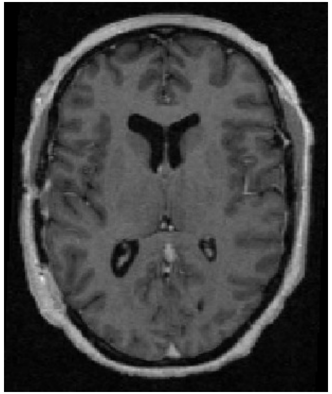
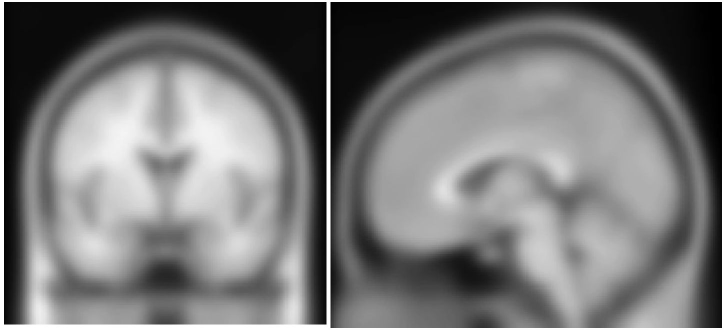
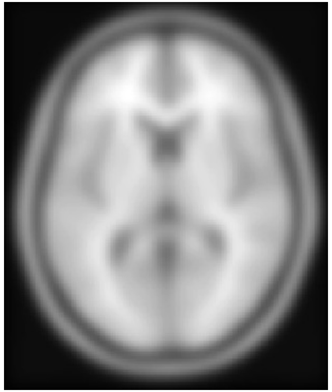



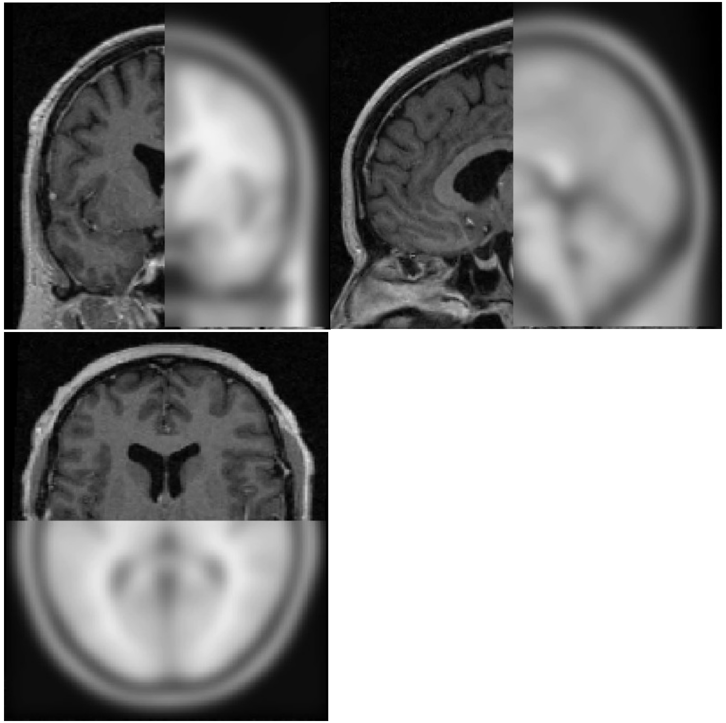
Example: population studies





average of 152 individuals

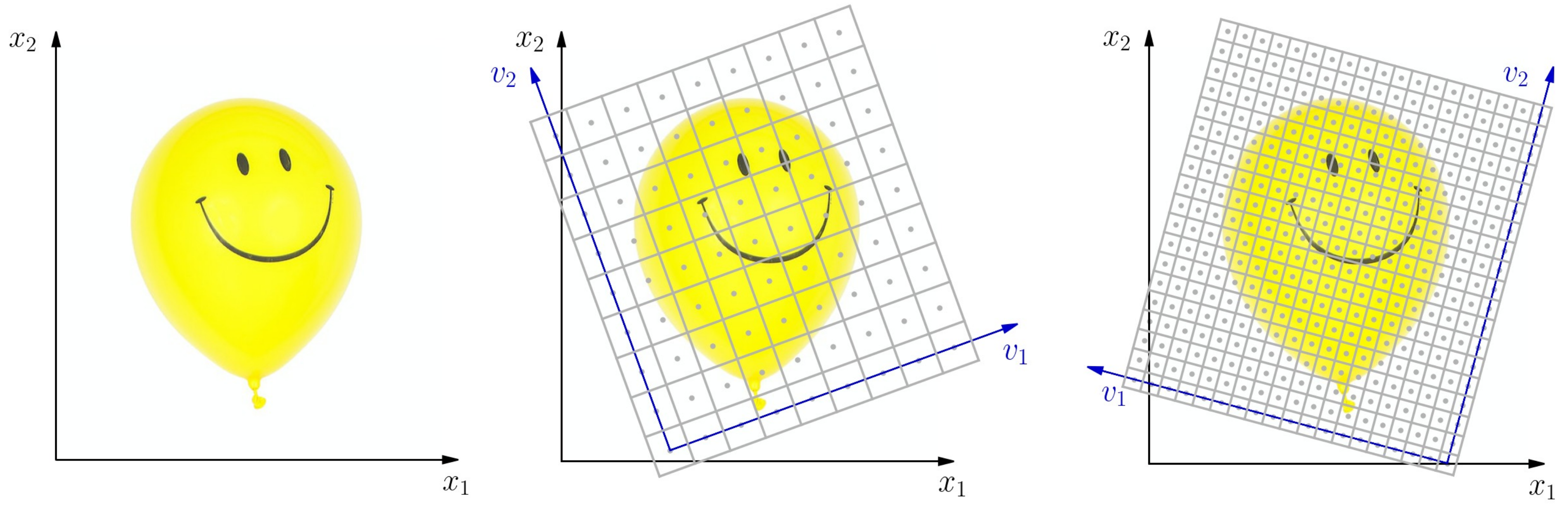
International Consortium for Brain Mapping



Coordinate systems

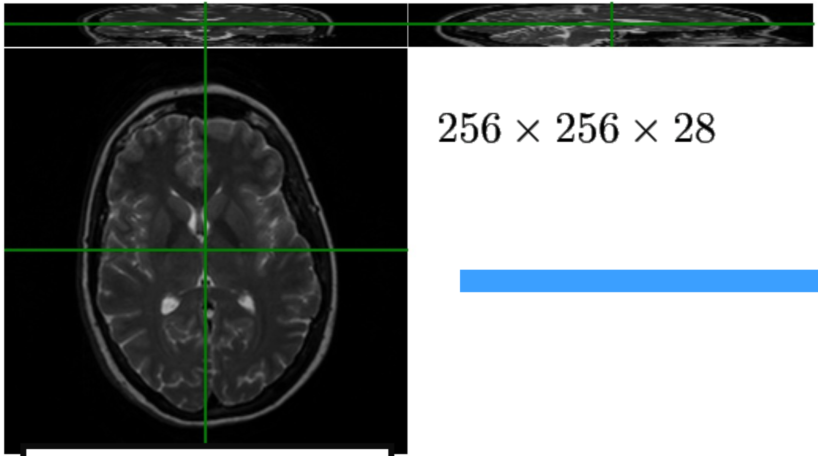
For each image, there are *__two__* coordinate systems:

- ✔ Voxel coordinates $\mathbf{v} = (v_1, v_2, v_3)^{\mathrm{T}}$ *(integer indices)*
- ✔ World coordinates $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ *(in mm)*



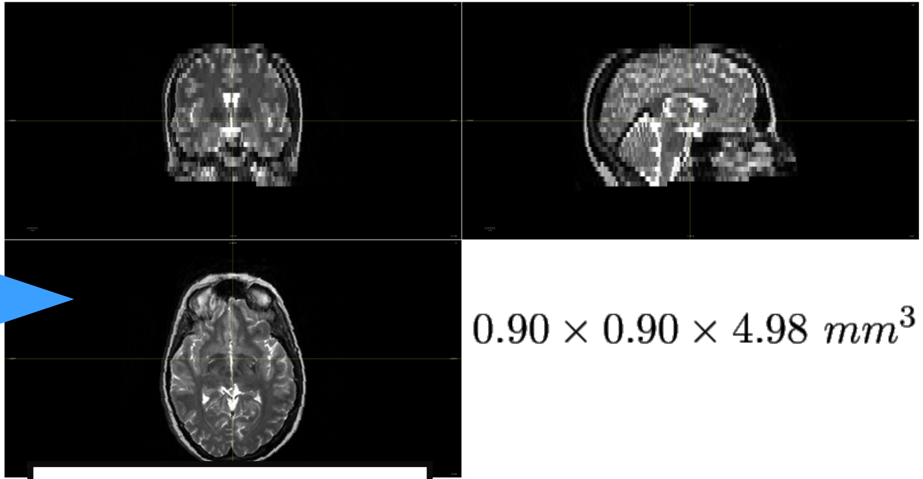
Conversion: $\mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{t}$

Coordinate systems



$256 \times 256 \times 28$

voxel indices $\mathbf{v}$



$0.90 \times 0.90 \times 4.98 \; mm^3$

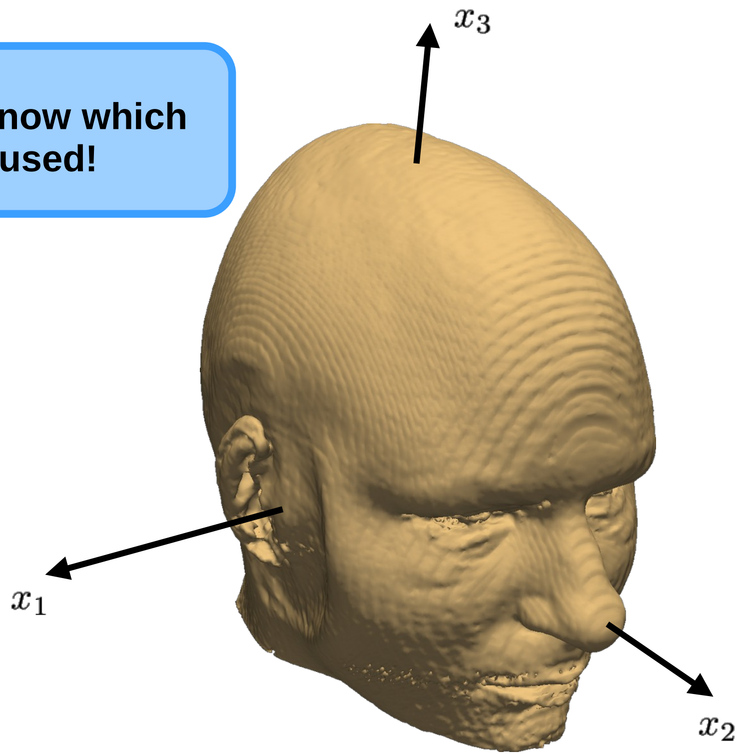
world positions $\mathbf{x}$

Conversion: $\mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{t}$

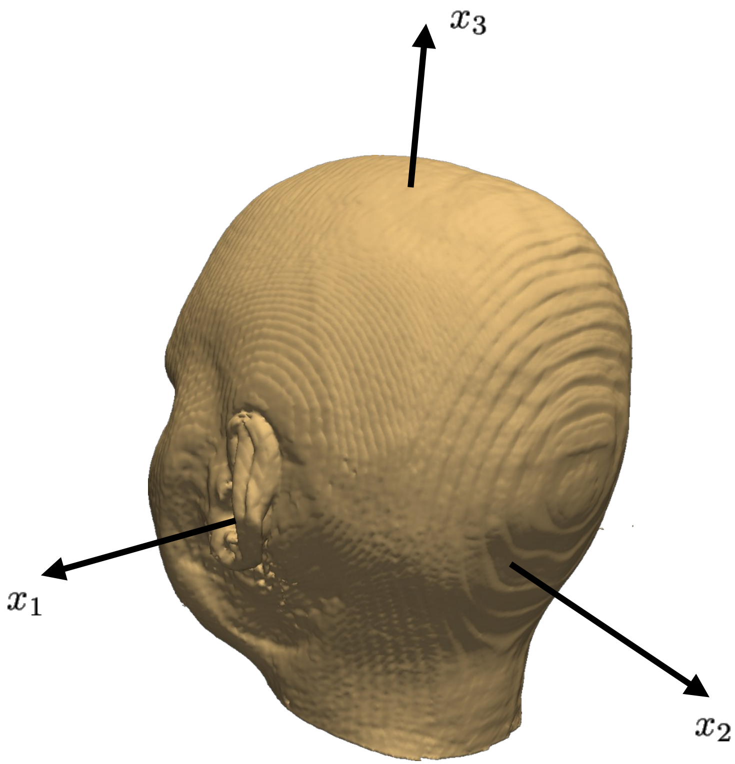
$$\mathbf{A} = \begin{pmatrix} -0.8923 & -0.0802 & -0.3732 \\ -0.0850 & 0.8921 & 0.3528 \\ -0.0612 & -0.0696 & 4.9512 \end{pmatrix} \qquad \mathbf{t} = \begin{pmatrix} 129.2834 \\ -98.7363 \\ -27.6911 \end{pmatrix}$$

World coordinates = convention

Important to know which convention is used!



Right – Anterior – Superior
(RAS)



Left – Posterior – Superior
(LPS)

Homogeneous coordinates

Vectors are augmented with a 1 at the end

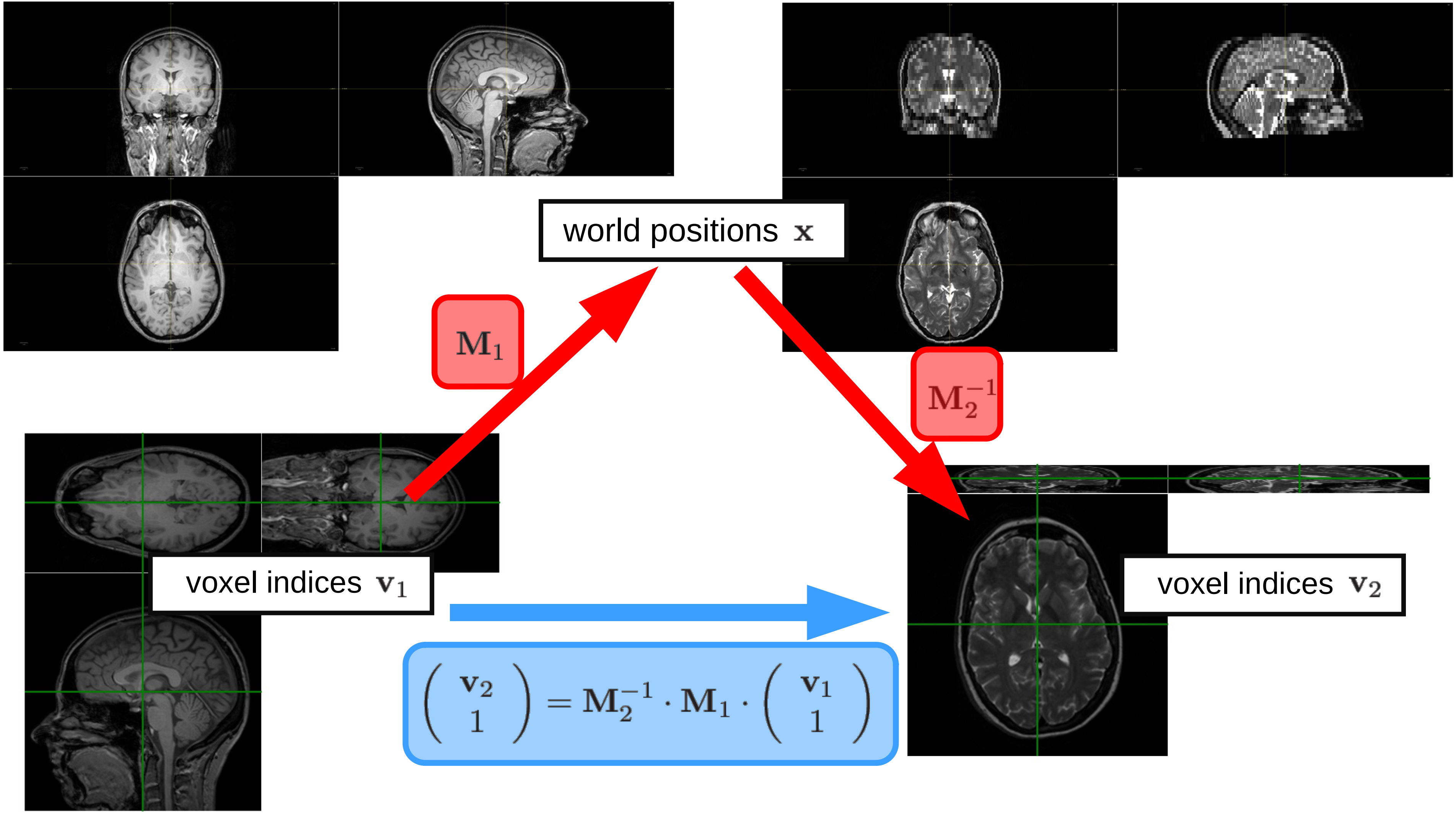
- ✔ **Idea:** Rewrite $\mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{t}$, i.e,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

as:

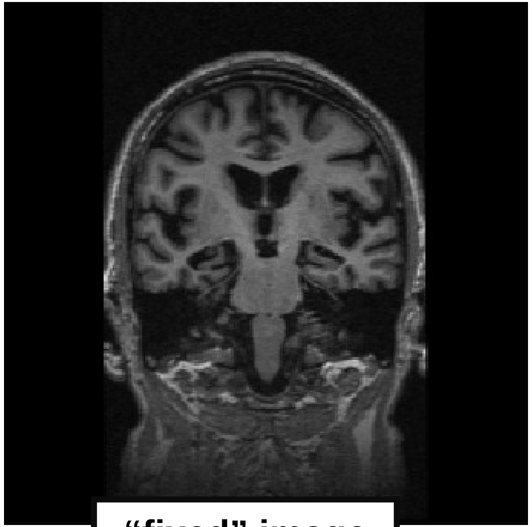
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & t_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & t_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 1 \end{pmatrix}$$

- ✔ **Benefit:** map voxel indices using only matrix multiplications



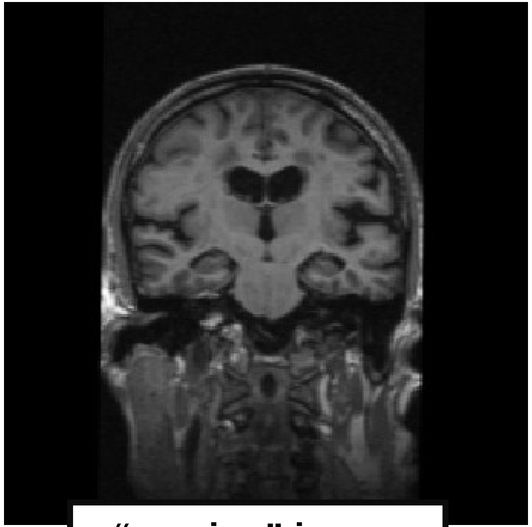
world positions $\mathbf{x}$
$\mathbf{M}_1$
$\mathbf{M}_2^{-1}$
voxel indices $\mathbf{v}_1$
voxel indices $\mathbf{v}_2$
$\begin{pmatrix} \mathbf{v}_2 \\ 1 \end{pmatrix} = \mathbf{M}_2^{-1} \cdot \mathbf{M}_1 \cdot \begin{pmatrix} \mathbf{v}_1 \\ 1 \end{pmatrix}$

Spatial transformations



“fixed” image

$$\mathbf{x} = (x_1, \ldots, x_D)^{\mathrm{T}}$$

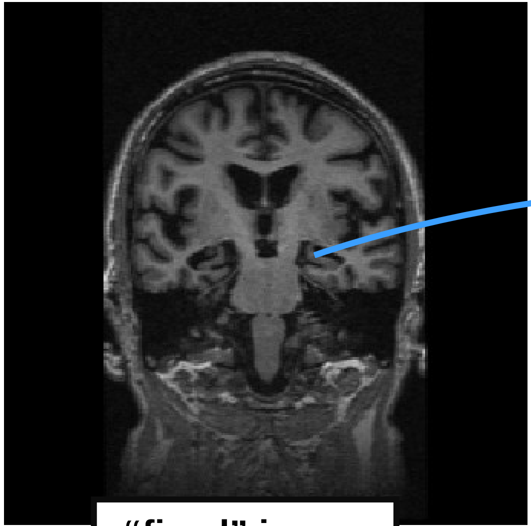


“moving” image

$$\mathbf{y} = (y_1, \ldots, y_D)^{\mathrm{T}}$$

Spatial transformations

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \begin{pmatrix} y_1(\mathbf{x}, \mathbf{w}) \\ \vdots \\ y_D(\mathbf{x}, \mathbf{w}) \end{pmatrix}$$



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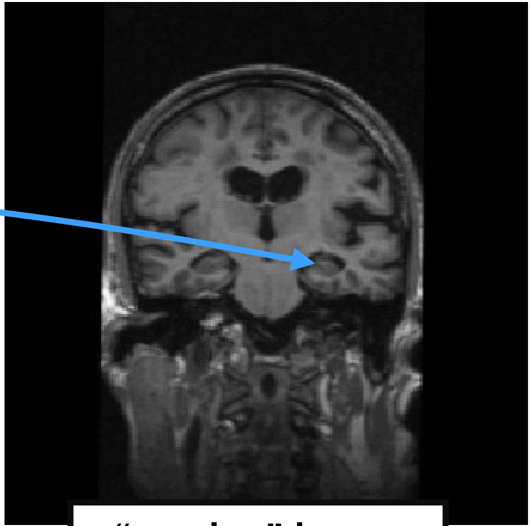

“moving” image

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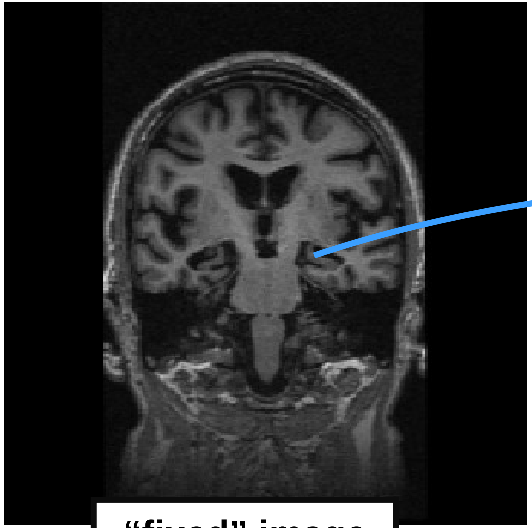
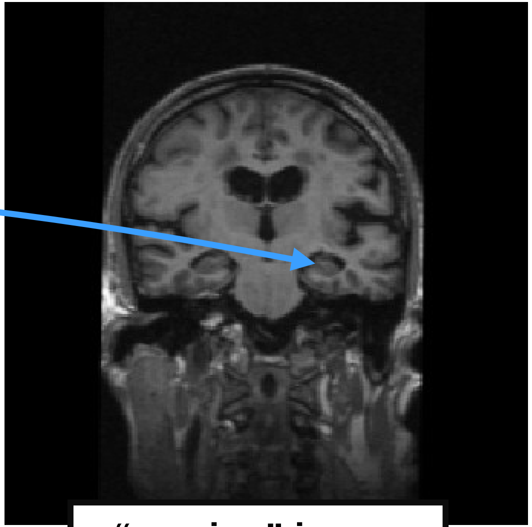


“fixed” image

$$\mathbf{x} = (x_1, \ldots, x_D)^{\mathrm{T}}$$

$y_d(\mathbf{x},\mathbf{w})$
controls how points $\mathbf{x}$ in the fixed image
move along the d-th direction in the moving image
as the parameters $\mathbf{w}$ are varied



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$$\mathbf{y} = (y_1, \ldots, y_D)^{\mathrm{T}}$$

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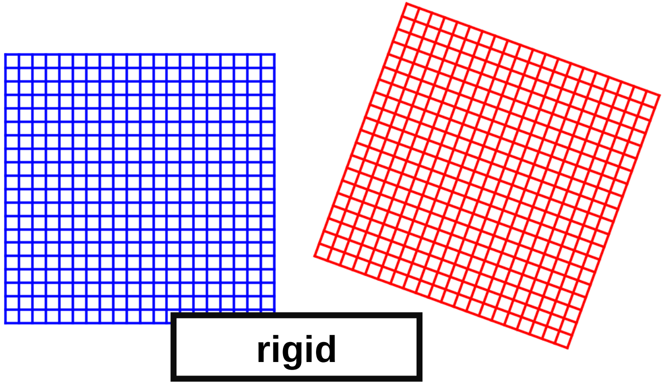
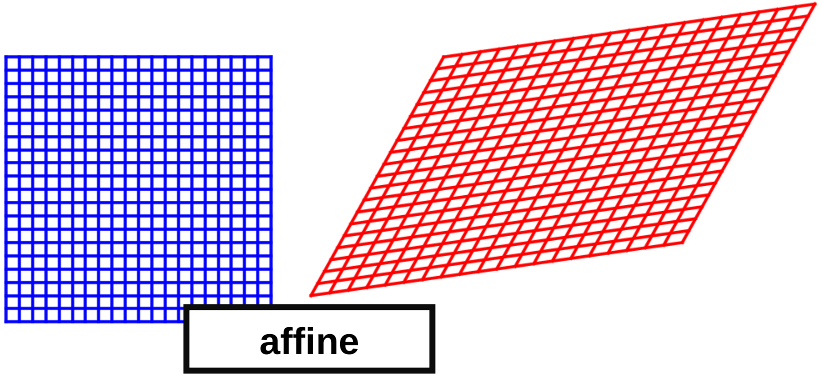
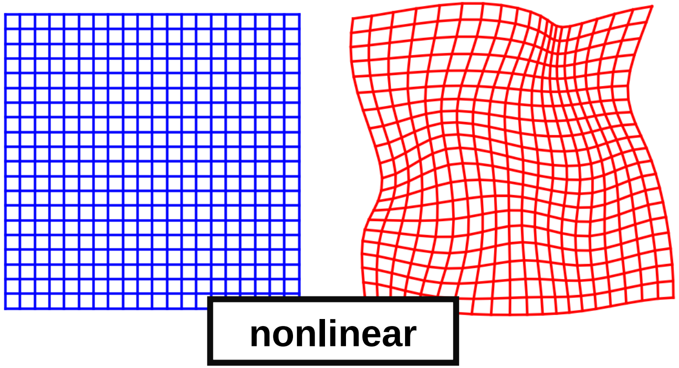
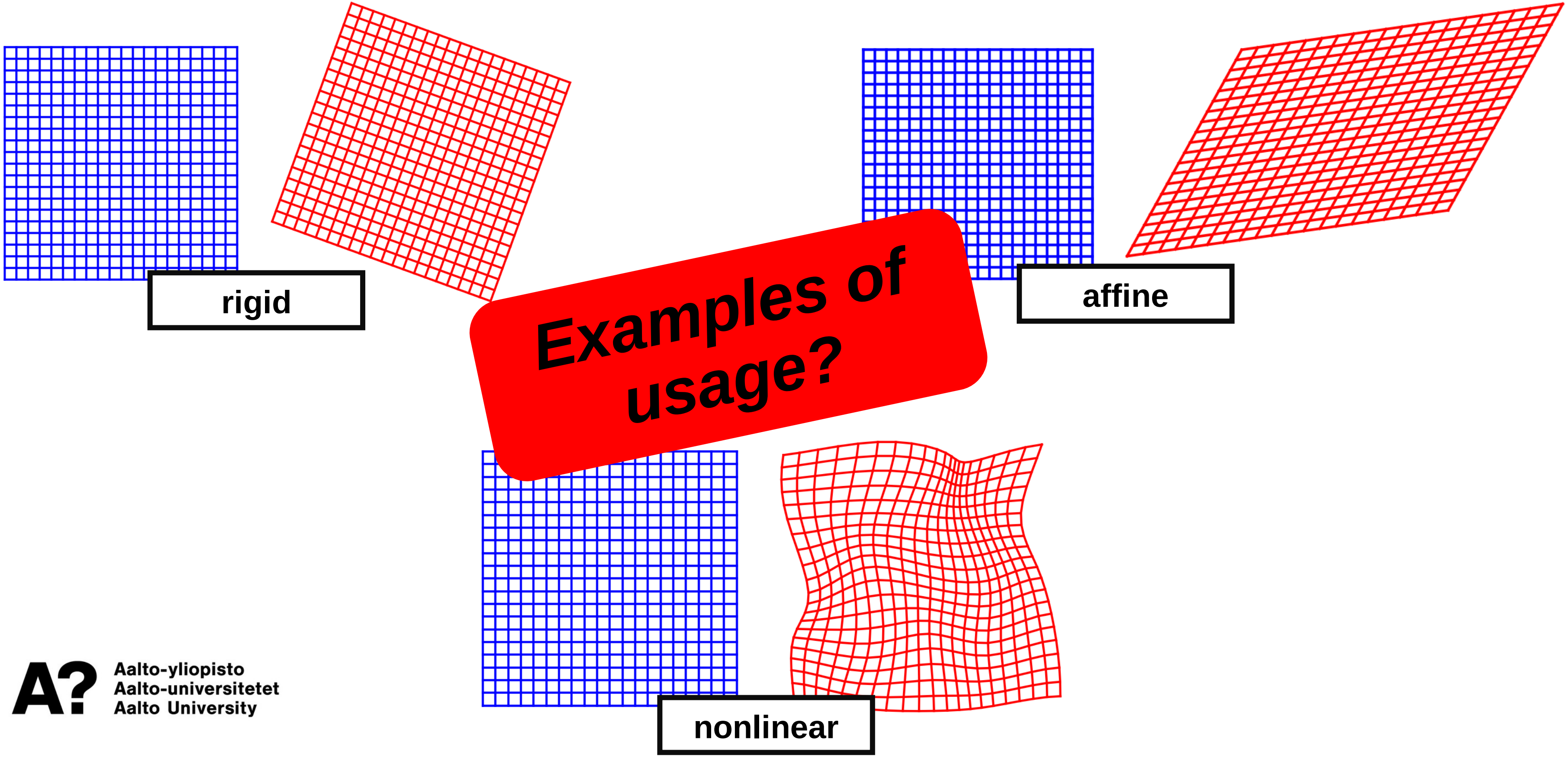
Spatial transformations







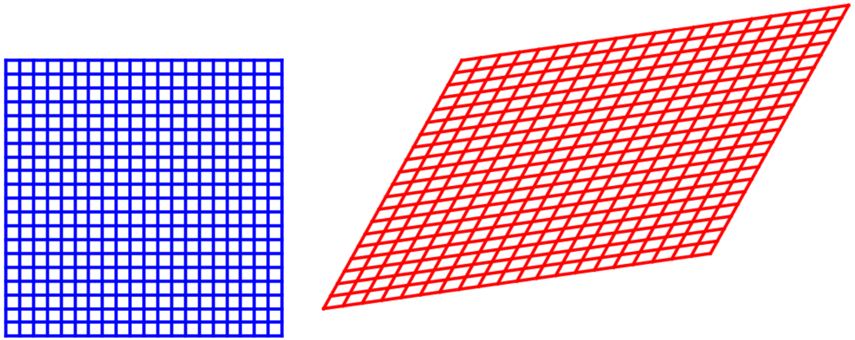
Spatial transformations



Affine transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad \text{and} \quad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$



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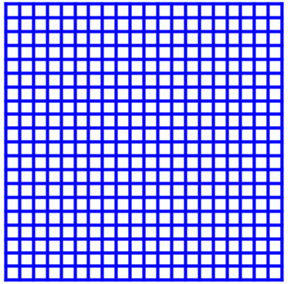
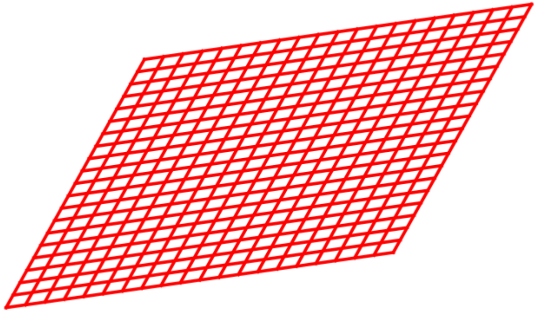




$y_d(\mathbf{x}, \mathbf{w})$
controls how points $\mathbf{x}$ in the fixed image
move along the d-th direction in the moving image
as the parameters $\mathbf{w}$ are varied

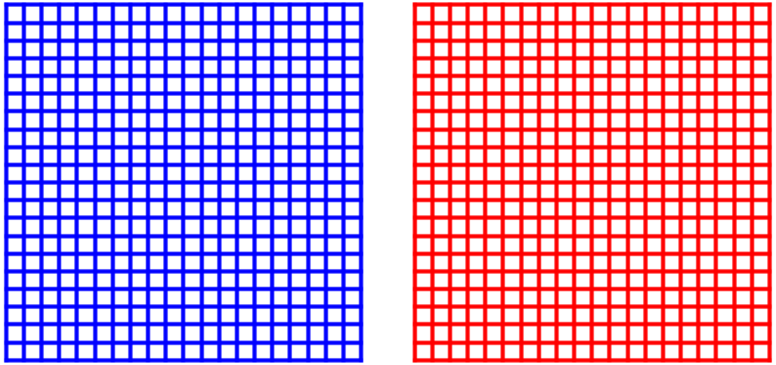
$$y_d(\mathbf{x}, \mathbf{w}_d) = t_d + a_{d,1}x_1 + \ldots + a_{d,D}x_D$$

$$\mathbf{w}_d = (t_d, a_{d,1}, \ldots, a_{d,D})^{\mathrm{T}}$$

$$\mathbf{w} = (\mathbf{w}_1^{\mathrm{T}}, \ldots, \mathbf{w}_D^{\mathrm{T}})^{\mathrm{T}}$$

Affine transformation

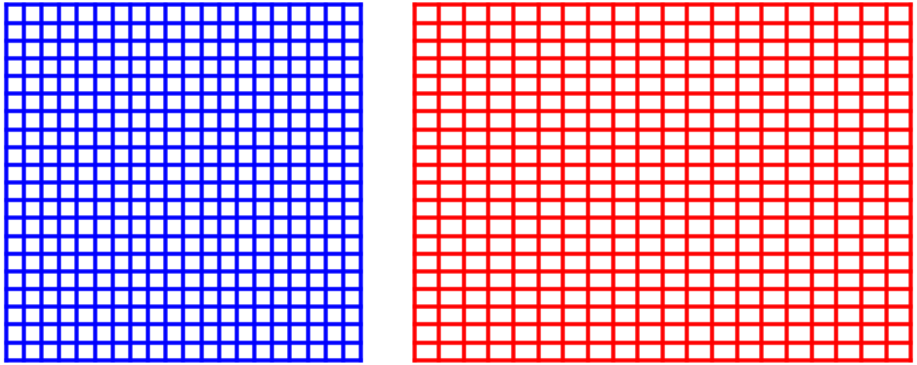
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 0 \end{pmatrix}$$

Affine transformation

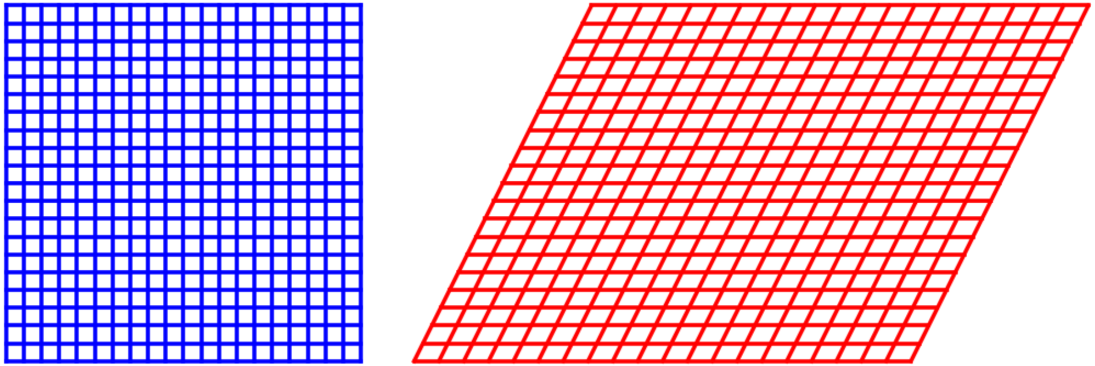
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.4 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 0 \end{pmatrix}$$

Affine transformation

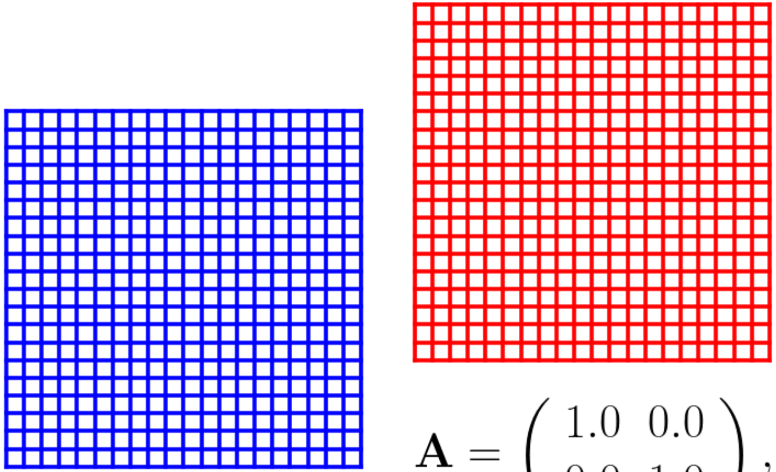
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.4 & 0.5 \\ 0.0 & 1.0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 0 \end{pmatrix}$$

Affine transformation

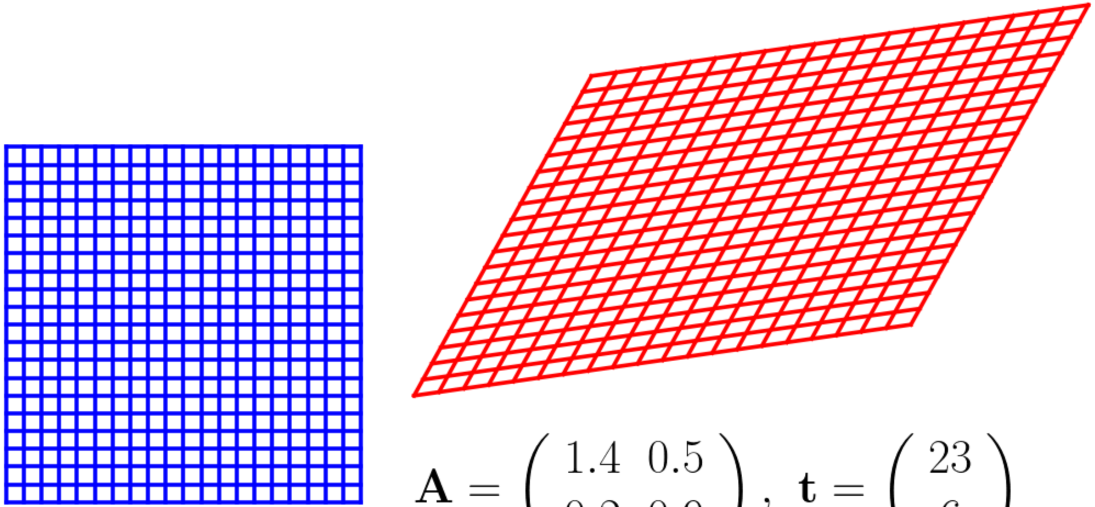
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 6 \end{pmatrix}$$

Affine transformation

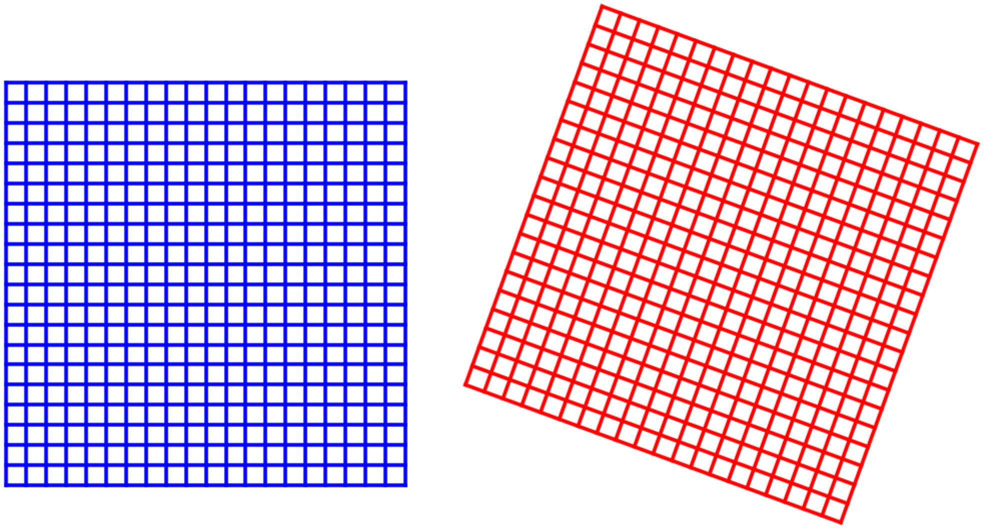
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.4 & 0.5 \\ 0.2 & 0.9 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 6 \end{pmatrix}$$

Rigid transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{R}\mathbf{x} + \mathbf{t}, \qquad \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I} \text{ and } \det(\mathbf{R}) = 1$$

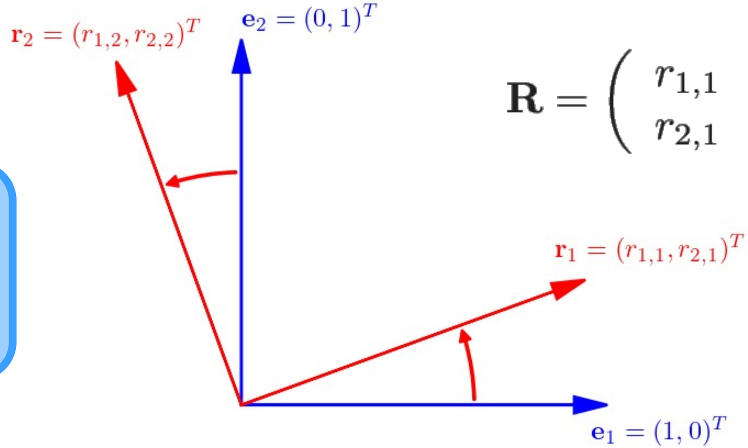


Rigid transformation

Task: why do $\mathbf{R}^T\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and $\det(\mathbf{R}) = 1$ impose a rotation?

$$\mathbf{R} = \begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}$$



Rigid transformation

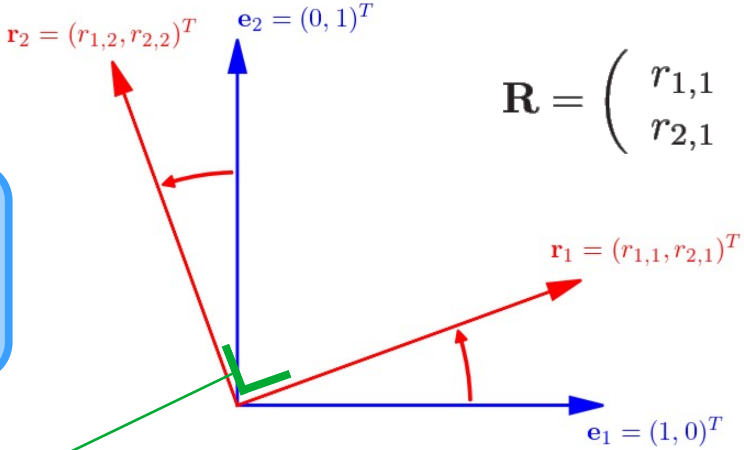
$$\|\mathbf{r}_1\| = \sqrt{\mathbf{r}_1^T \mathbf{r}_1} = 1$$

Task: why do $\mathbf{R}^T\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and $\det(\mathbf{R}) = 1$ impose a rotation?

$$\mathbf{r}_1^T \mathbf{r}_2 = 0$$

$$\|\mathbf{r}_2\| = \sqrt{\mathbf{r}_2^T \mathbf{r}_2} = 1$$



$$\mathbf{R} = \begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}$$

Rigid transformation

Task: why do $\mathbf{R}^T\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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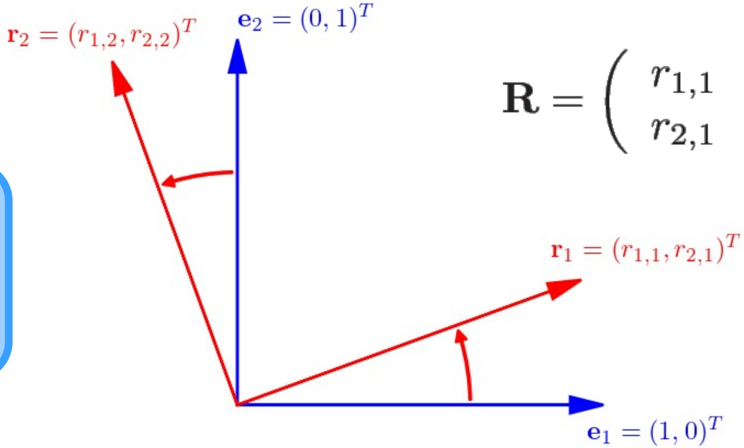
?



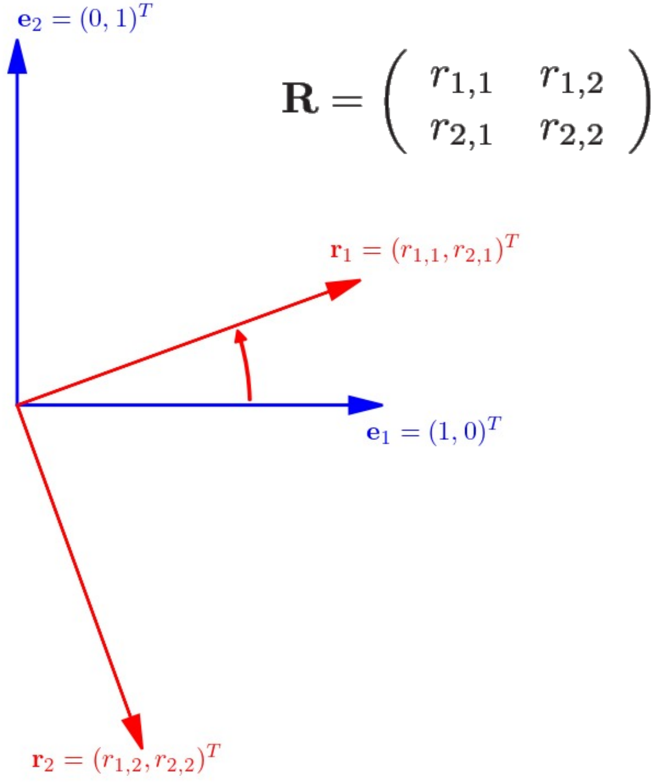
$$\mathbf{R} = \begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}$$

Rigid transformation

Task: why do $\mathbf{R}^T\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and $\det(\mathbf{R}) = 1$ impose a rotation?

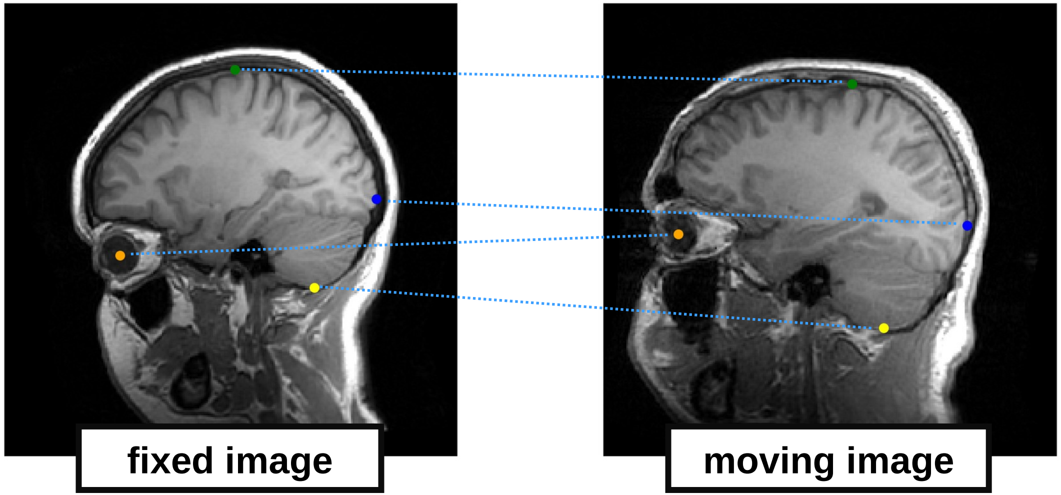
?



Landmark-based registration

✔ Manually annotate N corresponding points in two images:

$\{\mathbf{x}_n\}_{n=1}^N$



$\{\mathbf{y}_n\}_{n=1}^N$

✔ Register the images by minimizing the distance between matching point pairs:

$$E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{y}(\mathbf{x}_n, \mathbf{w})\|^2$$

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$

Task 1: if $\mathbf{A} = \mathbf{I}$, what is $\mathbf{t}$?

<u>Hint:</u> remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^{D}(a_d - b_d)^2$

Task 2: in the general case, what are $\mathbf{A}$ and $\mathbf{t}$?

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2 = \sum_{n=1}^{N}\sum_{d}^{D}(y_{n,d} - t_d - \sum_{d'=1}^{D} x_{n,d'} a_{d,d'})^2$

Task 1: if $\mathbf{A} = \mathbf{I}$, what is $\mathbf{t}$?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^{D}(a_d - b_d)^2$

Task 2: in the general case, what are $\mathbf{A}$ and $\mathbf{t}$?

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$ $= \sum_{n=1}^{N} \sum_{d}^{D} (y_{n,d} - t_d - \sum_{d'=1}^{D} x_{n,d'} a_{d,d'})^2$

Task 1: if $\mathbf{A} = \mathbf{I}$, what is $\mathbf{t}$?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^{D} (a_d - b_d)^2$

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Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$

$$= \sum_{n=1}^{N} \sum_{d}^{D} (y_{n,d} - t_d - \sum_{d'=1}^{D} x_{n,d'} a_{d,d'})^2$$

$$= \sum_{d}^{D} \sum_{n=1}^{N} (y_{n,d} - t_d - \sum_{d'=1}^{D} x_{n,d'} a_{d,d'})^2$$

Task 1: if $\mathbf{A} = \mathbf{I}$, what is $\mathbf{t}$?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^{D} (a_d - b_d)^2$

Task 2: in the general case, what are $\mathbf{A}$ and $\mathbf{t}$?

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$

$$= \sum_{n=1}^{N} \sum_{d}^{D} (y_{n,d} - t_d - \sum_{d'=1}^{D} x_{n,d'} a_{d,d'})^2$$

$$= \sum_{d}^{D} \sum_{n=1}^{N} (y_{n,d} - t_d - \sum_{d'=1}^{D} x_{n,d'} a_{d,d'})^2$$

$$\begin{pmatrix} t_d \\ a_{d,1} \\ \vdots \\ a_{d,D} \end{pmatrix} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}} \begin{pmatrix} y_{1,d} \\ \vdots \\ y_{N,d} \end{pmatrix}$$

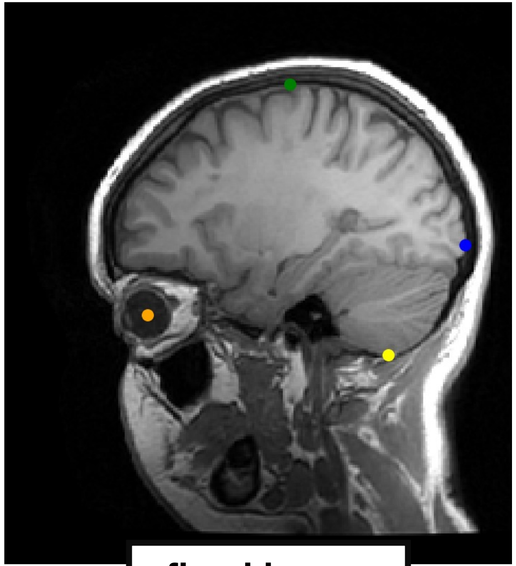
where $\mathbf{X} = \begin{pmatrix} 1 & x_{1,1} & \cdots & x_{1,D} \\ 1 & x_{2,1} & \cdots & x_{2,D} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N,1} & \cdots & x_{N,D} \end{pmatrix}$

Task 1: if $\mathbf{A} = \mathbf{I}$, what is $\mathbf{t}$?

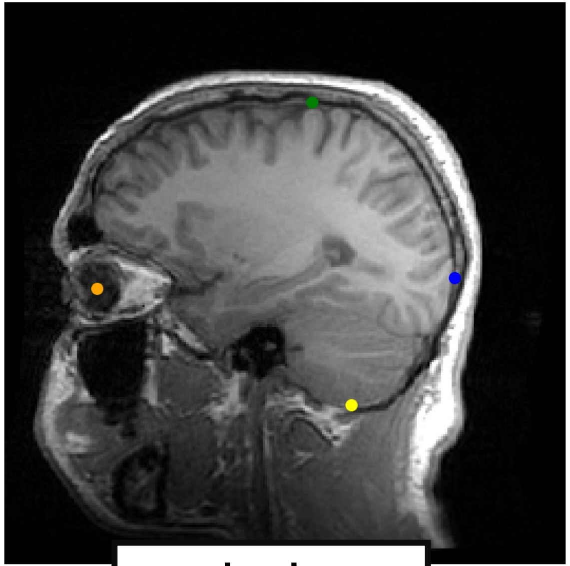
Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^{D} (a_d - b_d)^2$

Task 2: in the general case, what are $\mathbf{A}$ and $\mathbf{t}$?

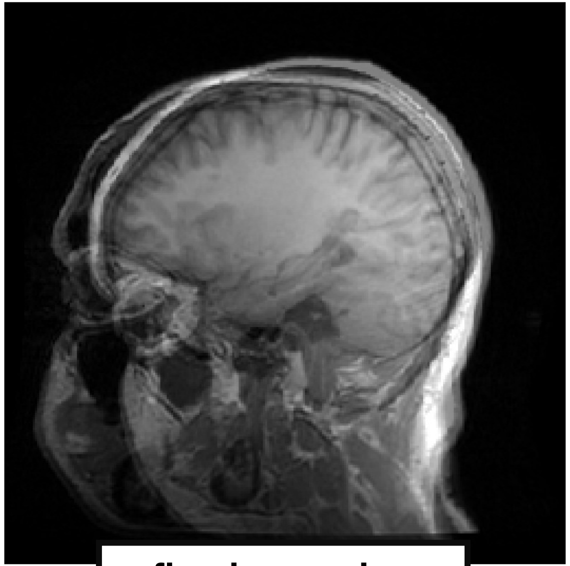
Landmark-based registration



fixed image



moving image

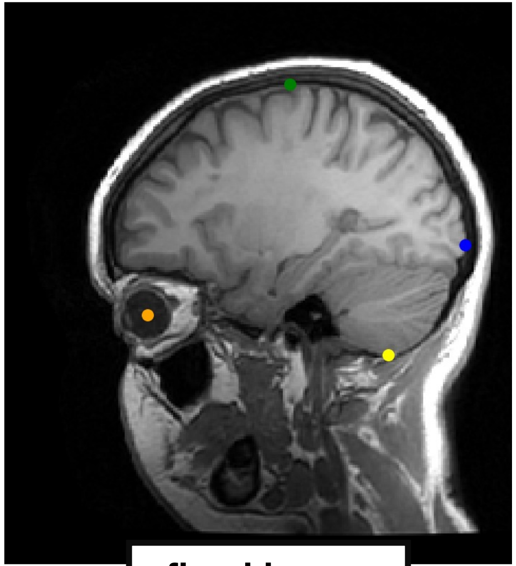


fixed + moving

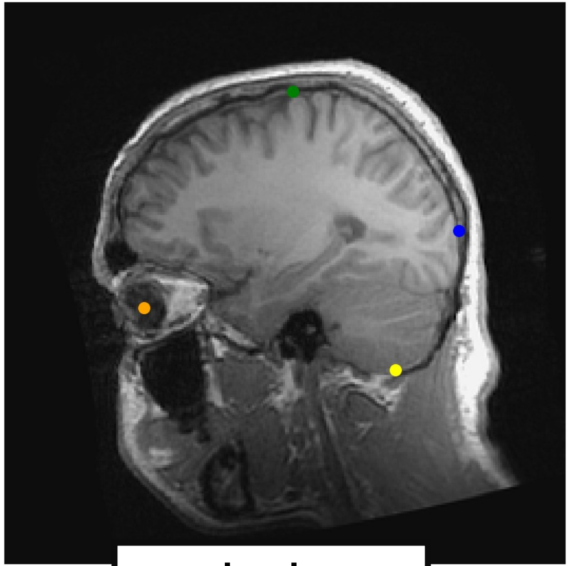
Before registration

Aalto-yliopisto
Aalto-universitetet
Aalto University

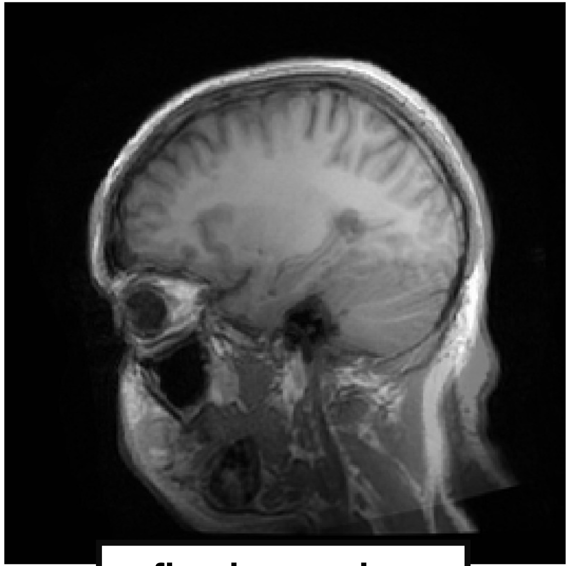
Landmark-based registration



fixed image



moving image

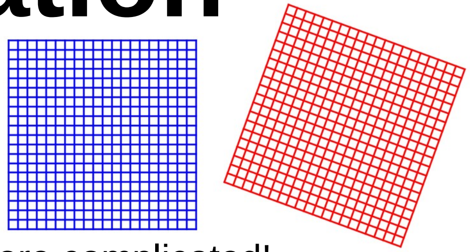


fixed + moving

After registration

Aalto-yliopisto
Aalto-universitetet
Aalto University

Landmark-based registration



Applied to *rigid* registration: $E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{R}\mathbf{x}_n - \mathbf{t}\|^2$

- ✔ Constraints $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ and $\det(\mathbf{R}) = 1$ make the math much more complicated!
- ✔ Solution:

$$\mathbf{R} = \mathbf{V}\mathbf{U}^{\mathrm{T}}, \quad \sum_{n=1}^{N} \tilde{\mathbf{x}}_n \tilde{\mathbf{y}}_n^{\mathrm{T}} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}, \quad \mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}, \quad \mathbf{V}^{\mathrm{T}}\mathbf{V} = \mathbf{I}$$

$$\mathbf{t} = \bar{\mathbf{y}} - \mathbf{R}\bar{\mathbf{x}},$$

where $\tilde{\mathbf{x}}_n = \mathbf{x}_n - \bar{\mathbf{x}}$ and $\tilde{\mathbf{y}}_n = \mathbf{y}_n - \bar{\mathbf{y}}$

with $\bar{\mathbf{x}} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{x}_n$ and $\bar{\mathbf{y}} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{y}_n$

(“flip” a column of $\mathbf{R}$ if $\det(\mathbf{R}) = -1$)