Princess Sumaya University for Technology
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# Curve Parametrization

# *Abstract*

***Curve parametrization is a fundamental concept in calculus, enabling the description of curves as functions of a parameter, typically time. This project explores the geometric and practical aspects of curve parametrization, focusing on the representation of curves in both two-dimensional ($\mathbb{R}^2$) and three-dimensional ($\mathbb{R}^3$) spaces. Through examples and code implementations, we analyze the velocity vector, arc length, and challenges in higher dimensions. The project highlights the importance of parametrization in understanding motion and geometry, providing insights into its applications in mathematics and beyond.***

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# 1- Introduction

Curves are fundamental objects in mathematics, playing a crucial role in various fields such as physics, engineering, and computer graphics. Understanding how to describe and analyze curves is essential for modeling real-world phenomena, from the trajectory of a projectile to the design of complex structures. In calculus, the concept of curve parametrization provides a powerful tool for studying the motion and geometry of curves. By representing a curve as a function of a parameter, typically time, we can analyze its properties, such as velocity, direction, and length, in a systematic way.

This project delves into the mathematical framework of curve parametrization, exploring its geometric interpretations and practical applications. We begin by defining curves and their types, followed by an in-depth analysis of how parametrization works in both two-dimensional ($\mathbb{R}^2$) and three-dimensional ($\mathbb{R}^3$) spaces. Through examples and code implementations, we demonstrate how to compute derivatives, visualize curves, and calculate arc lengths. Furthermore, we address the challenges of parametrizing curves in higher dimensions and discuss the implications of different parametrizations for the same curve. By the end of this exploration, we aim to provide a comprehensive understanding of curve theory and its significance in calculus.

# 2- Definition of curves

- Generally: A curve is a shape or a line which is smoothly drawn in a plane having a bent or turns in it.

- Mathematically: The path traced by a moving point. It can be described by a function that assigns a unique point in a coordinate system to each value of a parameter.

- Types of curves:

- Simple Curves:

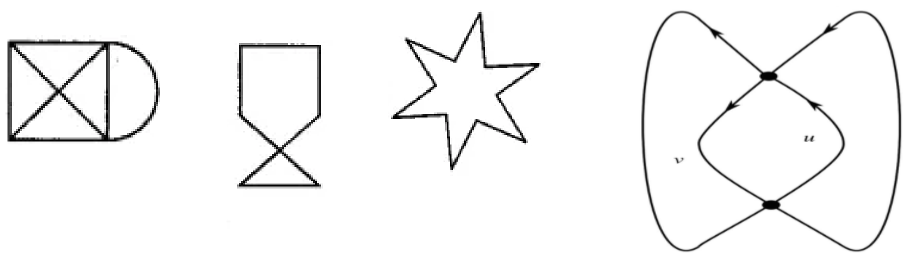
A curve that does not intersect (cross) itself while changing its direction except possibly at its end points.

Examples: 

- Closed Curves:

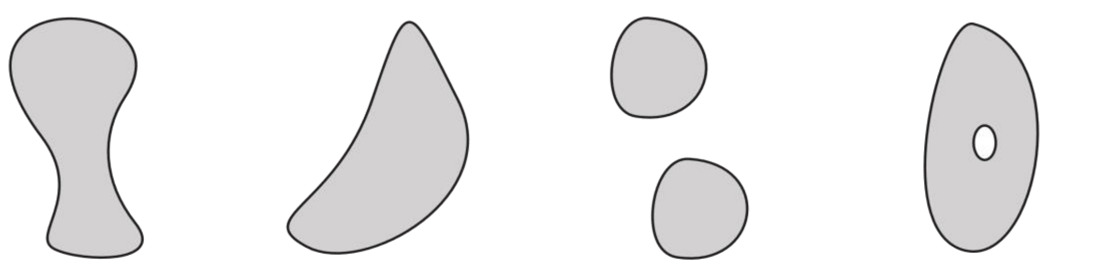
A curve that forms a closed loop with no end points, which means tracing this curve will return us to the starting point.
Note: closed curves that do not intersects themselves are simple curves.

Examples:




- Simply Connected Domain:

It is the region or area in a closed curve that doesn't contain any holes, or isolated points.



*simply connected* *simply connected* *not simply connected* *not simply connected*

## 3- Curve Parametrization

In curve parametrization, $\boldsymbol{r}(t)$ represents the **position vector** of a particle at time t$t$. The derivative $\boldsymbol{r}'(t)$ is the **velocity vector**, which describes how the particle's position changes over time.

- <u>Geometrically, $\boldsymbol{r}'(t)$ provides two key pieces of information:</u>

1. **Direction of Motion**: The direction of $\boldsymbol{r}'(t)$ indicates the instantaneous direction in which the particle is moving at time $t$. It is tangent to the curve $\boldsymbol{r}(t)$ at the point corresponding to $t$.
2. **Speed**: The magnitude of $\boldsymbol{r}'(t)$ represents the **speed** of the particle at time $t$. A larger magnitude corresponds to faster motion.

For example, consider the curve parametrized by $r(t) = \langle t, t^2 \rangle \boldsymbol{r}(t) = \langle t, t^2 \rangle$, which describes the coordinates of a point on the curve at any time $t$. By differentiating **r**($t$) with respect to t$t$, we obtain the velocity vector:

$$r'(t) = \langle 1{,}2t \rangle . \boldsymbol{r}'(t) = \langle 1{,}2t \rangle .$$

This vector represents both the **direction** and **speed** of the particle at time $t$.

Let's consider $t = 2$:

$$r'(2) = \langle 1{,}4 \rangle . \boldsymbol{r}'(2) = \langle 1{,}4 \rangle .$$

- The **magnitude** of $\boldsymbol{r}'(2)$ is $\sqrt{17}$, which represents the speed of the particle at $t = 2$.
- The **direction** of $\boldsymbol{r}'(2)$ is the direction in which the particle is moving at $t = 2$. The vector $\langle 1{,}4 \rangle \langle 1{,}4 \rangle$ points along the tangent to the curve at the point where the particle is located at $t = 2$.

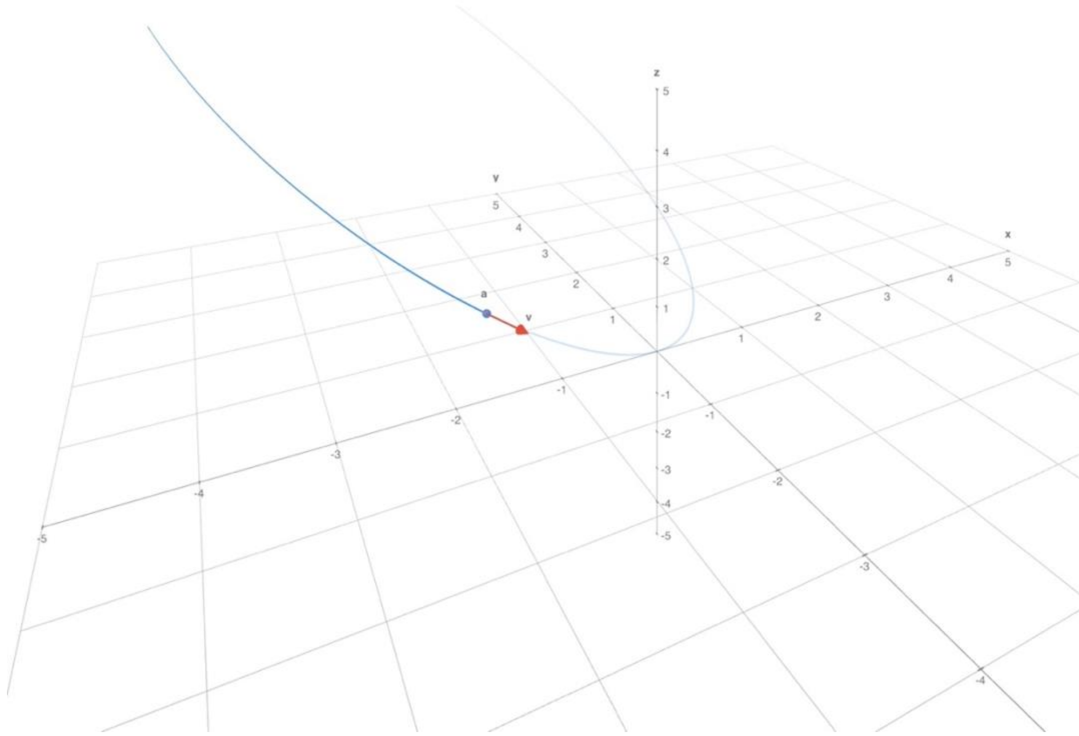



*Figure 1: Parametric Curve of r(t)*

- <u>Code Implementation in C++:</u>

Using C++ coding language for more explanation, here is a simple code that explain the idea geometrically by plotting a parametrization and showing how the derivative vector moves with the curve parametrized by the function: $r(t) =< t, t^2 >$ , when $1 \leq t \leq 5$.

This code must give us the output with:

1- The Parametric and Derivative Vector equations for the example above.
2- The Parametric Point r(t) and Derivative Vector r′(t) after substituting t from 1-5.

```cpp
#include <iostream>
#include <cmath>
using namespace std;

// Firstly we define the parametrization function r(t)
struct Point {
    float x, y;
};

Point parametrization(float t) {
    float x = t;           // this is the parametric equation for
x(t)
    float y = t * t;       // this is the parametric equation for
y(t)
    return {x, y};
}

// Now we define the derivative function r'(t)
Point derivative(float t) {
    float dx = 1.0;        // this is the derivative of x(t) with
respect to t
    float dy = 2 * t;      // this is the derivative of y(t) with
respect to t
    return {dx, dy};
}

int main() {
    float t;

     cout << "Parametric Equation r(t): \n x = t , y = t^2 "<<
endl;
     cout << "Derivative Vector Equation r'(t):\n dx/dt = 1 ,
dy/dt = 2t" << endl;
     cout<<"\n";

    for (float t = 1.0; t <= 5.0; t += 1.0) {
        Point point = parametrization(t);
        Point direction = derivative(t);

        cout<<"----------------------------\n";
        cout<< "t = "<< t <<" "<< endl;
        cout << "Parametric Point:  r(" << t << ") = " "<" <<
point.x << "," << point.y << ">" << endl;
        cout << "Derivative Vector: r'(" << t << ") = " "<" <<
direction.x << "," << direction.y << ">" << endl;
cout<<"\n";
    }

    return 0;
}
```

The output of the code:

```
Parametric Equation r(t):
 x = t , y = t^2
Derivative Vector Equation r'(t):
 dx/dt = 1 , dy/dt = 2t

----------------------------
t = 1
Parametric Point:  r(1) = <1,1>
Derivative Vector: r'(1) = <1,2>

----------------------------
t = 2
Parametric Point:  r(2) = <2,4>
Derivative Vector: r'(2) = <1,4>

----------------------------
t = 3
Parametric Point:  r(3) = <3,9>
Derivative Vector: r'(3) = <1,6>

----------------------------
t = 4
Parametric Point:  r(4) = <4,16>
Derivative Vector: r'(4) = <1,8>

----------------------------
t = 5
Parametric Point:  r(5) = <5,25>
Derivative Vector: r'(5) = <1,10>



...Program finished with exit code 0
Press ENTER to exit console.
```

## 4- Variability of Parametrizations

A curve can have infinitely many parametrizations. Here are the possible differences between the different parametrizations for the same curve:

### 1- Direction and Orientation:

Depending on the choice of parameterization, the curve may be traced in a clockwise or counterclockwise direction. For instance, the unit circle can be parametrized using either trigonometric functions , leading to counterclockwise motion, or by using their negatives, resulting in a clockwise motion.

### 2- Speed:

Parametrizations can cover the same curve at different speeds. For instance, a parameter might increase rapidly in one parametrization and slowly in another.

### 3- Starting Point and Parameter Domain:

Different parametrizations may start at different points on the curve. The parameter domain can vary, affecting how much of the curve is covered.

### 4- Piecewise Function:

A curve might be represented by different parametric equations in different intervals. This allows flexibility in describing complex curves with distinct segments.

# 5- Applications of Curve Parametrization

- <u>Perpendicularity of radius and tangent using the circle's parametric equations</u>:

$$\boldsymbol{x = r * cos(t)} \;\&\; \boldsymbol{y = r * sin(t)}$$

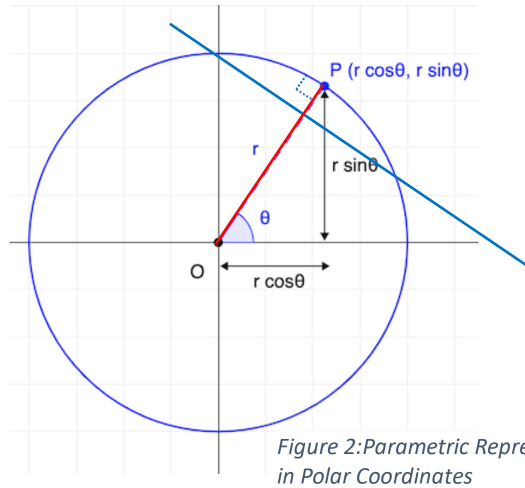



*Figure 2:Parametric Representation of a Circle in Polar Coordinates*

Assume the circle has a radius of r that is centred at the origin of the plane and where $0 \leq t \leq 2\pi$.
Now to conclude this assumption, our goal is finding that the dot product between the tangent vector and the radius vector equals to zero.

So now for finding the equation of the tangent vector:

$$T(t) \;= < \frac{d(r * cos(t))}{dt}, \frac{d(r * sin(t))}{dt} > = < -r * sin(t), r * cos(t) >$$

And now for finding the equation of the radius vector:

$$R(t) \;= < r * cos(t), r * sin(t) >$$

Now finally we will find the dot product of both the tangent and the radius vectors :

$$T(t) \cdot R(t) \;= < -r * sin(t), r * cos(t) > \cdot < r * cos(t), r * sin(t) > = \; 0$$

The result zero for the dot product results that the radius of the circle is always perpendicular to its tangent.

- <u>Arc length of curve $y = \; f(x)$</u>:

The formula for finding the length of a smooth curve in terms of cartesian coordinates involves the use of definite integrals.

Given a cure defined by the function y= f(x) on the interval [a, b], the length of the curve is given by the integral:

$$\boldsymbol{L=\int_a^b \sqrt{1+(f'(x))^2}\,dx}$$

By the Pythagoras theorem, the length of the segment is:

$$\sqrt{(\Delta x)^2+(\Delta y)^2}$$

$$f(x_i)-f(x_i-1)=f'(x_i)\big(x_i-(x_i-1)\big)$$

$$\Delta y_i=f'(x_i)(\Delta x)$$

$$\sqrt{(\Delta x)^2+(f'(x_i)(\Delta x))^2}$$

$$\Delta x\sqrt{1+\left(f'(x_i)\right)^2}$$

$$L=\lim_{n\to\infty}\sqrt{1+\left(f'(x_i)\right)^2}\,\Delta x$$

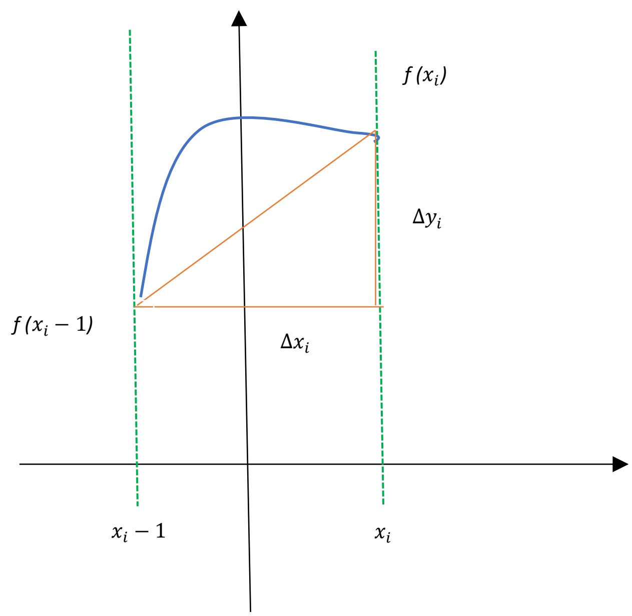


*Figure 3: Arc length of a curve*

Using the definition of the definite integral:

$$L=\int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{n=1}^{\infty} f(x_i)\,\Delta x$$

$$L=\int_a^b \sqrt{1+(f'(x_i))^2}\,dx=\int_a^b \sqrt{1+(\frac{dy}{dx})^2}\,dx$$

We can also derive a formula for x=h(y) on [c, d]:

$$L=\int_c^d \sqrt{1+\left(h'(y_i)\right)^2}\,dy=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

## 6- Challenges in $\mathbb{R}^3$

A curve can be parametrized by a series of functions of a single variable, each function represents one of the coordinates in the space we are talking about. In $R^2$ we describe a curve using one parameter (t) which represents the movement along the curve in one direction, the x-axis or the y-axis. While in $R^3$ we have to use two parameters (u,v) to represent the the movement in two directions along the surface, the x-axis with the y-axis or the x-axis with the z-axis or the y-axis with the z-axis. Curve parametrization becomes a bit harder in $R^3$ than in $R^2$ since there is a third dimension to consider, which means that we will have three functions to represent the three coordinates instead of two. Each time we increase the dimension there will be more possibilities for the shape and orientation of the curve in space, which makes the parametrization more complicated.

- Intersection of Surfaces:

Curves in $R^3$ are usually given as the intersection of surfaces, to show this we can have two functions $f(x, y, z) = 0$ and $g(x, y, z) = 0$; which represent two intersecting curves in $R^3$ we can find a parametrization for the intersection curve.

Firstly, lets solve for two variables in terms of the third, as solving for x,y in terms of z;

Secondly, we treat the two variables as parameters and explain them in terms of a third parameter which will result in giving you the parametrization of the curve.

- Difficulties in Parametrization:

1- If the surfaces of $f(x, y, z) = 0$ and $g(x, y, z) = 0$ do not intersect, then finding the parametrization is not possible because there will be no solution.
2- If there are multiple intersection curves or points, it will be difficult to find the parametrization curve.
3- Non-smooth and singular points both may result in difficulties too as some intersections may involve singular points where the parametrization isn't well defined or the curve not being smooth enough which makes the process of finding the parametrization harder.

An example to show one of the difficulties in finding the parametrization of a curve in $R^3$:

Consider the surfaces defined by

$\boldsymbol{f(x, y, z) = x^2 + y^2 + z^2 - 1}$; which is the equation of a sphere with radius 1.

$\boldsymbol{g}\ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{z} - \mathbf{2};$ which is a plane parallel to the xy-plane at z = 2.

1- <u>Express one variable in terms of the others:</u>
   Solve for z in the equation z - 2 = 0, resulting in z = 2.
2- <u>Substitute into the other equation:</u>
   Substitute z= 2 into the equation $x^2 + y^2 + z^2 - 1 = 0$ to get $x^2 + y^2 + 3 = 0.$
3- <u>Parametrize the remaining variables:</u>
   Attempt to parametrize the curve defined by $x^2 + y^2 + 3 = 0$. However, this equation has no real solutions, indicating that there is no intersection.

The difficulty here is that the surfaces given by the two equations do not intersect in $R^3$. Then finding the parametrization isn't possible.

## 7- Conclusion

In this project, we explored curve parametrization, a key concept in calculus for describing and analyzing the motion and geometry of curves. We began by defining curves and their types, such as simple and closed curves, and then examined how parametric equations represent curves mathematically. Through examples and code implementations, we demonstrated how to compute derivatives, visualize curves, and calculate arc lengths, deepening our understanding of their geometric and physical interpretations.

We also addressed the challenges of parametrizing curves in three-dimensional space, where the added complexity requires careful analysis, especially when dealing with the intersection of surfaces. This project highlights the importance of curve parametrization in fields like physics, engineering, and computer graphics, where modeling and analyzing curves is essential. The foundational knowledge gained here will serve as a steppingstone for further exploration in advanced calculus and related disciplines.